Mappings between distance sets or spaces

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Zusammenfassung

Die vorliegende Arbeit hat drei Ziele: Distanzfunktionen als wichtiges Werkzeug der allgemeinen Topologie wiedereinzuführen; den Gebrauch von Distanzfunktionen auf den verschiedensten mathematischen Objekten und eine Denkweise in Begriffen der Abstandstheorie anzuregen; und schliesslich spezifischere Beiträge zu leisten durch die Charakterisierung wichtiger Klassen von Abbildungen und die Verallgemeinerung einiger topologischer Sätze.

Zunächst werden die Konzepte des ‚Formelerhalts' und der ‚Übersetzung von Abständen' benutzt, um interessante ‚nicht-topologische' Klassen von Abbildungen zu finden, was zur Charakterisierung vieler bekannter Arten von Abbildungen mithilfe von Abstandsfunktionen führt. Nachdem dann eine ‚kanonische' Methode zur Konstruktion von Distanzfunktionen angegeben wird, entwickele ich einen geeigneten Begriff von ‚Distanzräumen', der allgemein genug ist, um die meisten topologischen Strukturen induzieren zu können. Sodann werden gewisse Zusammenhänge zwischen einigen Arten von Abbildungen bewiesen, wie z. B. dem neuen Konzept ‚streng gleichmässiger Stetigkeit'. Es folgt eine neuartige Charakterisierung der Ähnlichkeitsabbildungen zwischen Euklidischen Räumen. Die Dissertation schliesst mit einigen Verallgemeinerungen bekannter Vervollständigungskonstruktionen und wichtiger Fixpunktsätze, und einer kurzen Studie über Techniken der Visualisierung von Abständen.

Abstract

The aim of this thesis is threefold: to reinstate distance functions as a principal tool of general topology; to promote the use of distance functions on various mathematical objects and a thinking in terms of distances also in non-topological contexts; and to make more specific contributions by characterizing important classes of mappings and generalizing some important topological results.

I start by using the key concepts of 'preservation of formulae' and 'translation of distances' to extract interesting 'non-topological' classes of mappings, which leads to the characterization of many well-known types of mappings in terms of distance functions. After giving a 'canonical' method for constructing distance functions, a suitable notion of 'distance spaces' will be developed, general enough to induce most topological structures. Then certain relationships between many kinds of mappings are proved, including the new concept of 'strong uniform continuity', followed by a new characterization of the similarity maps between Euclidean spaces. The thesis closes with some generalizations of completions and fixed point theorems, and a short, self-contained study of distance visualization techniques.

Schlagworte: Distanzfunktion, Abbildung, Topologie
Key words: distance function, mapping, topology

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Contents

A GENERAL CONCEPT OF DISTANCE

BACK TO THE ROOTS OF TOPOLOGY

APPENDIX

INTRODUCTION

> *Railroad, telephone, bicycle, automobile, air plane, and cinema revolutionized the sense of distance. [...] Distances depended on the effect of memory, the force of emotions, and the passage of time.*
>
> *Stephen Kern,*
> *The Culture Of Time And Space 1880–1918*

In everyday language, 'distance' has always been something more general than the length of a segment in some geometrical space. Instead, the concept of 'near' and 'far' is one of the more important categories in human thinking. Extracting the abstract idea from the physical phenomenon, we speak of the growing distance to an old friend, of how near we are to reach a certain goal, or how far from being jealous. It is important that quite often the interesting question is not "how much is in between x and y" but rather "how much is needed to get from x to y". This somewhat *dynamical* interpretation of distance differs from the *geometrical* one in that it does not imply any symmetry, positivity, or strictness *a priori*.

It is only natural when mathematicians, too, think of their objects as being related by the one or other kind of distance—and how surprising is it that we still require mathematical distances to be real-valued, mostly symmetric, and non-negative? Before 1900, mathematical distances had beed used mainly in geometry and as a measure of difference between real numbers or functions. They had also played an important role for the clarification of the notion of 'real number' itself, which in turn was a strong impetus for the development of topology. In the beginning of the last century, when Fréchet [Fré05, Fré06, Fré28] and Hausdorff [Hau14, Hau27, Hau49] initiated the axiomatic study of distances in the general setting of metric instead of geometric spaces, the real numbers were therefore the natural candidates for the values of a distance function. Complex numbers or real vectors, being imaginable alternatives, would most certainly not have been considered suitable because of the difficulties in ordering such entities—given that partial orders had not received much attention at that time. On the other hand, rational numbers had already long been known to be too special because of their lacking completeness. Just as in case of measure theory, it is therefore not surprising that the theoretical treatment of distances was dominated by a paradigm of using real numbers.

Although, from the beginning, general topology was far more than the study of metric spaces, the question of which topological spaces can be endowed with a suitable metric, known as the 'metrization problem', remained important. This was not only because metric spaces had very nice topological properties, mostly inherited from even nicer properties of the real numbers themselves, but also since the idea of distance remained a principal intuition in building new topological concepts, and because topological spaces alone had not enough structure to formulate certain interesting notions. For example, Lipschitz- and uniform continuity, or completeness, being of great importance in real analysis, cannot be expressed in terms of open sets alone.

This motivated the search for suitable structural additives to general topological spaces, which could well have led to an early study of substantially more general distance functions than real metrics. But despite only a few attempts in the latter direction, the researchers in this field soon focused on systems of subsets instead, ending up with the notion of 'uniform space' (cf. [BHH98]). However, there were situations when distances had a great chance of being reconsidered—passing virtually unnoticed. Van Dantzig [vD32], for instance, defined fundamental sequences in a topological group, using Menger's 'Gruppenmetrik' [Men31] without recognizing it as a distance function. Even more surprisingly, Kelley essentially proved that every uniformity (even every quasi-uniformity) comes from a family of real-valued distance functions [Kel55], but despite the popularity of his classical textbook, the theory of uniform spaces did not yet enter a possibly fruitful engagement with a theory of vector-valued metrics.

The aim of this thesis is threefold: to reinstate distance functions as a principal tool of general topology; to promote the use of distance functions on various mathematical objects and a thinking in terms of distances also in non-topological contexts; and to make more specific contributions by characterizing important classes of mappings and generalizing some important topological results.

In order to present 'distance' as an interesting concept in its own right, independent from geometry and topology, I will start with the notion of 'distance sets', leaving aside all topological considerations until Part B. In

Chapter 1, examples from throughout mathematics are used to illustrate the frequent occurrence of natural non-real, non-symmetric, or non-positive distance functions, where the distance from x to y will often be expressing a 'least thing necessary to get y from x' instead of a measure of the 'space between x and y'. In Chapter 2, the key concepts of 'preservation of formulae' and 'translation of distances' are used to extract interesting classes of mappings between sets with the same or different type of distance function, leading to the characterization of many well-known types of mappings in terms of distance functions, such as affine maps, or homomorphisms between graphs, lattices, fields etc. Concluding Part A, I then give a 'canonical' method for constructing distance functions, illustrated with an application in logics.

Entering the realm of topology, Part B begins with the development of a suitable notion of 'distance spaces' which will be general enough to cover all at least moderately well-behaved topological structures. In particular, it is shown in Chapter 3 that all T_1 pre-topological spaces and all uniform frames can be induced from distance spaces. At the end of Chapter 4, these results will be joined by the proof that even most finite systems of quasi-uniformities on a set come from a single distance structure, this construction building the most technical section of the thesis. The remainder of Chapter 4 deals with all kinds of mappings between distance spaces, giving counter-examples and proving certain relationships, most notably between traditional forms of continuity and the new concept of 'strong uniform continuity'. Touching classical geometry, I also characterize the similarity maps between Euclidean spaces as those maps preserving the equality of distances. Chapters 5 and 6, finally, are dedicated to two traditionally central fields of topology: they contain generalizations of known completions and fixed point theorems.

As a supplement of a more applied nature, a self-contained chapter about the visualization of distances by means of different algorithms can be found in the appendix.

Terminology and notation are mostly standard and have been changed only in a few cases. For the reader's convenience, newly introduced notation and terminology is always indicated in the margin. To address a wider audience, I have abstained from using category theory as a language; it's usage would have shortened only a few arguments.[1] On the other hand, a certain amount of order-theory is used throughout (see [Ern82] for an introduction). Some proofs are structured by putting details into double square brackets $[\![\dots]\!]$, and these are also used for inline proofs. Moreover, the application of choice principles such as the Axiom of Choice has been made explicit by ending the affected proofs with a sign like $\boxed{\text{AC}}$.

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[1] For categorical aspects of distance sets, see [Hei02].

A GENERAL CONCEPT OF DISTANCE

1.

DISTANCE SETS

> *Entfernung, du, die über Herzen treuer*
> *Als Blick und Schwur belehren kann,*
> *Du bist der Liebe, was der Wind dem Feuer:*
> *Ein kleines löscht er aus, ein großes facht er an.*
>
> *Haug, Epigramme*

Definitions

d X $+$ $\leqslant$

A (general) distance function d assigns to each pair (x,y) of elements of a set X of "points" a distance $d(x,y)$ "from x to y" such that the triangle inequality holds, and such that the distance is zero in case that $x = y$. The distances need not be real numbers, but the co-domain of d must of course provide enough structure to state the *triangle inequality* $d(x,y) + d(y,z) \geqslant d(x,z)$. In this formula, $+$ is meant to be an "addition" and $\leqslant$ is meant to be some sort of "order", and these should clearly satisfy a certain amount of compatibility.

quasi-ordered monoid $\underline{M}$ 0 $\sim$ *p. o. m.*

A *quasi-ordered monoid* (or *q. o. m.* for short) is a quadruple $\underline{M} = (M,+,0,\leqslant)$ such that $+$ is an associative binary operation on M, 0 is a neutral element for $+$ (that is, $\alpha + 0 = 0 + \alpha = \alpha$ for all $\alpha \in M$), $\leqslant$ is a quasi-order on M (that is, a reflexive and transitive binary relation), and $+$ is isotone in both components (that is, $\alpha + \beta \leqslant \alpha' + \beta'$ whenever $\alpha \leqslant \alpha'$ and $\beta \leqslant \beta'$). Note that $+$ need not be commutative. The additive notation with the symbols $+$ and 0 instead of $\circ$ and 1 is used only because it resembles standard metric space notation. Also, $\leqslant$ need not be a *partial* order (that is, antisymmetric), and the symbol $\sim$ will denote its symmetric part, that is, $x \sim y :\Longleftrightarrow x \leqslant y \leqslant x$. If $\leqslant$ is antisymmetric, $\underline{M}$ will be called a *partially ordered monoid*, or *p. o. m.* for short.

$\underline{M}$-distance function

Now, given a set X and a q. o. m. $\underline{M}$, an *$\underline{M}$-distance function* on X is a map $d : X^2 \to \underline{M}$ with $d(x,x) = 0$ and $d(x,y) + d(y,z) \geqslant d(x,z)$. The triple $\underline{X} = (X, d, \underline{M})$ will then be called a *distance set*.[1]

$\underline{X}$

distance set

symmetric

positive

Note that d need not be *symmetric* (which would mean $d(x,y) = d(y,x)$ for all $x,y \in X$). Therefore, we shall not speak about the distance 'between x and y' but rather about that 'from x to y'. Also, distance functions need not be *positive* (which would mean $d(x,y) \geqslant 0$ for all $x,y \in X$).[2] There are many other special properties a distance set might have, some of which will be introduced in the examples below or even later in the text. At the beginning I only mention the following four separation axioms. The distance set $\underline{X}$ is said to be . . .

T_0

T_1

two-way separated

$$\left.\begin{array}{rl} T_0 & :\Longleftrightarrow\ d(x,y) \nleqslant 0 \text{ or } d(y,x) \nleqslant 0 \\ \textit{two-way separated} & :\Longleftrightarrow\ d(x,y) \not> 0 \text{ and } d(y,x) \not> 0 \\ T_1 & :\Longleftrightarrow\ d(x,y) \nleqslant 0 \text{ and } d(y,x) \nleqslant 0 \end{array}\right\} \begin{array}{c} \text{for all} \\ x,y \in X, \\ x \neq y. \end{array}$$

LEMMA 1.1. ([Hei98]) *For a distance function $d : X^2 \to \underline{M}$:*

1. *d is symmetric $\Longrightarrow 2d \geqslant 0$.*
2. *d is T_1 $\Longrightarrow$ d is two-way separated $\Longrightarrow$ d is T_0 $\Longleftrightarrow d(x,y) \not> 0$ or $d(y,x) \not> 0$ for all $x \neq y$. No other implications hold between these properties in general.*
3. *For symmetric distance functions, all three separation properties are equivalent and will be summarized under the name* separatedness.

separatedness

The remainder of this first chapter mainly contains a large number of examples of distance functions for various kinds of mathematical objects, beginning with the classical case of real-valued distances. Most of these examples will again be discussed in Chapter 2 under the aspect of mappings between distance sets. If nothing else is stated explicitly, Greek letters α, β, . . . will always refer to elements of M, while Roman letters x, y, . . . will always refer to elements of X.

Real distances

EUCLIDEAN DISTANCE

The oldest and most frequently used distance function is certainly Euclidean distance between points in the plane, which is nothing else than the length of segments. Using cartesian coordinates and generalizing to n dimensions, one has

e

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[1] The word 'space' is avoided here because it usually refers to topological structures which will not be considered in this first part. In Part B, however, I will define distance spaces as distance sets with an additional structural ingredient.

[2] Perhaps a more precise term would be the lengthy 'positively semidefinite'.

$$e(x,y) := \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} \quad \text{for } x,y \in \mathbb{R}^n.$$

$\underline{\mathbb{R}}^+$

The range of e is the interval $\mathbb{R}^+ := [0,\infty)$ of non-negative real numbers, and the triangle inequality[1] holds with respect to the ordinary addition $+$ and order $\leqslant$ on $\mathbb{R}^+$. In addition to being commutative and totally ordered, this p. o. m.

$$\underline{\mathbb{R}}^+ := \big([0,\infty),+,0,\leqslant\big)$$

$\underline{\mathbb{E}}_n$

$\underline{\mathbb{R}}$

has of course many other nice properties beyond the minimal requirements described above, and we will come across them frequently. The resulting distance set $(\mathbb{R}^n, e, \underline{\mathbb{R}}^+)$ will be designated by the symbol $\underline{\mathbb{E}}_n$.

Of course, e can also be interpreted as a distance function with co-domain

$$\underline{\mathbb{R}} := (\mathbb{R},+,0,\leqslant),$$

$\underline{\mathbb{R}}^\top$

the p. o. m. of real numbers, or even with co-domain

$$\underline{\mathbb{R}}^\top := (\mathbb{R} \cup \{\infty\},+,0,\leqslant),$$

extended real numbers

∞

top element

absorbing

$\underline{\mathbb{R}}^{+\top}$

the p. o. m. of *extended real numbers*. In the latter, the element ∞ behaves as usual: it is the *largest* (or *top*) element (that is, $\infty \geqslant \alpha$ for all α) and it is *absorbing* ($\alpha + \infty = \infty = \infty + \alpha$ for all α). The term '*real* distance function' will be used for all $\underline{\mathbb{R}}^\top$-distance functions in the following.

As usual, distance functions with co-domain

$$\underline{\mathbb{R}}^{+\top} := \big([0,\infty],+,0,\leqslant\big)$$

quasi-pseudometric

qp-metric

pseudometric

metric

spaces

are called *quasi-pseudometrics* (*qp-metrics*), the prefixes 'quasi' and 'pseudo' designating the potentially missing symmetry and separatedness. As a symmetric qp-metric, Euclidean distance is a *pseudometric*, and because it is also separated, it is even a *metric*. Following standard terminology, the corresponding distance sets are here also called qp-metric *spaces*.[2] For a "quasi-metric" one could either require T_0 or T_1, and because of the latter ambiguity, this term will not be used here. In concordance with most authors, the value ∞ will always be allowed for (qp-)metrics, while in the literature also the (potentially confusing) term 'extended metric' is used for such distance functions. As we will see in Chapter 2, it makes no essential difference to enlarge the domain of a distance function

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[1] Contrary to our modern "metric" approach, Eukleides of Alexandria did not seem to consider the triangle inequality so fundamentally evident as to postulate it *a priori*. In the first book of the traditional version of his *Elements* [Euk62], it is proved as Proposition 20, indirectly using four (of five) "postulates", six (of nine) "axioms", and fourteen of the preceding propositions. Looking at that proof, the triangle inequality appears as a corollary to Propositions 18 and 19 (which state that the largest angle of a triangle opposes the largest side). It is then used at least indirectly in ten of the remaining 28 propositions of Book I.

[2] It will become clear in Part B that certain distance sets like these may indeed be interpreted as distance spaces since they admit a canonical zero-filter.

by some additional elements, so we may also count all qp-metrics as real distance functions.

There is a large collection of literature on metric spaces (recent textbooks include [KK01, Cam00, Vra90, Köh88, Gil87, Rei82, Lim77, Kap72, Pit72, Cop68]), and so the following examples of real distance sets focus on non-symmetric ones.

REFLEXIVE RELATIONS OR DIRECTED GRAPHS

If R is a reflexive relation on X, we may define a T_1 qp-metric by

d_R

$$d_R(x,y) := \bigwedge\{n \in \omega \,|\, x\, R^n\, y\},$$

where ω is the set of natural numbers including zero, and $R^0 := \Delta_X := id_X = \{(x,x) \,|\, x \in X\}$ is the *diagonal* or *identity* relation on X. Since $x\,R\,y$ if and only if $d_R(x,y) \leqslant 1$, this is perhaps the simplest example of a mathematical structure which is completely determined by its distance function. Actually, d_R has only natural numbers and ∞ as its values, so it might also be interpreted as having as co-domain the submonoid

ω

Δ_X

diagonal

identity relation

$\underline{\omega}^\top$

$$\underline{\omega}^\top := (\omega \cup \{\infty\}, +, 0, \leqslant)$$

of $\underline{\mathbb{R}}^{+\top}$. The following lemma is an easy exercise and should give a feeling for the relationship between R and d_R.

LEMMA 1.2. *For a reflexive relation R on X:*

1. *R is symmetric* $\Longleftrightarrow$ *d_R is symmetric.*
2. *R is antisymmetric* $\Longleftrightarrow d_R(x,y) + d_R(y,x) \geqslant 3$ *whenever* $x \neq y$.
3. *R is acyclic* $\Longleftrightarrow d_R(x,y) + d_R(y,x) = \infty$ *whenever* $x \neq y$.
4. *R is transitive* $\Longleftrightarrow d_R(x,y) \in \{0,1,\infty\}$ *for all* x,y
$\Longleftrightarrow d_R(x,y) \vee d_R(y,z) \geqslant d_R(x,z)$ *for all* x,y,z.
5. *R is total* $\Longleftrightarrow d_R(x,y) \wedge d_R(y,x) \leqslant 1$ *for all* x,y.

In 2. and 3., a typical *modification* of an $\underline{M}$-distance function d is used: its *additive symmetrization*

additive symmetrization

$$d^S(x,y) := d(x,y) + d(y,x),$$

d^S

which is a symmetric $\underline{M}$-distance function at least when $+$ is commutative.

upper symmetrization
d^s
lower symmetrization
$s_{d,\wedge}$
d^0

In the non-commutative case, one might instead use the *upper symmetrization*

$$d^s(x,y) := d(x,y) \vee d(y,x),$$

which of course requires the existence of all these suprema. The dual concept of a *lower symmetrization*

$$s_{d,\wedge}(x,y) := d(x,y) \wedge d(y,x),$$

which is used in 5. above, fulfils the triangle inequality only when $s_{d,\wedge}(x,z) \leqslant d(x,y) + d(z,y)$ for all x,y,z (at least for sufficiently nice monoids, cf. [Hei98]). More generally, the supremum $\bigvee_{i \in I} d_i$ of a family $(d_i)_{i \in I}$ of $\underline{M}$-distance functions on X is again a distance function (if it exists), in particular, $d^0 := d \vee 0$ is a positive distance function.

ultra-$\underline{M}$-distance function

The last condition in 4. is nothing else than the triangle inequality with the operation $+$ replaced by the binary supremum operation in $\underline{M}$. Such an $\underline{M}$-distance function will be called an *ultra-$\underline{M}$-distance function.* Hence, R is transitive if and only if d_R is an *ultra-qp-metric.*

directed graph
$V(G)$
vertices
$E(G)$
edges
arrows

In discrete mathematics, a set X with a reflexive relation R is often interpreted as a *directed graph* (or *digraph* for short) $G = (V,E)$, consisting of a set $V = V(G)$ of *vertices* or *nodes* and a set $E = E(G) \subseteq V^2$ of *edges* or *arrows.* All digraphs here are understood to be *simple* (have at most one edge from x to y) and *loop-less* (have no arrow from x to x). The obvious translation of a relation R on X into a digraph is to put $V := X$ and $E := R \setminus \Delta_X$, that is, the arrows are exactly the related pairs of distinct elements of X.

walk
length
shortest walk
d_G
(undirected) graph
segment
$\overline{xy}^d$

Although reflexive (or, alternatively, irreflexive) relations and digraphs are essentially the same thing, graph theory usually focuses on different properties than relation or order theory, many of which are connected to the notion of a *walk* (always assumed finite here), which is just a tuple $(x_0,\dots,x_n)$ of vertices of which each successive pair x_i,x_{i+1} is joined by an edge $(x_i,x_{i+1}) \in E$. The *length* of the walk $(x_0,\dots,x_n)$ is then just n, the number of edges in it. A *shortest walk* from x to y is one whose length equals the *distance* $d_G(x,y)$ from x to y, which is defined as the infimum over all lengths of walks from x to y. Although this is obviously only a reformulation of the definition of d_R in terms of walks, the graph interpretation is much more natural as it corresponds to the most intuitive meaning of distance. For *(undirected) graphs* $G = (V,E)$ (that is, with $E \subseteq \{e \subseteq V \mid |e| = 2\}$ instead of $E \subseteq V^2$), all these notions are defined analogously. In this case d_G is symmetric. The following characterization of certain classes of undirected graphs makes use of the concept of a *segment*

$$\overline{xy}^d := \big\{z \in X \mid d(x,y) = d(x,z) + d(z,y)\big\}$$

in a distance set, which in case of graphs is also called an *interval*.[1]

PROPOSITION 1.3. *For an undirected graph G:*

1. *G is connected* $\iff d_G(x,y) < \infty$ *for all* x,y.
2. *If G is connected:*
   G is bipartite $\iff d_G(x,y) + d_G(y,z) + d_G(z,x)$ *is always even.*
3. *If G contains an edge but no circles of length 4:*
   G is a tree $\iff \overline{xy}^{d_G} \cap \overline{yz}^{d_G} \cap \overline{zx}^{d_G} \neq \emptyset$ *for all* x,y,z.

While 1. is trivial, the proofs of 2. and 3. can be found in the appendix.

QUASI-ORDERS

For a *transitive* reflexive relation (that is, a quasi-order) $\leqslant$ on X,[2] one can define an even more natural distance function by

$d_\leqslant$

$$d_\leqslant(x,y) := 1 - \chi_\leqslant(y,x) = \begin{cases} 0 & \text{if } x \geqslant y \\ 1 & \text{otherwise,} \end{cases}$$

where $\chi_\leqslant$ is the *characteristic* or *indicator function* of $\leqslant$, that is, $\chi_\leqslant(x,y)$ is 1 or 0 depending on whether $x \leqslant y$ or $x \nleqslant y$.

χ_A

indicator function

The idea is that when x dominates y, there should be no distance from x to y, while otherwise there should. This distance can not only be interpreted as an ultra-qp-metric that takes only two values, but also as one with values in the monoid

$\underline{2}$

$$\underline{2} := \big(\{0,1\}, \vee, 0, \leqslant\big).$$

Perhaps the most adequate distance function on a quasi-ordered set $(X, \leqslant)$ is in fact the characteristic function $\chi_\leqslant$ itself, interpreted as having the monoid $\underline{2}' := (\{0,1\}, \wedge, 1, \geqslant)$ of *binary truth values* as its co-domain. The reader's probable suspicion that the $\underline{2}$-distance function $d_\leqslant$ and the $\underline{2}'$-distance function $\chi_\leqslant$ are essentially the same thing is of course correct, and Chapter 2 will provide us with the suitable notions to make this statement precise. It will then also become clear that the choice between $\mathbb{R}$ and $\underline{2}$ as a co-domain *does* make a difference.

$\underline{2}'$

binary truth values

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[1] More definitions: a shortest walk is always a *path* (a walk whose vertices are all distinct). A *circle* is a *closed* walk (i. e., with $x_0 = x_n$) of length $\geqslant 3$ with $|\{x_0, \dots, x_n\}| = n$. An *odd* or *even* circle is one of odd or even length, respectively. A graph is *connected* if it contains a walk from x to y for all $x, y \in V$. It is *bipartite* or a *tree* if it contains no odd circles or no circles at all, respectively.

[2] If no confusion is likely, I will often use the same symbol for different entities, hence the quasi-order $\leqslant$ on X is of course different from the quasi-order $\leqslant$ on the reals in this example.

Table I. Equivalences between properties of a relation R and the function $1 - \chi_R$.

R	$1 - \chi_R$
reflexive	zero-distance condition
transitive	triangle inequality
quasi-order	qp-metric
symmetric	symmetric
equivalence relation	pseudometric
antisymmetric	T_0
identity	metric

Having only two values, $d_\leqslant$ is quite a "coarse" kind of distance for a quasi-ordered set, and we will learn later on that it can be generalized to a much "finer" and "internal" distance when $\leqslant$ has additional properties. Also, we will see in the next chapter how this idea of fineness can be made precise.

