Second Kind Integral Equations on the Real Line: Solvability and Numerical Analysis in Weighted Spaces

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Abstract

The exact and numerical solution of integral equations taking the form $\lambda x(s) - \int_{-\infty}^{\infty} v(s,t)x(t)\,dt = y(s)$ in certain weighted subspaces X_w of the space $X := BC(\mathbb{R})$ (of bounded and continuous functions over $\mathbb{R}$) is studied. Here, X_w denotes the weighted space of all functions $x \in X$ satisfying $|w(s)x(s)| = O(1)$ as $|s| \to \infty$, for some weight function $w \geq 1$. The kernel v is assumed to satisfy the simple condition $|v(s,t)| \leq |\kappa(s-t)|$, for some $\kappa \in L^1(\mathbb{R})$.

Conditions on v and w are obtained, which ensure that the integral operator K in above equation is bounded on X and X_w. These conditions are then strengthened to imply the equivalence of the spectrum (and essential spectrum) of K on X and X_w as well as several other statements about the solvability of the above integral equation.

Similar boundedness and spectral results are shown for the operators K^N arising from suitable quadrature approximations of the integral operator K. Nyström/product integration and finite section methods are studied and it is shown that, under certain conditions, whenever a method is stable on X it is also stable on X_w, with equivalence holding in many cases. Error estimates in the norm of X_w are given.

The class of kernels considered is large and contains, in particular, all kernels of the form $v(s,t) = \kappa(s-t)$, $\kappa \in L^1(\mathbb{R})$, leading to convolution or Wiener-Hopf equations. Special emphasis is laid on families of kernels of the form $v(s,t)k(s,t)$ or $\kappa(s-t)k(s,t)$, with k varying in a bounded and equicontinuous subset W of $BC(\mathbb{R}^2)$, for which the stability results hold uniformly in $k \in W$.

As an application, numerical methods for a family of kernels v with weak logarithmic singularity are analysed. For these methods, stability and convergence results in certain weighted spaces are obtained. The kernels considered arise, e.g., in boundary integral equations for rough surface scattering problems over unbounded domains, which are studied as practical examples. For a combined Nyström and finite section method, novel error estimates are obtained for the case of a point source.

Keywords: integral equations, Nyström method, weighted spaces

Zusammenfassung

Die exakte und numerische Lösbarkeit von Integralgleichungen der Form $\lambda x(s) - \int_{-\infty}^{\infty} v(s,t)x(t)\,dt = y(s)$ in gewichteten Unterräumen X_w des Raumes $X := BC(\mathbb{R})$ (der stetigen und beschränkten Funktionen über $\mathbb{R}$) wird untersucht. Dabei ist $X_w \subset X$ der Gewichtsraum aller Funktionen $x \in X$, die der Bedingung $|w(s)x(s)| = O(1)$, $|s| \to \infty$, für eine Gewichtsfunktion $w \geq 1$ genügen. Ferner wird angenommen, dass der Kern v die einfache Bedingung $|v(s,t)| \leq |\kappa(s-t)|$ für ein $\kappa \in L^1(\mathbb{R})$ erfüllt.

Es werden Anforderungen an v und w formuliert, die hinreichend dafür sind, dass der Integraloperator K in obiger Gleichung beschränkt auf X und X_w ist. Diese Bedingungen werden so verstärkt, dass sie die Übereinstimmung des Spektrums (wesentlichen Spektrums) von K auf X und X_w implizieren und weitere Aussagen über die Lösbarkeit obiger Integralgleichung angegeben werden können.

Ähnliche Beschränktheits- und Spektralergebnisse ergeben sich für die Operatoren K^N, die durch geeignete Quadraturapproximation des Integraloperators K entstehen. Nyström-/Produktintegrations- und Reduktionsverfahren (*finite section methods*) werden untersucht und es wird gezeigt, dass unter bestimmten Bedingungen die Stabilität auf X die Stabilität auf X_w impliziert und in vielen Fällen sogar Äquivalenz gilt. Fehlerschranken in den gewichteten Normen werden angegeben.

Die Klasse der betrachteten Kerne ist umfangreich und beinhaltet insbesondere sämtliche Kerne der Form $v(s,t) = \kappa(s-t)$, $\kappa \in L^1(\mathbb{R})$, die auf Faltungsgleichungen bzw. Wiener-Hopf-Gleichungen führen. Ein besonderer Schwerpunkt liegt auf Familien von Kernen der Form $v(s,t)k(s,t)$ oder $\kappa(s-t)k(s,t)$, wobei k in einer beschränkten und gleichstetigen Teilmenge W von $BC(\mathbb{R}^2)$ variiert; die Stabilitätsresultate gelten dann gleichmäßig für $k \in W$.

Als Anwendung werden numerische Verfahren für eine spezielle Familie von Kernen v mit schwacher logarithmischer Singularität analysiert, für die Stabilität in bestimmten Gewichtsräumen gezeigt wird. Die betrachteten Kerne ergeben sich u.a. bei der Randintegralmethode für Streuprobleme mit unbeschränkten Oberflächen, auf die besonders eingegangen wird. Für ein kombiniertes Nyström- und Reduktionsverfahren werden im Falle einer punktförmigen Quelle neuartige Fehlerabschätzungen angegeben.

Schlagwörter: Integralgleichungen, Nyströmverfahren, Gewichtsräume

Contents

Chapter 1

Introduction

The focus of this thesis is on the theoretical and numerical solution of Fredholm integral equations of the second kind over unbounded domains, which take the following general form:

$$\lambda x(s) - \int_\Omega v(s,t)x(t)\,dt = y(s), \quad s \in \Omega, \tag{1.1}$$

where the domain of integration Ω is one of the sets $\mathbb{R}$ or $\mathbb{R}_+$. In operator notation, (1.1) is written as

$$\lambda x - Kx = y. \tag{1.2}$$

Our study is centered around the investigation of (1.2) in the following class of weighted subspaces of $X := BC(\Omega)$ (the space of all bounded and continuous functions over Ω, equipped with the uniform norm $\|z\| := \sup_{s\in\Omega} |z(s)|$): For an even weight function $w : \mathbb{R} \to \mathbb{R}$ satisfying

$$w(0) = 1, \quad w(s) \geq w(t) \text{ for } s \geq t \geq 0, \quad \lim_{s\to\infty} w(s) = \infty, \tag{1.3}$$

we let X_w denote the weighted subspace $\{x \in X : \|x\|_w := \|xw\| < \infty\}$ of X. X_w is a Banach space when equipped with the norm $\|\cdot\|_w$.

We will show that many spectral properties of the operator K and suitable discretizations, obtained by quadrature methods, are essentially the same on X and X_w. Moreover, we will prove that, for a wide range of Nyström/finite section methods, stability on X is sufficient for stability to hold also on X_w, with equivalence holding for many Nyström methods. If stability holds, we provide estimates for the resulting error when y is contained in X_w.

Integral equations in weighted spaces

Since the ground-breaking work of Fredholm and Riesz at the beginning of the $20th$ century, second-kind Fredholm integral equations over finite and infinite intervals have been of continuing interest to mathematicians (see [11] for a historical account of the theory). A part of their importance stems from the fact that many problems in physics, electromagnetics and mechanics lead to such equations; the list of examples is long, and we refer the reader to [39, 41, 54, 29] and the references therein. Indeed, the reformulation of partial differential equations as boundary integral equations (see, e.g. [28]), in particular for the Helmholtz equation in two dimensions, is a rich source of practical applications for the results we present in this thesis (see, e.g., [25, 6, 23]; [27, 53] provide an introduction).

For second-kind integral equations over a *bounded* interval $[a, b]$ of the real-line, with the integral operator being compact on a suitable function space, the theory is mostly complete, with the Fredholm alternative theorem and the Riesz-Schauder theory being the most prominent and useful theoretical tools (see, e.g. [40]). From 1950 onwards, much effort was devoted to the development and analysis of suitable numerical methods for solving such equations, some of the most outstanding examples being Galerkin, Nyström, product integration, projection and degenerate kernel methods. An excellent overview of these can be found in the monograph [10] of Atkinson. In the preface of this book we are also informed that "this work is nearing a stage in which there will be few major additions to the theory".

However, if, as in (1.1), the domain of the integral equation is unbounded, we usually lose compactness of the integral operator K and the analysis of the numerical methods mentioned above requires much more subtlety than in the case of a finite interval.

The most prominent and well-studied examples of integral equations with a non-compact K are those in which the integral operator is a convolution operator, i.e.

$$v(s,t) = \kappa(s-t), \qquad s \in \Omega, \text{ a.e. } t \in \Omega, \tag{1.4}$$

holds, for some $\kappa \in L^1(\mathbb{R})$. In this case (1.1) is called a *convolution* ($\Omega = \mathbb{R}$) or *Wiener-Hopf* ($\Omega = \mathbb{R}_+$) integral equation:

$$\lambda x(s) - \int_\Omega \kappa(s-t)x(t)\,dt = y(s), \quad s \in \Omega. \tag{1.5}$$

The solvability of equation (1.5) is usually studied by Fourier transform methods, which give explicit expressions for the spectrum and essential spectrum of the operator K. The essential results for L^p-spaces, $1 \le p \le \infty$, and $BC(\Omega)$ can be found in [40] (we also mention the recent survey [12], an up-to-date overview of the L^p-theory for $1 < p < \infty$ for Wiener-Hopf operators with *discontinuous symbols*, arising when κ is not in $L^1(\mathbb{R})$).

Throughout this thesis, we will consider a much more general class of kernels. Let us make this precise: Our first pair of standing hypotheses on v are simple regularity conditions (Assumptions ***(A)*** and ***(B)*** on page 17), which ensure that the integral operator K is bounded on X. The main restriction on v throughout most of this thesis is that it is bounded by a convolution kernel, i.e. that

$$|v(s,t)| \le |\kappa(s-t)|, \qquad s \in \Omega, \text{ a.e. } t \in \Omega, \tag{1.6}$$

holds, for some $\kappa \in L^1(\Omega)$.

Until the last decade, not much was known about the solvability of (1.1) in this general setting. However, in a series of papers [17, 24, 26], Chandler-Wilde *et al.* developed a solvability theory for (1.1). The key idea in these papers is not to consider a single operator K, which on its own has insufficient properties, but to consider a whole family of operators $\{K_k : k \in W\}$. Here W is a set of functions $k \in L^\infty(\mathbb{R}^2)$ and K_k denotes the integral operator in (1.1), but with $v(s,t)$ replaced by $v(s,t)k(s,t)$. It is assumed that W possesses certain translation invariance properties and is such that the operators K_k enjoy sequential collective compactness in a weaker topology than the norm topology of X. One of the results obtained is that if $\lambda - K_k$ is injective, for all $k \in W$, then also $\lambda - K_k$ is surjective, for all $k \in W$, and the inverses $(\lambda - K_k)^{-1}$ are uniformly bounded in $k \in W$ (for a single, compact operator K on X, this is a well-known consequence of the Riesz-Schauder theory).

The arguments used to prove this result are a substantial generalisation of the previous analysis of Anselone and Sloan [9, 2, 3] for the Wiener-Hopf case. The key element of the theory in [3, 17, 24, 26] is the consideration that the operators K_k, while not being compact on X, still enjoy sequentially compactness in a weaker topology, namely, the strict topology of Buck [14]. In this text, we will also often employ this topology and similar arguments as in [3, 17, 24, 26].

Against this background, the first aim of this thesis is now to relate the solvability of (1.1) on X to its solvability on the weighted space X_w. After reviewing fundamental concepts and introducing some notation in Chapters 2 and 3, we commence this analysis in Chapter 4, beginning with the special case when $\Omega = \mathbb{R}_+$, i.e. when (1.1) is a equation on the half-line.

As a prerequisite, we establish general conditions on v and w, which ensure that $K : X_w \to X_w$ and is bounded. For kernels and weight functions which satisfy slightly stronger conditions, precisely

$$\textbf{\textit{(E')}} \qquad \sup_{s \ge 2A} \int_A^{s-A} \frac{w(s)}{w(t)} |\kappa(s-t)|\,dt \to 0, \quad \text{as } A \to \infty,$$

and

$$(\boldsymbol{F'}) \qquad \frac{w(s+1)}{w(s)} \to 1, \quad \text{as } s \to \infty,$$

we obtain the much stronger result that the spectra of K on X and X_w coincide and that the same holds for the essential spectrum, in symbols

$$\Sigma_X(K) = \Sigma_{X_w}(K), \quad \Sigma^e_X(K) = \Sigma^e_{X_w}(K). \tag{1.7}$$

With regard to the integral equation (1.1) the first of these equalities implies that the equation (1.1) has an unique solution $x \in X$ for every $y \in X$ if and only if it has an unique solution $x \in X_w$ for every $y \in X_w$.

Sufficient (and easier to check) conditions on v and w for $(\boldsymbol{E'})$ and $(\boldsymbol{F'})$ to hold and several examples are given in Section 4.4. In particular, we show that if v satisfies (1.4), for some $\kappa \in L^1(\mathbb{R})$, and $y \in X$ is a function such that $\lim_{s\to\infty} y(s) = 0$, then we can always construct a weight function w satisfying our main assumptions and such that $y \in X_w$. As an application of this construction we further establish that the spectra of K on X and X_0 coincide, where X_0 is the subspace of X consisting of those $x \in X$ vanishing at infinity.

Our study continues earlier investigations in [56] (see also the monograph [57] and [62]) which consider primarily the case $w(s) = (1+s)^r$ for some $r \in \mathbb{R}$. In [57, 56, 62] it is shown, using Banach algebra techniques, that in the Wiener-Hopf case, $v(s,t) = \kappa(s-t)$, it holds that $K \in \mathcal{B}(X_w)$ if

$$\int_{-\infty}^{\infty} (1+|t|)^r |\kappa(t)|\, dt < \infty \tag{1.8}$$

and that if (1.8) holds then

$$\Sigma^e_{X_w}(K) = \Sigma^e_X(K) = \{\hat\kappa(\xi) : \xi \in \mathbb{R}\} \cup \{0\} \tag{1.9}$$

and

$$\Sigma_{X_w}(K) = \Sigma_X(K) = \Sigma^e_X(K) \cup \{\lambda \in \mathbb{C} : [\arg(\lambda - \hat\kappa(\xi))]^{\infty}_{-\infty} \neq 0\}, \tag{1.10}$$

where $\hat\kappa$ is the Fourier transform of κ,

$$\hat\kappa(\xi) = \int_{-\infty}^{\infty} \kappa(t) e^{i\xi t}\, dt, \quad \xi \in \mathbb{R}. \tag{1.11}$$

In [43] (see also [42]) Karapetiants and Samko provide results for convolution kernels which include the result of [56] as a special case, based on a demonstration that $K - K_w$ is compact on X, where K_w is defined as the integral operator K, but with v replaced by the kernel $v_w(s,t) := (w(s)/w(t))\kappa(s-t)$.

In both [56] and [43] it is shown that the condition (1.8) guarantees that K is a bounded operator not just on X_w but also on the corresponding weighted L^p space, $1 \le p \le \infty$, and that (1.9) and (1.10) hold with X and X_w replaced by $L^p(\mathbb{R}_+)$ and the corresponding weighted L^p space.

In a series of papers [18, 20, 7] the case when v satisfies (1.6) is considered, with $w(s) = (1+s)^p$ for some $p > 0$ (so that w satisfies the conditions (1.3)). It is shown that if

$$\kappa(s) = O(s^{-q}), \quad s \to \infty, \tag{1.12}$$

for some $q > 1$ then $K \in \mathcal{B}(X_w)$ and (1.7) holds for $0 < p \le q$. A key component of the argument is the consideration, as in Samko [43], of properties of $K - K_w$. In the limiting case when $p = q$, $K - K_w$ may not be compact but is a sufficiently well-behaved operator to proceed by somewhat similar arguments to the case when $K - K_w$ is compact. We point out that for many applications the condition that (1.12) holds for some $q > 1$ with $q \ge p$ is a much less onerous condition than (1.8). In particular, in the case that $|\kappa(s)| \sim as^{-q}$ as $s \to \infty$, for some $a > 0$, in which case necessarily $q > 1$ given that $\kappa \in L^1(\mathbb{R})$, the results of [57] and [43] give that $K \in \mathcal{B}(X_w)$ and (1.7) holds for $0 < p \le q$, while (1.8) holds with $r = p$, and so the theory of [57] and [43] applies only if $|p| < q - 1$.

We also mention a paper [59] by Rejto and Taboada in which linear Volterra integral equations are considered (see also [60] for the non-linear case). Such equations are a special case of (1.1) when $v(s,t) = 0$ for $0 < s < t$. It is shown in [59] that if w is a positive and continuous weight function and a certain boundedness condition, similar to our condition (4.7) in Chapter 4, is satisfied then K has zero spectral radius on X_w. This result is proved by constructing the inverse of $\lambda - K$, $\lambda \neq 0$, on the weighted space with the aid of Neumann series, a technique that works well for Volterra equations but fails for most Fredholm equations.

In the Wiener-Hopf case the conditions we impose on k to obtain that $K \in B(X_w)$ and the main results (1.7) are, for many weight functions w, both necessary and sufficient. For example, consider the particular weight function

$$w(s) = \exp(as^\alpha)(1+s)^p\big(\ln(e+s)\big)^q, \quad s \in \mathbb{R}_+, \tag{1.13}$$

and suppose that the constants $\alpha \in (0,1)$, $a \geq 0, p, q \in \mathbb{R}$ are such that (1.3) holds and $\int_0^\infty w^{-1}(s)\,ds$ is finite. Then the results we obtain imply for the Wiener-Hopf case $v(s,t) = \kappa(s-t)$, with $\kappa \in L^1(\mathbb{R})$, that a necessary and sufficient condition for $K \in B(X_w)$ is

$$w(s)\int_s^{s+1} |\kappa(t)|\,dt = O(1), \quad s \to \infty. \tag{1.14}$$

Moreover this condition ensures the spectral equalities (1.7) hold. In the more general case that the kernel v satisfies $\boldsymbol{(A')}$ and $\boldsymbol{(B)}$ with $\kappa \in L^1(\mathbb{R})$, it remains true that (1.14) also ensures that $K \in B(X_w)$ and (1.7) hold.

Chapter 4 can be considered in large part as an attempt to sharpen and generalise the results and methods of argument of [18, 20, 7], establishing large classes of kernels v and weight functions w for which $K \in \mathcal{B}(X_w)$ and (1.7) holds. The special case referred to above for the weight function (1.13) contains many of the results of [18, 20, 7]. For the weight $w(s) = (1+s)^r$ with $r > 1$ and the Wiener-Hopf case $v(s,t) = \kappa(s-t)$, the general results of this chapter show that $K \in \mathcal{B}(X_w)$ if and only if

$$\int_s^{s+1} |\kappa(t)|\,dt = O(s^{-r}), \quad s \to \infty. \tag{1.15}$$

This condition does not imply that $K - K_w$ is compact but, nevertheless, ensures that (1.7) holds. Note that (1.15) is a considerably weaker condition than (1.8).

We also prove that, under the same assumptions on v and w, equivalences for semi-Fredholmness corresponding to (1.7) hold, i.e. $\lambda - K$ is semi-Fredholm on X if and only if $\lambda - K$ is semi-Fredholm on X_w, in which case the indices of both operators coincide.

Throughout most of Chapter 4, we restrict our attention to the case when K is an integral operator on the half-line, i.e. $\Omega = \mathbb{R}_+$. In Section 4.5 we then show how our conditions may be generalised to obtain similar results for the real line case when $\Omega = \mathbb{R}$.

The weighted space results we present for the equation (1.1) have numerous theoretical and practical applications. We show under assumption (1.6) the interesting result that if $\lambda - K$ is Fredholm on X then necessarily the null space of $\lambda - K$ on X is contained in the intersection of the spaces X_w, where w runs through the (non-empty) set of weight functions w satisfying (1.3), $\boldsymbol{(E')}$ and $\boldsymbol{(F')}$.

Some of the results presented in Chapter 4 have recently found an important application in the analysis of the finite section method for integral equations on the real line of the form

$$x(s) - \int_{-\infty}^{\infty} \kappa(s-t)z(t)x(t)\,dt = y(s), \quad s \in \mathbb{R}, \tag{1.16}$$

with $\kappa \in L^1(\mathbb{R})$, $z \in L^\infty(\mathbb{R})$. Let x^A denote the approximation to x obtained when (1.16) is solved with the range of integration reduced to $[-A, A]$. Then, using the results of Section 4.5 it is shown in [21] that, under certain conditions on z,

$$|x(s) - x^A(s)| \leq C\left(\frac{1}{w(s-A)} + \frac{1}{w(s+A)}\right) \operatorname*{ess.sup}_{|s|\geq A} |z(s)x(s)|, \quad |s| \leq A, \tag{1.17}$$

where C is a constant (depending only on κ and z) and w is a weight function, which can be specified in terms of κ.

Our results on the equivalence of spectra between X and X_w can also be exploited to shed light on the equivalence of spectra for other spaces. In particular, using the denseness of X_w in $L^1(\mathbb{R}_+)$ if $\int_0^\infty w^{-1}(s)ds < \infty$, it is possible to draw conclusions about the spectrum of K as an operator on $L^p(\mathbb{R}_+)$, for $p = 1$, and then, by interpolation, for $1 < p < \infty$. See [8] for results in this direction for the case when (1.6) holds with $|\kappa(s)| = O(|s|^{-q})$ as $|s| \to \infty$.

We remark that an earlier, less extensive version of Chapter 4 has been submitted for publication in [19], with Prof. Dr. Simon Chandler-Wilde as co-author.

Nyström/product integration and finite section methods in weighted spaces

The literature on the numerical solution of (1.1) when K is not compact is extensive. However, the vast majority of this literature (see, e.g. [3, 4, 32, 36, 48] and the references therein) is restricted to the case when $\Omega = \mathbb{R}_+$ and v is a Wiener-Hopf kernel, i.e. (1.4) holds. In some publications the slightly more general case when K is a compact perturbation of a Wiener-Hopf operator is considered [15, 1, 57].

In the Wiener-Hopf case, general results on stability and optimal convergence orders of Galerkin, collocation and Nyström quadrature methods using piecewise polynomial basis functions have been established [1, 15, 32, 31, 36]. In particular, provided a suitably graded mesh is used and the solution belongs to appropriate weighted Sobolev spaces, with sufficiently many derivatives decaying exponentially at infinity, one obtains in the L^q-norm ($1 \le q \le \infty$) the same (polynomial) rates of convergence that occur when the methods are applied to equations on finite intervals. In [33], even exponential convergence is obtained using *h-p*-spline approximation methods.

However, the results that have been obtained to date are in a number of respects unsatisfactory:

1. In many applications the requirement is to compute x with small *relative* error on the whole domain. If x vanishes at infinity, estimates in the L^q-norm say nothing about the size of the relative error when s is large.

2. In those papers where quantitative error estimates are obtained for the numerical solution of (1.5) (or a compact perturbation) it is invariably assumed that κ and sufficiently many of its derivatives vanish exponentially towards infinity. To our knowledge, the case when κ exhibits only a polynomial rate of decay (arising in many applications, particularly in scattering theory [23, 64] or radiative heat transfer [52]), has only been considered in [52, 50, 49].

3. Overwhelmingly, consideration has been focused on the Wiener-Hopf case and several slight generalisations. Apart from [17, 50] we know of no analysis of the case when $v(s,t)$ is merely *bounded* by a convolution kernel.

In Chapters 5 and 6, we make contributions to the numerical analysis of integral equations on the real line in all three of these directions. But, particularly, our concern is to make progress in regard to the first of these aspects, focusing on Nyström/product integration methods and their finite section variants for the approximate solution of (1.1).

Throughout these chapters, we will consider the real line case when $\Omega = \mathbb{R}$, but make the point that our results apply equally well to equations on the half-line. We will also assume that the kernel of (1.1) is replaced by the product $v(s,t)k(s,t)$, where $v(s,t)$ satisfies the assumptions of Chapter 4 and $k \in BC(\mathbb{R}^2)$. Then (1.1) becomes

$$\lambda x(s) - \int_\Omega v(s,t)k(s,t)x(t)\,dt = y(s), \quad s \in \mathbb{R}, \tag{1.18}$$

or, in operator from,

$$\lambda x - K_k x = y. \tag{1.19}$$

We note that the results obtained in Chapter 4, in particular (1.7), also hold, under the same assumptions on v, for the integral operators K_k, $k \in BC(\mathbb{R}^2)$.

We focus on a variant of the *Nyström method* known as *product integration approximation.* The idea is to approximate the integral operator K_k by so-called *quadrature* or *discretized integral* operators K_k^N, defined on X by

$$K_k^N x(s) := \sum_{j\in\mathbb{Z}} \omega_j^N(s) k(s, t_j^N) x(t_j^N), \qquad s \in \mathbb{R},\, x \in X,\, N \in \mathbb{N}. \tag{1.20}$$

In this definition, the $t_j^N := jh_N$, are the (equally spaced) abscissae of the Nth in a sequence of quadrature rules, with $h_N \to 0$ as $N \to \infty$, so that N is a parameter controlling the quality of the approximation. The corresponding weights $\omega_j^N(s)$ of the quadrature rule, are chosen appropriate to the kernel v and are usually constructed by integrating the product of $v(s, \cdot)$ with Lagrange interpolating functions (e.g. polynomials or trigonometric polynomials).

This form of approximating K_k by K_k^N is particularly suitable for badly behaved kernel functions that may be written as the product of a smooth or at least continuous function k and a discontinuous, possibly singular function v; see, e.g., [10, 44] and the references therein. A benefit of this approach is that it allows us to consider families of kernels $v(s,t)k(s,t)$, where k varies in a subset W of $BC(\mathbb{R}^2)$.

Once the quadrature operators K_k^N have been defined, it is hoped that for large N the solution x^N of the equation

$$\lambda x^N - K_k^N x^N = y, \tag{1.21}$$

obtained by solving a system of linear equations in which the unknowns are the values of x^N at the quadrature abscissae, is a good approximation of the solution x of equation (1.19).

Our standing assumptions on the quadrature weights $\omega_j^N(s)$ throughout this thesis will be two simple regularity conditions (***(QA)*** and ***(QB)*** on page 47), which ensure that $K_k^N : X \to X$ and is bounded for all $k \in BC(\mathbb{R}^2)$. From Section 5.2 onwards, we will also assume that the kernel v is such that (1.6) holds and $|\kappa(s)|$ is monotonic outside some finite interval and, moreover, that the quadrature weights $\omega_j^N(s)$ reflect the decay of both $|\kappa(s-t)|$ as $|s-t| \to \infty$ and h_N as $N \to \infty$ (this is made precise in the Assumptions ***(A″)*** and ***(QA″)*** of Section 5.2). Under these mild conditions we show that $K_k^N : X_w \to X_w$ and is bounded (for all weight functions satisfying (1.3) and real-line versions of ***(E′)*** and ***(F′)***) and, moreover, that analogues of (1.7) also hold for the operators K_k^N, $k \in BC(\mathbb{R}^2)$, $N \in \mathbb{N}$, namely that

$$\Sigma_X(K_k^N) = \Sigma_{X_w}(K_k^N), \quad \Sigma_X^e(K_k^N) = \Sigma_{X_w}^e(K_k^N). \tag{1.22}$$

Thus, if $\lambda \notin \Sigma_X(K_k) \cup \Sigma_X(K_k^N)$ and the right-hand side y of (1.19) and (1.21) is contained in X_w, the solution x and its approximation x^N are both contained in X_w.

We remark that, up to this point, we have assumed nothing about the convergence of the operators K_k^N to K_k. We take this into account in Section 5.3, where we are additionally assuming that the pointwise convergence assumption

$$\lim_{N\to\infty} K_k^N x(s) = K_k x(s), \quad x \in X,\ s \in \mathbb{R}, \tag{1.23}$$

holds for the constant function $k(s,t) = 1$ (it then also holds for every $k \in BC(\mathbb{R}^2)$). Provided that (1.19) is well-posed, for some $k \in BC(\mathbb{R}^2)$, we are then able to show that the Nyström method above is stable with respect to the uniform norm of X if and only if it is stable with respect to the norm of X_w (w satisfying (1.3), ***(E′)*** and ***(F′)***). An important feature of our stability theory is that this result remains valid if we simultaneously consider a whole family of operators K_k and K_k^N, with k varying in a bounded and equicontinuous subset W of $BC(\mathbb{R}^2)$. The methods used in the proof are based on a combination of the arguments as in [24, 26] with the weighted space theory developed in Section 5.2.

We then proceed to show that, if stability holds and the right-hand side y of (1.21) is contained in X_w, then the solutions of (1.19) and (1.21) (for N large) are contained in X_w and the error estimate

$$\|x - x^N\|_w \le C\|(K_k^N - K_k)x\|_w \tag{1.24}$$

holds, where the constant C does not depend on N (nor on $k \in W$ if a family of operators is considered). However, since K_k^N does not strongly converge to K_k in the norm of X_w, it will usually only be the case that the term on the right-hand side of (1.24), and thus $\|x - x^N\|_w$, tends to zero as $N \to \infty$ if x is smooth. Nevertheless, if $y \in X_w$ then we show that it will always be the case that, in the uniform norm,

$$\|x - x^N\| \to 0, \qquad \text{as } N \to \infty. \tag{1.25}$$

(Let us mention at this point that we obtain sharper error estimates in the norm of X_w than (1.24) for sufficiently well-behaved kernels in Chapter 6; these will be discussed later in this introduction.)

Of course, bounds in the weighted norm will guarantee that the *relative* error in calculating x is small only if $|xw(s)|$ is bounded below as well as above by a positive constant. But, we mention also that, for the Wiener-Hopf equation (1.5) (with $\Omega = \mathbb{R}_+$), it has been shown in [18] that, if $\kappa(s) \sim \kappa_\infty s^{-b}$ as $s \to \infty$, with $b > 1$, and $y(s) \sim y_\infty s^{-a}$ as $s \to \infty$, with $0 \le a \le b$, then

$$s^a x(s) \to \begin{cases} \dfrac{y_\infty}{1 - \int_{-\infty}^{\infty} \kappa(t)\,dt}, & \text{if } 0 \le a < b, \\ \dfrac{y_\infty + \kappa_\infty \int_0^\infty x(t)\,dt}{1 - \int_{-\infty}^{\infty} \kappa(t)\,dt}, & \text{if } a = b, \end{cases} \tag{1.26}$$

as $s \to \infty$. If the limit (1.26) is non-zero and and $x(s) \neq 0$, $s \ge 0$, then $\inf_{s \ge 0} |x(s)w(s)| > 0$, for the weight function $w(s) := (1 + |s|)^a$, in which case $\|x - x^N\|_w \to 0$ guarantees convergence in relative error as $N \to \infty$.

While Nyström methods have the advantage that they are usually easy to implement on a computer, a considerable difficulty in solving the discretized equation (1.21) is that, in general, one has to solve an infinite system of linear equations. If no periodicity is involved in the coefficients of these equations, this may be an onerous if not impossible task. Therefore, we will look, in Section 5.4, at the effect of truncating the summation in the definition of the quadrature operator K_k^N to a finite interval $[-A, A]$. We thereby obtain new quadrature operators, which we denote by $K_k^{N,A}$. Replacing K_k^N by $K_k^{N,A}$ in (1.21), we are looking at the new equation

$$\lambda x^{N,A} - K_k^{N,A} x^{N,A} = y. \tag{1.27}$$

The solution $x^{N,A}$ of (1.27), which is now an approximation of x^N (and thus x), may be obtained by solving a *finite* linear system.

We show that if the combined Nyström and truncation method is stable on X (meaning that the inverses of $\lambda - K^N$ and $\lambda - K_k^{N,A}$ exist on X and are uniformly bounded for all N and A large enough) then, as was the case for the pure Nyström method, it is also stable on X_w.

For the Wiener-Hopf case in which $\Omega = \mathbb{R}_+$, $v(s,t) = \kappa(s-t)$ and k is the constant function $k(s,t) = 1$, the approximation of (1.18) by (1.21) has been studied by Anselone and Sloan [4, 5]. In [4] it assumed that $\kappa \in L^1(\mathbb{R})$ is bounded and uniformly continuous on $\mathbb{R}$, $\lim_{|s| \to \infty} \kappa(s) = 0$, and $|\kappa(s)| \le \lambda(s)$, for some function $\lambda \in L^1(\mathbb{R})$ non-increasing on $\mathbb{R}_+$ and non-decreasing on $\mathbb{R}_-$. It is shown in [4] that if a certain condition of uniform convergence holds for the quadrature weights (satisfied by, e.g., the weights of simple compound rules) then, provided that (1.5) is well-posed on X_u^+ (the closed subspace of all uniformly continuous functions in $BC(\mathbb{R}_+)$), the operators $\lambda - K_k^{N,A}$ are uniformly invertible on X_u^+ for all A and N large enough. Moreover, for $x \in X_u^+$ the solutions x of (1.19) and $x^{N,A}$ of (1.27) satisfy

$$|x(s) - x^{N,A}(s)| \to 0, \qquad \text{as } A, N \to \infty, \text{ uniformly on finite intervals.} \tag{1.28}$$

In this thesis we consider in detail the case when the kernel $v(s,t) = \tilde{\kappa}(s-t)$ is an L^1-convolution kernel (bounded by a convolution kernel κ satisfying the monotonicity condition in $\boldsymbol{(A'')}$) and the quadrature weights satisfy a mild condition of translation invariance (satisfied for, e.g., the standard simple compound rules). However, in contrast to [4], we are only requiring the kernel function k to be bounded and uniformly equicontinuous and thus it need not be necessarily constant. We then show that if such a combined Nyström and finite section method is stable on X then it is also stable on X_w;

moreover, there always exists a weight function $\widetilde{w}$, defined in terms of κ, so that the error estimate

$$|x^N(s) - x^{N,A}(s)| \leq C\frac{1}{w(A)}\left(\frac{1}{\widetilde{w}(s-A)} + \frac{1}{\widetilde{w}(s+A)}\right)\|y\|_w, \qquad |s| \leq A, \tag{1.29}$$

holds for the solutions x^N of (1.21) and $x^{N,A}$ of (1.27) when the right-hand side y is contained in X_w. The constant $C > 0$ does not depend on the truncation parameter A, nor on N or y. (This error bound echoes the result (1.17) for the finite section method for the equation (1.16) obtained in [21].) We also note that inequality (1.29) shows that the error $|x^N(s) - x^{N,A}(s)|$ is particularly small when s is near the origin and increases as s approaches the endpoints of the truncation interval $[-A, A]$.

In particular, if $y \in X_w$ then combining (1.29) with (1.25) yields the bound

$$|x(s) - x^{N,A}(s)| \to 0, \qquad \text{as } A, N \to \infty, \text{ uniformly in } |s| \leq A. \tag{1.30}$$

Note that (1.29) and (1.30) are much sharper results than (1.28). We remark that our methods of argument with regard to the finite section method owe a lot to earlier work by Rahman [58, 21] on the finite-section method for integral equations (without discretization).

In the general case when $k(s,t)$ is not a constant function (or v is not a convolution kernel) the question of whether the well-posedness (i.e. unique solvability) of (1.21) is sufficient for the corresponding finite-section method to be stable on X is to a large extend still open (but see [46] for recent results in this direction). Nevertheless, in some cases stability on X might be obtained for a so-called *modified finite section method* in which, in addition to the truncation of the summation to a finite interval the function $k(s,t)$ is modified for values of t near the endpoints of the interval of truncation (see [51, 49]). We include this variant in our stability and error analysis.

We conclude Chapter 5 by pointing out how our results generalise to the case when $K = K_1 + \ldots + K_n$ is a finite sum of integral operators and different quadrature methods are used to approximate each of the K_i. We will make use of these generalisations in the applications we present in the final chapter of this thesis.

Applications

We have already indicated that our results have numerous practical applications. In Chapter 6 of this thesis, we discuss several examples of integral equations with kernels v satisfying (1.6), for some $\kappa \in L^1(\mathbb{R})$, which exhibits polynomial decay towards $\pm\infty$, i.e. when

$$|\kappa(s)| = O(|s|^{-b}), \qquad |s| \to \infty \tag{1.31}$$

for some $b > 1$. Throughout this chapter, we assume that the kernels v are (possibly) weakly singular at $s = t$ and partially differentiable up to order n for $s \neq t$, or at least if $|s - t|$ is large, and that these partial derivatives also exhibit polynomial decay as $|s - t| \to \infty$.

The resulting integral equations and their numerical solvability are then studied in the weighted spaces $X_a := X_{w_a}$, with norm $\|\cdot\|_a := \|\cdot\|_{w_a}$, where w_a is the power weight $w(s) = (1+|s|)^a$, $0 \leq a \leq b$ (this is the setting of [18, 20, 7]; see the discussion above).

Under the above assumptions, we decompose $v(s,t)$ into

$$v(s,t) = \widetilde{v}(s,t) + \widehat{v}(s,t),$$

where $\widetilde{v}$ is the smooth part of the kernel, supported outside the strip $|s-t| \geq A$, for some $A > 0$, and $\widehat{v}$ is the (possibly) weakly singular part of the kernel, supported inside the strip $|s-t| \leq A + \epsilon$, ϵ being a small constant. Then K_k can be written as the sum of the two integral operators $\widehat{K_k}$ and $\widetilde{K_k}$ corresponding to $\widehat{v}$ and $\widetilde{v}$, respectively.

The idea behind this decomposition is that the simple rectangle rule yields quadrature operators $\widetilde{K_k^N}$, which are already good approximations of the "well-behaved" integral operator $\widetilde{K_k}$. Particularly, we show that, for $k \in BC^n(\mathbb{R}^2)$ and $0 \leq a \leq b$, an error estimate of the form

$$\left\|\widetilde{K_k}x - \widetilde{K_k^N}x\right\|_a \leq CN^{-n}\|x\|_{BC_a^n(\mathbb{R})}, \qquad x \in BC_a^n(\mathbb{R}), \tag{1.32}$$

holds, where $C > 0$ does not depend on x or n; here $BC_a^n(\mathbb{R})$ denotes the weighted space of all functions $x \in X$, whose first n derivatives exist and are contained in X_a.

Since, for fixed $s \in \mathbb{R}$, $\widehat{v}(s,t)$, as a function of t, is compactly supported quadrature rules suitable for finite intervals may be used in the approximation of $\widehat{v}$. We prove that, if, for $a = 0$ and k the constant function $k(s,t) = 1$, the error estimate

$$\big\|\widehat{K_k}x - \widehat{K_k^N}x\big\|_a \leq CN^{-n}\|x\|_{BC_a^n(\mathbb{R})}, \qquad x \in BC_a^n(\mathbb{R}), \tag{1.33}$$

holds, for C not depending on N or x, and, moreover a simple condition on the quadrature weights is satisfied, then the estimate (1.33) also holds for every $0 < a \leq b$ and $k \in BC^n(\mathbb{R}^2)$ and does not depend on N.

If $0 \leq a \leq b$, (1.32) and (1.33) hold and $\widehat{v}$ fulfils a simple regularity assumption, we are able to show the following improved error bound for the solutions x and x^N of (1.19) and (1.21), provided that $k \in BC^n(\mathbb{R})$, well-posedness and stability hold on X and y is contained in $BC_a^n(\mathbb{R})$:

$$\|x - x^N\|_a \leq CN^{-n}\|y\|_{BC_a^n(\mathbb{R})}. \tag{1.34}$$

A strong motivation for the considerations in Chapter 6 are the results of Meier *et al.* [50], who have recently considered Nyström methods for a class of kernels v with logarithmic singularity at $s = t$. The idea in [50] is to perform a similar splitting of the kernel v to the one described above and then to use a quadrature method based on trigonometric interpolation (due to Kussmaul [45] and Martensen [47]) for the approximation of the kernel $\widehat{v}$. In particular, it is shown in [50] that this Nyström method is stable on X provided the equation (1.18) is uniquely solvable, and error bounds of the form (1.34) have been obtained for the case $a = 0$, i.e. in the uniform norm of X.

We show in Section 6.2 that the methods proposed in [50] match our general assumptions in Chapter 5. Thus we are able to employ our stability theory to extend the results of [50] to the weighted spaces X_a, $0 < a \leq b$.

A practical problem, that has been studied in [50], is to find the scattered field which arises when a wave is incident on an unbounded rough surface Γ. Using appropriate Green's functions and representing the scattered field as a combined single- and double layer potential, a reformulation of this problem leads to a boundary integral equation taking the form (1.18), where v is a convolution kernel satisfying (1.31), with $b = 3/2$, and the function k, depending on the shape of the scattering surface Γ, is contained in $BC^n(\mathbb{R})$ if the scattering surface is sufficiently smooth. In this formulation, the right-hand side y models the incident field on the boundary. The scattered field may then be determined by evaluating a boundary integral once the solution x of (1.18) is known. (We remark that the well-posedness of this integral equation formulation has recently been shown in [64, 22].)

Following on [50] (see also [51]), Meier proposed in [49] a combined Nyström and modified finite-section method in which the summation in the quadrature operators K_k^N is truncated to a finite interval $[-(A+\alpha_0), A+\alpha_0]$ and the values of $k(s,t)$ are modified for $A \leq |t| \leq A+\alpha_0$ (corresponding to a "flattening" of the surface Γ). In [49] it is shown that this method is stable and convergent on X. For y modelling an incident plane or cylindrical wave (then $y \in BC^n(\mathbb{R})$), an error estimate of the form

$$\big|x(s) - x^{N,A+\alpha_0}(s)\big| \leq C_1 N^{-n} + C_2\left(\frac{1}{(1+A+s)^{1/2}} + \frac{1}{(1+A-s)^{1/2}}\right), \tag{1.35}$$

for $|s| \leq A$, where C_1, $C_2 > 0$ do not depend on N or A, was given for the exact solution of (1.21) and the approximate solution $x^{N,A+\alpha_0}$ of the modified finite section equation

$$\lambda x^{N,A} - K_{k'}^{N,A+\alpha_0} x^{N,A} = y,$$

where k' is the modification of k mentioned above. The summands on the right-hand side of (1.35) are the respective errors introduced by the Nyström and finite section approximation.

In the special case when the scattered field of an incident cylindrical wave emanating from a point source above the surface Γ is sought, the right-hand side y can be shown to be contained in the weighted

$BC^n_{3/2}(\mathbb{R})$. Using our weighted space solvability results, in particular (1.7), we show in Section 6.3 that the exact solution x is then also contained in $BC^n_{3/2}(\mathbb{R})$. As a consequence, we can demonstrate that the approximate solutions $x^{N,A+\alpha_0}$ of (1.21), obtained by applying the combined Nyström and finite section method of [49], are contained in $X_{3/2}$ and that the following error estimate holds:

$$\left|x(s) - x^{N,A+\alpha_0}(s)\right| \le C_1 N^{-n} \frac{1}{(1+s)^{3/2}} + C_2 \frac{1}{(1+A)^{3/2}} \left(\frac{1}{(1+A+s)^{1/2}} + \frac{1}{(1+A-s)^{1/2}} \right),$$

for $|s| \le A$, where C_1, $C_2 > 0$ do not depend on N, A or α_0, Clearly, this inequality is a considerably sharper error estimate than (1.35).

For the convenience of the reader, an index of notations can be found at the end of this thesis.

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Chapter 2

Background material

If Y, Z are two normed spaces then we write $\mathcal{B}(Y,Z)$ and $\mathcal{K}(Y,Z)$ for the space of, respectively, all bounded and all compact linear operators from Y to Z. We write $\|\cdot\|_{Y\to Z}$ for the operator norm in $\mathcal{B}(Y,Z)$, and set $\|\cdot\|_Y := \|\cdot\|_{Y\to Y}$. The symbol I always denotes the identity operator, but we will frequently use the abbreviation $\lambda := \lambda I$. We will say that $L \in \mathcal{B}(Y,Z)$ is *invertible* and write $L^{-1} \in \mathcal{B}(Z,Y)$ if and only if L has a bounded inverse L^{-1} defined on all of Z. We denote by $\mathcal{GL}(Y,Z)$ the space of all invertible operators in $\mathcal{B}(Y,Z)$. We set $\mathcal{B}(Y) := \mathcal{B}(Y,Y)$, $\mathcal{K}(Y) := \mathcal{K}(Y,Y)$ and $\mathcal{GL}(Y) := \mathcal{GL}(Y,Y)$. If Y_1 is a subspace of Y and $L \in \mathcal{B}(Y,Z)$ then we write $L|_{Y_1}$ for the restriction of L to Y_1. For an operator $L : Y \to Z$ we denote by $L(U)$, $U \subset Y$, the set $\{Ly : y \in U\}$; $L(Y)$ is called the *range* or *image* of L and $\ker_Y L := \{y \in Y : Ly = 0\}$ the *null space* or *kernel* of L (most of the time, we will drop the index Y and just write $\ker L$). If $U \subset Y$ then we write $\operatorname{span} U$ for the set of all linear combinations of elements in U (with $\operatorname{span} U = \{0\}$ if $U = \varnothing$) and $\overline{\operatorname{span}}\, U$ for the closure of $\operatorname{span} U$ (in the norm of Y). For $U, U' \subset Y$ we define the *distance* between U and U' by $\operatorname{dist}(U,U') := \inf_{y\in U, y'\in U} \|y - y'\|$; for $y \in U$ we set $\operatorname{dist}(y,U') := \operatorname{dist}(\{y\},U')$.

If Y is a Banach space then we will write $Y = Y_1 \oplus_Y Y_2$ if Y is the direct sum of two closed subspaces Y_1, Y_2 of Y; in this case we say that Y_1 and Y_2 are *complemented* in Y. (In most cases we will omit the subscript and simply write $Y = Y_1 \oplus Y_2$.) It is a consequence of the closed graph theorem that a closed subspace of Y_1 is complemented in Y if and only if there exists a continuous projection $P \in \mathcal{B}(Y)$ such that $P(Y) = Y_1$. In a Banach space, every finite-dimensional subspace is closed and complemented.

We denote by $\mathbb{N}$ the set of natural numbers $\{1,2,\ldots\}$ and write $\mathbb{N}_0$ for the set $\{0\}\cup\mathbb{N}$. The set of all non-negative real numbers is denoted by $\mathbb{R}_+$. If S is an arbitrary set then 1_S denotes its characteristic function.

2.1 Normal solvability and Fredholm operators

Throughout this section we suppose that Y, Y' and Z are Banach spaces and that $L \in \mathcal{B}(Y,Z)$. Most of the following results are standard, see e.g. [55], [40] or [12].

The operator L is called *normally solvable* if and only if its range $L(Y)$ is closed. We give a number of useful characterizations of normal solvability in the following lemma.

Lemma 2.1. *For $L \in \mathcal{B}(Y,Z)$ the following are equivalent.*

1. *L is normally solvable.*

2. *There exists $C > 0$ such that for all $y \in Y$ there holds $\|Ly\| \geq C \operatorname{dist}(y, \ker L)$.*

3. *For every bounded sequence (z_n) in $L(Y)$, there exists a bounded sequence (y_n) in Y such that $Ly_n = z_n$, $n \in \mathbb{N}$.*

Normal solvability is a desirable property of L if we try to solve the equation

$$Ly = z, \quad z \in Z. \tag{2.1}$$

For every $z \in L(Y)$ this equation has a solution $y \in Y$, which may not be unique. If L is normally solvable then, whenever (z_n) is a convergent sequence in $L(Y)$, there exists a convergent sequence (y_n) in Y such that $Ly_n = z_n$, $n \in \mathbb{N}$; in this sense, the solution y of (2.1) depends continuously on z.

If we restrict a normally solvable operator to a closed subspace, the following lemma gives a sufficient condition for the normal solvability of this restriction.

Lemma 2.2. *([55]) Let $L \in \mathcal{B}(Y,Z)$ be normally solvable and suppose that Y_0 is a closed subspace of Y. Then the restriction $L|_{Y_0}$ of L to Y_0 is normally solvable if and only if $Y_0 + \ker_Y L$ is a closed set.*

The next lemma shows that compact operators are usually not normally solvable.

Lemma 2.3. *([38]) Let $L \in \mathcal{K}(Y,Z)$. Then L is normally solvable if and only if $L(Y)$ is finite-dimensional, in which case $Y = Y' \oplus \ker L$, for some finite dimensional subspace Y' of Y.*

For L we define the numbers

$$\alpha(L) := \dim \ker L, \quad \beta(L) := \operatorname{codim}_Z L(Y) := \dim Z/L(Y).$$

$\alpha(L)$ and $\beta(L)$ are called the *nullity* and *deficiency* of L and take values in $\mathbb{N}_0 \cup \{\infty\}$.

We call L *semi-Fredholm* if L is normally solvable and at least one of the numbers $\alpha(L)$ or $\beta(L)$ is finite; then L is called a Φ_+-operator if $\alpha(L) < \infty$ and a Φ_--operator if $\beta(L) < \infty$. Moreover, L is called a *Fredholm operator* or Φ*-operator* if both $\alpha(L)$ and $\beta(L)$ are finite. (We note that there is some redundancy in these definitions, for every continuous operator acting between two Banach spaces is normally solvable if its range has finite co-dimension; this follows from a theorem of Kato [38, Th. 55.4].)

We denote by $\Phi(Y,Z)$, $\Phi_+(Y,Z)$ and $\Phi_-(Y,Z)$ the set of all Fredholm, Φ_+- and Φ_--operators in $\mathcal{B}(Y,Z)$, respectively. If $Y = Z$ then we will use the simpler notations $\Phi(Y)$, $\Phi_+(Y)$ and $\Phi_-(Y)$, respectively.

If L is semi-Fredholm the *index* of L is defined as

$$\operatorname{ind} L := \begin{cases} \alpha(L) - \beta(L), & L \in \Phi(Y,Z), \\ +\infty, & L \in \Phi_-(Y,Z) \setminus \Phi(Y,Z), \\ -\infty, & L \in \Phi_+(Y,Z) \setminus \Phi(Y,Z). \end{cases}$$

Fredholm operators are an important tool in the study of equation (2.1). If L is Fredholm then we know that a solution of (2.1) exists for all z in a space of finite codimension in Z; also for every $z \in L(Y)$ the set of solutions is given by $y + N$, where $y \in Y$ and $N = \ker L$ is a finite dimensional subspace of Y. Moreover the spaces $\ker L$ and $L(Y)$ are complemented in Y and Z, respectively, so that, for some closed subspaces Y' of Y and Z' of Z, there holds

$$Y = \ker L \oplus Y', \qquad Z = L(Y) \oplus Z'.$$

Fredholm operators can be considered "almost invertible" operators. In fact, L is Fredholm if and only if there exists an operator $H \in \mathcal{B}(Z,Y)$ such that $HL - I$ and $LH - I$ are compact operators. In this case H is called a *regulariser* of L.

We collect in the following lemmas useful properties of semi-Fredholm operators, using here and throughout the symbols $\pm$ and $\mp$ whenever we wish to combine two similar statements into one: the first is formed by taking the upper part of the symbols $\pm$,$\mp$, the second by taking the lower parts.

Lemma 2.4. *The sets $\Phi(Y)$ and $\Phi_\pm(Y)$ are open subsets of $\mathcal{B}(Y)$ and the index function* ind *is constant on the connected components of $\Phi(Y) \cup \Phi_-(Y) \cup \Phi_+(Y)$.*

Lemma 2.5. *If $L \in \Phi_+(Y,Z)$ then there exists a compact projection $P \in \mathcal{B}(Y, \ker L)$ such that, for some constant $C > 0$,*

$$\|x\| \le C(\|Lx\| + \|Px\|), \quad x \in Y.$$

Lemma 2.6 (Atkinson). *If $L \in \Phi_\pm(Y, Y')$ and $H \in \Phi_\pm(Y', Z)$ then $LH \in \Phi_\pm(Y, Z)$ and $\operatorname{ind} LH = \operatorname{ind} L + \operatorname{ind} H$.*

Lemma 2.7. *Suppose that $L \in \mathcal{B}(Y, Y')$ and $H \in \mathcal{B}(Y', Z)$ and $C \in \mathcal{K}(Y, Y')$. Then the following statements hold:*

1. *If $L \in \Phi_\pm(Y, Y')$ then $L + C \in \Phi_\pm(Y, Y')$ and $\operatorname{ind} L = \operatorname{ind}(L + C)$.*
2. *$HL \in \mathcal{M}$ and $H \in \Phi(Y', Z)$ imply $L \in \mathcal{M}$, where $\mathcal{M}$ denotes one of the sets $\Phi(Y, Y')$, $\Phi_\pm(Y, Y')$.*
3. *If L is normally solvable and $H \in \Phi_+(Y', Z)$ then HL is normally solvable.*

Obviously, all invertible operators are Fredholm with index zero. The previous lemma thus shows that every operator of the form $\lambda + C$, with $\lambda \neq 0$ and $C \in \mathcal{K}(Y)$, is Fredholm of index zero.

If $Y = Z$ then we define the *spectrum* $\Sigma_Y(L)$ and *essential spectrum* $\Sigma^e_Y(L)$ of the operator L by

$$\Sigma_Y(L) := \{\lambda \in \mathbb{C} : \lambda - L \text{ is not invertible}\},$$
$$\Sigma^e_Y(L) := \{\lambda \in \mathbb{C} : \lambda - L \text{ is not Fredholm}\}.$$

Note that $\Sigma^e_Y(L)$ is empty if and only if Y is finite-dimensional [35, p. 191]. Obviously, $\Sigma^e_Y(L) \subset \Sigma_Y(L)$. We also introduce the non-standard notation $\Sigma^+_Y(L)$ and $\Sigma^-_Y(L)$ for the sets

$$\Sigma^\pm_Y(L) := \{\lambda \in \mathbb{C} : \lambda - L \text{ is not a } \Phi_\pm\text{-operator}\}.$$

2.2 Weighted spaces

In this section Ω is one of the sets $\mathbb{R}_+$ or $\mathbb{R}$ and $X := BC(\Omega)$, the space of all bounded and continuous functions mapping Ω into $\mathbb{C}$.

One major aim of this thesis is to relate, for $L \in \mathcal{B}(X)$, the solvability of the equation

$$(\lambda - L)x = y \tag{2.2}$$

in X to its solvability in *weighted spaces* of continuous functions. Our assumption throughout most of the thesis is that the *weight function* $w \in C(\mathbb{R})$ is even and satisfies

$$w(0) = 1, \quad w(s) \geq w(t) \text{ for } s \geq t \geq 0, \quad \lim_{s\to\infty} w(s) = \infty. \tag{2.3}$$

In Section 5.4 it will also be useful to consider slightly more general weight functions $w \in C(\mathbb{R})$, requiring only that $w(s) \geq 1$, for all $s \in \mathbb{R}$. The remarks in this section pertain in both cases.

We denote by X_w the subspace of X consisting of all functions $x \in X$ satisfying $xw \in X$. X and X_w are Banach spaces, if we equip them with the norms

$$\|x\| := \sup_{s\in\Omega} |x(s)|, \qquad \|x\|_w := \|xw\|,$$

respectively. Throughout we will write $\|\cdot\|$ and $\|\cdot\|_w$ for the operator norms on $\mathcal{B}(X)$ and $\mathcal{B}(X_w)$.

Note that (2.2) as an equation on X_w is equivalent to the following equation on X:

$$(\lambda - L_w)x_w = y_w, \tag{2.4}$$

where $x_w := wx$, $y_w := wy$ and $L_w := M_w L M_{w^{-1}}$. Here M_z denotes the operator of multiplication by a function $z : \Omega \to \mathbb{R}$. Since $w(s) \geq 1$, $s \in \mathbb{R}$, the operator $M_w : X_w \to X$ is an isometric isomorphism with inverse $M_{w^{-1}}$. Thus for an operator $L : X_w \to X_w$ and $\lambda \in \mathbb{C}$ there holds,

$$\lambda - L_w \in \mathcal{M} \quad \Longleftrightarrow \quad \lambda - L \in \mathcal{M}_w, \tag{2.5}$$

where $\mathcal{M}$ is one of the spaces $\mathcal{B}(X), \mathcal{GL}(X), \Phi(X), \Phi_\pm(X)$ and $\mathcal{M}_w$ its counterpart in $\mathcal{B}(X_w), \mathcal{GL}(X_w)$, $\Phi(X_w), \Phi_\pm(X_w)$. Moreover, if $L : X \to X$ then

$$\lambda - L_w \text{ injective on } X \quad \Longleftrightarrow \quad \lambda - L \text{ injective on } X_w \quad \Longleftarrow \quad \lambda - L \text{ injective on } X, \tag{2.6}$$
$$\lambda - L_w \text{ normally solvable on } X \quad \Longleftrightarrow \quad \lambda - L \text{ normally solvable on } X_w. \tag{2.7}$$

Further, the following implications hold:

$$L \in \mathcal{B}(X_w) \quad \Longrightarrow \quad \|\lambda - L\|_w = \|\lambda - L_w\|, \tag{2.8}$$
$$\lambda - L \in \mathcal{GL}(X_w) \quad \Longrightarrow \quad \|(\lambda - L)^{-1}\|_w = \|(\lambda - L_w)^{-1}\|. \tag{2.9}$$

2.3 The strict topology and equicontinuous sets

In this section Ω is one of the sets $\mathbb{R}_+$, $\mathbb{R}$ or $\mathbb{R}^m$. Then $X := BC(\Omega)$, equipped with the sup-norm, is a Banach space. Let X_0 be the subset of all functions in X that vanish at infinity; explicitly, $x \in X_0$ if for every $\epsilon > 0$ there exists a $A > 0$ such that $|s| > A$ implies $|x(s)| < \epsilon$. X_0 is a closed subspace of X and, with the same norm, thus a Banach space in its own right.