$\leqslant_d$

Despite this coarseness, the quasi-order $\leqslant$ is completely determined by $d_\leqslant$ as it was already the case with digraphs and reflexive relations, but here the relationship is in some respects stronger than that between R and d_R. While most $\underline{\omega}^\top$-distance functions (for example those that do not take the value 1) are not the d_R of some relation R, *every* $\underline{2}$-distance function d belongs to a quasi-order. More precisely, $\leqslant \, = \, \leqslant_{d_\leqslant}$ and $d = d_{\leqslant_d}$, where

$$x \geqslant_d y :\Longleftrightarrow d(x,y) \leqslant 0.$$

specialization

This quasi-order $\leqslant_d$ can of course be defined for all distance functions and will be called the *specialization* of d.[1] Secondly, there is a very strong correspondence between interesting properties of $\leqslant$ and those of $d_\leqslant$. Table I shows these equivalences in terms of an arbitrary relation R and the function $1 - \chi_R$. Thirdly, this bijection between the *objects* $(X, \leqslant)$ and $(X, d_\leqslant)$ is accompanied by one between some natural classes of *mappings*. Such relationships will also be described in the next chapter.

Multi-real distances

The first step of emancipation from the real number paradigm is quite easy. The idea is simply to replace the one real value of the distance function by a family or vector of real values, expressing disparate components of the distance. Although this kind of multi-valuedness could be expressed by sets or families

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[1] As an exception to the general rule of not changing common terminology, I deviate here from the usual definition. The reason for this negligible difference will become clear when we are concerned with topological structures in Part B.

of metrics instead of just one metric, doing so would give the treatment a new quality, suggesting that things become more complex and that new formalisms and methods would be required—which is not the case. It is much more simple: the co-domain $\underline{\mathbb{R}}^\top$ is replaced by a power monoid of the form

$(\underline{\mathbb{R}}^\top)^I$

$$(\underline{\mathbb{R}}^\top)^I = \big((\mathbb{R} \cup \{\infty\})^I, +, 0, \leqslant\big),$$

where I is some arbitrary index set, and $+$ and $\leqslant$ are the usual component-wise addition and order. Distance functions with such a co-domain will be called *multi-real*, and likewise the terms *multi-qp-metric* and [*multi-pseudometric*] will be used for the positive [and symmetric] ones among them.[1]

multi-real

multi-qp-metric

multi-pseudometric

EXAMPLE 1.4. In multivariate analysis of opinion polls, the set X of queried persons is often partitioned by means of a clustering algorithm that requires the definition of a distance function on X. At first, each of the different variables $v_1, \ldots, v_k$ (such as age, gender, educational level, etc.) leads to a different pseudometric d_i (for example, difference in age, or number of steps between educational levels, or 0 resp. 1 for same or different gender, respectively). In order to obtain a real metric, these components are usually combined to a weighted sum. However, in order to avoid the problem of finding suitable weights, one can also define clustering algorithms that directly work with the multi-real distance $(d_1, \ldots, d_k)$.

DISTANCES IN FUNCTION SPACES

The second example of a multi-qp-metric comes from functional analysis. In a function space such as $C([0,1])$, the real vector space of continuous real-valued functions on the unit interval, one traditionally uses a whole continuum of metrics defined via the so-called *L^p-norms*

L^p-norm

$\|f\|_p$

$$\|f\|_p := \left(\int_0^1 |f(x)|^p \, dx\right)^{1/p} \qquad \text{for } p \geqslant 1.$$

It is a natural idea to collect them together into the multi-pseudometric

d_L

$$d_L : \begin{cases} C([0,1])^2 & \rightarrow & (\underline{\mathbb{R}}^\top)^{[1,\infty)} \\ (f,g) & \mapsto & d_L(f,g) : \begin{cases} [1,\infty) & \rightarrow & \underline{\mathbb{R}}^\top \\ p & \mapsto & \|f-g\|_p. \end{cases} \end{cases}$$

Another idea is that the distance between two entities, in this case functions, could be an entity of the same kind:

$d_{\text{ptw.}}$

---

[1] The term 'multi-metric' should be avoided because of its ambiguity: it could either be used for an arbitrary separated multi-pseudometric or it could mean that all components are metrics.

$$d_{\text{ptw.}}\colon \begin{cases} C([0,1])^2 & \to \quad (\underline{\mathbb{R}}^\top)^{[0,1]} \\ (f,g) & \mapsto \quad d_{\text{ptw.}}(f,g)\colon \begin{cases} [0,1] & \to \quad \underline{\mathbb{R}}^\top \\ x & \mapsto \quad |f(x)-g(x)|. \end{cases} \end{cases}$$

When interpreted as having $C([0,1])$ itself as a co-domain, $d_{\text{ptw.}}$ becomes an example of an *internal* distance function.

internal distance

In the transition from $\underline{\mathbb{R}}^\top$ to $(\underline{\mathbb{R}}^\top)^I$, almost all nice properties of the reals are preserved, so that much of what can be proved about pseudometrics can also be proved about multi-pseudometrics. There are mainly two relevant differences. The first one is that the component-wise order of $(\underline{\mathbb{R}}^\top)^I$ is not total but partial, and there are certain applications of metrics where comparability of all distances is essential (as for example the generalized version of Brouwer's fixed point theorem in Chapter 6). The second difference only shows up when I is uncountably infinite as in case of $d_{\text{ptw.}}$ (see Chapter 3).

SET FUNCTIONS

set function

$\mathscr{P}(X)$

d_μ

The following construction can be used to deal with both closure systems and probability spaces, for example. Assume that on a set X, a non-negative *set function* $\mu : \mathscr{P}(X) \to [0,\infty)$ is given, where $\mathscr{P}(X)$ is the power set of X. For each subset A of X, define a quasi-order $A^{\Rightarrow} := \big((X \setminus A) \times X\big) \cup (X \times A)$, and put

$$d_\mu\colon \begin{cases} X^2 & \to \quad (\underline{\mathbb{R}}^\top)^{\mathscr{P}(X)} \\ (x,y) & \mapsto \quad d_\mu(x,y)\colon \begin{cases} \mathscr{P}(X) & \to \quad \underline{\mathbb{R}}^\top \\ A & \mapsto \quad \mu(A)d_{A^\Rightarrow}(x,y) = \\ & \qquad \begin{cases} 0 & \text{if } y \notin A \text{ or } x \in A \\ \mu(A) & \text{if } y \in A \text{ and } x \notin A. \end{cases} \end{cases} \end{cases}$$

In other words, $d_\mu(x,y)(A)$ is either 0 or $\mu(A)$, depending on whether the proposition "if y is in A then x is in A" is true or false, respectively. In still other words, the A-component $(d_\mu)_A$ of d_μ says that it "costs" $\mu(A)$ to get from outside of A into A. In this example, the original information μ is *almost* completely coded into d_μ since $\mu(A) = d_\mu(x,y)(A)$ for all $x \in X \setminus A$ and $y \in A$, but such a pair x,y only exists if $A \notin \{\emptyset, X\}$. The same information can also be extracted from d_μ without knowing which component of d_μ belongs to A, that is, from the set $D := \{(d_\mu)_A \mid A \subseteq X\}$ of components alone ⟦For all $e \in D$, put $\alpha_e := \max_{x,y} e(x,y)$ and $A_e := \{y \in X \mid e(x,y) > 0$ for some $x \in X\}$. Then, for all $A \notin \{\emptyset, X\}$, $\mu(A) = \alpha_e$ for the unique $e \in D$ with $A_e = A$ if such an e exists, or $\mu(A) = 0$ if no such component e exists⟧.[1]

---

[1] In this thesis, a pair of double square brackets ⟦ . . . ⟧ is used to parenthesize proof details for the claim preceding these brackets.

There is also a possibility to code μ into a multi-pseudometric:

d'_μ

$$d'_\mu\colon \left\{\begin{array}{lcl} X^2 & \to & \underline{\mathbb{R}}^{\top\mathscr{P}(X)} \\ (x,y) & \mapsto & d'_\mu(x,y)\colon \left\{\begin{array}{lcl} \mathscr{P}(X) & \to & \underline{\mathbb{R}}^{\top} \\ A & \mapsto & \left\{\begin{array}{ll} 0 & \text{if } x=y \\ \mu(A) & \text{if } x\neq y \text{ and } x,y\in A \\ \infty & \text{otherwise.} \end{array}\right. \end{array}\right. \end{array}\right.$$

Here $\mu(A)$ can be recovered from d'_μ if and only if $|A| \geqslant 2$, and again this is possible from $D' := \{(d'_\mu)_A \mid A \subseteq X\}$ alone $[\![$For all $e \in D'$, put $\alpha_e := \min_{x\neq y} e(x,y)$ and $A_e := \{y \in X \mid e(x,y) < \infty$ for some $x \in X$, $x \neq y\}$. Then, if $|A| \geqslant 2$, $\mu(A) = \alpha_e$ for the unique $e \in D'$ with $A_e = A]\!]$.

The latter construction also shows that when μ is *isotone* (that is, $A \subseteq B \Longrightarrow \mu(A) \leqslant \mu(B)$) and fulfils $\mu(\{x\}) = 0$ for all x, it is the infimum of a set of *diameter functions*, namely those corresponding to the components of d'_μ. Diameters will be discussed in Chapter 3.

diameter functions

Given a *probability space* $(X,\mathscr{A},P)$ (that is, with $\mathscr{A}$ a σ-algebra on X and $P : \mathscr{A} \to [0,1]$ a normalized σ-additive measure), we can define $\mu(A) := P(A)$ for $A \in \mathscr{A}$ and $\mu(A) := 0$ for $A \in \mathscr{P}(X) \setminus \mathscr{A}$, and then use d_μ as a distance on X. In many cases, $P(\{x\}) = 0$ holds for all $x \in X$, so that then also d'_μ can be used. Since $\mathscr{A}$ is closed under complements, $A \in \mathscr{A}$ holds if and only if $\mu(A) \vee \mu(X \setminus A) > 0$, hence the whole probability space structure can be recovered from d_μ resp. d'_μ.

probability space

Distances in classical algebraic structures

GROUPS

Abelian lattice-ordered groups. As we have already seen, the function $d_{\text{ptw.}}(f,g) = |f-g|$ on $C([0,1])$ can be interpreted as an internal distance function. This is because $C([0,1])$ is not only a vector space but also a p. o. m. under the pointwise order. But even more so, its addition is (i) a commutative group operation and its order provides (ii) binary suprema, since the supremum of finitely (in contrast to infinitely) many continuous functions is again continuous. Such a p. o. m. (having (i) and (ii)) also provides binary infima and is called an *abelian lattice-ordered group* or *abelian ℓ-group* for short (cf. [Goo86]). All lattice-ordered groups allow for the definition of a sub-additive *absolute value*

abelian lattice-ordered group

abelian ℓ-group

absolute value

$$|x| := x \vee (-x),$$

$|x|$

so that one has a symmetric operation $|x-y|$ with $|x-x|=0$ on every ℓ-group. In the commutative case, also the triangle inequality holds:

$$\begin{aligned}|x-y|+|y-z| &= \big((x-y)\vee(y-x)\big)+\big((y-z)\vee(z-y)\big)\\ &\geqslant (x-y+y-z)\vee(y-x+z-y)\\ &= (x-z)\vee(z-x)=|x-z|.\end{aligned}$$

On the other hand, it is easy to show that the triangle inequality in turn implies commutativity ⟦For $x,y\geqslant 0$, $x+y=|(-x)-0|+|0-y|\geqslant|(-x)-y|=|y+x|=y+x$, hence $x+y=y+x$. Since in an ℓ-group every element is a difference of positive ones, this suffices⟧. Holland [Hol85] shows that also $n|x-y|$ is a distance function if and only if the group is abelian.

Abelian partially ordered groups. In the theory of abelian *partially* ordered groups $(G,+,0,\leqslant)$, as presented by Goodearl [Goo86] for example, one also studies *real* distance functions on G whose definitions do not require the existence of suprema in G. Given any "unit" $u\in G$, one can define

$\|x\|_u$

$$\|x\|_u:=\bigwedge\left\{\frac{k}{n}\mid k,n\in\omega\setminus\{0\},\ -ku\leqslant nx\leqslant ku\right\}\in[0,\infty],$$

which always fulfils $\|x\|_u+\|y\|_u\geqslant\|x+y\|_u$ and $\|mx\|_u=|m|\|x\|_u$ for all $m\in\mathbb{Z}\setminus\{0\}$. In order that $\|\cdot\|_u$ is a pseudonorm, one only needs that also $\|0\|_u=0$, which is equivalent to $ku\geqslant 0$ for some $k>0$. When u is even an *order-unit*, that is, when each $x\in G$ is dominated by some ku with $k>0$, this *order-unit norm* is finite.[1] The induced pseudometric is $d_u(x,y):=\|x-y\|_u$., and we may combine all d_u with $u\geqslant 0$ to a multi-pseudometric $d_U:G^2\to(\underline{\mathbb{R}}^\top)^{G^+}$ by putting $d_U(x,y)(u):=d_u(x,y)$ for all $u\in G^+:=\{u\in G\,|\,u\geqslant 0\}$.

order-unit

order-unit norm

d_u

d_U

PROBLEM 1.5. For which groups can the group operation be recovered from this multi-pseudometric?

Arbitrary groups. A seemingly more trivial definition of an internal distance in an arbitrary group $(G,\circ)$ is

d_G

$$d_G(x,y):=x^{-1}y,$$

which is a two-way separated distance function in every group. Moreover, it is *skew-symmetric*, that is, the triangle inequality is in fact an equation (cf. [Hei98] for different characterizations of skew-symmetric distances). This distance was already introduced by Karl Menger in 1931 [Men31] and is probably the first example of a non-real distance function in the literature. But what is its value

skew-symmetric

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[1] Only these pseudonorms are actually considered by Goodearl.

monoid, or rather, what is its order relation? For a skew-symmetric distance function, the triangle equation holds irrespective of the order, so we might take any quasi-order on G that is compatible with $\circ$, for instance the identity relation. This gives the p. o. m.

$$(G,\circ,e,\Delta_G),$$

where e is the group's unit. Although this might seem quite ridiculous at the moment, we will however see in Example 3.11 that it leads to a "metrization" of all T_1 semigroup topologies of a group. For the moment, the following exercise might suffice as a motivation for choosing $x^{-1}y$ as a distance function.

EXERCISE 1.6. Express the conditions for upper and lower semi-continuity of a function $f:\mathbb{R}\to\mathbb{R}$ in terms of the two distance functions $e(x,y)=|x-y|$ and $d(x,y):=y-x$.

Note that when the group is abelian, and a lattice-order is used instead of the identity, the upper symmetrization of d_G is just the distance function $|x-y|$ defined above.

What about semigroups instead of groups? A commutative semigroup $(S,+)$ which is *cancellative*, that is, satisfies the cancellation law

cancellative

$$x+z=y+z\implies x=y,$$

can always be embedded into a group G so that d_G can be used. Another generalization of d_G to certain ordered semi-groups follows.

LOWER DISTRIBUTIVITY

Let me now introduce to you the perhaps most fruitful type of value monoid $\underline{M}$ for the general theory of distance functions: the *co-quantale*. It not only features *completeness*, that is, all subsets have both an infimum and a supremum, but at least the infima are also quite well behaved in that they fulfil[1]

co-quantale

completeness

$$\alpha+\bigwedge B=\bigwedge(\alpha+B)\quad\text{and}\quad\bigwedge A+\beta=\bigwedge(A+\beta)$$

for all $\alpha,\beta\in M$ and $A,B\subseteq M$, where $\alpha+B$ is short for $\{\alpha+\beta\,|\,\beta\in B\}$ of course. This property, which I will also call *lower distributivity* in the sequel, is of paramount importance for the success of many proofs. On the contrary, its dual (with suprema instead of infima), here called *upper distributivity*, only plays a minor role.[2] Most of us use the corresponding properties of the real numbers (which hold there only for nonempty and bounded subsets) like a duck takes

$\alpha+B$

lower distributivity

upper distributivity

---

[1] In general, only the left hand side of both equations is less than or equal to the right hand side.

[2] Completely ordered monoids that are upper distributive are usually called *quantales*. However, some authors use that term for what is here called co-quantales instead.

$d_\rightarrow, d_\leftarrow$

the water. The first application of lower distributivity in this thesis is the proof of the triangle inequality for the functions

$$\begin{aligned} d_\rightarrow(\alpha,\beta) &:= \bigwedge\{\delta \in M \mid \alpha+\delta \geqslant \beta\} \\ \text{and} \quad d_\leftarrow(\alpha,\beta) &:= \bigwedge\{\delta \in M \mid \alpha \leqslant \delta+\beta\} \end{aligned}$$

on M, which is easy enough ⟦ $d_\rightarrow(\alpha,\beta)+d_\rightarrow(\beta,\gamma) = \bigwedge\{\delta+\delta' \mid \alpha+\delta \geqslant \beta, \beta+\delta' \geqslant \gamma\} \geqslant \bigwedge\{\delta+\delta' \mid \alpha+\delta+\delta' \geqslant \gamma\} = d_\rightarrow(\alpha,\gamma)$, and likewise for $d_\leftarrow$ ⟧. However, for $d_\rightarrow$ to become a distance function it is also necessary that the implication $\alpha+\delta \geqslant \alpha \Longrightarrow \delta \geqslant 0$ holds, and likewise for $d_\leftarrow$. Otherwise, one would rather use the distance functions $d^0_\rightarrow$ and $d^0_\leftarrow$ which always fulfil the zero distance condition because of $d_\rightarrow(\alpha,\alpha), d_\leftarrow(\alpha,\alpha) \leqslant 0$. The function $d_\rightarrow$ is characterized by the equivalence

$$d_\rightarrow(\alpha,\beta) \leqslant \delta \Longleftrightarrow \beta \leqslant \alpha+\delta.$$

adjunction
lower adjoint
left adjoint
upper adjoint
right adjoint
(antitone, $\bigvee$-preserving)
($\bigwedge$-preserving, antitone)

For fixed α, this just says that the functions $f : \beta \rightarrow d_\rightarrow(\alpha,\beta)$ and $g : \delta \rightarrow \alpha+\delta$ build an *adjunction*, f being the *lower* or *left adjoint*, g being the *upper* or *right adjoint* (cf. [Ern82]). In such a situation, the upper adjoint necessarily preserves infima—which is half of the property of lower distributivity—while the lower must preserve suprema, that is, $d_\rightarrow(\alpha,\bigvee B) = \bigvee\{d_\rightarrow(\alpha,\beta) \mid \beta \in B\}$. One can summarize this and the fact that $\alpha \leqslant \alpha'$ implies $d_\rightarrow(\alpha,\beta) \geqslant d_\rightarrow(\alpha',\beta)$ by saying that $d_\rightarrow$ is *(antitone, $\bigvee$-preserving)*. Similarly, $d_\leftarrow$ is *($\bigwedge$-preserving, antitone)*. The upper symmetrizations $d_\leftrightarrow := d_\rightarrow \vee d_\leftarrow$ and $d^0_\leftrightarrow = d_\rightarrow \vee 0 \vee d_\leftarrow$ do not have such monotonicity properties.

$d_\leftrightarrow$

Note that $d_\rightarrow$ and $d_\leftarrow$ can also be defined for a non-complete p. o. m. provided that the relevant infima exist and that lower distributivity holds for all existing infima. In case of a group $(G,\circ,e,\Delta_G)$, for example, $d_\rightarrow$ is then just d_G ⟦ since $x \circ \delta = y$ is equivalent to $\delta = x^{-1}y$ ⟧.

MODULES AND RINGS

If $\underline{R} = (R,+,\cdot)$ is a ring and $(M,+,0,\cdot)$ an $\underline{R}$-module (which will here always mean a left module, that is, with ring elements multiplied from the left), then again $\underline{M} := (M,+,0,\Delta_M)$ is a p. o. m., and $d_r(x,y) := r(-x+y)$ is a skew-symmetric $\underline{M}$-distance function for each $r \in R$. Like before, we can combine all the d_r with $r \in R$ to a distance function d on M whose values are vectors of M-elements. This is done by using the p. o. m. $\underline{M}^R$ with component-wise addition and order as co-domain, just as in case of multi-real distances, and defining

d_M

$$d_M : \left\{ \begin{array}{rcl} M & \rightarrow & \underline{M}^R \\ (x,y) & \mapsto & d_M(x,y) : \left\{ \begin{array}{rcl} R & \rightarrow & \underline{M} \\ r & \mapsto & d_M(x,y)(r) := r(-x+y). \end{array} \right. \end{array} \right.$$

It is actually always possible to combine any family $(d_i)_{i\in I}$ of distance functions on a set X, even with different monoids $\underline{M}_i$, to a single distance function by using the direct product $\prod_{i\in I}\underline{M}_i$ of these monoids as the new co-domain.

The above definition of d works as well for a ring $(R,+,0,\cdot)$, being a module over itself, but there are also quite useful *real* distance functions on rings. For any $p\in R$, the *valuation* of $x\in R$ at p is

valuation

$w_p(x)$

$$w_p(x) := \bigvee\{n\in\omega \,|\, p^n a = x \text{ for some } a\in R\} \in [0,\infty],$$

which already leads to a symmetric distance function $w_p(x-y)$ whose co-domain is the co-quantale $([0,\infty],\wedge,\infty,\geqslant)$. More commonly, one defines the *p-adic "norm"*

p-adic "norm"

$\|x\|_p$

$$\|x\|_p := w_p(x)^{-1} \quad (\text{with } \infty^{-1} := 0 \text{ and } 0^{-1} := \infty)$$

which fulfils $\|0\|_p = 0$ and $\|x+y\|_p \leqslant \|x\|_p \vee \|y\|_p$, so that the induced real distance function $d_p(x,y) := \|-x+y\|_p$ is an ultra-pseudometric.[1] In case of $R=\mathbb{Z}$ and $p\geqslant 2$, d_p is separated[2] and a powerful tool in algebraic topology and geometry, especially when it is extended suitably to the field $\mathbb{Q}$ of rational numbers (see below).

d_p

For $p=0$, $d_p(x,y)$ is either 0 or ∞ depending on whether $x=y$ or not. Combining all the other p-adic distances to a multi-pseudometric, we get $d_\text{adic}(x,y) : R\setminus\{0\} \to \mathbb{R}^\top$, $p \to d_p(x,y)$. In contrast to a single d_p, d_adic is almost always separated:

d_adic

PROPOSITION 1.7. *For a ring R with* 1 *and without zero-divisors:* d_adic *is not separated* $\Longleftrightarrow$ d_adic *is constantly zero* $\Longleftrightarrow$ R *is a skew-field.*

Proof. In a skew-field, all non-zero elements are units, hence d_adic is constantly zero. On the other hand, assume that $\|x\|_p = 0$ for some $x\neq 0$ and all $p\neq 0$. In particular, $\|x\|_{x^2}\leqslant 1$, hence $x^2a = x$ for some $a\neq 0$, which implies that $xa=1$ since x is not a zero-divisor. Then x is a unit, and so is every $p\neq 0$ ⟦because $\|x\|_p\leqslant 1$ implies that p divides x⟧. ☐

This also shows that it does not make much sense to define d_adic on a (skew-)field. See below for a modification that works for quotient fields.

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[1] At least in the commutative case, one can also use an ideal I of R instead of an element p, and define an ultra-pseudometric d_I via the valuation $w_I(x) := \bigvee\{n\in\omega \,|\, x\in I^n\}$ and the "norm" $\|x\|_I := w_I(x)^{-1}$.

[2] More generally, d_p is separated for all non-units p in an integral domain (that is, a commutative ring with 1 and without zero-divisors) R that fulfils the ascending chain condition for principal ideals (accp) ⟦ $\|x\|_p = 0$ implies that, for all $n\in\omega$, we can choose $\lambda_n\neq 0$ with $x=\lambda_n p^n$. Then, in particular, $\lambda_n = \lambda_{n+1}p$ since R has no zero-divisors, and the ascending chain $(\lambda_n R)_{n\in\omega}$ of principal ideals must become stationary because of accp, that is, $\lambda_{n+1} = \lambda_n a$ for some $a\neq 0$. But then $p = \lambda_n/\lambda_{n+1}$ is a unit⟧.

In general, d_p might not be separated even when p is prime, for example, $\|x\|_2 = 0$ in $R := \mathbb{Z} + x\mathbb{Q}[x]$, the ring of polynomials in x with rational coefficients and integer constant term.

$w_p(x)$ d_{adic}

A look at the definition of w_p shows that it can also be applied to a module M over R, only that x and a are now elements of M instead of R. As above, $w_p(0) = 0$ and $w_p(x+y) \geqslant w_p(x) \wedge w_p(y)$, hence d_{adic} also exists for modules.

factorial domain

prime base

Factorial domains. Before we extend d_{adic} to quotient fields, I want to show how in a *factorial domain*, that is, in a commutative ring R with 1 in which each non-zero element has an (essentially) unique prime decomposition, the valuation w_p can also be used to define an internal distance rather than a multi-real one. In $\mathbb{Z}$, for instance, two elements $x, y > 0$ have a unique least common multiple $\operatorname{lcm}(x,y)$ and a unique greatest common divisor $\gcd(x,y)$ which are equal if and only if $x = y$. The idea is now to interpret their quotient $\operatorname{lcm}(x,y)/\gcd(x,y)$ as a distance. In a general factorial domain, there might be no natural order, or there may be units other than ±1, so that lcm and gcd might not be uniquely defined. Let us therefore assume that some *prime base* P of $R^\star$ has been fixed, that is, a maximal set of pairwise not divisible primes in $R^\star$ such as $\mathbb{P} = \{2,3,5,\dots\}$ in case of $\mathbb{Z}$. Then we have

$$x \sim \prod_{p \in P} p^{w_p(x)}$$

$d_{\text{div.}}$

for all $x \in R^\star$, where $\sim$ means mutual divisibility. Hence

$$d_{\text{div.}}(x,y) := \prod_{p \in P} p^{|w_p(x) - w_p(y)|}$$

T_0 classes

is a good candidate for a distance in $R^\star$. Indeed, $d_{\text{div.}}(x,x) = 1$, and $d_{\text{div.}}(x,z)$ divides $d_{\text{div.}}(x,y) \cdot d_{\text{div.}}(y,z)$. This is still true when we extend $d_{\text{div.}}$ to R by setting $d_{\text{div.}}(0,x) := d_{\text{div.}}(x,0) := 0$ and $d_{\text{div.}}(0,0) := 1$ for all $x \in R^\star$. Since $(R,\cdot,1)$ with divisibility as quasi-order builds a q. o. m. $\underline{M}$, we have defined a symmetric internal distance function on R that might be interpreted as a kind of symmetric "division". Note that $d_{\text{div.}}$ is separated if and only if R has at most one unit since for a unit u, $d_{\text{div.}}(x,ux)$ always equals 1. In particular, it must then have characteristic $\leqslant 2$ (that is, $1+1=0$). On the other hand, all its T_0 *classes* (that is, maximal subsets of zero diameter) have at most two elements if and only if R has no units other than ±1.

FIELDS AND VECTOR SPACES

prime

Quotient fields and vector spaces. Despite Proposition 1.7, d_p indirectly leads to interesting (multi-)real distances on a field when it is extended to the quotient field Q of a suitable ring. For Q to exist, R must be commutative with 1 and without zero-divisors, and for the extension to work, p must be *prime* and d_p must be separated. Then one first extends the valuation w_p to Q by putting $w_p(x/y) := w_p(x) - w_p(y)$. This is well-defined because $w_p(y) < \infty$ since $y \neq 0$ and d_p is separated, and because p is prime ⟦For prime p, w_p is additive on R, that is, $w_p(ab) = w_p(a) + w_p(b)$. If $x/y = x'/y'$ then $xy' = x'y$, hence $w_p(x) + w_p(y') = w_p(xy') = w_p(x'y) = w_p(x') + w_p(y)$⟧. Now $\|\cdot\|_p$ and d_p are easily extended to Q by setting $\|x/y\|_p := w_p(x/y)^{-1}$ and $d_p(x,y) := \|-x+y\|_p = \|x-y\|_p$ again. $w_p(x/y)$ d_p

It is interesting that almost the same construction is also possible in case of torsion-free modules. I did not find this fact in the literature, so the straightforward proofs are included in the appendix.

LEMMA 1.8. *Let M be a torsion-free module over a commutative ring R with* 1, *and $p \in R^\star$. Then p is M-*prime, *that is,* M*-prime*

$$rx \in pM,\ r \in R,\ x \in M \Longrightarrow r \in pR \text{ or } x \in pM,$$

if and only if M/pM is a torsion-free module over R/pR, in which case either $pM = M$ or p is prime in R. If, on the other hand, p is prime and R is a principal ideal domain then p is also M-prime. Finally, an M-prime p leads to an additive valuation $w_p : M \longrightarrow [0,\infty]$, that is,

$$w_p(rx) = w_p(r) + w_p(x)$$

for all $r \in R$ and $x \in M$.

THEOREM 1.9. (p-adic distance in quotient vector spaces). *With R and M as above, assume that $p \in R^\star$ is M-prime, d_p on R is separated, and $pM \neq M$. Then $(x,r) \sim (y,s) :\Longleftrightarrow sx = ry$ defines a congruence relation on the semigroup $(S,+)$ with $S := M \times (R \setminus \{0\})$ and $(x,r) + (y,s) := (sx+ry, rs)$. Moreover, $V := S/{\sim}$ becomes a vector space over the quotient field Q of R with the scalar multiplication $\frac{s}{t} \cdot \frac{x}{r} := \frac{sx}{tr}$, where $\frac{x}{r} := {\sim}(x,r)$. Finally, the valuation* w_p

$$w_p\left(\frac{x}{r}\right) := w_p(x) - w_p(r)$$

is additive, and $d_p(a,b) := w_p(a-b)^{-1}$ is a translation-invariant ultra-metric on V. d_p

Fields. On an arbitrary field $(F,+,\cdot,0,1)$, one can also define a two-component distance whose components are mainly the skew-symmetric distances in its additive and multiplicative groups. Put $\underline{M}_1 := (F,+,0,\Delta_F)$ and $d_1(x,y) := y-x$. For the second component, we need to deal with the special element 0. This is done by adjoining it to the p. o. m. $(F^\star,\cdot,1,\Delta_{F^\star})$ as an absorbing top element, resulting in $\underline{M}_2 := (F,\cdot,1,\leqslant)$, where $x \leqslant y \Longleftrightarrow (x-y)y = 0 \Longleftrightarrow y \in \{0,x\}$. Then $d_2(x,y) := x^{-1}y$ on $F^\star$ is extended to F by setting $d_2(0,0) := 1$ and $d_2(0,x) := d_2(x,0) := 0$ for $x \neq 0$. Now d_F

$$d_F(x,y) := \big(d_1(x,y), d_2(x,y)\big)$$

defines a distance function whose co-domain is the product $\underline{M} := \underline{M}_1 \times \underline{M}_2$.

BOOLEAN AND BROUWERIAN LATTICES

In logics, for instance, some algebraic objects are used which are quite different from groups in that they involve idempotent operations. The perhaps most important class is that of *Boolean lattices*, that is, lattices $(L,\wedge,\vee)$ with smallest and largest elements $\bot$ and $\top$, in which the *distributive laws*

Boolean lattices

distributive laws

$$\begin{aligned} x\wedge(y\vee z) &= (x\wedge y)\vee(x\wedge z) \\ \text{and}\quad x\vee(y\wedge z) &= (x\vee y)\wedge(x\vee z) \end{aligned}$$

complement operation

hold, and in which a unary *complement operation* $\neg$ is definable that fulfils

$$x\wedge\neg x = \bot \quad \text{and} \quad x\vee\neg x = \top.$$

The latter operation is then actually unique, and only one of the distributive laws is needed to prove the other. In such a structure, which can also be formalized as an algebra $(L,\wedge,\vee,\bot,\top,\neg)$, all of the "classical" laws of propositional calculus hold, for example DeMorgan's laws $\neg(x\wedge y) = \neg x\vee\neg y$ and $\neg(x\vee y) = \neg x\wedge\neg y$. An "implication" operation $\rightarrow$ is defined by $x\rightarrow y := \neg x\vee y$. Following the terminology of formal logics, I will call this operation *material implication* here to distinguish it from other kinds of implication. The most important law for $\rightarrow$ is the *cut rule*

$\rightarrow$

material implication

cut rule

$$(x\rightarrow y)\wedge(y\rightarrow z)\leqslant x\rightarrow z,$$

which is sometimes also called the 'law of transitivity' because of its resemblance to the transitive law $(x\,R\,y)\wedge(y\,R\,z)\implies x\,R\,z$ for relations. It should however rather be called the 'triangle inequality' because it is just that: $d_L(x,y) := x\rightarrow y$ is a positive, T_0 internal distance function in L with co-domain $(L,\wedge,\top,\geqslant)$. Its additive or upper (!) symmetrization is the *material equivalence* operation

d_L

material equivalence

$\leftrightarrow$

$$x\leftrightarrow y := (x\rightarrow y)\wedge(y\rightarrow x) = (x\wedge y)\vee(\neg x\wedge\neg y).$$

There are many more examples of distances that arise in the context of implication, entailment, or conditional sentences, some of which will be explored in Example 2.27. Another instance is the "intuitionistic version" of a Boolean lattice: a lattice $(L,\wedge,\vee)$ is called *Brouwerian* if and only if it has a least element $\bot$, and for all $x,y\in L$ the set $\{z\in L\,|\,x\wedge z\leqslant y\}$ has a greatest element $x\rightarrow y$. Again, $x\rightarrow y$ is a distance function: for all x,y,z we have $x\rightarrow x\geqslant y$, hence $x\rightarrow x$ is the largest element of L, and $x\wedge(x\rightarrow y)\wedge(y\rightarrow z)\leqslant y\wedge(y\rightarrow z)\leqslant z$ implies that $(x\rightarrow y)\wedge(y\rightarrow z)\leqslant x\rightarrow z$. Complete Brouwerian lattices (also called *frames* or *locales*) are special cases of quantales: when L is complete,

Brouwerian

frames

locales

the p. o. m. $(L,\wedge,\top,\leqslant)$ is upper distributive $\llbracket a \wedge b \leqslant \bigvee(a \wedge B)$ for all $b \in B$ implies that $b \leqslant a \to \bigvee(a \wedge B)$ for all $b \in B$, hence $\bigvee B \leqslant a \to \bigvee(a \wedge B)$, which in turn is equivalent to $a \wedge \bigvee B \leqslant \bigvee(a \wedge B)$ as required by upper distributivity$\rrbracket$. The dual p. o. m. $(L,\wedge,\top,\geqslant)$ (which is the correct co-domain for the distance function $\to$) is then lower distributive, hence a co-quantale, and $\to$ is then just the same as $d_{\to}$.

Some other concepts of 'generalized metric'

This list is not meant to be comprehensive since there have been many different approaches to generalize metric spaces. In particular, I only mention generalizations that use a type of distance function instead of a set system etc.

Value distributive lattices. In [Fla97], Flagg studies distance functions whose co-domains are what he calls *value distributive lattices*. Such a monoid is a commutative co-quantale that is also completely distributive (see Chapter 3 for the definition) and in which 0 is the smallest element. In particular, his distance functions are always positive.

value distributive lattices

Gauges and approach spaces. Every *approach space* (see Lowen [Low97], and Lowen and Windels [LW98]) may be interpreted as a multi-real distance set: one of the natural descriptions of approach spaces uses so-called *gauges*. These are certain ideals in the function lattice of all quasi-pseudometrics on a set X. Of course, such a gauge $\mathscr{G}$ may be identified with the multi-real distance function

approach space

gauges

$d_{\mathscr{G}}$

$$d_{\mathscr{G}} : X^2 \to [0,\infty]^{\mathscr{G}},\ d_{\mathscr{G}}(x,y)(g) := g(x,y) \text{ for } g \in \mathscr{G}.$$

For the context of (quasi-)uniformities (see Part B), Windels [Win97] weakened the approach space axioms and defined the notion of *uniform approach system*. This kind of "gauge" may also contain maps $\gamma : X^2 \to [0,\infty]$ that do not fulfil the triangle inequality.

uniform approach system

Distribution functions as distance values. Schweizer and Sklar [SS60, SS83] introduced a generalized concept of metric especially suitable for stochastics. The distance functions of their *probabilistic (pseudo-)metric spaces* have a value monoid whose elements are (lower semi-continuous) distribution functions on $[0,\infty)$. This monoid is equipped with the reverse pointwise partial order. As for the choice of the addition operation, they allowed every operation τ (called a *triangle function*) which is compatible to that order. Distances had to be symmetric and T_0.

probabilistic (pseudo-)metric spaces

triangle function

Stronger links between order and addition. Kopperman's [Kop81, Kop88, EK90] *(lattice) continuity spaces* require abelian monoids in which the partial order arises

(lattice) continuity spaces

from the addition by $\alpha \leqslant \beta :\Longleftrightarrow \beta = \alpha + \gamma$ for some $\gamma \in M$, together with some additional conditions. He did not require symmetry.

Totally ordered value monoids. Reichel [Rei78] considered positive distance functions into totally ordered (semi-)groups.

The most general approach so far can be found in Pouzet et. al. [JMP86, LSP87]. These authors consider distance functions into arbitrary partially ordered monoids in which 0 is the least element. They anticipate the notion of multi-pseudometric by interpreting a system of metrics as a vector-valued distance function. However, they require that $d(x,y) = d(y,x)^\star$ holds for a suitable involution * of $\underline{M}$, which is only slightly weaker than full symmetry.

2.

MAPPINGS

> *Why, look where he comes; and my good man too: he's as far from jealousy as I am from giving him cause; and that I hope is an unmeasurable distance.*
>
> *Shakespeare,*
> *The Merry Wives Of Windsor*

After all the examples of mathematical objects that allow for a meaningful definition of distance, we will now consider the question of how relationships *between* such objects might be connected with these distances. The most common method to compare two objects is to consider maps between the underlying sets and study their behaviour with respect to the structure of the objects. Important examples of such maps are the continuous functions between topological spaces and the homomorphisms between groups. As in case of the normed vector spaces $\mathbb{R}^n$, it is often the case that maps of a more topological nature, those that are more algebraic, and such with both flavours go hand in hand, and this will also be true in case of distance *spaces*. Having placed back all topological considerations until Part B, I will focus on the algebraic and order theoretic properties of maps between distance sets in this chapter.

Distance sets with the same value monoid

Let us again start with metric spaces. Already in a first course on analysis, different properties of maps (mostly from $\mathbb{R}$ to $\mathbb{R}$) are introduced, and many of them can be defined for maps between arbitrary metric spaces. Continuity for example is conveniently defined by the well-known ε-δ-criterion, but it does not need the whole structure of a distance set but only the information about which subsets are open — that is, it is a topological property.

A much more powerful property of a map $f : X \longrightarrow Y$ is Lipschitz-continuity, for whose definition one needs distance functions d on X and e on Y. The condition then says that $e\big(f(x), f(y)\big) \leqslant L \cdot d(x,y)$ must holds for some $L \geqslant 0$ and all x, y. It seems that this cannot be defined for general distance sets since it involves a multiplication. It can however be done when multiplication with L is replaced by the application of a suitable map between the value monoids, a technique that will lead to a generalization of one important application of Lipschitz-continuity: in Chapter 6, Banach's fixed point theorem will be shown to hold for very general distance spaces.

For $L = 1$, however, the condition reads $e\big(f(x), f(y)\big) \leqslant d(x,y)$, and this condition can be formulated without problems whenever domain and co-domain of f are equipped with the same value monoid. But first some simplifying notation for a distance function d on X and a map f into X:

df

$$\begin{aligned} fx &:= f(x), \\ df(x,y) &:= d(fx, fy), \\ d(x_1 y_1 \cdots x_n y_n) &:= d(x_1, y_1) + \cdots + d(x_n, y_n), \\ df(x_1 y_1 \cdots x_n y_n) &:= df(x_1, y_1) + \cdots + df(x_n, y_n). \end{aligned}$$

contractive *expansive* *exact homometry* *$\underline{M}$-homometry* *exact isometry* *$\underline{M}$-isometry*

Now a map $f : (X, d, \underline{M}) \longrightarrow (Y, e, \underline{M})$ between two $\underline{M}$-distance sets is *contractive*[1] or *expansive* if and only if $ef \leqslant d$ or $ef \geqslant d$, respectively. A contractive and expansive map, that is, one with $ef \sim d$, is an *exact homometry* (or *$\underline{M}$-homometry*), and a bijective exact homometry is an *exact isometry* (or *$\underline{M}$-isometry*). This must not be confused with the notion of 'isometric embedding' which is traditionally used for the exact homometries between metric spaces.

EXAMPLES

PROPOSITION 2.1.

relation-preserving

For reflexive relations R on X and S on Y, a map $f : (X, d_R) \longrightarrow (Y, d_S)$ is contractive if and only if it is relation-preserving, *that is, if $x\,R\,y \Longrightarrow fx\,S\,fy$.*

In that case: (i) if S is antisymmetric or acyclic, respectively, then so is R, and (ii) if R is total, so is S.

The *proof* is simple because $x\,R\,y \Longleftrightarrow d_R(x,y) \leqslant 1$. □

An example of interesting maps between graphs are colourings (cf. [KNS01]) or, more generally, edge-preserving maps:

---

[1] I avoid the term 'contraction' because it will be used for Lipschitz-continuous maps with $L < 1$. Also, the term 'non-expansive' is not a good replacement for 'contractive' because a contractive map might easily be expansive as well.

PROPOSITION 2.2. *Let* $f : V(G) \to V(H)$ *be a map between the vertex sets of two digraphs* G *and* H. *If* f *is a* graph homomorphism, *that is, if* $(x,y) \in E(G)$ *implies* $(fx,fy) \in E(H)$, *then* $f : (V(G),d_G) \to (V(H),d_H)$ *is contractive. If* f *is injective and contractive, it is a graph homomorphism and hence an isomorphism of* G *with some sub-digraph of* H. *The exact isometries are exactly the isomorphisms between* G *and* H.

graph homomorphism

Proof. f is contractive if and only if $(x,y) \in E(G)$ implies either $f(x) = f(y)$ or $(fx,fy) \in E(H)$. A sub-digraph of H is some digraph H' with $V(H') \subseteq V(H)$ and $E(H') \subseteq E(H)$. ☐

PROPOSITION 2.3. *For quasi-ordered sets* $(X,\leqslant)$ *and* $(Y,\leqslant')$, *the contractive maps* $(X,d_\leqslant) \to (Y,d'_\leqslant)$ *are exactly the isotone maps. If* $\leqslant$ *is antisymmetric, the exact homometries among them are exactly the order-isomorphic embeddings.*

The *proof* for this is trivial. ☐

PROPOSITION 2.4. *For set functions* μ *and* ν *on* X, *a map* $f : (X,d_\mu) \to (X,d_\nu)$ *is contractive [expansive] if and only if, for all* $A \subseteq X$,

$$\nu(A) = 0 \quad \textit{or} \quad \big(\nu(A) \leqslant \mu(A) \textit{ and } f^{-1}(A) = A\big) \quad \textit{or} \quad f^{-1}(A) \in \{\emptyset, X\}$$
$$\big[\ \mu(A) = 0 \quad \textit{or} \quad \big(\mu(A) \leqslant \nu(A) \textit{ and } A = f^{-1}(A)\big) \quad \textit{or} \quad A \in \{\emptyset, X\} \quad \big].$$

Proof (for contractive maps—the expansive case is strictly analogous). We have $d_\mu \geqslant d_\nu f$ if and only if for all $A \subseteq X$ and $x,y \in X$, (i) $d_\mu(x,y)(A) = 0$ implies $d_\nu f(x,y)(A) = 0$, and (ii) $d_\nu(x,y)(A) \leqslant \mu(A)$ holds. This is equivalent to

$$\begin{array}{rl} \text{(i)} & fy \notin A \text{ or } fx \in A \quad \text{or} \quad \nu(A) = 0 \quad \text{or} \quad y \in A \not\ni x \\ \text{and (ii)} & fy \notin A \text{ or } fx \in A \quad \text{or} \quad \nu(A) \leqslant \mu(A), \end{array}$$

that is, to

$$\text{(iii)} \quad fy \notin A \text{ or } fx \in A \text{ or } \nu(A) = 0 \text{ or } \big(y \in A \not\ni x \text{ and } \nu(A) \leqslant \mu(A)\big).$$

This is implied by the proposed condition since $f^{-1}(A) = A$ means that $fx \in A \iff x \in A$ for all x. On the other hand, assume that (iii) holds, $\nu(A) > 0$, and $f^{-1}(A) \notin \{\emptyset, X\}$. Then we can choose $x,y \in X$ with $fy \in A \not\ni fx$, so that $\nu(A) \leqslant \mu(A)$ by (iii). If also $f^{-1}(A) \neq A$, these x,y could even be chosen so that either $x \in A$ or $y \notin A$, in contradiction to (iii). ☐

COROLLARY 2.5. *For* $f : (X,d_\mu) \to (X,d_\nu)$*:*

1. *If* f *is surjective and contractive then* $\nu(A) \leqslant \mu(A)$ *for all* $A \notin \{\emptyset, X\}$.
2. *If, for all points* $x,y \in X$, *there is* $A \subseteq X$ *with* $x \in A \not\ni y$ *and* $\nu(A) > 0$, *then* f *is contractive if and only if it is either constant, or if it is the identity and* $\nu(A) \leqslant \mu(A)$ *holds for all* $A \notin \{\emptyset, X\}$.

Note that if ν is the indicator function of some topology without open singletons, the condition in 2. is just the T_1 property for that topology.

PROPOSITION 2.6. *For* $|X| \geqslant 3$ *and set functions* μ *and* ν *on* X*, a map* $f : (X, d'_\mu) \to (X, d'_\nu)$ *is contractive if and only if it is either constant, or if it is the identity on* X *and* $\nu(A) \leqslant \mu(A)$ *holds whenever* $|A| \geqslant 2$*. It is expansive if and only if it is the identity and* $\nu(A) \geqslant \mu(A)$ *holds whenever* $|A| \geqslant 2$.

Proof (for contractive maps). $d'_\nu(fx, fy)(A) \leqslant d'_\mu(x,y)(A)$ is equivalent to the implication $x, y \in A \Longrightarrow fx = fy$ or $\big(fx, fy \in A$ and $\nu(A) \leqslant \mu(A)\big)$. For fixed A, this holds for all x, y if and only if $(\star)$ $|f[A]| \leqslant 1$ or $\big(f[A] \subseteq A$ and $\nu(A) \leqslant \mu(A)\big)$. Suppose that $(\star)$ is true for all A, but f is neither constant nor the identity. Choose x, y, z with $f(x) = z \neq x$ and $f(y) \neq z$. For $v \in X$ with $f(v) \neq z$, put $A := \{x, v\}$, so that $|f[A]| = 2$, hence $f[A] \subseteq A$, that is, $z = v$. In particular, $z = y$. Because $|X| \geqslant 3$, there is $v \in X \setminus \{z, f(y)\}$, so that $f(v) = z$. With $A' := \{v, y\}$, again $|f[A']| = 2$, hence $f[A'] \subseteq A'$ in contradiction to $f(y) \notin A'$. Therefore f is either constant or the identity, and in the latter case $\nu(A) \leqslant \mu(A)$ holds whenever $|A| \geqslant 2$ because of $(\star)$. ☐

In case of algebraic objects, internal distances are often more easily handled than classical metrics on these objects. It is easy to show that each isotone or antitone group homomorphism $f : G \to H$ between partially ordered abelian groups is contractive w. r. t. the pseudometrics d_u and $d_{f(u)}$, for all $u \in G^+$. However, I know of no *characterization* of the contractive maps in this case. On the other hand, a characterization is very easy when internal distance functions are considered instead:

PROPOSITION 2.7. *With respect to the distance* $x^{-1}y$*, the contractive or expansive maps or exact homometries* $h : G \to G$ *on a group* G *are exactly the left translations* $x \mapsto ax$.

Proof. All three classes of maps are identical because the relevant order relation is the identity. Also $x^{-1}e = h(x)^{-1}h(e)$ implies $h(x) = h(e)x$ for all x. ☐

motions

group of motions

Aut($\underline{X}$)

In this example, all these maps are even bijective, that is, exact isometries. The exact isometries of a distance set $\underline{X}$ with itself will also be called the *motions* of $\underline{X}$.[1] They constitute the *group of motions* Aut($\underline{X}$), and the above proposition already shows that each group G is isomorphic to some group of motions. Extending a method by Caragiu [Car92], it was shown in [Hei98] that actually each group G is even isomorphic to the group of motions of some multi-pseudometric space, and also to that of some symmetric and positive distance set with a totally

---

[1] A more "categorical" name would be 'autometries'.

ordered abelian group as its value monoid. A possible construction for the proof of the first claim is this: put $X := G \times 2$, $\underline{M} := \mathbb{R}^{\top G}$, $i : G \to \underline{M}$, $i(a)(a) := 6$, and $i(a)(b) := 5$ for $b \neq a$. Then define $d(x,y)$ to be 0, 3, 4, $i(a^{-1}b)$, or $i(b^{-1}a)$, depending on whether $x = y$, $x \neq y$ and $x,y \in G \times \{0\}$, $x \neq y$ and $x,y \in G \times \{1\}$, $x = (a,0)$ and $y = (b,1)$, or $x = (a,1)$ and $y = (b,0)$, respectively (cf. [Hei98]).

MORE GENERAL CONTRACTIVITY

A slightly more general definition of contractive map would be this: a map $f : (X,d,\underline{M}) \to (Y,e,\underline{M}')$ is *contractive* or *expansive* if and only if $\underline{M}$ is a sub-q. o. m. of $\underline{M}'$ and $ef \leqslant d$ or $ef \geqslant d$, respectively, where $\leqslant$ is now the quasi-order of $\underline{M}'$.

contractive

expansive

The corresponding concept of *exact homometry* is characterized for the case of abelian ℓ-groups by a result of Holland. A *[dual] ℓ-homomorphism* is an additive map f with $f(a \vee b) = fa \vee fb$ [resp. $f(a \vee b) = fa \wedge fb$] for all a,b. An ℓ-group H is called a *cardinal product* of sub-ℓ-groups $A,B \leqslant H$ if there is an *ℓ-isomorphism* (= bijective ℓ-homomorphism) $\varphi : A \times B \to H$ with $\varphi(a,0) = a$ and $\varphi(0,b) = b$ for all $(a,b) \in A \times B$. In that case, every element $x \in H$ is a unique sum of elements $x_A \in A$ and $x_B \in B$.

exact homometry

ℓ-homomorphism

cardinal product

ℓ-isomorphism

THEOREM 2.8. (Holland [Hol85]). *With respect to the distance function $n|x - y|$ ($n \geqslant 1$), the exact homometries h from a sub-ℓ-group G of an abelian ℓ-group H into H are exactly the maps h that arise as follows: let G be the cardinal product of $A \leqslant G$ and $B \leqslant G$, $a \in G$, and put $h(x + y) := a + x - y$ for all $x \in A$ and $y \in B$.*

Actually, Holland stated this only for the case where $G = H$ and h is surjective. His proof however works literally also in this more general case. For $n = 1$, the theorem also follows from Theorem 2.15 below. ☐

Translating distances of different type

The examples of the first chapter show that, in contrast to metric spaces, two mathematical objects that are to be related in some way will often be equipped with natural distance functions that take their values in two different monoids. This is especially the case for internal distance functions. Moreover, one might want to study the relationship between distance functions with different co-domains on a *single* object, which will be the focus of the next section.

In both cases, we obviously need a nice class of mappings f between distance sets $\underline{X} = (X,d,\underline{M})$ and $\underline{Y} = (Y,e,\underline{N})$ with different monoids $\underline{M}$ and $\underline{N}$.