For a sequence (f_n) in X, we will write $f_n \stackrel{s}{\to} f \in X$ and say that (f_n) converges *strictly* or is *s-convergent* to f if (f_n) is bounded and $f_n(\xi) \to f(\xi)$ uniformly for ξ in compact subsets of Ω (this is convergence in the strict topology of Buck [14]). We note that $f_n \stackrel{s}{\to} f$ implies that

$$\|f\| \le \sup_{n\in\mathbb{N}} \|f_n\|. \tag{2.10}$$

Further, we will say that a set $W \subset BC(\mathbb{R}^m)$ is *relatively s-sequentially compact* if every sequence in W contains an s-convergent subsequence, and we will call W *s-sequentially-compact* if, additionally, W contains the limit of each s-convergent sequence.

Recall that a set $W \subset X$ is called *equicontinuous on a set* $\Omega' \subset \Omega$, or simply *equicontinuous* when $\Omega' = \Omega$, if for all ξ, η in Ω' and every $\epsilon > 0$ there exists some $\delta > 0$ such that, for all $f \in W$ and $|\xi - \eta| < \delta$,

$$|f(\xi) - f(\eta)| \le \epsilon, \tag{2.11}$$

which implies that W is *uniformly equicontinuous* on every compact $\Omega'' \subset \Omega'$, i.e. for every $\epsilon > 0$ there exists $\delta > 0$ such that, for all $\xi, \eta \in \Omega''$ with $|\xi - \eta| < \delta$ and all $x \in W$, inequality (2.11) holds. We say that a sequence (f_n) in X is (uniformly) equicontinuous (on Ω') if the set $\{f_n : n \in \mathbb{N}\}$ is.

We have the following important characterization of relatively s-sequentially compact sets in terms of equicontinuity, which can be proved with the aid of the Arzelá-Ascoli theorem and a diagonal argument (see [3]). We will often use this characterization without explicit mention.

Remark 2.8. *A set $W \subset X$ is relatively s-sequentially compact if and only if it is bounded and equicontinuous.*

We will call a subset Y of X *s-sequentially dense* in X if every element of X is the limit of an s-convergent sequence in Y. Examples of such subsets are X_0, every weighted space X_w, with w satisfying (2.3), or the space of all compactly supported functions in X (if $x \in X$ then there exists a bounded sequence (x_n) of compactly supported functions with the property that $x_n(s) = x(s)$ for $|s| \le n$ and thus $x_n \stackrel{s}{\to} x$).

We will call $L \in \mathcal{B}(X)$ *s-continuous* if it is sequentially continuous in the strict topology, i.e. if

$$x_n \stackrel{s}{\to} x \quad \Longrightarrow \quad Lx_n \stackrel{s}{\to} Lx$$

holds. We will call L *sn-continuous* if the stronger requirement

$$x_n \stackrel{s}{\to} x \quad \Longrightarrow \quad Lx_n \to Lx$$

holds, where, here and throughout this thesis, $\to$ denotes convergence in the norm topology of X, i.e. uniform convergence on Ω. If L has the property that the image (Lx_n) of every bounded sequence (x_n) in X has an s-convergent subsequence then we call L *s-sequentially compact.*

If $\lambda - H \in \mathcal{B}(X)$ is a Fredholm operator and $L \in \mathcal{K}(X)$ then we know that $\lambda - H + L$ is Fredholm of the same index as $\lambda - H$, but this may fail for an arbitrary $L \in \mathcal{B}(X)$. However, the following theorem shows that we can recover this situation to some extent if H and L are s-sequentially compact and L is sn-continuous. This perturbation result will become important at several stages in this thesis and is a generalisation of a recent result in [7].

Theorem 2.9. *(cf. La. 2.2 in [7]) Let Y denote one of the spaces X and X_0. If $H, L \in B(Y)$, H is s-sequentially compact and L is sn-continuous then LH is compact. If also L is s-sequentially compact and $\lambda \neq 0$, then $\lambda - L$ is Fredholm of index zero. Moreover the equivalences*

$$\lambda - H + L \in \Phi(Y) \iff \lambda - H \in \Phi(Y),$$
$$\lambda - H + L \in \Phi_\pm(Y) \iff \lambda - H \in \Phi_\pm(Y)$$

hold and, if $\lambda - H + L$ and $\lambda - H$ are both semi-Fredholm, their indices are the same.

Proof. Choose a bounded sequence (x_n) in Y. Then (Hx_n) has a strictly convergent subsequence, (Hx_{n_m}). Since L is sn-continuous (LHx_{n_m}) is convergent. Thus LH is compact. Therefore, if L is also s-sequentially compact, then L^2 is compact and so $\lambda^{-1} + \lambda^{-2}L$ is a regulariser for $\lambda - L$. Thus $\lambda - L$ is Fredholm for all $\lambda \neq 0$. But, for $|\lambda| > \|L\|$, $(\lambda - L)^{-1} \in B(Y)$, so $\lambda - L$ has index zero. It follows from Lemma 2.4 that $\lambda - L$ has index zero for all $\lambda \neq 0$.

Let $\mathcal{M}$ denote one of the sets $\Phi(Y)$, $\Phi_\pm(Y)$ and suppose that $\lambda - H \in \mathcal{M}$. Then, by Lemma 2.7a), we have that $\lambda - H + \lambda^{-1}L(H - L) \in \mathcal{M}$ as $L(H - L) \in \mathcal{K}(Y)$. Since

$$\lambda^{-1}(\lambda - L)(\lambda - H + L) = \lambda - H + \lambda^{-1}L(H - L)$$

we conclude that the operator on the left-hand side of this equation must be in $\mathcal{M}$ and further, since $(\lambda - L) \in \Phi(Y)$, we obtain from Lemma 2.7 b) that $\lambda - H + L$ must be in $\mathcal{M}$. Thus $\lambda - H \in \mathcal{M} \Rightarrow \lambda - H + L \in \mathcal{M}$. A reversal of the argument, using Atkinson's lemma, shows the other direction of this implication.

Lastly, if $\lambda - H + L \in \Phi_\pm(Y)$ and $\lambda - H \in \Phi_\pm(Y)$ then, by Lemma 2.7 a) and Atkinson's lemma, there holds $\mathrm{ind}(\lambda - H) = \mathrm{ind}(\lambda - H + \lambda^{-1}L(H - L)) = \mathrm{ind}(\lambda - L) + \mathrm{ind}(\lambda - H + L) = \mathrm{ind}(\lambda - H + L)$. □

Another important feature of s-sequentially compact operators is given in the next lemma. It allows us to derive information about the range $(\lambda - H)(X)$ from properties of $(\lambda - H)(Y)$, where Y is an s-sequentially dense subset of X.

Lemma 2.10. *Suppose that $H \in \mathcal{B}(X)$ is an s-sequentially continuous and s-sequentially compact operator and that $\lambda \neq 0$. Let $L := \lambda - H$ and assume further that at least one of the sets $L(X)$ or $L(X_0)$ is closed in X. Then, if $Y \subset X_0$ is an s-sequentially dense subset of X, the s-sequential closure of $L(Y)$ is the set $L(X)$. Moreover, provided that $L(X)$ is closed, the set $L(X)$ is s-sequentially closed.*

Proof. Denote the s-sequential closure of $L(Y)$ by $\overline{L(Y)}^{ss}$. We firstly show that $\overline{L(Y)}^{ss} \subset L(X)$. Let (y_n) be a s-convergent sequence in $L(Y)$, $y_n \xrightarrow{s} y \in X$ say. Then there exists a sequence (x_n) in Y so that $Lx_n = y_n$ for every $n \in \mathbb{N}$. (y_n) is contained in $L(X_0) \subset L(X)$ and since at least one of these sets is closed in X we may assume that the sequence (x_n) is bounded (Lemma 2.1). As H is s-sequentially compact (Hx_n) contains an s-convergent subsequence, $Hx_{n_m} \xrightarrow{s} z \in X$ say. There holds $\lambda x_{n_m} = Hx_{n_m} + y_{n_m}$, $m \in \mathbb{N}$, so that $x_{n_m} \xrightarrow{s} \lambda^{-1}(y + z) =: x$. Since H is s-sequentially continuous $y_{n_m} = Lx_{n_m} \xrightarrow{s} Lx = y$, whence $y \in L(X)$. This proves the desired inclusion. A similar argument shows that $L(X)$ coincides with its s-sequential closure, provided that $L(X)$ is closed.

On the other hand, if $y \in L(X)$ then there exists $x \in X$ such that $Lx = y$. Since Y is s-sequentially dense in X there exists a sequence (x_n) in Y such that $x_n \xrightarrow{s} x$, whence $Lx_n = (\lambda - H)x_n \xrightarrow{s} y$. Hence $L(X) \subset \overline{L(Y)}^{ss}$ and thus we have shown $L(X) = \overline{L(Y)}^{ss}$. □

Corollary 2.11. *Suppose that $H \in \mathcal{B}(X)$ is an s-sequentially continuous and s-sequentially compact operator and that $\lambda \neq 0$. Assume further that at least one of the sets $(\lambda - H)(X_0)$ and $(\lambda - H)(X)$ is closed and that $(\lambda - H)(X_0)$ contains an s-sequentially dense subset of X. Then $(\lambda - H)(X) = X$.*

Proof. Let X' denote one of the spaces X_0 and X and suppose that $(\lambda - H)(X')$ is closed and that $(\lambda - H)(X_0)$ contains an s-sequentially dense subset X, i.e. $\overline{(\lambda - H)(X_0)}^{ss} = X$. Then the corollary follows by applying the previous lemma, with $Y = X_0$, to see that $(\lambda - H)(X) = \overline{(\lambda - H)(X_0)}^{ss} = X$. □

At several stages of this thesis the following result will become useful. It shows that every s-convergent sequence, which is bounded in a weighted space X_w, is norm-convergent in X.

Lemma 2.12. *Let $X = BC(\mathbb{R})$ or $X = BC(\mathbb{R}_+)$. Assume that the weight function w is even and satisfies (2.3). If (x_n) is a bounded sequence in the weighted space X_w then*

$$x_n \xrightarrow{s} x \quad \Longrightarrow \quad x_n \to x.$$

Proof. We only prove the lemma for $X = BC(\mathbb{R}_+)$, for $X = BC(\mathbb{R})$ the proof is similar. Suppose that (x_n) is as in the assumption and $x_n \xrightarrow{s} x$. Then there holds

$$|w(s)x(s)| \le \sup_{n \in \mathbb{N}} |w(s)x_n(s)| = \sup_{n \in \mathbb{N}} \|x_n\|_w < \infty, \quad s \in \mathbb{R}_+,$$

and we see that $x \in X_w$. Now, for $A > 0$,

$$\|x_n - x\| \le \sup_{0 \le s \le A} |x_n(s) - x(s)| + \sup_{s \ge A} |x_n(s) - x(s)| \le \sup_{0 \le s \le A} |x_n(s) - x(s)| + w(A)^{-1}\big(\|x_n\|_w + \|x\|_w\big).$$

Given $\epsilon > 0$, we can firstly choose A large enough so that the second term on the right-hand side of this inequality is $< \epsilon/2$. Then, if n is large enough, the first term is also $< \epsilon/2$, since $x_n \xrightarrow{s} x$, whence x_n converges to x uniformly on the compact set $[0, A]$. Thus $\|x_n - x\| < \epsilon$ for all n large enough, as required. □

Chapter 3

Integral equations over unbounded domains

In this chapter we provide the framework for our subsequent analysis of second-kind Fredholm integral equations on the real-line. Throughout most of this chapter Ω denotes one of the sets $\mathbb{R}_+$ and $\mathbb{R}$, and X the Banach space $BC(\Omega)$. For $A > 0$, we define $\Omega_A := \Omega \cap [-A, A]$.

We consider classes of integral equations on the real-line where the kernel is the product of a fixed kernel function v and a variable kernel function k, more precisely, integral equations of the form

$$\lambda x(s) - \int_\Omega v(s,t)k(s,t)x(t)\,dt = y(s), \quad s \in \Omega, \tag{3.1}$$

where $\lambda \in \mathbb{C}$. Throughout this thesis, we will assume that $v(s,\cdot) \in L^1(\Omega)$ for every $s \in \Omega$ and most of the time that k varies in a bounded and equicontinuous subset of $BC(\Omega^2)$.

Define the integral operator $K = K_k$ on X by

$$K_k x(s) := \int_\Omega v(s,t)k(s,t)x(t)dt, \quad s \in \Omega, x \in X, \tag{3.2}$$

so that we can abbreviate (3.1) in operator notation to

$$\lambda x - K_k x = y. \tag{3.3}$$

Suppose that $v(s,t)$ satisfies the following two assumptions:

Assumption (*A*).

$$\sup_{s\in\Omega} \int_\Omega |v(s,t)|\,dt < \infty.$$

Assumption (*B*).

$$\int_\Omega |v(s,t) - v(s+h,t)|\,dt \to 0 \quad \textit{as } h \to 0, \quad \textit{for every } s \in \Omega.$$

Then, by a well-known result (e.g. [40]), the operator $K = K_k$, for $k(s,t) = 1$, maps $L^\infty(\Omega)$ to X and is bounded with norm

$$\|K\|_{L^\infty(\Omega)\to X} = \|K\| = \sup_{s\in\Omega} \int_\Omega |v(s,t)|\,dt. \tag{3.4}$$

In the general case where $k \in BC(\Omega^2)$, (***A***) and (***B***) are still sufficient to ensure that K is bounded on X as the next proposition shows.

Proposition 3.1. *If* $(\boldsymbol{A})$ *and* $(\boldsymbol{B})$ *are satisfied and* $k \in BC(\Omega^2)$ *then the kernel* $v'(s,t) := v(s,t)k(s,t)$ *satisfies* $(\boldsymbol{A})$ *and* $(\boldsymbol{B})$*, so that* $K_k : L^\infty(\Omega) \to X$ *and is bounded with norm*

$$\|K_k\|_{L^\infty(\Omega)\to X} = \|K_k\| \leq \|k\| \sup_{s\in\Omega} \int_\Omega |v(s,t)|\, dt.$$

Proof. We have that

$$\int_\Omega |v'(s,t)|\, dt \leq \|k\| \int_\Omega |v(s,t)|\, dt, \qquad s \in \Omega,$$

so that v' satisfies $(\boldsymbol{A})$.

The main task is to show that v' satisfies $(\boldsymbol{B})$. Let $s \in \Omega$, then, for $s' \in \Omega$,

$$\begin{aligned} \int_\Omega |v(s,t)k(s,t) - v(s',t)k(s',t)|\, dt &\leq \int_\Omega |v(s,t)||k(s,t) - k(s',t)|\, dt + \int_\Omega |v(s,t) - v(s',t)||k(s',t)|\, dt \\ &\leq 2\|k\| \int_{\Omega\setminus\Omega_A} |v(s,t)|\, dt + \max_{t\in\Omega_A} |k(s,t) - k(s',t)| \int_{\Omega_A} |v(s,t)|\, dt \\ &\quad + \|k\| \int_\Omega |v(s,t) - v(s',t)|\, dt, \end{aligned}$$

Given $\epsilon > 0$, we can choose $A > 0$ large enough so that the first term on the right-hand side is $< \epsilon/3$. Further, since k is uniformly continuous on compact sets and $(\boldsymbol{B})$ holds, the second and third term on the right are then $< \epsilon/3$ if $|s' - s|$ is small enough. We have thus shown that the term on the left-hand side becomes arbitrarily small if s' approaches s, whence v' satisfies $(\boldsymbol{B})$.

In view of (3.4) and the preceding discussion, $K_k : L^\infty(\Omega) \to X$ and the norm bound holds. □

We notice that the argument of the previous theorem does not depend on the choice of k, provided k varies in a bounded and equicontinuous subset W of $BC(\mathbb{R}^2)$ — for then W is uniformly equicontinuous on every compact subset of $\Omega \times \Omega$. As a consequence the operator K_k is s-sequentially compact. We collect this and some other features of K with regard to the strict topology in the next proposition.

Proposition 3.2. *Suppose that* v *satisfies* $(\boldsymbol{A})$ *and* $(\boldsymbol{B})$ *and* (x_n)*,* (k_n) *are sequences in* X *and* $BC(\mathbb{R}^2)$*, respectively. Then:*

a) If $B \subset X$ *is bounded and* $W \subset BC(\Omega^2)$ *is bounded and equicontinuous then the set*

$$V := \{K_k x : k \in W, x \in B\}$$

is bounded and equicontinuous and thus relatively s*-sequentially compact.*

b) If the sequence (k_n) *is bounded and equicontinuous then* $(K_{k_n} x_n)$ *contains an* s*-convergent subsequence. In particular, this is the case when* (k_n) *is* s*-convergent.*

c) If $x_n \overset{s}{\to} x$ *then* $K_k x_n \overset{s}{\to} K_k x$*, for every* $k \in BC(\mathbb{R}^2)$*, i.e.* K_k *is* s*-continuous.*

d) If $k_n \overset{s}{\to} k$ *and* $x_n \overset{s}{\to} x$ *then* $K_{k_n} x_n \overset{s}{\to} K_k x$.

Proof. To see a), we note that, for $k \in W$ and $x \in B$,

$$|K_k x(s) - K_k x(s')| \leq \|x\| \int_\Omega |v(s,t)k(s,t) - v(s',t)k(s',t)|\, dt,$$

and, as mentioned above, the term on the right-hand side converges to 0 as $s' \to s$, uniformly in $k \in W$. a) now follows from Remark 2.8.

The first part of b) is an immediate consequence of a). But so is the second, for $k_n \overset{s}{\to} k$ implies that every sequence in the set $W := \{k_n : n \in \mathbb{N}\}$ has an s-convergent subsequence and thus, again by Remark 2.8, that the sequence (k_n) is equicontinuous.

We now prove part d), of which c) is a special case. Suppose that $k_n \xrightarrow{s} k$ and $x_n \xrightarrow{s} x$. We may assume w.l.o.g. that $\sup_{n\in\mathbb{N}} \|x_n\| \leq 1$ and $\sup_{n\in\mathbb{N}} \|k_n\| \leq 1$, whence $\|x\|, \|k\| \leq 1$ by (2.10). Now, let $s \in \Omega$. Then, for $A > 0$ and $n \in \mathbb{N}$, $|K_{k_n} x_n(s) - K_k x(s)|$ is bounded above by

$$\int_{\Omega_A} |v(s,t)|\big(|k_n(s,t)x_n(t) - k(s,t)x(t)|\big)\,dt + 2\int_{\Omega\setminus\Omega_A} |v(s,t)|\,dt$$
$$\leq \Big(\sup_{t\in\Omega_A} |k_n(s,t) - k(s,t)| + \sup_{t\in\Omega_A} |x_n(t) - x(t)|\Big)\int_\Omega |v(s,t)|\,dt + 2\int_{\Omega\setminus\Omega_A} |v(s,t)|\,dt.$$

Given $\epsilon > 0$, we can first choose A large enough so that the last term on the right-hand side is $< \epsilon/2$ and then, using the uniform convergence of (x_n) and (k_n) over compact intervals, n large enough so that the second summand is also $< \epsilon/2$. Thus we have shown that

$$\lim_{n\to\infty} K_{k_n} x_n(s) = K_k x(s), \quad s \in \Omega. \tag{3.5}$$

By the remark above, the s-convergent sequence (k_n) must be equicontinuous. Thus, by part a), the set $\{K_{k_n} x_n : n \in \mathbb{N}\}$ is bounded and equicontinuous. It now follows from (3.5) that $K_{k_n} x_n \xrightarrow{s} K_k x$, for pointwise convergence on Ω of a bounded and equicontinuous sequence implies uniform convergence over compact subsets of Ω. □

Suppose for a moment that (3.1) were an integral equation over a *compact* subset Ω of $\mathbb{R}$. An argument as in the previous proposition shows that Assumptions ***(A)*** and ***(B)*** would then imply that K is a compact operator (as it maps bounded sets onto bounded and equicontinuous sets so that the Arzelá-Ascoli theorem applies). The Riesz theory would then give strong solvability results for the integral equation (3.3). For example, one fundamental result would be that if $\lambda \neq 0$ then

$$\lambda - K \text{ is injective} \quad \Longrightarrow \quad \lambda - K \text{ is surjective and } (\lambda - K)^{-1} \text{ is bounded,} \tag{3.6}$$

yielding the "uniqueness implies existence" criterion important in many applications. But if the domain Ω of the integral equation is the whole real line or half-line then Assumptions ***(A)*** and ***(B)*** are not enough to ensure that K is compact as an operator on X, as simple counterexamples (see [18, Ex. 3.1]) show. However, not everything is lost: we still have that the image of every bounded set is relatively s-sequentially compact. We will make heavy use of this fact throughout this thesis.

As is shown in [3] for the half-line case where $v(s,t) = 0$ for $t < 0$, a sufficient condition for the compactness of K on X is the following: v satisfies ***(A)***, ***(B)*** and

$$\int_\Omega |v(s,t)k(s,t)|\,dt \to 0, \quad \text{as } |s| \to \infty, \qquad (s \in \Omega).$$

The latter condition is certainly fulfilled if $k \in BC(\Omega^2)$ and ***(C)*** is satisfied:

Assumption *(C)*.

$$\int_\Omega |v(s,t)|\,dt \to 0, \quad \textit{as } |s| \to \infty, \qquad (s \in \Omega).$$

Together with ***(A)*** and ***(B)***, a necessary, yet not sufficient, condition for compactness of K on X is

$$\sup_{s\in\Omega} \int_{\Omega\setminus\Omega_A} |v(s,t)k(s,t)|\,dt \to 0, \quad \text{as } A \to \infty,$$

as is shown in [18] for the half-line case. This condition is certainly fulfilled if $k \in BC(\Omega^2)$ and v satisfies the following assumption:

Assumption *(D)*.

$$\sup_{s\in\Omega} \int_{\Omega\setminus\Omega_A} |v(s,t)|\,dt \to 0, \quad \textit{as } A \to \infty.$$

This condition will play an important role in this thesis. Its importance lies in the fact that it implies the *sn*-continuity of the operator K (together with $(\boldsymbol{A})$ and $(\boldsymbol{B})$). We show this in the next proposition.

Proposition 3.3. *Suppose that the kernel v satisfies $(\boldsymbol{A})$, $(\boldsymbol{B})$ and $(\boldsymbol{D})$ and that $k \in BC(\mathbb{R}^2)$. Then K_k is sn-continuous, i.e. if (x_n) is a bounded sequence in X then*

$$x_n \xrightarrow{s} x \quad \Longrightarrow \quad K_k x_n \to K_k x.$$

Proof. For $s \in \Omega$ and $A > 0$, the term $|K_k x_n(s) - K_k x(s)|$, is bounded above by

$$\|k\| \left(\sup_{t \in \Omega_A} |x_n(t) - x(t)| \int_{\Omega_A} |v(s,t)|\, dt + 2 \sup_{n \in \mathbb{N}} \|x_n\| \sup_{s \in \Omega} \int_{\Omega \setminus \Omega_A} |v(s,t)| dt \right).$$

Given any $\epsilon > 0$, the last term in the bracket on the right-hand side can be made $\leq \epsilon/2$, for all $s \in \mathbb{R}$, by choosing A large enough; this is possible since v satisfies $(\boldsymbol{D})$. Keeping this A fixed, the first term can be made $\leq \epsilon/2$, for all $s \in \mathbb{R}$, by choosing n large enough because we have assumed $(\boldsymbol{A})$ and $x_n \xrightarrow{s} x$. Thus $K_k x_n \to K_k x$, as desired. □

Throughout most of the thesis we assume that v is bounded by a convolution kernel, precisely, that v satisfies the following assumption

Assumption $(\boldsymbol{A'})$. *There exists $\kappa \in L^1(\mathbb{R})$ such that, for all $s \in \Omega$, the following inequality is satisfied for a.e. $t \in \Omega$:*

$$|v(s,t)| \leq |\kappa(s-t)|.$$

We note that $(\boldsymbol{A'})$ clearly implies that $(\boldsymbol{A})$ holds. Assumption $(\boldsymbol{A'})$ is satisfied in most practical applications, in particular, it is satisfied if $v(s,t)$ is a convolution kernel, i.e. if $v(s,t) = \kappa(s-t)$, for some $\kappa \in L^1(\mathbb{R})$, in which case v also satisfies Assumption $(\boldsymbol{B})$ and $\|K\| = \sup_{s \in \mathbb{R}_+} \|v_s\|_1 = \|\kappa\|_1$. [40]

As is common practice, we often act as if the elements of $L^1(\mathbb{R})$ were functions rather than equivalence classes of functions.

Chapter 4

Spectral properties of integral operators in weighted spaces

In this chapter we compare the spectral properties of the Fredholm integral operator (3.2) on $X = BC(\mathbb{R})$ and the weighted subspaces X_w of X as defined in Section 2.2. For most of the chapter we assume that $k(s,t) = 1$ and that $v(s,t) = 0$ for $t < 0$, so that (3.2) reduces to an integral equation on the half line. This makes the notation somewhat simpler and we will point out how our results generalise to the real-line case in Section 4.5.

We thus focus our attention on the integral equation

$$\lambda x(s) - \int_0^\infty v(s,t)x(t)\,dt = y(s), \quad s \in \mathbb{R}_+, \tag{4.1}$$

where the given right-hand side y and the sought solution x belong to the space $X = BC(\mathbb{R}_+)$.

We have already mentioned that a well-studied special case of some interest is that when $v(s,t) = \kappa(s-t)$ for some $\kappa \in L^1(\mathbb{R})$, in which case (4.1) is the integral equation of Wiener-Hopf type

$$\lambda x(s) - \int_0^\infty \kappa(s-t)x(t)\,dt = y(s), \quad s \in \mathbb{R}_+. \tag{4.2}$$

In this case ***(A)*** and ***(B)*** are satisfied and $\|K\| = \|\kappa\|_1$.

As in (3.2), we define the half-line integral operator K on X as

$$Kx(s) = \int_0^\infty v(s,t)x(t)dt, \quad s \in \mathbb{R}_+. \tag{4.3}$$

4.1 Boundedness in weighted spaces

A first major aim of this chapter is to derive conditions on v and w, which ensure that $K : X_w \to X_w$ and is bounded. To this end, let K_w denote the integral operator defined by

$$K_w = M_w K M_{w^{-1}}, \tag{4.4}$$

where, for $w \in C(\mathbb{R}_+)$, M_w is the operation of multiplication by w. K_w is an integral operator of the form (4.3) and has the kernel function v_w given by

$$v_w(s,t) := \frac{w(s)}{w(t)} v(s,t), \quad s,t \in \mathbb{R}_+. \tag{4.5}$$

We note that the equivalences and implication in (2.5)–(2.7) hold for $L = K$ and $L_w = K_w$. Combining (2.5) and (3.4), we obtain a characterization of the boundedness of K on X_w:

Proposition 4.1. *Suppose that the kernel v satisfies Assumptions* $(\boldsymbol{A})$ *and* $(\boldsymbol{B})$*. Then $K \in \mathcal{B}(X_w)$ if and only if*

$$\sup_{s\in\mathbb{R}_+}\int_0^s |v_w(s,t)|\,dt = \sup_{s\in\mathbb{R}_+}\int_0^s \left|\frac{w(s)}{w(t)}v(s,t)\right|\,dt < \infty, \tag{4.6}$$

in which case v_w also satisfies $(\boldsymbol{A})$ *and* $(\boldsymbol{B})$*.*

Proof. *Necessity:* The following inequality shows that (4.6) is a necessary condition for $K \in \mathcal{B}(X_w)$:

$$\|K\|_w \overset{(2.8)}{=} \|K_w\| \overset{(3.4)}{=} \sup_{s\in\mathbb{R}_+}\int_0^\infty \frac{w(s)}{w(t)}|v(s,t)|\,dt \;\geq\; \sup_{s\in\mathbb{R}_+}\int_0^s \frac{w(s)}{w(t)}|v(s,t)|\,dt.$$

Sufficiency: We note that $0 < w(s)/w(t) \leq 1$ if $0 \leq s \leq t$. Thus,

$$\sup_{s\in\mathbb{R}_+}\int_0^\infty \left|\frac{w(s)}{w(t)}v(s,t)\right|\,dt \;\leq\; \sup_{s\in\mathbb{R}_+}\int_0^s \left|\frac{w(s)}{w(t)}v(s,t)\right|\,dt + \sup_{s\in\mathbb{R}_+}\int_0^\infty |v(s,t)|\,dt.$$

The rightmost term in this inequality is finite since $(\boldsymbol{A})$ is satisfied by v. Hence v_w satisfies $(\boldsymbol{A})$, provided (4.6) holds. To see that v_w also satisfies $(\boldsymbol{B})$, note that, for $s, s' \in \mathbb{R}_+$,

$$\begin{aligned}\int_0^\infty |v_w(s,t) - v_w(s',t)|\,dt &\leq |w(s)|\int_0^\infty \left|\frac{v(s,t)-v(s',t)}{w(t)}\right|\,dt + |w(s)-w(s')|\int_0^\infty \left|\frac{v(s',t)}{w(t)}\right|\,dt\\ &\leq |w(s)|\int_0^\infty |v(s,t)-v(s',t)|\,dt + |w(s)-w(s')|\int_0^\infty |v(s',t)|\,dt.\end{aligned}$$

As w is continuous and $(\boldsymbol{A})$ and $(\boldsymbol{B})$ are satisfied by v, the summands on the right-hand side of this inequality tend to 0 as $s' \to s$. It follows that v_w satisfies $(\boldsymbol{B})$ and, by Proposition 3.1, $K \in \mathcal{B}(X_w)$. ☐

In the remainder of the chapter we will assume that v satisfies $(\boldsymbol{A'})$ and that the kernel bound κ satisfies the following assumption:

$$\int_0^s \frac{|\kappa(s-t)|}{w(t)}\,dt = \int_0^s \frac{|\kappa(t)|}{w(s-t)}\,dt = O\Big(\frac{1}{w(s)}\Big), \quad \text{as } s\to\infty. \tag{4.7}$$

Clearly, Proposition 4.1 has the following corollary.

Corollary 4.2. *Suppose that the kernel v satisfies* $(\boldsymbol{A'})$ *and* $(\boldsymbol{B})$*. Then $K \in \mathcal{B}(X_w)$ if (4.7) holds. In the Wiener-Hopf case $v(s,t) = \kappa(s-t)$, with $\kappa \in L^1(\mathbb{R})$, $K \in \mathcal{B}(X_w)$ if and only if (4.7) holds.*

We note some simple consequences of condition (4.7). Firstly, it follows from (4.7) that

$$w(s)\int_s^{s+1} |\kappa(t)|\,dt = O(1), \quad \text{as } s\to\infty, \tag{4.8}$$

holds and that, for every $A > 0$,

$$\frac{1}{w(s-A)}\int_A^{2A} |\kappa(t)|\,dt = O\Big(\frac{1}{w(s)}\Big), \quad \text{as } s\to\infty, \tag{4.9}$$

because there holds, for $s > 2A$,

$$\int_A^{2A} \frac{|\kappa(t)|}{w(s-A)}\,dt \leq \int_A^{2A} \frac{|\kappa(t)|}{w(s-t)}\,dt \leq \int_0^s \frac{|\kappa(t)|}{w(s-t)}\,dt.$$

Unless $\kappa(t) = 0$ for almost all $t > 0$, the integral in (4.9) is non-zero for some $A > 0$, so that (4.9) implies that

$$\frac{w(s)}{w(s-A)} = O(1), \quad s\to\infty, \tag{4.10}$$

for some $A > 0$. But it is clear that (4.10) must then hold for all $A > 0$.

Inspired by these observations, let us introduce at this point two additional assumptions which play a key role in this thesis:

Assumption $(\boldsymbol{E})$.

$$\sup_{s\geq 2A}\int_A^{s-A}\frac{w(s)}{w(t)}|\kappa(s-t)|\,dt = O(1), \quad \textit{as } A\to\infty.$$

Assumption $(\boldsymbol{F})$.

$$\frac{w(s+1)}{w(s)} = O(1) \quad \textit{as } s\to\infty.$$

For weight functions w satisfying $(\boldsymbol{F})$ and $A>0$, we introduce the notation

$$\Delta_w^A := \sup_{|s-t|\leq A}\frac{w(s)}{w(t)} < \infty. \tag{4.11}$$

Clearly, $(\boldsymbol{F})$ limits the growth of w, implying that

$$w(s)\leq Ce^{bs}, \quad s\in\mathbb{R}_+,$$

for some constants $C>0$ and $b>0$.

The next proposition, our first important result, shows that $(\boldsymbol{A'})$, $(\boldsymbol{B})$, $(\boldsymbol{E})$ and $(\boldsymbol{F})$ are sufficient conditions to ensure that $K\in\mathcal{B}(X_w)$.

Proposition 4.3. *Assumption* (4.7) *implies* $(\boldsymbol{E})$. *Unless $\kappa(s)=0$ for almost all $s>0$,* (4.7) *also implies* $(\boldsymbol{F})$. *Conversely,* $(\boldsymbol{E})$ *and* $(\boldsymbol{F})$ *together imply* (4.7). *Thus, if v satisfies* $(\boldsymbol{A'})$, *with $\kappa\in L^1(\mathbb{R})$, and* $(\boldsymbol{B})$, $(\boldsymbol{E})$ *and* $(\boldsymbol{F})$ *hold, then $K\in\mathcal{B}(X_w)$ and v_w satisfies* $(\boldsymbol{A})$ *and* $(\boldsymbol{B})$.

In the Wiener-Hopf case $v(s,t)=\kappa(s-t)$, with $\kappa\in L^1(\mathbb{R})$, it holds that $K\in\mathcal{B}(X_w)$ if and only if $(\boldsymbol{E})$ *and* $(\boldsymbol{F})$ *are satisfied or $\kappa(s)=0$ for almost all $s>0$.*

Proof. The first two assertions are immediate from the definitions and the discussion in the preceding paragraph. We thus start by proving that $(\boldsymbol{E})$ and $(\boldsymbol{F})$ imply (4.7). Note that $(\boldsymbol{E})$ implies that, for some $A>0$ and $C>0$,

$$\int_A^{s-A}\frac{|\kappa(s-t)|}{w(t)}\,dt\leq\frac{C}{w(s)}, \quad s\geq 2A. \tag{4.12}$$

From this inequality it follows that

$$\frac{1}{w(2A)}\int_A^{2A}|\kappa(s-t)|\,dt\leq\int_A^{2A}\frac{|\kappa(s-t)|}{w(t)}\,dt\leq\frac{C}{w(s)}, \quad s\geq 3A.$$

Thus, for $s\geq 2A$,

$$\int_0^A\frac{|\kappa(s-t)|}{w(t)}\,dt\leq\int_0^A|\kappa(s-t)|\,dt=\int_A^{2A}|\kappa(s+A-t)|\,dt\leq C\frac{w(2A)}{w(s+A)}\leq C\frac{w(2A)}{w(s)}. \tag{4.13}$$

Also, by Assumption $(\boldsymbol{F})$,

$$\int_{s-A}^{s}\frac{|\kappa(s-t)|}{w(t)}\,dt\leq\frac{1}{w(s-A)}\int_0^A|\kappa(t)|\,dt\leq\frac{\Delta_w^A\,\|\kappa\|_1}{w(s)}. \tag{4.14}$$

Combining inequalities (4.12) through (4.14) we see that $(\boldsymbol{E})$ and $(\boldsymbol{F})$ imply (4.7). The rest of the proposition follows from Corollary 4.2 and Proposition 4.1. □

4.2 Solvability in weighted spaces

Having established conditions for the boundedness of K on X_w, we now turn our attention to the Fredholm and invertibility properties of $\lambda-K$ on X_w. We show that if $(\boldsymbol{A'})$ and $(\boldsymbol{B})$ and two stronger versions of **(*E*)** and **(*F*)** are satisfied then the much stronger result that the spectrum of K is the same on X as on X_w holds, i.e.

$$\Sigma_X(K) = \Sigma_{X_w}(K), \tag{4.15}$$
$$\Sigma_X^e(K) = \Sigma_{X_w}^e(K), \tag{4.16}$$

We will also show that then $\Sigma_X^{\pm}(K) = \Sigma_{X_w}^{\pm}(K)$.

Because of equation (2.5) we are able to relate the invertibility and Fredholm properties of $\lambda - K$ on X to those of $\lambda - K$ on X_w by comparing the operators $\lambda - K$ and $\lambda - K_w$ acting on X. The difference between the second and the first of these operators is $K - K_w$, an integral operator of the form (4.3) with kernel $v - v_w$. In many cases, for example [20] if $\kappa(s) = O(s^{-q})$ as $s \to \infty$ for some $q > 1$ and $w(s) = (1+s)^p$, with $0 < p < q$, it holds that $K - K_w$ is compact on X, so that $\lambda - K$ is Fredholm if and only if $\lambda - K_w$ is Fredholm. To obtain the sharpest results, i.e. to show (4.15) and (4.16) for the widest class of weight functions w, it will prove important also to consider cases when $K - K_w$ is not compact but $v - v_w$ satisfies **(*A*)**, **(*B*)** and **(*D*)**. For such operators we have the following perturbation result as an immediate consequence of Theorem 2.9.

Theorem 4.4. *Suppose K, K' are two integral operators of the form* (4.3) *with kernels v, v' satisfying conditions* **(*A*)**, **(*B*)** *and v' also satisfying* **(*D*)**. *Then $K'K$ is a compact operator on X. If, in addition, $\lambda \neq 0$, then $\lambda - K' \in \Phi(X)$ with index zero,*

$$\lambda - K + K' \in \Phi(X) \quad \Longleftrightarrow \quad \lambda - K \in \Phi(X), \tag{4.17}$$

and

$$\lambda - K + K' \in \Phi_\pm(X) \quad \Longleftrightarrow \quad \lambda - K \in \Phi_\pm(X),$$

and if the operators in (4.17) are both Fredholm then their indices are the same. Moreover, if $K, K' \in \mathcal{B}(X_0)$ then the theorem also holds with X replaced by X_0.

Proof. The assumptions imply that K and K' are s-sequentially compact (Proposition 3.2) and that K' is sn-continuous (Proposition 3.3). Theorem 2.9 now yields the desired result. □

Clearly, we set $K' := K - K_w$ and hope to find conditions on v so that $v - v_w$ satisfies **(*A*)**, **(*B*)** and **(*D*)**. Let us consider first the Wiener-Hopf case when $v(s,t) = \kappa(s-t)$ for some $\kappa \in L^1(\mathbb{R})$. Since

$$\frac{w(s)}{w(t)} = \left|1 - \frac{w(s)}{w(t)}\right| + 1, \quad 0 \le t \le s,$$

we have that

$$\sup_{s \ge 2A} \int_A^{s-A} \frac{w(s)}{w(t)} |\kappa(s-t)|\, dt \le \sup_{s \ge 2A} \int_A^{\infty} \left|1 - \frac{w(s)}{w(t)}\right| |\kappa(s-t)|\, dt + \sup_{s \ge 2A} \int_A^{s-A} |\kappa(s-t)|\, dt.$$

Now, for $s \ge 2A$,

$$\int_A^{s-A} |\kappa(s-t)|\, dt \le \int_A^{\infty} |\kappa(u)|\, du \to 0,$$

as $A \to \infty$. Thus, in the Wiener-Hopf case, if $v - v_w$ satisfies **(*D*)** then the following stronger version of **(*E*)** holds:

Assumption (*E′*).

$$\sup_{s \ge 2A} \int_A^{s-A} \frac{w(s)}{w(t)} |\kappa(s-t)|\, dt \to 0, \quad \text{as } A \to \infty.$$

Moreover, if κ does not vanish a.e., and $v - v_w$ satisfies **(*D*)**, then a stronger version of **(*F*)** also holds, namely

Assumption $(\boldsymbol{F'})$.

$$\frac{w(s+1)}{w(s)} \to 1, \quad \textit{as } s \to \infty.$$

This assumption limits the growth of w still further, implying that for all $b > 0$,

$$w(s) = o(e^{bs}), \quad s \to \infty.$$

To see that $(\boldsymbol{D})$ implies $(\boldsymbol{F'})$ in the Wiener-Hopf case, suppose that $(\boldsymbol{F'})$ does not hold. Then since, for all $\delta > 0$, $w(s+1)/w(s) \to 1$ as $s \to \infty$ if and only if $w(s+\delta)/w(s) \to 1$ as $s \to \infty$, it follows that for every $\delta > 0$ there exists $\epsilon > 0$ and a sequence (s_n) of positive numbers with $s_n \to \infty$ as $n \to \infty$ such that

$$\frac{w(s_n+\delta)}{w(s_n)} \geq 1+\epsilon, \quad n \in \mathbb{N}.$$

It follows that, for every n,

$$\frac{w(s_n+\delta)}{w(t)} \geq 1+\epsilon, \quad 0 \leq t \leq s_n, \qquad \frac{w(s_n)}{w(t)} \leq \frac{1}{1+\epsilon}, \quad t \geq s_n+\delta.$$

Now, if $v(s,t) = \kappa(s-t)$ and $v - v_w$ satisfies $(\boldsymbol{D})$ then for every $\eta > 0$ there exists $A > 0$ such that

$$\sup_{s \in \mathbb{R}_+} \int_A^\infty \left|1 - \frac{w(s)}{w(t)}\right| |\kappa(s-t)|\, dt < \eta.$$

This implies that, for every n for which $s_n > A$, we have

$$\epsilon \int_A^{s_n} |\kappa(s_n+\delta-t)|\, dt < \eta, \qquad \frac{\epsilon}{1+\epsilon} \int_{s_n+\delta}^\infty |\kappa(s_n-t)|\, dt < \eta.$$

Since $s_n \to \infty$ as $n \to \infty$, it follows that

$$\left(\int_{-\infty}^{\delta} + \int_{\delta}^{\infty}\right) |\kappa(t)|\, dt < \eta \frac{1+\epsilon}{\epsilon},$$

for all $\eta > 0$. Thus $\kappa(t) = 0$ for almost all t with $|t| > \delta$ and, since this holds for every $\delta > 0$, we have that $\kappa = 0$.

In the proof of the following theorem, we show that, conversely, $(\boldsymbol{E'})$ and $(\boldsymbol{F'})$ are sufficient conditions to ensure that $v - v_w$ satisfies $(\boldsymbol{D})$ whenever $(\boldsymbol{A'})$ holds with $\kappa \in L^1(\mathbb{R})$.

Theorem 4.5. *Suppose v and w satisfy Assumptions $(\boldsymbol{A'})$, $(\boldsymbol{B})$, $(\boldsymbol{E'})$ and $(\boldsymbol{F'})$, with $\kappa \in L^1(\mathbb{R})$. Then the difference kernel $v - v_w$ satisfies conditions $(\boldsymbol{A})$, $(\boldsymbol{B})$ and $(\boldsymbol{D})$, so that $K - K_w$ is an sn-continuous operator. In the Wiener-Hopf case $v(s,t) = \kappa(s-t)$, with $\kappa \in L^1(\mathbb{R})$, $v - v_w$ satisfies $(\boldsymbol{A})$, $(\boldsymbol{B})$ and $(\boldsymbol{D})$ if and only if κ and w satisfy $(\boldsymbol{E'})$ and $(\boldsymbol{F'})$ or $\kappa = 0$.*

Proof. If v and w satisfy $(\boldsymbol{A'})$, $(\boldsymbol{B})$, $(\boldsymbol{E'})$ and $(\boldsymbol{F'})$, then from Proposition 4.3 and Corollary 4.2 we have that v_w satisfies $(\boldsymbol{A})$ and $(\boldsymbol{B})$, so $v - v_w$ must also satisfy $(\boldsymbol{A})$ and $(\boldsymbol{B})$. It remains to check whether $v - v_w$ fulfils $(\boldsymbol{D})$.

Let $s \geq 0$ and $0 < A^* < A/2$. We have

$$\begin{aligned} \int_A^\infty \left|\left(1 - \frac{w(s)}{w(t)}\right) v(s,t)\right| dt &\leq \int_A^\infty \left|1 - \frac{w(s)}{w(t)}\right| |\kappa(s-t)|\, dt \\ &\leq \left(\int_A^{\max\{s-A^*,A\}} + \int_{\max\{s-A^*,A\}}^{\max\{A,s+A^*\}} + \int_{s+A^*}^\infty\right) \left|1 - \frac{w(s)}{w(t)}\right| |\kappa(s-t)|\, dt. \end{aligned} \quad (4.18)$$

We use $(\boldsymbol{E'})$ to bound the first integral on the right-hand side of equation (4.18). Note that it is non-zero only if $s \geq A + A^*$. Further, if $s \geq A + A^* > 2A^*$ then

$$\int_A^{\max\{s-A^*,A\}} \Big|1 - \frac{w(s)}{w(t)}\Big| |\kappa(s-t)|\, dt \quad \leq \quad \int_A^{s-A^*} \frac{w(s)}{w(t)} |\kappa(s-t)|\, dt \leq E_{A^*},$$

where

$$E_{A^*} := \sup_{s \geq 2A^*} \int_{A^*}^{s-A^*} \frac{w(s)}{w(t)} |\kappa(s-t)|\, dt \to 0$$

as $A^* \to \infty$, as a consequence of Assumption $(\boldsymbol{E'})$.

The second integral in (4.18) vanishes for $s \leq A - A^* < A/2$. So

$$\int_{\max\{s-A^*,A\}}^{\max\{A,s+A^*\}} \Big|1 - \frac{w(s)}{w(t)}\Big| |\kappa(s-t)|\, dt \leq c_{A^*}(A/2)\|\kappa\|_1,$$

where

$$c_{A^*}(A) := \sup_{s \geq A} \max\left\{1 - \frac{w(s)}{w(s+A^*)}, \frac{w(s)}{w(s-A^*)} - 1\right\} \to 0, \quad \text{as } A \to \infty,$$

as follows by Assumption $(\boldsymbol{F'})$.

Lastly, since $0 \leq 1 - w(s)/w(t) \leq 1$ for $s \leq t$, we have for the third integral in (4.18) that

$$\int_{s+A^*}^{\infty} \Big|1 - \frac{w(s)}{w(t)}\Big| |\kappa(s-t)|\, dt \leq \int_{-\infty}^{-A^*} |\kappa(u)|\, du \to 0,$$

as $A^* \to \infty$. Thus

$$\sup_{s \geq 0} \int_A^{\infty} \left| \left(1 - \frac{w(s)}{w(t)}\right) v(s,t) \right| \leq E_{A^*} + \int_{-\infty}^{-A^*} |\kappa(u)|\, du + c_{A^*}(A/2)\|\kappa\|_1,$$

and, given any $\epsilon > 0$, we can choose A^* such that the sum of the first two terms on the right-hand side of this inequality is less than ϵ, and then $c_{A^*}(A/2)\|\kappa\|_1 < \epsilon$ for all sufficiently large A. Therefore, $v - v_w$ satisfies $(\boldsymbol{D})$ and, by Proposition 3.3, $K - K_w$ is thus *sn*-continuous.

The results for the Wiener-Hopf case follow from the paragraphs preceding Theorem 4.5 and as a special case of the general result, since $(\boldsymbol{A'})$ and $(\boldsymbol{B})$ are then automatically satisfied. □

If the conditions of Theorem 4.5 hold we may invoke Theorem 4.4 with $K' := K - K_w$ to obtain the following central theorem of the present chapter. Here and henceforth we will use the following notation: If $\kappa \in L^1(\mathbb{R})$ then we let $\mathcal{W}(\kappa)$ stand for the collection of all even $w \in C(\mathbb{R})$ fulfilling (2.3) and Assumptions $(\boldsymbol{E'})$ and $(\boldsymbol{F'})$. (We mention the important fact that $\mathcal{W}(\kappa)$ is never the empty set, but defer its proof until the end of Section 4.4.)

Theorem 4.6. *Suppose that v satisfies $(\boldsymbol{A'})$, $(\boldsymbol{B})$, with $\kappa \in L^1(\mathbb{R})$ in $(\boldsymbol{A'})$, and that $w \in \mathcal{W}(\kappa)$. Then, for every $\lambda \in \mathbb{C}$, there holds*

$$(\lambda - K) \in \mathcal{M} \iff (\lambda - K_w) \in \mathcal{M} \iff (\lambda - K) \in \mathcal{M}_w, \tag{4.19}$$

where $\mathcal{M}$ denotes one of the spaces $\mathcal{GL}(X), \Phi(X), \Phi_\pm(X)$ and $\mathcal{M}_w$ its counterpart in $\mathcal{GL}(X_w)$, $\Phi(X_w)$, $\Phi_\pm(X_w)$. The indices of $\lambda - K$ on X and $\lambda - K$ on X_w coincide if $\lambda - K$ is semi-Fredholm. Moreover,

$$0 \in \Sigma_X^e(K) = \Sigma_X^e(K_w) = \Sigma_{X_w}^e(K), \tag{4.20}$$
$$0 \in \Sigma_X(K) = \Sigma_X(K_w) = \Sigma_{X_w}(K), \tag{4.21}$$
$$0 \in \Sigma_X^\pm(K) = \Sigma_X^\pm(K_w) = \Sigma_{X_w}^\pm(K). \tag{4.22}$$

In order to prove this theorem, we need the following auxiliary proposition, which shows that 0 is contained in all the spectra in (4.20)–(4.22).

Proposition 4.7. *Suppose that v satisfies ($\boldsymbol{A'}$), ($\boldsymbol{B}$), with $\kappa \in L^1(\mathbb{R})$ in ($\boldsymbol{A'}$), and that $w \in \mathcal{W}(\kappa)$. Then $K \notin \Phi_\pm(X)$ and $K \notin \Phi_\pm(X_w)$. In particular K is not Fredholm and not invertible on X and X_w.*

Proof. We will prove all claims by contradiction.

(i) Suppose first that $K \in \Phi_-(X)$. Then $X = K(X) \oplus Z$ for some finite dimensional subspace Z of X. Let P denote the projection from X onto $K(X)$ along Z and define the sequence (x_n) by $x_n(s) := \exp(ins)$, $n \in \mathbb{N}$. This sequence is obviously not equicontinuous. Now, we let $(z_n) := (Px_n)$. Since (x_n) is bounded and P is continuous and has finite dimensional range (z_n) is a bounded and equicontinuous sequence. Thus the sequence in $K(X)$ given by $(y_n) := (x_n - z_n)$ is bounded but not equicontinuous. However, K is normally solvable and, by Lemma 2.1, there exists a bounded sequence (x'_n) in X such that $Kx'_n = y_n$, $n \in \mathbb{N}$. Since v satisfies ($\boldsymbol{A}$) and ($\boldsymbol{B}$) Proposition 3.2 applies, showing that the sequence $(Kx'_n) = (y_n)$ must be equicontinuous, a contradiction. Hence, our assumption was wrong and $K \notin \Phi_-(X)$. The assumptions of the proposition imply, by Proposition 4.3, that v_w satisfies ($\boldsymbol{A}$) and ($\boldsymbol{B}$) so that a similar argument shows that $K_w \notin \Phi_-(X)$ and thus, by (2.7), $K \notin \Phi_-(X_w)$.

(ii) Now suppose that $K \in \Phi_+(X)$. We choose a sequence of functions (x_n) in X so that

$$\|x_n\| = 1, \quad \operatorname{supp} x_n \subset \big[n, n + \tfrac{1}{n}\big), \qquad n \in \mathbb{N}. \tag{4.23}$$

Let $X' := \overline{\operatorname{span}}\{x_n : n \in \mathbb{N}\}$, the closure of the linear space spanned by these functions. Since $n \neq m$ implies that x_n and x_m have disjoint supports it follows that each $x \in X'$ must take the form $x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots$, where (α_n) is a sequence in $\mathbb{C}$; moreover, x is then the limit of a sequence in the closed subspace X_0 of X, for each element of $\operatorname{span}\{x_n : n \in \mathbb{N}\}$ has compact support and so lies in X_0. But this means $x \in X_0$ and thus $\alpha_n \to 0$ as $n \to \infty$. On the other hand, if (α_n) is a sequence with $\alpha_n \to 0$ as $n \to \infty$ then $x := \alpha_1 x_1 + \alpha_2 x_2 + \ldots$ lies in X'. Thus

$$X' = \Big\{ \sum_{n \in \mathbb{N}} \alpha_n x_n : \lim_{n \to \infty} \alpha_n = 0 \Big\} \subset X_0.$$

It is clear that, if $U \subset X'$ is bounded then U is equicontinuous. Hence each bounded subset of X' is relatively s-sequentially compact (see Remark 2.8).

Since X' is closed and $\ker_X(K)$ is finite-dimensional, $X' + \ker_X(K)$ is closed. It follows from Lemma 2.2 that the restriction $K' := K|_{X'}$ of K to X' is normally solvable. Moreover, $N := \ker K'$ is finite-dimensional, whence $K' \in \Phi_+(X', X)$. By Lemma 2.5, there exists a compact projection P from X' onto N which satisfies

$$\|x\| \leq C(\|K'x\| + \|Px\|), \quad x \in X', \tag{4.24}$$

where $C > 0$ is a constant. Since P is compact and (x_n) is bounded, we obtain $Px_{n_m} \to x_0$ for some $x_0 \in N$ and a subsequence (x_{n_m}) of (x_n). By (4.24), it holds that

$$\|x_{n_m} - x_0\| \leq C\big(\|K'(x_{n_m} - x_0)\| + \|P(x_{n_m} - x_0)\|\big), \quad m \in \mathbb{N}. \tag{4.25}$$

Observe that, since $x_0 \in N \subset X_0$ and (4.23) holds, the left-hand side converges to 1 as $m \to \infty$. But the terms in the bracket on the right-hand side converge to 0 as $m \to \infty$: since $x_0 \in N$ holds $K'(x_{n_m} - x_0) = K'x_{n_m}$ and, as v satisfies ($\boldsymbol{A'}$),

$$|Kx_{n_m}(s)| \leq \int_0^\infty |\kappa(s-t)||x_{n_m}(t)|\, dt \leq \int_{n_m}^{n_m + 1/n_m} |\kappa(s-t)|\, dt \to 0,$$

as $m \to \infty$, uniformly in $s \geq 0$. This contradicts (4.25), so that $K' \notin \Phi_+(X', X)$. Thus our initial assumption was wrong, i.e. $K \notin \Phi_+(X)$.

(iii) Next, we assume that $K_w \in \Phi_+(X)$. Then the argument in (ii) yields that $K'_w \in \Phi_+(X', X)$, where $K'_w := K_w|_{X'}$ and X' is defined as above. It follows from the assumptions that $(K - K_w)$ is sn-continuous, see Theorem 4.5. Since each bounded set in X' is relatively s-sequentially compact we obtain that $(K' - K'_w) \in \mathcal{K}(X', X)$. But, by Lemma 2.7, this means that $K' = K'_w + (K' - K'_w) \in \Phi_+(X', X)$, which cannot be true as (ii) has shown. Thus $K_w \notin \Phi_+(X)$ and, by (2.5), $K \notin \Phi_+(X_w)$. □

With this proposition at hand, we now commence the proof of Theorem 4.6.

Proof. Proposition 4.7 has just shown us that 0 is contained in all of the sets in (4.20)–(4.22). Note that, by (2.5), we only need to show the first equalities in equations (4.20)–(4.22).

If $\lambda \neq 0$ and $\mathcal{M}$ is one of the sets $\Phi(X)$ and $\Phi_\pm(X)$ then (4.19) and the statement about the indices immediately follow from Theorem 4.4 (applied with $K' = K - K_w$) and Theorem 4.5. Together with the discussion of the case $\lambda = 0$ this shows that (4.20) and (4.22) hold.

To establish (4.19) when $\mathcal{M} = \mathcal{GL}(X)$, and thus (4.21), note that, by what we have just shown, $(\lambda - K)^{-1} \in \mathcal{B}(X)$ implies that $\lambda - K$ is injective and Fredholm of index zero on $X_w \subset X$. But this means that $(\lambda - K) : X_w \to X_w$ is also surjective, and thus, by Banach's inverse theorem, $(\lambda - K)^{-1} \in \mathcal{B}(X_w)$.

For the other direction, if $(\lambda - K)^{-1} \in \mathcal{B}(X_w)$, then $X_w \subset (\lambda - K)(X)$ and also, by what has already been shown, $\lambda - K$ is Fredholm of index zero on X, so that $(\lambda - K)(X)$ is closed in X. From Corollary 2.11 it follows that $\lambda - K : X \to X$ is surjective, for $(\lambda - K)(X)$ contains the s-sequentially dense subset X_w of X. Since the surjective operator $\lambda - K$ has index zero on X, it must also be injective, whence $\lambda \notin \Sigma_X(K)$ by Banach's inverse theorem. □

As an immediate consequence of Theorem 4.6 we have the following corollary on the solvability of the integral equation (4.1).

Corollary 4.8. *Suppose v satisfies Assumptions **(A′)** and **(B)**, with $\kappa \in L^1(\mathbb{R})$ in **(A′)**. Assume further that, for* some *$w \in \mathcal{W}(\kappa)$ or for $w(s) = 1$, the integral equation* (4.1) *has an unique solution $x \in X_w$ for every $y \in X_w$. Then, for* all *$w \in \mathcal{W}(\kappa)$ and for $w(s) = 1$, the integral equation* (4.1) *has an unique solution $x \in X_w$ for every $y \in X_w$ and*

$$\sup_{s \in \mathbb{R}_+} |w(s)x(s)| = \|x\|_w \leq C\|y\|_w = C \sup_{s \in \mathbb{R}_+} |w(s)y(s)|,$$

where C is a positive constant depending only on w, v and λ.

Apart from this straightforward interpretation, we can derive more subtle results from Theorem 4.6. One of them is: whenever the assumptions of Theorem 4.6 are satisfied and $\lambda - K$ is Fredholm, then the null space of $\lambda - K$ (on X) is contained in the intersection of all X_w, $w \in \mathcal{W}(\kappa)$. This result is a consequence of the next proposition, which shows that the range $(\lambda - K)(X)$ has a complement (with respect to the space X) that is contained in X_w, provided $\lambda - K \in \Phi_-(X)$.

Proposition 4.9. *Suppose that v satisfies **(A′)**, **(B)**, with $\kappa \in L^1(\mathbb{R})$ in **(A′)**, and that $w \in \mathcal{W}(\kappa)$. Let $\lambda \in \mathbb{C}$. Then, if one of the operators $L := \lambda - K$ on X and $L^w := L|_{X_w}$ on X_w is a Φ_--operator then so is the other. Moreover, if both operators are Φ_- then there exist finite-dimensional subspaces N_w and N of X_w such that N is contained in N_w and*

$$X = L(X) \oplus N, \quad X_w = L(X_w) \oplus_{X_w} N_w. \tag{4.26}$$

In particular, there holds $\beta(L) \leq \beta(L^w)$.

Proof. The equivalence has already been shown in Theorem 4.6. It remains to prove (4.26). To this end, suppose that L and L^w are both Φ_--operators. Then we have the following decomposition

$$X_w = L(X_w) \oplus_{X_w} N_w,$$

where $L(X_w)$ is closed in X_w and N_w a finite-dimensional subspace X_w.

The subspace $L(X) \cap N_w$, which might be 0-dimensional, is complemented in N_w. Let N denote one of its complementary spaces. Then $X_w = L(X_w) \oplus_{X_w} \big(L(X) \cap N_w\big) \oplus_{X_w} N$; moreover

$$X_w \subset L(X) + N. \tag{4.27}$$

We now show that N is the sought complementary space of $L(X)$, i.e. that there holds

$$X = L(X) \oplus N. \tag{4.28}$$

By construction, $N \cap L(X) = \{0\}$. Moreover, both $L(X)$ (by assumption) and N (since it is finite-dimensional) are closed subspaces of X, where here and in the remainder of the proof the terms *closed* and *bounded* are always to be understood with respect to the norm topology of X. Hence, we only need to show that $L(X) + N$ is indeed the whole space X.

Since $L(X) + N$ is the sum of a closed and a finite-dimensional subspace of X it is closed and, equipped with the norm of X, a Banach space in its own right. Thus the projection P from $L(X) + N$ onto N along $L(X)$ is continuous and, since it has finite-dimensional range, also compact.

Choose an arbitrary $y \in X$. We have to show that $y = y' + y''$ with $y' \in N$ and $y'' \in L(X)$. Since X_w is s-sequentially dense in X there exists a sequence (y_n) in X_w such that $y_n \overset{s}{\to} y$. Note that by, (4.27), this sequence is contained in $L(X) + N$. Since this sequence must be bounded and P is compact, $(y'_n) := (Py_n)$ has a norm convergent subsequence, $y'_{n_m} \to y' \in N$ say. Thus $y_{n_m} - y'_{n_m} \overset{s}{\to} y - y' =: y''$. But the sequence $(y_{n_m} - y'_{n_m})$ is contained in $L(X)$. Since v satisfies $\boldsymbol{(A')}$ and $\boldsymbol{(B)}$ K is s-sequentially continuous and s-sequentially compact and thus, by Lemma 2.10, we know that $L(X)$ is sequentially closed with respect to the strict topology. Thus the limit y'' must be in $L(X)$, whence $y = y' + y'' \in N + L(X)$ follows. Thus (4.28) holds and the theorem is shown. □

As a corollary we obtain in the next proposition, under the same assumptions on v and w, that if $\lambda - K$ is Fredholm then the kernel of $\lambda - K$ is contained in X_w.

Proposition 4.10. *Suppose that v satisfies $\boldsymbol{(A')}$, $\boldsymbol{(B)}$, with $\kappa \in L^1(\mathbb{R})$ in $\boldsymbol{(A')}$, and that $w \in \mathcal{W}(\kappa)$. Then, for every $\lambda \neq 0$, if one of the operators $L := \lambda - K$ on X and $L^w := L|_{X_w}$ on X_w is Fredholm then so is the other and moreover,*

$$\alpha(L) = \alpha(L_w), \quad \beta(L) = \beta(L_w), \tag{4.29}$$

so that $\ker L = \ker L^w \subset X_w$.

Proof. By previous results, we already know if one of the operators L and L^w is Fredholm then so is the other and the indices of both coincide. The null space $\ker L^w$ is contained in $\ker L$, and hence $\alpha(L^w) \leq \alpha(L)$ and $\beta(L^w) \leq \beta(L)$. But Proposition 4.9 shows that $\beta(L^w) \geq \beta(L)$, whence $\beta(L^w) = \beta(L)$. From the equality of the indices we then also get that (4.29) holds and the corollary follows. □

This proposition has a noteworthy consequence for the solvability of the integral equation (4.1), when $\lambda - K$ is Fredholm: for a given $y \in X$, any two solutions $x_1(s), x_2(s)$ of (4.1), if they exist, show the same behaviour as $s \to \infty$.

Corollary 4.11. *Suppose that v satisfies $\boldsymbol{(A')}$, $\boldsymbol{(B)}$, with $\kappa \in L^1(\mathbb{R})$ in $\boldsymbol{(A')}$, and that $w \in \mathcal{W}(\kappa)$. Further, assume that $\lambda \notin \Sigma^{\mathrm{e}}_X(K) = \Sigma^{\mathrm{e}}_{X_w}(K)$. Then the integral equation* (4.1) *has at least one solution $x_1 \in X$ if and only if $y \in (\lambda - K)(X)$; in this case the set S of* all *solutions of* (4.1) *with right-hand side y takes the form*

$$S = x_1 + \ker_{X_w}(\lambda - K).$$

Moreover,

$$\lim_{s\to\infty} \left|x_1(s) - x_2(s)\right| = 0, \quad x_2 \in S,$$

so that, provided the limit $\lim_{s\to\infty} x_1(s)$ exists, there holds

$$\lim_{s\to\infty} x_1(s) = \lim_{s\to\infty} x_2(s), \quad x_2 \in S.$$

4.3 Sharpness of Assumptions

In the special case that $v(s,t) = 0$ for $0 \leq t \leq s$, by Proposition 4.1, $K \in \mathcal{B}(X_w)$ for every w satisfying (2.3), as we have observed already for the Wiener-Hopf case in Proposition 4.3. Slightly more can be said about the relationship between $\Sigma_X(K)$ and $\Sigma_{X_w}(K)$ in this case.

Theorem 4.12. *If v satisfies **(A)** and **(B)** and $v(s,t)=0$ for $0 \le t \le s$, then $K \in \mathcal{B}(X_w)$ and*

$$\Sigma_{X_w}(K) \subset \Sigma_X(K). \tag{4.30}$$

*If also **(F′)** holds and v satisfies **(A′)** for some $\kappa \in L^1(\mathbb{R})$, in which case κ can be chosen with $\kappa(s)=0, s>0$, then Assumption **(E′)** holds so that Theorem 4.6 applies and, in particular,* (4.20) *and* (4.21) *hold.*

Remark 4.13. *This result shows that, if v satisfies **(A′)** and **(B)**, with $\kappa(s)=0$, $s>0$, then* (4.30) *holds, and that if also w satisfies **(F′)** then* (4.20) *and* (4.21) *hold. Example 4.14 below shows that, if w satisfies **(F)** but not **(F′)**, then no stronger relationship between spectra than* (4.30) *need hold. In particular, it need not hold that $\Sigma_{X_w}(K)=\Sigma_X(K)$ nor that $\Sigma^{\mathrm{e}}_{X_w}(K) \subset \Sigma^{\mathrm{e}}_X(K)$.*

Proof. Let $u>0$ and define for every $y \in X$ the function $y_u \in X$ by setting $y_u(s)=y(s)$ for $s \ge u$ and $y_u(s)=y(u)$ for all $0 \le s < u$. Then $\|y_u\| = \sup_{s\ge u}|y(s)|$.

Suppose v satisfies the assumptions of the theorem. Then $K \in \mathcal{B}(X)$ so that for every $x \in X$ we have

$$|Kx(s)| = |Kx_s(s)| \le \|K\|\|x_s\| \le \|K\| \sup_{t\ge s}|x(t)|, \quad s \in \mathbb{R}_+.$$

Hence $K \in \mathcal{B}(X_w)$ with norm not larger than $\|K\|$.

To prove (4.30) let us assume that $\lambda \notin \Sigma_X(K)$, i.e. $(\lambda-K)^{-1} \in \mathcal{B}(X)$. Then, for every $y \in X$ the integral equation

$$\lambda x(s) - \int_s^\infty v(s,t)x(t)\,dt = y(s), \quad s \in \mathbb{R}_+. \tag{4.31}$$

has an unique solution $x \in X$ and $\|x\| \le C\|y\|$.

Let $u>0$ and $y \in X$. Denote by x, x^u the unique solution of (4.31) with right-hand side y, y_u, respectively. We shall see in a moment that

$$x(s) = x^u(s), \quad s \ge u, \tag{4.32}$$

holds, so that

$$\sup_{s\ge u}|x(s)| \le \|x^u\| \le C\|y_u\| = C\sup_{s\ge u}|y(u)|.$$

Thus, if $y \in X_w$ then $x \in X_w$ with $\|x\|_w \le C\|y\|_w$. Hence $(\lambda-K)^{-1} \in \mathcal{B}(X_w)$, i.e. $\lambda \notin \Sigma_{X_w}(X)$ which is what we set out to show.

It remains to prove that (4.32) is true. To this end let us show that the integral equation

$$\lambda\tilde{x}(s) - \int_{\max\{s,u\}}^\infty v(s,t)\tilde{x}(t)\,dt = \tilde{y}(s), \quad s \in \mathbb{R}_+. \tag{4.33}$$

has an unique solution $\tilde{x} \in X$ for every $\tilde{y} \in X$. Denote the kernel of the integral operator K_+ in (4.33) by v_+, so that

$$v_+(s,t) = \begin{cases} 0, & 0 \le t < u, \\ v(s,t), & u \le t, \end{cases} \quad s \in \mathbb{R}_+.$$

Also, set $v_- := v - v_+$. It is not hard to see that v_- satisfies Assumptions **(A)**, **(B)** and **(D)**. We apply Theorem 4.4 with $K' = K - K_+$ to see that $\lambda - K_+$ is Fredholm of index 0 since $\lambda - K$ (as an invertible operator) is Fredholm of index 0. To see that $\lambda - K_+$ is also surjective and thus invertible, choose any $\tilde{y} \in X$ and let $x := (\lambda-K)^{-1}\tilde{y}$ and set

$$\tilde{x}(s) := \begin{cases} x(s) - \frac{1}{\lambda}\int_s^u v(s,t)x(t)\,dt, & 0 \le s < u, \\ x(s), & s \ge u. \end{cases}$$

Then $\tilde{x} \in X$ and $(\lambda - K_+)\tilde{x} = (\lambda-K)x = \tilde{y}$ and thus $\lambda - K_+$ is surjective, whence $(\lambda-K_+)^{-1} \in \mathcal{B}(X)$.

For the last step, we define the function z by

$$z(s) := \frac{1}{\lambda} \int_{\max\{s,u\}}^{\infty} v(s,t)\big(x(t) - x^u(t)\big)\, dt, \quad s \in \mathbb{R}_+.$$

Then, by the definition of x and x^u,

$$z(s) = x(s) - x^u(s), \quad s \geq u. \tag{4.34}$$

Thus $\lambda z = K_+ z$ and, since $(\lambda - K_+)$ is injective, $z = 0$; now (4.34) implies that (4.32) must indeed be true and the theorem follows. □

We now comment further on the necessity of the requirement $(\boldsymbol{F'})$ in the Wiener-Hopf case $v(s,t) = \kappa(s-t)$. We have seen in Proposition 4.3 that, unless κ vanishes on the positive half line, necessarily $(\boldsymbol{F})$ holds in this case if $K \in \mathcal{B}(X_w)$. We have seen also that our method of argument, based on Theorem 4.4 applied with $K' = K - K_w$, so that $v - v_w$ must satisfy $(\boldsymbol{D})$, requires that w satisfies the stronger condition $(\boldsymbol{F'})$. Thus $(\boldsymbol{F'})$ is a necessary condition for $K - K_w$ to be compact, though not, as discussed above, a sufficient condition. But the question arises as to whether, in the Wiener-Hopf case, Assumption $(\boldsymbol{F'})$ is also necessary for the results of Theorem 4.6 to hold.

We can give a partial answer to this question by considering the weight function $w(s) = \exp(bs)$, $b > 0$, which satisfies $(\boldsymbol{F})$ but not $(\boldsymbol{F'})$. In this case, if $v(s,t) = \kappa(s-t)$ with $\kappa \in L^1(\mathbb{R})$, then $v_w(s,t) = \kappa_b(s-t)$ with $\kappa_b(s) := \kappa(s)\exp(bs)$. Thus

$$K \in \mathcal{B}(X_w) \quad \Longleftrightarrow \quad \int_0^{\infty} |\kappa(t)| e^{bt}\, dt < \infty. \tag{4.35}$$

Further, if (4.35) holds, then, from (1.9) and (1.10) applied with $\kappa = \kappa_b$, we deduce that

$$\Sigma^e_{X_w}(K) = \{\hat{\kappa}(\xi - ib) : \xi \in \mathbb{R}\} \cup \{0\} \tag{4.36}$$

and

$$\Sigma_{X_w}(K) = \Sigma^e_{X_w}(K) \cup \{\lambda : [\arg(\lambda - \hat{\kappa}(\xi - ib))]^{\infty}_{-\infty} \neq 0\}, \tag{4.37}$$

with $\hat{\kappa}$ defined by (1.11). Thus we have explicit expressions in this case for the spectrum and essential spectrum of K as an operator on both X and X_w and can check for a particular choice of κ whether these spectra coincide, i.e. whether (4.20) and (4.21) hold. We point out that, if (4.35) holds, then

$$\sup_{s \geq 2A} \int_A^{s-A} \frac{w(s)}{w(t)} |\kappa(s-t)|\, dt = \int_A^{\infty} e^{bt} |\kappa(t)|\, dt \to 0$$

as $A \to \infty$, so that $(\boldsymbol{E'})$ holds. Thus all the conditions of Theorem 4.6 are satisfied in this case, except that $(\boldsymbol{F'})$ is replaced by the weaker $(\boldsymbol{F})$.

The following examples illustrate the range of possible behaviour. The first example shows that there exists a large class of κ for which (4.20) and (4.21) do hold, while the second example shows that (4.20) and (4.21) do not hold for a large class of κ. The third example is a case in which $\kappa(s) = 0$, $s > 0$, and (4.20) and (4.21) do not hold, although, by Theorem 4.12, (4.30) applies.

Example 4.14. *Suppose that f is real and even and that*

$$\int_0^{\infty} e^{bs/2}\big(|f(s)| + |f(-s)|\big)\, ds < \infty.$$

Then $\hat{f}(\xi)$ is analytic in the strip $|\mathrm{Im}\,\xi| < b/2$ and continuous in $|\mathrm{Im}\,\xi| \leq b/2$. Further $\hat{f}(\xi) = \hat{f}(-\xi)$, $|\mathrm{Im}\,\xi| \leq b/2$. Define $\kappa(s) := e^{-bs/2} f(s)$. Then (4.35) holds and

$$\hat{\kappa}(\xi) = \hat{f}(\xi + ib/2), \quad \hat{\kappa}(\xi - ib) = \hat{f}(\xi - ib/2), \quad \xi \in \mathbb{R}.$$

Thus, and from (1.9)*,* (1.10)*,* (4.36) *and* (4.37) *it follows that* (4.20) *and* (4.21) *in Theorem 4.6 hold.*

If $\lambda - K$ is Fredholm on X then its index (see e.g. [40, 57]) is

$$\gamma := \frac{1}{2\pi}[\arg(\lambda - \hat{\kappa}(\xi))]_{-\infty}^{\infty}$$

so that the index of $\lambda - K$ on X_w is

$$\frac{1}{2\pi}[\arg(\lambda - \hat{\kappa}(\xi - ib))]_{-\infty}^{\infty} = \frac{1}{2\pi}[\arg(\lambda - \hat{\kappa}(-\xi))]_{-\infty}^{\infty} = -\gamma.$$

Thus the other conclusion of Theorem 4.6 does not hold in this case since, if $\lambda - K$ is Fredholm on X and X_w, its index on X is the negative of its index on X_w.

Example 4.15. *Suppose that κ is real and even and that* (4.35) *holds. Then $\hat{\kappa}(\xi)$ is real and even so that*

$$\Sigma_X(K) = \Sigma_X^e(K) = [\kappa_-, \kappa_+],$$

where $\kappa_- = \inf_{\xi\in\mathbb{R}} \hat{\kappa}(\xi)$, $\kappa_+ = \sup_{\xi\in\mathbb{R}} \hat{\kappa}(\xi)$. But

$$\hat{\kappa}(\xi - ib) = \int_{-\infty}^{\infty} \kappa(s)e^{bs}\cos(\xi s)\,ds + 2i\int_0^{\infty} \kappa(s)\sinh(bs)\sin(\xi s)\,ds, \quad \xi \in \mathbb{R}.$$

The imaginary part of $\hat{\kappa}(\xi - ib)$ is the sine transform of $2\kappa(s)\sinh(bs)$. By the injectivity of the sine transform, unless $\kappa = 0$, $\operatorname{Im}\hat{\kappa}(\xi - ib) \neq 0$ for at least one $\xi \in \mathbb{R}$, so that

$$\Sigma_X^e(K) \neq \Sigma_{X_w}^e(K), \quad \Sigma_X(K) \neq \Sigma_X(K).$$

Example 4.16. *Define κ by*

$$\kappa(s) = \begin{cases} 0, & s \geq 0 \\ e^s, & s < 0. \end{cases}$$

Then (4.35) *holds for all $b > 0$ so that $K \in \mathcal{B}(X_w)$. Also*

$$\hat{\kappa}(\xi) = \frac{1}{1 + i\xi}, \quad \hat{\kappa}(\xi - ib) = \frac{1}{1 + b + i\xi}, \quad \xi \in \mathbb{R},$$

so that

$$\Sigma_X^e(K) = \left\{\lambda : \left|\lambda - \frac{1}{2}\right| = \frac{1}{2}\right\}, \quad \Sigma_X(K) = \left\{\lambda : \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}$$

and

$$\Sigma_{X_w}^e(K) = \left\{\lambda : \left|\lambda - \frac{1}{2(1+b)}\right| = \frac{1}{2(1+b)}\right\}, \quad \Sigma_{X_w}(K) = \left\{\lambda : \left|\lambda - \frac{1}{2(1+b)}\right| \leq \frac{1}{2(1+b)}\right\}.$$

Thus $\Sigma_X(K) \neq \Sigma_{X_w}(K)$ and $\Sigma_X^e(K) \neq \Sigma_{X_w}^e(K)$; in fact $\Sigma_X^e(K) \cap \Sigma_{X_w}^e(K) = \{0\}$. But note that $\Sigma_{X_w}(K) \subset \Sigma_X(K)$, in agreement with Theorem 4.12.

4.4 Sufficient conditions on kernels and examples

While in applications Assumption ***(F′)*** is often easily verified, Assumption ***(E′)*** is typically much harder to check. In this section we derive simpler conditions which imply that ***(E′)*** holds, and give examples of kernels and weights which satisfy ***(E′)*** and ***(F′)***. Further, we provide and discuss examples of kernels and weight functions to which our results apply.

In most cases of practical interest it holds that $w(s)$ is continuously differentiable, at least for all sufficiently large s, say $s \geq s_0$. In this case we have that

$$\frac{w(s)}{w(t)} = \exp\left(\int_t^s \frac{w'(u)}{w(u)}\,du\right), \quad s_0 \leq t \leq s, \tag{4.38}$$

so that, if

$$\frac{w'(s)}{w(s)} \to 0, \quad \text{as } s \to \infty,$$

then $(\boldsymbol{F'})$ holds. Of course, not every w satisfying (2.3) is differentiable. But for every $w \in C(\mathbb{R}_+)$ satisfying (2.3) the function

$$\tilde{w}(s) := \frac{\int_s^{s+1} w(t)\,dt}{\int_0^1 w(t)\,dt}, \quad s \in \mathbb{R}_+,$$

satisfies (2.3) and is continuously differentiable. Further, we have the following result.

Lemma 4.17. *Assumption* $(\boldsymbol{F})$ *holds if and only if*

$$\frac{\tilde{w}'(s)}{\tilde{w}(s)} = O(1), \quad \textit{as } s \to \infty. \tag{4.39}$$

Assumption $(\boldsymbol{F'})$ *holds if and only if*

$$\frac{\tilde{w}'(s)}{\tilde{w}(s)} \to 0, \quad \textit{as } s \to \infty. \tag{4.40}$$

If w *satisfies* $(\boldsymbol{F})$ *then, for some* $C > 0$,

$$\frac{w(s)}{w(1)} \le \tilde{w}(s) \le Cw(s), \quad s \ge 0. \tag{4.41}$$

Proof. For $s \ge 0$,

$$\frac{w(s)}{w(1)} \le \frac{w(s)}{\int_0^1 w(t)\,dt} \le \tilde{w}(s) \le \frac{w(s+1)}{\int_0^1 w(t)\,dt} \le w(s+1),$$

so that

$$\frac{\tilde{w}'(s)}{\tilde{w}(s)} = \frac{w(s+1) - w(s)}{\tilde{w}(s)\int_0^1 w(t)\,dt} \le \frac{w(s+1)}{w(s)} - 1$$

and, for $s \ge 1$,

$$\frac{w(s+1)}{w(s)} \le w(1)\frac{\tilde{w}(s+1)}{\tilde{w}(s-1)} = w(1)\exp\left(\int_{s-1}^{s+1} \frac{\tilde{w}'(t)}{\tilde{w}(t)}\,dt\right).$$

From these inequalities the equivalence of $(\boldsymbol{F})$ and (4.39) and also that of $(\boldsymbol{F'})$ and (4.40) follows. Further, if $(\boldsymbol{F})$ holds then, $w(s+1) \le \Delta_w^1 w(s)$, $s \ge 0$, so that (4.41) is true. □

In view of this result, in order to check that $(\boldsymbol{E})$ and $(\boldsymbol{F})$ hold, or that $(\boldsymbol{E'})$ and $(\boldsymbol{F'})$ hold, it is sufficient to check that $\tilde{w}$ satisfies (4.39) or (4.40), respectively, and that $(\boldsymbol{E})$ or $(\boldsymbol{E'})$, respectively, hold with w replaced by $\tilde{w}$. We will assume in the remainder of this section, when deriving conditions which ensure that $(\boldsymbol{E'})$ and $(\boldsymbol{F'})$ hold, that $w(s)$ is continuously differentiable for all sufficiently large s. The reader should bear in mind that if $\tilde{w}$, which is necessarily continuously differentiable, satisfies the conditions we require in the various propositions below, then $\tilde{w}$ satisfies (4.40), (4.41) and $(\boldsymbol{E'})$ and hence, by Lemma 4.17, w satisfies $(\boldsymbol{E'})$ and $(\boldsymbol{F'})$.

Our first two propositions deal with the case when $w'(s)/w(s)$ is bounded by θ/s for some $\theta > 0$ and all sufficiently large s. Note that we have then the bound

$$1 \le \frac{w(s)}{w(t)} \le \exp\left(\int_t^s \frac{\theta}{u}\,du\right) = \left(\frac{s}{t}\right)^\theta \tag{4.42}$$

if $s \ge t$ and t is sufficiently large. Keeping t fixed in this equation, we see that in this case necessarily $w(s) = O(s^\theta)$, $s \to \infty$.

Proposition 4.18. *Suppose that v satisfies $(\boldsymbol{A'})$, with $\kappa \in L^1(\mathbb{R})$, and that there exists $\theta > 0$ such that for all sufficiently large s the inequality*

$$\frac{w'(s)}{w(s)} \leq \frac{\theta}{s} \tag{4.43}$$

holds. Further, suppose that either

$$w^{-1} \in L^1(\mathbb{R}_+) \quad \text{and} \quad \lambda(s) := \int_s^{s+1} |\kappa(t)|\, dt = O\Big(\frac{1}{w(s)}\Big), \qquad \text{as } s \to \infty, \tag{4.44}$$

or, alternatively,

$$w(s) \int_s^\infty |\kappa(t)|\, dt = O(1), \qquad \text{as } s \to \infty, \tag{4.45}$$

holds. Then Assumptions $(\boldsymbol{E'})$ and $(\boldsymbol{F'})$ are satisfied.

Proof. That $(\boldsymbol{F'})$ holds follows from (4.38). Note that, for $1 \leq u \leq s$,

$$\int_{u-1}^u \frac{|\kappa(t)|}{w(s-t)}\, dt \leq \frac{\lambda(u-1)}{w(s-u)} \leq \lambda(u-1) \int_u^{u+1} \frac{dt}{w(s-t)}. \tag{4.46}$$

Thus, if (4.44) holds, then, for some $C > 0$, $w(s)\lambda(s) \leq C$ for $s \geq 0$, and we obtain, for A sufficiently large and $s \geq 2A$, the bound

$$\begin{aligned}\int_A^{s/2} \frac{w(s)}{w(t)} |\kappa(s-t)|\, dt = w(s) \int_{s/2}^{s-A} \frac{|\kappa(t)|}{w(s-t)}\, dt &\leq w(s) \sup_{t \geq s/2-1} \lambda(t) \int_{s/2}^{s-A+1} \frac{dt}{w(s-t)} \\ &\leq C \frac{w(s)}{w(s/2-1)} \int_{A-1}^\infty \frac{dt}{w(t)}.\end{aligned} \tag{4.47}$$

Note that, by our assumption (4.43), the inequality (4.42) holds for $s \geq t$ and t large enough. Hence, and from (4.47), for all sufficiently large A,

$$\sup_{s \geq 2A} \int_A^{s/2} \frac{w(s)}{w(t)} |\kappa(s-t)|\, dt \leq C \left(\frac{2}{1 - A^{-1}} \right)^\theta \int_{A-1}^\infty \frac{dt}{w(t)} \to 0, \quad \text{as } A \to \infty. \tag{4.48}$$

In the other case, when assumption (4.45) holds, inequality (4.42) implies that for all sufficiently large A and $s \geq 2A$

$$\int_A^{s/2} \frac{w(s)}{w(t)} |\kappa(s-t)|\, dt \leq \frac{w(s)}{w(A)} \int_A^{s/2} |\kappa(s-t)|\, dt \leq 2^\theta \frac{w(s/2)}{w(A)} \int_{s/2}^\infty |\kappa(t)|\, dt \to 0 \tag{4.49}$$

as $A \to \infty$, uniformly in $s \geq 2A$.

Further, in both cases, for all sufficiently large A it holds that

$$\sup_{s \geq 2A} \int_{s/2}^{s-A} \frac{w(s)}{w(t)} |\kappa(s-t)|\, dt \leq 2^\theta \int_A^\infty |\kappa(t)|\, dt \to 0,$$

as $A \to \infty$. Thus $(\boldsymbol{E'})$ holds. □

If the constant θ in the bound for $w'(s)/w(s)$ is in the interval $(0, 1]$, then $1 \leq w(s) = O(s)$ as $s \to \infty$, so that w^{-1} is not integrable. Thus condition (4.44) of the previous proposition is not satisfied, and Proposition 4.18 applies only if (4.45) holds. Consider now the example when $\kappa(s) = (1 + |s|)^{-3/2}$ and $w(s) = (1 + s)^{3/4}$. Then

$$w(s) \int_s^\infty |\kappa(t)|\, dt = 2(1 + s)^{1/4},$$

which is clearly unbounded as $s \to \infty$, so that neither of the two conditions on κ in Proposition 4.18 is applicable. The next proposition gives alternative conditions on κ when $\theta \leq 1$ which apply to this example.

Proposition 4.19. *Suppose that v satisfies Assumption* $(\boldsymbol{A'})$*, with $\kappa \in L^1(\mathbb{R})$, and that, for some $\theta \in (0,1]$,*

$$\frac{w'(s)}{w(s)} \leq \frac{\theta}{s}, \tag{4.50}$$

for all sufficiently large s, and

$$\lambda(s) := \int_s^{s+1} |\kappa(t)|\, dt = \begin{cases} O(s^{-1}) & , \text{ if } \theta < 1, \\ o\big((s \ln s)^{-1}\big) & , \text{ if } \theta = 1, \end{cases} \qquad \text{as } s \to \infty.$$

Then Assumptions $(\boldsymbol{E'})$ *and* $(\boldsymbol{F'})$ *are satisfied.*

Proof. Since (4.50) holds for all sufficiently large s, it follows that, for some $M > 0$, (4.42) holds for $s \geq t \geq M$. Further, if $\theta < 1$, then, for some $C > 0$,

$$s\lambda(s) \leq C, \quad s \in \mathbb{R}_+. \tag{4.51}$$

Suppose $A > M + 1$ and $\eta \in (0, 1/2]$. Then, for $s \geq 2A$,

$$\int_{\max\{A,\eta s\}}^{s-A} \frac{w(s)}{w(t)} |\kappa(s-t)|\, dt \leq \left(\frac{s}{\eta s}\right)^{\theta} \int_{\max\{A,\eta s\}}^{s-A} |\kappa(s-t)|\, dt \leq \eta^{-\theta} \int_A^{\infty} |\kappa(t)|\, dt. \tag{4.52}$$

Further, for $\eta s \geq A$, using (4.46) with $w(s) = s^{\theta}$ to obtain (4.54) from (4.53), we see that

$$\begin{aligned} \int_A^{\eta s} \frac{w(s)}{w(t)} |\kappa(s-t)|\, dt &\leq \int_{s(1-\eta)}^{s-A} \frac{s^{\theta} |\kappa(t)|}{(s-t)^{\theta}}\, dt && (4.53) \\ &\leq s^{\theta} \Big(\sup_{t \geq s(1-\eta)-1} \lambda(t) \Big) \int_{s(1-\eta)}^{s-A+1} \frac{dt}{(s-t)^{\theta}} && (4.54) \\ &\leq \frac{C s^{\theta}}{s(1-\eta)} \int_{A-1}^{\eta s} \frac{dt}{t^{\theta}}. && (4.55) \end{aligned}$$

In the case $\theta < 1$, since $0 < \eta \leq \frac{1}{2}$ and $\eta s \geq A$, this expression is bounded above by

$$\frac{2Cs^{\theta}}{s-2} \int_0^{\eta s} \frac{dt}{t^{\theta}} = \frac{2sC}{(s-2)(1-\theta)} \eta^{1-\theta} \leq \frac{2AC}{(A-1)(1-\theta)} \eta^{1-\theta} \leq \frac{2(M+1)C}{M(1-\theta)} \eta^{1-\theta}. \tag{4.56}$$

Combining the inequalities (4.52) through (4.56), we see that, for some $C_1 > 0$ and all sufficiently large A,

$$\sup_{s \geq 2A} \int_A^{s-A} \frac{w(s)}{w(t)} |\kappa(s-t)|\, dt \leq \eta^{-\theta} \int_A^{\infty} |\kappa(t)|\, dt + C_1 \eta^{1-\theta}.$$

For every $\epsilon > 0$ we can choose first η small enough so that $\eta^{1-\theta} C_1 < \epsilon/2$ and then, for all sufficiently large A,

$$\sup_{s \geq 2A} \int_A^{s-A} \frac{w(s)}{w(t)} |\kappa(s-t)|\, dt < \eta^{-\theta} \int_A^{\infty} |\kappa(t)|\, dt + \frac{\epsilon}{2} < \epsilon.$$

so that $(\boldsymbol{E'})$ follows.

In the case $\theta = 1$, we set $\eta = 1/2$ and find from (4.54) that, for $A \geq 2$ and $s \geq 2A$,

$$\int_A^{s/2} \frac{w(s)}{w(t)} |\kappa(s-t)|\, dt \leq s \Big(\sup_{t \geq \frac{s}{2}-1} \lambda(t) \Big) \int_{A-1}^{s/2} \frac{dt}{t} \leq s \Big(\sup_{t \geq \frac{s}{2}-1} \lambda(t) \Big) \ln \frac{s}{2} \to 0$$

as $s \to \infty$. Combining this bound with (4.52) we see that $(\boldsymbol{E'})$ holds. □

Next is an example of a particular weight function for which we are forced to check condition (4.45) of Proposition 4.18, because (4.44) does not apply and neither does Proposition 4.19.

Example 4.20. *Choose a monotonic increasing weight function which satisfies* $w(s) \in C(\mathbb{R}_+)$, $w(0) = 1$ *and* $w(s) := s\ln(s)$, *for all* $s \geq e$. *Clearly,* w *is unbounded. Moreover,*

$$\frac{w'(s)}{w(s)} = \frac{1 + \ln s}{s \ln s}, \quad s \geq e,$$

so that, for every $\theta > 1$,

$$\frac{w'(s)}{w(s)} \leq \frac{\theta}{s}$$

for all sufficiently large s. *On the other hand, since*

$$\frac{w'(s)}{w(s)} > \frac{1}{s}, \quad s \geq e,$$

Proposition 4.19 does not apply. Further w^{-1} *is not integrable over* $\mathbb{R}$ *and thus* (4.44) *in Proposition 4.18 does not hold. In order to find kernels* v *satisfying Assumptions* $(\boldsymbol{A'})$ *and* $(\boldsymbol{E'})$ *for this choice of* w, *one would have to check the second condition* (4.45) *of Proposition 4.18.*

The following example considers the important special case of the power weight $w(s) = (1+s)^p$, sharpening, as discussed in the introduction, the results of [42, 57, 18].

Example 4.21. *Suppose* $w(s) := (1+s)^p$, *for some* $p > 0$, *and the kernel* v *satisfies Assumption* $(\boldsymbol{A'})$ *with* $\kappa \in L^1(\mathbb{R})$. *Then Assumption* $(\boldsymbol{F'})$ *holds,*

$$\frac{w'(s)}{w(s)} = \frac{p}{1+s}, \quad s \in \mathbb{R}_+,$$

and, by Propositions 4.19 and 4.18, Assumption $(\boldsymbol{E'})$ *holds if*

$$\int_s^{s+1} |\kappa(t)|\, dt = \begin{cases} O(s^{-p}), & p > 1, \\ o\big((s\ln s)^{-1}\big), & p = 1, \\ O(s^{-1}), & 0 < p < 1, \end{cases} \qquad \text{as } s \to \infty. \tag{4.57}$$

Thus, if (4.57) *is satisfied and* v *also satisfies* $(\boldsymbol{B})$, *then, by Proposition 4.3 and Theorem 4.6,* $K \in \mathcal{B}(X_w)$ *and the spectral equivalences* (4.20) *and* (4.21) *hold.*

In the Wiener-Hopf case $v(s,t) = \kappa(s-t)$, with $\kappa \in L^1(\mathbb{R})$, it follows from Example 4.21 that, if $w(s) = (1+s)^p$, for some $p > 0$, and (4.57) holds, then $K \in \mathcal{B}(X_w)$ and (4.20) and (4.21) hold. As a consequence of Corollary 4.2 and since (4.7) implies (4.8), we have also that $K \in \mathcal{B}(X_w)$ implies that (1.15) holds for $r = p$. Thus the statement

$$\int_s^{s+1} |\kappa(t)|\, dt = O(s^{-q}) \text{ as } s \to \infty \implies K \in \mathcal{B}(X_w) \implies \int_s^{s+1} |\kappa(t)|\, dt = O(s^{-r}) \text{ as } s \to \infty \tag{4.58}$$

holds for $r = q = p$ if $p > 1$, for $r = 1$ and every $q > 1$ if $p = 1$, and for $r = p$ and $q = 1$ if $0 < p < 1$. In the case $0 < p < 1$ the implications (4.58) do not hold for any values of q and r with $r > p$ or $q < 1$ as shown by the following examples.

Example 4.22. *Suppose that* $v(s,t) = \kappa(s-t)$ *and that, for some* $p > 0$,

$$\kappa(t) = \begin{cases} t^{-p}, & e^n \leq t < e^n + 1, \quad n \in \mathbb{N}, \\ 0, & \textit{otherwise}. \end{cases}$$

Then $\kappa \in L^1(\mathbb{R})$, in fact, for $s > 0$, where $\lfloor \ln s \rfloor$ denotes the largest integer $\le \ln s$,

$$\int_s^\infty |\kappa(t)|\,dt \le \int_{e^{\lfloor \ln s \rfloor}}^\infty |\kappa(t)|\,dt < \sum_{m=\lfloor \ln s \rfloor}^\infty e^{-pm} = \frac{e^{-p\lfloor \ln s \rfloor}}{1-e^{-p}} \le \frac{e^p s^{-p}}{1-e^{-p}}.$$

*Thus, if $w(s) = (1+s)^p$, then (4.45) is satisfied and, by Proposition 4.18, **(E′)** and **(F′)** hold. It follows from Proposition 4.3 that $K \in \mathcal{B}(X_w)$. But note that, for $s = e^n, n \in \mathbb{N}$,*

$$\int_s^{s+1} |\kappa(t)|\,dt > (1+s)^{-p},$$

so that (1.15) holds only for $r \le p$.

Example 4.23. *Suppose that $v(s,t) = \kappa(s-t)$ and that, for some $q \in (0,1)$ and some positive sequences $(a_n), (b_n)$, with $0 < a_1 < a_1 + b_1 < a_2 < a_2 + b_2 < a_3 < \dots$ it holds that*

$$\kappa(t) = \begin{cases} t^{-q}, & a_n \le t < a_n + b_n, n \in \mathbb{N}, \\ 0, & \textit{otherwise.} \end{cases}$$

Further, suppose that $a - 1 > b \ge 0$, $a_n \sim n^a$, $b_n \sim n^b$ as $n \to \infty$, and $p \in (0,q)$. Then

$$\|\kappa\|_1 = \sum_{n=1}^\infty \int_{a_n}^{a_n+b_n} t^{-q}\,dt \le \sum_{n=1}^\infty b_n {a_n}^{-q} < \infty,$$

provided $aq - b > 1$. Moreover, where $w(s) = (1+s)^p$, it holds that

$$\begin{aligned} w(a_n+b_n)\int_0^{a_n+b_n} \frac{|\kappa(t)|}{w(a_n+b_n-t)}\,dt &\ge \frac{w(a_n+b_n)}{w(b_n)} \int_{a_n}^{a_n+b_n} |\kappa(t)|\,dt \\ &> \frac{w(a_n+b_n)}{w(b_n)}(a_n+b_n)^{-q} b_n \sim n^{ap-aq-bp+b} \end{aligned}$$

as $n \to \infty$. Now, suppose that we choose (a_n) and (b_n) so that $a > (1-p)/((1-q)p)$ (which ensures that $a(q-p)/(1-p) < aq-1$) and so that $a(q-p)/(1-p) < b < aq-1$. Then $aq - b > 1$, so that $\kappa \in L^1(\mathbb{R})$, and $ap - aq - bp + b > 0$, so that (4.7) does not hold, and so, by Corollary 4.2, $K \notin \mathcal{B}(X_w)$. But note that (1.15) holds with $r = q$.

Having dealt with the case when $w'(s)/w(s)$ is bounded by a multiple of $1/s$, we now turn our attention to the case when $w'(s)/w(s)$ decays at a slower rate.

Proposition 4.24. *Suppose that v satisfies **(A′)**, with $\kappa \in L^1(\mathbb{R})$, that $w'(s)/w(s)$ is monotonic decreasing for all sufficiently large s and, for some $\alpha \in (0,1)$, we have that*

$$\frac{sw'(s)}{w(s)} \to \infty, \quad \frac{w'(s)}{w(s)} = O(s^{\alpha-1}),$$

*as $s \to \infty$. Then w satisfies **(F′)**. If also*

$$\lambda(s) := \int_s^{s+1} |\kappa(t)|\,dt = O\Big(\frac{1}{w(s)}\Big), \quad s \to \infty, \tag{4.59}$$

*then Assumption **(E′)** is fulfilled.*

Proof. Choose $\beta > 1/(1-\alpha)$. By the assumptions of the proposition we have, for some $q > 0$ and all sufficiently large s,

$$\frac{\beta}{s} \le \frac{w'(s)}{w(s)} \le \frac{q}{s^{1-\alpha}}. \tag{4.60}$$

Thus, for $s \geq t$ and t large enough,

$$\left(\frac{s}{t}\right)^{\beta} = \exp\left(\int_t^s \frac{\beta}{u}\,du\right) \leq \exp\left(\int_t^s \frac{w'(u)}{w(u)}\,du\right) = \frac{w(s)}{w(t)}$$
$$\leq \exp\left(\int_t^s \frac{q}{u^{1-\alpha}}\,du\right) \leq \exp\bigl(q(s-t)t^{\alpha-1}\bigr). \quad (4.61)$$

Keeping t fixed in this equation, we see that $s^{1/(1-\alpha)}/w(s) \to 0$ as $s \to \infty$ so that $w^{-1} \in L^1(\mathbb{R}_+)$ and

$$\frac{s}{w(s^{1-\alpha})} \to 0, \qquad \text{as } s \to \infty. \quad (4.62)$$

Now, for all u sufficiently large and $s \geq u$, we get from (4.46), our assumption (4.59) on κ and the fact that $w(s)/w(t)$ is bounded for $|s-t| \leq 1$ when s is large enough, the bound

$$\int_{u-1}^{u} \frac{|\kappa(t)|}{w(s-t)}\,dt \leq \int_u^{u+1} \frac{\lambda(u-1)}{w(s-t)}\,dt \leq \int_u^{u+1} \frac{C}{w(t-1)w(s-t)}\,dt \leq \int_u^{u+1} \frac{C_1}{w(t)w(s-t)}\,dt,$$

where $C > 0$ is some constant and $C_1 := \Delta_w^1 C$. Then, if $A > 0$ is large enough and $s \geq 2A$, we obtain

$$\int_A^{s-A} \frac{w(s)}{w(t)}|\kappa(s-t)|\,dt \leq C_1 w(s) \int_{A-1}^{s-A+1} \frac{dt}{w(t)w(s-t)} = 2C_1 w(s) \int_{A-1}^{s/2} \frac{dt}{w(t)w(s-t)}. \quad (4.63)$$

Now, for all sufficiently large s, from (4.61),

$$\frac{w(s)}{w(s-s^{1-\alpha})} \leq C_2, \quad (4.64)$$

where C_2 is some positive constant. Thus, if A is large enough and $s^{1-\alpha} \geq A-1$,

$$w(s)\int_{A-1}^{s^{1-\alpha}} \frac{dt}{w(s-t)w(t)} \leq \frac{w(s)}{w(s-s^{1-\alpha})}\int_{A-1}^{\infty} \frac{dt}{w(t)} \leq C_2 \int_{A-1}^{\infty} \frac{dt}{w(t)}. \quad (4.65)$$

Further, by the monotonicity of $w'(s)/w(s)$ for large argument we get that

$$\frac{d}{dt}\bigl(w(t)w(s-t)\bigr) = w(t)w(s-t)\left(\frac{w'(t)}{w(t)} - \frac{w'(s-t)}{w(s-t)}\right) \geq 0, \quad s^{1-\alpha} \leq t \leq s/2,$$

when s is large enough. Thus, for all sufficiently large s,

$$w(s)\int_{s^{1-\alpha}}^{s/2} \frac{dt}{w(t)w(s-t)}\,dt \leq \frac{s}{2}\,\frac{w(s)}{w(s^{1-\alpha})w(s-s^{1-\alpha})} \to 0, \quad (4.66)$$

as $s \to \infty$ from (4.64) and (4.62). From (4.63), (4.65) and (4.66) we conclude that $\boldsymbol{(E')}$ is satisfied. □

As an application of the lemmas we have just proved, we now give an example of an important class of weight functions for which $\boldsymbol{(E')}$ is satisfied for many kernels v.

Example 4.25. *Choose $\alpha \in (0,1)$, $a \geq 0$ and $p, q \in \mathbb{R}$ and define*

$$w(s) = \exp(as^{\alpha})(1+s)^p\bigl(\ln(e+s)\bigr)^q, \quad s \in \mathbb{R}_+. \quad (4.67)$$

Moreover, assume α, a, p, q are such that $w^{-1} \in L^1(\mathbb{R}_+)$ (i.e. $a > 0$ or $p > 1$ or $p = 1$ and $q > 1$) and (2.3) holds. Then

$$\ln w(s) = as^{\alpha} + p\ln(1+s) + q\ln\ln(e+s),$$

so that

$$\frac{w'(s)}{w(s)} = \frac{d}{ds}\ln w(s) = a\alpha s^{\alpha-1} + \frac{p}{1+s} + \frac{q}{(e+s)\ln(e+s)}$$

and

$$\frac{d}{ds}\frac{w'(s)}{w(s)} = -a\alpha(1-\alpha)s^{\alpha-2} - \frac{p}{(1+s)^2} - \frac{q}{(e+s)^2\ln(e+s)} - \frac{q}{(\ln(e+s))^2(e+s)^2} \leq 0,$$

for all sufficiently large s. Thus, if

$$\int_s^{s+1} |\kappa(t)|\,dt = O\Big(\frac{1}{w(s)}\Big), \quad \text{as } s \to \infty, \tag{4.68}$$

the assumptions of Proposition 4.24 (in case $a \neq 0$) and Proposition 4.18 (in case $a = 0$) are satisfied, so that ***(E′)*** *and* ***(F′)*** *hold. On the other hand, as has been shown in the previous sections, in the Wiener-Hopf case $v(s,t) = \kappa(s-t)$, with $\kappa \in L^1(\mathbb{R})$,* ***(E′)*** *and* ***(F′)*** *imply that* (4.8) *holds.*

The following proposition can be seen as a generalisation of the second case of Proposition 4.18.

Proposition 4.26. *Suppose that v satisfies* ***(A′)*** *with $\kappa \in L^1(\mathbb{R})$. Assume further that $g \in C^1(0,\infty)$ satisfies*

$$g(s) > 0, \quad 0 < \frac{g'(s)}{g(s)} \leq \frac{1}{s}, \quad \text{for } s > 0, \tag{4.69}$$

and that

$$g(s)\frac{w'(s)}{w(s)} = O(1), \quad w(s)\int_{g(s)}^\infty |\kappa(t)|\,dt = O(1), \tag{4.70}$$

as $s \to \infty$. Then Assumptions ***(E′)*** *and* ***(F′)*** *are satisfied.*

Proof. Note that (4.69) implies that g is monotonic increasing and that

$$1 \leq \frac{g(s)}{g(t)} = \exp\left(\int_t^s \frac{g'(u)}{g(u)}\,du\right) \leq \exp\left(\int_t^s \frac{1}{u}\,du\right) = \frac{s}{t}, \quad 0 < t \leq s. \tag{4.71}$$

Note also that the second equation in (4.70) implies $g(s) \to \infty$ as $s \to \infty$ and that the first of equations (4.70) implies, for some $C > 0$ and all $s \geq t$ with t sufficiently large,

$$\frac{w(s)}{w(t)} = \exp\left(\int_t^s \frac{w'(u)}{w(u)}\,du\right) \leq \exp\left(\int_t^s \frac{C}{g(u)}\,du\right) \leq \exp\left(\frac{C(s-t)}{g(t)}\right), \tag{4.72}$$

so that ***(F′)*** holds.

Let us now first suppose that for some $\theta \in (0,1)$ the inequality $g(s) \leq \theta s$ is true for all sufficiently large s. It follows from (4.71) and the inequality (4.72) that, for all sufficiently large s,

$$\frac{w(s)}{w(s-g(s))} \leq \exp\left(\frac{Cg(s)}{g(s-g(s))}\right) \leq \exp\left(\frac{Cg(s)}{g((1-\theta)s)}\right) \leq \exp\left(\frac{C}{1-\theta}\right).$$

Thus, for sufficiently large A and all $s \geq 2A$,

$$\int_{\min\{s-g(s),s-A\}}^{s-A} \frac{w(s)}{w(t)}|\kappa(s-t)|\,dt \leq \exp\left(\frac{C}{1-\theta}\right)\int_A^\infty |\kappa(t)|\,dt,$$

while

$$\int_A^{s-g(s)} \frac{w(s)}{w(t)}|\kappa(s-t)|\,dt \leq \frac{w(s)}{w(A)}\int_{g(s)}^\infty |\kappa(t)|\,dt.$$

Combining the last two inequalities and noting (4.70) we see that ***(E′)*** must be satisfied.

If it is not true that for some $\theta \in (0,1)$ the inequality $g(s) \le \theta s$ holds for all sufficiently large s, then there exists sequences $\theta_n \to 1$ and $s_n \to \infty$ such that $g(s_n) \ge \theta_n s_n$. From (4.71) it follows that $g(t) \ge tg(s_n)/s_n \ge \theta_n t$, $0 < t \le s_n$, and hence that $g(t) \ge t$, $t > 0$, so that, in view of (4.70), (4.43) holds for some $\theta > 0$ and all sufficiently large s. But also, from (4.71), $g(s) \le g(1)s$, $s \ge 1$. Thus $g(s/g(1)) \le s$, for $s \ge g(1)$, and so, by (4.70),

$$w\Big(\frac{s}{g(1)}\Big) \int_s^\infty |\kappa(t)|\, dt \le w\Big(\frac{s}{g(1)}\Big) \int_{g(s/g(1))}^\infty |\kappa(t)|\, dt = O(1), \quad \text{as } s \to \infty.$$

Further, from (4.72) and since $g(s/g(1)) \ge s/g(1)$, for $s > 0$, it holds that

$$\frac{w(s)}{w\Big(\dfrac{s}{g(1)}\Big)} \le \exp(C(g(1)-1))$$

for all sufficiently large s. Combining both inequalities, we see that (4.45) holds. It follows from Proposition 4.18 that $\boldsymbol{(E')}$ is satisfied. □

We now use this proposition to show that, for every kernel v satisfying $\boldsymbol{(A')}$ with $\kappa \in L^1(\mathbb{R})$, there exists a weight function w such that Assumption $\boldsymbol{(E')}$ holds. (The construction is based on [18, p.58].)

Suppose we are given a kernel v which satisfies $\boldsymbol{(A')}$ with $\kappa \in L^1(\mathbb{R})$. Then, provided $\mu(s) > 0$ for all $s \in \mathbb{R}_+$, a first guess at such a weight function might be $w(s) := \mu(0)/\mu(s)$, $s \in \mathbb{R}_+$, where

$$\mu(s) := \int_s^\infty |\kappa(t)|\, dt, \quad s \in \mathbb{R}_+. \tag{4.73}$$

Then, at least for almost all $s \in \mathbb{R}_+$ (or even for all $s \in \mathbb{R}_+$ if κ is continuous), the derivative $w'(s)$ exists and $w'(s) = |\kappa(s)|/\mu(s)^2$, so that Proposition 4.18 shows that $\boldsymbol{(E')}$ holds if

$$\frac{sw'(s)}{w(s)} = \frac{s|\kappa(s)|}{\mu(s)} = O(1), \quad s \to \infty. \tag{4.74}$$

Alternatively, if, for some $\alpha \in (0,1)$,

$$\frac{s^\alpha |\kappa(s)|}{\mu(s)} = O(1), \quad s \to \infty, \quad \text{and} \quad \frac{s|\kappa(s)|}{\mu(s)} \to \infty, \quad s \to \infty, \tag{4.75}$$

and $w'(s)/w(s)$ is monotonic increasing for all sufficiently large s, then Proposition 4.24 implies that $\boldsymbol{(E')}$ holds.

Conditions (4.74) and (4.75) contain rather strong pointwise estimates of κ. It therefore makes sense to introduce some averaging process in the definition of w. We also augment the definition of w to make the point that, given any $y \in X_0 := \{x \in X : x(s) \to 0 \text{ as } s \to \infty\}$, we can construct w such that $y \in X_w$. Let $y \in X_0 \setminus \{0\}$ and, for some $\beta \in (0,1)$,

$$q(s) := \min\left\{\frac{\mu(0)}{\mu(s^\beta)}, \frac{\|y\|}{\sup_{t \ge s} |y(t)|}, (1+s)^{1-\beta}\right\}, \quad s \in \mathbb{R}_+, \tag{4.76}$$

and define the weight function $w \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ by

$$w(-s) := w(s) := \begin{cases} 1, & s = 0, \\ \frac{2}{s}\int_{s/2}^s q(t)\, dt, & s > 0. \end{cases} \tag{4.77}$$

Note that

$$(1+s)^{1-\beta} \ge q(s) \ge w(s) \ge q(s/2) \ge 1, \quad s \ge 0.$$

We also have that

$$w'(s) = \frac{2q(s) - q(s/2) - w(s)}{s} \ge 0, \quad s > 0.$$

Thus (2.3) holds and

$$\frac{w'(s)}{w(s)} = \frac{2q(s) - q(s/2) - w(s)}{sw(s)} \le \frac{2q(s)}{s} \le 2\frac{(1+s)^{1-\beta}}{s}, \quad s > 0.$$

Thus, setting $g(s) := s^\beta$, $g(s)w'(s)/w(s) = O(1)$ as $s \to \infty$ and

$$w(s)\mu(s) \le w(s)\mu(g(s)) \le q(s)\mu(s^\beta) \le \mu(0),$$

so that our last proposition applies. Further, for $s \in \mathbb{R}_+$, $|y(s)|w(s) \le |y(s)|q(s) \le \|y\|$, so that $y \in X_w$. We have thus obtained the following theorem.

Theorem 4.27. *Suppose the kernel v satisfies* (***A′***)*, with $\kappa \in L^1(\mathbb{R})$, and $y \in X_0$. Then there exists a weight function $w \in \mathcal{W}(\kappa)$, defined by equations* (4.73)*,* (4.76) *and* (4.77)*, so that $y \in X_w$ and*

$$w(s)\int_s^\infty |\kappa(s)| = O(1), \qquad \text{as } s \to \infty. \tag{4.78}$$

Corollary 4.28. *If $\kappa \in L^1(\mathbb{R})$ then $\mathcal{W}(\kappa) \neq \varnothing$.*

As an interesting consequence of this result we relate the solvability of (4.1) in X_0 to its solvability in X in the following theorem, which forms an extension of Theorem 5.3 in [18], where the special case $\kappa(s) = O(s^{-q})$ as $s \to \infty$, for some $q > 1$, has been considered and only "$\subset$" in (4.79) has been shown.

Theorem 4.29. *Suppose that the kernel v satisfies* (***A′***)*, with $\kappa \in L^1(\mathbb{R})$, and* (***B***)*. Then $K \in \mathcal{B}(X)$ and $K \in \mathcal{B}(X_0)$. Moreover,*

$$\Sigma_{X_0}(K) = \Sigma_X(K). \tag{4.79}$$

Proof. By Theorem 4.27, given any $y \in X_0$ there exists $w = w(y) \in \mathcal{W}(\kappa)$ and $y \in X_w$. From Proposition 4.3 it follows that $Ky \in X_w \subset X_0$. Thus, and since we have $\|Kx\| \le \|\kappa\|_1\|x\|$, for all $x \in X$, it holds that K is bounded on both X and X_0.

Now, suppose that $\lambda \notin \Sigma_X(K)$. Then, by Theorem 4.6, for every $y \in X_0$, $\lambda \notin \Sigma_{X_{w(y)}}(K)$. In particular, for every $y \in X_0$, it follows that there exists $x \in X_{w(y)} \subset X_0$ such that $(\lambda - K)x = y$, so that $\lambda - K : X_0 \to X_0$ is surjective. Moreover, $\lambda - K$ is injective on $X_0 \subset X$ since it is injective on X. Thus $\lambda \notin \Sigma_X(K)$ implies that $(\lambda - K) : X_0 \to X_0$ is bijective. Hence, since X_0 is a Banach space, it follows from Banach's inverse theorem that $(\lambda - K)^{-1} \in \mathcal{B}(X_0)$, i.e. that $\lambda \notin \Sigma_{X_0}(K)$. We have thus shown that $\Sigma_{X_0}(K) \subset \Sigma_X(K)$.

For the other inclusion in (4.79) suppose that $\lambda \notin \Sigma_{X_0}(K)$. Choose an arbitrary weight function $w \in \mathcal{W}(\kappa)$; in view of Corollary 4.28, this is always possible. By the assumption on v and since $(\lambda - K)(X_0) = X_0$ is s-sequentially dense in X, K satisfies the Assumptions of Corollary 2.11 (with $H = K$). This corollary shows that $(\lambda - K)(X) = X$, whence $(\lambda - K) \in \Phi_-(X)$. Thus, by Theorem 4.6, $(\lambda - K) \in \Phi_-(X_w)$. But this implies $(\lambda - K) \in \Phi(X_w)$, for $\lambda - K$ is injective on X_w as it is injective on $X_0 \supset X_w$. We apply Theorem 4.6 again to obtain $(\lambda - K) \in \Phi(X)$. But, by Proposition 4.10, $\ker(\lambda - K) = \ker(\lambda - K)|_{X_w} = \{0\}$, so that $\lambda - K$ is injective on X. But we have already seen that $(\lambda - K)(X) = X$, whence $\lambda \notin \Sigma_X(K)$ by Banach's inverse theorem. □

4.5 The real line case

All our results obtained so far in this chapter were concerned with the integral equation (4.1) and the corresponding integral operators defined on the half line $\mathbb{R}_+$. However, many practical applications lead to integral equations on the real line of the form

$$\lambda x(s) - \int_{-\infty}^{\infty} v(s,t)x(t)\,dt = y(s), \quad s \in \mathbb{R},$$

to be solved in $X = BC(\mathbb{R})$ and its weighted subspaces X_w as defined in Section 2.2. The aim of this section is to emulate the analysis in the half line case to show that generalised versions of the main assumptions yield similar solvability results in the real line case.