Thinking categorically, there is a suggesting way to overcome the problem that the distances $d(x,y)$ and $ef(x,y)$ cannot be compared directly as in the case where $\underline{M} = \underline{N}$. The "original" distance $d(x,y)$ only needs to be "translated" into an element of $\underline{N}$ prior to the comparison with the "image" distance $ef(x,y)$. Assume that, most simply, this translation is done by an arbitrary mapping $t : \underline{M} \to \underline{N}$ between the monoids, and that the map f preserves the distance up to this translation, that is, $ef(x,y) = t\bigl(d(x,y)\bigr)$, or $ef = td$ for short. When no additional requirements are imposed on t, this results in the very broad class of *distance equality preserving* maps. For some purposes, it is however more appropriate to require that the translation map t has some nice properties, and it will soon turn out which they are.

distance equality preserving

Preservation of formulae. Another approach to the problem of different monoids is of a more model-theoretic nature. It starts with the selection of a certain set of "interesting" formulae in the language under consideration, such as the formula '$v_1 \leqslant v_2$' in case of posets, or '$v_1 \cdot v_2 = v_3$' in case of groups (where the v_i are variables). Then one considers those maps $f : X \to Y$ for which every true interpretation of such a formula in object X translates into a true interpretation when every occurrence of some $x \in X$ in the original interpretation is replaced by its image $f(x)$. I will say that f *preserves* the formula in this case. Without stating this formally, the point will become clear from the following examples. (i) An isotone map $f : (X,\leqslant_X) \to (Y,\leqslant_Y)$ between posets translates a true interpretation $x \leqslant_X y$ of the formula '$v_1 \leqslant v_2$' in X (that is, with $x,y \in X$) into a true interpretation $f(x) \leqslant_Y f(y)$ of the same formula in Y, because this is just a different way of saying that $f(x) \leqslant_Y f(y)$ for all $x,y \in X$ with $x \leqslant_X y$. (ii) A ring homomorphism $f : (R,+_R,\cdot_R) \to (S,+_S,\cdot_S)$ translates a true interpretation $x +_R y = z$ of '$v_1 + v_2 = v_3$' into a true interpretation $f(x) +_S f(y) = f(z)$, and it translates a true interpretation $x \cdot_R y = z$ of '$v_1 \cdot v_2 = v_3$' into a true interpretation $f(x) \cdot_S f(y) = f(z)$. This is equivalent to saying that $f(x +_R y) = f(x) +_S f(y)$ and $f(x \cdot_R y) = f(x) \cdot_S f(y)$ for all $x,y \in R$. It is clear that the larger the set of formulae that are preserved by a mapping, the more structure is "transported" by it.

preservation of formulae

In case of distance sets, the possible formulae are made up of the structural ingredients d, $+$, 0, $=$, and $\leqslant$, together with variables for elements of X and Y, variables for elements of M and N, and logical symbols. However, variables for elements of M and N must be avoided since f is between X and Y, so that is cannot be used to replace elements of M by elements of N in the way described above.

Here come the definitions:

A map $f : \underline{X} \to \underline{Y}$ is. . .	if it preserves (for all $n,m \geqslant 0$). . .
specialization preserving	$d(x,y) \leqslant 0$
distance equivalence preserving	$d(x,y) \sim d(z,w)$
distance inequality preserving	$d(x,y) \leqslant d(z,w)$
an *order representation*	$d(x,y) < d(z,w)$
a *local order representation*	$d(x,y) < d(x,z)$, and also $d(y,x) < d(z,x)$
a *weak homometry*	$d(x,y) \leqslant d(z_1w_1 \cdots z_mw_m)$
a *dually weak homometry*	$d(x_1y_1 \cdots x_ny_n) \leqslant d(z,w)$
a *(set) homometry*	$d(x_1y_1 \cdots x_ny_n) \leqslant d(z_1w_1 \cdots z_mw_m)$

Some of the above classes of maps can be derived also in the "categorical" way, at least with a slight modification: the translating maps need not be definable on all of $\underline{M}$, but only on some subset like the *induced submonoid* $\underline{S}_d := \langle d[X^2]\rangle_{\underline{M}} = \{d(x_1,x_2) + \cdots + d(x_n,y_n) \mid x_i,y_i \in X\}$ of d. By a *q. o. m.-morphism* I mean an isotone and additive map c between quasi-ordered monoids with $c(0) = 0$.

induced submonoid

$\underline{S}_d$

q. o. m.-morphism

PROPOSITION 2.9. *For a map $f : \underline{X} \to \underline{Y}$:*

1. *f is distance equivalence preserving $\Longleftrightarrow$ there is some map $t : M \to N$ with $ef \sim td$.*
2. *f is distance inequality preserving $\Longleftrightarrow$ there is an isotone map $t : d[X^2] \to \underline{N}$ with $ef \sim td$ which is unique when $\underline{N}$ is partially ordered.*
3. *f is a weak homometry $\Longleftarrow$ there is an isotone and sub-additive $t : \underline{M} \to \underline{N}$ with $ef \sim td$. If $\underline{N}$ is completely lattice-ordered and lower distributive, the converse holds, too.*
4. *f is a dually weak homometry $\Longleftarrow$ there is an isotone and super-additive map $t : \underline{M} \to \underline{N}$ with $ef \sim td$. If $\underline{N}$ is completely lattice-ordered and upper distributive, the converse holds, too.*
5. *f is a homometry $\Longleftrightarrow$ there is a q. o. m.-morphism $c : \underline{S}_d \to \underline{N}$ with $ef \sim cd$ which is unique when $\underline{N}$ is partially ordered.*

The *proof* is almost straightforward (but needs the Axiom of Choice when $\underline{N}$ is not partially ordered). In the second part of 3., lower distributivity comes into play in a very typical way, so let us have a look at that proof. Given a weak homometry, we first see that $t(\alpha) := \bigwedge \{ef(x_1y_1 \cdots x_ny_n) \mid \alpha \leqslant d(x_1y_1 \cdots x_ny_n)\}$ defines an isotone map $t : \underline{M} \to \underline{N}$ with $ef = td$. Then, for sub-additivity, we have

$$\begin{aligned} t(\alpha) + t(\beta) &= \bigwedge \{ef(x_1y_1\cdots x_ny_n) \mid \alpha \leqslant d(x_1y_1\cdots x_ny_n)\} \\ &\quad + \bigwedge \{ef(z_1w_1\cdots z_mw_m) \mid \beta \leqslant d(z_1w_1\cdots z_mw_n)\} \\ &=^\star \bigwedge \{ef(x_1y_1\cdots x_ny_nz_1w_1\cdots z_mw_m) \mid \\ &\qquad \alpha \leqslant d(x_1y_1\cdots x_ny_n) \text{ and } \beta \leqslant d(z_1w_1\cdots z_mw_n)\} \\ &\geqslant t(\alpha+\beta), \end{aligned}$$

which relies on lower distributivity at the position marked by *. $\boxed{\text{AC}}$[1]

As the reader might guess already, the most fruitful class is that of homometries, and these maps will usually be designated by the letter h. Their distance translation functions c will be called *calibrations*, and $eh \sim cd$ could be called the *homometry equation*. The fact that homometries can either be thought of as preserving inequalities between sums of distances or as preserving the distances themselves up to a calibration will be quite convenient in the sequel. The above proposition also shows that weak homometries are a generalization of so-called "metric transforms" (cf. [DM90]).

calibrations

homometry equation

The first example is about real distances again. Here the homometries are most easily characterized by determining the possible calibrations:

LEMMA 2.10. *If $\underline{S}$ is a submonoid of $\underline{\mathbb{R}}^\top$, $S' = S \setminus \{\infty\}$, and $c : \underline{S} \to \underline{\mathbb{R}}^\top$ is a q. o. m.-morphism, then either $c(\infty) = \infty$ and $c|_{S'} \equiv 0$, or c is a multiplication with a non-negative constant $\gamma \in [0,\infty]$.*

archimedean

Proof. Note that $\underline{\mathbb{R}}$ is *archimedean*, that is, for each $\varepsilon > 0$ and $\beta \in \mathbb{R}$, there is $n \in \omega$ such that $\beta \leqslant n\varepsilon$. If $0 > \alpha \in S'$ then $\infty \notin c[S']$, because $c(\beta) = \infty$ would imply that $0 = c(0) \geqslant c(n\alpha + \beta) = nc(\alpha) + \infty = \infty$ for sufficiently large $n \in \omega$. Hence, by contraposition, $c(\beta) = \infty$ for at least one $\beta \in S'$ implies that $S' \subseteq [0,\infty)$ and thus $c(\alpha) = \infty$ for all $\alpha \in S \setminus \{0\}$ because $\underline{\mathbb{R}}$ is archimedean and c is isotone. Now assume that $\infty \notin c[S']$ and $\alpha, \beta \in S'$ with $c(\beta) \neq 0$. Then, for all integers z, z' and positive integers n, n',

$$\frac{z}{n} \leqslant \frac{\alpha}{\beta} \leqslant \frac{z'}{n'} \text{ implies } \frac{z}{n} \leqslant \frac{c(\alpha)}{c(\beta)} \leqslant \frac{z'}{n'},$$

hence $\alpha/\beta = c(\alpha)/c(\beta)$, so that $\gamma := c(\beta)/\beta \in [0,\infty)$ is constant for $0 \neq \beta \in S'$. That is, $c|_{S'}$ is just multiplication with γ, and $c(\infty) = 0$ is only possible if $\gamma = 0$. □

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[1] The letters AC are placed inside the box to indicate that this proof utilizes the Axiom of Choice. All proofs which—to my awareness—use a variant of this axiom will be marked in this way.

Homometries between graphs. To illustrate this, consider two (undirected) graphs G and H and a homometry $h : (G, d_G) \to (H, d_H)$ with calibration c. Let us assume that G is not connected but has at least one edge, so that $S_{d_G} = \underline{\omega}^\top$. Then c is from $\underline{\omega}^\top$ into $\underline{\omega}^\top \leqslant \underline{\mathbb{R}}^\top$, and we can distinguish four cases.

(i) If $c \equiv 0$ then h is constant.

(ii) If $c|_\omega \equiv 0$ and $c(\infty) = \infty$, h is constant on every component of G, but the images of two different components are always in two different components of H.

(iii) If $c(\alpha) = \infty$ for all $\alpha \neq 0$, all vertices of G are mapped into different components of H.

(iv) $c(\alpha) = \gamma\alpha$ for some positive integer γ. Then, for all $x, y \in V(G)$ with $e = \{x, y\} \in E(G)$, there is a shortest path p_e in H between $h(x)$ and $h(y)$ that has length γ, and between every pair hx, hy of image vertices there is a shortest path in H which is a union of such paths p_e.

EXAMPLE 2.11. Let $G = (V, E)$ be a complete graph, and H be the star-shaped graph $(V \cup \{V\}, E')$ with $E' = \big\{\{x, V\} \,|\, x \in V\big\}$. Then $id_V : (V, d_G) \to (V \cup \{V\}, d_H)$ is a homometry with calibration $r \mapsto 2r$.

triangle-free

An interesting situation arises when, to the contrary, G is *triangle-free*, that is, contains no circles of length three. Then case (iv) implies that the graph G' that results from G when all its edges are subdivided into γ parts (that is, replaced by paths of length γ) is isomorphic to an induced subgraph of H. This is because then the paths p_e must be disjoint up to common end-points ⟦assume that $e = \{x, y\}$ and $e' = \{z, w\}$ are distinct edges in G, say with $z \notin \{x, y\}$, and p_e and $p_{e'}$ share a vertex $v \notin \{h(x), h(y)\}$. Then there is a path from $h(x)$ to $h(z)$ of length less than 2γ, so that $d_G(x, z)$ must be 1. Similarly, $d_G(y, z) = 1$, hence $\{x, y, z\}$ is a triangle in G⟧. Also, there can be no additional edges between their interior vertices ⟦proved similarly⟧. This implies:

PROPOSITION 2.12. *Let $h : (V(G), d_G) \to (V(H), d_H)$ be a non-constant homometry between connected, triangle-free graphs. Then H contains a subdivision of G as an induced subgraph.*

HOMOMETRIES BETWEEN (ℓ-)GROUPS

PROPOSITION 2.13. *With respect to the distance $x^{-1}y$, the distance equivalence preserving maps between two groups G and H are exactly the group homomorphisms composed with left translations.*

Proof. For a homomorphism $c : G \to H$ and some $a \in H$, the map $f(x) := ac(x)$ is a homometry with calibration c ⟦$f(x)^{-1}f(y) = c(x)^{-1}c(y) = c(x^{-1}y)$⟧, hence distance equivalence preserving ⟦since the value monoids are partially ordered⟧. On the other hand, $c(x) := f(e)^{-1}f(x)$ is a group homomorphism when f is distance equivalence preserving ⟦$c(x^{-1}y) = d_H f(e,x^{-1}y) = d_H f(x,y) = f(e)^{-1}f(x^{-1}y) = f(x)^{-1}f(y) = c(x)^{-1}c(y)$⟧, and $f(x)$ equals $f(e)c(x)$. ☐

Consequently, these maps are also exactly the distance inequality preserving maps, (dually) weak homometries, and homometries. Surprisingly, the homometries between abelian ℓ-groups can be characterized quite similarly. We start with the case where the second group is totally ordered.

LEMMA 2.14. *For a homometry* $h : G \longrightarrow H$ *from an abelian* ℓ*-group* G *into an abelian totally ordered group* H*, both equipped with the distance* $|x-y|$*, the map* $f(x) := h(x) - h(0)$ *is either an* ℓ*-homomorphism or a dual* ℓ*-homomorphism.*

Proof. If $c : G^+ \longrightarrow H^+$ is the calibration of h then f is again a homometry with calibration c. Moreover, $c(x) = |fx|$ and thus $f(x) \in \{cx, -cx\}$ for all $x \in G^+$. Now either (i) $f|_{G^+} = c$ or (ii) $f|_{G^+} = -c$ ⟦Assume that, to the contrary, $f(x) > 0 > f(y)$ for some $x,y \in G^+$, so that $f(x) = c(x)$ and $f(y) = -c(y)$, and put $z := x \vee y \geqslant 0$. Then either $f(z) \geqslant 0$, that is, $f(z) = c(z)$ and $c(z) + c(y) = |fz - fy| = c|z-y| = c(z) - c(y)$, or $f(z) \leqslant 0$, that is, $f(z) = -c(z)$ and $c(z) + c(x) = |fz - fx| = c|z-x| = c(z) - c(x)$. Both would be a contradiction to $c(y) \neq 0 \neq c(x)$⟧.

$\hat{c}$

In both (i) and (ii), $c|x-y| = |fx - fy| = \big||fx| - |fy|\big| = |cx - cy|$ for all $x,y \in G^+$. Hence, the unique extension of c to a group homomorphism $\hat{c}$ on $G = G^+ - G^+$, defined by $\hat{c}(x-y) = \hat{c}(x) - \hat{c}(y)$, preserves the absolute value. Consequently, it is an ℓ-homomorphism ⟦First, $\hat{c}$ is isotone: $x - y \leqslant x' - y' \Longrightarrow x + y' \leqslant x' + y \Longrightarrow cx + cy' \leqslant cx' + cy \Longrightarrow \hat{c}(x-y) \leqslant \hat{c}(x'-y')$. Note that abelian ℓ-groups fulfil $2(x \vee y) = 2x \vee 2y$ and $x + |y| = (x+y) \vee (x-y)$ (cf. [AF88]). Now let $x,y \in G$ and put $z := x - y$. Then $\hat{c}(x \vee y) \geqslant \hat{c}x \vee \hat{c}y$, and

$$2\hat{c}(x \vee y) = \hat{c}(2x \vee 2y) = \hat{c}(2y + z + |z|) = 2\hat{c}y + \hat{c}z + |\hat{c}z| = 2(\hat{c}x \vee \hat{c}y).$$

Since this excludes $\hat{c}(x \vee y) > \hat{c}x \vee \hat{c}y$, $\hat{c}$ preserves binary suprema⟧.

Finally, f is additive. Indeed, for $z = x - y \in G$ with $x,y \in G^+$, we have $|fz - fx| = c|z-x| = c(y) = \pm f(y)$, hence $f(z) = f(x) \pm f(y)$ since H is totally ordered. In both cases, $f(z) = f(x) - f(y)$ ⟦since $f(z) = f(x) + f(y)$ implies $|2fy| = c(2y) = c|(x+y) - z| = |f(x+y) - f(z)| = 0$, hence $2f(y) = 0$, that is, $f(z) = f(x) + f(y) - 2f(y) = f(x) - f(y)$ also⟧. Therefore, either $f = \hat{c}$ or $f = -\hat{c}$ because the extension of c to G is unique. ☐

Although the above proof strongly depends on the total orderedness of H, it leads to a complete characterization of homometries between arbitrary ℓ-groups.

This is because the totally ordered groups "generate" the abelian ℓ-groups in the following sense. The direct product of totally ordered groups, that is, the cartesian product with pointwise addition and order, is always an ℓ-group, and at least every *abelian* ℓ-group is isomorphic to a sub-ℓ-group of such a product (this property is called *representability*) in which all factors are also abelian (cf. [AF88]).

representability

THEOREM 2.15. *For a map $h : G \longrightarrow H$ between abelian ℓ-groups, and a map $c : G^+ \longrightarrow H^+$, the following are equivalent:*

(a) h is a homometry with respect to $|x - y|$ on both sides, and with calibration c.

(b) c extends uniquely to an ℓ-homomorphism $\hat{c} : G \longrightarrow H$, and H is the cardinal product of sub-ℓ-groups $A, B \leqslant H$ such that $h(x) = h(0) + \hat{c}(x)_A - \hat{c}(x)_B$ for all $x \in G$.

Proof. (b)$\Longrightarrow$(a) is quite straightforward:

$$\begin{aligned} |hx - hy| &= |(\hat{c}x)_A - (\hat{c}y)_A - (\hat{c}x)_B + (\hat{c}y)_B| \\ &= |\hat{c}(x-y)_A| + |\hat{c}(x-y)_B| = \hat{c}|x-y| \end{aligned}$$

since $|a - b| = |a| + |b| = |a + b|$ for all $(a, b) \in A \times B$.

Now for (a)$\Longrightarrow$(b). Let $\varepsilon : H \longrightarrow H' = \prod_{i \in I} H_i$ be an ℓ-embedding into a product of abelian totally ordered groups H_i, and put $h_i := \pi_i \circ \varepsilon \circ h$ for all $i \in I$, where π_i is the i-th projection map. Since ε and π_i are homometries, so is $h_i : G \longrightarrow H_i$. Hence, the above lemma shows that for each i there is a unique ℓ-homomorphism $c_i : G \rightarrow H_i$ with $h_i = h_i(0) \pm c_i$ and $c_i(x) = |h_i x - h_i 0|$ for all $x \in G^+$. These c_i combine to a unique ℓ-homomorphism $c' : G \longrightarrow H'$ with $\pi_i \circ c' = c_i$ for all $i \in I$.

ε

H_i, h_i

c_i, c'

The restriction $c'|_{G^+}$ must equal $\varepsilon \circ c$ ⟦for $x \in G^+$ and $i \in I$, $\pi_i c'(x) = c_i(x) = |h_i x - h_i 0| = |\pi_i \varepsilon h x - \pi_i \varepsilon h 0| = \pi_i \varepsilon |hx - h0| = \pi_i \varepsilon c(x)$⟧. Because c' is additive, the unique extension of c to an additive map $\hat{c} : G \longrightarrow H$ must fulfil $c' = \varepsilon \circ \hat{c}$ and is thus also an ℓ-homomorphism ⟦since ε is an embedding⟧.

$\hat{c}$

Now let $I_+ \subseteq I$ consist of all i with $h_i = h_i(0) + c_i$. For $i \in I_+$, put $A_i := H_i$ and $B_i := \{0\} \leqslant H_i$. Likewise, for $i \in I \setminus I_+$, put $A_i := \{0\} \leqslant H_i$ and $B_i := H_i$. Then each H_i is the cardinal product of A_i and B_i, hence H is the cardinal product of $A := \varepsilon^{-1}\left[\prod_{i \in I} A_i\right]$ and $B := \varepsilon^{-1}\left[\prod_{i \in I} B_i\right]$ ⟦since ε is an embedding⟧.

I_+, A_i, B_i

A, B

Finally, $h(x) = h(0) + (\hat{c}x)_A - (\hat{c}x)_B$ ⟦for $i \in I_+$, we have $h_i(x) = h_i(0) + (c_i x)_{A_i} \pm (c_i x)_{B_i}$ since $(c_i x)_{B_i} \in B_i = \{0\}$, and otherwise $h_i(x) = h_i(0) \pm (c_i x)_{A_i} - (c_i x)_{B_i}$ since $(c_i x)_{A_i} \in A_i = \{0\}$⟧. ☐

As a consequence of Theorem 2.8, this result has also an analogue for non-abelian ℓ-groups. In that case, however, c must be known to preserve absolute values *a priori*. Hence the above theorem is not just a special case of Holland's results.

The following characterization is perhaps even more interesting than the previous. Unfortunately, the proof is quite technical.

THEOREM 2.16. *Let* F *be a field of characteristic* $p \neq 2$ *and* F' *a field of characteristic* $q \neq 2$. *Then a map* $h : (F, d_F) \longrightarrow (F', d_{F'})$ *is a homometry if and only if it is constant or if there is some* $a \in {F'}^{\star}$ *and a field homomorphism* $f : F \longrightarrow F'$ *with* $h = af$.

Proof. Put $d := d_F$, $d' := d_{F'}$, and designate the operation of the monoids $\underline{M} = (F, +, 0, \Delta_F) \times (F, \cdot, 1, \leqslant)$ and $\underline{M}' = (F', +, 0, \Delta_{F'}) \times (F', \cdot, 1, \leqslant)$ by the symbol $\oplus$. Note that constant maps are always homometries.

For $a \in {F'}^{\star}$ and a field homomorphism f, the map $c : \underline{M} \longrightarrow \underline{M}'$, $c(\alpha,\beta) := (af\alpha, f\beta)$ becomes additive $[\![\, c\big((\alpha,\beta) \oplus (\alpha',\beta')\big) = c(\alpha+\alpha', \beta\cdot\beta') = (af\alpha + af\alpha', f\beta \cdot f\beta') = (af\alpha, f\beta) \oplus (af\alpha', f\beta') \,]\!]$ and isotone $[\![\, (\alpha,\beta) < (\alpha',\beta')$ implies $\alpha = \alpha'$ and $\beta' = 0$, hence $af\alpha = af\alpha'$ and $f\beta' = 0$, that is, $c(\alpha,\beta) \leqslant c(\alpha',\beta') \,]\!]$. Moreover, c is a calibration for $h := af$ $[\![$For $x, y \in F^{\star}$, we have $hx, hy \neq 0$, so that

$$\begin{aligned} d'h(x,y) &= (hy - hx, (hx)^{-1} \cdot hy) = (af(y-x), f(x^{-1}y)) \\ &= c(y - x, x^{-1}y) = cd(x,y). \end{aligned}$$

Also, $d'h(0,x) = (hx, 0) = (afx, f0) = c(x,0) = cd(0,x)$, and similarly for $d'h(x,0)\,]\!]$.

On the other hand, assume that h is a non-constant homometry with
b calibration $c : \underline{S}_d \longrightarrow \underline{M}'$, put $b := h(0)$, and define $g : F \longrightarrow F'$ by $g(x) :=$
g $h(x) - b$. Because $g(x)$ is the first component of $d'h(0,x) = cd(0,x) = c(x,0)$, and since c is additive, so is g.

y Since h is not constant, we can choose some $y \in F^{\star}$ with $h(y) \neq b$. In case that $q \neq 3$, we can also assume that $h(y) \neq 3b$ $[\![$if $h(y) = 3b$ then $b \neq 0$ and we have $c(-y, 0) = cd(y,0) = d'h(y,0) = (-2b, \frac{1}{3})$, hence $d'h(2y,0) = cd(2y,0) = c(-2y,0) = (-4b, \frac{1}{9})$ by additivity of c. But then $h(2y) \notin \{b, 3b\}$ since $d'h(2y,0) \notin \{(0,1), (-2b, \frac{1}{3})\} = \{d'(b, h0), d'(3b, h0)\}$. This means that we can take $2y$ instead of $y\,]\!]$.

a Put $a := g(y) \neq 0$. I will now show that (1) $b = 0$, that is, $g = h$, (2) the map $x \mapsto h(xy)/a$ is multiplicative, and (3) we can assume that $y = 1$ without loss of generality. It will then be clear by (1) and (2) that $f := h/h(1)$ is both additive and multiplicative, that is, a field homomorphism.

Proof of (1). Assume that $b \neq 0$. There must be some $n \in \omega$ which is not a multiple of p so that ($\star$) in F', $n \cdot 1$ does not equal one of the at most six values $-b/a$, $b/2a$, $b/8a$, $9b/16a$, or $2b/(9 \pm \sqrt{21})a$ (some of which might not exist
A at all). Indeed, let $A \subseteq F'$ be the set of these at most six different "forbidden" values.

(i) If $q = 0$ or $q \geqslant 11$, the set $\{0,\dots,10\} \subseteq F'$ has eleven elements so that condition ($\star$) holds for at least five different values of n in $\{0,\dots,10\}$. Since $p \neq 2$, at most four of them are multiples of p.

(ii) If $q = 3$ then $A = \{2b/a, 0\}$ so that either $n = 1$ or $n = 2$ works.

(iii) If $q = 7$ then $A = \{6b/a, 4b/a, b/a\}$, which cannot contain all of the three distinct elements $1,2,4 \in F'$, so that at least one of these values works for n (since $p \neq 2$).

(iv) Finally, assume that $q = 5$. Then $A = \{4b/a, 3b/a, 2b/a\}$, which cannot contain all of the four distinct elements $1,2,3,4 \in F'$, so that one of these values for n fulfils ($\star$). If $p \neq 3$, none of $1,\dots,4$ is a multiple of p. Therefore, assume that $p = 3$ and $1,2,4 \in A$. In that case $2b/a$ must be the element of A that equals 1, hence $h(y) = g(y) + b = a + b = 3b$, in contradiction to the choice of y.

Now we possess a suitable n. In F, we have $n \cdot 1 \neq 0 \neq 2$. Hence none of ny, $-ny$, $-2ny$, $-4ny$, and $-8ny$ is zero, and

$$(0,1) = 2\big(d(ny,-ny) \oplus d(-2ny,-4ny) \oplus d(-8ny,-4ny)\big).$$

Since h is a homometry, it follows that also

$$\begin{aligned}(0,1) &= 2\big(d'h(ny,-ny) \oplus d'h(-2ny,-4ny) \oplus d'h(-8ny,-4ny)\big) \\ &= 2\big(d'(b+g(ny), b-g(ny)) \oplus d'(b-g(2ny), b-g(4ny)) \oplus \\ &\qquad \oplus d'(b-g(8ny), b-g(4ny))\big).\end{aligned}$$

By choice of n, the second component of the last line is

$$e := \left(\frac{b-na}{b+na} \cdot \frac{b-4na}{b-2na} \cdot \frac{b-4na}{b-8na}\right)^2$$

since none of the three denominators $b+na$, $b-2na$, and $b-8na$ vanishes. Given n and a, the possible solutions of the equation $e = 1$ are $b = 0$, $b = 16na/9$, and $b = (9 \pm \sqrt{21})na/2$, of which all but the first have been excluded by the choice of n. Hence $b = 0$ and thus $g = h$.

Proof of (2). For all $x \in F$, I will show that $h(xy)/a = \pi_2 \circ c(0,x)$, where π_2 is the second projection map. Because of $c(0,1) = (0,1) = (0, h(y)/a)$, we may assume that $x \neq 1$. For each positive integer k with $k \cdot 1 \neq x$ and $h(xy) \neq ka$, we have (with $z := (x-k)y$)

$$\begin{aligned}c(0,x/k) &= d'h(ky,xy) \oplus d'h(z,2z) \oplus d'h(4z,2z) \\ &= \left(0, \frac{h(xy)}{ka} \cdot \frac{2h(xy)-2ka}{h(xy)-ka} \cdot \frac{2h(xy)-2ka}{4h(xy)-4ka}\right) = \left(0, \frac{h(xy)}{ka}\right).\end{aligned}$$

If $h(xy) \neq a$, this implies that $c(0,x) = (0, h(xy)/a)$. Otherwise choose $k \in \{2,3\}$ with $k \cdot 1 \neq x$ and $h(xy) \neq ka$ ⟦this is possible since $q \neq 2$⟧, and put $x' := k \cdot 1 \in F \setminus \{1\}$ and $k' := 1$. Then $k' \cdot 1 \neq x'$ and $h(x'y) = ka \neq k'a$, so that $c(0,x) = c(0,x/k) \oplus c(0,x'/k') = \big(0,h(xy)/ka\big) \oplus \big(0,h(x'y)/k'a\big) = \big(0,h(xy)/ka \cdot ka/a\big) = \big(0,h(xy)/a\big)$. Since $\pi_2 \circ c$ is multiplicative, this proves (2).

Proof of (3). By (2),

$$\begin{aligned} h(1)h(yy)/a^2 &= h(y^{-1}y)/a \cdot h(yy)/a \\ &= h\big((y^{-1}y)y\big)/a = h(y)/a = 1 \neq 0, \end{aligned}$$

hence $h(1) \neq 0 = b$. This means that we may have chosen $y := 1$ in the first place. ☐

CO-QUANTALES, BROUWERIAN LATTICES, AND BOOLEAN LATTICES

Between two co-quantales with either $d_{\rightarrow}$ or $d_{\leftarrow}$ on both sides, a map is specialization preserving if and only if it is isotone (that is, order preserving). When one of the co-quantales is equipped with $d_{\rightarrow}$ and the other with $d_{\leftarrow}$, the specialization preserving maps are exactly the *antitone* ones (that is, order reversing).

LEMMA 2.17. *With respect to $d^0_{\rightarrow}$ on both sides, a weak homometry f between co-quantales is isotone and satisfies the following form of weak sub-additivity:*

$$f(\alpha_1 + \cdots + \alpha_n) \leqslant f0 + (0 \vee f\alpha_1) + \cdots + (0 \vee f\alpha_n).$$

Proof. One has $\sum_i d^0_{\rightarrow}(0,\alpha_i) = \sum_i (0 \vee \alpha_i) \geqslant 0 \vee (\sum_i \alpha_i) = d^0_{\rightarrow}(0,\sum_i \alpha_i)$, hence $\sum_i d^0_{\rightarrow} f(0,\alpha_i) \geqslant d^0_{\rightarrow} f(0,\sum_i \alpha_i)$ since f is a weak homometry. Thus $f(\sum_i \alpha_i) \leqslant f0 + d^0_{\rightarrow} f(0,\sum_i \alpha_i) \leqslant f0 + \sum_i (0 \vee f\alpha_i)$. ☐

Note that for Brouwerian lattices with distance $\rightarrow$, the specialization preserving maps are again just the isotone ones.

LEMMA 2.18. *With respect to the distance $\rightarrow$ on both sides, a weak homometry f between Brouwerian lattices preserves binary infima.*

Proof. As above, $(\top \rightarrow x) \wedge (\top \rightarrow y) = x \wedge y = \top \rightarrow (x \wedge y)$, hence $(f\top \rightarrow fx) \wedge (f\top \rightarrow fy) \leqslant f\top \rightarrow f(x \wedge y)$ since f is a weak homometry (note that orders are dual here). Thus $f(x \wedge y) \geqslant f\top \wedge \big(f\top \rightarrow f(x \wedge y)\big) \geqslant f\top \wedge (f\top \rightarrow fx) \wedge (f\top \rightarrow fy) = fx \wedge fy$. ☐

In the special case of Boolean lattices, a similar argument applies to suprema instead of infima, which leads to the following characterization. Note that a lattice homomorphism need not preserve $\top$ or $\bot$.

PROPOSITION 2.19. *With respect to the distance $x \rightarrow y = \neg x \vee y$ on both sides, as well the homometries and the weak homometries coincide with the lattice homomorphisms.*

Proof. For a weak homometry f, it remains to show that $f(x \vee y) \leqslant fx \vee fy$. Because of $(x \to \bot) \wedge (y \to \bot) = (x \vee y) \to \bot$, we know that also $(fx \to f\bot) \wedge (fy \to f\bot) \leqslant f(x \vee y) \to f\bot$. Therefore,

$$\begin{aligned} &\neg f\bot \to \neg f(x \vee y) = f(x \vee y) \to f\bot \\ &\quad \geqslant (fx \to f\bot) \wedge (fy \to f\bot) = \neg f\bot \to \neg(fx \vee fy), \text{ thus} \\ &\neg f(x \vee y) \geqslant \neg f\bot \wedge (\neg f\bot \to \neg f(x \vee y)) \\ &\quad \geqslant \neg f\bot \wedge (\neg f\bot \to \neg(fx \vee fy)) = \neg f\bot \wedge \neg(fx \vee fy) \end{aligned}$$

and finally $f(x \vee y) \leqslant f\bot \vee fx \vee fy = fx \vee fy$.

On the other hand, every lattice homomorphism f is already a homometry with calibration $c(x) := fx \vee \neg f\top$. Indeed, $c\top = f\top \vee \neg f\top = \top$, and $c(x \wedge y) = f(x \wedge y) \vee \neg f\top = (fx \vee \neg f\top) \wedge (fy \vee \neg f\top) = cx \wedge cy$, that is, c is a q. o. m.-morphism. Moreover, $\neg fx \vee f\bot \leqslant \neg f\top \vee f(\neg x)$ $[\![$ since $f\bot \leqslant f(\neg x)$ and $fx \vee (\neg f\top \vee f(\neg x)) = f\top \vee \neg f\top = \top]\!]$ and also $\neg fx \vee f\bot \geqslant \neg f\top \vee f(\neg x)$ $[\![$ since $\neg f\top \leqslant \neg fx$ and $\neg f(\neg x) \vee (\neg fx \vee f\bot) = \neg f\bot \vee f\bot = \top]\!]$. Finally, $fx \to fy = \neg fx \vee f\bot \vee fy = \neg f\top \vee f(\neg x) \vee fy = \neg f\top \vee f(x \to y) = c(x \to y)$. ☐

When we consider the additive symmetrization $\leftrightarrow$ of $\to$ instead, there is a characterization which is similar to the case of ℓ-groups. Note that when $f : (X, d, \underline{M}) \to (Y, e, \underline{N})$ is a homometry, it is also a homometry between $(X, d^S, \underline{M})$ and $(Y, e^S, \underline{N})$ since calibrations are additive.

PROPOSITION 2.20. *For maps $f, c : L \to L'$ between Boolean lattices with $c(\top) = \top$, the following are equivalent.*

a) f is a homometry w. r. t. the distance $\leftrightarrow$ on both sides, and with calibration c.

b) c is a lattice homomorphism, and $f(x) = a \leftrightarrow c(x)$ for some $a \in L'$ and all $x \in L$.

Proof. Let $f(x) = a \leftrightarrow cx$ with a homomorphism c, and $a \in L'$. Proposition 2.19 implies that c is a homometry w. r. t. $\to$ which is its own calibration $[\![$ since $c(\top) = \top]\!]$, in particular, it is a homometry w. r. t. $\leftrightarrow$. Now $fx \leftrightarrow fy = cx \leftrightarrow cy = c(x \leftrightarrow y)$ shows that also f is a homometry with calibration c.

On the other hand, let f be a homometry with calibration c and put $g(x) := fx \leftrightarrow f\top$. Then $g(\top) = \top$ and, as above, $gx \leftrightarrow gy = fx \leftrightarrow fy = c(x \leftrightarrow y)$ shows that also g is a homometry with calibration c. Because of $g(x) = gx \leftrightarrow g\top = c(x \leftrightarrow \top) = c(x)$, we have $f(x) = f\top \leftrightarrow cx$. Since, as a calibration, c preserves binary infima, it remains to show that it also preserves binary suprema. Because $c(\bot) = c(x \leftrightarrow \neg x) = cx \leftrightarrow c(\neg x)$ and $cx \geqslant c\bot$

imply that $c(\neg x) = cx \leftrightarrow c\bot = cx \rightarrow c\bot$, we have

$$\begin{aligned} c(x \vee y) &= c\big(\neg(\neg x \wedge \neg y)\big) = c(\neg x \wedge \neg y) \rightarrow c\bot \\ &= \big(c(\neg x) \rightarrow c\bot\big) \vee \big(c(\neg y) \rightarrow c\bot\big) = cx \vee cy. \end{aligned}$$

□

Comparing distance functions on the same set

The first chapter already contained some examples of mathematical objects which are endowed with more than one natural distance function, such as the $\underline{2}$- or $\underline{\mathbb{R}}$-distance $d_{\leqslant}$ and the $\underline{2}'$-distance $\chi_{\leqslant}$ on a quoset, which have already been discussed briefly. Somewhat more interesting are the pair $d_{\leqslant}$ and $d^0_{\rightarrow}$ on a co-quantale, the three types of distance functions d_p, d_L, and $d_{\text{ptw.}}$ on a function space, or d_G, $|x-y|$, d_u, and d_U on an abelian ℓ-group. On a factorial domain, there are even five different types of distances: d_G, d_M, d_p, d_{adic}, and $d_{\text{div.}}$. While some of the latter are defined by so different means that there seems to be virtually no connection between them that would not involve the structure of the factorial domain, we are in other cases able to compare the different distance functions on an object independently of the structure of the object.

Such comparisons can most easily be made by extracting the algebraic and order-theoretic "information" which is contained in the distance functions. Let $\underline{X} = (X, d, \underline{M})$ be fixed for the moment. When $s = x_1y_1\cdots x_ny_n$ is a word consisting of an even number of *letters* in X, we have already introduced the notation $d(s)$ as a shorthand for the sum $d(x_1,y_1) + \cdots + d(x_n,y_n)$. Now this shorthand notation can easily be turned into a formal definition by considering the *free monoid* $X^{2\star}$ of words of even length over X, that is, the set

letters

$d(s)$

free monoid

$X^{2\star}$

$$X^{2\star} := \{x_1y_1\cdots x_ny_n \,|\, n \in \omega,\ x_i,y_i \in X\}$$

$\circ$

concatenation

together with the operation $\circ$ of *concatenation*,

$$x_1y_1\cdots x_ny_n \circ z_1w_1\cdots z_mw_m := x_1y_1\cdots x_ny_nz_1w_1\cdots z_mw_m.$$

To be precise, one should also define what the sequence of symbols $x_1y_1\cdots x_ny_n$ stands for, for example by putting $x_1y_1\cdots x_ny_n := (x_1,y_1,\cdots,x_n,y_n)$. The neutral element of $X^{2\star}$ is of course the empty word which will be identified with the empty set and accordingly designated by the symbol $\emptyset$. What was so far a shorthand notation now defines a map $d : X^{2\star} \rightarrow \underline{M}$. Although this d differs from the original distance function, it can be interpreted as an extension of it by identifying X^2 with the subset of two-letter words in $X^{2\star}$. Hence we can use the same symbol d for this extension without causing confusion.

$\emptyset$

Now since $\leqslant$ is a quasi-order on M, also R_d

$$s\, R_d\, t :\Longleftrightarrow d(s) \leqslant d(t)$$

defines a quasi-order on $X^{2\star}$, and since addition in M is compatible with $\leqslant$, also concatenation in $X^{2\star}$ is compatible with R_d. In other words, $\underline{X}^{2\star}$

$$\underline{X}^{2\star} := (\underline{X}^{2\star}, \circ, \emptyset, R_d)$$

is a quasi-ordered monoid.

Likewise, each mapping $f : X \to Y$ can be extended to a mapping $f^{2\star} : X^{2\star} \to Y^{2\star}$ by simply putting $f^{2\star}$

$$f^{2\star}(x_1 y_1 \cdots x_n y_n) := f(x_1) f(y_1) \cdots f(x_n) f(y_n).$$

Note that $f^{2\star}$ is additive, that is, $f^{2\star}(s \circ t) = f^{2\star}(s) \circ f^{2\star}(t)$. But is it also isotone w. r. t. R_d and R_e when $e : Y^2 \to \underline{N}$ is a distance function on Y? By definition of homometries, the answer is of course this:

LEMMA 2.21. *A mapping $f : \underline{X} \to \underline{Y}$ is a homometry if and only if $f^{2\star} : \underline{X}^{2\star} \to \underline{Y}^{2\star}$ is isotone (and hence a q. o. m.-morphism).*

This should suffice as a motivation for the following definition. A second distance function $e : X^2 \to \underline{N}$ is *finer* than d (and d is *coarser* than e) if and only *finer* if $R_e \subseteq R_d$, that is, if and only if the identity map id_X is a homometry from *coarser* $(X, e, \underline{N})$ to $(X, d, \underline{M})$. When e is both finer and coarser than d, it is *equivalent* *equivalent* to d.

EXAMPLE 2.22. For a quasi-order $\leqslant$ on X, the distance functions $d_{\leqslant} : X^2 \to \underline{2}$ and $\chi_{\leqslant} : X^2 \to \underline{2}'$ are equivalent since $c : \underline{2} \to \underline{2}'$, $\alpha \mapsto 1 - \alpha$ is an isomorphism so that id_X is an *isometry*, that is, a homometry whose inverse is *isometry* also a homometry. Also, $id_X : (X, d_{\leqslant}, \underline{\mathbb{R}}) \to (X, d_{\leqslant}, \underline{2})$ is a homometry with calibration $\alpha \mapsto \min\{\alpha, 1\}$, but its inverse cannot be a homometry since the additive maps $\underline{2} \to \underline{\mathbb{R}}$ are constant.

More generally, the identity id_X is a weak homometry $(X, d, \underline{M}) \to (X, d_{\leqslant d}, \underline{2})$ for any distance set $\underline{X}$. Indeed, $d_{\leqslant_d}(x, y)$ is zero if $d(x, y) \leqslant 0$ and one otherwise. Therefore, the isotone map $t(\alpha) := 0$ for $\alpha \leqslant 0$ and $t(\alpha) := 1$ for $\alpha \not\leqslant 0$ is a translation map for id_X, and the latter is distance inequality preserving. Since $t(\alpha) \vee t(\beta) = 0$ implies $\alpha + \beta \leqslant 0$, t is also subadditive, hence id_X is a weak homometry. In case that d is positive, also $t(\alpha + \beta) = 0$ implies $\alpha, \beta \leqslant 0$, so that in this case t is even additive, and id_X a homometry.

In particular, $id_M : (M, d^0_{\rightarrow}, \underline{M}) \to (M, d_{\leqslant}, \underline{2})$ is a homometry for every co-quantale $\underline{M} = (M, +, 0, \leqslant)$, and $id_X : (X, d_{\leqslant}, \underline{\mathbb{R}}) \to (X, d_{\leqslant}, \underline{2})$ is a weak homometry.

EXAMPLE 2.23. Let G be a lattice ordered group and $e(x,y) := |x-y|$. Then $id_G : (G,d_G) \to (G,e)$ is a weak homometry with translation map $t(\alpha) = |\alpha|$. Since t is extensive but not additive, id_G is expansive but not a homometry. Hence its inverse is contractive but not even distance equivalence preserving since t is not injective. This means that d_G and e are incomparable w. r. t. fineness.

Also $id_G : (G,e) \to (G,d_u)$ is a weak homometry for all $u \in G^+$, with subadditive translation map $t(\alpha) = \|\alpha\|_u$. Again t is neither additive nor injective, hence all of d_G, e, and d_u are incomparable. On the other hand, d_u and d_{mu} are equivalent for all $m \in \omega \setminus \{0\}$. Indeed, $\|x\|_u = m\|x\|_{mu}$, and $\alpha \mapsto m\alpha$ is a q. o. m.-isomorphism on $\underline{\mathbb{R}}^\top$. Finally, d_U is finer than each d_u since the projection map $\pi_u : (\mathbb{R}^\top)^{G^+} \to \mathbb{R}^\top$, $\alpha \mapsto \alpha(u)$ is a q. o. m.-morphism.

The latter observation is of course always true for multi-$\underline{M}$-distance functions: they are always finer than each of their components.

Weak homometries occur most naturally in connection with norms like the u-norm in partially ordered groups or the p-norm in factorial domains. Another example is $id : (C([0,1]),d_{\text{ptw.}}) \to (C([0,1]),d_p)$ for all $1 \leqslant p \leqslant \infty$.

CANONICAL DISTANCE FUNCTIONS AND GENERATING QUASI-ORDERS

$\bar{d}$

Can we perhaps use the q. o. m. $\underline{X}^{2\star}$ to define a new distance function $\bar{d}$ on X that is somehow related to d? Of course we can: simply put $\bar{d}(x,y) := xy$ with the only exception that $\bar{d}(x,x)$ must be $\emptyset$ because of the zero distance condition. Note that still $xx\, R_d\, \bar{d}(x,x)\, R_d\, xx$ because of $d(x,x) = 0$, and that the triangle inequality is trivially true here. By definition, $R_{\bar{d}} = R_d$, hence $\bar{d}$ and d are equivalent. Therefore, $\bar{d}$ will be called the *canonical modification* of d.

canonical modification

G_X G^0_X G^s_X G^{0s}_X

For the following, let

$$\begin{aligned} G_X &:= \{(\emptyset,xx),(xx,\emptyset),(xz,xyyz)\mid x,z \in X,\ y \in X \setminus \{x,z\}\},\\ G^0_X &:= G_X \cup \{(\emptyset,xy)\mid x,y \in X\},\\ G^s_X &:= G_X \cup \{(xy,yx)\mid x,y \in X\},\\ \text{and}\quad G^{0s}_X &:= G^0_X \cup G^s_X. \end{aligned}$$

LEMMA 2.24. *For a relation R on $X^{2\star}$, the function $d_R : X^2 \to (X^{2\star},\circ,\emptyset,R)$ is an arbitrary, positive, symmetric, or positive and symmetric distance function if and only if R is a quasi-order on $X^{2\star}$ that is compatible with $\circ$ and includes G_X, G^0_X, G^s_X, or G^{0s}_X, respectively.*

d_R

canonical

generating quasi-order

The *proof* is trivial. These distance functions d_R will be called *canonical*, and the corresponding $R = R_{d_R}$ is the *generating quasi-order* of d_R. Note that since two canonical distance functions on X are equivalent only if they are equal, the system of all canonical distance functions on X is a set (!) of representatives for the proper class of all distance functions on X w. r. t. equivalence.

The following lemma has a straightforward proof as well:

LEMMA 2.25. *For every* $A \subseteq (X^{2\star})^2$*, the smallest generating quasi-order* R *including* A *is the transitive hull of the relation*

$$B := \{(sut,svt) \mid s,t \in X^{2\star},\ (u,v) \in A \cup G_X \cup \Delta_{X^{2\star}}\}.$$

This means that in order to get from a word $s \in X^{2\star}$ to a "larger" word $t \in X^{2\star}$ with $s\,R\,t$, one must perform a finite number of "replacement steps" $u \longrightarrow v$, that is, a subword u that occurs after an even number of letters is replaced by a word v with $(u,v) \in A \cup G_X$. Accordingly, I will speak of 'steps in A' or 'in G_X'. In this context, it is also convenient to call a subword of length two that occurs after an even number of letters in s a *syllable* of s. A *null-syllable* is one of the form xx with $x \in X$. The steps in G_X, for example, are then the insertion or deletion of a null-syllable, or the replacement of one syllable xz by two non-null syllables, that is, by some word $xyyz$.

syllable

null-syllable

The smallest generating quasi-order for X, that is, the transitive hull of

$R_X^\perp$

$$\bigl\{(s,s),(st,sxxt),(sxxt,st),(sxzt,sxyyzt) \mid s,t \in X^{2\star},\ x,y,z \in X\bigr\},$$

will be designated by $R_X^\perp$ because it leads to "the" (up to equivalence) finest distance function $d_X^\perp$ on X. In Chapter 4 we will see that this distance function suffices to induce all T_1 quasi-uniformities on X.

$d_X^\perp$

Here comes a first example of reasoning with syllables.

PROPOSITION 2.26. *For every family* $Q = (\leqslant_x)_{x\in X}$ *of quasi-orders on a set* X*, there is a finest distance function* d_Q *on* X *with* $y \leqslant_x z \iff d_Q(y,x) \leqslant d_Q(z,x)$ *for all* $x,y,z \in X$.

In that case, d_Q *is positive resp.* T_1 *resp.* T_0 *if and only if, for all* $y \neq z$*, we have* $y \leqslant_y z$ *resp.* $y <_y z$ *resp. (*$y <_y z$ *or* $z <_z y$*).*

Moreover, there is also a finest symmetric such distance function if and only if the implications

$$\begin{aligned}
y \leqslant_{x_0} x_1,\ x_0 \leqslant_{x_1} x_2,\ \ldots,\ x_{n-1} \leqslant_{x_n} x_0,\ x_n \leqslant_{x_0} z \ &\Longrightarrow\ y \leqslant_{x_0} z,\\
y \leqslant_{x_0} x_1,\ x_0 \leqslant_{x_1} x_2,\ \ldots,\ x_{n-1} \leqslant_{x_n} x_n \ &\Longrightarrow\ y \leqslant_{x_0} x_0,\\
\textit{and}\quad x_1 \leqslant_{x_1} x_2,\ \ldots,\ x_{n-1} \leqslant_{x_n} x_0,\ x_n \leqslant_{x_0} z \ &\Longrightarrow\ x_0 \leqslant_{x_0} z
\end{aligned}$$

hold for all $n \geqslant 1$ *and all* $y,z,x_0,\ldots,x_n \in X$.

Proof. Let R be the smallest generating quasi-order on X including $A := \{(yx,zx) \mid y \leqslant_x z\}$, and put $d_Q := d_R$. Assuming that $yx\,R\,zx$, we have to prove that also $yx\,A\,zx$. Consider the steps in $A \cup G_X$ that lead from yx to zx. Since all $\leqslant_x$ are transitive, so is A. Note that no step in $A \cup G_X$ diminishes the number of non-null syllables, hence in no step from yx to zx the number of non-null syllables can be larger than one. In particular, all steps that affect non-null syllables are steps in A, so that each non-null syllable has a well-defined "trace". Now the trace of yx has the form $yx\,A\,y'x\,A\,y''x\cdots$ and either ends with zx, in which case we are done, or ends with xx and the subsequent removal of xx. Likewise, the trace of zx has the form $\cdots z''x\,A\,z'x\,A\,zx$ and either begins with yx (and we are done) or some freshly inserted xx. In the latter case, one has $yx\,A\,xx\,A\,zx$, hence also $yx\,A\,zx$.

The proposed equivalences are now clear from $d(y,y) \leqslant d(z,y) \iff y \leqslant_y z$ etc.

For the last claim, assume that $y \leqslant_x z \iff d(y,x) \leqslant d(z,x)$ holds with a symmetric d, and that $y \leqslant_{x_0} x_1, x_0 \leqslant_{x_1} x_2, \dots, x_{n-2} \leqslant_{x_{n-1}} x_n, x_{n-1} \leqslant_{x_n} x_0$, and $x_n \leqslant_{x_0} z$. Then $d(y,x_0) \leqslant d(x_0,x_1) \leqslant \cdots \leqslant d(x_n,x_0) \leqslant d(z,x_0)$, that is, $y \leqslant_{x_0} z$. Likewise, the premises of the second and third implications are equivalent to $d(y,x_0) \leqslant d(x_0,x_1) \leqslant \cdots \leqslant d(x_{n-1},x_n) \leqslant d(x_n,x_n) = d(x_0,x_0)$ and $d(x_0,x_0) = d(x_1,x_1) \leqslant d(x_1,x_2) \leqslant \cdots \leqslant d(x_n,x_0) \leqslant d(z,x_0)$.