In fact, we will devote the rest of this chapter to such equations on the real line. So, from now on, we will refer to the real-line variants of the integral operator K, the weighted spaces X and X_w and Assumptions **(A)**, **(A′)**, **(B)**, **(C)** and **(D)**, with $\Omega = \mathbb{R}$ in the respective definition (see Chapters 2 and 3; notice that our assumption that $k(s,t) = 1$ is still in force). Moreover, K_w will now denote the integral operator $M_w K M_{w^{-1}}$, whose kernel is given by $v_w(s,t) := (w(s)/w(t))v(s,t)$ for $s, t \in \mathbb{R}$.

Towards boundedness of K in X_w we can use the symmetry of w to obtain without difficulty the following variant of Proposition 4.1 and Corollary 4.2.

Proposition 4.30. *Suppose that the kernel k satisfies Assumptions* **(A)** *and* **(B)**. *Then $K \in B(X_w)$ if and only if*

$$\sup_{s\in\mathbb{R}} \int_{-|s|}^{|s|} |v_w(s,t)|\, dt = \sup_{s\in\mathbb{R}} \int_{-|s|}^{|s|} \frac{w(s)}{w(t)}\, |v(s,t)|\, dt < \infty,$$

in which case v_w also satisfies **(A)** *and* **(B)**. *If $v(s,t) = \kappa(s-t)$ for some $\kappa \in L^1(\mathbb{R})$, then $K \in B(X_w)$ if and only if*

$$\int_{-|s|}^{|s|} \frac{|\kappa(s-t)|}{w(t)}\, dt = \int_{s-|s|}^{s+|s|} \frac{|\kappa(t)|}{w(s-t)}\, dt = O\Big(\frac{1}{w(s)}\Big), \quad \text{as } |s| \to \infty. \tag{4.80}$$

If k satisfies **(A′)** *for some $\kappa \in L^1(\mathbb{R})$ then $K \in B(X_w)$ if* (4.80) *holds.*

For $\kappa \in L^1(\mathbb{R})$, we introduce the following functions on $[0, \infty)$, which we will use throughout the remainder of this thesis.

$$\lambda(A) := \int_A^{A+1} |\kappa(t)|\, dt + \int_{-A-1}^{-A} |\kappa(t)|\, dt, \tag{4.81}$$

$$\mu(A) := \int_{\mathbb{R}\setminus[-A,A]} |\kappa(t)|\, dt = \int_A^{\infty} |\kappa(t)|\, dt + \int_{-\infty}^{-A} |\kappa(t)|\, dt. \tag{4.82}$$

Arguing as in Section 4.1, if $\kappa \neq 0$ then it follows from (4.80) that **(F)** holds and, on the other hand, if **(F)** holds and $\int_{-\infty}^0 |\kappa(t)|\, dt \neq 0$ and $\int_0^\infty |\kappa(t)|\, dt \neq 0$ then (4.80) implies

$$w(s)\lambda(s) = O(1) \quad \text{as } s \to \infty.$$

Concerning the boundedness of K on X_w, we have the following partial generalisation of Proposition 4.3, which can be shown using slightly modified arguments. For the formulation of the proposition, we need the following real-line variant of **(E)**.

Assumption (E).

$$\sup_{|s|\geq 2A} \left(\int_{-|s|+A}^{-A} + \int_A^{|s|-A} \right) \frac{w(s)}{w(t)} |\kappa(s-t)|\, dt = O(1), \quad \text{as } A \to \infty.$$

Proposition 4.31. *If the kernel v satisfies* **(A′)**, *with $\kappa \in L^1(\mathbb{R})$,* **(B)** *and, further the real-line variants of Assumption* **(E)** *and* **(F)** *are satisfied then the kernel $v_w(s,t) = (w(s)/w(t))v(s,t)$ satisfies Assumptions* **(A)** *and* **(B)** *so that $K \in B(X_w)$.*

Theorem 4.4 remains valid in the real line case. We now introduce a real-line variant of **(E′)**.

Assumption (E′).

$$\sup_{|s|\geq 2A} \left(\int_{-|s|+A}^{-A} + \int_A^{|s|-A} \right) \frac{w(s)}{w(t)} |\kappa(s-t)|\, dt \to 0, \quad \text{as } A \to \infty.$$

From now we use the notation $(\boldsymbol{E_+})$ and $(\boldsymbol{E'_+})$ to refer to the half line variants of $(\boldsymbol{E})$ and $(\boldsymbol{E'})$. Moreover, we will use the notation $\mathcal{W}(\kappa)$, with $\kappa \in L^1(\mathbb{R})$, for the collection of all even weight functions w satisfying (2.3) and for which this modified version of $(\boldsymbol{E'})$ and also $(\boldsymbol{F'})$ are satisfied. (Note that, since w is assumed even, we do not need to modify $(\boldsymbol{F'})$.)

Note that $(\boldsymbol{E'})$ is satisfied by w and κ if and only if κ and its reflection around the origin $\breve{\kappa}$, defined almost everywhere by $\breve{\kappa}(t) := \kappa(-t)$, both satisfy $(\boldsymbol{E'_+})$ and the following two conditions also hold:

$$\sup_{s \geq 2A} \int_{-s+A}^{-A} \frac{w(s)}{w(t)} |\kappa(s-t)|\, dt \to 0, \quad \text{as } A \to \infty, \tag{4.83}$$

$$\sup_{s \leq -2A} \int_{A}^{|s|-A} \frac{w(s)}{w(t)} |\kappa(s-t)|\, dt \to 0, \quad \text{as } A \to \infty. \tag{4.84}$$

The next lemma shows that in many cases it suffices to check that both κ and $\breve{\kappa}$ satisfy $(\boldsymbol{E'_+})$, for a given w, in order to check whether κ and w satisfy $(\boldsymbol{E'})$. We note that in many applications κ can be chosen to be symmetric around the origin, in which case it suffices to check that κ and w satisfy $(\boldsymbol{E'_+})$.

Lemma 4.32. *Let $\kappa \in L^1(\mathbb{R})$ and w be a weight function satisfying* (2.3). *Assume that $(\boldsymbol{E'_+})$ is satisfied by both κ and $\breve{\kappa}$. Then $(\boldsymbol{E'})$ holds if one of the following conditions is satisfied:*

a) $w^{-1} \in L^1(\mathbb{R})$ and $w(s)\lambda(s) = O(1)$, as $s \to \infty$.

b) $w(s)\mu(s) = O(1)$, as $s \to \infty$.

c) for some $M > 0$, the following functions are monotonic decreasing on the interval $[M, \infty)$:

$$\lambda_+(s) := \int_s^{s+1} |\kappa(t)|\, dt, \quad \lambda_-(s) := \int_{-s-1}^{-s} |\kappa(t)|\, dt, \qquad s \in \mathbb{R}_+.$$

Remark 4.33. *Assumption c) is satisfied if κ and $\breve{\kappa}$ are monotonic decreasing on $[M, \infty)$, $M > 0$.*

Proof. In view of the discussion above, we only have to check if (4.83) and (4.84) hold whenever κ and $\breve{\kappa}$ satisfy $(\boldsymbol{E'_+})$ and one of the conditions a), b) or c) is satisfied. In the following we restrict ourselves to proving (4.83), for (4.84) can then be shown by symmetric arguments.

a) Firstly, let us assume that a) holds. Then there exists some constant C_1 such that

$$\int_s^{s+1} |\kappa(-t)|\, dt \leq \frac{C_1}{w(s)} = \frac{C_1}{w(-s)}, \quad s \in \mathbb{R}_+. \tag{4.85}$$

Let $A > 1$. For $s \leq -2A$ we have that $w(s)/w(s-n) = w(|s|)/w(|s|+n) \leq 1$, $n \in \mathbb{N}$, and thus

$$\begin{aligned}\int_A^{|s|-A} \frac{w(s)}{w(t)} |\kappa(s-t)|\, dt &\leq \sum_{n=\lfloor A \rfloor}^{\lfloor |s|-A \rfloor} \frac{w(s)}{w(n)} \int_n^{n+1} |\kappa(s-t)|\, dt \leq \sum_{n=\lfloor A \rfloor}^{\lfloor |s|-A \rfloor} \frac{C_1 w(s)}{w(n) w(s-n)} \\ &\leq \sum_{n=\lfloor A \rfloor}^{\lfloor |s|-A \rfloor} \frac{C_1}{w(n)} \leq \sum_{n=\lfloor A \rfloor - 1}^{\infty} C_1 \int_n^{n+1} \frac{1}{w(t)}\, dt = \int_{\lfloor A \rfloor - 1}^{\infty} \frac{C_1}{w(t)}\, dt \to 0,\end{aligned}$$

as $A \to \infty$, uniformly in $s < -2A$ (here, we have use the notation $\lfloor \cdot \rfloor$ introduced in Example 4.22). It follows that (4.83) holds in case a).

b) If b) is true then, there exists $C_2 > 0$ such that

$$\int_s^{\infty} |\kappa(-t)|\, dt \leq \frac{C_2}{w(s)} = \frac{C_2}{w(-s)}, \quad s \in \mathbb{R}_+.$$

Thus, if $A > 0$ and $s < -2A$,

$$\int_A^{|s|-A} \frac{w(s)}{w(t)} |\kappa(s-t)|\,dt \le \frac{w(s)}{w(A)} \int_A^{|s|-A} |\kappa(s-t)|\,dt \le \frac{C_2 w(|s|)}{w(A)w(|s|+A)} \le \frac{C_2}{w(A)} \to 0,$$

as $A \to \infty$, uniformly in $s < -2A$, whence (4.83) holds also in case b).

c) Finally, suppose that $\lambda_-(s)$ is monotonic decreasing on the interval $[M,\infty)$. Then, for $A > M+2$ and $s < -2A$,

$$\begin{aligned}\int_A^{|s|-A} \frac{w(s)}{w(t)} |\kappa(s-t)|\,dt &\le \sum_{n=\lfloor A\rfloor}^{\lfloor |s|-A\rfloor} \frac{w(s)}{w(n)} \int_n^{n+1} |\kappa(s-t)|\,dt = \sum_{n=\lfloor A\rfloor}^{\lfloor |s|-A\rfloor} \frac{w(s)\lambda_-(|s|+n)}{w(n)} \\ &\le \sum_{n=\lfloor A\rfloor}^{\lfloor |s|-A\rfloor} \frac{w(s)\lambda_-(|s|-n)}{w(-n)} \le \sum_{n=\lfloor A\rfloor}^{\lfloor |s|-A\rfloor} \int_{-n}^{-n+1} \frac{w(s)}{w(t)} |\kappa(s-t)|\,dt \le \int_{s+A}^{-A+2} \frac{w(s)}{w(t)} |\kappa(s-t)|\,dt.\end{aligned}$$

Since, by assumption, $\breve{\kappa}$ satisfies $(\boldsymbol{E}_+')$, the last term tends to 0 as $A \to \infty$, uniformly in $s < -2A$. Hence (4.83) holds whenever the conditions in c) are satisfied. □

It is straightforward to verify that Theorem 4.5 remains valid in the real line case, with the assumptions replaced by their real line variants. Thus we can state the following version of the main result of Section 4.2, Theorem 4.6.

Theorem 4.34. *Suppose that v satisfies (the real-line variants of) $(\boldsymbol{A}')$, $(\boldsymbol{B})$, with $\kappa \in L^1(\mathbb{R})$ in $(\boldsymbol{A}')$, and that $w \in \mathcal{W}(\kappa)$. Then, for every $\lambda \in \mathbb{C}$, there holds*

$$(\lambda - K) \in \mathcal{M} \iff (\lambda - K_w) \in \mathcal{M} \iff (\lambda - K) \in \mathcal{M}_w, \tag{4.86}$$

where $\mathcal{M}$ denotes one of the spaces $\mathcal{GL}(X), \Phi(X), \Phi_\pm(X)$ and $\mathcal{M}_w$ its counterpart in $\mathcal{GL}(X_w)$, $\Phi(X_w)$, $\Phi_\pm(X_w)$. The indices of $\lambda - K$ on X and $\lambda - K$ on X_w coincide if $\lambda - K$ is semi-Fredholm. Moreover,

$$0 \in \Sigma_X^e(K) = \Sigma_X^e(K_w) = \Sigma_{X_w}^e(K), \tag{4.87}$$

$$0 \in \Sigma_X(K) = \Sigma_X(K_w) = \Sigma_{X_w}(K), \tag{4.88}$$

$$0 \in \Sigma_X^\pm(K) = \Sigma_X^\pm(K_w) = \Sigma_{X_w}^\pm(K). \tag{4.89}$$

The proof of this theorem proceeds analogously to that of Theorem 4.6 and relies on a real line version of Proposition 4.7. But this proposition, as well as the remaining results of Section 4.2 also hold in the real-line case with the straightforward modifications to their statements and proofs.

We finish this chapter by stating results which, in accordance with Section 4.4, specify simpler conditions on w, v and κ that ensure that the conditions of Theorem 4.34 are satisfied. We start with a variant of Propositions 4.18 and 4.19.

Proposition 4.35. *Suppose that v satisfies $(\boldsymbol{A}')$, with $\kappa \in L^1(\mathbb{R})$, and that there exists $\theta > 0$ such that for all sufficiently large $s > 0$ the inequality*

$$\frac{w'(s)}{w(s)} \le \frac{\theta}{s} \tag{4.90}$$

holds. Further, suppose that either

$$w^{-1} \in L^1(\mathbb{R}) \quad \text{and} \quad w(s)\lambda(s) = O(1), \quad \text{as } s \to \infty, \tag{4.91}$$

or, alternatively,

$$w(s)\mu(s) = O(1), \quad \text{as } s \to \infty, \tag{4.92}$$

holds. Then Assumptions $(\boldsymbol{E}')$ and $(\boldsymbol{F}')$ are satisfied.

Alternatively, if w satisfies (4.90) *for some $\theta \le 1$ and all sufficiently large $s > 0$, and also*

$$\lambda(s) = \begin{cases} O(s^{-1}) & , \text{ if } \theta < 1, \\ o\big((s\ln s)^{-1}\big) & , \text{ if } \theta = 1, \end{cases} \qquad \text{as } s \to \infty, \tag{4.93}$$

holds then $(\boldsymbol{E'})$ *is satisfied.*

Proof. If any of the three assumptions (4.91)–(4.93) is satisfied then it follows from Proposition 4.18 or 4.19 that κ and $\breve{\kappa}$ both satisfy $(\boldsymbol{E'_+})$. If (4.91) or (4.92) is satisfied then part a) or b) of Lemma 4.32 applies and shows that $(\boldsymbol{E'})$ holds. Thus we concentrate on the case when $\theta \le 1$ and (4.93) holds.

Similar to the first part of the proof of Proposition 4.19, we can then choose positive constants M and C so that

$$\frac{w(s)}{w(t)} \le \left(\frac{|s|}{|t|}\right)^{\theta}, \quad |s| \ge |t| \ge M,$$
$$\lambda(s) \le \begin{cases} C|s|^{-1}, & \text{if } \theta < 1, \\ C|s|\ln|s|^{-1}, & \text{if } \theta = 1, \end{cases} \qquad |s| \ge M.$$

Now, suppose that $A > M + 1$, $\eta \in (0, 1/2]$ and $s \le -2A$. Then, arguing as in (4.52),

$$\int_{\max\{A,\eta|s|\}}^{|s|-A} \frac{w(s)}{w(t)}|\kappa(s-t)|\,dt \le \eta^{-\theta}\int_A^{\infty} |\kappa(-t)|\,dt. \tag{4.94}$$

If $\theta < 1$ then we obtain, for $A > M + 1$ and $\eta s \le -A$,

$$\begin{aligned}\int_A^{\eta|s|} \frac{w(s)}{w(t)}|\kappa(s-t)|\,dt &\le \int_A^{\eta|s|}\Big(\frac{|s|}{t}\Big)^{\theta}|\kappa(s-t)|\,dt \le C\sum_{n=\lfloor A\rfloor}^{\lfloor\eta|s|\rfloor}\Big(\frac{|s|}{n}\Big)^{\theta}\int_n^{n+1}|\kappa(s-t)|\,dt \\ &\le C\sum_{n=\lfloor A\rfloor}^{\lfloor\eta|s|\rfloor}\Big(\frac{|s|}{n}\Big)^{\theta}\frac{C}{|s-n|} \le C|s|^{\theta}\int_{\lfloor A\rfloor-1}^{\lfloor\eta|s|\rfloor}\frac{dt}{t^{\theta-1}} \le C|s|^{\theta}\int_0^{\eta|s|}\frac{dt}{t^{\theta-1}} \le C\eta^{1-\theta}.\end{aligned}$$

Combining this inequality with (4.94) there holds, for $A > M + 1$ and $\eta \in (0, 1/2]$,

$$\int_A^{|s|-A}\frac{w(s)}{w(t)}|\kappa(s-t)|\,dt \le \eta^{-\theta}\int_A^{\infty}|\kappa(-t)|\,dt + C\eta^{1-\theta}, \quad s \le -2A.$$

The term on the right-hand side of this inequality can be made arbitrarily small, uniformly in s, by choosing first η small enough and then A large enough. Thus (4.83) holds when $\theta < 1$.

If $\theta = 1$ then we set $\eta = 1/2$ and obtain, for $A > M + 2$ and $s < -2A$,

$$\int_A^{|s|-A}\frac{w(s)}{w(t)}|\kappa(s-t)|\,dt \le |s|\Big(\sup_{t\ge|s|}\lambda(t)\Big)\int_{\lfloor A\rfloor-1}^{|s|}\frac{dt}{t} \le |s|\ln|s|\Big(\sup_{t\ge|s|}\lambda(t)\Big) \to 0$$

as $A \to \infty$, so that (4.83) holds also when $\theta = 1$.

By symmetric arguments we show that (4.84) also holds when (4.93) is true, whence $(\boldsymbol{E'})$ must be satisfied in both cases of (4.93). □

Proposition 4.36. *Suppose w fulfils the conditions of Proposition 4.24 and that v satisfies* $(\boldsymbol{A'})$*, with $\kappa \in L^1(\mathbb{R})$, and, moreover,*

$$w(s)\lambda(s) = O(1), \quad s \to \infty.$$

Then Assumptions $(\boldsymbol{E'})$ *and* $(\boldsymbol{F'})$ *are satisfied.*

Proof. If the assumptions of the proposition are satisfied then it follows from Propositions 4.24 that κ and $\breve{\kappa}$ both satisfy $(\boldsymbol{E'_+})$.

Let $C_1 > 0$ be the constant in (4.85). Now, for $A > 2$ and $s \le -2A$, we use the bound $w(s-t) \le w(|s|-t)$, for $0 \le t \le |s|$, to see that

$$\int_A^{|s|-A} \frac{w(s)}{w(t)} |\kappa(s-t)|\,dt \le \sum_{n=\lfloor A\rfloor}^{\lfloor |s|-A\rfloor} \frac{w(s)}{w(n)} \int_n^{n+1} |\kappa(s-t)|\,dt \le \sum_{n=\lfloor A\rfloor}^{\lfloor |s|-A\rfloor} \frac{C_1 w(s)}{w(n)w(s-n)}$$
$$\le \sum_{n=\lfloor A\rfloor-1}^{\lfloor |s|-A\rfloor} C_1 \int_n^{n+1} \frac{w(s)}{w(t)w(|s|-t)}\,dt \le C_1 \int_{A-2}^{|s|-A+1} \frac{w(|s|)}{w(t)w(|s|-t)}\,dt.$$

It now follows from the proof of Proposition 4.24 (in particular (4.63), (4.65) and (4.66)) that the integral on the right-hand side of this inequality tends to 0 as $A \to \infty$, uniformly in $s \le -2A$. Thus (4.83) and, by symmetry, also (4.84) holds, whence $(\boldsymbol{E'})$ is satisfied. □

Proposition 4.37. *Suppose that v satisfies $(\boldsymbol{A'})$ with $\kappa \in L^1(\mathbb{R})$. Assume further that $g \in C^1(0,\infty)$ is a positive function which satisfies condition (4.69) of Proposition 4.26. Moreover, assume that*

$$g(s)\frac{w'(s)}{w(s)} = O(1), \quad w(s)\mu(g(s)) = O(1),$$

as $s \to \infty$. Then Assumptions $(\boldsymbol{E'})$ and $(\boldsymbol{F'})$ are satisfied.

Proof. If the assumptions of the proposition are satisfied then it follows from Propositions 4.26 that κ and $\breve{\kappa}$ both satisfy $(\boldsymbol{E'_+})$.

As in the proof of Proposition 4.26, let us firstly assume that $g(s) \le \theta s$ for some $\theta \in (0,1)$ and all sufficiently large s. Then, for A sufficiently large and $s \le -2A$ there holds

$$\int_A^{|s|-A} \frac{w(s)}{w(t)} |\kappa(s-t)|\,dt \le \frac{w(s)}{w(A)} \int_{|s|+A}^{\infty} |\kappa(-t)|\,dt \le \frac{w(s)}{w(A)} \int_{g(|s|)}^{\infty} |\kappa(-t)|\,dt, \tag{4.95}$$

By assumption there holds $w(s)\mu(g(|s|)) = w(|s|)\mu(g(|s|) \le C$, $|s| > 1$, for some constant $C > 0$, whence the term on the right-hand side of this inequality vanishes as $A \to \infty$, uniformly in $s \le -2A$. Thus (4.83) and, by symmetry, also (4.84) holds, whence $(\boldsymbol{E'})$ is satisfied.

If $g(s) \le \theta s$ is not satisfied, then the last part of the proof of Proposition 4.26 shows that w and κ satisfy assumptions (4.91) and (4.92) of Proposition 4.35, which in turn shows that $(\boldsymbol{E'})$ must hold. □

Finally, we show that, as in the half line case, for a given kernel v satisfying $(\boldsymbol{A'})$, for some $\kappa \in L^1(\mathbb{R})$, and given $y \in X_0$, we can always construct a suitable weight function w so that $y \in X_w$ and $w \in \mathcal{W}(\kappa)$.

Theorem 4.38. *Suppose the kernel v satisfies $(\boldsymbol{A'})$, with $\kappa \in L^1(\mathbb{R})$, and that $y \in X_0$. Then there exists an even weight function $w \in \mathcal{W}(\kappa)$ such that $w(s)\mu(s) = O(1)$, as $s \to \infty$, and $y \in X_w$ holds.*

Proof. Let κ and y be given as in the assumption and denote the reflection around the origin of κ and y by $\breve{\kappa}$ and $\breve{y}$, respectively. By Theorem 4.27 there exist even weight functions w_1, w_2 satisfying $(\boldsymbol{F'})$ and $(\boldsymbol{E'_+})$ with κ and $\breve{\kappa}$, respectively, and such that

$$\sup_{s\in\mathbb{R}_+} |w_1(s)y(s)| < \infty, \qquad \sup_{s\in\mathbb{R}_+} |w_2(s)\breve{y}(s)| < \infty,$$

and, as $s \to \infty$,

$$w_1(s)\int_s^{\infty} |\kappa(t)|\,dt = O(1), \qquad w_2(s)\int_s^{\infty} |\kappa(-t)|\,dt = O(1).$$

Let $w(s) := \min\{w_1(s), w_2(s)\}$, $s \in \mathbb{R}$. Then w is even, satisfies (2.3), $w(s)\mu(s) = O(1)$ as $s \to \infty$ and $y \in X_w$. Moreover, part b) of Lemma 4.32 shows that $w \in \mathcal{W}(\kappa)$ and the proof is complete. □

Corollary 4.39. *If $\kappa \in L^1(\mathbb{R})$ then $\mathcal{W}(\kappa) \neq \varnothing$.*

Chapter 5

Numerical methods in weighted spaces

5.1 Nyström and product integration methods

In the previous chapter we have investigated the theoretical solvability of the integral equation

$$\lambda x(s) - \int_{-\infty}^{\infty} v(s,t)k(s,t)x(t)\,dt = y(s), \quad s \in \mathbb{R}, \tag{5.1}$$

(and its half line variant), in operator form

$$\lambda x - K_k x = y, \tag{5.2}$$

on the weighted subspaces X_w of X defined in Chapter 2. In this chapter we now focus on the practical solution of (5.1), employing variants of the Nyström method, in which we replace the integral operator K_k in (5.2) by a discretized integral operator K_k^N, obtained by quadrature approximation, with N being a parameter controlling the quality of the approximation. We then try to solve the discretized equation

$$\lambda x^N - K_k^N x^N = y. \tag{5.3}$$

Our methods for the numerical solution of (5.1) are based on an application of a quadrature approximation, sometimes known as a *product integration method*, taking the form

$$\int_{-\infty}^{\infty} v(s,t)f(t)\,dt \approx \sum_{j\in\mathbb{Z}} \omega_j^N(s) f(t_j^N), \qquad f \in X. \tag{5.4}$$

In this equation, for $N \in \mathbb{N}$ and $j \in \mathbb{Z}$, $t_j^N := jh_N$, are the abscissae of the Nth in a sequence of quadrature rules. The abscissae are equally spaced with distance $h_N > 0$, where $h_N \to 0$ as $N \to \infty$. The corresponding weights of the quadrature rule, appropriate to the weight function $v(s,\cdot)$ in the integrand, are $\omega_j^N(s)$, $j \in \mathbb{Z}$, $N \in \mathbb{N}$.

As we will make clear soon, the approximation in (5.4) is well-defined for every $f \in X$ and depends continuously on s if we make the following two assumptions,

Assumption (*QA*).

$$C_Q := \sup_{s\in\mathbb{R}} \sup_{N\in\mathbb{N}} \sum_{j\in\mathbb{Z}} |\omega_j^N(s)| < \infty.$$

Assumption (*QB*).

$$\sup_{N\in\mathbb{N}} \sum_{j\in\mathbb{Z}} |\omega_j^N(s) - \omega_j^N(s+h)| \to 0, \quad \textit{as } h \to 0, \textit{ for every } s \in \mathbb{R}.$$

We note that $\boldsymbol{(QB)}$ is the same as Assumption (3.2) in [63], except for the fact that the summation in [63] is finite.

We cannot, in general, expect convergence as $N \to \infty$ of the approximation to the integral in (5.4) that is uniform in s or f for $s \in \mathbb{R}$ and $f \in X$ (but there is a large class of kernels and quadrature rules where uniform convergence in $s \in \mathbb{R}$ and f varying in a bounded and uniformly equicontinuous subset of X, is possible; we will discuss this phenomenon later in Section 5.3.1). Thus we will be more modest and assume throughout most of this chapter that the quadrature rule used to approximate the integral in (5.4) satisfies a weaker pointwise convergence condition:

Assumption $\boldsymbol{(Q)}$.

$$\sum_{j\in\mathbb{Z}} \omega_j^N(s) f(t_j^N) \to \int_{-\infty}^{\infty} v(s,t) f(t)\, dt \quad \textit{as } N \to \infty, \qquad f \in X, s \in \mathbb{R}.$$

The above assumptions are closely related to $\boldsymbol{(A)}$ and $\boldsymbol{(B)}$. In particular, we note that if $\boldsymbol{(Q)}$ and $\boldsymbol{(QA)}$ hold then, for all $s, s' \in \mathbb{R}$ and $f \in X$,

$$\begin{aligned} \left|\int_{-\infty}^{\infty} v(s,t) f(t)\, dt\right| &\leq C_Q \|f\|, \\ \left|\int_{-\infty}^{\infty} (v(s,t) - v(s',t)) f(t)\, dt\right| &\leq \|f\| \sup_{n\in\mathbb{N}} \sum_{j\in\mathbb{Z}} |w_j^N(s) - w_j^N(s')|. \end{aligned}$$

For every $z \in L^1(\mathbb{R})$, it follows from (3.4), applied with $v(s,t) := z(t)$, that

$$\int_{-\infty}^{\infty} |z(t)|\, dt = \sup_{f\in X,\, \|f\|\leq 1} \left|\int_{-\infty}^{\infty} z(t) f(t)\, dt\right|.$$

Thus, there holds

$$\boldsymbol{(Q)},\ \boldsymbol{(QA)} \quad \Rightarrow \quad \boldsymbol{(A)},\ \sup_{s\in\mathbb{R}} \int_{-\infty}^{\infty} |v(s,t)|\, dt \leq C_Q, \tag{5.5}$$

$$\boldsymbol{(Q)},\ \boldsymbol{(QA)},\ \boldsymbol{(QB)} \quad \Rightarrow \quad \boldsymbol{(A)},\ \boldsymbol{(B)}. \tag{5.6}$$

In the context of numerical methods for integral equations, the process of decomposing a given kernel into the product of functions v and k is not new. The product-integration approximation

$$\int_{-\infty}^{\infty} v(s,t) k(s,t) x(t)\, dt \approx \sum_{j\in\mathbb{Z}} \omega_j^N(s) k(s, t_j^N) x(t_j^N), \quad N \in \mathbb{N}, \tag{5.7}$$

is suitable for badly behaved kernel functions that may be written as the product of a smooth or at least continuous function k and a discontinuous, possibly singular function v (see, e.g. [10, 44] and the references therein). The quadrature weights $\omega_j^N(s)$ in (5.7) are usually constructed by integrating the product of $v(s,\cdot)$ with Lagrange interpolating functions (e.g. polynomials or trigonometric polynomials).

In analogy to the definition of the integral operators K_k, we define the *discretized integral operators* K_k^N by setting

$$K_k^N x(s) := \sum_{j\in\mathbb{Z}} \omega_j^N(s) k(s, t_j^N) x(t_j^N), \qquad s \in \mathbb{R},\ x \in X,\ N \in \mathbb{N}. \tag{5.8}$$

We will see in our first proposition that, under the conditions $\boldsymbol{(Q)}$, $\boldsymbol{(QA)}$ and $\boldsymbol{(QB)}$, the operator K_k^N is a bounded operator on the space X and maps bounded sets onto equicontinuous sets. In order to prove this result, we need the following technical lemma that will also be useful later on.

Lemma 5.1. *Suppose that the quadrature weights $\omega_j^N(s)$ satisfy $\boldsymbol{(Q)}$, $\boldsymbol{(QA)}$ and $\boldsymbol{(QB)}$. Then, for every compact set $\Omega' \subset \mathbb{R}$ and every $\epsilon > 0$, there exists a constant $A > 0$ such that*

$$\sup_{s\in\Omega'} \sup_{N\in\mathbb{N}} \sum_{|t_j^N|>A} |\omega_j^N(s)| < \epsilon. \tag{5.9}$$

Proof. We firstly show the lemma for the special case when Ω' contains just a single point $s \in \mathbb{R}$. Suppose that the lemma does not hold in this case. Then there exists an $\epsilon > 0$ such that, for every $A > 0$ and $N_0 \in N$, there holds

$$\sup_{N \geq N_0} \sum_{|t_j^N| > A} |\omega_j^N(s)| \geq \epsilon. \tag{5.10}$$

We will show, in three steps, that this leads to a contradiction.
(i) Let us choose $A_0 > 0$ such that

$$\int_{\mathbb{R}\setminus[-A_0,A_0]} |v(s,t)|\, dt \leq \frac{\epsilon}{4}. \tag{5.11}$$

Starting with A_0, we can choose $N_1 \in \mathbb{N}$ and then $A_1 > A_0$ such that

$$\sum_{|t_j^{N_1}| > A_0} |\omega_j^{N_1}(s)| \geq \epsilon, \qquad \sum_{|t_j^{N_1}| > A_1} |\omega_j^{N_1}(s)| < \frac{\epsilon}{4},$$

and we repeat this process to obtain N_2, A_2, N_3, A_3, ... (using the procedure in (i) to obtain N_n from N_{n-1} and A_{n-1}) so that (A_n) and (N_n) are strictly monotonic increasing sequences with the property

$$\sum_{|t_j^{N_n}| > A_{n-1}} |\omega_j^{N_n}(s)| \geq \epsilon, \qquad \sum_{|t_j^{N_n}| > A_n} |\omega_j^{N_n}(s)| < \frac{\epsilon}{4}, \qquad n \in \mathbb{N}. \tag{5.12}$$

(ii) We now construct inductively a function $x \in X$ with $\|x\| \leq 1$ as follows:

- We set $x(t) = 0$, for every $|t| \leq A_0$.
- Provided $x(t)$ is already defined on the interval $[-A_{n-1}, A_{n-1}]$, for some $n \in \mathbb{N}$, we define $x(t_j^{N_n})$, at all quadrature nodes $t_j^{N_n}$ with $A_{n-1} < |t_j^{N_n}| \leq A_n$ and $\omega_j^{N_n}(s) \neq 0$, implicitly by

$$x(t_j^{N_n})\omega_j^{N_n}(s) = \frac{C_n}{|C_n|} \cdot |\omega_j^{N_n}(s)|.$$

Here, C_n is given by

$$C_n := \sum_{|t_j^{N_n}| \leq A_{n-1}} \omega_j^{N_n}(s) x(t_j^{N_n}) \tag{5.13}$$

if this sum is non-zero, and $C_n := 1$ otherwise. All remaining values of $x(t)$, for $A_{n-1} < |t| \leq A_n$, are then chosen so that $|x(t)| \leq 1$ and x is continuous on $[-A_n, A_n]$.

(iii) For the function x constructed in (iii) we obtain from ***(Q)*** that

$$\lim_{n\to\infty} \Big|\sum_{j\in\mathbb{Z}} \omega_j^{N_n}(s) x(t_j^{N_n})\Big| \;=\; \Big|\int_{-\infty}^{\infty} v(s,t)x(t)\, dt\Big| \;\leq\; \int_{\mathbb{R}\setminus[-A_0,A_0]} |v(s,t)|\, dt \;\leq\; \frac{\epsilon}{4}. \tag{5.14}$$

For all those $n > 1$, for which C_n was defined by (5.13), we see, by (5.11), (5.12) and our choice of x, that

$$\begin{aligned}
\Big|\sum_{j\in\mathbb{Z}} \omega_j^{N_n}(s)x(t_j^{N_n})\Big| &= \Big|\sum_{|t_j^{N_n}|\leq A_{n-1}} \omega_j^{N_n}(s)x(t_j^{N_n}) + \frac{C_n}{|C_n|} \sum_{A_{n-1}<|t_j^{N_n}|\leq A_n} |\omega_j^{N_n}(s)| + \sum_{|t_j^{N_n}|>A_n} \omega_j^{N_n}(s)x(s)\Big| \\
&\geq \Big|C_n + \frac{C_n}{|C_n|} \sum_{A_{n-1}<|t_j^{N_n}|\leq A_n} |\omega_j^{N_n}(s)|\Big| - \frac{\epsilon}{4} \;\geq\; |C_n|\left(1 + \frac{1}{|C_n|}\frac{3\epsilon}{4}\right) - \frac{\epsilon}{4} \;\geq\; \frac{\epsilon}{2},
\end{aligned}$$

and, for all other $n > 1$,

$$\Big|\sum_{j\in\mathbb{Z}} \omega_j^{N_n}(s)x(t_j^{N_n})\Big| \;\geq \sum_{A_{n-1}<|t_j^{N_n}|\leq A_n} |\omega_j^{N_n}(s)| - \frac{\epsilon}{4} \;\geq\; \frac{3\epsilon}{4} - \frac{\epsilon}{4} = \frac{\epsilon}{2}.$$

Thus (5.14) is contradicted, so that the lemma holds when Ω' is a singleton.

Finally, we prove the general case. To this end, let Ω' and ϵ be as in the assumption of the lemma. Then, using $(\boldsymbol{QB})$, we can find a finite set of points Ω'' in Ω' such that, for every $s \in \Omega'$, there exists a point $s' \in \Omega''$ such that

$$\sup_{N\in\mathbb{N}} \sum_{j\in\mathbb{Z}} |\omega_j^N(s) - \omega_j^N(s')| < \frac{\epsilon}{2}. \tag{5.15}$$

By the first part of the proof, we may choose $A > 0$ large enough such that for all s' in the finite set Ω''

$$\sup_{N\in\mathbb{N}} \sum_{|t_j^N|>A} |\omega_j^N(s')| < \frac{\epsilon}{2}. \tag{5.16}$$

Combining (5.15) and (5.16), we see that for every $s \in \Omega'$ there is some $s' \in \Omega''$ such that

$$\sup_{N\in\mathbb{N}} \sum_{|t_j^N|>A} |\omega_j^N(s)| \leq \sup_{N\in\mathbb{N}} \sum_{|t_j^N|>A} |\omega_j^N(s) - \omega_j^N(s')| + \sup_{N\in\mathbb{N}} \sum_{|t_j^N|>A} |\omega_j^N(s')| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

proving the lemma. □

Proposition 5.2. *Denote the unit ball in X by B and suppose $W \subset BC(\mathbb{R}^2)$ is a bounded set. Suppose that the quadrature weights $\omega_j^N(s)$ satisfy Assumptions $(\boldsymbol{QA})$ and $(\boldsymbol{QB})$. Then the following statements are true:*

a) $K_k^N : X \to X$ and is uniformly bounded in $N \in \mathbb{N}$ and $k \in W$.

b) For every $k \in BC(\mathbb{R}^2)$ and $N \in \mathbb{N}$ the set $K_k^N(B)$ is bounded and equicontinuous, so that K_k^N is s-sequentially compact.

c) If, moreover, Assumption $(\boldsymbol{Q})$ is satisfied and the set W is equicontinuous then the set

$$V := \bigcup_{N\in\mathbb{N}} \bigcup_{k\in W} K_k^N(B) = \{K_k^N x : x \in X, \|x\| \leq 1,\, k \in W,\, N \in \mathbb{N}\}$$

is bounded and equicontinuous.

Proof. We will see in the proof of b) that $K_k^N x \in X$, for every $x \in X$. Assumption $(\boldsymbol{QA})$ implies that, for $s \in \mathbb{R}$, $x \in B$, $k \in W$, $N \in \mathbb{N}$,

$$|K_k^N x(s)| \leq \sum_{j\in\mathbb{Z}} |\omega_j^N(s)k(s,t_j^N)x(t_j^N)| \leq \big(\sup_{k\in W}\|k\|\big) \sum_{j\in\mathbb{Z}} |\omega_j^N(s)| \leq C_Q \sup_{k\in W}\|k\|,$$

so that $\|K_k^N\| \leq C_Q \sup_{k\in W}\|k\|$, where C_Q is the constant in $(\boldsymbol{QA})$. This proves the uniform boundedness of the operators K_k^N, $k \in W$, $N \in \mathbb{N}$, i.e. part a) of the proposition.

To prove c), choose $s \in \mathbb{R}$. Then, given $\epsilon > 0$, we can use Lemma 5.1 with $\Omega' = [s-1, s+1]$ to find a constant A such that (5.9) holds. Then, for all $k \in W$, $x \in B$ and $N \in \mathbb{N}$, we have by (5.9) and $(\boldsymbol{QA})$

$$\begin{aligned} |K_k^N x(s) - K_k^N x(s')| \leq& \Big(\sum_{|t_j^N|\leq A} + \sum_{|t_j^N|>A}\Big) |\omega_j^N(s)k(s,t_j^N) - \omega_j^N(s')k(s',t_j^N)||x(t)| \\ \leq& \sum_{|t_j^N|\leq A} |\omega_j^N(s) - \omega_j^N(s')||k(s',t_j^N)| + \sum_{|t_j^N|\leq A} |\omega_j^N(s)||k(s,t_j^N) - k(s',t_j^N)| + 2\epsilon\|k\| \\ \leq& \|k\| \sum_{j\in\mathbb{Z}} |\omega_j^N(s) - \omega_j^N(s')| + C_Q \max_{|t_j^N|<A} |k(s,t_j^N) - k(s',t_j^N)| + 2\epsilon\|k\|, \end{aligned} \tag{5.17}$$

provided s' is such that $|s-s'| \le 1$. If $s' \to s$ then the first summand on the right-hand side of (5.17) converges to 0, uniformly in $k \in W$ and $N \in \mathbb{N}$, because v satisfies **(QB)** and W is bounded. The same is true for the second summand, for W is, by assumption, uniformly equicontinuous over compact subsets of $\mathbb{R}^2$. Since ϵ was arbitrary, we see that $|K_k^N x(s) - K_k^N x(s')|$ becomes arbitrarily small as $s' \to s$, uniformly in $k \in W$ and $N \in \mathbb{N}$, proving the uniform equicontinuity of V.

Finally, to see that b) holds, let $s \in \mathbb{R}$, $N \in \mathbb{N}$ and $k \in BC(\mathbb{R}^2)$. Given $\epsilon > 0$, there exists $A > 0$ such that, for all $s' \in [s-1, s+1]$,

$$\sum_{|t_j^N|>A} |\omega_j^N(s')| < \epsilon. \tag{5.18}$$

Using a similar argument as in the final part of the proof of Lemma 5.1, we obtain from **(QA)** and **(QB)** (only) $A > 0$ so large that (5.18) also holds uniformly for *all* $s' \in [s-1, s+1]$. With this choice of A, inequality (5.17) holds for all $s' \in [s-1, s+1]$. Arguing similarly to the proof of c), we see that $|K_k^N x(s) - K_k^N x(s')| \to 0$ as $|s-s'| \to 0$. □

We have shown in Chapter 3 that, if v satisfies **(A)** and **(B)** and $k \in BC(\mathbb{R}^2)$, then the integral operator K_k maps bounded onto equicontinuous sets. Under the assumptions of the previous theorem this is also true for the discretized integral operator K_k^N, $k \in BC(\mathbb{R}^2)$, $N \in \mathbb{N}$. We prove some other features of the discretized integral operators in the next proposition.

Proposition 5.3. *Suppose that the quadrature weights $\omega_j^N(s)$ satisfy* **(Q)**, **(QA)** *and* **(QB)**. *Further, let (x_n) be a bounded sequence in X, (N_n) a sequence in $\mathbb{N}$ and (k_n) a bounded sequence in $BC(\mathbb{R}^2)$. Then the following assertions hold:*

a) For every $N \in \mathbb{N}$ and $k \in BC(\mathbb{R}^2)$ the operator K_k^N is s-continuous.

b) If $x_n \xrightarrow{s} x \in X$, $k_n \xrightarrow{s} k \in BC(\mathbb{R}^2)$ and $N_n \to N \in \mathbb{N}$ then $K_{k_n}^{N_n} x_n \xrightarrow{s} K_k^N x$.

If, further, Assumption **(Q)** *is satisfied then the following statements are true:*

c) If $x_n \xrightarrow{s} x \in X$, $k_n \xrightarrow{s} k \in BC(\mathbb{R}^2)$ and $N_n \to \infty$ then $K_{k_n}^{N_n} x_n \xrightarrow{s} K_k x$.

d) If the sequence (k_n) is equicontinuous then the sequence $(K_{k_n}^{N_n} x_n)$ contains an s-convergent subsequence.

e) If $W \subset BC(\mathbb{R}^2)$ is bounded and equicontinuous, then the following set is relatively s-sequentially compact:

$$V := \bigcup_{N\in\mathbb{N}} \bigcup_{k\in W} K_k^N(B) = \{K_k^N x : x \in X, \|x\| \le 1,\ k \in W,\ N \in \mathbb{N}\}.$$

Remark 5.4. *Note that if $k_n \xrightarrow{s} k$ then each sequence in the (bounded) set $\{k_n : n \in \mathbb{N}\}$ has an s-convergent subsequence. By Remark 2.8, this means that $\{k_n : n \in \mathbb{N}\}$ is bounded and equicontinuous. Thus part d) shows that, if (x_n) is a bounded sequence in X, (N_n) a sequence in $\mathbb{N}$ and (k_n) a sequence in $BC(\mathbb{R}^2)$ so that $k_n \xrightarrow{s} k \in BC(\mathbb{R}^2)$ then $(K_{k_n}^{N_n} x_n)$ contains an s-convergent subsequence.*

Proof. We begin with the proof of part b), for a) is a special case of b). Assume we are given convergent sequences (x_n), (k_n) and (N_n) as in the assumption. Since $N_n \to N \in \mathbb{N}$ implies that (N_n) is eventually constant, we may assume w.l.o.g. that $N_n = N$, for every $n \in \mathbb{N}$.

Given a compact set $\Omega' \subset \mathbb{R}$ we now have to show that $K_{k_n}^N x_n(s) \to K_k^N x(s)$, uniformly in $s \in \Omega'$.

Let $s \in \Omega'$. Bearing in mind (2.10), we have, for all $A > 0$,

$$\begin{aligned} |K^N_{k_n} x_n(s) - K^N_k x(s)| &\le \sum_{|t^N_j| \le A} |\omega^N_j(s)| \Big(|k_n(s,t^N_j) - k(s,t^N_j)||x_n(t^N_j)| + |k(s,t^N_j)||x_n(t^N_j) - x(t^N_j)| \Big) \\ &\quad +2 \big(\sup_{n\in\mathbb{N}} \|x_n\|\big) \big(\sup_{n\in\mathbb{N}} \|k_n\|\big) \sum_{|t^N_j| > A} |\omega^N_j(s)| \\ &\le C_Q \Big(\big(\sup_{n\in\mathbb{N}} \|x_n\|\big) \max_{|t|\le A} |k_n(s,t) - k(s,t)| + \|k\| \max_{|t|\le A} |x_n(t) - x(t)| \Big) \\ &\quad + 2 \big(\sup_{n\in\mathbb{N}} \|x_n\|\big) \big(\sup_{n\in\mathbb{N}} \|k_n\|\big) \sum_{|t^N_j| > A} |\omega^N_j(s)|. \end{aligned} \tag{5.19}$$

Given $\epsilon > 0$, we can choose A large enough to make the last summand on the right-hand side $< \epsilon/2$, for all $s \in \Omega'$ (see the argument in the last step in the proof of Lemma 5.1). Next, we use the uniform convergence of (k_n) and (x_n) on compact intervals to find that for $n \in \mathbb{N}$ large enough the first summand is also less than $\epsilon/2$ for all $s \in \Omega'$. Since ϵ was arbitrary $K^N_{k_n} x_n \xrightarrow{s} K^N_k x$ follows.

Now, additionally suppose that Assumption $\boldsymbol{(Q)}$ holds. Then, using Lemma 5.1 to bound the series in the very last term in (5.19) we see that the argument in b) does not depend on the choice of $N \in \mathbb{N}$, and, hence, $x_n \xrightarrow{s} x \in X$, $k_n \xrightarrow{s} k \in BC(\mathbb{R}^2)$ imply that $K^{N_n}_{k_n} x_n - K^{N_n}_k x \xrightarrow{s} 0$.

We continue with part c). We write

$$K^{N_n}_{k_n} x_n - K_k x = \big(K^{N_n}_{k_n} x_n - K^{N_n}_k x\big) + \big(K^{N_n}_k x - K_k x\big),$$

and note that we have just shown that the first bracket on the right-hand side is strictly converging to 0. We now show that this is also true for the second bracket. Since $\boldsymbol{(Q)}$ holds we already know that $K^{N_n}_k x(s) \to K_k x(s)$ pointwise for all $s \in \mathbb{R}$. By Proposition 5.2, $\{K^{N_n}_k x : n \in \mathbb{N}\}$ is bounded and equicontinuous on $\mathbb{R}$. But, over compact sets, pointwise convergence of an equicontinuous sequence implies uniform convergence. Thus, for every compact $\Omega' \subset \mathbb{R}$, $K^{N_n}_k x_n(s) \to K_k x(s)$ uniformly in $s \in \Omega'$. This proves $K^{N_n}_k x_n \xrightarrow{s} K_k x$ and c) follows.

Part e) is a consequence of Remark 2.8 and Proposition 5.2, part c). Part d) is immediate from e). □

5.2 Boundedness and spectral properties in weighted spaces

Suppose now that v is a kernel satisfying Assumptions $\boldsymbol{(A')}$ and $\boldsymbol{(B)}$ of the previous chapter, with $\kappa \in L^1(\mathbb{R})$ in $\boldsymbol{(A')}$. We have then denoted by $\mathcal{W}(\kappa)$ the (non-empty) set of all weight functions for which Assumptions $\boldsymbol{(E')}$ and $\boldsymbol{(F')}$ are satisfied. For $w \in \mathcal{W}(\kappa)$ and $k \in BC(\mathbb{R}^2)$, we have shown that $K_k \in \mathcal{B}(X_w)$ and that the spectral equivalences $\Sigma_X(K_k) = \Sigma_{X_w}(K_k)$, $\Sigma^e_X(K_k) = \Sigma^e_{X_w}(K_k)$ hold.

Suppose we are given a set of quadrature weights $\omega^N_j(s)$ and an even weight function w satisfying (2.3). Similar to the definition of the operator K_w in the previous chapter, we now define the operators $K_{k,w} := M_w K_k M_{w^{-1}}$ and $K^N_{k,w} := M_w K^N_k M_{w^{-1}}$, for $k \in BC(\mathbb{R}^2)$ and $N \in \mathbb{N}$, where K_k and K^N_k are defined by (3.2) and (5.8), respectively. (We recall that M_w and $M_{w^{-1}}$ denote the operators of multiplication with the functions w, w^{-1}, respectively.) In the sequel, we will sometimes drop the index k and simply write K^N_w when the dependence on k is clear.

$K^N_{k,w}$ is then an approximation of the integral operator K_k with quadrature weights

$$\omega^N_{j,w}(s) := \frac{w(s)}{w(t^N_j)} \omega^N_j(s), \qquad s \in \mathbb{R}, N \in \mathbb{N}, j \in \mathbb{Z}. \tag{5.20}$$

Assume further that we have found a set of quadrature weights $\omega^N_j(s)$ satisfying $\boldsymbol{(Q)}$, $\boldsymbol{(QA)}$ and $\boldsymbol{(QB)}$. We now seek additional conditions on the quadrature weights, which ensure that $K^N_k \in \mathcal{B}(X_w)$ (equivalently, $K^N_{k,w} \in \mathcal{B}(X)$) for all $w \in \mathcal{W}(\kappa)$, and, moreover, that the spectral equivalences

$$\Sigma_X(K^N_k) = \Sigma_{X_w}(K^N_k), \qquad \Sigma^e_X(K^N_k) = \Sigma^e_{X_w}(K^N_k)$$

hold, for all $w \in \mathcal{W}(\kappa)$. To this end, let us suppose, from now on, that the kernel v fulfils the following assumption.

Assumption $(\boldsymbol{A''})$. *For every $s \in \mathbb{R}$ there holds, for a.e. $t \in \mathbb{R}$,*

$$|v(s,t)| \le |\kappa(s-t)|,$$

where $\kappa \in L^1(\mathbb{R})$ is such that, for some $A_0 > 0$, $|\kappa(t)|$ and $|\kappa(-t)|$ are monotonic decreasing on $[A_0, \infty)$.

Clearly, $(\boldsymbol{A''})$ is a slightly stronger requirement than $(\boldsymbol{A'})$. However, $(\boldsymbol{A''})$ is still simple enough to be easily checked in applications and to hold for most kernels of practical interest. The usefulness of the monotonicity condition imposed on κ in $(\boldsymbol{A''})$ will become evident soon. We also remark that, for kernels satisfying $(\boldsymbol{A''})$, the third condition in Lemma 4.32 is satisfied, so that, in order to decide whether a particular weight function w is contained in $\mathcal{W}(\kappa)$, one has to check whether $(\boldsymbol{E'_+})$ holds for κ and $\breve{\kappa}$, its reflection around the origin.

If a quadrature approximation as in (5.4) is defined for a particular kernel v satisfying $(\boldsymbol{A''})$, then two reasonable assumptions on the quadrature weights $\omega_j^N(s)$ are that they reflect the decay of $v(s,t)$ as $|s-t| \to \infty$ and the decay of h_N as $N \to \infty$. We express these requirements in the following assumption on the quadrature weights, where κ is the kernel bound in $(\boldsymbol{A''})$.

Assumption $(\boldsymbol{QA''})$. *For some $C^* > 0$ and $A_1 > 0$, the quadrature weights $\omega_j^N(s)$ satisfy*

$$|\omega_j^N(s)| \le C^* h_N |\kappa(s - t_j^N)|, \qquad N \in \mathbb{N},\ j \in \mathbb{Z},\ |s - t_j^N| \ge A_1.$$

$(\boldsymbol{QA''})$ is satisfied for many sensible approximations of kernels satisfying $(\boldsymbol{A''})$; in particular it is satisfied for standard rules such as the (compound) trapezium or Simpson's rule.

The next lemma shows the usefulness of $(\boldsymbol{A''})$ and $(\boldsymbol{QA''})$, because it will allow us to use the results of Chapter 4 in our investigation of the boundedness and spectral properties of the operators K_k^N on X and X_w.

Lemma 5.5. *Suppose that $N \in \mathbb{N}$, that the kernel v satisfies $(\boldsymbol{A''})$, for some $\kappa \in L^1(\mathbb{R})$ and $A_0 \ge 0$, and that the quadrature weights $\omega_j^N(s)$ satisfy $(\boldsymbol{QA''})$, for some $C^*, A_1 > 0$. Assume, further, that either $w(s) = 1$, $s \in \mathbb{R}$, or that w is an even weight function satisfying* (2.3) *and $(\boldsymbol{F'})$. Then, for M_1, M_2 such that* $\max\{A_0 + h_N, A_1\} \le M_1 < M_2 \le \infty$, *the estimates*

$$\begin{aligned} \sum_{M_1 \le s - t_j^N \le M_2} \frac{1}{w(t_j^N)} |\omega_j^N(s)| &\le C \int_{s-M_2}^{s-M_1+h_N} \frac{1}{w(t)} |\kappa(s-t)|\, dt, \\ \sum_{-M_2 \le s - t_j^N \le -M_1} \frac{1}{w(t_j^N)} |\omega_j^N(s)| &\le C \int_{s+M_1-h_N}^{s+M_2} \frac{1}{w(t)} |\kappa(s-t)|\, dt, \end{aligned}$$

hold, for every $s \in \mathbb{R}$, where $C > 0$ is some positive constant not depending on s, M_1, M_2 or N.

Proof. Let $M := \max_{N \in \mathbb{N}} h_N$ and choose $N \in \mathbb{N}$ and M_1, M_2 as in the assumption. Then, for every

$s \in \mathbb{R}$, we obtain from $\boldsymbol{(QA'')}$ that

$$\begin{aligned}\sum_{M_1 \le s - t_j^N \le M_2} \frac{1}{w(t_j^N)} |\omega_j^N(s)| &\le C^* h_N \sum_{s-M_2 \le t_j^N \le s-M_1} \frac{1}{w(t_j^N)} |\kappa(s - t_j^N)| \\ &= C^* \sum_{s-M_2 \le t_j^N \le s-M_1} \int_{t_{j-1}^N}^{t_j^N} \frac{1}{w(t_j^N)} |\kappa(s - t_j^N)|\, dt \\ &\le C^* \sum_{s-M_2 \le t_j^N \le s-M_1} \int_{t_j^N}^{t_{j+1}^N} \frac{1}{w(t_j^N)} |\kappa(s - t)|\, dt \\ &\le C^* \Delta_w^M \sum_{s-M_2 \le t_j^N \le s-M_1} \int_{t_j^N}^{t_{j+1}^N} \frac{1}{w(t)} |\kappa(s - t)|\, dt \\ &\le C^* \Delta_w^M \int_{s-M_2}^{s-M_1+h_N} \frac{1}{w(t)} |\kappa(s-t)|\, dt,\end{aligned}$$

where Δ_w^M has been defined in (4.11). Setting $C := C^* \Delta_w^M$, we see that the first of the desired inequalities holds. A symmetric argument shows that second inequality holds as well. $\square$

After these preparations we now prove that Assumptions $\boldsymbol{(A'')}$, $\boldsymbol{(QA)}$, $\boldsymbol{(QA'')}$ and $\boldsymbol{(QB)}$ together imply that K_k^N is a bounded operator on X_w, for every $w \in \mathcal{W}(\kappa)$. In the following theorem and subsequently we will say that the quadrature weights $\omega_{j,w}^N(s)$ satisfy Assumption $\boldsymbol{(Q)}$, if Assumption $\boldsymbol{(Q)}$ holds when ω_j^N and v are replaced by $\omega_{j,w}^N$ and v_w, respectively.

Theorem 5.6. *Suppose that the kernel v satisfies $\boldsymbol{(A'')}$, with $\kappa \in L^1(\mathbb{R})$, and $\boldsymbol{(B)}$ and that $w \in \mathcal{W}(\kappa)$. Further, assume that the quadrature weights $\omega_j^N(s)$ satisfy $\boldsymbol{(QA)}$, $\boldsymbol{(QA'')}$ and $\boldsymbol{(QB)}$. Then, the quadrature weights $\omega_{j,w}^N(s)$ satisfy $\boldsymbol{(QA)}$ and $\boldsymbol{(QB)}$, so that, for all $N \in \mathbb{N}$ and $k \in BC(\mathbb{R}^2)$, the operator K_k^N is bounded on X_w. Moreover, if W is a bounded subset of $BC(\mathbb{R}^2)$ then*

$$\sup_{N \in \mathbb{N}} \sup_{k \in W} \|K_k^N\|_w < \infty.$$

If the quadrature weights $\omega_j^N(s)$ satisfy $\boldsymbol{(Q)}$ then the quadrature weights $\omega_{j,w}^N(s)$ also satisfy $\boldsymbol{(Q)}$.

Proof. Assumption $\boldsymbol{(Q)}$ for the quadrature weights $\omega_{j,w}^N(s)$ is easy to verify if weights $\omega_j^N(s)$ satisfy $\boldsymbol{(Q)}$; because then, for every $s \in \mathbb{R}$ and $x \in X$, there holds

$$\sum_{j \in \mathbb{Z}} \omega_{j,w}^N(s) x(t_j^N) = w(s) \sum_{j \in \mathbb{Z}} \omega_j^N(s) \frac{x(t_j^N)}{w(t_j^N)} \to w(s) \int_{-\infty}^{\infty} v(s,t) \frac{x(t)}{w(t)}\, dt, \quad N \to \infty,$$

showing that the quadrature weights $\omega_{j,w}^N(s)$ indeed satisfy $\boldsymbol{(Q)}$.

We now show that the quadrature weights $\omega_{j,w}^N(s)$ satisfy Assumptions $\boldsymbol{(QA)}$ and $\boldsymbol{(QB)}$. The conclusion of the theorem then follows from Proposition 5.2 and equivalence (2.8). We also note that the following arguments do not depend on the value of the parameter $N \in \mathbb{N}$. Further, all bounds can be chosen such that they are independent of $N \in \mathbb{N}$.

To see that the quadrature weights $\omega_{j,w}^N(s)$ satisfy $\boldsymbol{(QB)}$, observe that, for all $s, s' \in \mathbb{R}$,

$$\sum_{j \in \mathbb{Z}} |\omega_{j,w}^N(s) - \omega_{j,w}^N(s')| \le w(s) \sum_{j \in \mathbb{Z}} |\omega_j^N(s) - \omega_j^N(s')| + |w(s) - w(s')| \sum_{j \in \mathbb{Z}} |\omega_j^N(s')|.$$

By the assumptions on the quadrature weights $\omega_j^N(s)$ and the continuity of w, both summands on the right-hand side of this inequality vanish as $s' \to s$, uniformly in N, proving that the quadrature weights $\omega_{j,w}^N(s)$ satisfy $\boldsymbol{(QB)}$ as desired.

To see that the quadrature weights $\omega^N_{j,w}(s)$ satisfy $\boldsymbol{(QA)}$, we need to prove that

$$\sup_{s\in\mathbb{R}}\sup_{N\in\mathbb{N}}\sum_{j\in\mathbb{Z}}|\omega^N_{j,w}(s)|<\infty. \tag{5.21}$$

We now choose $A\geq\max\{A_0+2M,A_1\}$, where A_0, A_1 denote the constants in $\boldsymbol{(A'')}$ and $\boldsymbol{(QA'')}$, respectively, and $M:=\max_{N\in\mathbb{N}}h_N$. We then set, for $s\in\mathbb{R}$,

$$S_1(s)+S_2(s)+S_3(s):=\left(\sum_{t^N_j<s-A}+\sum_{|s-t^N_j|\leq A}+\sum_{t^N_j>s+A}\right)\frac{w(s)}{w(t^N_j)}|\omega^N_j(s)|=\sum_{j\in\mathbb{Z}}|\omega^N_{j,w}(s)|, \tag{5.22}$$

and bound each term separately. For $S_2(s)$ we have the bound

$$S_2(s)\;=\;\sum_{|s-t^N_j|\leq A}\frac{w(s)}{w(t^N_j)}\omega^N_j(s)\;\leq\;\Delta^A_w\sum_{|s-t^N_j|\leq A}|\omega^N_j(s)|\;\leq\;\Delta^A_wC_Q,\quad s\in\mathbb{R}, \tag{5.23}$$

where C_Q is the constant from Assumption $\boldsymbol{(QA)}$. Thus $S_2(s)$ is uniformly bounded in $s\in\mathbb{R}$.

By choice of A and Lemma 5.5 we have, for every $s\in\mathbb{R}$ and some constant $C>0$,

$$S_1(s)+S_3(s)\leq C\left(\int_{-\infty}^{s-A+M}+\int_{s+A-M}^{\infty}\right)\frac{w(s)}{w(t)}|\kappa(s-t)|\,dt\leq C\int_{-\infty}^{\infty}\frac{w(s)}{w(t)}|\kappa(s-t)|\,dt. \tag{5.24}$$

The term on the right-hand side of this inequality is uniformly bounded in s, because the kernel $(w(s)/w(t))\kappa(s-t)$ satisfies Assumption $\boldsymbol{(A)}$; this was shown in Proposition 4.31.

Combining (5.22), (5.23) and (5.24), we see that (5.21) holds. We have shown that the quadrature weights $\omega^N_{j,w}(s)$ satisfy $\boldsymbol{(QA)}$ and $\boldsymbol{(QB)}$. As indicated above, the first part of the theorem is thus established. □

Recall that we have seen in Proposition 5.3 that Assumptions $\boldsymbol{(QA)}$ and $\boldsymbol{(QB)}$ ensure the s-continuity of the operator K^N_k, for every $N\in\mathbb{N}$ and $k\in BC(\mathbb{R}^2)$, i.e. there holds

$$x_n\xrightarrow{s}x\qquad\Longrightarrow\qquad K^N_kx_n\xrightarrow{s}K^N_kx. \tag{5.25}$$

We now seek conditions on the quadrature weights such that also sn-continuity holds.

A stronger version of (5.9) in Lemma 5.1, key element in the proof of Proposition 5.3, is the following assumption on the quadrature weights $\omega^N_j(s)$:

Assumption $\boldsymbol{(QD)}$.

$$\sup_{s\in\mathbb{R}}\sup_{N\in\mathbb{N}}\sum_{|t^N_j|\geq A}|\omega^N_j(s)|\to 0,\qquad \textit{as } A\to\infty.$$

Assumption $\boldsymbol{(QD)}$ is a discretized version of $\boldsymbol{(D)}$. The following lemma shows that K^N_k will be sn-continuous if $\boldsymbol{(QA)}$, $\boldsymbol{(QB)}$ and $\boldsymbol{(QD)}$ are satisfied and, moreover, implication (5.27) below holds, an important tool for our stability analysis later in Section 5.3.

Lemma 5.7. *Suppose Assumptions* $\boldsymbol{(QA)}$, $\boldsymbol{(QB)}$ *and* $\boldsymbol{(QD)}$ *are satisfied. Then*

$$x_n\xrightarrow{s}x\qquad\Longrightarrow\qquad K^N_kx_n\to K^N_kx, \tag{5.26}$$

for every sequence (x_n) *in* X, $k\in BC(\mathbb{R}^2)$ *and* $N\in\mathbb{N}$, *i.e. each such* K^N_k *is sn-continuous. Further, if* (k_n) *is a bounded sequence in* $BC(\mathbb{R}^2)$ *and* (N_n) *a sequence in* $\mathbb{N}$ *then*

$$x_n\xrightarrow{s}0\quad\Longrightarrow\quad K^{N_n}_{k_n}x_n\to 0. \tag{5.27}$$

Proof. To show (5.26), we repeat the argument used to show part b) of Proposition 5.3, but with $k_n := k$, $n \in \mathbb{N}$, the set $\mathbb{R}$ taking the role of the set Ω' and using $(\boldsymbol{QD})$ to bound the very last term in inequality (5.19), uniformly in $N \in \mathbb{N}$.

For the second implication, assume that (k_n), (N_n) and (x_n) are chosen as in the assumption. We then have for all $s \in \mathbb{R}$ and $A > 0$, using the constant C_Q from $(\boldsymbol{QA})$, that

$$\left|K_{k_n}^{N_n} x_n(s)\right| \leq \sup_{n\in\mathbb{N}} \|k_n\| \Big(C_Q \sup_{|t|\leq A} |x_n(t)| + \sup_{n\in\mathbb{N}} \|x_n\| \sum_{|t_j^N|>A} |\omega_j^N(s)| \Big). \tag{5.28}$$

The sequences (k_n) and (x_n) are bounded in $BC(\mathbb{R}^2)$ and X, respectively. Given $\epsilon > 0$, we can thus use $(\boldsymbol{QD})$ to choose $A > 0$ so that the second product in the bracket is $< \epsilon$, irrespective of $n \in \mathbb{N}$ and $s \in \mathbb{R}$. Keeping this A fixed, we see that the first summand in the bracket is $< \epsilon$, for all $n \in \mathbb{N}$ large enough, as $x_n \stackrel{s}{\rightarrow} 0$. Since ϵ was arbitrary, we must have $K_{k_n}^{N_n} x_n \to 0$ as $n \to \infty$. ☐

In the next proposition we show that $(\boldsymbol{QA})$, $(\boldsymbol{QA''})$ and $(\boldsymbol{QB})$ ensure the sn-continuity of the difference operator $K_k^N - K_{k,w}^N$.

Proposition 5.8. *Suppose that the kernel v satisfies $(\boldsymbol{A''})$, $(\boldsymbol{B})$ and $w \in \mathcal{W}(\kappa)$. Further, assume that the quadrature weights $\omega_j^N(s)$ satisfy $(\boldsymbol{QA})$, $(\boldsymbol{QA''})$ and $(\boldsymbol{QB})$. Then the quadrature weights $\omega_j^N(s) - \omega_{j,w}^N(s)$ satisfy $(\boldsymbol{QA})$, $(\boldsymbol{QB})$ and $(\boldsymbol{QD})$, so that the operator $K_k^N - K_{k,w}^N$ is sn-continuous.*

Proof. That the quadrature weights $\omega_j^N(s) - \omega_{j,w}^N(s)$ satisfy $(\boldsymbol{QA})$ and $(\boldsymbol{QB})$ follows from Theorem 5.6. Throughout the remainder of the proof we assume that $A^* \geq \max\{A_0 + 2M, A_1\}$ where A_0, A_1 denote the constants in $(\boldsymbol{A''})$ and $(\boldsymbol{QA''})$ and $M := \max_{N\in\mathbb{N}} h_N$.

We define, for $s \in \mathbb{R}$ and $A > 0$,

$$S_1^N(s) + S_2^N(s) := \left(\sum_{t_j^N \leq -A} + \sum_{t_j^N \geq A} \right) \left| 1 - \frac{w(s)}{w(t_j^N)} \right| |\omega_j^N(s)| \;=\; \sum_{|t_j^N| \geq A} |\omega_j^N(s) - \omega_{j,w}^N(s)|.$$

In order to prove that $\omega_j^N(s) - \omega_{j,w}^N(s)$ satisfies $(\boldsymbol{QD})$, it suffices to show that, for $i = 1, 2$,

$$\sup_{s\in\mathbb{R}} \sup_{N\in\mathbb{N}} S_i^N(s) \to 0, \quad \text{as } A \to \infty. \tag{5.29}$$

For every $s \in \mathbb{R}$, we can bound $S_2^N(s)$ as follows:

$$S_2^N(s) \leq \left(\sum_{A \leq t_j^N < s - A^*} + \sum_{\substack{\max\{A, s-A^*\} \leq t_j^N \\ t_j^N \leq \max\{s+A^*, A\}}} + \sum_{\max\{s+A^*, A\} < t_j^N} \right) \left| 1 - \frac{w(s)}{w(t_j^N)} \right| |\omega_j^N(s)|$$

$$=: S_{2,1}^N(s) + S_{2,2}^N(s) + S_{2,3}^N(s). \tag{5.30}$$

Now, $S_{2,2}^N(s)$ can only be non-zero when $s \geq A - A^*$, whence

$$S_{2,2}^N(s) \leq c_{A^*}(A) \sum_{j\in\mathbb{Z}} |\omega_j^N(s)| \leq c_{A^*}(A) C_Q, \quad s \in \mathbb{R}, N \in \mathbb{N},$$

where C_Q is the constant from $(\boldsymbol{QA})$ and $c_{A^*}(A)$ is defined as in the proof of Theorem 4.5. For fixed A^*, $c_{A^*}(A)$, and thus also $S_{2,2}^N(s)$, tends to 0 as $A \to \infty$, uniformly in $N \in \mathbb{N}$ and $s \in \mathbb{R}$.

Next, suppose that $s > A + A^* > 2A^*$. Then $|1 - w(s)/w(t)| \leq w(s)/w(t)$ holds, for $A \leq t < s - A^*$. By Lemma 5.5, there exists a constant $C > 0$, not depending on s or N, such that, for all $N \in \mathbb{N}$,

$$S_{2,1}^N(s) \leq \sum_{A \leq t_j^N < s - A^*} \frac{w(s)}{w(t)} |\omega_j^N(s)| \leq C \int_A^{s-A^*+M} \frac{w(s)}{w(t)} |\kappa(s-t)| \, dt \leq C \int_{A^*-M}^{s-(A^*-M)} \frac{w(s)}{w(t)} |\kappa(s-t)| \, dt.$$

The term on the right-hand side, and thus $S_{2,1}^N(s)$, tends to 0 as $A^* \to \infty$, uniformly in $s \geq A^* + A$ since $w \in \mathcal{W}(\kappa)$ implies that $\boldsymbol{(E')}$ holds. Since $S_{2,1}^N(s) = 0$ if $s \leq A + A^*$ we conclude that $S_{2,1}^N(s) \to 0$ as $A^* \to \infty$, uniformly in $s \in \mathbb{R}$ and $N \in \mathbb{N}$.

Since $|1 - w(s)/w(t)| \leq 1$, $0 \leq s \leq t$, and again by Lemma 5.5, there holds, for all $s \in \mathbb{R}$ and $N \in \mathbb{N}$,

$$S_{2,3}^N(s) \leq \sum_{\max\{s+A^*,A\}<t_j^N} |\omega_j^N(s)| \leq C \int_{\max\{s+A^*,A\}-M}^{\infty} |\kappa(s-t)|\,dt \leq \int_{A^*-M}^{\infty} C|\kappa(-t)|\,dt,$$

for some constant $C > 0$. The term on the right-hand side of this inequality vanishes as $A^* \to \infty$ so that $S_{2,3}^N(s) \to 0$ as $A^* \to \infty$, uniformly in s and $N \in \mathbb{N}$.

Given $\epsilon > 0$, we can now choose A^* large enough so that $S_{2,1}^N(s), S_{2,3}^N(s) < \epsilon/3$ for all $s \in \mathbb{R}$, $N \in \mathbb{N}$. Keeping this A^* fixed, we can then choose A sufficiently large so that $S_{2,2}^N(s) < \epsilon/3$, and thus $S_2^N(s) < \epsilon$, for all $s \in \mathbb{R}$, $N \in \mathbb{N}$. By virtue of (5.30) it follows that $S_2^N(s) \to 0$ as $A \to \infty$, uniformly in $s \in \mathbb{R}$ and $N \in \mathbb{N}$. A symmetric argument shows that the same is true of $S_1^N(s)$. Thus (5.29) holds, whence the quadrature weights $\omega_j^N(s) - \omega_{j,w}^N(s)$ satisfy $\boldsymbol{(QD)}$ and, in view of Lemma 5.7, the theorem is shown. □

After these preparations, we now present the two main results on the solvability of the discretized integral equation (5.3) in weighted spaces. We show that if the quadrature weights $\omega_j^N(s)$ satisfy Assumptions $\boldsymbol{(QA)}$, $\boldsymbol{(QA'')}$ and $\boldsymbol{(QB)}$ and $w \in \mathcal{W}(\kappa)$ then the operator K_k^N is invertible (Fredholm) on X if and only if it is invertible (Fredholm) on X_w.

Theorem 5.9. *Suppose that the kernel v satisfies $\boldsymbol{(A'')}$, with $\kappa \in L^1(\mathbb{R})$, $\boldsymbol{(B)}$ and that $w \in \mathcal{W}(\kappa)$. Then, if the quadrature weights $\omega_j^N(s)$ satisfy $\boldsymbol{(QA)}$, $\boldsymbol{(QA'')}$ and $\boldsymbol{(QB)}$, there holds, for all $\lambda \in \mathbb{C}$ and $k \in BC(\mathbb{R}^2)$,*

$$(\lambda - K_k^N) \in \Phi(X) \Leftrightarrow (\lambda - K_{k,w}^N) \in \Phi(X) \Leftrightarrow (\lambda - K_{k,w}^N) \in \Phi(X_w), \tag{5.31}$$

$$(\lambda - K_k^N)^{-1} \in \mathcal{B}(X) \Leftrightarrow (\lambda - K_{k,w}^N)^{-1} \in \mathcal{B}(X) \Leftrightarrow (\lambda - K_{k,w}^N)^{-1} \in \mathcal{B}(X_w), \tag{5.32}$$

and, if these operators are all Fredholm, their indices are the same. Further,

$$0 \in \Sigma_X^{\mathrm{e}}(K_k^N) = \Sigma_X^{\mathrm{e}}(K_{k,w}^N) = \Sigma_{X_w}^{\mathrm{e}}(K_k^N), \tag{5.33}$$

$$0 \in \Sigma_X(K_k^N) = \Sigma_X(K_{k,w}^N) = \Sigma_{X_w}(K_k^N). \tag{5.34}$$

Proof. Let us first consider the case $\lambda = 0$. If $N \in \mathbb{N}$ then $K_k^N x = 0$ if $x \in X$ and $x(t_j^N) = 0$ for all $j \in \mathbb{Z}$. But the space of functions in X enjoying this property is infinite-dimensional. Thus the kernel of K_k^N is infinite-dimensional, whence K_k^N is neither Fredholm nor invertible on X. The same argument works for $K_{k,w}^N$ (on X) or K_k^N (on X_w). Thus (5.31) and (5.32) hold for $\lambda = 0$ and the statements in (5.33) and (5.34) will follow if we can show that (5.31) and (5.32) are also true for $\lambda \neq 0$.

It follows from the assumptions on the quadrature weights that the operators K_k^N and $K_{k,w}^N$ are s-continuous (Theorem 5.6, Proposition 5.3), s-sequentially compact (Theorem 5.6, Proposition 5.2) and that the difference operator $K_k^N - K_{k,w}^N$ is sn-continuous (Proposition 5.8). We can now apply Theorem 2.9 with $H := K_k^N$ and $L := K_k^N - K_{k,w}^N$ to see that (5.31) holds, when $\lambda \neq 0$.

It remains to show that (5.32) holds. But this follows from a repetition of the argument to show that (4.21) is true in Theorem 4.6, with K, K_w replaced by K_k^N and $K_{k,w}^N$, respectively. □

5.3 Stability and uniform stability in weighted spaces

Throughout this section, we will assume that the kernel v satisfies the Assumptions $\boldsymbol{(A'')}$, $\boldsymbol{(B)}$ and that w is a weight function in $\mathcal{W}(\kappa)$.

Suppose that we have found quadrature weights satisfying $\boldsymbol{(Q)}$, $\boldsymbol{(QA)}$, $\boldsymbol{(QA'')}$ and $\boldsymbol{(QB)}$, and that the resulting Nyström/product integration method for the numerical solution of equation (5.1) is stable with respect to the uniform norm; precisely, for some $N' \in \mathbb{N}$, $(\lambda - K_k^N)^{-1} \in \mathcal{B}(X)$ for all $N \geq N'$, with

$$C := \sup_{N \geq N'} \|(\lambda - K_k^N)^{-1}\| < \infty. \tag{5.35}$$

From the results in the previous subsection we may then conclude that $(\lambda - K_k^N)^{-1} \in \mathcal{B}(X_w)$, $N \geq N'$, but we do not know if the uniform bound (5.35) also holds with respect to operator norm on $\mathcal{B}(X_w)$. The theorems in this subsection show that this is indeed true, even if we consider simultaneously a whole class of operators, where k varies in some bounded and equicontinuous set W.

Before we proceed to these key results of this chapter, we prove a preliminary proposition. It is convenient to use, in this and the following results, the notations K_k^N and $K_{k,w}^N$, with $N = \infty$, to denote K_k and $K_{k,w}$, respectively.

Proposition 5.10. *Suppose the quadrature weights $w_j^N(s)$ satisfy **(Q)**, **(QA)** and **(QB)**. Let (k_n) be a sequence in $BC(\mathbb{R}^2)$, $k \in BC(\mathbb{R}^2)$, (N_n) be a sequence in $\mathbb{N}$, (y_n) be a sequence in X, $y \in X$ and $\lambda \neq 0$. If $k_n \xrightarrow{s} k$, $N_n \to N \in \mathbb{N} \cup \{\infty\}$, $(\lambda - K_{k_n}^{N_n})^{-1} \in \mathcal{B}(X)$, for all $n \in \mathbb{N}$,*

$$C := \sup_{n \in \mathbb{N}} \|(\lambda - K_{k_n}^{N_n})^{-1}\| < \infty, \tag{5.36}$$

and $\lambda - K_k^N$ is injective on X then $(\lambda - K_k^N)^{-1} \in \mathcal{B}(X)$ and $\|(\lambda - K_k^N)^{-1}\| \leq C$. If also $y_n \xrightarrow{s} y$, then $(\lambda - K_{k_n}^{N_n})^{-1} y_n \xrightarrow{s} (\lambda - K_k^N)^{-1} y$.

Proof. If $y_n \xrightarrow{s} y$ then, by (5.36), the sequence (x_n), defined by $x_n := (\lambda - K_{k_n}^{N_n})^{-1} y_n$, $n \in \mathbb{N}$, is contained in X and is bounded. But then, by Remark 5.4, $(K_{k_n}^{N_n} x_n)$ contains an s-convergent subsequence. Since

$$x_n = \lambda^{-1}\big(K_{k_n}^{N_n} x_n + y_n\big), \qquad n \in \mathbb{N},$$

it follows that (x_n) has an s-convergent subsequence, denoted again by (x_n). Let $x \in X$ be the limit of this subsequence. But then, by Proposition 5.3, $K_{k_n}^{N_n} x_n \xrightarrow{s} K_k^N x$ and thus $y_n = (\lambda - K_{k_n}^{N_n}) x_n \xrightarrow{s} (\lambda - K_k^N) x$, $\quad n \to \infty$. But $y_n \xrightarrow{s} y$ and hence $(\lambda - K_k^N) x = y$.

Suppose that we are given arbitrary $y \in X$. Then we define the constant sequence (y_n), by setting $y_n = y$, $n \in \mathbb{N}$, and use the argument of the preceding paragraph to show that there exists $x \in X$ such that $(\lambda - K_k^N) x = y$. Hence $\lambda - K_k^N$, which is injective by assumption, is also surjective and thus invertible on X by Banach's inverse theorem. The bound on the inverse follows, for we have seen that, for every $y \in X$ with $\|y\| = 1$, $(\lambda - K_{k_n}^{N_n})^{-1} y \xrightarrow{s} (\lambda - K_k^N)^{-1} y$, whence inequality (2.10) yields $\|(\lambda - K_k^N)^{-1} y\| \leq \sup_{n \in \mathbb{N}} \|(\lambda - K_{k_n}^{N_n})^{-1} y\| \leq C$. □

If also **(QA″)** holds then we can remove the assumption that $\lambda - K_k^N$ be injective in the above proposition and relate the uniform boundedness of the inverses on X to that on X_w, $w \in \mathcal{W}(\kappa)$. The following theorem is our first central result on the stability of the Nyström method.

Theorem 5.11. *Suppose that the kernel v satisfies **(A″)** and **(B)** and that the quadrature weights $\omega_j^N(s)$ satisfy **(Q)**, **(QA)**, **(QA″)** and **(QB)**. Let (k_n) be a sequence in $BC(\mathbb{R}^2)$, $k \in BC(\mathbb{R}^2)$, (N_n) be a sequence in $\mathbb{N}$ and $\lambda \neq 0$. If $w \in \mathcal{W}(\kappa)$, $k_n \xrightarrow{s} k$, $N_n \to N \in \mathbb{N} \cup \{\infty\}$ and $(\lambda - K_{k_n}^{N_n})^{-1} \in \mathcal{B}(X)$, for all $n \in \mathbb{N}$, then*

$$C := \sup_{n \in \mathbb{N}} \|(\lambda - K_{k_n}^{N_n})^{-1}\| < \infty$$

if and only if

$$C_w := \sup_{n \in \mathbb{N}} \|(\lambda - K_{k_n}^{N_n})^{-1}\|_w < \infty$$

and $\lambda - K_k^N$ is injective on X_w. Further, if $C < \infty$, then $\lambda \notin \Sigma_X(K_k^N) \cup \Sigma_{X_w}(K_k^N)$, with

$$\|(\lambda - K_k^N)^{-1}\| \leq C, \quad \|(\lambda - K_k^N)^{-1}\|_w \leq C_w.$$

Proof. Note first that if the assumptions of the theorem hold then it follows from Theorem 5.6 that the quadrature weights $\omega_{j,w}^N(s)$ satisfy Assumptions $(\boldsymbol{Q})$, $(\boldsymbol{QA})$ and $(\boldsymbol{QB})$ and, further, that the operators $K_{k_n}^{N_n}$ on X_w and $K_{k_n,w}^{N_n}$ on X are uniformly bounded in n. Also, $(\lambda - K_{k_n}^{N_n})^{-1} \in \mathcal{B}(X_w)$, $n \in \mathbb{N}$, by Theorem 5.9. The proof of the present theorem now proceeds in a number of steps.

(i) We note first that if $C < \infty$ then $\lambda - K_k^N$ is injective on X_w. To see this, suppose that $C < \infty$, $x \in X_w$ and $\lambda x = K_k^N x$. Let $y_n := \lambda x - K_{k_n}^{N_n} x$, $n \in \mathbb{N}$. The operators $K_{k_n}^{N_n}$ are uniformly bounded on X_w and hence the sequence (y_n) is bounded in X_w. Moreover, by Proposition 5.3, $y_n \xrightarrow{s} \lambda x - K_k^N x = 0$ as $n \to \infty$. It now follows from Lemma 2.12 that $y_n \to 0$ and, since

$$\|x\| = \|(\lambda - K_{k_n}^{N_n})^{-1} y_n\| \leq C\|y_n\|, \quad n \in \mathbb{N},$$

we must have $x = 0$.

(ii) We next point out that, if $\lambda - K_k^N$ is injective on X_w and $C_w < \infty$, then $\lambda - K_k^N$ is injective on X. As $\lambda - K_k^N$ is injective on X_w if and only if $\lambda - K_{k,w}^N$ is injective on X and, moreover, by

$$\sup_{n\in\mathbb{N}} \|(\lambda - K_{k_n,w}^{N_n})^{-1}\| \overset{(2.9)}{=} \sup_{n\in\mathbb{N}} \|(\lambda - K_{k_n}^{N_n})^{-1}\|_w = C_w,$$

it follows from Proposition 5.10 that if $\lambda - K_k^N$ is injective on X_w and $C_w < \infty$ then $(\lambda - K_{k,w}^N)^{-1} \in \mathcal{B}(X)$, so that, by Theorem 5.9, $(\lambda - K_k^N)^{-1} \in \mathcal{B}(X)$; in particular $\lambda - K_k^N$ is injective on X.

(iii) We next show that if $\lambda - K_k^N$ is injective on X_w (which implies that $\lambda - K_{k_n,w}^{N_n}$ is injective on X) then $C < \infty$ if and only if $C_w < \infty$.

Suppose first that $C < \infty$ but $C_w = \infty$. By passing to appropriate subsequences, denoted again by (k_n) and (N_n), we may assume, that $\lim_{n\to\infty} \|(\lambda - K_{k_n}^{N_n})\|_w = \infty$. Thus there exists a sequence (z_n) in X_w with $\|z_n\|_w = 1$ such that $\|(\lambda - K_{k_n}^{N_n})z_n\|_w \to 0$. Defining $x_n := wz_n$, $n \in \mathbb{N}$, we then have

$$(\lambda - K_{k_n,w}^{N_n})x_n \to 0, \quad n \to \infty. \tag{5.37}$$

It follows from Proposition 5.3, part d), that $(K_{k_n,w}^{N_n} x_n)$ has an s-convergent subsequence, so by passing to subsequences, we may assume that $K_{k_n,w}^{N_n} x_n \xrightarrow{s} \lambda x$ for some $x \in X$. But then, by (5.37), $x_n \xrightarrow{s} x$. Now Proposition 5.3 yields $K_{k_n,w}^{N_n} x_n \xrightarrow{s} K_{k,w}^N x$. But this means that $\lambda x = K_{k,w}^N x$. But $\lambda - K_{k,w}^N$ is injective, so that $x = 0$, which implies $x_n \xrightarrow{s} 0$. By Proposition 5.8 and Lemma 5.7, we thus see that $(K_{k_n}^{N_n} - K_{k_n,w}^{N_n})x_n \to 0$. Combined with equation (5.37), this proves $(\lambda - K_{k_n}^{N_n})x_n \to 0$. Since $\|x_n\| = 1$, $n \in \mathbb{N}$, this contradicts $C < \infty$ and thus it must hold that $C_w < \infty$.

If, on the other hand, if $C_w < \infty$ and $\lambda - K_k^N$ is injective on X_w then we can reverse the roles of $K_{k,w}^N$ and K_k^N in above argument to prove $C < \infty$ (using (ii) to show that $\lambda - K_k^N$ is injective on X).
(iv) From (i)-(iii) two implications follow:

$$C < \infty \implies \big(C_w < \infty, \lambda - K_k^N \text{ injective on } X_w\big) \implies \lambda - K_k^N \text{ injective on } X.$$

Thus, if $C < \infty$, it follows from Proposition 5.10 that $(\lambda - K_k^N)^{-1} \in \mathcal{B}(X)$ with $\|(\lambda - K_k^N)^{-1}\| \leq C$. Applying Proposition 5.10 a second time, as in the proof of (ii), we have that $(\lambda - K_{k,w}^N)^{-1} \in \mathcal{B}(X)$, with $\|(\lambda - K_{k,w}^N)^{-1}\| \leq C_w$, so that $(\lambda - K_k^N)^{-1} \in \mathcal{B}(X_w)$ with $\|(\lambda - K_k^N)^{-1}\|_w \leq C_w$. □

The above theorem has a number of important corollaries. The first concerns the stability of the Nyström/product integration method for a single fixed k, and shows, in particular, that if the Nyström method is stable on X, i.e. (5.38) holds for some $N' \in \mathbb{N}$, then it is stable on X_w, for all $w \in \mathcal{W}(\kappa)$.

Corollary 5.12. *Suppose that the kernel v satisfies $(\boldsymbol{A''})$ and $(\boldsymbol{B})$ and that the quadrature rule* (5.7) *satisfies $(\boldsymbol{Q})$, $(\boldsymbol{QA})$, $(\boldsymbol{QA''})$ and $(\boldsymbol{QB})$. Further, if, for some $N' \in \mathbb{N}$, $k \in BC(\mathbb{R}^2)$, $w \in \mathcal{W}(\kappa)$ and*

$$\lambda \notin \bigcup_{N \geq N'} \Sigma_X(K_k^N),$$

then

$$\sup_{N \geq N'} \|(\lambda - K_k^N)^{-1}\| < \infty \tag{5.38}$$

if and only if

$$\sup_{N \geq N'} \|(\lambda - K_k^N)^{-1}\|_w < \infty$$

and $\lambda x = K_k x$ *has only the trivial solution in* X_w*. Further, if* (5.38) *holds, then* $\lambda \notin \Sigma_X(K_k) \cup \Sigma_{X_w}(K_k)$*.*

The following theorem, our next corollary, shows that the Nyström/product integration method is even uniformly stable on X_w, for $k \in W$, if it is uniformly stable on X and $W \subset BC(\mathbb{R}^2)$ is bounded and equicontinuous.

Theorem 5.13. *Suppose that the kernel* v *satisfies* **(*A″*)** *and* **(*B*)** *and that the quadrature weights* $\omega_j^N(s)$ *satisfy* **(*Q*)**, **(*QA*)**, **(*QA″*)** *and* **(*QB*)**. *Assume, further, that* $W \subset BC(\mathbb{R}^2)$ *is bounded and equicontinuous and* $\mathbb{N}' \subset \mathbb{N}$*. Then*

$$\lambda \notin \bigcup_{N \in \mathbb{N}'} \bigcup_{k \in W} \Sigma_X(K_k^N), \qquad \text{and} \qquad C := \sup_{N \in \mathbb{N}'} \sup_{k \in W} \|(\lambda - K_k^N)^{-1}\| < \infty \tag{5.39}$$

imply, for every $w \in \mathcal{W}(\kappa)$,

$$C_w := \sup_{N \in \mathbb{N}'} \sup_{k \in W} \|(\lambda - K_k^N)^{-1}\|_w < \infty. \tag{5.40}$$

If (5.39) *holds and* $\mathbb{N}'$ *is unbounded then* $\lambda \notin \Sigma_X(K_k) \cup \Sigma_{X_w}(K_k)$*,* $k \in W$*, and*

$$\sup_{k \in W} \|(\lambda - K_k)^{-1}\| \leq C, \qquad \sup_{k \in W} \|(\lambda - K_k)^{-1}\|_w \leq C_w.$$

Proof. If (5.39) holds but (5.40) does not, then there exist sequences (k_n) in W and (N_n) in $\mathbb{N}'$ such that $\sup_{n \in \mathbb{N}} \|(\lambda - K_{k_n}^{N_n})^{-1}\| < \infty$ but $\sup_{n \in \mathbb{N}} \|(\lambda - K_{k_n}^{N_n})^{-1}\|_w = \infty$. Furthermore, since W is bounded and equicontinuous and so, by Remark 2.8, relatively s-sequentially compact we can choose these sequences such that $k_n \xrightarrow{s} k \in BC(\mathbb{R}^2)$ and $N_n \to N \in \mathbb{N} \cup \{\infty\}$. But this contradicts Theorem 5.11. □

From this theorem we draw the following corollary, the last theorem of this subsection, which gives a first estimate of the error in the weighted norm $\|\cdot\|_w$.

Theorem 5.14. *Suppose that the assumptions of the previous theorem are satisfied with* $\mathbb{N}'$ *unbounded. Then, for every* $N \in \mathbb{N}'$*,* $k \in W$ *and* $y \in X_w$*, unique solutions* $x \in X_w$ *and* $x^N \in X_w$ *of the equations*

$$\lambda x - K_k x = y, \qquad \lambda x - K_k^N x = y, \tag{5.41}$$

exist and satisfy

$$\|x - x^N\|_w \leq C_w \|(K_k - K_k^N)x\|_w. \tag{5.42}$$

where C_w *is the constant in* (5.40)*. Moreover, if* (N_n) *is a sequence in* $\mathbb{N}'$ *and* $N_n \to \infty$ *then*

$$\|x - x^{N_n}\| \to 0, \quad n \to \infty. \tag{5.43}$$

Proof. Theorem 5.13 shows that $(\lambda - K_k)^{-1}, (\lambda - K_k^N)^{-1} \in \mathcal{B}(X_w)$, for all $k \in W$, $N \in \mathbb{N}'$. Thus, for every $N \in \mathbb{N}'$, $k \in W$ and $y \in X_w$, unique solutions $x, x^N \in X_w$ of (5.41) exist. Further, we obtain from $\lambda x - y = K_k x$ that

$$(\lambda - K_k^N)x - y = (K_k - K_k^N)x, \qquad N \in \mathbb{N}',$$

and, by applying the operator $(\lambda - K_k^N)^{-1}$,

$$x - x^N = (\lambda - K_k^N)^{-1}(K_k - K_k^N)x, \quad N \in \mathbb{N}', \tag{5.44}$$

holds. Hence, in view of (5.40), (5.42) holds.

To see that (5.43) holds, observe that from Proposition 5.3 we obtain that $(K_k - K_k^{N_n})x \xrightarrow{s} 0$ as $n \to \infty$. On the other hand, since $x \in X_w$ and the operators K_k and $K_k^{N_n}$, for $n \in \mathbb{N}$, are uniformly bounded by Theorem 5.6 we obtain that the sequence $(K_k - K_k^{N_n})x$ is bounded in X_w. But, by Lemma 2.12, this means that $K_k^{N_n} x \to K_k x$ so that, taking into account (5.39) and (5.44), $x_N \to x$. □

5.3.1 Sufficient conditions for stability and uniform stability on X

We still assume that the kernel v satisfies the Assumptions $(\boldsymbol{A''})$ and $(\boldsymbol{B})$.

The results shown in the previous section can be used to prove stability of the Nyström method on X_w, for all $w \in \mathcal{W}(\kappa)$, once its stability on X is known. In this short subsection, we now give sufficient conditions, which ensure the stability of the Nyström method on X and X_w for a certain class of kernels and quadrature methods.

To this end, we require the following uniform versions of $(\boldsymbol{B})$ and $(\boldsymbol{QB})$.

$$(\boldsymbol{B_u}) \quad \int_{-\infty}^{\infty} |v(s,t) - v(s+h,t)|\,dt \to 0 \quad \text{as } h \to 0, \quad \text{uniformly in } s \in \mathbb{R},$$

$$(\boldsymbol{QB_u}) \quad \sup_{N\in\mathbb{N}} \sum_{j\in\mathbb{Z}} |\omega_j^N(s) - \omega_j^N(s+h)| \to 0 \quad \text{as } h \to 0, \quad \text{uniformly in } s \in \mathbb{R}.$$

Clearly $(\boldsymbol{B_u})$ implies $(\boldsymbol{B})$ and $(\boldsymbol{QB_u})$ implies $(\boldsymbol{QB})$. We also note that $(\boldsymbol{B_u})$ is satisfied if v is a convolution kernel, i.e. $v(s,t) = \kappa(s-t)$, for some $\kappa \in L^1(\mathbb{R})$, in which case many sensible quadrature rules for the approximation of v satisfy $(\boldsymbol{QB_u})$.

We obtain the following stronger variants of the statements in Proposition 3.2a) and 5.2c) if $(\boldsymbol{B})$ and $(\boldsymbol{B_u})$ are satisfied.

Proposition 5.15. *Suppose that the kernel v satisfies $(\boldsymbol{A})$ and $(\boldsymbol{B_u})$ and that the weights $\omega_j^N(s)$ satisfy $(\boldsymbol{QA})$ and $(\boldsymbol{QB_u})$. Further, let $W \subset BC(\mathbb{R}^2)$ be bounded and uniformly equicontinuous and let B denote the unit ball in X. Then the sets*

$$V_1 := \bigcup_{k\in W} K_k(B) = \{K_k x : x \in X, \|x\| \le 1,\, k \in W\}, \tag{5.45}$$

$$V_2 := \bigcup_{N\in\mathbb{N}} \bigcup_{k\in W} K_k^N(B) = \{K_k^N x : x \in X, \|x\| \le 1,\, k \in W,\, N \in \mathbb{N}\} \tag{5.46}$$

are bounded and uniformly equicontinuous.

Proof. The boundedness of V_1 and V_2 has already been shown in Proposition 3.2 and 5.2. For $k \in W$, $x \in B$ and $s, s' \in \mathbb{R}$, we see that

$$\begin{aligned} |K_k x(s) - K_k x(s')| &\le \int_{-\infty}^{\infty} |v(s,t) - v(s',t)||k(s,t)|\,dt + \int_{-\infty}^{\infty} |v(s',t)||k(s,t) - k(s',t)|\,dt \\ &\le \sup_{k\in W} \|k\| \int_{-\infty}^{\infty} |v(s,t) - v(s',t)|\,dt + C \sup_{k\in W, t\in\mathbb{R}} |k(s,t) - k(s',t)|, \end{aligned}$$

where C denotes the supremum in $(\boldsymbol{A})$. The first term on the right-hand side converges to 0 as $|s - s'| \to 0$ since W is bounded and $(\boldsymbol{B_u})$ holds, the convergence being uniform in s. The second term converges to 0 as $|s - s'| \to 0$, uniformly in $s \in \mathbb{R}$ and $k \in W$, since W is uniformly equicontinuous. The uniform equicontinuity of V_1 follows.

A similar argument shows that V_2 also has the desired properties. □

We also require a modified version of $(\boldsymbol{Q})$.

$$(\boldsymbol{Q_u}) \quad \forall U \in \mathcal{U} \quad \sum_{j\in\mathbb{Z}} \omega_j^N(s) x(t_j^N) \to \int_{-\infty}^{\infty} v(s,t)x(t)\,dt \quad \text{as } N \to \infty, \quad \text{uniformly in } s \in \mathbb{R} \text{ and } x \in U,$$

where $\mathcal{U}$ denotes the collection of bounded and uniformly equicontinuous subsets of X.

It is quite obvious that $(\boldsymbol{Q})$ does not imply $(\boldsymbol{Q_u})$, as one might use, for different values of s, different quadrature weights $\omega_j^N(s)$ belonging to quadrature rules with arbitrarily slow rate of convergence so that the necessary uniform convergence in s required in $(\boldsymbol{Q_u})$ cannot be achieved.

On the other hand, one might surmise that $(\boldsymbol{Q_u})$ is a stronger condition than $(\boldsymbol{Q})$, but this is not the case; however, $(\boldsymbol{Q})$ follows from $(\boldsymbol{Q_u})$ if $(\boldsymbol{QA''})$ is satisfied. We will prove both facts in the next example and lemma.

Example 5.16. *Let the kernel v be given by $v(s,t) = 1_{[-1,1]}(t)$. Define the abscissae $t_j^N := j/N$, for $j \in \mathbb{Z}$ and $N \in \mathbb{N}$, and let the quadrature weights $\omega_j^N(s)$ be given by*

$$\omega_j^N(s) := \begin{cases} \frac{1}{N}, & |t_j^N| \le 1, \\ 1, & |t_j^N| = N, \\ -1, & |t_j^N| = N + \frac{1}{N}, \\ 0, & \textit{otherwise}, \end{cases} \qquad s \in \mathbb{R},\ j \in \mathbb{Z},\ N \in \mathbb{N}.$$

Then, for every $x \in X$, $N \in \mathbb{N}$ and $s \in \mathbb{R}$, we see that

$$\Big|\sum_{j\in\mathbb{Z}} \omega_j^N(s) x(t_j^N) - \int_{-\infty}^{\infty} v(s,t)x(t)\,dt\Big| \le \big|x(N) - x(N+\tfrac{1}{N})\big| + \Big|\sum_{|t_j^N|\le 1} \frac{x(t_j^N)}{N} - \int_{-1}^{1} x(t)\,dt\Big|.$$

In order to verify $(\boldsymbol{Q_u})$, we choose an arbitrary $U \in \mathcal{U}$. The second summand on the right-hand side of this inequality converges to 0 as $N \to \infty$, uniformly in $s \in \mathbb{R}$ and $x \in U$, as the compound rectangle rule for finite intervals is uniformly convergent on bounded, uniformly equicontinuous sets [30]. Since U is bounded and uniformly equicontinuous the first summand on the right-hand side also converges to 0 as $N \to \infty$, uniformly in $s \in \mathbb{R}$ and $x \in U$, whence $(\boldsymbol{Q_u})$ holds.

On the other hand, to see that $(\boldsymbol{Q})$ is not satisfied, we can choose $f \in X$ so that f has support outside the interval $[-1,1]$ and $f(N) = 1$ and $f(N+\frac{1}{N}) = -1$ for every $N \ge 2$. Then

$$\Big|\sum_{j\in\mathbb{Z}} \omega_j^N(s) f(t_j^N) - \int_{-\infty}^{\infty} v(s,t)f(t)\,dt\Big| = |f(N) - f(N+\tfrac{1}{N})| = 2, \quad s \in \mathbb{R}, N \ge 2,$$

so that $(\boldsymbol{Q})$ cannot hold.

Lemma 5.17. *If the quadrature weights $\omega_j^N(s)$ satisfy $(\boldsymbol{Q_u})$ and $(\boldsymbol{QA''})$ then they also satisfy $(\boldsymbol{Q})$.*

Proof. For every $A > 0$, we choose a "cut-off" function $\chi_A \in X$ such that $\|\chi_A\| = 1$, $\chi_A(s) = 1$ if $|s| \le A$ and $\chi_A(s) = 0$ if $|s| \ge A+1$. Let $f \in X$ and $s \in \mathbb{R}$. Then, for every $A > 0$ and $N \in \mathbb{N}$, the term

$$\Big|\sum_{j\in\mathbb{Z}} \omega_j^N(s) f(t_j^N) - \int_{-\infty}^{\infty} v(s,t)f(t)\,dt\Big| \tag{5.47}$$

is bounded above by

$$\Big|\sum_{j\in\mathbb{Z}} \omega_j^N(s)(\chi_A f)(t_j^N) - \int_{-\infty}^{\infty} v(s,t)(\chi_A f)(t)\,dt\Big| + \|(1-\chi_A)f\|\Big(\sum_{|t_j^N|>A} |\omega_j^N(s)| + \int_{\mathbb{R}\setminus[-A,A]} |v(s,t)|\,dt\Big).$$

Given $\epsilon > 0$, we choose first $A > 0$ large enough so that the second summand is $\le \epsilon/2$, irrespective of $N \in \mathbb{N}$ (this is possible as $v(s,\cdot)$ is integrable and Lemma 5.5, applied with $w(s) = 1$ and $M_2 = \infty$, shows that the sum converges to 0 as $A \to \infty$, uniformly in $N \in \mathbb{N}$). The function $\chi_A f$ is continuous and has compact support and thus must be uniformly continuous. By $(\boldsymbol{Q_u})$, the first summand in (5.47) is thus $\le \epsilon/2$ for all N large enough. We have thus shown that (5.47) can be made $\le \epsilon$ for all N large enough. But this is all that is required to prove $(\boldsymbol{Q})$. □

The key element of our stability proof is the following proposition, a corollary to Theorem 3.6 in [50].

Proposition 5.18. *Let Y be an arbitrary Banach space and suppose that $L \in \mathcal{B}(Y)$. Assume further that $(\lambda - L)^{-1} \in \mathcal{B}(Y)$, for some $\lambda \ne 0$, and that (L^N) is a bounded sequence in $\mathcal{B}(Y)$ for which*

$$\|(L^N - L)L\| \to 0, \quad \|(L^N - L)L^N\| \to 0, \qquad \textit{as } N \to \infty, \tag{5.48}$$

holds. Then there exists some $N' \in \mathbb{N}$ such that, for all $N \geq N'$, the estimates

$$\|(\lambda - L)^{-1}(L^N - L)L^N\| < |\lambda|, \qquad \|(L^N - L)(\lambda - L)^{-1}L^N\| < |\lambda|, \tag{5.49}$$

hold and the operator $(\lambda - L^N)$ is invertible on X with inverse bounded by

$$\|(\lambda - L^N)^{-1}\| \leq \frac{1 + \|(\lambda - L)^{-1}\|\|L^N\|}{|\lambda| - \|(\lambda - L)^{-1}(L^N - L)L^N\|}.$$

Remark 5.19. *Note that, under the assumptions of the previous proposition, it is easy to see that* (5.48) *implies that* (5.49) *holds for all N large enough: For the first inequality in* (5.49) *this is clear by the triangle inequality, for the second this follows from the representation $(\lambda - L)^{-1} = \lambda^{-1}(I + L(\lambda - L)^{-1})$ and the estimate*

$$\|(L^N - L)(\lambda - L)^{-1}L^N\| \leq |\lambda|^{-1}\|(L^N - L)L^N\| + |\lambda|^{-1}\|(L^N - L)L\|\|(\lambda - L)^{-1}\|\|L^N\|.$$

We now prove the announced stability result for quadrature weights satisfying $(\boldsymbol{Q_u})$.

Theorem 5.20. *Suppose that $\lambda \neq 0$, that the kernel v satisfies $(\boldsymbol{A})$ and $(\boldsymbol{B_u})$ and the quadrature weights $\omega_j^N(s)$ satisfy $(\boldsymbol{Q})$, $(\boldsymbol{Q_u})$, $(\boldsymbol{QA})$ and $(\boldsymbol{QB_u})$. Assume, further, that $W \subset BC(\mathbb{R}^2)$ is a bounded and uniformly equicontinuous set and that $(\lambda - K_k)^{-1}$ exists for every $k \in W$ with*

$$\sup_{k \in W} \|(\lambda - K_k)^{-1}\| =: C_W < \infty, \tag{5.50}$$

for some positive constant C_W. Then

$$\|(K_k^N - K_k)K_k\| \to 0, \quad \|(K_k^N - K_k)K_k^N\| \to 0, \qquad N \to \infty, \tag{5.51}$$

uniformly in $k \in W$, and there exists $N' \in \mathbb{N}$ so that $\lambda - K_k^N$ is invertible on X for all $N \geq N'$, $k \in W$, and there holds

$$\sup_{N \geq N'} \sup_{k \in W} \|(\lambda - K_k^N)^{-1}\| < \infty.$$

Proof. Let B denote the unit ball in X. From the assumptions and Proposition 5.15 we learn that

$$V := \bigcup_{N \in \mathbb{N}} \bigcup_{k \in W} \Big(K_k^N(B) \cup K_k(B)\Big)$$

is bounded and uniformly equicontinuous. Since W is also bounded and uniformly equicontinuous, so must be the set $U \subset X$, defined by

$$U := \{k(s,\cdot)x(\cdot) : s \in \mathbb{R}, k \in W, x \in V\},$$

i.e. $U \in \mathcal{U}$. By $(\boldsymbol{Q_u})$ we thus have that

$$\Big|\sum_{j \in \mathbb{Z}} \omega_j^N(s)x(t) - \int_{-\infty}^{\infty} v(s,t)x(t)\,dt\Big| \to 0,$$

as $N \to \infty$, uniformly in $x \in U$ and $s \in \mathbb{R}$. But this entails that (5.51) holds.

Since we have assumed (5.50) we can now apply Proposition 5.18. We obtain that there exists $N' \in \mathbb{N}$ such that, for all $N \geq N'$ and $k \in W$, $(\lambda - K_k^N)^{-1} \in \mathcal{B}(X)$ and there holds

$$\sup_{N \geq N'} \sup_{k \in W} \|(\lambda - K_k^N)^{-1}\| \leq \sup_{N \geq N'} \sup_{k \in W} \frac{1 + \|(\lambda - K_k)^{-1}K_k^N\|}{|\lambda| - \|(\lambda - K_k)^{-1}(K_k^N - K_k)K_k\|} < \infty.$$

$\square$

If we additionally assume that $(\boldsymbol{A''})$ and $(\boldsymbol{QA''})$ are satisfied then we can use the weighted space theory to obtain that stability also holds on X_w. We thus obtain the following theorem, a corollary to the previous theorem and Theorem 5.13.

Theorem 5.21. *Suppose that the assumptions of the previous theorem are satisfied and that, for some $\kappa \in L^1(\mathbb{R})$, the kernel v and the quadrature weights $\omega_j^N(s)$ satisfy $(\boldsymbol{A''})$ and $(\boldsymbol{QA''})$, respectively. Let $w \in \mathcal{W}(\kappa)$. Then $\lambda - K_k$ and $\lambda - K_k^N$ are invertible on X_w, for all $N \geq N'$ and $k \in W$; moreover,*

$$\sup_{k\in W} \|(\lambda - K_k)^{-1}\|_w \leq \sup_{k\in W, N\geq N'} \|(\lambda - K_k^N)^{-1}\|_w < \infty.$$

An application of this result to a large class of kernels and quadrature rules satisfying its assumptions can be found in Section 6.2.

5.4 The finite section method

Throughout most of this section, we will assume that the kernel v is a convolution kernel satisfying Assumption $(\boldsymbol{A''})$, so that $v(s,t) = \kappa(s-t)$, for some $\kappa \in L^1(\mathbb{R})$ satisfying the monotonicity condition in $(\boldsymbol{A''})$. Moreover, we suppose in this section that the quadrature weights $\omega_j^N(s)$ satisfy $(\boldsymbol{Q})$, $(\boldsymbol{QA})$, $(\boldsymbol{QA''})$ and $(\boldsymbol{QB})$.

The previous sections were devoted to the study of Nyström/product integration methods for the integral equation

$$\lambda x - K_k x = y. \tag{5.52}$$

This has led us to relate the solvability of the discretized equations

$$\lambda x^N - K_k^N x^N = y \tag{5.53}$$

on X to their solvability in the weighted spaces X_w, $w \in \mathcal{W}(\kappa)$. In particular, we have seen that, for a large class of kernel functions v and k and quadrature weights $\omega_j^N(s)$, the invertibility of $\lambda - K_k$ on X is sufficient for the invertibility of $\lambda - K_k^N$ on a class of weighted spaces X_w for large values of $N \in \mathbb{N}$.

As has been explained earlier, the solution x^N of (5.53) may be obtained by solving an infinite system of linear equations. However, in many cases, solving this infinite system exactly, or at least approximately, will be an onerous if not impossible task. Therefore we will now consider the effect of truncating the summation in the definition of the discretized integral operator K_k^N to a finite interval $[-A, A]$, where $A > 0$, i.e. we replace the quadrature operator K_k^N in (5.53) by the operator $K_k^{N,A}$, defined by

$$K_k^{N,A} x(s) := \sum_{|t_j^N| \leq A} \omega_j^N(s) k(s, t_j^N) x(t_j^N), \qquad s \in \mathbb{R},\ x \in X. \tag{5.54}$$

Clearly $K_k^{N,A}$ is the operator K_k^N defined in (5.8), but with the quadrature weights

$$\omega_j^{N,A}(s) := \omega_j^N(s) 1_{[-A,A]}(t_j^N), \qquad s \in \mathbb{R}, j \in \mathbb{Z}, \tag{5.55}$$

instead of the quadrature weights $\omega_j^N(s)$. It will be convenient to use the notation $K_k^{N,A} := K_k^N$ and $\omega_j^{N,A}(s) := \omega_j^N(s)$ for $A = \infty$. We recall also our notational convention $K_k^N := K_k$ for $N = \infty$.

We note that, for every $A \in (0, \infty]$, $K_k^{N,A}(B) \subset K_k^N(B)$, where B denotes the unit ball of X or X_w, $w \in \mathcal{W}(\kappa)$, whence the inequalities

$$\|K_k^{N,A}\| \leq \|K_k^N\|, \qquad \|K_k^{N,A}\|_w \leq \|K_k^N\|_w, \qquad 0 < A \leq \infty, \tag{5.56}$$

hold. Another noteworthy fact is that the operators $K_k^{N,A}$, $A \in (0, \infty)$, have finite dimensional range and hence are compact on X and X_w; more precisely, $K_k^{N,A}(X_w) = K_k^{N,A}(X) = \mathrm{span}\{\omega_j^N(\cdot) k(\cdot, t_j^N) : |t_j^N| \leq A\}$.

Replacing K_k^N by $K_k^{N,A}$ in (5.53), we obtain the equation

$$\lambda x^{N,A} - K_k^{N,A} x^{N,A} = y. \tag{5.57}$$

Provided it exists, a solution $x^{N,A}$ of this equation can be obtained by solving the *finite* linear system

$$\lambda x^{N,A}(t_{j'}^N) - \sum_{|t_j^N| \le A} \omega_j^{N,A}(t_{j'}^N) k(t_{j'}^N, t_j^N) x^{N,A}(t_j^N) = y(t_{j'}^N), \qquad |t_{j'}^N| \le A, \tag{5.58}$$

for the values of $x^{N,A}(t_{j'}^N)$ at the quadrature nodes $t_{j'}^N \in [-A, A]$ and then setting

$$x^{N,A}(s) := \frac{1}{\lambda} \left(\sum_{|t_j^N| \le A} \omega_j^{N,A}(s) k(s, t_j^N) x^{N,A}(t_j^N) + y(s) \right), \quad s \in \mathbb{R}.$$

This method of obtaining an approximate solution $x^{N,A}$ of (5.57) is known as the *finite-section method*.

Naturally, two questions arise: the first concerns the applicability and stability of the finite section method, i.e. the question whether the inverses of $\lambda - K_k^{N,A}$ exist and are uniformly bounded for large values of A; once this has been answered in the affirmative, the second question asks for estimates of the accuracy of the approximate solutions. In this section our focus will be on the latter problem, for the weighted space theory developed earlier in this chapter contributes to its answer.

By definition, it is immediately clear that the procedure of replacing the quadrature weights $\omega_j^N(s)$ by $\omega_j^{N,A_N}(s)$, where (A_N) is a sequence in $(0, \infty]$, is consistent with Assumptions **(*QA*)**, **(*QA''*)**, **(*QB*)** and also **(*Q*)** if $A_N \to \infty$. We note this fact in the next lemma.

Lemma 5.22. *Suppose that the quadrature weights $\omega_j^N(s)$ satisfy Assumptions* **(*QA*)**, **(*QA''*)** *and* **(*QB*)** *and that (A_N) is a sequence in $(0, \infty]$. Then the quadrature weights $\omega_j^{N,A_N}(s)$ also satisfy Assumptions* **(*QA*)**, **(*QA''*)** *and* **(*QB*)**. *If the quadrature weights $\omega_j^N(s)$ additionally satisfy* **(*Q*)** *then the quadrature weights $\omega_j^{N,A_N}(s)$ also satisfy* **(*Q*)**.

Proof. Suppose the assumptions of the lemma are fulfilled. It is immediately clear that the quadrature weights $\omega_j^{N,A_N}(s)$ satisfy **(*QA*)**, **(*QA''*)** and **(*QB*)**, so we are left with proving that they also satisfy **(*Q*)** if $A_N \to \infty$. To this end, let $s \in \mathbb{R}$ and $f \in X$; then we obtain

$$\left| \sum_{j \in \mathbb{Z}} \omega_j^{N,A_N}(s) f(t_j^N) - \int_{-\infty}^{\infty} v(s,t) f(t) \right| \le \left| \sum_{j \in \mathbb{Z}} \omega_j^N(s) f(t_j^N) - \int_{-\infty}^{\infty} v(s,t) f(t) \right| + \|f\| \sum_{|t_j^N| > A_N} |\omega_j^N(s)|. \tag{5.59}$$

Since the quadrature weights $\omega_j^N(s)$ satisfy **(*Q*)** we know that the first term on the right-hand side of this inequality converges to zero as $N \to \infty$. But, since $A_N \to \infty$ and in view of Lemma 5.1 (applied with $\Omega' = \{s\}$), the same is true of the second term. Thus the term on the left-hand side of (5.59) converges to 0 as $N \to \infty$, as required in **(*Q*)**. □

This lemma shows that, if $A_N \to \infty$ the weighted space stability theory of Section 5.3, in particular Theorems 5.11–5.14, also applies to the finite-section method, provided we can show that show that the resulting finite section method is stable on X.