On the other hand, assume that the required implication holds, let R be the smallest generating quasi-order on X including A and $S := \{(vw,wv) \mid v,w \in X\}$, and suppose that $yx_0\,R\,zx_0$. As above, all steps from yx_0 to zx_0 that affect non-null syllables are steps in $A \cup S$. In the traces of yx and zx, subsequent steps in A can be replaced by single steps in A (since A is transitive), and each pair of subsequent steps in S can be left out (since they have the form $vw\,S\,wv\,S\,vw$). After doing so, there is either a common trace of the form

$$yx_0\,A\,x_1x_0\,S\,x_0x_1\,A\,x_2x_1\,S\,x_1x_2\cdots x_{n-1}x_n\,A\,x_0x_n\,S\,x_nx_0\,A\,zx_0,$$

in which case the premise implies $yx_0\,A\,zx_0$. Or there are traces

$$\begin{aligned} & yx_0\,A\,x_1x_0\,S\,x_0x_1\,A\,x_2x_1\,S\,x_1x_2\cdots x_{n-1}x_n\,A\,x_nx_n \\ \text{and}\quad & v_1v_1\,A\,v_2v_1\,S\,v_1v_2\cdots v_{n-1}v_n\,A\,x_0v_n\,S\,v_nx_0\,A\,zx_0. \end{aligned}$$

in which case the premise implies $yx_0\,A\,x_0x_0\,A\,zx_0$. □

Families of quasi-orders are used in logics, for instance:

EXAMPLE 2.27. *Possible worlds and counterfactual sentences.*

propositions
sentences

Let X be a set of "possible worlds" or "information states", and call the subsets $A \subseteq X$ *propositions* or *sentences*. The intended interpretation of $x \in A$ is that the sentence A is "true" in world x. When $X \setminus A$, $A \cap B$, $A \cup B$, and $(X \setminus A) \cup B$ are interpreted as negation, conjunction, disjunction, and material implication, respectively, the usual laws of Boolean propositional calculus obviously hold.

modal operators

However, in addition to these classical connectives, the possible worlds approach also allows us to define *modal operators* $\Box$ and $\Diamond$ of "necessity" and "possibility". This is usually done by assuming a reflexive *accessibility relation* $S \subseteq X^2$ and putting

accessibility relation

$$\Box A := \{x \in X \,|\, xS \subseteq A\} \quad \text{and} \quad \Diamond A := SA = \{x \in X \,|\, xS \cap A \neq 0\}.$$

In other words, the sentence that 'A is necessary [or possible]' is true in world x if A is true in every [or at least one] world accessible from x. One can try to formalize also counterfactual sentences of the form 'if A would be true, also B would be true' in a similar manner, using some more structure on X than just the relation S. Now, the so far most intuitive of these approaches is that of Lewis [Lew73] and Veltman [Vel85]. It requires a certain family $(\leqslant_x)_{x \in X}$ of quasi-orders on X, where $y \leqslant_x z$ has the intended meaning that world y is "more similar" to x than z is. Lewis' favourite type of *counterfactual operator* $\mathrel{\Box\!\!\rightarrow}$ can then be defined by means of this family:

$\Box$

$\Diamond$

counterfactual operator

$\mathrel{\Box\!\!\rightarrow}$

$$A \mathrel{\Box\!\!\rightarrow} B := \{x \in X \,|\, \forall w \in A \; \exists z \in A, z \leqslant_x w \; \forall y \in A \setminus B : y \not\leqslant_x z\}.$$

This operation fulfils $A \mathrel{\Box\!\!\rightarrow} B \subseteq (X \setminus A) \cup B$, which is equivalent to the rule of *modus ponens* $A \cap (A \mathrel{\Box\!\!\rightarrow} B) \subseteq B$, preserves binary intersections in its second argument, that is, $A \mathrel{\Box\!\!\rightarrow} (B \cap C) = (A \mathrel{\Box\!\!\rightarrow} B) \cap (A \mathrel{\Box\!\!\rightarrow} C)$, but is not in general antitone in the first argument. Also, $\mathrel{\Box\!\!\rightarrow}$ does not fulfil the cut rule $(A \mathrel{\Box\!\!\rightarrow} B) \cap (B \mathrel{\Box\!\!\rightarrow} C) \subseteq A \mathrel{\Box\!\!\rightarrow} C$, but its symmetrization $A \mathrel{\leftarrow\!\!\Box\!\!\rightarrow} B := (A \mathrel{\Box\!\!\rightarrow} B) \cap (B \mathrel{\Box\!\!\rightarrow} A)$ does and is thus an internal $(\mathscr{P}(X), \cap, X, \supseteq)$-distance function.

modus ponens

$\mathrel{\leftarrow\!\!\Box\!\!\rightarrow}$

The above proposition shows that in fact all such possible worlds semantics for counterfactuals come from a suitable distance function on the set of possible worlds, to be interpreted as a *dissimilarity* measure, by way of the definition $(\star)$ $y \leqslant_x z :\Longleftrightarrow d(y,x) \leqslant d(z,x)$. The usual heuristics for counterfactual semantics is that the sentence 'if it were the case that A, it would be the case that B' should be true in a world x if and only if, whenever x is changed in some way only so much that A becomes true, also B becomes true. Given a positive, T_1 distance function d on X, this heuristics is also (and perhaps better) met when $\leqslant_x$ is defined in terms of betweenness instead of $(\star)$. Indeed,

dissimilarity

$$y \leqslant_x z :\Longleftrightarrow xyyz \, R_d \, xz$$

also defines a quasi-order on X for every x $[\![$ since $y = z \Longrightarrow d(xyyz) = d(xz)$, and $xyyz \, R_d \, xz$ and $xzzw \, R_d \, xw$ imply $xyyw \, R_d \, xyyzzw \, R_d \, xw$ $]\!]$. It is a nice exercise to determine the set $A \mathrel{\Box\!\!\rightarrow} B$ defined by means of these quasi-orders in case of, say, two closed subsets A, B of the Euclidean plane $\mathbb{E}_2$ that share some of their boundary.

We will come back to this idea in Chapter 6.

Alexandroff completion
$\underline{M}_\downarrow$
lower sets
$\downarrow A$
$\check{+}$
$A+B$
$\downarrow\alpha$
$\bigvee$*-dense*

Before we enter the realm of topology in Part B, let me recall that like quasi-ordered sets, also every quasi-ordered monoid $\underline{M}$ possesses at least two natural completions. The *Alexandroff completion* $\underline{M}_\downarrow$ is the system $M_\downarrow$ of all *lower sets* $\downarrow A := \{\beta \in M \,|\, \beta \leqslant \alpha \text{ for some } \alpha \in A\}$, with the partial order $\subseteq$ of set inclusion and the addition operation

$$\downarrow A \check{+} \downarrow B := \downarrow(\downarrow A + \downarrow B) = \downarrow(A+B),$$

where $A+B$ is of course short for $\{\alpha+\beta \,|\, \alpha \in A, \beta \in B\}$. Its neutral element is the principal ideal $\downarrow\{0\}$, and the original monoid $\underline{M}$ is embedded into $\underline{M}_\downarrow$ via $\lambda : \alpha \mapsto \downarrow\alpha := \downarrow\{\alpha\}$. The term 'completion' means in this case that $\underline{M}_\downarrow = (M_\downarrow, \check{+}, \downarrow 0, \subseteq)$ is in fact a complete lattice-ordered monoid, λ is an injective q. o. m.-morphism that preserves arbitrary infima, and $\lambda[M]$ is $\bigvee$*-dense* in $\underline{M}_\downarrow$, that is, every element in $\underline{M}_\downarrow$ is a supremum of embedded elements $\lambda(\alpha)$. Indeed, the supremum of a set in $\underline{M}_\downarrow$ is just its union, hence $\underline{M}_\downarrow$ is upper distributive, that is, a quantale.

dual Alexandroff completion
$\underline{M}_\uparrow$
upper sets
$\uparrow A$
$\hat{+}$
$\uparrow\alpha$
$\bigwedge$*-dense*

Although in case of quasi-ordered sets, the above completion is more usual than its dual, in our case lower distributivity is more desirable than upper. The *dual Alexandroff completion* $\underline{M}^\uparrow$ consists of all *upper sets* $\uparrow A := \{\beta \in M \,|\, \beta \geqslant \alpha$ for some $\alpha \in A\}$, with reverse set inclusion $\supseteq$ as its partial order, and with the addition operation

$$\uparrow A \hat{+} \uparrow B := \uparrow(\uparrow A + \uparrow B) = \uparrow(A+B).$$

In complete analogy to the first completion, the map $\nu : \alpha \mapsto \uparrow\alpha := \uparrow\{\alpha\}$ is an injective q. o. m.-morphism from $\underline{M}$ into the co-quantale $\underline{M}^\uparrow := (M^\uparrow, \hat{+}, \uparrow 0, \supseteq)$ that preserves arbitrary suprema, and $\nu[M]$ is $\bigwedge$*-dense* in $\underline{M}^\uparrow$. Note that infima in $\underline{M}^\uparrow$ are again unions, not intersections.

$\check{d}$
$\hat{d}$
$\underline{\check{M}}_d$
$\underline{\hat{M}}_d$
upper/lower canonical modifications

Now let d be an $\underline{M}$-distance function. Since λ and ν are order-isomorphic q. o. m.-morphisms, both $\lambda \circ d$ and $\nu \circ d$ are distance functions equivalent to d. In other words, every distance function is equivalent to one with an upper and one with a lower distributive monoid. The distance functions $\check{d} := \lambda \circ \bar{d}$ and $\hat{d} := \nu \circ \bar{d}$, having as co-domains the partially ordered monoids $\underline{\check{M}}_d := (X^{2\star}, \circ, \emptyset, R_d)_\downarrow$ and $\underline{\hat{M}}_d := (X^{2\star}, \circ, \emptyset, R_d)^\uparrow$, respectively, will be called the *upper* and *lower canonical modification* of d, corresponding to upper and lower distributivity of their monoids, respectively.

BACK TO THE ROOTS OF TOPOLOGY

3.

CONVERGENCE AND CLOSURE

> *Motion does not exist because the moving body must go half the distance before it goes the whole distance.*
>
> *Zenon of Elea*

Converging to the right class of structures

OVERTURE: SEQUENTIAL CONVERGENCE STRUCTURES

In 1905, Maurice Fréchet [Fré05] was the first to introduce the concept of what we now call a metric space under the name 'écart'.[1] The remarkable thing about this is not only that he chose exactly the nowadays usual conditions of zero distance, triangle inequality, positivity, symmetry, and separatedness, but that he introduced his concept in full abstractness, not imposing any considerable restriction on the objects being related by his distance functions. It was a first journey into the realm of abstract *spaces* of things, that is, arbitrary sets equipped with some sort of topological structure.

spaces

One year later in his thesis [Fré06], Fréchet introduced another class of abstract spaces whose topological structure consists of the information about which sequences converge to which points. Following this historical line, let us consider a set X with a *sequential convergence structure* $\mathscr{L}$, that is, a subset of $X^\omega \times X$, and call a sequence $S \in X^\omega$ *convergent to x,* in symbols $S \longrightarrow_{\mathscr{L}} x$, if and

sequential convergence structure

$\mathscr{L}$

$\longrightarrow_{\mathscr{L}}$

---

[1] This word was used by Jordan for distances. Only in 1914, Hausdorff introduced the term 'metrischer Raum' for this concept, a choice that was criticized by Fréchet because of its then different meaning in geometry: "Il peut en résulter une confusion regrettable que nous préférons éviter. [. . .] on pourrait, suivant une suggestion de M. Bouligand, employer un néologisme et les appeler 'espaces distanciés' " [Fré28].

only if $(S,x) \in \mathscr{L}$. When we designate the set of all isotone injections $\sigma : \omega \to \omega$ by Σ, the *subsequences* of S are exactly the sequences $S \circ \sigma$ for $\sigma \in \Sigma$. General sequences are also written as $(x_i)_i$, whereas *constant sequences* are written as $(x)_i$. The *specialization relation* of $\mathscr{L}$ is the relation $x \geqslant_{\mathscr{L}} y :\Longleftrightarrow (x)_i \to_{\mathscr{L}} y$, which, however, is *not* a quasi-order in general. Among the many possible conditions on sequential convergence structures, the following are of special importance: by their means, it is possible to characterize those types of convergence that can be induced by other kinds of topological concepts such as "closure" or "neighbourhood".

Σ

subsequences

constant sequences

specialization relation

$\geqslant_{\mathscr{L}}$

($\mathrm{C_s}$1) Constant sequences converge to their constant value, that is, $(x)_i \to_{\mathscr{L}} x$.

($\mathrm{C_s}$2) Subsequences inherit all limits, that is, $S \to_{\mathscr{L}} x$ and $\sigma \in \Sigma$ imply $S \circ \sigma \to_{\mathscr{L}} x$.

($\mathrm{C_s}$3) *Urysohn's axiom:* If, for all $\sigma \in \Sigma$, there is $\tau \in \Sigma$ with $S \circ \sigma \circ \tau \to_{\mathscr{L}} x$, then $S \to_{\mathscr{L}} x$.

($\mathrm{C_s}$p) If $x_i \geqslant_{\mathscr{L}} y$ for all $i \in \omega$, then $(x_i)_i \to_{\mathscr{L}} y$.

($\mathrm{C_s}$z) If $x_i \geqslant_{\mathscr{L}} y_i$ for all $i \in \omega$, and $(y_i)_i \to_{\mathscr{L}} y \geqslant_{\mathscr{L}} z$, then $(x_i)_i \to_{\mathscr{L}} z$.

Note that ($\mathrm{C_s}$1) and ($\mathrm{C_s}$z) together are stronger than ($\mathrm{C_s}$p) and imply that $\geqslant_{\mathscr{L}}$ is a quasi-order.

In a distance set, of course, one would try to define convergence in a way similar to classical analysis by saying that $(x_i)_i$ converges to x if and only if the distances $d(x_i,x)$ from the elements of the sequence to the prospective limit become arbitrarily small as i grows.[1] But what does 'small' mean in a q. o. m. $\underline{M}$? Our way to make this notion precise will be to specify a set $D \subseteq M$ and say that $(x_i)_i$ converges to x if and only if, for all $\delta \in D$, there is $n \in \omega$ such that $d(x_i,x) \leqslant \delta$ for all $i \geqslant n$. The resulting sequential convergence structure will be designated by $\mathscr{L}(d,D)$, with $\to_{\mathscr{L}(d,D)}$ and $\geqslant_{\mathscr{L}(d,D)}$ abbreviated by $\to_{d,D}$ and $\geqslant_{d,D}$. In case of a real distance function, the natural choice for D is of course the set of all elements > 0, in which case it makes no difference whether we require $d(x_i,x) \leqslant \delta$ or $< \delta$ instead.[2] For general distance sets, however,

D

$\mathscr{L}(d,D)$

$\to_{d,D}$

$\geqslant_{d,D}$

---

[1] Most authors who consider non-symmetric distance functions require that $d(x,x_i)$ becomes small instead, since then the correspondence to quasi-uniform spaces is a bit more direct than it will be in this thesis. However, I do not consider this enough motivation to change the intuitive understanding that a sequence converges *towards* its limit, and that therefore the distances *from* the sequence *to* the limit should be the relevant ones. This is also underlined by the fact that one usually writes $(x_i)_i \to x$ and not $x \leftarrow (x_i)_i$.

[2] The latter remark is always true when D has no minimal elements. Otherwise, it makes a difference, but I will always use the $\leqslant$-version here since that is both more natural from an order-theoretic point of view and more convenient when working with generating quasi-orders.

there are some reasons (which will become clear soon) not to restrict our choice of D to that set. Instead, let us study the relationship between properties of D and those of $\mathscr{L}(d,D)$. By definition, the latter always fulfils ($\mathrm{C_s}2$), ($\mathrm{C_s}3$) $\llbracket$ if $(x_i)_i \not\to_{d,D} x$, there is $\delta \in D$, and for all $n \in \omega$, there is a smallest $i(n) \geqslant n$ with $d(x_{i(n)},x) \nleqslant \delta$. Then $(x_{i(n)})_n$ has no subsequence converging to x $\rrbracket$, and ($\mathrm{C_s}$p) $\llbracket$ since $x_i \geqslant_{d,D} y \iff d(x_i,y) \leqslant \delta$ for all $\delta \in D$ $\rrbracket$. Axiom ($\mathrm{C_s}1$) holds if and only if $D \subseteq \uparrow 0$. Obviously, $\mathscr{L}(d,D) = \mathscr{L}(d,D')$ whenever $\uparrow D = \uparrow D'$.

positive filter

Since $(\omega,\leqslant)$ is an up-directed set, it also makes no difference to close D under finite infima that might exist in $\underline{M}$. Indeed, it will turn out that any convergence $\mathscr{L}(d,D)$ that comes from a pair (d,D) with $D \subseteq \uparrow 0$ already comes from one with a *positive filter*, that is, a down-directed set $D = \uparrow D \subseteq \uparrow 0 \subseteq M$. This is because a positive filter can still be quite tailor-made for a specific purpose, for example to produce or disable certain convergences:

LEMMA 3.1. *Let* $\mathscr{L}$ *be a sequential convergence structure with* ($\mathrm{C_s}1$), ($\mathrm{C_s}2$), ($\mathrm{C_s}3$)*, and* ($\mathrm{C_s}$p)*, and* $(S,z) \in (X^\omega \times X) \setminus \mathscr{L}$. *Then there is a distance function* $d_{S,z} : X^2 \to \underline{M}$ *and a positive filter* $D_{S,z}$ *in* $\underline{M}$ *with* $(S,z) \notin \mathscr{L}(d_{S,z},D_{S,z}) \supseteq \mathscr{L}$.

$\leqslant$

Proof. Define a partial order on $M_0 := \{0\} \cup X^2 \setminus \Delta_X$ by putting $0 \leqslant (x,y)$ for all $(x,y) \in M_0$, that is, $\leqslant := (\{0\} \times M_0) \cup \Delta_{M_0}$.

$+$ $\underline{M}$

On $M := M_{0\downarrow}$, define an addition $+$ by putting $\alpha + \beta := M_0$ for all $\alpha,\beta \in M \setminus \{\downarrow 0\}$, and $\downarrow 0 + \alpha := \alpha + \downarrow 0 := \alpha$, thus making $\underline{M} := (M,+,\downarrow 0,\subseteq)$ a partially ordered monoid with absorbing largest element M_0.

S'

Choose some $\sigma \in \Sigma$ such that z is not an $\mathscr{L}$-limit of any subsequence of $S \circ \sigma$, which is possible by ($\mathrm{C_s}3$), then choose some $\sigma' \in \Sigma$ such that $S \circ \sigma \circ \sigma'(i) \ngeqslant_{\mathscr{L}} z$ for all $i \in \omega$, which is possible by ($\mathrm{C_s}$p) $\llbracket$ if for all $\sigma' \in \Sigma$, $S \circ \sigma \circ \sigma'(i) \geqslant_{\mathscr{L}} z$ for some $i \in \omega$, then there would be a $\sigma' \in \Sigma$ with $S \circ \sigma \circ \sigma'(i) \geqslant_{\mathscr{L}} z$ for all $i \in \omega$. But then z would be a limit of $S \circ \sigma \circ \sigma'$ by ($\mathrm{C_s}$p) $\rrbracket$. Put $S' := S \circ \sigma \circ \sigma'$ and $A := \{(S'(i),z) \mid i \in \omega\}$.

A δ $D_{S,z}$ $d_{S,z}$

Because of ($\mathrm{C_s}1$), $(z,z) \notin A$. Since A is an upper set of $M_0 \setminus \{0\}$, $\delta := M_0 \setminus A$ is a lower set of M containing 0, that is, $\downarrow 0 \subseteq \delta \in M$. Hence the upper set $D := D_{S,z} := \uparrow\{\delta\}$ of $\underline{M}$ which is generated by $\{\delta\}$ is a positive filter. Furthermore, putting $d(x,x) := \downarrow 0$ and $d(x,y) := \downarrow(x,y)$ for $x \neq y$ defines an $\underline{M}$-distance function $d_{S,z} := d$ on X $\llbracket$ the triangle inequality holds by definition of $\leqslant$ and $+$ $\rrbracket$. Note that $d(x,y) \nsubseteq \delta \iff (x,y) \in A$. Since $d(S'(i),z) \nsubseteq \delta$ for all $i \in \omega$, we have $(S,z) \notin \mathscr{L}(d,D)$.

Finally, assume that $\big((y_i)_i,y\big) \in (X^\omega \times X) \setminus \mathscr{L}(d,D)$. Then, for all $n \in \omega$, there is a smallest $i \geqslant n$ with $d(y_i,y) \nsubseteq \delta$. Hence, there is $\tau \in \Sigma$ and $f \in \omega^\omega$ with $(y_{\tau(n)},y) = \big(S'(f(n)),z\big)$ for all $n \in \omega$. In particular, $y = z$. Also, there is some $\tau' \in \Sigma$ so that either $f \circ \tau' \in \Sigma$ (if f is unbounded) or $f \circ \tau'$ is constant (if f is bounded).

In the ‘bounded’ case, $S'(f\circ\tau'(i)) = v$ for all $i\in\omega$ and some $v\in X$, hence $y_{\tau\circ\tau'(i)} = v \not\geqslant_{\mathscr{L}} z = y$ for all $i\in\omega$. Therefore, $(y_i)_i \not\to_{\mathscr{L}} y$ by (C$_\mathrm{s}$2).

In the ‘unbounded’ case, the sequence $(y_{\tau\circ\tau'(i)})_i = S'\circ(f\circ\tau')$ is a common subsequence of $(y_i)_i$ and $S\circ\sigma$, and $z = y$ is not an $\mathscr{L}$-limit of some subsequence of $S\circ\sigma$. Hence $(y_i)_i \not\to_{\mathscr{L}} y$ by (C$_\mathrm{s}$2). ☐

Note that since ω is well-ordered, no choice principle is needed in the proof. By taking categorical suprema of several pairs (d_i,D_i), the above construction can be used to characterize those sequential convergence structures that come from distances. The *categorical supremum* of a (set-indexed) family $(d_i : X^2 \to \underline{M}_i)_{i\in I}$ of distance functions is the distance function

categorical supremum

$\sup_{i\in I} d_i$

$$\sup\nolimits_{i\in I} d_i \colon \left\{ \begin{array}{lcl} X^2 & \to & \prod_{i\in I}\underline{M}_i \\ (x,y) & \mapsto & \sup_{i\in I} d_i(x,y) : i \mapsto d_i(x,y), \end{array} \right.$$

which is “the” (up to equivalence) coarsest distance function finer than all d_i $\llbracket$ the projection map $\pi_j : \prod_{i\in I}\underline{M}_i \to M_j$ is a calibration for $id : (X,d) \to (X,d_j)$, and for every family of calibrations $(c_i : \underline{S}_e \to \underline{M}_i)_{i\in I}$ for a common homometry $id : (X,e) \to (X,d_i)$, the map $c : \underline{S}_e \to \prod_{i\in I}\underline{M}_i$, $\alpha \mapsto (c_i(\alpha))_{i\in I}$ is a calibration for $id : (X,e) \to (X,\sup_{i\in I} d_i)$ $\rrbracket$.

In order to be able to define the supremum of pairs (d_i,D_i), we must first adjoin a new largest absorbing element $\top$ to all the filtered monoids that do not already possess such an element. That is, put $(\underline{M}_i^\top, D_i^\top) := (\underline{M}_i, D_i)$ if $\underline{M}_i$ has an absorbing largest element, otherwise let $\underline{M}_i^\top$ be the monoid $\underline{M}_i$ with a new $\top$ adjoined, and put $D_i^\top := D_i \cup \{\top\}$. The *categorical supremum* of a family $(d_i : X^2 \to \underline{M}_i, D_i)_i$ is then the pair $\sup_{i\in I}(d_i,D_i) := \big(\sup^\top_{i\in I} d_i, \coprod_{i\in I} D_i^\top\big)$, where $\sup^\top_{i\in I} d_i$ is defined exactly as $\sup_{i\in I} d_i$, only with the larger co-domain $\prod_{i\in I}\underline{M}_i^\top$, and with

$\underline{M}^\top$

$D^\top$

categorical supremum

$\sup_{i\in I}(d_i,D_i)$

$\sup^\top_{i\in I} d_i$

$\coprod_{i\in I} D_i^\top$

$$\coprod_{i\in I} D_i^\top := \Big\{ \delta \in \prod_{i\in I} D_i^\top \,\big|\, \delta(i) = \top \text{ for all but finitely many } i \in I \Big\}.$$

Note that if all D_i are positive, idempotent, or filters, respectively, then so is $\coprod_{i\in I} D_i^\top$.

LEMMA 3.2. *A sequence converges to a point x with respect to* $\sup_{i\in I}(d_i,D_i)$ *if and only if it does so w. r. t. each* (d_i,D_i)*:* $\mathscr{L}\big(\sup_{i\in I}(d_i,D_i)\big) = \bigcap_{i\in I}\mathscr{L}(d_i,D_i)$.

Proof. Given $i\in I$ and $\delta_i \in D_i$, put $(d,D) := \sup_{j\in I}(d_j,D_j)$ and define $\delta \in D = \coprod_{j\in I} D_j^\top$ by putting $\delta(i) := \delta_i$ and $\delta(j) := \top$ for all $j \in I\setminus\{i\}$. Then $d(x,y) \leqslant \delta \iff d_i(x,y) \leqslant \delta_i$. Hence (d,D)-convergence implies (d_i,D_i)-convergence. On the other hand, let $\delta \in D$, that is, $\delta(i) = \top$ for all $i \in I\setminus F$ with some finite set $F \subseteq I$. If $(x_j)_j \to_{d_i,D_i} x$ for all $i\in F$, there are numbers $n_i \in \omega$ such that $d_i(x_j,x) \leqslant \delta(i)$ for all $i \in F$ and $j \geqslant n_i$. Taking the maximum

n of the finitely many n_i, we finally get $d(x_j, x) \leqslant \delta$ for all $j \geqslant n$ as required for $(x_j)_j \to_{d,D} x$. $\square$

THEOREM 3.3. *For a sequential convergence structure* $\mathscr{L}$*, there is a distance function* d *and a positive filter* D *with* $\mathscr{L} = \mathscr{L}(d, D)$ *if and only if* $\mathscr{L}$ *fulfils* ($\mathrm{C_s}$1)*,* ($\mathrm{C_s}$2)*,* ($\mathrm{C_s}$3)*, and* ($\mathrm{C_s}$p)*.*

Proof. The categorical supremum $(d, D) := \sup_{(S,z)\in(X^\omega \times X)\setminus\mathscr{L}}(d_{S,z}, D_{S,z})$ (defined as in Lemma 3.1) fulfils $\mathscr{L} = \mathscr{L}(d, D)$ by Lemma 3.2. $\boxed{\text{AC}}$[1]

Although most common types of convergence fulfil the above requirements, there is a prominent example that violates Urysohn's axiom:

EXAMPLE 3.4. *Almost sure convergence.* In probability theory, one studies probability spaces $(\Omega, \mathscr{A}, P)$ (where $\mathscr{A} \subseteq \mathscr{P}(\Omega)$ is a σ-algebra on Ω, that is, nonempty and closed under complements and countable unions, and P is a probability measure on $\mathscr{A}$, that is, a measure with $P(\Omega) = 1$) and (real) random variables (= measurable functions $x : \Omega \to \mathbb{R}$) on them. It is then quite often the case that one cannot assure the pointwise convergence of a sequence $(x_i)_i$ of random variables for every $a \in \Omega$ but only for all $a \in A$, where A is a subset of measure 1. In other words, if

$$P\big((x_i)_i \to x\big) = P\big(a \in \Omega \,|\, (x_i(a))_i \to x(a)\big) = 1,$$

almost sure convergence $\to_{\text{a.s.}}$

one says that $(x_i)_i$ *converges almost surely* to x, written as $(x_i)_i \to_{\text{a.s.}} x$. An important case of this kind of convergence is the strong law of large numbers.