However, we will now consider a variant of the finite-section method in which, in addition to the truncation of the quadrature weights, the function k is modified. The first major result of this section addresses the stability of these modified finite-section method in weighted spaces. As we have done in our stability analysis for the Nyström method, we consider simultaneously a class of discretized integral operators K_k^N, with the kernel function k varying in a set W_∞, but we also allow k to vary in different (possibly empty) sets W_A depending on the truncation level A. We will discuss possible applications of this theorem below.

Theorem 5.23. *Suppose that the kernel* v *satisfies* $(\boldsymbol{A''})$ *and* $(\boldsymbol{B})$ *and that the quadrature weights* $\omega_j^N(s)$ *satisfy* $(\boldsymbol{Q})$, $(\boldsymbol{QA})$, $(\boldsymbol{QA''})$ *and* $(\boldsymbol{QB})$. *Moreover, assume that* $\mathbb{N}' \subset \mathbb{N}$, W *is a bounded and equicontinuous subset of* $BC(\mathbb{R}^2)$ *and that,* $W_A \subset W$, *for every* $A \in (0,\infty]$.

If

$$\lambda \notin \bigcup_{N\in\mathbb{N}'} \bigcup_{A\in(0,\infty]} \bigcup_{k\in W_A} \Sigma_X(K_k^{N,A}) \tag{5.60}$$

and

$$C := \sup_{N\in\mathbb{N}'} \sup_{A\in(0,\infty]} \sup_{k\in W_A} \|(\lambda - K_k^{N,A})^{-1}\| < \infty \tag{5.61}$$

then, for every $w \in \mathcal{W}(\kappa)$, *the operators* $\lambda - K_k^{N,A}$ *in* (5.61) *are also invertible on* X_w *and*

$$C_w := \sup_{N\in\mathbb{N}'} \sup_{A\in(0,\infty]} \sup_{k\in W_A} \|(\lambda - K_k^{N,A})^{-1}\|_w < \infty.$$

Let us comment on possible applications of this theorem before we begin its proof. In the simplest case the sets $\mathbb{N}'$ and W are singletons, $W_A = W_\infty = W$ holds, for all A larger than some $A_0 > 0$, and $W_A = \varnothing$ otherwise. This is the classical finite section method for the equation 5.52. The theorem then states that the stability of the finite section method on X implies its stability on X_w, $w \in \mathcal{W}(\kappa)$.

However, proving the stability on X of the finite-section method for the equation 5.52, i.e. the existence and uniform boundedness of the inverses of $\lambda - K_k^{N,A}$ on X, for A large, is difficult and a general theory, that includes the case when v is not a convolution kernel and k is not the constant function $k(s,t) = 1$, does not seem to exist (but see [46] for recent results in this direction).

Nevertheless, sometimes it is possible to obtain the existence and uniform boundedness of the inverses $(\lambda - K_{k_A}^{N,A})^{-1}$ on X, for all A sufficiently large, where k_A is obtained by modifying k near the endpoints of the interval $[-A, A]$. This procedure is sometimes called a *modified finite section method* and has been considered in, e.g., [51]. In this case, one would choose $W_\infty = \{k\}$, $W_A = \{k_A\}$, for all A large enough, and $W_A = \varnothing$ otherwise.

Finally, the assumptions allow us to consider the (modified) finite section method for families of operators $\lambda - K_k^N$, with N and k varying in $\mathbb{N}' \subset \mathbb{N}$ and $W \subset BC(\mathbb{R}^2)$. We will consider a specific application of this variant of the modified finite section method later in this thesis, and now commence the proof of Theorem 5.23.

Proof. Let $A \in (0,\infty]$. If $A < \infty$ then, by Lemma 5.22 (applied with $A_N = A$, $N \in \mathbb{N}$), the quadrature weights $\omega_j^{N,A}(s)$ satisfy $(\boldsymbol{QA})$, $(\boldsymbol{QA''})$ and $(\boldsymbol{QB})$; if $A = \infty$ this follows immediately from the assumptions of the theorem. In both cases, we may thus deduce from Theorem 5.9 and (5.60) that $\lambda \notin \Sigma_{X_w}(K_k^{N,A})$ for all $k \in W_A$ and $N \in \mathbb{N}'$, so that (5.60) holds with X replaced by X_w, $w \in \mathcal{W}(\kappa)$.

We are left with proving that C_w is finite, which we will do by contradiction. To this end, assume that $C_w = \infty$. Then we could find sequences (N_n) in $\mathbb{N}'$, (A_n) in $(0,\infty]$ and (k_n) in W such that

$$\lim_{n\to\infty} \|(\lambda - K_{k_n}^{N_n,A_n})^{-1}\|_w = \infty. \tag{5.62}$$

By passing to appropriate subsequences, we may assume w.l.o.g. that $A_n \to A \in [0,\infty]$ and, since W is bounded and equicontinuous and thus relatively s-sequentially compact, $k_n \stackrel{s}{\to} k \in BC(\mathbb{R}^2)$. We distinguish two cases and will show that each leads to a contradiction.

$A = \infty$: For every $n \in \mathbb{N}$, we choose a function $k_n' \in BC(\mathbb{R}^2)$ with the following properties: (i) $\|k_n'\| \le \|k_n\|$; (ii) $k_n'(s,t) = k_n(s,t)$, for all $s \in \mathbb{R}$ and $|t| \le A_n$; (iii) $k_n'(s,t) = 0$, for all $s \in \mathbb{R}$ and $|t| \ge \min\{|t_j^{N_n}| : t_j^{N_n} \notin [-A_n, A_n]\}$. (We note that $k_n' = k_n$ if $A_n = \infty$.) Then $K_{k_n'}^{N_n} = K_{k_n}^{N_n,A_n}$, for every $n \in \mathbb{N}$, so that, by virtue of (5.62), $\lim_{n\to\infty} \|(\lambda - K_{k_n'}^{N_n})^{-1}\|_w = \infty$. However, (i), (ii), $A_n \to \infty$ and $k_n \stackrel{s}{\to} k$ ensure that $k_n' \stackrel{s}{\to} k$; moreover, (5.61) shows that $\sup_{n\in\mathbb{N}} \|(\lambda - K_{k_n'}^{N_n})^{-1}\| < \infty$. Thus and by the assumptions on the quadrature weights, Theorem 5.11 applies to the sequences (k_n') and (N_n), implying that the values of $\|(\lambda - K_{k_n'}^{N_n})^{-1}\|_w = \|(\lambda - K_{k_n}^{N_n,A_n})^{-1}\|$ must be uniformly bounded, contradicting (5.62).

$A < \infty$: If $A_n \to A < \infty$ then $A_n < A+1$, for all n large enough. Thus, by passing to appropriate subsequences, we may assume that $\sup_{n\in\mathbb{N}} A_n < A+1$. In view of (5.62), we can find a sequence (z_n) in X_w such that $\|z_n\|_w = 1$, $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \|(\lambda - K_{k_n}^{N_n,A_n})z_n\|_w = 0. \tag{5.63}$$

Then also $\lim_{n\to\infty} \|(\lambda - K_{k_n}^{N_n,A_n})z_n\| = 0$ and from (5.61), we get $z_n \to 0$. We now choose a non-negative "cut-off" function $\chi \in X$ such that $\|\chi\| = 1$, $\chi(s) = 1$, $|s| < A+1$ and $\chi(s) = 0$, $|s| > A+2$. Let $(z_n') := (\chi z_n)$ and $(z_n'') := (z_n - z_n')$. Clearly, (z_n') and (z_n'') are sequences in X_w. Since χ is compactly supported $z_n \to 0$ ensures that $\|z_n'\|_w \to 0$, and hence, by choice of (z_n), $\|z_n''\|_w \to 1$. Moreover, we have that $K_{k_n}^{N_n,A_n} z_n'' = 0$, $n \in \mathbb{N}$, since z_n'' is always supported outside the interval $[-A_n, A_n]$. Thus

$$\|(\lambda - K_{k_n}^{N_n,A_n})z_n\|_w \geq \|(\lambda - K_{k_n}^{N_n,A_n})z_n''\|_w - \|(\lambda - K_{k_n}^{N_n,A_n})z_n'\|_w = \|\lambda z_n''\|_w - \|(\lambda - K_{k_n}^{N_n,A_n})z_n'\|_w.$$

But, since $\|z_n'\|_w \to 0$ and, by (5.56) and Theorem 5.6, the operators $\lambda - K_{k_n}^{N_n,A_n}$ are uniformly bounded on X_w, the right-hand side of this inequality converges to $|\lambda| > 0$ as $n \to \infty$, contradicting (5.63). $\square$

We are also interested in estimating the error of the (modified) finite section method. To this end, we now start with the following general situation: We choose $N \in \mathbb{N}$, $A > 0$, $k, k' \in BC(\mathbb{R}^2)$ and suppose that the inverses of the operators $\lambda - K_k^N$ and $\lambda - K_{k'}^{N,A}$ exist on X. Further, let us assume that k, k' satisfy, for some $D \in (0, A]$,

$$k(s,t) = k'(s,t), \quad |s|, |t| \leq D. \tag{5.64}$$

For a given $y \in X$, we now compare the difference $x^N - x^{N,A}$ between the unique solutions $x^N, x^{N,A} \in X$ of the equations

$$(\lambda - K_k^N)x^N = y, \qquad (\lambda - K_{k'}^{N,A})x^{N,A} = y,$$

i.e. the error introduced by the truncation and the replacement of k by k'. To this end, we will investigate the *residual* r, defined by

$$r := y - (\lambda - K_k^N)x^{N,A} = (K_k^N - K_{k'}^{N,A})x^{N,A}. \tag{5.65}$$

Clearly, $r \in X$ and we observe that, since $x^N - x^{N,A} = (\lambda - K_k^N)^{-1}(K_k^N - K_{k'}^{N,A})x^{N,A}$, there holds

$$x^N - x^{N,A} = (\lambda - K_k^N)^{-1} r,$$

and hence

$$\|x^N - x^{N,A}\| \leq \|(\lambda - K_k^N)^{-1}\| \, \|r\|. \tag{5.66}$$

With the aid of our assumption (5.64), we can bound r in terms of the values of $x^{N,A}$ outside the interval $[-D, D]$. To achieve this, we split the residual, $r = r_+ + r_-$, where, bearing in mind the second equality in (5.65), $r_\pm$ is given by

$$r_\pm(s) := \begin{cases} \sum_{\pm t_j^N \geq -D} \big(\omega_j^N(s)k(s,t) - \omega_j^{N,A}(s)k'(s,t)\big)x^{N,A}(t_j^N), & \pm s > D \\ \sum_{\pm t_j^N \geq D} \big(\omega_j^N(s)k(s,t) - \omega_j^{N,A}(s)k'(s,t)\big)x^{N,A}(t_j^N), & \pm s \leq D. \end{cases} \tag{5.67}$$

Since (5.64) holds and the quadrature weights $\omega_j^{N,A}(s)$ satisfy $\boldsymbol{(QA)}$, $\boldsymbol{(QB)}$ we easily see that $r_\pm \in X$. We now bound $r_\pm(s)$ as follows:

$$|r_\pm(s)| \quad \leq \quad \big(\|k\| + \|k'\|\big)\, \mu_{\pm D}^N(s) \sup_{|t|>D} |x^{N,A}(t)|, \qquad s \in \mathbb{R}, \tag{5.68}$$

where $\mu_{\pm D}^N(s)$ are the functions defined by

$$\mu_{\pm D}^N(s) := \begin{cases} \sum_{\pm t_j^N > -D} |\omega_j^N(s)|, & \pm s > D, \\ \sum_{\pm t_j^N > D} |\omega_j^N(s)|, & \pm s \leq D. \end{cases}$$

We observe that, for every $N \in \mathbb{N}$, the function $\mu^N_{\pm D}$ is a member of $L^\infty(\mathbb{R})$ with $\|\mu^N_{\pm D}\|_\infty \le C_Q$ (the constant in **(*QA*)**), and the same bound holds for the function $\mu^N_D := \mu^N_{+D} + \mu^N_{-D}$. However, we can find much stronger bounds on $\mu^N_{\pm D}$, but before we formulate these, we introduce a suitable class of weight functions.

Given an even weight function $\widetilde{w}(s)$ satisfying (2.3) and $D \ge 0$, let us define the lateral weight functions

$$\widetilde{w}^D_+(s) := \begin{cases} \widetilde{w}(s+D), & s > -D, \\ 1, & s \le -D, \end{cases} \qquad \widetilde{w}^D_-(s) := \begin{cases} 1, & s \ge D, \\ \widetilde{w}(D-s), & s < D. \end{cases}$$

If $D = 0$ then we will also use the simpler notation $\widetilde{w}_\pm(s)$ instead of $\widetilde{w}^D_\pm(s)$.

The motivation for this definition is the following lemma, which shows that, under condition (5.69) below, the functions $r_\pm$ are contained in the weighted spaces $X_{\widetilde{w}^D_\mp}$.

Lemma 5.24. *Suppose that $v(s,t) = \tilde{\kappa}(s-t)$ satisfies Assumption **(A′′)** and that $\widetilde{w} \in \mathcal{W}(\kappa)$ is such that*

$$\widetilde{w}(s)\mu(s) = O(1), \qquad \text{as } s \to \infty, \tag{5.69}$$

where $\mu(s)$ is defined in (4.82)*. Further, assume that the quadrature weights $\omega^N_j(s)$ satisfy Assumptions **(QA)** and **(QA′′)**. Then, for every $D > 0$, the following inequalities hold,*

$$\widetilde{w}^D_\pm(s)\mu^N_{\mp D}(s) \le C, \qquad s \in \mathbb{R},\ N \in \mathbb{N},\ D \ge 0, \tag{5.70}$$

where $C > 0$ is a constant not depending on s, D or N.

Remark 5.25. *Given a convolution kernel v as in the lemma, we can always find a weight function $\widetilde{w} \in \mathcal{W}(\kappa)$ satisfying* (5.69)*; such a $\widetilde{w}$ is constructed in Theorem 4.38.*

Proof. We only prove the inequality (5.70) for the lower subscripts, for the other then follows by symmetric arguments. We let C_Q, A_0, A_1, C^* denote the constants in **(*QA*)**, **(*A′′*)** and **(*QA′′*)** and define $M := \max\{h_N : N \in \mathbb{N}\}$, $A_2 := \max\{A_0 + M, A_1\}$. We distinguish two cases. If $s \ge D - A_2$ then $\widetilde{w}^D_-(s) \le \widetilde{w}(A_2)$, and hence there holds

$$\widetilde{w}^D_\pm(s)\mu^N_{\mp D} \le \widetilde{w}(A_2)C_Q, \qquad s \ge D - A_2. \tag{5.71}$$

If, on the other hand, $s < D - A_2$ then Lemma 5.5 (applied with $w(s) = 1$, $M_1 = D - s$ and $M_2 = \infty$) yields

$$\mu^N_{\mp D}(s) = \sum_{t^N_j > D} |\omega^N_j(s)| = \sum_{-\infty < s - t^N_j < s - D} |\omega^N_j(s)| \le C_1 \int_{D-h_N}^\infty |\kappa(s-t)|\,dt \le C_1 \int_{D-M}^\infty |\kappa(s-t)|\,dt, \tag{5.72}$$

where C_1 is some constant not depending on D, M or N. Moreover, by (5.69), there exists some constant $C_\mu > 0$ such that $\widetilde{w}(s)\mu(s) \le C_\mu$, for every $s \ge 0$. Hence

$$\int_{D-M}^\infty |\kappa(s-t)|\,dt = \int_{-\infty}^{s-(D-M)} |\kappa(t)|\,dt \le \frac{C_\mu}{\widetilde{w}\big((D-M)-s\big)} \le \frac{C_\mu \Delta^M_{\widetilde{w}}}{\widetilde{w}(D-s)} = \frac{C_\mu \Delta^M_{\widetilde{w}}}{\widetilde{w}^D_-(s)}.$$

Combining this inequality with (5.72), we see that

$$\widetilde{w}^D_\pm(s)\mu^N_{\mp D}(s) = \widetilde{w}^D_-(s) \sum_{t^N_j > D} |\omega^N_j(s)| \le C_\mu C_1 \Delta^M_{\widetilde{w}}, \qquad s < D - A_2.$$

Together with (5.71), this inequality shows that (5.70) holds. □

The error $x^N - x^{N,A}$ of the finite-section method is given by $(\lambda - K_k^N)^{-1}(r_+ + r_-)$. If (5.69) holds then we deduce from (5.70) and (5.68) that $r_\pm \in X_{\widetilde{w}_\mp^D}$. To carry over this bound on the residual to a bound on the error of the finite section method, we will now study the invertibility of the operators $\lambda - K_k^N$ on the weighted spaces $X_{\widetilde{w}_\pm^D}$.

As a first step in this direction, we are now going to demonstrate that many of the boundedness, invertibility and stability results for the discretized integral operators K_k^N remain valid when X_w is replaced by $X_{\widetilde{w}_\pm}$. The next proposition (cf. Theorems 5.6, 5.9, Proposition 5.8) collects the specific results needed in our error analysis of the finite-section method.

In the formulation of the proposition and in the remainder of the section, we will use the following notation: given a kernel $v(s,t)$, quadrature weights $\omega_j^N(s)$ and a weight function $\widetilde{w}$, we define the kernel $v_\pm(s,t) := v_{\widetilde{w}_\pm}(s,t)$ as in (4.5) and the quadrature weights $\omega_{j,\widetilde{w}_\pm}^N(s)$ as in (5.20), but with w replaced by $\widetilde{w}_\pm$ in both cases. The corresponding integral and quadrature operators are $K_{k,\widetilde{w}_\pm} := M_{\widetilde{w}_\pm} K_k M_{\widetilde{w}_\pm}^{-1}$ and $K_{k,\widetilde{w}_\pm}^N := M_{\widetilde{w}_\pm} K_k^N M_{\widetilde{w}_\pm}^{-1}$, for $k \in BC(\mathbb{R}^2)$.

Proposition 5.26. *Suppose that the kernel v satisfies **(A″)**, with $\kappa \in L^1(\mathbb{R})$, and **(B)**. Assume, further, that the quadrature weights $\omega_j^N(s)$ satisfy **(QA)**, **(QA″)** and **(QB)**. Then, for every $\widetilde{w} \in \mathcal{W}(\kappa)$, the quadrature weights $\omega_{j,\widetilde{w}_\pm}^N(s)$ satisfy **(QA)** and **(QB)**, and the quadrature weights $\omega_j^N(s) - \omega_{j,\widetilde{w}_\pm}^N(s)$ satisfy **(QA)**, **(QB)** and **(QD)**.*

Moreover, if $W \subset BC(\mathbb{R}^2)$ is bounded, the operators K_k^N, $k \in W$, $N \in \mathbb{N}$, are uniformly bounded on $X_{\widetilde{w}_\pm}$. Moreover, for every $N \in \mathbb{N}$, $k \in BC(\mathbb{R}^2)$, the following statements hold

$$0 \in \Sigma_X(K_k^N) = \Sigma_{X_{\widetilde{w}_\pm}}(K_k^N), \qquad 0 \in \Sigma_X^e(K_k^N) = \Sigma_{X_{\widetilde{w}_\pm}}^e(K_k^N). \tag{5.73}$$

*If, additionally, the quadrature weights $\omega_j^N(s)$ satisfy **(Q)** then the quadrature weights $\omega_{j,\widetilde{w}_\pm}^N(s)$ also satisfy **(Q)** (with $v = v_\pm$).*

Proof. To see that the operators K_k^N, $k \in W$, $N \in \mathbb{N}$, are uniformly bounded on $X_{\widetilde{w}_\pm}$, it suffices to show that the quadrature weights $\omega_{j,\widetilde{w}_\pm}^N(s)$ satisfy Assumptions **(QA)** and **(QB)**, for then Proposition 5.2 applies and shows the equivalent statement that the operators $K_{k,\widetilde{w}_\pm}^N$, $k \in W$, $N \in \mathbb{N}$ are uniformly bounded on X (see equation (2.8)).

To verify **(Q)** and **(QB)**, we use the respective arguments of Theorem 5.6, but with w replaced by $\widetilde{w}_\pm$. That **(QA)** holds follows from the bound

$$\sum_{j\in\mathbb{Z}} |\omega_{j,\widetilde{w}_\pm}^N(s)| \le \sum_{j\in\mathbb{Z}} |\omega_j^N(s)| + \sum_{j\in\mathbb{Z}} |\omega_{j,\widetilde{w}}^N(s)|, \qquad s \in \mathbb{R},$$

and the fact that, by Theorem 5.6, the quadrature weights $\omega_j^N(s)$ and $\omega_{j,\widetilde{w}}^N(s)$ both satisfy **(QA)**.

For all $s \in \mathbb{R}$ and $A \ge 0$, the inequality

$$\sum_{|t_j^N|\ge A} \left|1 - \frac{\widetilde{w}_\pm(s)}{\widetilde{w}_\pm(t_j^N)}\right| |\omega_j^N(s)| = \sum_{\pm t_j^N \ge A} \left|1 - \frac{\widetilde{w}_\pm(s)}{\widetilde{w}_\pm(t_j^N)}\right| |\omega_j^N(s)| \le \sum_{|t_j^N|\ge A} \left|1 - \frac{\widetilde{w}(s)}{\widetilde{w}(t_j^N)}\right| |\omega_j^N(s)|$$

holds. By Proposition 5.8, the quadrature weights $\omega_j^N(s) - \omega_{j,\widetilde{w}}^N(s)$ satisfy **(QD)** and, hence, so must the quadrature weights $\omega_j^N(s) - \omega_{j,\widetilde{w}_\pm}^N(s)$. Lemma 5.7 now shows that the operators $K_k^N - K_{k,\widetilde{w}_\pm}^N$ are sn-continuous. Moreover, by Proposition 5.2, each of the operators K_k^N, $K_{k,\widetilde{w}_\pm}^N$ is s-continuous and s-sequentially compact. Now (5.73) follows using a similar argument as in Theorem 5.9, but with $K_{k,w}^N$ replaced by $K_{k,\widetilde{w}_\pm}^N$. □

We define the translation operators $T_a^{(1)}$ and $T_a^{(2)}$, $a \in \mathbb{R}$, by setting, for functions $x : \mathbb{R} \to \mathbb{C}$ and $k : \mathbb{R}^2 \to \mathbb{C}$,

$$T_a^{(1)}x(s) := x(s-a), \qquad T_a^{(2)}k(s,t) := k(s-a,t-a), \qquad s,t \in \mathbb{R}.$$

We note that $T_a^{(1)}$ and $T_a^{(2)}$, acting on X and $BC(\mathbb{R}^2)$, respectively, are isometric isomorphisms with inverses $T_{-a}^{(1)}$ and $T_{-a}^{(2)}$, respectively.

A characteristic feature of convolution kernels v is that $T_a^{(2)}v = v$ holds, for every $a \in \mathbb{R}$. In most cases of practical interest, reasonable quadrature weights $\omega_j^N(s)$ for the approximation of the convolution kernel v will also be translation invariant in the following sense:

$$\omega_j^N(s) = \omega_{j+j'}^N(s + j'h_N), \qquad s \in \mathbb{R},\ j, j' \in \mathbb{Z},\ N \in \mathbb{N}. \tag{5.74}$$

These properties are crucial for our subsequent analysis, for they enable us to relate the invertibility of $\lambda - K_k^N$ on X to that on $X_{\widetilde{w}_\pm^D}$.

Lemma 5.27. *Suppose that the quadrature weights $\omega_j^N(s)$ satisfy Assumptions **(QA)**, **(QA′′)**, **(QB)** and* (5.74)*, and that $k \in BC(\mathbb{R}^2)$ and $\widetilde{w} \in \mathcal{W}(\kappa)$. Then, if $D := t_j^N$, for some $N \in \mathbb{N}$ and $j \in \mathbb{N}_0$, the following equivalence holds for every $\lambda \in \mathbb{C}$:*

$$\lambda - K_k^N \in \mathcal{GL}(X) \iff \lambda - K_{T_{\pm D}^{(2)}k}^N \in \mathcal{GL}(X) \iff \lambda - K_k^N \in \mathcal{GL}(X_{\widetilde{w}_\pm^D}). \tag{5.75}$$

Further, if $\lambda - K_k^N$ is invertible on X for some $\lambda \in \mathbb{C}$, then

$$\|(\lambda - K_k^N)^{-1}\| = \|(\lambda - K_{T_{\pm D}^{(2)}k}^N)^{-1}\| \quad \text{and} \quad \|(\lambda - K_{T_{\pm D}^{(2)}k}^N)^{-1}\|_{\widetilde{w}_\pm} = \|(\lambda - K_k^N)^{-1}\|_{\widetilde{w}_\pm^D}. \tag{5.76}$$

Proof. From (5.74) we obtain, by straight-forward calculation, $T_{\pm D}^{(1)}K_k^N = K_{T_{\pm D}^{(2)}k}^N T_{\pm D}^{(1)}$, and thus

$$\lambda - K_{T_{\pm D}^{(2)}k}^N = T_{\pm D}^{(1)}(\lambda - K_k^N)T_{\mp D}^{(1)}. \tag{5.77}$$

Since the mappings $T_{\mp D}^{(1)} : X \to X$ and $T_{\mp D}^{(1)} : X_{\widetilde{w}_\pm} \to X_{\widetilde{w}_\pm^D}$ are isometric isomorphisms and Proposition 5.26 applies we conclude that

$$\lambda - K_k^N \in \mathcal{GL}(X) \overset{(5.77)}{\iff} \lambda - K_{T_{\pm D}^{(2)}k}^N \in \mathcal{GL}(X) \overset{(5.73)}{\iff} \lambda - K_{T_{\pm D}^{(2)}k}^N \in \mathcal{GL}(X_{\widetilde{w}_\pm}) \overset{(5.77)}{\iff} \lambda - K_k^N \in \mathcal{GL}(X_{\widetilde{w}_\pm^D}).$$

Moreover, if all the inverses exist, then

$$\|(\lambda - K_{T_{\pm D}^{(2)}k}^N)^{-1}\| \overset{(5.77)}{=} \|T_{\pm D}^{(1)}(\lambda - K_k^N)^{-1}T_{\mp D}^{(1)}\| = \|(\lambda - K_k^N)^{-1}\|,$$

$$\|(\lambda - K_{T_{\pm D}^{(2)}k}^N)^{-1}\|_{\widetilde{w}_\pm} \overset{(5.77)}{=} \|T_{\pm D}^{(1)}(\lambda - K_k^N)^{-1}T_{\mp D}^{(1)}\|_{\widetilde{w}_\pm} = \|(\lambda - K_k^N)^{-1}\|_{\widetilde{w}_\pm^D}.$$

proving the two remaining assertions. □

Combing this Lemma with the proposition preceding it, we obtain the following stability result, a central element of our error analysis of the finite-section method.

Proposition 5.28. *Suppose that $v(s,t) = \tilde{\kappa}(s-t)$ satisfies **(A′′)**. Assume, further, that the quadrature weights $\omega_j^N(s)$ satisfy **(Q)**, **(QA)**, **(QA′′)**, **(QB)** and also* (5.74)*. Moreover, let $W \subset BC(\mathbb{R}^2)$ be bounded and uniformly equicontinuous and $\mathbb{N}' \subset \mathbb{N}$. If*

$$\lambda \notin \bigcup_{N \in \mathbb{N}'} \bigcup_{k \in W} \Sigma_X(K_k^N) \qquad \text{and} \qquad C := \sup_{N \in \mathbb{N}'} \sup_{k \in W} \|(\lambda - K_k^N)^{-1}\| < \infty \tag{5.78}$$

then, for every $\tilde{w} \in \mathcal{W}(\kappa)$, $N \in \mathbb{N}'$, $k \in W$ and $D = t_j^N$, $j \in \mathbb{N}_0$, the operator $\lambda - K_k^N$ is invertible on $X_{\widetilde{w}_\pm^D}$. Moreover, the following inequalities hold

$$C_\pm := \sup_{N \in \mathbb{N}'} \sup_{k \in W} \sup_{D = t_j^N,\, j \in \mathbb{N}_0} \|(\lambda - K_k^N)^{-1}\|_{\widetilde{w}_\pm^D} < \infty. \tag{5.79}$$

Proof. The existence of the inverses in (5.79) follows from (5.78) by an application of Lemma 5.27. This lemma also shows that

$$C \stackrel{(5.76)}{=} \sup_{N\in\mathbb{N}'} \sup_{k\in W} \sup_{D=t_j^N,\, j\in\mathbb{N}_0} \|(\lambda - K^N_{T^{(2)}_{\pm D} k})^{-1}\| < \infty. \tag{5.80}$$

We are left with showing that $C_\pm$ is finite. In view of the second inequality in (5.76), it suffices to prove

$$\sup_{N\in\mathbb{N}'} \sup_{k\in W} \sup_{D=t_j^N,\, j\in\mathbb{N}_0} \|(\lambda - K^N_{T^{(2)}_{\pm D} k})^{-1}\|_{\widetilde{w}_\pm} < \infty. \tag{5.81}$$

Suppose this were not the case. Then we could find sequences (N_n) in $\mathbb{N}'$, $(\widetilde{k}_n)$ in W, (j_n) in $\mathbb{N}_0$ such that, for $(D_n) := (t^{N_n}_{j_n})$ and $(k_n) := (T^{(2)}_{\pm D_n} \widetilde{k}_n)$, there holds $\lim_{n\to\infty} \|(\lambda - K^{N_n}_{k_n})^{-1}\|_{\widetilde{w}_\pm} = 0$. By passing to appropriate subsequences, we may assume w.l.o.g. that $N_n \to N \in \mathbb{N}' \cup \{\infty\}$ and $k_n \stackrel{s}{\to} k \in BC(\mathbb{R}^2)$ (since (k_n) is bounded and uniformly equicontinuous as a sequence of translates of functions in W, see Remark 2.8). Moreover, there exists a sequence (z_n) in $X_{\widetilde{w}_\pm}$ such that $\|z_n\|_{\widetilde{w}_\pm} = 1$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} \|(\lambda - K^{N_n}_{k_n}) z_n\|_{\widetilde{w}_\pm} = 0$.

The argument leading to a contradiction is now that in part (iii) of the proof of Theorem 5.11, with $\widetilde{w}_\pm$ instead of w, once we have shown that this replacement is allowed. To this end, we need to verify that: (i) the quadrature weights $\omega^N_{j,\widetilde{w}_\pm}(s)$ satisfy **(Q)**, **(QA)**, **(QB)** (to see that Proposition 5.3 applies); (ii) the quadrature weights $\omega^N_j(s) - \omega^N_{j,\widetilde{w}_\pm}(s)$ satisfy **(QA)**, **(QB)** and **(QD)** (to see that Lemma 5.7 and Proposition 5.8 may be invoked); (iii) the operator $\lambda - K^N_k$ is injective on $X_{\widetilde{w}_\pm}$ (we recall our notational convention that $K^\infty_k := K_k$).

(i) and (ii) have already been shown in Proposition 5.26. Moreover, in view of (5.80) and our assumptions on the quadrature weights $\omega^N_j(s)$, we may appeal to Theorem 5.11, showing that $\lambda - K^N_k$ is invertible on X and thus, in particular, injective on $X_{\widetilde{w}_\pm} \subset X$, whence (iii) also holds. Now, as indicated above, a contradiction of (5.81) arises and the proposition follows. □

We are now ready to prove our main error estimate for the (modified) finite-section method, when v is a convolution kernel and the quadrature weights $\omega^N_j(s)$ satisfy (5.74).

Theorem 5.29. *Suppose that $v(s,t) = \tilde{\kappa}(s-t)$ satisfies **(A″)**. Assume, further, that the quadrature weights $\omega^N_j(s)$ satisfy **(Q)**, **(QA)**, **(QA″)**, **(QB)** and also* (5.74). *Moreover, assume that W is a bounded and uniformly equicontinuous subset of $BC(\mathbb{R}^2)$, $\mathbb{N}' \subset \mathbb{N}$ and that, for every $A \in (0,\infty]$, $W_A \subset W$. Then the following statements hold: If*

$$\lambda \notin \bigcup_{N\in\mathbb{N}'} \bigcup_{A\in(0,\infty]} \bigcup_{k\in W_A} \Sigma_X(K^{N,A}_k) \tag{5.82}$$

and

$$C := \sup_{N\in\mathbb{N}'} \sup_{A\in(0,\infty]} \sup_{k\in W_A} \|(\lambda - K^{N,A}_k)^{-1}\| < \infty \tag{5.83}$$

then, for every $w \in \mathcal{W}(\kappa)$, the operators $\lambda - K^{N,A}_k$ in (5.83) *are also invertible on X_w and*

$$C_w := \sup_{N\in\mathbb{N}'} \sup_{A\in(0,\infty]} \sup_{k\in W_A} \|(\lambda - K^{N,A}_k)^{-1}\|_w < \infty. \tag{5.84}$$

Moreover, if $k \in W_\infty, k' \in W_A$, for some $A \in (0,\infty]$, and $N \in \mathbb{N}'$ then the equations

$$(\lambda - K^N_k) x^N = y, \qquad (\lambda - K^{N,A}_{k'}) x^{N,A} = y \tag{5.85}$$

have unique solutions $x^N, x^{N,A} \in X_w$ for every $y \in X_w$. If, additionally, $\widetilde{w} \in \mathcal{W}(\kappa)$ satisfies

$$\widetilde{w}(s)\Big(\int_A^\infty |\kappa(t)|\,dt + \int_{-\infty}^{-A} |\kappa(t)|\,dt\Big) = O(1), \qquad \text{as } s \to \infty, \tag{5.86}$$

and, for some $D \le A$, given by $D = t_j^N$, for some $j \in \mathbb{N}$, the equality

$$k(s,t) = k'(s,t), \quad |s|, |t| \le D, \tag{5.87}$$

holds then the error of the finite section method may be estimated by

$$|x^N(s) - x^{N,A}(s)| \le C \frac{1}{w(D)} \left(\frac{1}{\widetilde{w}(s-D)} + \frac{1}{\widetilde{w}(s+D)} \right) \|y\|_w, \qquad |s| \le D, \tag{5.88}$$

where $C > 0$ is some constant not depending on k, k', N, D or y.

Proof. Provided the assumptions up to (5.83) are satisfied, the existence of the inverses in (5.84) and their uniform boundedness follows from Theorem 5.23. It is clear that then the statement about the solvability of (5.85) holds.

Thus we are left with showing (5.88) if also the remaining assumptions of the theorem hold. Let us assume that these are satisfied and we are given $y \in X_w$ and solutions x^N and $x^{N,A}$ of (5.85). We define $C_W := 2 \sup_{\widetilde{k} \in W} \|\widetilde{k}\|$ and $r_\pm$ as in (5.67). Then

$$x^N - x^{N,A} = (\lambda - K_k^N)^{-1}(r_+ + r_-), \tag{5.89}$$

Then, for $s \in \mathbb{R}$, we obtain from (5.68)

$$|r_\pm(s)| \;\le\; C_W \mu_{\pm D}^N(s) \sup_{|t|>D} |x^{N,A}(t)| \;\le\; C_W \mu_{\pm D}^N(s) w(D)^{-1} \|x^{N,A}\|_w,$$

whence, by (5.86) and Lemma 5.24, there exists some constant C_0 such that

$$\|r_\pm\|_{\widetilde{w}_\mp^D} \le C_0 C_W w(D)^{-1} \|x^{N,A}\|_w.$$

We note that C_0 and all constants later in this proof do not depend on k, k', N, D or y. We have already shown that (5.84) holds, whence $\|x^{N,A}\|_w \le C_w \|y\|_w$. It follows that

$$\|r_\pm\|_{\widetilde{w}_\mp^D} \le C_1 w(D)^{-1} \|y\|_w,$$

for some positive constant $C_1 > 0$. Proposition 5.28 applies and shows that

$$\|(\lambda - K_k^N)^{-1}\|_{\widetilde{w}_\mp^D} < C_{\widetilde{w}},$$

for some $C_{\widetilde{w}} > 0$. Let $e_\pm := (\lambda - K_k^N)^{-1} r_\pm$. Then

$$\|e_\pm\|_{\widetilde{w}_\mp^D} \le C_{\widetilde{w}} \|r_\pm\|_{\widetilde{w}_\mp^D} \le C_{\widetilde{w}} C_1 w(D)^{-1} \|y\|_w,$$

and hence

$$|e_\pm(s)| \le C_{\widetilde{w}} C_1 \big(w(D) \widetilde{w}_\mp^D(s)\big)^{-1}, \quad s \in \mathbb{R}$$

Since $x^N - x^{N,A} = e_+ + e_-$, and in view of (5.89), adding these two inequalities yields, for $|s| \le D$

$$|x^N(s) - x^{N,A}(s)| \;\le\; |e_+(s)| + |e_-(s)| \;\le\; C_{\widetilde{w}} C_1 \|y\|_w \frac{1}{w(D)} \left(\frac{1}{\widetilde{w}_+^D(s)} + \frac{1}{\widetilde{w}_-^D(s)} \right),$$

from which (5.88) follows, by the definition of the weight functions $\widetilde{w}_\pm^D$. □

For an example of how this theorem may be applied, we refer to the end of Section 6.3.

5.5 Sums of integral operators and their approximation

In some applications it is desirable to consider integral operators $K_{\mathbf{k}}$, which may be written as a finite sum of integral operators of the form (3.2), i.e. integral operators of the form

$$K_{\mathbf{k}} := K_{1,k_1} + \cdots + K_{n,k_n}, \tag{5.90}$$

where, for $i = 1, \dots n$, K_{i,k_i} is defined as in (3.2) with $v = v_i$, $k = k_i$, and $\mathbf{k} = (k_1, \dots, k_n) \in (BC(\mathbb{R}^2))^n$. Here, we are assuming that the kernels v_i, $i = 1, \dots, n$, satisfy Assumption $(\boldsymbol{A''})$, each with the same $\kappa \in L^1(\mathbb{R})$, and also $(\boldsymbol{B})$.

The advantage of the decomposition (5.90) is that different quadrature formulae may be used for each of the integral operators $K_{1,k_1}, \dots, K_{n,k_n}$. Thus, let us suppose that $K^N_{1,k_1}, \dots, K^N_{n,k_n}$ are quadrature operators, defined as in (5.8), with the quadrature weights $\omega^N_j(s)$, $j \in \mathbb{Z}$, $N \in \mathbb{N}$, replaced by $\omega^N_{1,j}(s)$, $\dots$, $\omega^N_{n,j}(s)$, respectively, where, for $i = 1, \dots, n$, the i-set of quadrature weights satisfies $(\boldsymbol{Q})$ (with $v = v_i$), $(\boldsymbol{QA})$, $(\boldsymbol{QA''})$ and $(\boldsymbol{QB})$.

We combine these operators to define, for $\mathbf{k} = (k_1, \dots, k_n) \in (BC(\mathbb{R}^2))^n$, the operator

$$K^N_{\mathbf{k}} := K^N_{1,k_1} + \cdots + K^N_{n,k_n}, \qquad N \in \mathbb{N},$$

as an approximation of the integral operator $K_{\mathbf{k}}$.

It then follows from the theory in Chapters 4 and 5 that the operators $K_{\mathbf{k}}$ and $K^N_{\mathbf{k}}$ are bounded on the spaces X and X_w, $w \in \mathcal{W}(\kappa)$. Also, the operators $K_{\mathbf{k}}$ and $K^N_{\mathbf{k}}$ inherit the continuity and compactness properties in the strict topology from the operators $K^N_{1,k_1}, \dots, K^N_{n,k_n}$, e.g. if each of the operators $K^N_{1,k_1}, \dots, K^N_{n,k_n}$ is s-sequentially compact, s-continuous, sn-continuous then, respectively, so is $K^N_{\mathbf{k}}$. Bearing this observation in mind, it is not hard to modify the proofs of the results in Chapters 4 and 5 to obtain the following variants of Theorems 5.13, 5.14 and Theorem 5.29.

Theorem 5.30. *Let $v_1, \dots, v_n$ be kernels satisfying Assumptions $(\boldsymbol{A''})$, each with the same $\kappa \in L^1(\mathbb{R})$, and $(\boldsymbol{B})$. Suppose that the quadrature weights $\omega^N_{1,j}(s), \dots, \omega^N_{n,j}(s)$, satisfy $(\boldsymbol{Q})$ (with $v = v_i$ for the i-th set of weights), $(\boldsymbol{QA})$, $(\boldsymbol{QA''})$ and $(\boldsymbol{QB})$. Assume, further, that $\mathbf{W} \subset W_1 \times \cdots \times W_n$, for some bounded and equicontinuous subsets $W_i \subset BC(\mathbb{R}^2)$, and $\mathbb{N}' \subset \mathbb{N}$. Then*

$$\lambda \notin \bigcup_{\mathbf{k}\in\mathbf{W}} \bigcup_{N\in\mathbb{N}'} \Sigma_X(K^N_{\mathbf{k}}) \quad \text{and} \quad C := \sup_{\mathbf{k}\in\mathbf{W}} \sup_{N\in\mathbb{N}'} \|(\lambda - K^N_{\mathbf{k}})^{-1}\| < \infty.$$

imply, for each $w \in \mathcal{W}(\kappa)$, that

$$C_w := \sup_{\mathbf{k}\in\mathbf{W}} \sup_{N\in\mathbb{N}'} \|(\lambda - K^N_{\mathbf{k}})^{-1}\|_w < \infty.$$

If, further, $\mathbb{N}'$ is unbounded then the inverse of $(\lambda - K_{\mathbf{k}})^{-1}$ exists on X and X_w, for every $\mathbf{k} \in \mathbf{W}$, and

$$\sup_{\mathbf{k}\in\mathbf{W}} \|(\lambda - K_{\mathbf{k}})^{-1}\| \le C, \quad \sup_{\mathbf{k}\in\mathbf{W}} \|(\lambda - K_{\mathbf{k}})^{-1}\|_w \le C_w.$$

Moreover, for every $N \in \mathbb{N}'$ and $y \in X_w$, unique solutions $x, x^N \in X_w$ of the equations

$$(\lambda - K_{\mathbf{k}})x = y, \quad (\lambda - K^N_{\mathbf{k}})x^N = y$$

exist and satisfy

$$\|x - x^N\|_w \le C_w \|(K_{\mathbf{k}} - K^N_{\mathbf{k}})x\|_w.$$

Theorem 5.31. *For $i = 1, \dots, n$, suppose that $v_i(s,t) = \kappa_i(s-t)$ and that, additionally, each of the kernels v_i satisfies $(\boldsymbol{A''})$, for some $\kappa \in L^1(\mathbb{R})$ (the same in each occurrence). Moreover, assume that the quadrature weights $\omega^N_{1,j}(s), \dots, \omega^N_{n,j}(s)$, satisfy $(\boldsymbol{Q})$ (with $v = v_i$ for the i-th set of weights), $(\boldsymbol{QA})$, $(\boldsymbol{QA''})$, $(\boldsymbol{QB})$ and (5.74). Assume, further, that $\mathbf{W} \subset W_1 \times \cdots \times W_n$, for some bounded and uniformly equicontinuous subsets $W_i \subset BC(\mathbb{R}^2)$, that, for every $A \in (0, \infty]$, $\mathbf{W}_A \subset \mathbf{W}$, and $\mathbb{N}' \subset \mathbb{N}$.*

Then

$$\lambda \notin \bigcup_{N\in\mathbb{N}'} \bigcup_{A\in(0,\infty]} \bigcup_{k\in\mathbf{W}_A} \Sigma_X(K_{\mathbf{k}}^{N,A}) \tag{5.91}$$

and

$$C := \sup_{N\in\mathbb{N}'} \sup_{A\in(0,\infty]} \sup_{\mathbf{k}\in\mathbf{W}_A} \|(\lambda - K_{\mathbf{k}}^{N,A})^{-1}\| < \infty \tag{5.92}$$

imply, for every $w \in \mathcal{W}(\kappa)$, that the operators $\lambda - K_{\mathbf{k}}^{N,A}$ in (5.92) are also invertible on X_w and

$$C_w := \sup_{N\in\mathbb{N}'} \sup_{A\in(0,\infty]} \sup_{\mathbf{k}\in\mathbf{W}_A} \|(\lambda - K_{\mathbf{k}}^{N,A})^{-1}\|_w < \infty. \tag{5.93}$$

Moreover, if $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbf{W}_\infty, \mathbf{k}' = (k_1', \ldots, k_n') \in \mathbf{W}_A$, for some $A \in (0,\infty]$, and $N \in \mathbb{N}'$ then the equations

$$(\lambda - K_{\mathbf{k}}^{N})x^N = y, \qquad (\lambda - K_{\mathbf{k}'}^{N,A})x^{N,A} = y \tag{5.94}$$

have unique solutions $x^N, x^{N,A} \in X_w$ for every $y \in X_w$. If, additionally, $\widetilde{w} \in \mathcal{W}(\kappa)$ satisfies

$$\widetilde{w}(s)\Big(\int_A^\infty |\kappa(t)|\,dt + \int_{-\infty}^{-A} |\kappa(t)|\,dt\Big) = O(1), \qquad s \to \infty, \tag{5.95}$$

and, for some $D \le A$ satisfying $D = t_j^N$, $j \in \mathbb{N}$,

$$k_i(s,t) = k_i'(s,t), \quad |s|, |t| \le D \tag{5.96}$$

holds, for $i = 1, \ldots, n$, then the error of the finite section method may be estimated by

$$|x^N(s) - x^{N,A}(s)| \le C\frac{1}{w(D)}\left(\frac{1}{\widetilde{w}(s-D)} + \frac{1}{\widetilde{w}(s+D)}\right)\|y\|_w, \qquad |s| \le D, \tag{5.97}$$

where $C > 0$ is some constant not depending on $\mathbf{k}$, $\mathbf{k}'$, N, D or y.

Chapter 6

Applications

A common practical situation is that the kernel $v(s,t)$ of the integral equation (5.1) satisfies $\boldsymbol{(A')}$, $\boldsymbol{(B)}$ and is smooth for $s \neq t$, or at least if $|s-t|$ is sufficiently large. In many cases, $v(s,t)$ will then, for some $n \in \mathbb{N}$ and $b > 1$, satisfy the following assumption, which is the *leitmotiv* of this chapter.

Assumption $\boldsymbol{(A'_n)}$. *The kernel v satisfies Assumptions $\boldsymbol{(A')}$, with $\kappa \in L^1(\mathbb{R})$, $\boldsymbol{(B)}$ and there exist constants $C_n, A_0 \geq 0$ such that v is n times partially differentiable on the set $\{s,t \in \mathbb{R} : |s-t| \geq A_0\}$, and for all $i,j \in \mathbb{N}_0$ with $i+j \leq n$, the derivatives satisfy the bound*

$$\left|\frac{\partial^i}{\partial s^i}\frac{\partial^j}{\partial t^j} v(s,t)\right| \leq \frac{C_n}{(1+|s-t|)^b}, \quad |s-t| \geq A_0. \tag{6.1}$$

In this chapter we will always assume that v satisfies this assumption and investigate Nyström methods for the numerical solution of the corresponding integral equations

$$\lambda x(s) - \int_{-\infty}^{\infty} v(s,t)k(s,t)x(t)\,dt = y(s), \quad s \in \mathbb{R}.$$

As in the previous chapter, for $k \in BC(\mathbb{R}^2)$, we let K_k denote the integral operator in this equation. We will show that the operators K_k are bounded (on X and) the weighted spaces X_{w_a}, where $w_a(s) := (1+|t|)^a$ and $0 < a \leq b$.

We restrict our attention to this family of weighted spaces in the remainder of this thesis. We recall that we have seen in Example 4.21 that these weight functions all satisfy Assumption $\boldsymbol{(F')}$. To avoid double subscripts, we will use the notation X_a, $\|\cdot\|_a$, instead of X_{w_a}, $\|\cdot\|_{w_a}$, respectively. For $a = 0$ we set $X_a := X$, $\|\cdot\|_a := \|\cdot\|$. Moreover, for $A > 0$, we introduce the notation $\Delta_a^A := \Delta_{w_a}^A$, i.e. $\Delta_a^A := \sup_{|s-t|\leq A}\big((1+|s|)/(1+|t|)\big)^a < \infty$.

6.1 A few technical prerequisites

In this short section, we provide some definitions and easy facts, which will be useful in our subsequent analysis of quadrature and integral operators acting on X_a.

In the sequel, we will work with the special class of weighted spaces we are going to define now: For $n \in \mathbb{N}_0$, we let $BC^n(\mathbb{R})$, $BC^n(\mathbb{R}^2)$, denote the Banach space of all functions, which, together with their (partial) derivatives up to order n, are bounded and continuous on $\mathbb{R}$, $\mathbb{R}^2$, respectively, equipped with the norms

$$\|f\|_{BC^n(\mathbb{R})} := \max\left\{\|f^{(i)}\| : \ 0 \leq i \leq n\right\},$$
$$\|f\|_{BC^n(\mathbb{R}^2)} := \max\left\{\|\partial_1^i \partial_2^j f\| : \ 0 \leq i,j \leq n, \ i+j \leq n\right\},$$

where, here and throughout, ∂_1, ∂_2 denote the operators of partial differentiation with respect to the first and second coordinate, respectively. For $a \geq 0$ we will use the non-standard notation $BC^n_a(\mathbb{R})$ for the Banach space

$$BC^n_a(\mathbb{R}) := \left\{ f \in BC^n(\mathbb{R}) : \|f\|_{BC^n_a(\mathbb{R})} := \max\{\|w_a f^{(i)}\| : 0 \leq i \leq n\} < \infty \right\}.$$

Moreover, we will write $BC^n_a(\mathbb{R}^2)$ for the Banach space

$$BC^n_a(\mathbb{R}^2) := \left\{ f \in BC^n(\mathbb{R}^2) : \|f\|_{BC^n_a(\mathbb{R}^2)} := \max\{\|w^*_a \partial_1^i \partial_2^j f\| : \ 0 \leq i, j \leq n, \ i+j \leq n\} < \infty \right\},$$

where $w^*_a(s,t) := w_a(s-t)$.

If two functions $x, y : \Omega' \to \mathbb{C}$, $\Omega' \subset \mathbb{R}$ open, are n times differentiable ($n \in \mathbb{N}_0$) then so is their product $xy : \Omega' \to \mathbb{C}$, with derivatives given by the *Leibniz product rule*

$$xy^{(j)} = \sum_{m=0}^{j} \binom{j}{m} x^{(m)} y^{(j-m)}, \qquad 0 \leq j \leq n. \tag{6.2}$$

From this fact we draw a number of simple but useful conclusions and collect them in the next lemma.

Lemma 6.1. *Suppose that $n \in \mathbb{N}_0$ and $a \geq 0$.*

a) If $x \in BC^n_a(\mathbb{R})$ and $y \in BC^n(\mathbb{R})$ then $xy \in BC^n_a(\mathbb{R})$ and

$$\|xy\|_{BC^n_a(\mathbb{R})} \leq 2^n \|x\|_{BC^n_a(\mathbb{R})} \|y\|_{BC^n(\mathbb{R})}. \tag{6.3}$$

b) If, further, for some $s \in \mathbb{R}$ and $A > 0$, $y(t) = 0$ for all t satisfying $|s-t| > A$ then

$$\|xy\|_{BC^n(\mathbb{R})} \leq 2^n \Delta^A_a (1+|s|)^{-a} \|x\|_{BC^n_a(\mathbb{R})} \|y\|_{BC^n(\mathbb{R})}. \tag{6.4}$$

c) If $v \in BC^n_a(\mathbb{R}^2)$ and $k \in BC^n(\mathbb{R}^2)$, then $vk \in BC^n_a(\mathbb{R}^2)$ and

$$\|vk\|_{BC^n_a(\mathbb{R}^2)} \leq 2^n \|v\|_{BC^n_a(\mathbb{R}^2)} \|k\|_{BC^n(\mathbb{R}^2)}. \tag{6.5}$$

Proof. **a)** If the assumptions of the lemma are satisfied then, for $0 \leq j \leq n$, $w_a(s)|x^{(j)}(s)| \leq \|x\|_{BC^n_a(\mathbb{R})}$, $s \in \mathbb{R}$. Thus, and, by the Leibniz product rule, we obtain, for $t \in \mathbb{R}$, $0 \leq j \leq n$,

$$|(xy)^{(j)}(t)| = \left| \sum_{m=0}^{j} \binom{j}{m} x^{(m)}(t) y^{(j-m)}(t) \right| \leq 2^n (1+|t|)^{-a} \|x\|_{BC^n_a(\mathbb{R})} \|y\|_{BC^n(\mathbb{R})}. \tag{6.6}$$

Multiplying these inequalities with $(1+|t|)^a$ and taking the supremum over $s \in \mathbb{R}$ yields (6.3).

b) If, in addition, y satisfies the assumptions in b) then $xy(t) = 0$ for $|s-t| > A$, and for $|s-t| \leq A$, arguing similarly to a), we see that

$$|(xy)^{(j)}(t)| \leq 2^n (1+|t|)^{-a} \|x\|_{BC^n_a(\mathbb{R})} \|y\|_{BC^n(\mathbb{R})} \leq 2^n \Delta^A_a (1+|s|)^{-a} \|x\|_{BC^n_a(\mathbb{R})} \|y\|_{BC^n(\mathbb{R})},$$

holds for $0 \leq j \leq n$. The inequality in b) follows.

c) Under the assumptions of part c), we obtain, similarly, for $0 \leq i+j \leq n$,

$$\begin{aligned} \left| \frac{\partial^i}{\partial s^i} \frac{\partial^j}{\partial t^j} \big(v(s,t)k(s,t)\big) \right| &= \left| \sum_{m=0}^{i} \binom{i}{m} \frac{\partial^m}{\partial s^m} \frac{\partial^j}{\partial t^j} v(s,t) \frac{\partial^{i-m}}{\partial s^{i-m}} \frac{\partial^j}{\partial t^j} k(s,t) \right| \\ &\leq 2^n (w_a(s-t))^{-1} \|v\|_{BC^n_a(\mathbb{R}^2)} \|k\|_{BC^n(\mathbb{R}^2)}, \end{aligned}$$

from which the desired inequality follows. □

A particular feature of kernels v satisfying $(\boldsymbol{A'_n})$ is that they satisfy Assumption $(\boldsymbol{A'})$, for some $\kappa \in L^1(\mathbb{R})$ such that $\kappa(t) = O(|t|^{-b})$ as $|t| \to \infty$ (b is the constant in $(\boldsymbol{A'_n})$). Our next proposition shows that such kernels yield integral operators bounded on the weighted spaces X_a, $0 \le a \le b$. In the proof, we use the weighted space theory developed in Chapter 4.

Proposition 6.2. *Suppose that the kernel v satisfies $(\boldsymbol{A'})$, with $\kappa \in L^1(\mathbb{R})$, and $(\boldsymbol{B})$, and further that, for some $b > 1$, $A_0 > 0$ and $C > 0$, κ satisfies*

$$|\kappa(s)| \le C(1 + |s|)^{-b}, \qquad |s| \ge A_0. \tag{6.7}$$

Then, for every $0 \le a \le b$, $w_a \in \mathcal{W}(\kappa)$. Moreover, for every $k \in BC^m(\mathbb{R}^2)$, $m \in \mathbb{N}_0$, and $0 \le a \le b$, the integral operator K_k, defined by (3.2)*, is bounded on X_a, with norm bounded by*

$$\|K_k\|_a \le C\|k\| \le C\|k\|_{BC^m(\mathbb{R}^2)}, \tag{6.8}$$

where C is a constant depending only on v and a.

Proof. The second inequality in (6.8) is trivial, hence we concentrate on proving the first.

Let $a = 0$. Since $(\boldsymbol{A'})$ implies $(\boldsymbol{A})$, the boundedness of K_k on X and the first inequality in (6.8) are immediate from Proposition 3.1. Thus we now focus on the case when $a > 0$.

Note first, that we may assume w.l.o.g. that κ is chosen such that κ is non-negative and equality holds in (6.7). Then κ is monotonic outside the interval $[-A_0, A_0]$ and both κ and its reflection around the origin, $\breve{\kappa}$, defined by $\breve{\kappa}(t) := \kappa(-t)$, satisfy condition (4.57) with $p = b > 1$. Hence both κ and $\breve{\kappa}$ satisfy Assumption $(\boldsymbol{E'_+})$ with $w = w_a$; moreover, w_a satisfies $(\boldsymbol{F'})$ (see Example 4.21). It thus follows from (6.7) and Lemma 4.32, part c), that κ and w_a satisfy Assumptions $(\boldsymbol{E'})$ and $(\boldsymbol{E})$, and hence $w_a \in \mathcal{W}(\kappa)$. Hence, by Proposition 4.31, the kernel $v_{w_a}(s,t) := (w_a(s)/(w_a(t))v(s,t)$ satisfies $(\boldsymbol{A})$ and $(\boldsymbol{B})$. From Proposition 3.1, we now obtain that the operators $K_{w_a,k} := M_{w_a} K_k M_{w_a^{-1}}$ are bounded on X and satisfy $\|K_{w_a,k}\| \le \|\kappa\|_1 \|k\|$. Using (2.5) and (2.8), we infer that $K_k : X_a \to X_a$ and that the first inequality in (6.8) holds. □

In a corollary to this proposition, we define two quantities, $\Theta_{a,b}$ and $\theta^A_{a,b}$, which we will use occasionally later on.

Corollary 6.3. *Suppose that $b > 1$, $A > 0$ and $0 \le a \le b$. Then the following two numbers are finite:*

$$\Theta_{a,b} := \sup_{s \in \mathbb{R}} \int_{-\infty}^{\infty} \Big(\frac{1+|s|}{1+|t|}\Big)^a (1 + |s-t|)^{-b}\, dt, \tag{6.9}$$

$$\theta^A_{a,b} := \sup_{0<h\le A} \sup_{s\in\mathbb{R}} h \sum_{j\in\mathbb{Z}} \bigg(\Big(\frac{1+|s|}{1+|jh|}\Big)^a (1 + |s-jh|)^{-b}\bigg). \tag{6.10}$$

We remark that sum in (6.10) is the rectangle rule approximation of the integral in (6.9), with distance h between the quadrature abscissae.