Almost sure convergence fulfils ($\mathrm{C_s}$1), ($\mathrm{C_s}$2) $[\![\, P(S \circ \sigma \to x) \geqslant P(S \to x) \,]\!]$, and ($\mathrm{C_s}$z) $[\![\, (x)_i \to_{\text{a.s.}} y \iff P(x = y) = 1$, and $P(x_i = y) = 1$ for all $i \in \omega$ implies $P(x_i = y$ for all $i \in \omega) = 1 \,]\!]$. However, it highly violates ($\mathrm{C_s}$3). Let $\mathscr{A}$ be any σ-algebra on $\Omega = [0, 1]$ (for example the Borel-sets) that contains all singletons and includes the countable system $\mathscr{B}$ of all intervals of the form $[k/2^n, (k+1)/2^n]$ with non-negative integers k, n and $0 \leqslant k < 2^n$. Let P be any probability measure on $\mathscr{A}$ with $P(\{a\}) = 0$ for all singletons (for example Lebesgue-measure). Choose a bijection $f : \omega \to \mathscr{B}$ and let $g(a) := \{i \in \omega \,|\, a \in f(i)\}$. Then $(g(a))_a \in \omega$ is an *almost disjoint family*, that is, (i) each $g(a)$ is infinite, while (ii) $g(a) \cap g(b)$ is finite for $a \neq b$ $[\![\,$only finitely many of the intervals have length $\geqslant |a - b| \,]\!]$. Each of the characteristic functions x_i with $x_i(a) := 1$ for $a \in f(i)$ and $x_i(a) := 0$ for $a \notin f(i)$ is measurable, hence a random variable. Also, $x :\equiv 0$ defines a random variable. Now, each subsequence of $(x_i)_i$ has a subsequence which converges almost surely to x. Let $A \subseteq \omega$ be

almost disjoint family

---

[1] Although the proof relies on it, the Axiom of Choice is not needed for this and the following results: the class-indexed supremum of *all* (d, D) which fulfil $\mathscr{L} \subseteq \mathscr{L}(d, D)$ can be constructed without choice principles and is even a *finest* pair that induces $\mathscr{L}$.

infinite. If $A \cap g(a)$ is finite for all $a \in \Omega$ then $(x_i)_{i\in A} \to x$ even pointwise. Otherwise, choose a such that $B := A \cap g(a)$ is infinite. Then (ii) implies that $(x_i(b))_{i\in B} \to 0$ for all $b \neq a$, in particular $(x_i)_{i\in B} \to_{\text{a.s.}} x$ since $P(\{a\}) = 0$. But (i) implies that $(x_i(a))_i \not\to 0$ for *all* a, in particular $P((x_i)_i \to x) = 0$ instead of 1 as required by ($\mathrm{C_s}$3).

However, the following uniform version of almost sure convergence does come from a distance.

EXAMPLE 3.5. Write $(x_i)_i \to_{\text{u.a.s.}} x$ if and only if there is some $A \in \mathscr{A}$ with $P(A) = 1$ so that the sequence $(x_i|_A)_i$ of random variables restricted to A converges uniformly to $x|_A$. Then this type of convergence is induced by the usual internal distance function $d(x,y) := |x - y|$ (as is the case for pointwise convergence), together with the quasi-order $\alpha \leqslant_{\text{a.s.}} \beta \iff P(\alpha \leqslant \beta) = 1$ and the zero-filter D generated by the countable base $\{\Omega \times \{2^{-n}\} \mid n \in \omega\}$. Indeed,

$\to_{\text{u.a.s.}}$

$\leqslant_{\text{a.s.}}$

$$\begin{aligned}
&(x_i)_i \to_{d,D} x\\
&\iff \forall n\, \exists k\, \forall i \geqslant k:\ P\big(|x_i - x| \leqslant 2^{-n}\big) = 1\\
&\iff \forall n\, \exists k:\ P\big(\forall i \geqslant k: |x_i - x| \leqslant 2^{-n}\big) = 1\\
&\iff \exists A \in \mathscr{A}\, \forall n\, \exists k\, \forall \omega \in A\, \forall i \geqslant k:\ |x_i - x| \leqslant 2^{-n} \text{ and } P(A) = 1\\
&\iff \exists A \in \mathscr{A}:\ (x_i|_A)_i \to x|_A \text{ uniformly, and } P(A) = 1\\
&\iff (x_i)_i \to_{\text{u.a.s.}} x.
\end{aligned}$$

DISTANCE SPACES: SPECIALIZATION TO ZERO-FILTERS

In Fréchets thesis, the limit of a sequence was required to be unique, this property being of particular importance in classical analysis. In a metric space, uniqueness is assured by symmetry and separatedness: if $S \to x$, $S \to y$, and $\delta > 0$, there is $i \in \omega$ such that $d(S(i),x), d(S(i),y) \leqslant \delta$, hence $d(x,y) \leqslant 2\delta$. Therefore, $d(x,y), d(y,x) \leqslant 0$ and thus $x = y$. This argument, of course, relies on the fact that $\bigwedge\{\delta \mid \delta > 0\} = 0$. In the general setting, the same proof obviously works if $\underline{M}$ is lower distributive and D is a *zero-filter*, that is, a (positive) filter in $\underline{M}$ for which 0 is an infimum $[\![$since then $\bigwedge_{\delta\in D} 2\delta = 2\bigwedge D = 0]\!]$. Also, in case of a zero-filter, the specializations of d and $\mathscr{L}(d,D)$ coincide, and the latter fulfils ($\mathrm{C_s}$z) $[\![$since then $d(x_i,z) \leqslant d(y_i,y)]\!]$. Let us therefore consider this a minimal requirement for a sensible choice of D and introduce the name *distance space* for a quadruple $(X,d,\underline{M},D)$ with $(X,d,\underline{M})$ a distance set and D a zero-filter in $\underline{M}$. The triple $(d,\underline{M},D)$ (often abbreviated by (d,D)) will then be called a *distance structure* on X, and $(\underline{M},D)$ a *filtered monoid.*

zero-filter

distance space

$(X,d,$
$\underline{M},D)$

$(d,\underline{M},D)$

distance structure

$(\underline{M},D)$

filtered monoid

Only a minor modification of Lemma 3.1 shows that sequential convergence in a distance space is characterized by condition ($\mathrm{C_s}$z):

LEMMA 3.6. *Let* $\mathscr{L}$ *be a sequential convergence structure with* ($\mathrm{C_s}$1)*,* ($\mathrm{C_s}$2)*,* ($\mathrm{C_s}$3)*, and* ($\mathrm{C_s}$z)*, and* $(S,z) \in (X^\omega \times X) \setminus \mathscr{L}$*. Then there is a distance structure* (d,D) *on* X *with* $(S,z) \notin \mathscr{L}(d,D) \supseteq \mathscr{L}$*.*

$\leqslant$ *Proof.* Define a quasi-order $\leqslant$ on $M_0 := \{0\} \cup \{(x,y) \in X^2 \mid x \ngeqslant_{\mathscr{L}} y\}$ by putting $0 \leqslant \alpha$ for all $\alpha \in M_0$, and

$$(v,w) \leqslant (x,y) :\Longleftrightarrow v \geqslant_{\mathscr{L}} x \text{ and } w \leqslant_{\mathscr{L}} y,$$

where transitivity follows from ($\mathrm{C_s}$z). Using this partially ordered set $(M_0, \leqslant)$,
$\underline{M}, S', A$ define $\underline{M}$, S', and A as in Lemma 3.1.

For each finite set $F \subseteq M_0 \setminus \{0\}$, let $\delta_F := M_0 \setminus {\uparrow}(F \cup A)$. Obviously, $E := \{\delta_F \mid F \subseteq M_0 \setminus \{0\} \text{ finite}\}$ is a filter-base in $\underline{M}$. Moreover, $\bigcap E = {\downarrow}0$
D since $(x,y) \notin \delta_{\{(x,y)\}}$ for all $(x,y) \in M_0$. Let $D := {\uparrow}E$ be the zero-filter in
d $\underline{M}$ generated by E. Now, $d(x,y) := {\downarrow}0$ for $x \geqslant_{\mathscr{L}} y$ and $d(x,y) := {\downarrow}(x,y)$ for $x \ngeqslant_{\mathscr{L}} y$ defines an $\underline{M}$-distance function on X $\llbracket$the triangle inequality holds since $\alpha + \beta = M_0$ for all $\alpha, \beta \neq {\downarrow}0$, and since $d(x,y) = d(y,z) = {\downarrow}0$ implies $d(x,z) = {\downarrow}0$ because of ($\mathrm{C_s}$z)$\rrbracket$. Since $d(S'(i),z) \nsubseteq \delta_F$ for all $i \in \omega$ and all finite F, we have $(S,z) \notin \mathscr{L}(d,D)$.

Assume that $\big((y_i)_i, y\big) \in (X^\omega \times X) \setminus \mathscr{L}(d,D)$, and choose some F such that, for all $n \in \omega$, there is $i \geqslant n$ with $d(y_i,y) \nsubseteq \delta_F$, that is, $y_i \ngeqslant_{\mathscr{L}} y$ and $(y_i,y) \in {\uparrow}(F \cup A)$. The latter implies that either (i) there is $(v,w) \in F$ and $\tau \in \Sigma$ such that $(y_{\tau(i)}, y) \geqslant (v,w)$ for all $i \in \omega$, or (ii) there is $\tau \in \Sigma$ and $f \in \omega^\omega$ with $(y_{\tau(i)}, y) \geqslant \big(S'(f(i)), z\big)$ for all $i \in \omega$. In case (i), $y_{\tau(i)} \leqslant_{\mathscr{L}} v \ngeqslant_{\mathscr{L}} w \leqslant_{\mathscr{L}} y$, hence $(y_i)_i \not\to_{\mathscr{L}} y$ by ($\mathrm{C_s}$z) and ($\mathrm{C_s}$2). In case (ii), we have $z \leqslant_{\mathscr{L}} y$, and there is $\tau' \in \Sigma$ with either $f \circ \tau' \in \Sigma$ (if f is unbounded) or $f \circ \tau'$ is constant (if f is bounded). In the latter case, $S'(f \circ \tau'(i)) = v$ for all $i \in \omega$ and some $v \in X$ with $(v,z) \in A$, hence $y_{\tau \circ \tau'(i)} \leqslant_{\mathscr{L}} v \ngeqslant_{\mathscr{L}} z \leqslant_{\mathscr{L}} y$ and thus $(y_i)_i \not\to_{\mathscr{L}} y$ as in (i). And in the former case, the sequences $(y_{\tau \circ \tau'(i)})_i$ and $(x_i)_i := S' \circ (f \circ \tau')$ fulfil $x_i \geqslant_{\mathscr{L}} y_{\tau \circ \tau'(i)}$ and $(x_i)_i \not\to_{\mathscr{L}} z$, hence $(y_i)_i \not\to_{\mathscr{L}} y$ by ($\mathrm{C_s}$z). ☐

THEOREM 3.7. *A sequential convergence structure can be induced by a distance structure if and only if it fulfils* ($\mathrm{C_s}$1)*,* ($\mathrm{C_s}$2)*,* ($\mathrm{C_s}$3)*, and* ($\mathrm{C_s}$z)*.*

Because the categorical supremum of distance structures is again a distance structure, this can be proved exactly like Theorem 3.3. ☐

Open and closed; filters and nets

While sequential convergence is mostly about *points* of a space, set-theoretic topology is often more interested in properties of *subsets* of a space. In a distance space, such properties can be most easily defined either by means of balls, or *α-ball* $B_{d,\alpha}y$ *D-ball* *ball-system*

using the notion of convergence. For all $\alpha \in M$ and $y \in X$, the *α-ball about y* is the set $B_{d,\alpha}y := \{x \in X \mid d(x,y) \leqslant \alpha\}$, which contains y if and only if $\alpha \geqslant 0$. A *D-ball* is one with $\alpha \in D$, and the *ball-system* $\mathscr{B}_{d,D}y$ about y is the system of all D-balls about y.

$\mathscr{B}_{d,D}y$ *ball-open* *kernel system*

Now, a *ball-open* set includes some D-ball about each of its points, that is, it is a subset $O \subseteq X$ such that for all $y \in O$, there is $\delta \in D$ with $x \in O$ whenever $d(x,y) \leqslant \delta$. It is obvious by this definition that for a given distance d and an arbitrary set $D \subseteq M$, the resulting ball-open sets build a *kernel system*, that is, are closed under arbitrary unions. Modifying a construction by Pervin [Per62], we obtain

PROPOSITION 3.8. *For every kernel system* $\mathscr{O}$*, there is a multi-real distance function* $d : X^2 \longrightarrow (\mathbb{R}^\top)^I$ *on* $X := \bigcup \mathscr{O}$ *such that* $\mathscr{O}$ *is the system of ball-open sets of* (X, d, D)*, where* $D := \{\delta \in (0,\infty]^I \mid \delta(i) < \infty$ *for at most one* $i \in I\}$.

Proof. Similar to the construction for set functions in Chapter 1, put $I := \mathscr{O}$, $d(x,y)(O) := 1$ for $(x,y) \in O^{\Rightarrow}$, and $d(x,y) := 0$ otherwise, so that d_O is the characteristic function of $O^{\Rightarrow} = (X \setminus O) \times O$. Then $B_{d,\delta}y = O$ for each $\delta = \{(O,\varepsilon)\} \cup \big((\mathscr{O} \setminus \{O\}) \times \{\infty\}\big) \in D$ with $\varepsilon < 1$ and $y \in O$. The rest is routine. ☐

topology $\mathscr{T}(d,D)$

The above D consists of all elements that are *long-way-above* 0, a concept that will be used in the next section. Note that it fulfils $\bigwedge D = 0$ but is not a filter. In a distance space, that is, when D is down-directed, the ball-open sets build a *topology* $\mathscr{T}(d,D)$, that is, a kernel system which is also closed under binary (and hence finite) intersections. If a topology $\mathscr{O}$ is given, one can modify the above construction and put $D := \coprod_{i \in I}(0,\infty]$, which is the smallest zero-filter for $(\mathbb{R}^\top)^I$ and consists of all elements *way-above* 0 (see also the next section). Since then still all ball-open sets are members of $\mathscr{O}$, this is the easiest way to show that every topology comes from a distance structure.[1]

sequentially closed *topological hull system*

Closed sets. Given a sequential convergence structure $\mathscr{L}$ on X, call a subset $C \subseteq X$ *sequentially closed* if it contains all limits of sequences in C. Without any conditions on $\mathscr{L}$, these sets always build a hull system, that is, are closed under arbitrary intersections. In case of (C$_s$2), it is a *topological hull system*, that is, also closed under finite unions ⟦because a sequence in $A \cup B$ has a subsequence in either A or B⟧.

In a metric space, the two notions of (ball-)openness and (sequential) closedness are nicely related by the fact that the open sets are just the

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[1] Kopperman's article [Kop88] is essentially based on that fact, which however was known long before. Flagg [Fla97] requires the long-way-above set of 0 to be a filter in his value distributive lattices, hence his theory is not applicable to multi-real distances. Another disadvantage of this requirement is that it is not preserved when taking products.

complements of the closed ones. In the general case of an arbitrary positive filter D, only half of this is true: complements of ball-open sets are sequentially closed. Indeed, when A is ball-open and S a sequence in $X \setminus A$, no limit x of S can belong to A. Otherwise, there would be a ball $B_{d,\delta}x \subseteq A$ in which the sequence stays eventually[1] (that is, $S(i) \in B_{d,\delta}x$ for all i larger than some $n \in \omega$).

EXAMPLE 3.9. *Multi-real distance spaces.* If nothing else is specified, a multi-metric distance set $(X, d : X^2 \longrightarrow (\underline{\mathbb{R}}^\top)^I)$ will always be interpreted as a distance space having the smallest possible zero-filter $\coprod_{i\in I}(0,\infty]$.

Now consider the function space $X := \mathbb{R}^\mathbb{R}$ together with the pointwise Euclidean multi-pseudometric, that is, $d(f,g) := d_{\text{ptw.}}(f,g) = |f - g|$. The induced sequential convergence structure is that of pointwise convergence. The set C of all functions $f \in X$ with countable support (that is, those with $f(r) = 0$ for all but countably many r) is sequentially closed $[\![$ a limit f of such functions f_i has $f(r) = 0$ wherever $f_i(r) = 0$ for all i $]\!]$ but its complement is not ball-open. More precisely, every ball $B_{d,\delta}f$ with $\delta \in D$ and $f \in X$ contains an element of A $[\![$put $f'(r) = f(r)$ if $\delta(r) < \infty$ (which is only finitely often the case) and $f'(r) = 0$ otherwise. Then $f'(r) \in A \cap B_{d,\delta}f$ $]\!]$, that is, A is *ball-dense* in $\underline{X}$.

ball-dense

That metric spaces are better behaved in this respect is because their zero-filter D has a *base* (that is, a subset $E \subseteq D$ with $\uparrow E = D$) which is countable $[\![$ for example $D \cap \mathbb{Q}$ or, which is more frequently used, the base $\{2^{-n} \mid n \in \omega\}$ $]\!]$. In contrast to this, $\coprod_{i\in I}(0,\infty]$ does not have a countable base when I is uncountable as in the example above.

base

LEMMA 3.10. *If D is a positive filter with countable base, the complements of sequentially closed subsets of (X, d, D) are exactly the ball-open sets.*

Proof. Let $E = \{\delta_i \mid i \in \omega\}$ be a countable base of D with $\delta_i \leqslant \delta_j$ whenever $i \leqslant j$ $[\![$ such a base can be chosen since D is a filter $]\!]$. Assuming that every D-ball about some $x \in X \setminus A$ intersects A, choose some $(x_i)_i$ with $x_i \in B_{d,\delta_i}x \cap A$. But this is a sequence in A with limit $x \notin A$. $\boxed{\text{CC}}$[2]

FILTER CONVERGENCE

The above proof shows that the discrepancy between ball-openness and sequential closedness for general positive filters just lies in the fact that the minimal cardinality of a base of D and that of ω (the domain of all sequences) may differ. One solution to this problem would be to consider "sequences" with

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[1] See below for the general meaning of 'eventually' and 'frequently'.

[2] 'CC' means the Axiom of Countable Choice is needed here, requiring the existence of choice functions only for countable families.

other domains than ω, that is, nets. However, there are some problems finding a suitable notion of "subnet" (cf. [Suh80]), so I will first use filters here.

For a sequence S, call every set of the form $S[\omega \setminus n] = \{S(i) \mid i \geqslant n\}$ an *end* of S. Suppose that D is a positive filter. Then the definition of sequential convergence can be simplified to this: $S \longrightarrow_{d,D} x$ if and only if every D-ball about x includes an end of S. The ends of S build a *filter base on* X, that is, a nonempty down-directed set in the partially ordered set $(\mathscr{P}(X) \setminus \{\emptyset\}, \subseteq)$, and also the ball system $\mathscr{B}_{d,D}x$ is a filter base since D is a filter. Convergence now just means that the ends of S build a *finer* base than the D-balls about x. The generated filters $\mathscr{E}S := \uparrow\{S[\omega \setminus \{n\}] \mid n \in \omega\}$ and $\mathscr{C}_{d,D}x := \uparrow\mathscr{B}_{d,D}x$ on X are called the *end filter* of S and the *neighbourhood filter* of x, respectively.[1] By their means, the statement that S converges to x reduces to $\mathscr{E}S \supseteq \mathscr{C}_{d,D}x$.

end

filter base on X

$\mathscr{E}S$

$\mathscr{C}_{d,D}x$

end filter

neighbourhood filter

If we designate the system of all filters on X by $\mathrm{Fil}(X)$, a *filter convergence structure* on X is a relation $\mathscr{C} \subseteq \mathrm{Fil}(X) \times X$, and again $\mathscr{F} \longrightarrow_{\mathscr{C}} x$ means $(\mathscr{F}, x) \in \mathscr{C}$. The induced sequential convergence structure is $\mathscr{L}(\mathscr{C}) := \{(S, x) \mid (\mathscr{E}S, x) \in \mathscr{C}\}$. A *cluster point* of a filter is a limit of a finer filter, and since subsequences have finer end filters, every limit of a subsequence of a sequence S is a cluster point of $\mathscr{E}S$. A constant sequence $S = \omega \times \{x\}$ has the end filter $\check{x} := \{F \subseteq X \mid x \in F\}$,[2] called the *principal ultra-filter* of x, and again the *specialization relation* $x \geqslant_{\mathscr{C}} y :\Longleftrightarrow \check{x} \longrightarrow_{\mathscr{C}} y$ need not be a quasi-order in general. However, $\uparrow_{\mathscr{C}}A$ will be short for $\{x \mid x \geqslant_{\mathscr{C}} y$ for some $y \in A\}$. As in case of sequential convergence, there are several natural conditions $\mathscr{C}$ might satisfy:

$\mathrm{Fil}(X)$

filter convergence structure

$\mathscr{F} \longrightarrow_{\mathscr{C}} x$

$\mathscr{C}(\mathscr{L})$

cluster point

$\check{x}$

principal ultra-filter

specialization relation

$\geqslant_{\mathscr{C}}$

($\mathrm{C_f}1$) Principal ultra-filters converge to their "base" point: $\check{x} \longrightarrow_{\mathscr{C}} x$.

($\mathrm{C_f}2$) Finer filters inherit all limits, that is, $\mathscr{F} \longrightarrow_{\mathscr{C}} x$ and $\mathscr{G} \supseteq \mathscr{F}$ imply $\mathscr{G} \longrightarrow_{\mathscr{C}} x$.

($\mathrm{C_f}3$) A common cluster point of all finer filters is already a limit, that is, if for all $\mathscr{G} \supseteq \mathscr{F}$ there exists $\mathscr{H} \supseteq \mathscr{G}$ with $\mathscr{H} \longrightarrow_{\mathscr{C}} x$, then $\mathscr{F} \longrightarrow_{\mathscr{C}} x$.

($\mathrm{C_f}\ell$) Binary intersections of filters convergent to x converge to x, that is, $\mathscr{F} \longrightarrow_{\mathscr{C}} x$ and $\mathscr{G} \longrightarrow_{\mathscr{C}} x$ imply $\mathscr{F} \cap \mathscr{G} \longrightarrow_{\mathscr{C}} x$.

($\mathrm{C_f}$p) For each $x \in X$, there is a smallest filter $\mathscr{C}x$ that converges to x.

($\mathrm{C_f}$z) $\mathscr{F} \longrightarrow_{\mathscr{C}} x \geqslant_{\mathscr{C}} y$ implies $\uparrow\{\uparrow_{\mathscr{C}}F \mid F \in \mathscr{F}\} \longrightarrow_{\mathscr{C}} y$.

($\mathrm{C_f}$t) For each $F \in \mathscr{C}y$, there is $G \in \mathscr{C}y$ with $F \in \mathscr{C}x$ for all $x \in G$.

Note that ($\mathrm{C_f}1$) is equivalent to ($\mathrm{C_s}1$) for $\mathscr{L}(\mathscr{C})$, and ($\mathrm{C_f}2$) entails ($\mathrm{C_s}2$) for $\mathscr{L}(\mathscr{C})$. Moreover, ($\mathrm{C_f}2$) and ($\mathrm{C_f}$p) together mean that $\mathscr{F} \longrightarrow_{\mathscr{C}} x \Longleftrightarrow \mathscr{F} \supseteq \mathscr{C}x$,

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[1] Here the symbol $\uparrow$ refers to the partial order $\subseteq$ of course.

[2] I reserve the more common symbol $\dot{x}$ for constant nets instead.

which already implies ($\mathrm{C_f}\ell$) and ($\mathrm{C_f}3$) $\llbracket$If $\mathscr{F} \not\to_{\mathscr{C}} x$, choose $F \in \mathscr{C}x \setminus \mathscr{F}$ and let $\mathscr{G}$ be the filter generated by $\mathscr{F} \cup \{X \setminus F\}$. Then $\mathscr{H} \not\to_{\mathscr{C}} x$ for all $\mathscr{H} \supseteq \mathscr{G}$ since otherwise $F \in \mathscr{C}x \subseteq \mathscr{H}$ would imply $\emptyset = F \cap (X \setminus F) \in \mathscr{H}\rrbracket$. Furthermore, ($\mathrm{C_f}1$) and ($\mathrm{C_f}$z) together imply that $\leqslant_{\mathscr{C}}$ is a quasi-order. On the other hand, if $\leqslant_{\mathscr{C}}$ equals the diagonal Δ_X, the space is called T_1, and ($\mathrm{C_f}$z) is fulfilled.