Proof. Given a, b as in the assumption, let us consider the convolution kernel v, given by

$$v(s,t) := (1 + |s-t|)^{-b}, \qquad s, t \in \mathbb{R}.$$

Then $v(s,t) = \kappa(s-t)$, with $\kappa \in L^1(\mathbb{R})$, given by $\kappa(t) = (1+|t|)^{-b}$, so that v satisfies $(\boldsymbol{A'})$ and $(\boldsymbol{B})$. We have seen in the proof of the previous proposition that then the kernel $v_{w_a}(s,t) := (w_a(s)/(w_a(t))v(s,t)$ satisfies $(\boldsymbol{A})$, which is equivalent to saying that $\Theta_{a,b} < \infty$.

(ii) For the second part of the lemma, we choose $A > 0$. For $s \in \mathbb{R}$ and $0 < h \le A$, we define

$$S_1(s,h) + S_2(s,h) := h\Big(\sum_{jh<s} + \sum_{jh\ge s}\Big) \frac{w_a(s)}{w_a(jh) w_b(s-jh)} = h \sum_{j\in\mathbb{Z}} \frac{w_a(s)}{w_a(jh) w_b(s-jh)}.$$

We obtain, for every $s \in \mathbb{R}$ and $0 < h \le A$,

$$S_2(s,h) = h \sum_{jh \ge s} \frac{w_a(s)}{w_a(jh)w_b(s-jh)} = \sum_{jh \ge s} \int_{jh}^{(j+1)h} \frac{w_a(s)}{w_a(jh)w_b(s-jh)}\,dt$$
$$\le \Delta_a^A \Delta_b^A \sum_{jh \ge s} \int_{jh}^{(j+1)h} \frac{w_a(s)}{w_a(t)w_b(s-t)}\,dt \le \Delta_a^A \Delta_b^A \int_s^\infty \frac{w_a(s)}{w_a(t)w_b(s-t)}\,dt.$$

A symmetric argument shows that, for every $s \in \mathbb{R}$ and $0 < h \le A$, there also holds

$$S_1(s,h) \le \Delta_a^A \Delta_b^A \int_{-\infty}^s \frac{w_a(s)}{w_a(t)w_b(s-t)}\,dt.$$

Adding these inequalities and taking the suprema yields that $\theta_{a,b}^A \le \Delta_a^A \Delta_b^A \Theta_{a,b} < \infty$. □

6.2 Kernels with polynomial decay and error estimates for smooth inhomogeneities

If the kernel v satisfies Assumption $(\boldsymbol{A'_n})$ then we immediately obtain from the bound (6.1) that the kernel bound κ may be chosen such that

$$\kappa(s) = C_n(1+|s|)^{-b}, \quad |s| \ge A_0. \tag{6.11}$$

holds. Therefore we will, throughout this thesis, often tacitly assume that κ satisfies (6.11) whenever we say that v satisfies Assumption $(\boldsymbol{A'_n})$. As a consequence, the kernel v then satisfies $(\boldsymbol{A''})$.

Given a kernel v satisfying Assumption $(\boldsymbol{A'_n})$, we now choose a small $\eta > 0$ and a "cut-off" function χ satisfying

$$\chi \in C^\infty(\mathbb{R}), \quad 0 \le \chi(t) \le 1, \quad \chi(t) = \begin{cases} 1, & |t| \le A_0 + \eta, \\ 0, & |t| \ge A_0 + 2\eta, \end{cases} \tag{6.12}$$

unless the constant A_0 in $(\boldsymbol{A'_n})$ is 0, in which case we simply set $\chi(t) := 0$, $t \in \mathbb{R}$.

We now split the kernel into a smooth part $\widetilde{v}$ and a non-smooth, possibly weakly singular, part $\widehat{v}$. We let $\chi^*(s,t) := \chi(s-t)$, notice that χ^* and $1-\chi^*$ are functions in $BC^n(\mathbb{R})$ and define the kernels

$$\widehat{v}(s,t) := v(s,t)\chi^*(s,t), \quad \widetilde{v}(s,t) := v(s,t)(1-\chi^*(s,t)). \tag{6.13}$$

We note that if $A_0 = 0$ then $\widehat{v} = 0$ and $v = \widetilde{v}$. We also see that, since $\|\chi^*\| \le 1$, $\widehat{v}$ and $\widetilde{v}$ both satisfy $(\boldsymbol{A})$, $(\boldsymbol{A'})$, $(\boldsymbol{A''})$, and $(\boldsymbol{B})$, with the same kernel bound κ as v in $(\boldsymbol{A'})$ and $(\boldsymbol{A''})$.

If $A_0 = 0$, $v = \widetilde{v} \in BC_b^n(\mathbb{R}^2)$ is immediate from (6.1). In the case $A_0 > 0$, $(1-\chi^*(s,t))$ vanishes whenever $|s-t| \le A_0 + \eta$, and, again, it follows from (6.1) that $\widetilde{v} \in BC_b^n(\mathbb{R}^2)$ when $A_0 > 0$ by an argument, which is essentially that used to show (6.5).

The decomposition (6.13) allows us to split K_k into the integral operators $\widehat{K_k}$ and $\widetilde{K_k}$ defined by (3.2), with v replaced by $\widehat{v}$ and $\widetilde{v}$, respectively; then we may write

$$K_k = \widehat{K_k} + \widetilde{K_k}.$$

The next proposition addresses the boundedness of these operators on the weighted spaces X_a.

Proposition 6.4. *Suppose that the kernel v satisfies $(\boldsymbol{A'_n})$. Then v also satisfies $(\boldsymbol{A''})$ (not necessarily with the same $\kappa \in L^1(\mathbb{R})$). Moreover, for every $0 \le a \le b$ and every bounded $W \subset BC(\mathbb{R}^2)$, the operators K_k, $\widetilde{K_k}$ and $\widehat{K_k}$, $k \in W$, are uniformly bounded on X_a.*

Proof. If v satisfies $(\boldsymbol{A'_n})$ with κ as in the assumption, then we are automatically assuming that v also satisfies $(\boldsymbol{A'})$ and $(\boldsymbol{B})$. Moreover, we may modify κ outside the interval $[-A_0, A_0]$ so that (6.11) holds and v satisfies $(\boldsymbol{A''})$ for this modified κ. The uniform boundedness of the operators K_k, $k \in W$, now follows from Proposition 6.2. Since, by definition, $|\widehat{v}(s,t)| \le |v(s,t)|$ and $|\widetilde{v}(s,t)| \le |v(s,t)|$ we see that $\|\widetilde{K_k}\|_a, \|\widehat{K_k}\|_a \le \|K_k\|_a$, proving the proposition. □

The motivation for the splitting of the kernel is that a simple quadrature rule for the approximation of $\widetilde{v}$, e.g. a compound Newton-Cotes formula, already yields a sufficiently good order of convergence (and is often very easy to implement on a computer). The kernel $\widehat{v}$ may require a more sophisticated quadrature rule, but has the advantage that, for fixed $s \in \mathbb{R}$, $\widehat{v}(s,\cdot)$ is compactly supported.

In the following, we will consider what is perhaps the simplest approximation for $\widetilde{v}$: the compound rectangle rule. However, we make the point that our arguments are easily adapted to work with the other compound rules such as Simpson's rule.

We define, for $N \in \mathbb{N}$ and $k \in BC(\mathbb{R}^2)$, the discretized integral operator $\widetilde{K_k^N}$ by

$$\widetilde{K_k^N} x(s) := h_N \sum_{j\in\mathbb{Z}} \widetilde{v}(s, t_j^N) k(s, t_j^N) x(t_j^N), \quad s \in \mathbb{R},\ x \in X, \tag{6.14}$$

taking $t_j^N = jC/N$ for some constant $C > 0$, so that $h_N = C/N$. This means that $\widetilde{K_k^N}$ is the operator defined in (5.8) with the quadrature weights $\omega_j^N(s)$ replaced by the quadrature weights

$$\widetilde{\omega}_j^N(s) := h_N \widetilde{v}(s, t_j^N), \quad s \in \mathbb{R},\ j \in \mathbb{Z},\ N \in \mathbb{N}. \tag{6.15}$$

The next proposition shows that the quadrature weights $\widetilde{\omega}_j^N(s)$ for the rectangular rule are compatible with our assumptions in Chapter 5 and yield bounded quadrature operators on X_a, $0 \le a \le b$.

Proposition 6.5. *Suppose that the kernel v satisfies Assumption $(\boldsymbol{A'_n})$. The quadrature weights $\widetilde{\omega}_j^N(s)$, defined by (6.15), then satisfy Assumptions $(\boldsymbol{Q_u})$ (with $v = \widetilde{v}$), $(\boldsymbol{QA})$, $(\boldsymbol{QA''})$, $(\boldsymbol{QB})$ and $(\boldsymbol{QB_u})$. Moreover, for every $0 \le a \le b$ and every bounded set $W \subset BC(\mathbb{R}^2)$, the quadrature operators $\widetilde{K_k^N}$, $k \in W$, are uniformly bounded on X_a.*

Proof. In the first part of the proof, we show that the quadrature weights $\widetilde{\omega}_j^N(s)$ satisfy the desired assumptions.

$(\boldsymbol{QA''})$: We may assume w.l.o.g. that κ satisfies $\kappa(s) = C_n(1+|s|)^{-b}$, $|s| \ge A_0$, so that

$$|\widetilde{\omega}_j^N(s)| = h_N |\widetilde{v}(s, t_j^N)| \le h_N |v(s, t_j^N)| \le C_n h_N (1 + |s - t_j^N|)^{-b},$$

for all $s \in \mathbb{R}$, $j \in \mathbb{Z}$, $N \in \mathbb{N}$ satisfying $|s - t_j^N| \ge A_0$. Hence $(\boldsymbol{QA''})$ is satisfied (with $A_1 := A_0$).

$(\boldsymbol{QA})$: Since $\widetilde{\omega}_j^N(s) = 0$, if $|s - t_j^N| < A_0$, we obtain, in view of the above inequality and Corollary 6.3,

$$\sum_{j\in\mathbb{Z}} |\widetilde{\omega}_j^N(s)| \;\le\; C_n h_N \sum_{j\in\mathbb{Z}} (1 + |s - t_j^N|)^{-b} \;\le\; C_n \theta_{0,b}^M \;<\; \infty, \qquad s \in \mathbb{R},\ N \in \mathbb{N},$$

where $M := \max\{h_N : N \in \mathbb{N}\}$. But this means that the quadrature weights $\widetilde{\omega}_j^N(s)$ satisfy $(\boldsymbol{QA})$.

$(\boldsymbol{QB_u})$: Since $\widetilde{v} \in BC_b^n(\mathbb{R})$ we can use the mean value theorem to see that, for all $s, t \in \mathbb{R}$, $|h| < 1$,

$$|\widetilde{v}(s,t) - \widetilde{v}(s+h, t)| \le \Delta_b^1 |h| \|\widetilde{v}\|_{BC_b^1(\mathbb{R}^2)} (1 + |s-t|)^{-b}. \tag{6.16}$$

From this inequality we obtain, for all $|h| < 1$ and $s \in \mathbb{R}$,

$$\sum_{j\in\mathbb{Z}} \big|\widetilde{\omega}_j^N(s) - \widetilde{\omega}_j^N(s+h)\big| = h_N \sum_{j\in\mathbb{Z}} \big|\widetilde{v}(s, t_j^N) - \widetilde{v}(s+h, t_j^N)\big| \le \Delta_b^1 |h| \|\widetilde{v}\|_{BC_b^1(\mathbb{R}^2)} \sum_{j\in\mathbb{Z}} (1 + |s - jh_N|)^{-b}.$$

Irrespective of $s \in \mathbb{R}$, the series in the last term is bounded above by $\theta_{0,b}^M$ (with M as above), whence the term on the left must tend to 0 as $h \to 0$, so that $(\boldsymbol{QB_u})$ and $(\boldsymbol{QB})$ must be satisfied.

$(\boldsymbol{Q_u})$: Let $U \in \mathcal{U}$, i.e. U is a bounded and uniformly equicontinuous subset of X. Since $\widehat{v} \in BC^n_b(\mathbb{R}^2)$, we may apply Theorem 3.3 in [50] to obtain that, for k the constant function $k(s,t) = 1$, there holds $\|(\widetilde{K_k} - \widetilde{K^N_k})x\| \to 0$, as $N \to \infty$, uniformly in $x \in U$. This is the convergence required in $(\boldsymbol{Q_u})$. Since U was arbitrary $(\boldsymbol{Q_u})$ holds. □

If v satisfies $(\boldsymbol{A'_n})$ with $A_0 = 0$ then we may use this proposition to show that the Nyström method based on replacing K_k by K^N_k in the equation $(\lambda - K_k)x = y$ is stable on X_a, $0 \le a \le b$, provided that the equation $(\lambda - K_k)x = y$ is uniquely solvable for every $y \in X$.

Theorem 6.6. *Let v be a kernel satisfying $(\boldsymbol{A'_n})$ with $A_0 = 0$ and assume that, for every k in some bounded and uniformly equicontinuous set $W \subset BC(\mathbb{R}^2)$, $\lambda \notin \Sigma_X(K_k)$, and the inverses $(\lambda - K_k)^{-1}$ are uniformly bounded. Then the Nyström method of replacing the integral operators K_k by the rectangle rule operators $K^N_k = \widetilde{K^N_k}$, $k \in W$, is uniformly stable on X and X_a, $0 \le a \le b$, i.e. for some $N' \in \mathbb{N}$, the operators $\lambda - K^N_k$, $N \ge N'$, $k \in W$, are invertible on X_a and*

$$\sup_{N \ge N'} \sup_{k \in W} \|(\lambda - K^N_k)^{-1}\|_a < \infty.$$

Proof. The previous theorem shows that the quadrature weights $\widetilde{\omega}^N_j(s)$ satisfy $(\boldsymbol{Q_u})$, $(\boldsymbol{QA})$, $(\boldsymbol{QA''})$ and $(\boldsymbol{QB_u})$. In view of (6.16), there holds, for $s \in \mathbb{R}$ and $|h| < 1$,

$$\int_{-\infty}^{\infty} |\widetilde{v}(s,t) - \widetilde{v}(s+h,t)|\,dt \le C|h| \int_{-\infty}^{\infty} (1 + |s-t|)^{-b}\,dt < |h| C \Theta_{0,b},$$

where $C := \Delta^1_b \|\widetilde{v}\|_{BC^1_b(\mathbb{R}^2)}$. Letting $h \to 0$, we see that the kernel $v = \widetilde{v}$ satisfies $(\boldsymbol{B_u})$. By assumption, v also satisfies $(\boldsymbol{A''})$, with $\kappa(s) := C_n(1 + |s|)^{-b}$ and C_n being the constant in $(\boldsymbol{A'_n})$.

We have seen that all the assumptions of Theorems 5.20 and 5.21 are satisfied. Let $0 \le a \le b$; then, since $w_a \in \mathcal{W}(\kappa)$, these theorems show that the Nyström method is stable on X_a. □

To estimate the error of the rectangle rule approximation, we will use the following lemma, which generalises [50, La. 3.10].

Lemma 6.7. *Let $0 \le a \le b$ and suppose that the kernel v satisfies $(\boldsymbol{A'_n})$, for some $n \in \mathbb{N}$. Then, for every $x \in BC^n_a(\mathbb{R})$ and $k \in BC^n(\mathbb{R}^2)$, the error estimate*

$$\left| \sum_{j \in \mathbb{Z}} h_N \widetilde{v}(s, t^N_j) k(s, t^N_j) x(t^N_j) - \int_{-\infty}^{\infty} \widetilde{v}(s,t) k(s,t) x(t)\,dt \right| \le C C_n \|k\|_{BC^n(\mathbb{R}^2)} N^{-n} (1 + |s|)^{-a} \|x\|_{BC^n_a(\mathbb{R})}$$

holds, for every $s \in \mathbb{R}, N \in \mathbb{N}$, where $C > 0$ is a constant not depending on k, x or N, and C_n is the constant from $(\boldsymbol{A'_n})$.

Proof. Throughout this proof $C > 0$ denotes a generic constant, not necessarily the same at each occurrence. We will first show the theorem for the case when k is the constant function $k(s,t) = 1$.

We choose $\phi \in BC^n(\mathbb{R})$ such that $\phi(s) = -1/2$, $s \le -1/2$, and $\phi(s) = 1/2$, $s \ge 1/2$. Let $\phi_0(s) := \phi(s) - \phi(s-1)$ and $\phi_i(s) := \phi_0(s - i)$ for every $i \in \mathbb{Z}$. Then $\phi_i \in BC^n(\mathbb{R})$, $\operatorname{supp} \phi_i \subset [-1/2 + i, 3/2 + i]$, $i \in \mathbb{Z}$, and $\{\phi_i : i \in \mathbb{Z}\}$ is a partition of unity, because $\sum_{i \in \mathbb{Z}} \phi_i(s) = 1$ holds for every $s \in \mathbb{R}$.

Let, for $i \in \mathbb{Z}$ and $N \in \mathbb{N}$,

$$e^N_i(s) := \sum_{j \in \mathbb{Z}} h_N \phi_i(t^N_j) \widetilde{v}(s, t^N_j) x(t^N_j) - \int_{-\infty}^{\infty} \phi_i(t) \widetilde{v}(s,t) x(t)\,dt, \quad s \in \mathbb{R},$$

so that

$$e^N(s) := \sum_{j \in \mathbb{Z}} h_N \widetilde{v}(s, t^N_j) x(t^N_j) - \int_{-\infty}^{\infty} \widetilde{v}(s,t) x(t)\,dt = \sum_{i \in \mathbb{Z}} e^N_i(s), \quad s \in \mathbb{R}.$$

The next step is to estimate $e_i^N(s)$. Note that, for $s \in \mathbb{R}$ and $i \in \mathbb{Z}$, the summands and integrands in the definition of $e_i^N(s)$ are zero when $t, t_j^N \notin [s-1/2, s+3/2]$. Lemma 3.9 in [50], an application of the Euler-Maclaurin sum formula, provides an error estimate for the rectangle rule over finite intervals:

$$|e_i^N(s)| \le CN^{-n} \max_{0\le m\le n} \sup_{s-\frac{1}{2}\le t\le s+\frac{3}{2}} \left| \left(\phi_i(t)\widetilde{v}(s,t)x(t)\right)^{(m)}\right|, \quad s \in \mathbb{R}.$$

where the differentiation in the last term is with respect to t and the constant $C > 0$ only depends on n. Bearing in mind that $\widetilde{v} \in BC_b^n(\mathbb{R}^2)$, $x \in BC_a^n(\mathbb{R})$ and $\phi \in BC^n(\mathbb{R})$, we obtain, by repeated application of the Leibniz product rule, that

$$|e_i^N(s)| \le CN^{-n}\|\widetilde{v}\|_{BC_b^n(\mathbb{R}^2)}\|x\|_{BC_a^n(\mathbb{R})} \max_{-\frac{1}{2}+i\le t\le \frac{3}{2}+i} \left((1+|t|)^{-a}(1+|s-t|)^{-b}\right), \quad s \in \mathbb{R}.$$

where $C > 0$ is a constant depending only on n and ϕ. Observing that $\|\widetilde{v}\|_{BC_b^n(\mathbb{R}^2)} \le CC_n$, where $C > 0$ depends only on χ and n, and C_n is the constant in $(\boldsymbol{A'_n})$, we see that, for every $s \in \mathbb{R}$,

$$\begin{aligned}
|e^N(s)| \le \sum_{i\in\mathbb{Z}} |e_i^N(s)| &\le CC_nN^{-n}\|x\|_{BC_a^n(\mathbb{R})} \sum_{i\in\mathbb{Z}} \max_{-\frac{1}{2}+i\le t\le \frac{3}{2}+i} \left((1+|t|)^{-a}(1+|s-t|)^{-b}\right) \\
&\le CC_nN^{-n}\|x\|_{BC_a^n(\mathbb{R})}\Delta_a^{3/2}\Delta_b^{3/2} \sum_{i\in\mathbb{Z}} \left((1+|i|)^{-a}(1+|s-i|)^{-b}\right) \\
&\overset{(6.10)}{\le} CC_nN^{-n}\|x\|_{BC_a^n(\mathbb{R})}\Delta_a^{3/2}\Delta_b^{3/2}(1+|s|)^{-a}\theta_{a,b}^1.
\end{aligned}$$

where $C > 0$ depends only on n, ϕ and χ. The desired inequality now follows.

To prove the general case, we only need to replace $\widetilde{v}$ by $\widetilde{v}k$ in the above arguments and remember the statement in part c) of Lemma 6.1. □

An immediate corollary of the previous lemma is the following proposition, which bounds the error for the approximation of $\widetilde{K_k}$ by $\widetilde{K_k^N}$.

Proposition 6.8. *Suppose that the kernel v satisfies $(\boldsymbol{A'_n})$, for some $n \in \mathbb{N}$, and that $0 \le a \le b$. Then, for every $k \in BC^n(\mathbb{R}^2)$, $x \in BC_a^n(\mathbb{R})$ and $N \in \mathbb{N}$, the error estimate*

$$\left\|\widetilde{K_k}x - \widetilde{K_k^N}x\right\|_a \le CN^{-n}\|k\|_{BC^n(\mathbb{R}^2)}\|x\|_{BC_a^n(\mathbb{R})},$$

holds, where $C > 0$ is a constant not depending on k, x or N.

If the constant A_0 in $(\boldsymbol{A'_n})$ is not 0 then, for the (possibly weakly singular) part $\widehat{v} = v - \widetilde{v}$ of the kernel, one needs, in general, a better quadrature rule than the rectangle rule to achieve a high order of convergence (we will later, as a practical example, consider the case where $\widehat{v}(s,t)$ has a logarithmic singularity at $s = t$ and a quadrature rule based on trigonometric interpolation is used).

For the time being, we consider the following generic product integration approximation (cf. (5.7)):

$$\int_{-\infty}^{\infty} \widehat{v}(s,t)f(t)\,dt \approx \sum_{j\in\mathbb{Z}} \widehat{\omega}_j^N(s)f(t_j^N), \quad f \in X, \tag{6.17}$$

for quadrature weights $\widehat{\omega}_j^N(s)$ appropriate to the function $\widehat{v}(s,\cdot)$. Let us assume we are using the same quadrature abscissae ($t_j^N = jC/N$) as in the definition of the rectangle rule approximation. This leads to the definition of the corresponding discretized integral operator. For $k \in BC(\mathbb{R}^2)$, we define

$$\widehat{K_k^N}x(s) := \sum_{j\in\mathbb{Z}} \widehat{\omega}_j^N(s)k(s,t_j^N)x(t_j^N), \qquad s \in \mathbb{R},\, x \in X,\, N \in \mathbb{N}.$$

If $A_0 = 0$ then we will assume that $\widehat{K_k^N}$ is the zero operator.

In many cases of practical interest it is possible to choose the weights $\widehat{\omega}_j^N(s)$ to obtain a high order of convergence when the integrand f in (6.17) is smooth. Since $\widehat{v}(s,t) = 0$, for $|s-t|$ sufficiently large, it will also be the case that $\widehat{\omega}_j^N(s) = 0$ if $|s - t_j^N|$ is sufficiently large for many choices of quadrature weights. These two requirements are encapsulated in the following assumption:

Assumption $(\boldsymbol{Q_n})$. *$A_0 \neq 0$ and the quadrature weights $\widehat{\omega}_j^N(s)$ for the approximation of $\widehat{v}$ satisfy*

1. *For some $A_0^* > 0$, $|s - t_j^N| > A_0^*$ implies $\widehat{\omega}_j^N(s) = 0$.*
2. *For some $C > 0$, the following error estimate holds: for all $N \in \mathbb{N}$ and $s \in \mathbb{R}$,*

$$\left|\sum_{j\in\mathbb{Z}} \widehat{\omega}_j^N(s)x(t_j^N) - \int_{-\infty}^{\infty} \widehat{v}(s,t)x(t)\,dt\right| \leq CN^{-n}\|x\|_{BC^n(\mathbb{R})}, \quad x \in BC^n(\mathbb{R}).$$

The next lemma relates $(\boldsymbol{Q_n})$ to the assumptions made on the quadrature weights in the previous chapter. It shows that $(\boldsymbol{Q_n})$ implies $(\boldsymbol{QA''})$ – and also $(\boldsymbol{Q})$, provided that $(\boldsymbol{QA})$ is satisfied.

Lemma 6.9. *Suppose that, for some $n \in \mathbb{N}$, the kernel v satisfies $(\boldsymbol{A'_n})$. Then the following implications hold for the quadrature weights $\widehat{\omega}_j^N(s)$:*

$$(\boldsymbol{Q_n}) \implies (\boldsymbol{QA''}), \tag{6.18}$$

$$(\boldsymbol{Q_n}),\ (\boldsymbol{QA}) \implies (\boldsymbol{Q}),\ \textit{with } v = \widehat{v}. \tag{6.19}$$

Proof. Assume that $(\boldsymbol{Q_n})$ is true. The kernel bound κ in $(\boldsymbol{A'_n})$ may be chosen such that (6.11) holds. The first requirement in $(\boldsymbol{Q_n})$ makes it clear that $(\boldsymbol{QA''})$ holds for this choice of κ, $A_1 := A_0^*$ and arbitrary $C^* > 0$.

Now, we suppose that also $(\boldsymbol{QA})$ is satisfied. Let us choose an arbitrary s in $\mathbb{R}$ and denote by Ω'_s the union of the intervals $[s-(A_0+2\eta), s+(A_0+2\eta)]$ and $[s-A_0^*, s+A_0^*]$, where A_0, η and A_0^* are the constants from $(\boldsymbol{A'_n})$, (6.12) and $(\boldsymbol{Q_n})$, respectively. Then there hold $\widehat{v}(s,t) = 0$ and $\widehat{\omega}_j^N(s) = 0$ whenever $t, t_j^N \notin \Omega'_s$.

Given $x \in X$ and $\epsilon > 0$, we can choose $\hat{x} \in BC^n(\mathbb{R})$ such that $|x(t) - \hat{x}(t)| < \epsilon$ for all $t \in \Omega'_s$. (This is possible since the Weierstraß approximation theorem allows us to find a polynomial p so that $|x(t) - p(t)| < \epsilon$, $t \in \Omega'_s$; we then set $\hat{x} := pq$, where q is a $C^\infty(\mathbb{R})$ function with compact support satisfying $q(t) = 1$, $t \in \Omega'_s$.) Then:

$$\left|\sum_{j\in\mathbb{Z}} \widehat{\omega}_j^N(s)x(t_j^N) - \int_{-\infty}^{\infty} \widehat{v}(s,t)x(t)\,dt\right| = \left|\sum_{t_j^N\in\Omega'_s} \widehat{\omega}_j^N(s)x(t_j^N) - \int_{\Omega'_s} \widehat{v}(s,t)x(t)\,dt\right| \leq S_1^N + S_2^N, \tag{6.20}$$

with S_1^N, S_2^N given by

$$S_1^N := \left|\sum_{t_j^N\in\Omega'_s} \widehat{\omega}_j^N(s)(x(t_j^N) - \hat{x}(t_j^N)) - \int_{\Omega'_s} \widehat{v}(s,t)(x(t_j^N) - \hat{x}(t_j^N))\,dt\right|,$$

$$S_2^N := \left|\sum_{j\in\mathbb{Z}} \widehat{\omega}_j^N(s)\hat{x}(t_j^N) - \int_{-\infty}^{\infty} \widehat{v}(s,t)\hat{x}(t)\,dt\right|.$$

Since $(\boldsymbol{Q_n})$ is satisfied and $\hat{x} \in BC^n(\mathbb{R})$ we infer that $S_2^N < \epsilon$ for all $N \in \mathbb{N}$ large enough. Moreover, since $\|x - \hat{x}\| < \epsilon$ there holds

$$S_1^N \leq \epsilon(C_Q + \|\kappa\|_1), \qquad N \in \mathbb{N},$$

where C_Q denotes the supremum in $(\boldsymbol{QA})$. Since ϵ was arbitrary we have thus shown that the term on the left-hand side of (6.20) tends to 0 as $N \to \infty$. But this means that the quadrature weights $\widehat{\omega}_j^N(s)$ satisfy $(\boldsymbol{Q})$. □

For $s \in \mathbb{R}$, $N \in \mathbb{N}$, $j \in \mathbb{Z}$, let us now define the combined weights (if $A_0 = 0$ then we set $\widehat{\omega}_j^N(s) = 0$)

$$\omega_j^N(s) := \widehat{\omega}_j^N(s) + \widetilde{\omega}_j^N(s). \tag{6.21}$$

Then the operator K_k^N, defined for $k \in BC(\mathbb{R}^2)$ by

$$K_k^N := \widehat{K_k^N} + \widetilde{K_k^N}, \tag{6.22}$$

is a sensible approximation of the operator $K_k = \widehat{K_k} + \widetilde{K_k}$. Clearly, the operator K_k^N is the quadrature operator defined by (5.8) for the combined quadrature weights given by (6.21).

Proposition 6.10. *Suppose that the kernel v satisfies Assumption* $(\boldsymbol{A'_n})$*, for some $n \in \mathbb{N}$. If the quadrature weights $\widehat{\omega}_j^N(s)$ satisfy Assumptions* $(\boldsymbol{Q_n})$*,* $(\boldsymbol{QA})$ *and* $(\boldsymbol{QB})$ *then they also satisfy* $(\boldsymbol{Q})$ *(with $v = \widehat{v}$) and* $(\boldsymbol{QA''})$*. Moreover, for every $0 \le a \le b$ and every bounded set $W \subset BC(\mathbb{R}^2)$, the operators $\widehat{K_k^N}$, $k \in W$, are uniformly bounded on X_a.*

Proof. The first statement follows from Lemma 6.9; the kernel bound in $(\boldsymbol{QA''})$ may be chosen such that (6.11) holds. Now, the second statement is a consequence of Theorem 5.6, bearing in mind that $w_a \in \mathcal{W}(\kappa)$, $0 \le a \le b$ □

Corollary 6.11. *Suppose that the kernel v satisfies Assumption* $(\boldsymbol{A'_n})$*, for some $n \in \mathbb{N}$. If the quadrature weights $\widehat{\omega}_j^N(s)$ satisfy Assumptions* $(\boldsymbol{Q_n})$*,* $(\boldsymbol{QA})$ *and* $(\boldsymbol{QB})$ *then the combined weights $\omega_j^N(s)$, defined in* (6.21)*, satisfy* $(\boldsymbol{Q})$*,* $(\boldsymbol{QA})$*,* $(\boldsymbol{QA''})$ *and* $(\boldsymbol{QB})$*.*

Proof. We conclude from Proposition 6.5 that the quadrature weights $\widetilde{\omega}_j^N(s)$ satisfy Assumptions $(\boldsymbol{Q})$ (with $v = \widetilde{v}$), $(\boldsymbol{QA})$, $(\boldsymbol{QA''})$ and $(\boldsymbol{QB})$. Moreover, Proposition 6.10 shows that the quadrature weights $\widehat{\omega}_j^N(s)$ satisfy Assumptions $(\boldsymbol{Q})$ (with $v = \widehat{v}$), $(\boldsymbol{QA})$, $(\boldsymbol{QA''})$ and $(\boldsymbol{QB})$. It is then easily seen that the combined weights satisfy $(\boldsymbol{Q})$, $(\boldsymbol{QA})$, $(\boldsymbol{QA''})$ and $(\boldsymbol{QB})$. □

Assumption $(\boldsymbol{Q_n})$ bounds the error in the approximation of $K_k x$ by $K_k^N x$ when $x \in BC^n(\mathbb{R})$. We now show that a similar bound in the weighted norm $\|\cdot\|_a$ holds when $x \in BC_a^n(\mathbb{R})$, for some $a > 0$.

Proposition 6.12. *Suppose that $a \ge 0$, that the kernel v satisfies Assumption* $(\boldsymbol{A'_n})$*, for some $n \in \mathbb{N}$, and that the quadrature weights $\widehat{\omega}_j^N(s)$ satisfy Assumption* $(\boldsymbol{Q_n})$*. Then, for all $k \in BC^n(\mathbb{R}^2)$, $x \in BC_a^n(\mathbb{R})$ and $N \in \mathbb{N}$, there holds*

$$\big\|\widehat{K_k^N} x - \widehat{K_k} x\big\|_a \le CN^{-n}\|k\|_{BC^n(\mathbb{R}^2)}\|x\|_{BC_a^n(\mathbb{R})}.$$

for some constant $C > 0$, not depending on k, x or N.

Proof. Let A_0 and A_0^* denote the constants in $(\boldsymbol{Q_n})$ and $(\boldsymbol{A'_n})$. We set $A := \max\{A_0 + 2\eta, A_0^*\}$ and choose a "cut-off" function ϕ with the following properties

$$\phi \in BC^n(\mathbb{R}), \qquad 0 \le \phi(t) \le 1, \qquad \phi(t) = \begin{cases} 1, & |t| \le A, \\ 0, & |t| \ge A+1. \end{cases} \tag{6.23}$$

Let $k \in BC^n(\mathbb{R}^2)$ and define, for all $s \in \mathbb{R}$, the function $z_s \in X$ by $z_s(t) := \phi(s-t)k(s,t)$, $t \in \mathbb{R}$. Then $z_s(t) = 0$ whenever $|s-t| > A+1$; moreover, for every $x \in BC_a^n(\mathbb{R})$, $s \in \mathbb{R}$, $N \in \mathbb{N}$, we have

$$|\widehat{K_k^N} x(s) - \widehat{K_k} x(s)| = \Big|\sum_{j\in\mathbb{Z}} \widehat{\omega}_j^N(s) z_s(t_j^N) x(t_j^N) - \int_{-\infty}^{\infty} \widehat{v}(s,t) z_s(t) x(t)\, dt\Big| \le CN^{-n}\|z_s x\|_{BC^n(\mathbb{R})},$$

where the equality holds since $\widehat{\omega}_j^N(s) = 0$, for $|s - t_j^N| > A$, and $\widehat{v}(s,t) = 0$, for $|s-t| > A$; the constant $C > 0$ on the right-hand side is the constant in $(\boldsymbol{Q_n})$. By Lemma 6.1 there holds, for all $s \in \mathbb{R}$,

$$\|z_s x\|_{BC^n(\mathbb{R})} \overset{(6.3)}{\le} 2^n\|k(s,\cdot)\|_{BC^n(\mathbb{R})}\|\phi(s-\cdot)x\|_{BC^n(\mathbb{R})} \overset{(6.4)}{\le} C'(1+|s|)^{-a}\|k\|_{BC^n(\mathbb{R}^2)}\|x\|_{BC_a^n(\mathbb{R}^2)}.$$

with $C' := 2^{2n}\|\phi\|_{BC^n(\mathbb{R})}\Delta_a^{A+1}$. Combining the previous two inequalities and taking the supremum over $s \in \mathbb{R}$ completes the proof. □

Let us stop here for a moment and summarize what we have done so far.

1. We started with a kernel v satisfies Assumption $(\boldsymbol{A'_n})$, for some $n \in \mathbb{N}$, $b > 1$ and $\kappa \in L^1(\mathbb{R})$ satisfying (6.11); we split the corresponding integral operator $K_k = \widetilde{K_k} + \widehat{K_k}$ into a "nice" operator $\widetilde{K_k}$ and a "singular" operator $\widehat{K_k}$.

2. We introduced, in (6.15), the rectangle rule quadrature weights $\widetilde{\omega}_j^N(s)$ and *assumed* that we have found a set of quadrature weights $\widehat{\omega}_j^N(s)$ satisfying $(\boldsymbol{Q_n})$, $(\boldsymbol{QA})$ and $(\boldsymbol{QB})$.

3. We have shown the boundedness on X_a, $0 \le a \le b$, of the corresponding quadrature operators $\widetilde{K_k^N}$,$\widehat{K_k^N}$ and obtained error bounds in X_a for the approximation of $\widetilde{K_k}$ by $\widetilde{K_k^N}$ and of $\widehat{K_k}$ by $\widehat{K_k^N}$.

Under these assumptions, by virtue of Corollary 6.11, we see that the combined weights $\omega_j^N(s)$, defined in (6.21), satisfy $(\boldsymbol{Q})$, $(\boldsymbol{QA})$, $(\boldsymbol{QA''})$, and $(\boldsymbol{QB})$. Moreover, for every $0 \le a \le b$ and every bounded set $W \subset BC(\mathbb{R}^2)$, the combined operators K_k^N, $k \in W$, defined in (6.22), are uniformly bounded on X_a. Propositions 6.8 and 6.12 yield an error estimate for the approximation of K_k by K_k^N. Precisely, for $x \in BC_a^n(\mathbb{R})$, $0 \le a \le b$, $k \in BC^n(\mathbb{R})$, there holds

$$\|K_k x - K_k^N x\|_a \le \|\widehat{K_k^N} x - \widehat{K_k} x\|_a + \|\widetilde{K_k^N} x - \widetilde{K_k} x\|_a \le CN^{-n}\|k\|_{BC^n(\mathbb{R}^2)}\|x\|_{BC_a^n(\mathbb{R})}, \tag{6.24}$$

for some constant $C > 0$ not depending on k, N or x. We now insert this bound into the outcome of Theorem 5.14 and obtain the following theorem, which is our second major result in this section.

Theorem 6.13. *Suppose that the kernel v of the integral equation* (5.1) *satisfies* $(\boldsymbol{A'_n})$*, for some $n \in \mathbb{N}$. Assume, further, that the quadrature weights $\widehat{\omega}_j^N(s)$ satisfy Assumptions* $(\boldsymbol{Q_n})$*,* $(\boldsymbol{QA})$ *and* $(\boldsymbol{QB})$*. Moreover, assume that $W \subset BC^n(\mathbb{R}^2)$ is bounded, that $\mathbb{N}' \subset \mathbb{N}$ is unbounded, and that, for every $k \in W$ and $N \in \mathbb{N}'$, $(\lambda - K_k^N)^{-1} \in \mathcal{B}(X)$ and*

$$\sup_{N\in\mathbb{N}'}\sup_{k\in W}\|(\lambda - K_k^N)^{-1}\| < \infty. \tag{6.25}$$

Let $0 \le a \le b$, $N \in \mathbb{N}'$ and $k \in W$. Then, the equations

$$\lambda x - K_k x = y \quad \text{and} \quad \lambda x^N - K_k^N x^N = y \tag{6.26}$$

both have unique solutions $x \in X_a$ and $x^N \in X_a$ for every $y \in X_a$. Moreover, if $y \in X_a$ is such that the solution x is in $BC_a^n(\mathbb{R})$, then

$$\|x - x^N\|_a \le CN^{-n}\|x\|_{BC_a^n(\mathbb{R})}, \quad N \in \mathbb{N}', \tag{6.27}$$

where $C > 0$ is a constant that does not depend on y, k or N.

Remark 6.14. *In the special case when $A_0 = 0$, the assumptions about the quadrature weights $\widehat{\omega}_j^N(s)$ may be dropped. A sufficient condition for* (6.25) *to hold, for $\mathbb{N}' := \{N \in \mathbb{N} : N \ge N'\}$, for some $N' \in \mathbb{N}$, is then that the inverses $(\lambda - K_k)^{-1}$, $k \in W$, exist and are uniformly bounded (see Theorem 6.6).*

Proof. The proof proceeds in a number of steps.

(i) Suppose that the assumptions in the first part of the proposition are satisfied. We firstly show that then the assumptions of Theorem 5.13 are fulfilled: a) v satisfies $(\boldsymbol{A''})$, for some $\kappa \in L^1(\mathbb{R})$ satisfying (6.11), and also $(\boldsymbol{B})$. Then $w_a \in \mathcal{W}(\kappa)$, $0 \le a \le b$. b) Proposition 6.5 ($A_0 = 0$) or Corollary 6.11 ($A_0 > 0$) shows that the combined quadrature weights $\omega_j^N(s)$ satisfy $(\boldsymbol{Q})$, $(\boldsymbol{QA})$, $(\boldsymbol{QA''})$ and $(\boldsymbol{QB})$. c) If W is a bounded subset of $BC^n(\mathbb{R}^2)$, $n \ge 1$, then W is equicontinuous and bounded in $BC(\mathbb{R}^2)$. d) Inequality (5.39) is the same as (6.25).

(ii) We have shown in (i) that Theorem 5.13 applies; hence, for all $k \in W$, the inverses $(\lambda - K_k)^{-1}$ and $(\lambda - K_k^N)^{-1}$, $N \in \mathbb{N}'$, exist on X_a and are uniformly bounded. Thus the statement about the solvability of the equations (6.26) is true.

(iii) Next, suppose that $y \in X_a$ and x, x^N are as in the second part of the theorem. Since $\mathbb{N}'$ is unbounded we note that Theorem 5.14 applies and gives

$$\|x - x^N\|_a \le C' \|(K_k - K_k^N)x\|_a,$$

where $C' > 0$ is a positive constant not depending on k, y or N. Thus, if we additionally suppose that $x \in BC_a^n(\mathbb{R})$, inequality (6.24) shows that

$$\|(K_k - K_k^N)x\|_a \le C'' N^{-n} \|k\|_{BC^n(\mathbb{R}^2)} \|x\|_{BC_a^n(\mathbb{R})},$$

where $C'' > 0$ is a positive constant not depending on k, y or N. We combine the previous two inequalities and, bearing in mind that W is bounded in $BC^n(\mathbb{R}^2)$, obtain that (6.27) holds. □

6.2.1 Regularity

The previous theorem gives us an idea of the convergence order of x^N to x, provided that we know that $x \in BC_a^n(\mathbb{R})$. If the kernel v is sufficiently well-behaved then the associated operators K_k, $k \in BC^n(\mathbb{R}^2)$ have certain *regularity* properties and we may deduce that the solution x is in $BC_a^n(\mathbb{R})$ if we know that the right-hand side y is in $BC_a^n(\mathbb{R})$. For $n \in \mathbb{N}$ and $a \ge 0$, we introduce the following regularity assumption.

Assumption ($\boldsymbol{R_n^a}$). *There exists $m \in \mathbb{N}$ and a constant $C > 0$ such that for all $k \in BC^n(\mathbb{R}^2)$ $(K_k)^m \in \mathcal{B}(X_a, BC_a^n(\mathbb{R}))$ and $K_k \in \mathcal{B}(BC_a^n(\mathbb{R}))$ and moreover*

$$\|(K_k)^m\|_{X_a \to BC_a^n(\mathbb{R})} \le C(\|k\|_{BC^n(\mathbb{R}^2)})^m \quad \text{and} \quad \|K_k\|_{BC_a^n(\mathbb{R}) \to BC_a^n(\mathbb{R})} \le C\|k\|_{BC^n(\mathbb{R}^2)}.$$

The following proposition shows that if ($\boldsymbol{A_n'}$) holds then the smooth part $\widetilde{v}$ of the kernel v automatically satisfies Assumption ($\boldsymbol{R_n^a}$), for every $0 \le a \le b$.

Proposition 6.15 (cf. [16, Th. 12]). *Suppose that the kernel v satisfies ($\boldsymbol{A_n'}$) and that $0 \le a \le b$. Then there exists a constant $C > 0$ such that, for every $k \in BC^n(\mathbb{R}^2)$, $\widetilde{K_k} : X_a \to BC_a^n(\mathbb{R})$ and is bounded with norm*

$$\|\widetilde{K_k}\|_{X_a \to BC_a^n(\mathbb{R})} \le C\|k\|_{BC^n(\mathbb{R}^2)}, \qquad \|\widetilde{K_k}\|_{BC_a^n(\mathbb{R}) \to BC_a^n(\mathbb{R})} \le C\|k\|_{BC^n(\mathbb{R}^2)}, \tag{6.28}$$

i.e. $\widetilde{v}$ satisfies ($\boldsymbol{R_n^a}$), with $m = 1$.

Proof. Choose $0 \le a \le b$ and $k \in BC^n(\mathbb{R}^2)$. We firstly show that, for every $x \in X_a$, the function $\widetilde{K_k}x$ is n times differentiable with j-th derivative given by

$$\big(\widetilde{K_k}x\big)^{(j)}(s) = \int_{-\infty}^{\infty} \frac{\partial^j}{\partial s^j}\big(\widetilde{v}(s,t)k(s,t)\big)x(t)\,dt, \quad s \in \mathbb{R}. \tag{6.29}$$

To do so, we first note that ($\boldsymbol{A_n'}$) implies that $\widetilde{v} \in BC_b^n(\mathbb{R}^2)$ and hence $\widetilde{v}k \in BC_b^n(\mathbb{R}^2)$ (Lemma 6.1). Next, we see that, for $s, t \in \mathbb{R}$, $h \in [-1, 1]$ and $0 \le j \le n$,

$$\Big|\big(\partial_1^j(\widetilde{v}k)\big)(s+h,t)\Big| \le \|\widetilde{v}k\|_{BC_b^n(\mathbb{R}^2)}(1 + |(s+h) - t|)^{-b} \le \|\widetilde{v}k\|_{BC_b^n(\mathbb{R}^2)}\Delta_b^1 C_n(1 + |s-t|)^{-b}.$$

Given $s \in \mathbb{R}$, we thus see that the partial derivatives of $\widetilde{v}k$ with respect to the first variable exist up to order n on the interval $[s-1, s+1]$. Moreover, all these derivatives, as functions of t only, are majorized by the integrable function $t \to \|\widetilde{v}k\|_{BC_b^n(\mathbb{R}^2)}\Delta_b^1 C_n(1 + |s-t|)^{-b}$. Repeated application of [37, Th. 128.2] now shows that, for our chosen s, we may exchange differentiation and integration as in (6.29). Since

$s \in \mathbb{R}$ was arbitrary, (6.29) holds. Hence, we obtain the bound

$$\begin{aligned}
\left\|\big(\widetilde{K_k}x\big)^{(j)}\right\|_a &= \sup_{s\in\mathbb{R}}\left|(1+|s|)^a\int_{-\infty}^{\infty}\frac{\partial^j}{\partial s^j}\big(\widetilde{v}(s,t)k(s,t)x(t)\big)\,dt\right| \\
&\le \|x\|_a\sup_{s\in\mathbb{R}}\int_{-\infty}^{\infty}\left(\frac{1+|s|}{1+|t|}\right)^a\left|\frac{\partial^j}{\partial s^j}\big(\widetilde{v}(s,t)k(s,t)\big)\right|\,dt \\
&\le \|x\|_a\|\widetilde{v}k\|_{BC_b^n(\mathbb{R}^2)}\sup_{s\in\mathbb{R}}\int_{-\infty}^{\infty}\left(\frac{1+|s|}{1+|t|}\right)^a(1+|s-t|)^{-b}\,dt \\
&\overset{(6.5)}{\le} 2^n\|\widetilde{v}\|_{BC_b^n(\mathbb{R}^2)}\|k\|_{BC^n(\mathbb{R}^2)}\|x\|_a\Theta_{a,b}.
\end{aligned}$$

From these $(n+1)$ inequalities we deduce that there exists $C > 0$ so that the first inequality in (6.28) holds. But so does the second, since $\|x\|_{X_a} \le \|x\|_{BC_a^n(\mathbb{R})}$ for all $x \in BC_a^n(\mathbb{R})$. □

This preceding proposition tells us that, if v satisfies $\boldsymbol{(A_n')}$ with constant $A_0 = 0$ (so that $K_k = \widetilde{K_k}$) then v automatically satisfies $\boldsymbol{(R_n^a)}$, $0 \le a \le b$. If $A_0 \ne 0$ then we cannot repeat the argument for $\widehat{v}(s,\cdot)$ might have (integrable) singularities near $t = s$ and thus, in general, an equivalent of equation (6.29) does not hold.

The next proposition, in combination with the previous proposition, shows that, in practice, one would have to verify that the non-smooth part $\widehat{v}$ of the kernel v satisfies $\boldsymbol{(R_n^0)}$ to obtain that v satisfies $\boldsymbol{(R_n^a)}$, for $0 < a \le b$.

Proposition 6.16. *Let v be a kernel satisfying $\boldsymbol{(A_n')}$ and assume that $\widehat{v}$ satisfies $\boldsymbol{(R_n^0)}$, for some $m \in \mathbb{N}$. Then, for $0 < a \le b$, $\widehat{v}$ also satisfies $\boldsymbol{(R_n^a)}$, i.e.*

$$\|(\widehat{K_k})^m\|_{X_a\to BC_a^n(\mathbb{R})} < \widehat{C}(\|k\|_{BC^n(\mathbb{R}^2)})^m, \quad \|\widehat{K_k}\|_{BC_a^n(\mathbb{R})\to BC_a^n(\mathbb{R})} < \widehat{C}\|k\|_{BC^n(\mathbb{R}^2)}, \tag{6.30}$$

where $\widehat{C} > 0$ is a constant not depending on $k \in BC^n(\mathbb{R}^2)$.

Proof. We begin with the following observation: Let $x \in X$. Then the values of $\widehat{K_k}x(s)$ only depend on the values of x on the interval $[s-(A_0+2\eta), s+A_0+2\eta]$, with A_0 and η the constants in $\boldsymbol{(A_n')}$ and (6.12). Let $\delta > 0$. By induction, it follows that the values of $\big((\widehat{K_k})^m x\big)(s+h)$, for $|h| < \delta$, only depend on the values of $x(t)$ on an interval of the form $[s-A, s+A]$, where $A := (A_0+2\eta+\delta)^m$.

For this choice of A, we now choose ϕ as in (6.23) and define, for every $s \in \mathbb{R}$, the function z_s by $z_s(t) := \phi(t-s)$, $t \in \mathbb{R}$. By the above remarks, there holds $\big((\widehat{K_k})^m x\big)(s+h) = \big((\widehat{K_k})^m(xz_s)\big)(s+h)$ for $|h| < \delta$, $s \in \mathbb{R}$. Thus we obtain, for $0 \le j \le n$ and $x \in X_a$, the inequality

$$\left|\big((\widehat{K_k})^m x\big)^{(j)}(s)\right| = \left|\big((\widehat{K_k})^m(xz_s)\big)^{(j)}(s)\right| \le \|\widehat{K_k}\|_{X\to BC^n(\mathbb{R})}\|z_sx\| \le C(\|k\|_{BC^n(\mathbb{R})})^m\|z_sx\|$$

where $C > 0$ is the positive constant in $\boldsymbol{(R_n^0)}$. But, by (6.4), $\|z_sx\| \le \Delta_a^A(1+|s|)^{-a}\|x\|_a\|\phi\|$. Combining these inequalities and taking the supremum, we see that

$$\|(\widehat{K_k})^m\|_{X_a\to BC_a^n(\mathbb{R})} < C'(\|k\|_{BC^n(\mathbb{R}^2)})^m, \tag{6.31}$$

for some constant $C' > 0$ not depending on $k \in BC^n(\mathbb{R}^2)$.

A similar argument, using the inequality

$$\|z_sx\|_{BC^n(\mathbb{R})} \overset{(6.4)}{\le} 2^n\Delta_a^A(1+|s|)^{-a}\|x\|_{BC_a^n(\mathbb{R})}\|z_s\|_{BC^n(\mathbb{R})}, \qquad x \in BC_a^n(\mathbb{R}),\ s \in \mathbb{R},$$

proves that, for some $C'' > 0$, also

$$\|\widehat{K_k}\|_{BC_a^n(\mathbb{R})\to BC_a^n(\mathbb{R})} < C''\|k\|_{BC^n(\mathbb{R}^2)}, \tag{6.32}$$

holds. Together, (6.31) and (6.32) show that (6.30) hold thus that $\widehat{v}$ satisfies Assumption $\boldsymbol{(R_n^a)}$. □

In those cases when the assumptions of Theorem 6.13 are satisfied and, in addition, $\widehat{v}$ satisfies $(\boldsymbol{R_n^0})$ then we can improve the error bounds in Theorem 6.13, provided that the right-hand side y of (5.1) is contained in $BC_a^n(\mathbb{R})$.

Theorem 6.17. *Suppose that the assumptions of Theorem 6.13 are satisfied and that* $0 \le a \le b$. *Then, for every* $N \in \mathbb{N}'$ *and* $y \in X_a$, *the equations*

$$\lambda x - K_k x = y \quad \text{ and } \quad \lambda x^N - K_k^N x^N = y \tag{6.33}$$

both have unique solutions $x \in X_a$ *and* $x^N \in X_a$ *for every* $N \in \mathbb{N}'$, $k \in W$.

If, additionally, the constant A_0 *in* $(\boldsymbol{A_n'})$ *is* 0 *or the kernel* $\widehat{v}$ *satisfies the regularity assumption* $(\boldsymbol{R_n^0})$ *then, for every* $y \in BC_a^n(\mathbb{R})$, *the following error bound holds for the solutions* x *and* x^N *of* (6.33):

$$\|x - x^N\|_a \le CN^{-n}\|y\|_{BC_a^n(\mathbb{R})}. \tag{6.34}$$

Here, $C > 0$ *is a constant not depending on* y, k *or* N.

Proof. Suppose that the assumptions of the first part of the theorem are satisfied. Then (see the proof of Theorem 6.13), for every $k \in W$, the operators $\lambda - K_k$ and $\lambda - K_k^N$, $N \in \mathbb{N}'$, are invertible on X_a and, moreover, there holds

$$C_a := \sup_{k \in W} \|(\lambda - K_k)^{-1}\|_a < \infty. \tag{6.35}$$

Also, it is shown that, when the $y \in X_a$ is such that $x := (\lambda - K_k)^{-1}y$ is contained in $BC_a^n(\mathbb{R})$ then

$$\|x - x^N\|_a \le C_0 N^{-n}\|x\|_{BC_a^n(\mathbb{R})}, \tag{6.36}$$

holds for the unique solutions x, x^N of (6.33). Here, $C_0 > 0$ is a constant not depending on k, N or y.

If $A_0 \ne 0$ suppose that the kernel $\widehat{v}$ additionally satisfies the regularity assumption $(\boldsymbol{R_n^0})$, for some $m \in N$. We set $C_W := \sup\{\|k\|_{BC^n(\mathbb{R}^2)} : k \in W\}$. Then, the previous proposition and Proposition 6.15 shows that, for all $k \in W$,

$$\begin{aligned} &\|(\widehat{K_k})^m\|_{X_a \to BC_a^n(\mathbb{R})} \le C_1 {C_W}^m, && \|\widehat{K_k}\|_{BC_a^n(\mathbb{R}) \to BC_a^n(\mathbb{R})} \le C_1 C_W, \\ &\|(\widetilde{K_k})^m\|_{X_a \to BC_a^n(\mathbb{R})} \le C_2 {C_W}^m, && \|\widetilde{K_k}\|_{BC_a^n(\mathbb{R}) \to BC_a^n(\mathbb{R})} \le C_2 C_W, \end{aligned}$$

for some constants $C_1, C_2 > 0$ (if $A_0 = 0$ then the inequalities involving $\widehat{K_k} = 0$ are trivial). Setting $\widehat{C} := C_1 + C_2$, we conclude from (6.22) that there holds

$$\|(K_k)^m\|_{X_a \to BC_a^n(\mathbb{R})} \le \widehat{C}{C_W}^m, \qquad \|K_k\|_{BC_a^n(\mathbb{R}) \to BC_a^n(\mathbb{R})} \le \widehat{C} C_W, \qquad k \in W. \tag{6.37}$$

Now suppose that $y \in BC_a^n(\mathbb{R})$, $k \in W$ and that x, x^N are the unique solutions of (6.33). Then $\lambda x - K_k x = y$. Let us define $H_k := \lambda^{-1} K_k$. Then $x = \lambda^{-1} y + H_k x$. We repeatedly insert this equation into itself and obtain

$$\begin{aligned} x &= \lambda^{-1}\big(I + H_k + \cdots + (H_k)^{m-1}\big)y + (H_k)^m x \\ &= \lambda^{-1}\big(I + H_k + \cdots + (H_k)^{m-1}\big)y + (H_k)^m (\lambda - K_k)^{-1} y. \end{aligned} \tag{6.38}$$

In view of (6.37), the operators $(I + H_k + \cdots + (H_k)^{m-1})$, $k \in W$, are uniformly bounded in $\mathcal{B}(BC_a^n(\mathbb{R}))$. From (6.37) and (6.35), we get the information that the operators $(H_k)^m (I - K_k)^{-1}$, $k \in W$, are also uniformly bounded in $\mathcal{B}(BC_a^n(\mathbb{R}))$. Thus (6.38) shows that

$$\|x\|_{BC_a^n(\mathbb{R})} \le C^* \|y\|_{BC_a^n(\mathbb{R})},$$

where $C^* > 0$ is a positive constant that does not depend on y on $k \in W$. If we combine this inequality with (6.36) we obtain (6.34) and the theorem is shown. □

6.2.2 Kernels with logarithmic singularities

In this section we investigate the integral equation

$$x(s) - \int_{-\infty}^{\infty} l(s,t)x(t)\,dt = y(t), \quad s \in \mathbb{R}, \tag{6.39}$$

for a class of kernels l with logarithmic singularities at $s = t$. Precisely, we will consider kernels l that satisfy the following assumption, for some $n \in \mathbb{N}$ and $b > 1$ kept fixed throughout most of this section.

Assumption (L_n). *The kernel l satisfies $l(s,t) = a^*(s,t)\ln|s-t| + b^*(s,t)$, where $a^*, b^* \in C^n(\mathbb{R}^2)$ and there exist constants $C > 0$ and $b > 1$ such that for all $i, j \in \mathbb{N}_0$ with $i + j \le n$*

$$\left|\frac{\partial^{i+j} a^*(s,t)}{\partial s^i \partial t^j}\right|, \left|\frac{\partial^{i+j} b^*(s,t)}{\partial s^i \partial t^j}\right| \le C, \qquad s,t \in \mathbb{R}, |s-t| \le \pi, \tag{6.40}$$

and

$$\left|\frac{\partial^{i+j} l(s,t)}{\partial s^i \partial t^j}\right| \le \frac{C}{(1+|s-t|)^b}, \qquad s,t \in \mathbb{R}, |s-t| \ge \pi. \tag{6.41}$$

Let us denote by L_l the integral operator on X, defined by

$$L_l x(s) := \int_{-\infty}^{\infty} l(s,t)x(t)\,dt, \qquad s \in \mathbb{R}, x \in X. \tag{6.42}$$

We will see soon that this operator is a bounded operator on X if l satisfies (L_n).

For $\gamma > 0$, we denote by Λ_γ the collection of all kernels l satisfying (L_n) with $C = \gamma$ and by Λ'_γ its subset $\Lambda'_\gamma := \{l \in \Lambda_\gamma : I - L_l$ is invertible on X and $\|(I - L_l)^{-1}\| < \gamma\}$.

Kernels of the form above arise in numerous practical applications. In particular, in two dimensions, the fundamental solutions to elliptic partial differential equations, such as the Laplace and Helmholtz equation and the equations of linear elasticity, contain logarithmic singularities, and the reformulation of the associated boundary value problems as boundary integral equations leads to logarithmic singularities in the kernel of the resulting integral operators (see, e.g. the discussion [44]). We will consider one specific boundary value problem arising in rough surface scattering in the last section of this chapter.

Kernels satisfying (L_n) and the corresponding integral equations (6.39) have been considered by Meier et al. in [50]. In [50] certain Nyström/product integration methods for the numerical solution of (6.39) are suggested and stability and convergence results are established in the unweighted spaces X and $BC^n(\mathbb{R}^2)$. One aim of the following discussion is to make use of our weighted space theory to generalise the results of [50] to the weighted spaces X_a and $BC^n_a(\mathbb{R})$, $0 < a \le b$.

To formulate a Nyström method when l satisfies (L_n), we need a "cut-off"-function χ to split the kernel into a singular and a smooth part. We choose χ so that it has the following properties:

$$\chi \in C^\infty(\mathbb{R}), \quad 0 \le \chi(t) \le 1, \quad \chi(t) = \begin{cases} 1, & |t| \le 1, \\ 0, & |t| \ge \pi - \epsilon_0, \end{cases} \tag{6.43}$$

where $\epsilon_0 > 0$ is a small fixed constant. We set $\chi^*(s,t) := \chi(s-t)$, and note that $\chi^* \in BC^n(\mathbb{R}^2)$ for every $n \in \mathbb{N}$. We keep these two functions fixed throughout the remainder of this section.

Now, we define two kernels v_1 and v_2 by

$$v_1(s,t) := \frac{\chi(s-t)}{2\pi} \ln\left(4\sin^2\left(\frac{s-t}{2}\right)\right), \quad v_2(s,t) := (1+(s-t)^2)^{-b/2}, \quad s \ne t, \tag{6.44}$$

and assume that v_1 and v_2 are given by this definition throughout the rest of the thesis. We note that both $v_1(s,t)$ and $v_2(s,t)$ depend only on the difference $s - t$ of the arguments, i.e. v_1 and v_2 are convolution kernels.

Proposition 6.18. *Suppose that l is a kernel satisfying Assumption $(\boldsymbol{L_n})$, for some $n \in \mathbb{N}$, $b > 1$ and $C > 0$. Then there exists $\mathbf{k}_l = (k_1, k_2) \in (BC^n(\mathbb{R}^2))^2$ such that*

$$l(s,t) = k_1(s,t)v_1(s,t) + k_2(s,t)v_2(s,t), \qquad s,t \in \mathbb{R}, s \neq t, \tag{6.45}$$

$$\|k_1\|_{BC^n(\mathbb{R}^2)}, \|k_2\|_{BC^n(\mathbb{R}^2)} < C'C, \tag{6.46}$$

where $C' > 1$ is a constant depending only on χ, b and n.

Proof. Suppose that we are given a kernel l is as in the assumption, i.e.

$$l(s,t) = a^*(s,t) \ln |s-t| + b^*(s,t), \qquad s,t \in \mathbb{R}, s \neq t,$$

for some functions a^* and b^* as described in $(\boldsymbol{L_n})$. It follows from the proof of Theorem 2.1 in [50], in particular equations (2.6) and (2.7) that l may be written as

$$l(s,t) = l_1(s,t)\frac{1}{2\pi} \ln\left(4\sin^2\Big(\frac{s-t}{2}\Big)\right) + l_2(s,t), \qquad s,t \in \mathbb{R}, s \neq t, \tag{6.47}$$

where $l_1 \in BC^n(\mathbb{R}^2)$, $l_2 \in BC^n_b(\mathbb{R}^2)$ are such that

$$l_1(s,t) = \pi a^*(s,t)\chi^*(s,t),\ s,t \in \mathbb{R} \qquad \text{and} \qquad \|l_2\|_{BC^n_b(\mathbb{R}^2)} \leq C_1 C, \tag{6.48}$$

where $C_1 > 0$ is a constant depending only on χ, b and n and C is the constant in $(\boldsymbol{L_n})$.

Let $\widetilde{\chi}$ be a second "cut-off"-function with

$$\widetilde{\chi} \in C^\infty(\mathbb{R}), \quad 0 \leq \widetilde{\chi}(t) \leq 1, \quad \widetilde{\chi}(t) = \begin{cases} 1, & |t| \leq \pi - \epsilon_0, \\ 0, & |t| \geq \pi, \end{cases}$$

where ϵ_0 is the constant in (6.43). We set $\widetilde{\chi}^*(s,t) := \widetilde{\chi}(s-t)$, notice that $\widetilde{\chi}^* \in BC^n(\mathbb{R}^2)$ and introduce the functions k_1 and k_2 by setting

$$k_1(s,t) := \pi a^*(s,t)\widetilde{\chi}^*(s,t), \quad k_2(s,t) := l_2(s,t)v_2(s,t)^{-1}, \qquad s,t \in \mathbb{R}.$$

These functions satisfy (6.45), because, in view of the equality $\chi^*\widetilde{\chi}^* = \chi^*$, there holds, for all $s,t \in \mathbb{R}$,

$$\begin{aligned} l(s,t) &\overset{(6.47)}{=} \pi a^*(s,t)\chi(s-t)\frac{1}{2\pi}\ln\left(4\sin^2\Big(\frac{s-t}{2}\Big)\right) + l_2(s,t) \\ &= k_1(s,t)\chi(s-t)\frac{1}{2\pi}\ln\left(4\sin^2\Big(\frac{s-t}{2}\Big)\right) + k_2(s,t)v_2(s,t) \\ &= k_1(s,t)v_1(s,t) + k_2(s,t)v_2(s,t). \end{aligned}$$

Moreover, from the definition of $\widetilde{\chi}^*$ and (6.40), we deduce

$$\|k_1\|_{BC^n(\mathbb{R}^2)} = \pi\|a^*\widetilde{\chi}^*\|_{BC^n(\mathbb{R}^2)} \overset{(6.5)}{\leq} 2^n\pi\|\widetilde{\chi}^*\|_{BC^n(\mathbb{R}^2)}\|a^*\|_{BC^n(\mathbb{R}^2)} \leq 2^n\pi\|\widetilde{\chi}^*\|_{BC^n(\mathbb{R}^2)}C. \tag{6.49}$$

Moreover, for all $i,j \in \mathbb{N}_0$ with $i+j \leq n$, there holds

$$|\partial_1^i\partial_2^j(v_2(s,t)^{-1})| \leq C_2(1+|s-t|)^b, \quad s,t \in \mathbb{R},$$

where C_2 is a positive constant depending only on n and b. From this bound one obtains, by (6.48) and the Leibniz product rule, that $k_2 = l_2 {v_2}^{-1} \in BC^n(\mathbb{R}^2)$; moreover,

$$\|k_2\|_{BC^n(\mathbb{R}^2)} \leq 2^n C_2\|l_2\|_{BC^n_b(\mathbb{R}^2)} \leq 2^n C_1 C_2 C.$$

Together with (6.49) this proves (6.46) when we set $C' := 2^n \max\{C_1C_2, \pi\|\widetilde{\chi}^*\|_{BC^n(\mathbb{R}^2)}\}$. □

Corollary 6.19. *Let $\gamma > 0$. Then there exist two bounded subsets $W_1, W_2 \subset BC^n(\mathbb{R}^2)$ with the following properties: Every $l \in \Lambda_\gamma$ may be written in the form (6.46) with $k_1 \in W_1$, $k_2 \in W_2$, and the sets W_1 and W_2 are uniformly equicontinuous and bounded in $BC(\mathbb{R}^2)$.*

Proof. Both statements are an immediate consequence of the previous proposition, the second holding since every bounded subset of $BC^n(\mathbb{R}^2)$ is uniformly equicontinuous and bounded in $BC(\mathbb{R}^2)$. □

Now, if a kernel l satisfies $\boldsymbol{(L_n)}$ then, we can always choose k_1, k_2 as in Proposition 6.18. In analogy to the definition of the operator K_k in (3.2), we then define the integral operators K_{1,k_1} and K_{2,k_2} by replacing $v(s,t)k(s,t)$ in (3.2) by $v_1(s,t)k_1(s,t)$ and $v_2(s,t)k_2(s,t)$, respectively. Every l satisfying $\boldsymbol{(L_n)}$ we thus associate, here and in the rest of the section, with the operator

$$K_{\mathbf{k}_l} := K_{1,k_1} + K_{2,k_2}, \quad \mathbf{k}_l = (k_1, k_2),$$

which, by (6.45), is the integral operator L_l, i.e.

$$K_{\mathbf{k}_l} x(s) = L_l x(s) = \int_{-\infty}^{\infty} l(s,t)x(t)\,dt, \quad s \in \mathbb{R},\ x \in X. \tag{6.50}$$

Proposition 6.20. *The kernels v_1 and v_2, defined in (6.44), both satisfy $\boldsymbol{(A'_n)}$ and $\boldsymbol{(A'')}$, for some $\kappa \in L^1(\mathbb{R})$ (the same in each occurrence). If $\gamma > 0$, $0 \le a \le b$ and $l \in \Lambda_\gamma$ then the integral operators $K_{\mathbf{k}_l}$, K_{1,k_1}, K_{2,k_2}, are bounded on X_a (with $\mathbf{k}_l = (k_1, k_2)$ defined as above). Moreover, the following three quantities are finite*

$$\sup_{l \in \Lambda_\gamma} \|K_{1,k_1}\|_a, \quad \sup_{l \in \Lambda_\gamma} \|K_{2,k_2}\|_a, \quad \sup_{l \in \Lambda_\gamma} \|K_{\mathbf{k}_l}\|_a.$$

Proof. Let $\kappa \in L^1(\mathbb{R})$ be defined by

$$\kappa(t) := 2^{b/2}(1+|t|)^{-b} + \begin{cases} 2\big|\ln(|t|/2)\big|, & 0 < |t| < \pi/3, \\ 2, & \pi/3 \le |t| \le \pi, \\ 0, & |t| > \pi. \end{cases} \tag{6.51}$$

We firstly note that (since $0 \le t/2 \le \sin t \le 1/2$, for $0 \le t \le \pi/6$) there holds

$$\left|\ln\left(4\sin^2\Big(\frac{t}{2}\Big)\right)\right| = 2\left|\ln\left(2\sin\Big(\frac{|t|}{2}\Big)\right)\right| \le \begin{cases} 2\ln(|t|/2), & 0 < |t| < \pi/3, \\ 2, & \pi/3 \le |t| \le \pi. \end{cases}$$

Thus, we see that $|v_1(s,t)| \le \kappa(s-t)$, $s,t \in \mathbb{R}$.

Using the inequality $(1+|t|^2)^{-1} \le 2(1+|t|)^{-2}$, $t \in \mathbb{R}$, we observe that

$$v_2(s,t) = (1+(s-t)^2)^{-b/2} = (1+|s-t|^2)^{-b/2} \le 2^{b/2}(1+|s-t|)^{-b}, \qquad s,t \in \mathbb{R},$$

so that $v_2(s,t) \le \kappa(s-t)$, $s,t \in \mathbb{R}$. Since $\kappa(t)$ is monotonic outside the interval $[-\pi,\pi]$ it follows that both v_1 and v_2 satisfy $\boldsymbol{(A'')}$ and $\boldsymbol{(A')}$. As convolution kernels, v_1 and v_2 also satisfy $\boldsymbol{(A)}$ and $\boldsymbol{(B)}$.

Moreover, v_1 also satisfies $\boldsymbol{(A'_n)}$ because $v_1(s,t) = 0$, for $|s-t| \ge \pi$, so that (6.1) holds for $A_0 := \pi$. By straightforward calculations, we also see that all partial derivatives $\partial_1^i \partial_2^j v_2$, with $i,j \in \mathbb{N}_0$, $i+j \le n$, satisfy the inequality

$$|\partial_1^i \partial_2^j v_2(s,t)| \le C_n(1+|s-t|)^{-b}, \qquad s,t \in \mathbb{R},$$

for some constant $C_n > 0$. Hence, the kernel v_2 also satisfies (6.1), with $A_0 = \pi$, and thus $\boldsymbol{(A'_n)}$.

Since they satisfy Assumptions $\boldsymbol{(A'_n)}$ the kernels v_1 and v_2 both satisfy the assumptions of Proposition 6.4. Given $0 \le a \le b$ and $\gamma > 0$, we thus obtain constants $C_1, C_2 > 0$ so that, for every $l \in \Lambda_\gamma$ and $\mathbf{k}_l = (k_1, k_2)$, there holds

$$\|K_{1,k_1}\|_a < C_1\|k_1\|_{BC^n(\mathbb{R}^2)}, \qquad \|K_{2,k_2}\|_a < C_2\|k_2\|_{BC^n(\mathbb{R}^2)}.$$

The uniform boundedness of the operators $K_{\mathbf{k}_l}$, K_{1,k_1}, K_{2,k_2}, $l \in \Lambda_\gamma$, is now a consequence of Corollary 6.19 and the inequality $\|K_{\mathbf{k}_l}\|_a \le \|K_{1,k_1}\|_a + \|K_{2,k_2}\|_a$. □

Suppose now that we are given a kernel l satisfying $(\boldsymbol{L_n})$ and that $\mathbf{k}_l$ is defined as in Proposition 6.18. We will now define a quadrature approximation for the integral operator K_{1,k_1}. Therefore, observe that, for $s \in \mathbb{R}$, $x \in X$, $k_1 \in BC(\mathbb{R}^2)$, there holds (see also [50, p. 294])

$$\begin{aligned}\int_{-\infty}^{\infty} v_1(s,t)k_1(s,t)x(t)\,dt &= \frac{1}{2\pi}\int_{s-\pi}^{s+\pi} \ln\left(4\sin^2\Big(\frac{s-t}{2}\Big)\right)\chi(s-t)k_1(s,t)x(t)\,dt,\\ &= \frac{1}{2\pi}\int_{s-\pi}^{s+\pi} \ln\left(4\sin^2\Big(\frac{s-t}{2}\Big)\right)\widetilde{k}_1(s,t)\widetilde{x}(t)\,dt\\ &= \frac{1}{2\pi}\int_{0}^{2\pi} \ln\left(4\sin^2\Big(\frac{s-t}{2}\Big)\right)\widetilde{k}_1(s,t)\widetilde{x}(t)\,dt. \end{aligned} \tag{6.52}$$

Here, for $s,t \in \mathbb{R}$, the functions $\widetilde{k}_1$ and $\widetilde{x}$ are defined implicitly by

$$\widetilde{k}_1(s,t) := \begin{cases}\chi(s-t)k_1(s,t), & s-\pi \le t < s+\pi,\\ \widetilde{k}_1(s,t+2\pi), & t \in \mathbb{R},\end{cases} \quad \text{and} \quad \widetilde{x}(t) := \begin{cases} x(t), & s-\pi \le t < s+\pi,\\ \widetilde{x}(t+2\pi), & t\in\mathbb{R}.\end{cases}$$

The last equality in (6.52) holds since the integrand is 2π-periodic integrand with respect to t.

As in [50] (following [45] and [47]; see also [44])), we approximate the integral in equation (6.52) with a quadrature rule based on trigonometric interpolation,

$$\frac{1}{2\pi}\int_0^{2\pi} \ln\left(4\sin^2\Big(\frac{s-t}{2}\Big)\right)\widetilde{k}_1(s,t)\widetilde{x}(t)\,dt \approx \sum_{j=0}^{2N-1} R_j^N(s)\widetilde{k}_1(s,t_j^N)\widetilde{x}(t_j^N), \tag{6.53}$$

where the quadrature nodes are given by $t_j^N := j\pi/N$ and the quadrature weights are derived from the Lagrange basis for trigonometric interpolation (see [44, Sec. 11.3]):

$$R_j^N(s) := -\frac{1}{N}\left(\sum_{m=1}^{N-1}\frac{1}{m}\cos m(s-t_j^N) + \frac{1}{2N}\cos N(s-t_j^N)\right), \quad s\in\mathbb{R},\ N\in\mathbb{N},\ j\in\mathbb{Z}.$$

Note that, for every $s \in \mathbb{R}$, the function $t \mapsto \widetilde{k}_1(s,t)\widetilde{x}(t)$ is 2π-periodic and, since $\chi(s\pm\pi) = 0$, also continuous and hence contained in X. It now follows from [44, Sec. 12.3] that,

$$\lim_{N\to\infty}\sum_{j=0}^{2N-1} R_j^N(s)\widetilde{k}_1(s,t_j^N)\widetilde{x}(t_j^N) = \frac{1}{2\pi}\int_0^{2\pi}\ln\left(4\sin^2\Big(\frac{s-t}{2}\Big)\right)\widetilde{k}_1(s,t)\widetilde{x}(t)\,dt, \quad s\in[0,2\pi], \tag{6.54}$$

$$\sup_{s\in[0,2\pi]}\ \sup_{N\in\mathbb{N}} \sum_{j=0}^{2N-1}\left|R_j^N(s)\right| < \infty, \tag{6.55}$$

$$\lim_{h\to 0}\ \sup_{N\in\mathbb{N}}\sum_{j=0}^{2N-1}\left|R_j^N(s) - R_j^N(s+h)\right| = 0, \qquad s\in[0,2\pi]. \tag{6.56}$$

To be consistent with our definition of a general quadrature rule (5.4), we need to transpose this quadrature rule from the interval $[0,2\pi]$ to the interval $[s-\pi, s+\pi]$, in order to find a quadrature rule of the form

$$\int_{-\infty}^{\infty} v_1(s,t)k_1(s,t)x(t)\,dt \approx \sum_{j\in\mathbb{Z}}\omega_{1,j}^N(s)k_1(s,t_j^N)x(t_j^N), \tag{6.57}$$

where the quadrature nodes are given by $t_j^N = j\pi/N$ (as for the rectangle rule above) and that does not involve the periodic extension of $k_1(s,\cdot)x(\cdot)\chi(s-\cdot)$. We set, for $s\in\mathbb{R}$, $N\in\mathbb{N}$ and $j\in\mathbb{Z}$,

$$\omega_{1,j}^N(s) := R_j^N(s)\chi(s-t_j^N), \tag{6.58}$$

and define, for $k_1 \in BC(\mathbb{R}^2)$ and $x \in X$, the discretized integral operator K_{1,k_1}^N by

$$K_{1,k_1}^N x(s) := \sum_{j\in\mathbb{Z}} \omega_{1,j}^N(s) k_1(s,t_j^N) x(t_j^N), \quad s \in \mathbb{R}. \tag{6.59}$$

The point of this definition is that, because $R_j^N(s) = R_{j+2N}^N(s)$ holds for all $s \in \mathbb{R}$ and $N \in \mathbb{N}$, we obtain, for $x \in X$, (cf. [50, p. 296])

$$\begin{aligned} K_{1,k_1}^N x(s) = \sum_{j\in\mathbb{Z}} R_j^N(s)\chi(s-t_j^N) k_1(s,t_j^N) x(t_j^N) &= \sum_{|s-t_j^N|\le\pi} R_j^N(s)\widetilde{k_1}(s,t_j^N)\widetilde{x}(t_j^N) \\ &= \sum_{j=0}^{2N-1} R_j^N(s)\widetilde{k_1}(s,t_j^N)\widetilde{x}(t_j^N), \quad s\in\mathbb{R}. \end{aligned} \tag{6.60}$$

Remark 6.21. *In the notation of [50], the operators K_{1,k_1} and K_{1,k_1}^N defined above are denoted by K^A and K_N^A, respectively, where the function A is given by $A(s,t) := \chi^*(s,t)k_1(s,t)$, $s,t \in \mathbb{R}$.*

We now show that the quadrature weights $\omega_{1,j}^N(s)$ fit the theory developed in Chapter 5.

Proposition 6.22. *The quadrature weights $\omega_{1,j}^N(s)$ for the approximation of the kernel v_1, defined by* (6.58)*, satisfy* $\boldsymbol{(Q)}$ *(with $v = v_1$),* $\boldsymbol{(QA)}$*,* $\boldsymbol{(QA'')}$ *(with κ given by* (6.51)*),* $\boldsymbol{(QB)}$ *and* (5.74)*.*

Proof. We set $k_1(s,t) = 1$, $s,t \in \mathbb{R}$ so that $\widetilde{k_1} = k_1$. Bearing in mind the equalities (6.52) and (6.60), we observe that $\boldsymbol{(Q)}$ follows from (6.54). Moreover, since (6.60) also holds with $R_j^N(s)$ replaced by $|R_j^N(s)|$ Assumptions $\boldsymbol{(QA)}$ and $\boldsymbol{(QB)}$ are satisfied because of (6.55) and (6.56), respectively.

$\boldsymbol{(QA'')}$ is satisfied since $\omega_{1,j}^N(s) = 0$ whenever $|s-t_j^N| \ge \pi$, by choice of χ, and $\omega_{1,j}^N(s) \le C$, for $|s-t_j^N| < \pi$, where C denotes the supremum in (6.55). Finally, (5.74) holds, since, by definition, the value of $\omega_{1,j}^N(s)$ depends only on the difference $s - t_j^N$ if $N \in \mathbb{N}$ is kept fixed. □

One consequence of this result is that the operators K_{1,k_1} are bounded on X_a, $0 \le a \le b$. We can also give an error estimate for the approximation of K_{1,k_1} by K_{1,k_1}^N.

Proposition 6.23. *Let $0 \le a \le b$ and $\gamma > 0$. Suppose that, for every $l \in \Lambda_\gamma$, k_1 is chosen as in Proposition 6.18. Then the following statements are true:*

a) For every $l \in \Lambda_\gamma$, the corresponding operators K_{1,k_1} and K_{1,k_1}^N are bounded on X_a. Moreover, $\|K_{1,k_1}\| \le C_a$ and $\|K_{1,k_1}^N\|_a \le C_a$, for some constant C_a not depending on l or N.

b) For every $l \in \Lambda_\gamma$ and $x \in BC_a^n(\mathbb{R})$, the error estimate

$$\|K_{1,k_1}^N x - K_{1,k_1} x\|_a \le C_a' N^{-n} \|x\|_{BC_a^n(\mathbb{R})}$$

holds, where C_a' is a constant not depending on l, x or N.

Proof. **a)** The uniform boundedness of the operators K_{1,k_1} has already been shown in the proof of Proposition 6.20. We will show in b) below that the quadrature weights $\omega_{1,j}^N(s)$ satisfy $\boldsymbol{(Q_n)}$. Since they also satisfy $\boldsymbol{(QA)}$ and $\boldsymbol{(QB)}$, the uniform boundedness of the operators K_{1,k_1}^N, now follows from Proposition 6.10 and Corollary 6.19.

b) We prove that the quadrature weights $\omega_{1,j}^N(s)$ satisfy $\boldsymbol{(Q_n)}$. Condition 1) in $\boldsymbol{(Q_n)}$ is satisfied since, by definition, $|s-t_j^N| \ge A_0^* := \pi$ implies that $\omega_{1,j}^N(s) = 0$. To show that Condition 2) in $\boldsymbol{(Q_n)}$ is satisfied, we apply [50, La. 3.12] (with $A(s,t) = \chi^*(s,t)$, see Remark 6.21) and obtain, for the constant function $k_1(s,t) = 1$,

$$\|K_{1,k_1}^N x - K_{1,k_1} x\| \le C N^{-n} \|\chi^*\|_{BC^n(\mathbb{R}^2)} \|x\|_{BC^n(\mathbb{R})}, \qquad x \in BC^n(\mathbb{R}),$$

for some constant C depending only on n. Thus $\boldsymbol{(Q_n)}$ is satisfied by the quadrature weights $\omega_{1,j}^N(s)$. Now the statement in b) now follows from Proposition 6.12 and Corollary 6.19. □

Let us now consider the approximation of K_{2,k_2} by discretized integral operators. The kernel v_2 satisfies Assumption $(\boldsymbol{A'_n})$ with $A_0 = 0$ (see the proof of Proposition 6.20). As suggested in Section 6.2, we choose the rectangle rule to approximate the integral operator K_{2,k_2} (this is also the approach taken in [50]).

For $N \in \mathbb{N}$ and $k_2 \in BC^n(\mathbb{R})$, let K^N_{2,k_2} denote the discretized integral operator $\widetilde{K^N_k}$ defined in (6.14), but with $\widetilde{v} = v_2$, $k = k_2$ and the quadrature abscissae be given by $t^N_j = j\pi/N$, $j \in \mathbb{Z}$. Then

$$K^N_{2,k_2}x(s) = \frac{\pi}{N}\sum_{j\in\mathbb{Z}} v_2(s,t^N_j)k(s,t^N_j)x(t^N_j), \qquad x \in X, \tag{6.61}$$

and the corresponding quadrature weights are given by

$$\omega^N_{2,j}(s) := \frac{\pi}{N}v_2(s,t^N_j), \quad s \in \mathbb{R}, j \in \mathbb{Z}, N \in \mathbb{N}.$$

Proposition 6.24. *The quadrature weights $\omega^N_{2,j}(s)$ satisfy Assumptions $(\boldsymbol{Q})$ (with $v = v_2$), $(\boldsymbol{QA})$, $(\boldsymbol{QA''})$ (with κ given by (6.51)), $(\boldsymbol{QB})$ and also (5.74).*

Moreover, for every $l \in \Lambda_\gamma$ and $x \in BC^n_a(\mathbb{R})$, the error estimate

$$\|K^N_{2,k_2}x - K_{2,k_2}x\|_a \le C''_a N^{-n}\|x\|_{BC^n_a(\mathbb{R})}$$

holds, where C''_a is a constant not depending on l, x or N.

Proof. The kernel v_2 satisfies $(\boldsymbol{A'_n})$ with $A_0 = 0$. Thus Proposition 6.5 shows that $(\boldsymbol{Q})$ (with $v = v_2$), $(\boldsymbol{QA})$, $(\boldsymbol{QA''})$ and $(\boldsymbol{QB})$ hold. Since v_2 is a convolution kernel the value of $\omega^N_{2,j}(s)$, for fixed $N \in \mathbb{N}$, depends only on the difference $s - t^N_j$, whence the quadrature weights $\omega^N_{2,j}(s)$ satisfy (5.74). The error estimate now follows from Corollary 6.19 and Proposition 6.8. □

We have seen in the previous two propositions that, for every $k_1, k_2 \in BC^n(\mathbb{R}^2)$, the discretized integral operator

$$K^N_{\mathbf{k}_l} := K^N_{1,k_1} + K^N_{2,k_2}$$

is a reasonable approximation of the integral operator $L_l = K_{\mathbf{k}_l} = K_{1,k_1} + K_{2,k_2}$.

Remark 6.25. *Suppose that the kernel l satisfies Assumption $(\boldsymbol{L_n})$, for some $n \in \mathbb{N}$, and that $\mathbf{k}_l = (k_1, k_2)$ is defined as in Proposition 6.18. Then, in the notation of [50], the operators K_{2,k_2}, K^N_{2,k_2}, $K^N_{\mathbf{k}_l}$ defined above are denoted by K^B, K^B_N and K_N with $B := k_2 v_2$, respectively.*

For the Nyström method we have defined, we now derive the following stability result on the weighted space X_a, $0 \le a \le b$. In the proof, we combine our weighted space stability theory with the main stability theorem (for the unweighted space X) in [50].

Theorem 6.26. *Suppose that the kernel l satisfies $(\boldsymbol{L_n})$ and that $0 \le a \le b$. Assume further that $(I - K_{\mathbf{k}_l})^{-1} \in \mathcal{B}(X)$. Then there exist constants $N' \in \mathbb{N}$ and $C_a, C'_a > 0$ such that, for every $N \ge N'$, there holds $(I - K^N_{\mathbf{k}_l})^{-1} \in \mathcal{B}(X_a)$ and*

$$\sup_{N\ge N'} \|(I - K^N_{\mathbf{k}_l})^{-1}\|_a = C_a < \infty, \tag{6.62}$$

so that, for every $N \ge N'$ and $y \in X_a$, the equations

$$(I - K_{\mathbf{k}_l})x = y, \quad (I - K^N_{\mathbf{k}_l})x^N = y, \tag{6.63}$$

have unique solutions $x, x^N \in X_a$, for which the estimates $\|x\|_a, \|x^N\|_a \le C_a\|y\|_a$, and

$$\|x - x^N\|_a \le C_a\|(K^N_{\mathbf{k}_l} - K_{\mathbf{k}_l})x\|_a. \tag{6.64}$$

hold. Moreover, if $y \in BC^n_a(\mathbb{R})$ then the solutions x, x^N of satisfy the error estimate

$$\|x - x^N\|_a \le C'_a N^{-n}\|y\|_{BC^n_a(\mathbb{R})}. \tag{6.65}$$

For every $\gamma > 0$, the constants C_a and C'_a may be chosen independently of l, for $l \in \Lambda'_\gamma$.

Proof. **(a=0)** We first treat the case $a = 0$. Given $\gamma > 0$, Corollary 6.19 yields bounded subsets W_1, W_2 of $BC^n(\mathbb{R})$ so that every $l \in \Lambda'_\gamma$ may be written as

$$l(s,t) = k_1(s,t)v_1(s,t) + k_2(s,t)v_2(s,t), \qquad s,t \in \mathbb{R}, s \neq t,$$

with $k_1 \in W_1$ and $k_2 \in W_2$. We now define $A(s,t) := k_1(s,t)\chi^*(s,t)$ and $B(s,t) := k_2(s,t)v_2(s,t)$. With this choice of A and B, $K^N_{\mathbf{k}_l} = K_N = K^A_N + K^B_N$, in the notation of [50].

Theorem 2.1 in [50] shows that the kernels in Λ'_γ all satisfy Assumption $(\mathbf{C''_n})$ of [50] and that $\|A\|_{BC^n(\mathbb{R})} + \|B\|_{BC^n_b(\mathbb{R})} < \beta$, for some $\beta > 0$ not depending on the choice of l. We may thus invoke Theorem 3.8 and 3.13 of [50] to obtain that there exists $N' \in \mathbb{N}$ and $C_0 > 0$ so that, for every $l \in \Lambda'_\gamma$, the inequalities (6.62)–(6.65) hold.

$(\mathbf{0 < a \leq b})$ We have shown above that the kernels v_1 and v_2 satisfy $(\boldsymbol{A''})$, with κ given by (6.51), and that $w_a \in \mathcal{W}(\kappa)$, $0 < a \leq b$ (see Proposition 6.20, 6.2). Moreover, the quadrature weights $\omega^N_{1,j}(s)$ and $\omega^N_{2,j}(s)$ both satisfy $(\boldsymbol{Q})$ (with $v = v_1$ and $v = v_2$, respectively), $(\boldsymbol{QA})$, $(\boldsymbol{QA''})$ (with κ given by (6.51)) and $(\boldsymbol{QB})$. Hence, we may invoke Theorem 5.30 with W_1, W_2 as above, $\mathbf{W} := \{\mathbf{k}_l : l \in \Lambda'_\gamma\}$ and $\mathbb{N}' := \{N \in \mathbb{N} : N \geq N'\}$. This theorem yields constants C_a, C'_a such that inequalities (6.62)–(6.64) are satisfied.

Now, only inequality (6.65) remains to be shown. To this end, we choose $l \in \Lambda'_\gamma$ and $y \in BC^n_a(\mathbb{R})$ and let x, x^N denote the solutions of (6.63). Then (6.64) holds and, further, we have

$$\|(K^N_{\mathbf{k}_l} - K_{\mathbf{k}_l})x\|_a \leq \|K^N_{1,k_1}x - K_{1,k_1}x\|_a + \|K^N_{2,k_2}x - K_{2,k_2}x\|_a. \tag{6.66}$$

We now show that $x \in BC^n_a(\mathbb{R})$. To this end, observe that, the kernel v_1 satisfies $(\boldsymbol{A'_n})$ with $A_0 = 0$ and thus, by Proposition 6.15,

$$\|K_{2,k_2}\|_{X_a \to BC^n_a(\mathbb{R})} \leq C_1\|k_1\|_{BC^n(\mathbb{R}^2)}, \|K_{2,k_2}\|_{BC^n_a \to BC^n_a(\mathbb{R})} \leq C_1\|k_1\|_{BC^n(\mathbb{R}^2)}, \tag{6.67}$$

for some constant $C_1 > 0$. We remark that this constant, and the constants we introduce below can all be chosen so that they do not depend on $l \in \Lambda'_\gamma$. It follows from Corollary 2.8 and Theorem 2.1 in [50] (we apply these with $k := v_1k_1$, a kernel which satisfies Assumption $(\mathbf{C''_n})$ of [50]) that the kernel v_1 satisfies $(\boldsymbol{R^0_n})$ with $m := 2n$, so that, for $p = 0$,

$$\|(K_{1,k_1})^m\|_{X_p \to BC^n_p(\mathbb{R})} < C_2(\|k_2\|_{BC^n(\mathbb{R}^2)})^m, \quad \|K_{1,k_1}\|_{BC^n_p(\mathbb{R}) \to BC^n_p(\mathbb{R})} < C_2\|k_2\|_{BC^n(\mathbb{R}^2)}, \tag{6.68}$$

for some constant $C_2 > 0$ (note that we have tacitly used the following inequality: $\|k_1\chi^*\|_{BC^n(\mathbb{R}^2)} \leq 2^n\|k_1\|_{BC^n(\mathbb{R}^2)}\|\chi^*\|_{BC^n(\mathbb{R}^2)}$). The argument of Proposition 6.16 then shows that (6.68) also holds for $p = a$, (perhaps) with a larger value of C_2. Since $y \in BC^n_a(\mathbb{R})$ and there holds (cf. (6.38))

$$x = \big(I + K_{\mathbf{k}_l} + \cdots + (K_{\mathbf{k}_l})^{m-1}\big)y + (K_{\mathbf{k}_l})^m(I - K_{\mathbf{k}_l})^{-1}y,$$

we thus obtain from (6.62), (6.67) and (6.68) that $x \in BC^n_a(\mathbb{R})$.

Using the fact that $\|k_1\|_{BC^n(\mathbb{R}^2)}$, $i = 1, 2$, are uniformly bounded for $l \in \Lambda'_\gamma$, there holds

$$\|x\|_{BC^n_a(\mathbb{R})} \leq C_3\|y\|_{BC^n_a(\mathbb{R})}, \tag{6.69}$$

for some constant $C_3 > 0$ not depending on the choice of l.

Now, in view of Propositions 6.23 and 6.24, we see that, for some constant $C_4 > 0$

$$\|K^N_{1,k_1}x - K_{1,k_1}x\|_a \leq C_4N^{-n}\|x\|_{BC^n_a(\mathbb{R})}, \qquad \|K^N_{2,k_2}x - K_{2,k_2}x\|_a \leq C_4N^{-n}\|x\|_{BC^n_a(\mathbb{R})}. \tag{6.70}$$

Now the desired inequality follows from (6.64), (6.66) and (6.70). □

6.3 A problem in rough surface scattering

As an application of our error estimates in the weighted norms $\|\cdot\|_a$, we will now consider a class of integral equations in scattering theory. The problem we consider arises in the scattering of time-harmonic waves by unbounded rough surfaces and leads to a boundary integral equation over $\mathbb{R}$. It was used before in [50, 49] as a model problem to illustrate the error bounds for the Nyström method proposed in the same paper, which is, as we have seen the same as the Nyström method considered in the previous section.

To illustrate the behaviour of the error in the weighted norm, we will consider the special case when the incident wave is emanating from a point source, which, as we will see soon, implies that the inhomogeneity of the integral equation is contained in the weighted space $X_{3/2}$.

We begin with a description of the problem. The propagation of time-harmonic acoustic waves with wave number $\bar{\kappa}$ for a domain Ω is governed by the Helmholtz equation

$$\Delta u(\mathbf{x}) + \bar{\kappa}^2 u(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega. \tag{6.71}$$

We consider two-dimensional domains that can be described as the area above the graph of a smooth function, namely, domains of the form $\Omega = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_2 > f(x_1)\}$, where the function f is contained in $BC^{n+2}(\mathbb{R})$ for some $n \in \mathbb{N}_0$ and there are positive constants c_1, c_2 such that $0 < c_1 \leq f(s) \leq c_2$ for all $s \in \mathbb{R}$. Let $\Gamma := \partial\Omega$ denote the boundary of Ω and Φ denote the free-field Green's function for the Helmholtz equation,

$$\Phi(\mathbf{x}, \mathbf{y}) := \frac{i}{4} H_0^{(1)}(\bar{\kappa}|\mathbf{x} - \mathbf{y}|), \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^2, \quad \mathbf{x} \neq \mathbf{y},$$

where $H_0^{(1)}$ denotes the first-kind Hankel function of order zero. We will also make use of the notation $U_H := \{\mathbf{x} \in \mathbb{R}^2 : x_2 > h\}$ and $\Gamma_H := \{\mathbf{x} \in \mathbb{R}^2 : x_2 = h\}$, for $H \geq 0$.

We now consider the following scattering problem:

Scattering Problem. *Given the incident field u^i, find the scattered field $u^s \in C^2(\Omega) \cap C(\overline{\Omega})$ such that*

1. *u^s satisfies the Helmholtz equation* (6.71),
2. *$u^s = -u^i$ on Γ,*
3. *the* upwards propagating radiation condition *of [25] holds in Ω, i.e. for some $h > \sup f$ and some $\phi \in L^\infty(\Gamma_H)$*

$$u^s(\mathbf{x}) = 2 \int_{\Gamma_H} \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial y_2} \phi(\mathbf{y})\, ds(\mathbf{y}), \quad \mathbf{x} \in U_h,$$

4. *u^s is bounded in the horizontal strip $\Omega \setminus U_H$ for every $H > 0$.*

We only consider the sound-soft case where the total field $u = u^i + u^s$ vanishes on the boundary Γ. To illustrate our error bounds in weighted spaces we will consider the particular incident field given by

$$u^i(\mathbf{x}) = \Phi(\mathbf{x}, \mathbf{z}), \quad \mathbf{x} \in \mathbb{R}^2$$

where $\mathbf{z} = (z_1, z_2) \in \mathbb{R}^2$ is some point above the boundary, i.e. $z_2 > f(z_1)$. This choice of u^i models a monopole point source located at $\mathbf{z}$, so that, in the real two-dimensional acoustic problem, the incoming wave is a cylindrical wave emanating from the point $\mathbf{z}$ (see Figure 6.1).

We now reformulate the scattering problem as a boundary integral equation. To this end, we need the Green's function for the Helmholtz equation in the half-plane U_0 with Dirichlet boundary conditions, namely

$$G(\mathbf{x}, \mathbf{y}) := \Phi(\mathbf{x}, \mathbf{y}) - \Phi(\mathbf{x}, \mathbf{y}'), \quad \mathbf{x}, \mathbf{y} \in \overline{U_0}, \mathbf{x} \neq \mathbf{y},$$

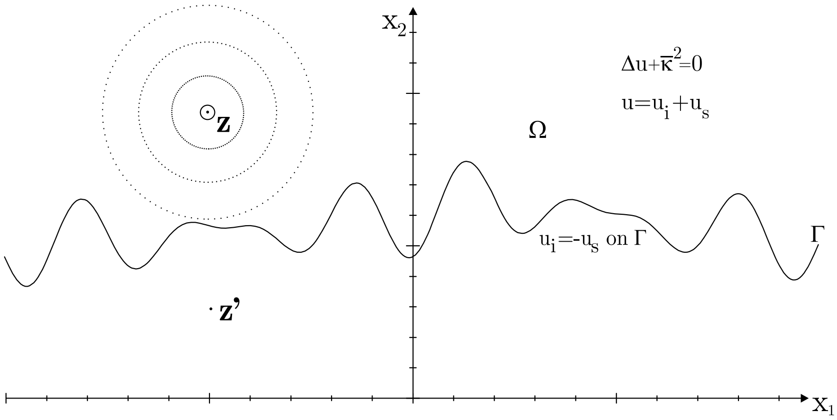

Figure 6.1: A rough surface scattering problem for a monopole sound source located at $\mathbf{z}$

where $\mathbf{y}' := (y_1, -y_2)$ corresponds to $\mathbf{y} = (y_1, y_2)$. Let $\mathbf{z}' := (z_1, 2f(z_1) - z_2) \in \mathbb{R}^2 \setminus \overline{\Omega}$. We make the following modified Brakhage and Werner ansatz [13] for the scattered field

$$u^s(\mathbf{x}) = -\Phi(\mathbf{x}, \mathbf{z}') + \int_\Gamma \left(\frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} - i\eta G(\mathbf{x}, \mathbf{y}) \right) \psi(\mathbf{y})\, ds(\mathbf{y}), \tag{6.72}$$

where $\eta > 0$ is some fixed constant, $\mathbf{n}(\mathbf{y})$ denotes the unit normal to Γ at $\mathbf{y}$, directed into Ω, and the function $\psi \in BC(\Gamma)$ is called a *density*. We note that $-\Phi(\mathbf{x}, \mathbf{z}')$ is the scattered field in the special case when $\Gamma = \Gamma_H$ with $H = f(z_1)$.

It follows from the results in [64] that a scattered field of this type is a solution to the scattering problem if and only if the density ψ satisfies the boundary integral equation

$$\psi + \mathcal{D}\psi - i\eta \mathcal{S}\psi = -2\big(\Phi(\cdot, \mathbf{z}') - \Phi(\cdot, \mathbf{z})\big) \quad \text{on } \Gamma, \tag{6.73}$$

where $\mathcal{S}$ and $\mathcal{D}$ are the single- and double-layer potential operators, acting on $BC(\Gamma)$, given by

$$\mathcal{S}\psi(\mathbf{x}) := 2 \int_\Gamma G(\mathbf{x}, \mathbf{y}) \psi(\mathbf{y})\, ds(\mathbf{y}), \quad \mathcal{D}\psi(\mathbf{x}) := 2 \int_\Gamma \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} \psi(\mathbf{y})\, ds(\mathbf{y}), \quad \mathbf{x} \in \Gamma.$$

We parameterize Γ as $\{(s, f(s)) : s \in \mathbb{R}\}$ and set

$$x(s) := \psi(s, f(s)), \qquad y(s) := -2\big(\Phi((s, f(s)), \mathbf{z}) - \Phi((s, f(s)), \mathbf{z}')\big), \quad s \in \mathbb{R}. \tag{6.74}$$

Then (6.73) is equivalent to

$$x - L_{l_f} x = y \tag{6.75}$$

where L_{l_f} is the integral operator defined by (6.42), with kernel l_f given by

$$l_f(s, t) := 2 \left(i\eta G(\mathbf{x}, \mathbf{y}) - \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} \right) \sqrt{1 + f'(t)^2}, \qquad s, t \in \mathbb{R},$$

where $\mathbf{x} = (s, f(s))$, $\mathbf{y} = (t, f(t))$. [50, Th. 4.3] shows that, for every $n \in \mathbb{N}_0$ and $f \in BC^{n+2}(\mathbb{R})$, l satisfies assumption $(\boldsymbol{L_n})$ with $b = 3/2$. The same shows, further, that, for every $c_1, M > 0$, there exists a constant $\gamma > 0$ such that for all

$$f \in B^n_{c_1, M} := \{f \in BC^{n+2}(\mathbb{R}) : c_1 \leq \inf f, \|f\|_{BC^{n+2}(\mathbb{R})} \leq M\} \tag{6.76}$$

there holds

$$l_f \in \Lambda_\gamma, \tag{6.77}$$

with the same constant γ for every $f \in B^n_{c_1,M}$. Thus, in view of Proposition 6.20, for $f \in B^n_{c_1,M}$, the operators $L_{l_f} : X_a \to X_a$ and are uniformly bounded, $0 \le a \le 3/2$.

The next theorem tells us that the problem of finding a solution of equation (6.75) (which is a first step towards the solution of the scattering problem) is well-posed for a large class of surfaces. It also shows that we may replace Λ_γ by Λ'_γ in (6.77).

Theorem 6.27 ([22, Th. 3.2]). *For every $f \in BC^{n+2}(\mathbb{R})$, $n \in \mathbb{N}_0$ the operator $I - L_{l_f}$ has a bounded inverse on X. Further, for every $c_1, M > 0$ there exists $\beta > 0$ such that*

$$\sup_{f \in B^n_{c_1,M}} \|(I - L_{l_f})^{-1}\| \le \beta.$$

It follows from results in [61] that — and this is the motivation for our modified Brakhage/Werner ansatz (6.72) with $\Phi(\mathbf{x}, \mathbf{z}')$ added to the combined potential — the function $\Phi(\mathbf{x}, \mathbf{z}) - \Phi(\mathbf{x}, \mathbf{z}')$ satisfies the following estimate: For given $\epsilon > 0$ there holds

$$|\Phi(\mathbf{x}, \mathbf{z}) - \Phi(\mathbf{x}, \mathbf{z}')| \le \frac{C(1 + x_2)(1 + y_2)}{|\mathbf{x} - \mathbf{z}|^{3/2}}, \quad |\mathbf{x} - \mathbf{z}| \ge \epsilon > 0, \tag{6.78}$$

where C is a constant, not depending on $\mathbf{x}$. Since $\Phi(\mathbf{x}, \mathbf{z}) - \Phi(\mathbf{x}, \mathbf{z}')$ satisfies the Helmholtz equation it follows from standard regularity estimates for elliptic partial differential equations in [34] that bounds such as (6.78) also hold for every partial derivative of $\Phi(\mathbf{x}, \mathbf{z}) - \Phi(\mathbf{x}, \mathbf{z}')$ up to any order. As a consequence, the right-hand side y of (6.75), given by (6.74), is contained in the weighted space $BC^n_{3/2}(\mathbb{R})$. Further, it follows that if $\mathbf{z}$ is kept fixed, then the collection of all such right-hand sides y, as f runs through $B^n_{c_1,M}$, is uniformly bounded in $BC^n_{3/2}(\mathbb{R})$.

We can thus appeal to the error estimates in weighted spaces, which we have obtained in the previous section for the Nyström method introduced in [50]. To this end, we associate, with every $f \in B^n_{c_1,M}$ and the corresponding kernel $l = l_f$, the quadrature operator $K^N_{\mathbf{k}_f}$, $N \in \mathbb{N}$, as defined in the previous section; more precisely $K^N_{\mathbf{k}_f} := K^N_{1,k_1} + K^N_{2,k_2}$, where $\mathbf{k}_f = (k_1, k_2) \in BC^n(\mathbb{R}^2)$ is defined in Proposition 6.18 (applied with $l = l_f$), and K^N_{1,k_1}, K^N_{2,k_2} are defined in (6.59) and (6.61), respectively. We note that, for improved readability, we have written $K^N_{\mathbf{k}_f}$ and $\mathbf{k}_f$ instead of $K^N_{\mathbf{k}_{l_f}}$ and $\mathbf{k}_{l_f}$, respectively.

The next theorem shows that if the surface Γ is smooth then the approximate solutions x^N of (6.79) below always exist when N is large enough and, moreover, x^N is rapidly converging to the exact solution x (with superalgebraic convergence if $f \in BC^\infty(\mathbb{R})$). Moreover, for fixed n, c_1, M, the convergence does not depend on $f \in B^n_{c_1,M}$, in particular not on the amplitude and slope of the surface Γ.

Theorem 6.28. *Let $n \in \mathbb{N}$, $c_1, M > 0$ and y be given by (6.74), so that y models a cylindrical wave emanating from a point $\mathbf{z} = (z_1, z_2)$ with $z_2 > M$. Then there exists some $N' \in \mathbb{N}$ such that, provided that $N \ge N'$ and $f \in B^n_{c_1,M}$, the equations*

$$(I - L_{l_f})x = y \qquad \text{and} \qquad (I - K^N_{\mathbf{k}_f})x^N = y \tag{6.79}$$

have unique solutions $x, x^N \in X_{3/2}$.

Further, there exists a constant $C > 0$ such that, for every $f \in B^n_{c_1,M}$, the numerical solution x^N and the exact solution x of (6.79) satisfy the estimate

$$\|x - x^N\|_{3/2} \le C N^{-n},$$

where the constant C does not depend on the choice of f.

Proof. Theorem 6.27 shows that $\{l_f : f \in B^n_{c_1,M}\} \subset \Lambda'_\gamma$, for some $\gamma > 0$ large enough. Since $y \in BC^n_{3/2}(\mathbb{R})$ (with a norm that remains bounded as f varies in $B^n_{c_1,M}$), the theorem is now immediate from Theorem 6.26. □

The approximate solution of $(I - L_{l_f})x = y$ by a modified finite-section method and its stability on X has been considered in [51]; in [49] these results have then been combined with the stability analysis of [50]. We will now combine the results in [49] with our analysis of the modified finite section method in Section 5.4. As an outcome we present novel error estimates, improving those in [49] in the special case when the right-hand side y is given by (6.74).

As yet, there seems no way of proving the stability of the unmodified finite section method for the equation $(I - K^N_{\mathbf{k}_f})y = x$ in (6.79) (or for the integral equation in (6.79)), but it is shown in [49] that stability holds for a modified finite-section method, in which the scattering surface is "flattened" near the endpoints of the truncation interval.

To describe this flattening, we choose a "cut-off" function ν: let $\alpha > 0$ and $\nu \in \mathbb{C}^\infty(\mathbb{R})$ satisfy $0 \leq \nu(s) \leq 1$, $s \in \mathbb{R}$, and $\nu(s) = 0$ if $s \geq \alpha$ and $\nu(s) = 1$ if $s \leq 0$. We keep α and ν fixed throughout the remainder of this section.

Using the cut-off function ν, we approximate the function f by f^A, defined by

$$f^A(s) = f(s)\nu(s-A)\nu(-s-A) + f(A)\big(1 - \nu(s-A)\big) + f(-A)\big(1 - \nu(-s-A)\big), \qquad s \in \mathbb{R}. \quad (6.80)$$

Then $f^A(s) = f(s)$, $|s| \leq A$, $f^A(s) = f(A)$, $s \geq A + \alpha$ and $f^A(s) = f(-A)$, $s \leq -A - \alpha$. Moreover, it is not hard to see that

$$f \in B^n_{c_1,M} \quad \Longrightarrow \quad f^A \in B^n_{c_1,M'}, \quad (6.81)$$

for some constant $M' \geq M$, depending only on M and ν but not on A.

With every $f \in B$, $\alpha_0 \geq \alpha$ and $A > 0$, we can thus associate the operator $K^{N,A+\alpha_0}_{\mathbf{k}_{fA}}$, defined as $K^N_{\mathbf{k}_{fA}}$, but with the summations reduced to the interval $[-A + \alpha_0, A + \alpha_0]$. The following stability theorem is shown in [49].

Theorem 6.29. *Let $n \in \mathbb{N}$, $c_1, M > 0$. Then there exists some $N' \in \mathbb{N}$ and $\alpha_0' \geq 0$ such that, provided that $N \geq N'$, $\alpha_0 \geq \alpha_0'$, $A > 0$ and $f \in B := B^n_{c_1,M}$, the operator $I - K^{N,A+\alpha_0}_{\mathbf{k}_{fA}}$ is invertible on X; moreover*

$$\sup_{N \geq N'} \sup_{f \in B} \sup_{A>0, \alpha_0 \geq \alpha_0'} \|(I - K^{N,A+\alpha_0}_{\mathbf{k}_{fA}})^{-1}\| < \infty. \quad (6.82)$$

This theorem allows us to invoke Theorem 5.31, which gives the following error estimate for approximate solution of the boundary integral equation (6.75) by the modified finite section method.

Theorem 6.30. *Let $n \in \mathbb{N}$, $c_1, M > 0$ and y be given by (6.74), so that y models a cylindrical wave emanating from a point $\mathbf{z} = (z_1, z_2)$ with $z_2 > M$. Then there exists some $N' \in \mathbb{N}$ and $\alpha_0' \geq 0$ such that, provided that $N \geq N'$, $A > 0$, $\alpha_0 \geq \alpha_0'$ and $f \in B^n_{c_1,M}$, the equations*

$$(I - L_{l_f})x = y \qquad (I - K^{N,A+\alpha_0}_{\mathbf{k}_{fA}})x^{N,A+\alpha_0} = y \quad (6.83)$$

have unique solutions $x, x^{N,A+\alpha_0} \in X_{3/2}$.

Further, if $A = t^N_j$, for some $j \in \mathbb{N}$, there exists a constant $C > 0$ such that, for every $f \in B^n_{c_1,M}$, the numerical solution $x^{N,A+\alpha_0} \in X_{3/2}$ and the exact solution $x \in X_{3/2}$ of (6.83) satisfy the estimate

$$|x(s) - x^{N,A+\alpha_0}(s)| \leq C_1 N^{-n}(1+|s|)^{-3/2} + C_2(1+A)^{-3/2}\Big((1+A+s)^{-1/2} + (1+A-s)^{-1/2}\Big), \quad |s| \leq A. \quad (6.84)$$

where the constants $C_1, C_2 > 0$ do not depend on A, f, α_0 or N.

Proof. As mentioned before, we wish to invoke Theorem 5.31, thus we need to check that the assumptions of the theorem are satisfied. That the kernels v_1 and v_2 and the quadrature weights $\omega^N_{1,j}(s)$ and $\omega^N_{2,j}(s)$ satisfy the requirements of Theorem 5.31, apart from (5.74), has already been shown in the proof of Theorem 6.26. However, that the quadrature weights $\omega^N_{1,j}(s)$ and $\omega^N_{2,j}(s)$ fulfil (5.74) has been proved in Propositions 6.22 and 6.24, respectively.

For $0 < A \le \alpha_0'$, we let $W_A := \varnothing$, moreover, we define

$$\mathbf{W}_{A+\alpha_0} := \{\mathbf{k}_{f^A} : f \in B^n_{c_1,M}\}, \quad A > 0, \quad \text{and} \quad \mathbf{W}_\infty := \{\mathbf{k}_f : f \in B^n_{c_1,M}\}.$$

Let $\mathbf{W} := \mathbf{W}_\infty \cup \bigcup_{A>0} \mathbf{W}_A$. Then, by (6.81), (6.76)/(6.77) and Corollary 6.19, the set $\mathbf{W}$ satisfies the boundedness and equicontinuity assumptions of of Theorem 5.31.

Further, it then follows from Theorem 6.29, the proof of Theorem 6.28 and Theorem 6.26 that (5.91) and (5.92) are satisfied (for $\lambda = 1$), where $\mathbb{N}' := \{N \in \mathbb{N} : N \ge N'\}$, with N' chosen as in Theorem 6.26.

Thus all assumptions up to (5.95) of Theorem 5.31 are satisfied and the statement about the solvability of (6.83) follows.

We have already remarked that the kernels v_1, v_2 satisfy $(\boldsymbol{A_n'})$, for $b = 3/2$, some $C_n > 0$ and κ given by (6.51). For this κ, the function $\mu(s)$, defined in (4.82), satisfies $\mu(s) \le C'(1+|s|)^{-1/2}$, $s \ge 0$, for some constant $C' > 0$, and hence (5.95) holds for the weight function $\widetilde{w} := w_{1/2}$.

If we are given $f \in B^n_{c_1,M}$, $A = t_j^N > 0$ and $\alpha_0 \ge \alpha_0'$ then $\mathbf{k}_f$ and $\mathbf{k}_{f^A}$, satisfy (5.96) for $D := A$. We can now apply Theorem 5.31 (with A taking the role of D and $A + \alpha_0$ taking the role of A in that theorem). Since $y \in BC^n_{3/2}(\mathbb{R})$, $n \in \mathbb{N}$, we thus obtain, from (6.84) and (5.97), the following error estimates for the solutions x, $x^{N,A}$ and $x^{N,A+\alpha_0}$ of the equations (6.79) and (6.83).

$$|x(s) - x^N(s)| \le C_1 N^{-n}(1+|s|)^{-3/2}, \qquad |s| \le A,$$

$$|x^N(s) - x^{N,A+\alpha_0}(s)| \le C_2(1+A)^{-a}\left((1+A+s)^{-1/2} + (1+A-s)^{-1/2}\right), \qquad |s| \le A,$$

where $C_1, C_2 > 0$ are constants not depending on f, A, α_0 or y. Since

$$|x(s) - x^{N,A+\alpha_0}(s)| \le |x(s) - x^N(s)| + |x^N(s) - x^{N,A+\alpha_0}(s)|, \quad |s| \le A,$$

combining these two inequalities yields (6.84) and the theorem is shown. $\square$

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