T_1

limit space

pseudo-topological

pre-topological

topological

If $\mathscr{C}$ fulfils ($\mathrm{C_f}1$), ($\mathrm{C_f}2$), and ($\mathrm{C_f}\ell$), the pair $(X,\mathscr{C})$ is called a *limit space* [Fis59]. If also ($\mathrm{C_f}3$) or ($\mathrm{C_f}$p) is fulfilled, $\mathscr{C}$ is called *pseudo-topological* [Cho48] or *pre-topological* [Cho48, Pre88], respectively. A pre-topological limit space that satisfies ($\mathrm{C_f}$t) is a *topological* one and fulfils also ($\mathrm{C_f}$z) $\llbracket$For $F \in \mathscr{C}y$, choose G as in ($\mathrm{C_f}$t). Then $\uparrow_{\mathscr{C}}G \subseteq F$ since $z \geqslant_{\mathscr{C}} x \in G$ implies $F \in \mathscr{C}x \subseteq \check{z}$, that is, $z \in F$. Hence every $\mathscr{C}y$ has a base of upper sets of $\leqslant_{\mathscr{C}}$. Now, $\mathscr{F} \to_{\mathscr{C}} x \geqslant_{\mathscr{C}} y$ implies

$$\mathscr{C}x = \uparrow\{\uparrow_{\mathscr{C}}F \mid F \in \mathscr{C}x\} \subseteq \uparrow\{\uparrow_{\mathscr{C}}F \mid F \in \mathscr{F}\}$$

and $\mathscr{C}y \subseteq \check{x}$. For $F \in \mathscr{F}_y$, choose G as in ($\mathrm{C_f}$t). Then $x \in G$ and thus $F \in \mathscr{C}x \subseteq \mathscr{F}$, so that finally $\mathscr{C}y \subseteq \mathscr{C}x$ implies $\uparrow\{\uparrow_{\mathscr{C}}F \mid F \in \mathscr{F}\} \to_{\mathscr{C}} y\rrbracket$.

EXAMPLE 3.11. Let $(G,\circ,e)$ be a group, $\underline{M} := (\mathscr{P}(G),\circ,\{e\},\subseteq)$, and $d(x,y) := \{d_G(x,y)\} = \{x^{-1}y\}$. Then the zero-filters D in $\underline{M}$ are exactly the filters $\mathscr{C}e$ of $\mathrm{T_1}$ translation-invariant filter convergence structures $\mathscr{C}$ on G.

$\mathscr{C}(d,D)$

right convergence structure

left convergence structure

Surprisingly, the construction from Lemma 3.1 has an analogue for filters which is even more simple. The filter convergence structure induced by a distance function d and a positive filter D is $\mathscr{C}(d,D) := \{(\mathscr{F},x) \mid \mathscr{F} \supseteq \mathscr{C}_{d,D}x\}$. More precisely, let us call $\mathscr{C}(d,D)$ the *right convergence structure* of (d,D) since limits occur as the right argument of d, while $\mathscr{C}(d^{\mathrm{op}},D)$ will be called the *left convergence structure* of (d,D). Note that $\mathscr{C}_{d,D}x$ is indeed the smallest filter converging to x, in other words, $\mathscr{C}_{d,D}x = \mathscr{C}(d,D)x$.

LEMMA 3.12. $\mathscr{C}\big(\sup_{i\in I}(d_i,D_i)\big) = \bigcap_{i\in I}\mathscr{C}(d_i,D_i)$.

Proof. Let $(d,D) := \sup_{i\in I}(d_i,D_i)$. Then each ball $B_{d,\delta}x$ is a finite intersection of balls $B_{d_i,\delta(i)}x$ with $\delta(i) < \top$. Hence $\mathscr{C}_{d,D}x$ is the smallest filter including all $\mathscr{C}_{d_i,D_i}x$ with $i \in I$, so that $\mathscr{F} \to_{d,D} x$ if and only if $\mathscr{F}$ includes all $\mathscr{C}_{d_i,D_i}x$. $\square$

THEOREM 3.13. *For a filter convergence structure $\mathscr{C}$, there is a distance function e and a set $E \subseteq \uparrow 0$ with $\mathscr{C} = \mathscr{C}(e,E)$ if and only if $(X,\mathscr{C})$ is a pre-topological limit space, that is, fulfils* ($\mathrm{C_f}1$), ($\mathrm{C_f}2$), *and* ($\mathrm{C_f}$p). *Again, E can always be chosen as a positive filter.*

Proof. In $\mathscr{C}(e,E)$, all conditions hold. On the other hand, assume that $\mathscr{C}$ fulfils them, let $z \in X$ and $Z \in \mathscr{C}z$, and put $A := (X \setminus Z) \times \{z\}$. Using this A, construct $\underline{M}$, $D_{z,Z} := D$, and $d_{z,Z} := d$ as in Lemma 3.1. Then $B_{d,\delta}z = Z$ and $B_{d,\delta}x = X$ for $x \neq z$. By definition, $\mathscr{C} \subseteq \mathscr{C}(d,D)$.

Now, take $(e,E) := \sup_{z\in X}\sup_{Z\in\mathscr{C}z}(d_{z,Z},D_{z,Z})$. Then still $\mathscr{C}\subseteq\mathscr{C}(e,E)$ by Lemma 3.12. For $(\mathscr{F},z)\notin\mathscr{C}$, there is $Z\in\mathscr{C}z\setminus\mathscr{F}$ by (C$_{\mathrm{f}}$p), hence $B_{d_{z,Z},\delta_F}\notin\mathscr{F}$ for all F, that is, $(\mathscr{F},z)\notin\mathscr{C}(d_{z,Z},D_{z,Z})\supseteq\mathscr{C}(e,E)$. Therefore, $\mathscr{C}=\mathscr{C}(e,E)$. $\square$

For a positive filter D, the filter convergence structure $\mathscr{C}(d,D)$ leads to a notion of closedness which perfectly fits that of ball-openness. A set $A\subseteq X$ is *filter-closed* if and only if it contains the limits of every filter $\mathscr{F}$ with $F\cap A\neq\emptyset$ for all $F\in\mathscr{F}$. In case of a pre-topological space like $\big(X,\mathscr{C}(d,D)\big)$, this is equivalent to the condition that, for all $x\in X$, $X\setminus A\in\mathscr{C}x$ or $x\in A$.

filter-closed

LEMMA 3.14. *For positive filters D, the filter-closed sets of $\mathscr{C}(d,D)$ are exactly the complements of the ball-open sets of (d,D).*

Proof. $X\setminus A$ is ball-open if and only if, for all $x\in X\setminus A$, some D-ball about x is included in $X\setminus A$, in other words, if $X\setminus A\in\mathscr{C}_{d,D}x$. $\square$

In particular, for positive filters D, the filter-closed sets of (d,D) always build a topological hull-system, that is, they are closed under arbitrary intersections and binary unions.[1] Similarly to the sequential case, those filter convergence structure that come from distance spaces are characterized by condition (C$_{\mathrm{f}}$z).

THEOREM 3.15. *A filter convergence structure can be induced by a distance structure if and only if it fulfils* (C$_{\mathrm{f}}$1)*,* (C$_{\mathrm{f}}$2)*,* (C$_{\mathrm{f}}$p)*, and* (C$_{\mathrm{f}}$z).

Proof. As always, necessity is checked easily.

On the other hand, let $z\in X$ and assume that $Z\in\mathscr{C}z$ is an upper set of $\leqslant_{\mathscr{C}}$, that is, $Z=\uparrow_{\mathscr{C}}Z:=\{x\in X\mid x\geqslant_{\mathscr{C}}y \text{ for some } y\in Z\}$. Put $A:=(X\setminus Z)\times\{z\}$ and define $\underline{M}$, $D_{z,Z}:=D$, and $d_{z,Z}:=d$ as in Lemma 3.6, but using $\leqslant_{\mathscr{C}}$ instead of $\leqslant_{\mathscr{L}}$. Note that $A\subseteq M_0\setminus\{0\}$ since $Z=\uparrow_{\mathscr{C}}Z$. Moreover, $B_{d,\delta_F}z\subseteq Z$ for all finite $F\subseteq M_0\setminus\{0\}$.

Suppose $\mathscr{F}\to_{\mathscr{C}}y$ and $B_{d,\delta_F}y\notin\mathscr{F}$ for some $\mathscr{F}$. Then each $G\in\mathscr{F}$ intersects $X\setminus B_{d,\delta_F}y$. By definition of $\delta_F=M_0\setminus\uparrow(F\cup A)$, we know that $x\in X\setminus B_{d,\delta_F}y\iff(x,y)\in\uparrow(F\cup A)$. Hence each $G\times\{y\}$ intersects either $\uparrow A$, in which case $y\geqslant_{\mathscr{C}}z$ must hold, or it intersects one of the finitely many sets $\uparrow(v,w)$ with $(v,w)\in F$. As $\mathscr{F}$ is a filter, either (i) $y\geqslant_{\mathscr{C}}z$, and all $G\in\mathscr{F}$ intersect $\downarrow_{\mathscr{C}}(X\setminus Z)=X\setminus Z$, or (ii) there is $(v,w)\in F$ with $y\geqslant_{\mathscr{C}}w$ so that all $G\in\mathscr{F}$ intersect $\downarrow_{\mathscr{C}}v$.

In case (i), $\mathscr{F}\to_{\mathscr{C}}z$ by (C$_{\mathrm{f}}$z), in particular $Z\in\mathscr{F}$ by (C$_{\mathrm{f}}$p), in contradiction to (i). In case (ii), in particular each $G\in\mathscr{C}y\subseteq\mathscr{F}$ intersects $\downarrow_{\mathscr{C}}v$, hence $v\in\uparrow_{\mathscr{C}}G$. Since by (C$_{\mathrm{f}}$z), $\mathscr{C}y$ has a base of upper sets of $\leqslant_{\mathscr{C}}$, we have $\check{v}\supseteq\mathscr{C}y$, that

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[1] Hence, the ball-open sets build a topology and their complements, the filter-closed sets, a topological hull system, without $\mathscr{C}(d,D)$ being topological in general!

is, $v \geqslant_{\mathscr{C}} y \geqslant_{\mathscr{C}} w$. Again by ($\mathrm{C_f z}$), this implies $v \geqslant_{\mathscr{C}} w$ in contradiction to $(v,w) \in M_0$. Consequently, $\mathscr{C} \subseteq \mathscr{C}(d_{z,Z}, D_{z,D})$.

Having defined $(d_{z,Z}, D_{z,Z})$ for all choices of z and Z, the proof is now completed exactly as in Theorem 3.13. □

CLOSED AND "CLOSURE"

Čech-closure operator u

An equivalent way to determine a pre-topological filter convergence structure $\mathscr{C}$ on X is to specify a *Čech-closure operator* $u : \mathscr{P}(X) \to \mathscr{P}(X)$ with $u\emptyset = \emptyset$, $uA \supseteq A$, and $u(A \cup B) = uA \cup uB$ for all $A, B \subseteq X$, that is, an extensive operator that preserves finite (including empty) unions (cf. [Čec66]). $\mathscr{C}$ and u are related by

$$x \in uA \iff X \setminus A \notin \mathscr{C}x \quad \text{and} \quad y \in u\{x\} \iff x \geqslant_{\mathscr{C}} y.$$

As the first equivalence is equivalent to

$$x \in uA \iff \big(F \cap A \neq \emptyset \text{ for all } F \in \mathscr{C}x\big),$$

uA
Čech-closure
idempotency
closure operator
u_0

the set uA is the set of all limits of filters whose elements all intersect A, and is called the *Čech-closure* of A. However, it usually is *not* closed, that is, neither is uA filter-closed w. r. t. $\mathscr{C}$, nor does the *idempotency* law $uuA = uA$ hold in general. In particular, u is not a *closure operator* in the order-theoretic sense, that is, extensive, isotone, and idempotent. Putting $u_0 A := \bigcup_{x \in A} u\{x\} = \downarrow_{\mathscr{C}} A$, we can easily characterize those Čech closure operators that come from distance spaces.

LEMMA 3.16. *$\mathscr{C}$ fulfils* ($\mathrm{C_f z}$) *if and only if* $u_0 \circ u \circ u_0 = u$.

Proof. Let $\mathscr{C}$ fulfil ($\mathrm{C_f z}$) and assume that $y \in u_0 u u_0 A \setminus uA$, that is, $y \leqslant_{\mathscr{C}} x$ for some $x \in uu_0 A$, and $G := X \setminus A \in \mathscr{C}y$. By ($\mathrm{C_f z}$), there is $F \in \mathscr{C}x$ with $F = \uparrow_{\mathscr{C}} F \subseteq G$. Then $A \subseteq X \setminus F$ and thus $u_0 A \subseteq u_0(X \setminus F) = X \setminus \uparrow_{\mathscr{C}} F = X \setminus F$. Hence $x \in uu_0 A \subseteq u(X \setminus F)$ in contradiction to $F \in \mathscr{C}x$.

On the other hand, let $u_0 u u_0 = u$, and assume that (x,y) violates ($\mathrm{C_f z}$). That is, $x \geqslant_{\mathscr{C}} y$ and $\uparrow_{\mathscr{C}} F \nsubseteq G$ for some $G \in \mathscr{C}y$ and all $F \in \mathscr{C}x$. Then $A := X \setminus G$ intersects all $\uparrow_{\mathscr{C}} F$ with $F \in \mathscr{C}x$, that is, $u_0 A = \downarrow_{\mathscr{C}} A$ intersects all $F \in \mathscr{C}x$. Now $x \in uu_0 A$, and thus $y \in u_0 u u_0 A = uA$. In contradiction to that, $G = X \setminus A \in \mathscr{C}y$ implies that $y \notin uA$. □

COROLLARY 3.17. *A Čech closure operator* u *comes from a distance structure* (d, D) *if and only if* $u_0 \circ u \circ u_0 = u$.

NET CONVERGENCE

net

Definitions. A *net* $\mathsf{x} = (x_i)_{i \in I_{\mathsf{x}}}$ on a set X is a map $\mathsf{x} : I_{\mathsf{x}} \to X, i \mapsto x_i$, defined on a nonempty, up-directed quasi-ordered[1] *index set* $(I_{\mathsf{x}},\leqslant)$. The images of upper sets of I_{x} are the *ends* of the net, and $\mathscr{E}\mathsf{x} := \uparrow\{\mathsf{x}[\uparrow n] \mid n \in I_{\mathsf{x}}\}$ is the *end filter* of x. The constant net $\dot{x} := (x)_{i\in\{x\}}$ is called the *principal net* of x.

index set
I_{x}
end
$\mathscr{E}\mathsf{x}$
end filter
$\dot{x}$
principal net

A proposition about points x_i of x is said to hold *eventually* if and only if it holds for all $i \geqslant n$, for some choice of $n \in I_{\mathsf{x}}$. It is said to hold *frequently* if and only if, for all $n \in I_{\mathsf{x}}$, it holds for some $i \geqslant n$.

eventually
frequently

In a distance space (X,d,D), a net x *converges* to a point x, in symbols $\mathsf{x} \to_{d,D} x$, if and only if, for all $\delta \in D$, eventually $d(x_i,x) \leqslant \delta$. As in case of the more special sequences, this is equivalent to $\mathscr{E}\mathsf{x} \supseteq \mathscr{C}_{d,D}x$, that is, to $\mathscr{E}\mathsf{x} \to_{d,D} x$. A point x is a *cluster point* of x, in symbols $\mathsf{x} \succ_{d,D} x$, if and only if, for all $\delta \in D$, frequently $d(x_i,x) \leqslant \delta$, which is equivalent to being a cluster point of $\mathscr{E}\mathsf{x}$. A detailed comparison of nets and filters can be found in [Suh80].

$\to_{d,D}$
cluster point
$\succ_{d,D}$

Convergence in filtered monoids. For sequences in metric spaces, the equivalence $(x_i)_i \to x \iff \big(d(x_i,x)\big)_i \to 0$ is frequently used. This is also possible for nets in a distance set (X,d) with positive filter D—we only need the correct notion of convergence in $\underline{M}$.

For a positive filter D of a q. o. m. $\underline{M}$ and a net $\mathsf{a} = (\alpha_i)_{i\in I}$ in M, let $\mathsf{a} \to_D \alpha$ if and only if, for all $\delta \in D$, eventually $\alpha_i \leqslant \delta + \alpha$. This definition corresponds to the pre-topological filter convergence structure $\mathscr{C}_D$ with

$\to_D$
$\mathscr{C}_D$

$$\mathscr{C}_D\alpha := \uparrow\{\downarrow\{\delta+\alpha\} \mid \delta \in D\}.$$

Now it is a triviality to see that

$$\begin{aligned}(x_i)_{i\in I} \to_{d,D} x &\iff \big(d(x_i,x)\big)_{i\in I} \to_D 0\\ &\iff \big(d(x_i,y)\big)_{i\in I} \to_D d(x,y) \text{ for all } y \in X.\end{aligned}$$

In other words, in their first argument, distance functions are *continuous*.

continuous

We know that $\mathscr{C}_D$ must come from a distance function on M. Also, from Chapter 1, we know that there are at least two *internal* distance functions on $\underline{M}$ if the latter is a co-quantale. And indeed, $d^0_{\leftarrow}(\alpha,\beta) = \bigwedge\{\gamma \geqslant 0 \mid \alpha \leqslant \gamma+\beta\}$ obviously fulfils $\mathscr{C}_{d_{\to},D} = \mathscr{C}_D$. Using the above equivalence, this can be rewritten as

$$(\alpha_i)_{i\in I} \to_D \alpha \iff \big(d^0_{\leftarrow}(\alpha_i,\alpha)\big)_{i\in I} \to_D 0.$$

WHAT MAKES TOPOLOGICAL LIFE SUFFICIENTLY EASY

In a number of respects, general distance spaces cannot be considered to provide enough good properties for convenient topological reasoning. For example: (i) A *dual D-ball* $xB_{d,\delta} := \{y \in X \mid d(x,y) \leqslant \delta\}$ need not be closed ⟦on $[0,1]$,

dual D-ball

---

[1] It is not necessary to require a partially ordered index set. In fact, when one seeks to associate a natural net to a filter, the construction is far easier when quasi-ordered index sets are allowed.

let $d(x,y) := |x-y|$ except that $d(0,1) := \infty$, put $\alpha+\beta := \infty$ for $\alpha,\beta > 0$, and $D := (0,\infty]$. Then $1B_{d,1} = (0,1]$ but $1/n \to 0$⟧. (ii) A D-ball need not include any nonempty ball-open set ⟦ on $\mathbb{R}^2$, let $d(x,y) := |x-y|$ when x and y coincide in one coordinate, and $d(x,y) := \infty$ otherwise. Again, $\alpha+\beta := \infty$ for $\alpha,\beta > 0$, and $D := (0,\infty]$. But no ball $B_{d,\delta}x$ with $\delta < \infty$ includes a nonempty ball-open set ⟧.[1]

idempotent

All this cannot happen when D is *idempotent* as an element of $\underline{M}^{\uparrow}$, that is, $D \hat{+} D = D$ or, equivalently, for all $\varepsilon \in D$, there is $\delta \in D$ with $2\delta \leqslant \varepsilon$. Then the induced filter convergence structure is topological as in case of metric spaces ⟦ For $F \in \mathscr{C}_{d,D}y$, there is $\varepsilon \in D$ with $B_{d,\varepsilon}y \subseteq F$ and $\delta \in D$ with $2\delta \leqslant \varepsilon$. Then, for all $x \in G := B_{d,\delta}y \in \mathscr{C}_{d,D}y$, we have $B_{d,\delta}x \subseteq F$ so that $F \in \mathscr{C}_{d,D}x$ ⟧. On the other hand, every topological convergence structure $\mathscr{C}$ is already determined by its induced topology of open sets (hence the name 'topological') since $\mathscr{C}x$ is then just the system of all $F \subseteq X$ with $x \in O \subseteq F$ for some open O. We have already seen that every topology comes from a distance space with idempotent zero-filter, hence:

PROPOSITION 3.18. *Those filter convergence structures that come from a distance structure with idempotent zero-filter are exactly the topological ones.*

Čech's monograph [Čec66] is a good starting point for a comparison of pre-topological spaces (there called closure spaces) and topological spaces.

Distances in point-free situations, and hyperspaces

point-free topology

frame

locale

In *point-free topology*, the topological structure of an object is coded without reference to points of a space. Rather, the basic elements of a point-free topological object can be considered as extended things or "spots". The natural point-free object associated with a topological space $(X,\mathscr{T})$ is most often considered to be the topology $\mathscr{T}$ itself. In other words, the "spots" of the space are just the open sets, which build a *frame* or *locale*, that is, a complete Brouwerian lattice. Here the infinite distributive law

$$a \wedge \bigvee B = \bigvee\{a \wedge b \,|\, b \in B\},$$

frame law

hyperspace

also called the *frame law*, holds because binary infima and arbitrary suprema in that lattice are just intersections and unions, respectively. For some reasons, it may sometimes be more appropriate to consider the system $C_0(\underline{X}) = \{X \setminus A \,|\, A \in \mathscr{T}\}$ of closed sets instead. For example, general topologists have defined many different topological structures on the so-called *hyperspace* $C(\underline{X}) := C_0(\underline{X}) \setminus \{\emptyset\}$.

---

[1] A word on notation: as usual in order theory, xR and Ry stand for $\{y \,|\, x\,R\,y\}$ and $\{x \,|\, x\,R\,y\}$, respectively, hence the notation $xB_{d,\delta}$ and $B_{d,\delta}y$. The alternative form $R(x)$ for xR will not be used here.

The system $C_0(\underline{X})$ of course is a co-quantale, that is, fulfils the dual law

$$a \vee \bigwedge B = \bigwedge\{a \vee b \,|\, b \in B\}$$

and is thus called a *co-frame* or *co-locale.*

co-frame

co-locale

The theory of metric spaces has as well been developed into a point-free direction to some extent by Pultr and Banaschewski (cf. [Pul84] and [BP89]). Their approach was to consider *diameter functions* on frames which generalize the usual definition of diameter in a metric space. The aim of this section is to show that one does not need to change the basic notion in this way but can still use distance functions between "spots" even in a very general point-free context. Like the one-sided Hausdorff distance $d_{H+}(A,B) := \bigvee_{y\in B}\bigwedge_{x\in A} d(x,y)$ between closed sets of a metric space, these distances will be non-symmetric in a natural way that reflects the fact that, contrary to points, "spots" are *extended* things.

diameter functions

d_{H+}

The following lattice-theoretic concepts will prove useful. Let $F \Subset C$ mean that F is a finite subset of C. For elements α,β of a complete lattice M, the *[long-]way-below relations* $\ll$ [resp. $\lll$] and the *[long-]way-above relations* $\gg$ [resp. $\ggg$] are defined like this:

$\Subset$

[long-]way-below relations

[long-]way-above relations

$$\begin{aligned}
\alpha \ll \beta \;&:\Longleftrightarrow\; \forall C \subseteq M\Big(\bigvee C \geqslant \beta \Longrightarrow \exists F \Subset C : \bigvee F \geqslant \alpha\Big),\\
\alpha \gg \beta \;&:\Longleftrightarrow\; \forall C \subseteq M\Big(\bigwedge C \leqslant \beta \Longrightarrow \exists F \Subset C : \bigwedge F \leqslant \alpha\Big),\\
\alpha \lll \beta \;&:\Longleftrightarrow\; \forall C \subseteq M\Big(\bigvee C \geqslant \beta \Longrightarrow \exists \gamma \in C : \gamma \geqslant \alpha\Big),\\
\alpha \ggg \beta \;&:\Longleftrightarrow\; \forall C \subseteq M\Big(\bigwedge C \leqslant \beta \Longrightarrow \exists \gamma \in C : \gamma \leqslant \alpha\Big).
\end{aligned}$$

Note that $\alpha \ll \beta$ is *not* equivalent to $\beta \gg \alpha$, nor is $\alpha \lll \beta$ equivalent to $\beta \ggg \alpha$. For the corresponding upper and lower sets, the notation

$$\begin{aligned}
{\twoheaddownarrow}\beta &:= \{\alpha \in M \,|\, \alpha \ll \beta\}, & {\twoheaduparrow}\beta &:= \{\alpha \in M \,|\, \alpha \gg \beta\},\\
{\twoheaddownarrow\!\!\!\downarrow}\beta &:= \{\alpha \in M \,|\, \alpha \lll \beta\}, & {\twoheaduparrow\!\!\!\uparrow}\beta &:= \{\alpha \in M \,|\, \alpha \ggg \beta\}
\end{aligned}$$

is used. In particular, ${\twoheaduparrow}0$ is a positive filter in M. Now M is called *completely distributive* if and only if

completely distributive

$$\forall \beta \in M : \beta = \bigvee {\twoheaddownarrow\!\!\!\downarrow}\beta,$$

$$\text{which is equivalent to} \quad \forall \beta \in M : \beta = \bigwedge {\twoheaduparrow\!\!\!\uparrow}\beta,$$

the equivalence being shown in [Ran53] for instance.

Pre-diameter spaces. In direct generalization of Pultr's [Pul84] notion of pre-diameter, let us call a quadruple $(\underline{L}, d, \underline{M}, D)$ a *pre-diameter space* if and only if $\underline{L} = (L, \vee, 0)$ is a (supremum-)semilattice with least element 0, $\underline{M}$ is a complete

pre-diameter space

lattice-ordered commutative monoid with zero-filter D, and d fulfils

$$d(0) = 0 \quad \text{and} \quad d(a) \leqslant d(a \vee b) \leqslant d(a) + d(b)$$

intersect for all $a, b \in L$ which *intersect*, that is, for which there is $c \in L$ with $0 < c \leqslant a, b$. If, additionally,

$$d(c) = \bigvee \{d(a \vee b) \mid a, b \leqslant c,\ d(a), d(b) \leqslant \varepsilon\}$$

metric for all $c \in L$ and all $\varepsilon \in D$, the space is called *metric.*

THEOREM 3.19. *Let* $(\underline{L}, d, \underline{M}, D)$ *be a metric pre-diameter space such that* $\underline{M}$ *is a completely distributive co-quantale,* D *is idempotent, and* $\underline{L}$ *fulfils*[1]

$$\text{(I)} \quad \forall\, a, b, c \in L : c \leqslant a \vee b,\ c \nleqslant b \Longrightarrow c, a \textit{ intersect.}$$

e_d *Then*

$$e(a, b) := e_d(a, b) := \bigwedge_{\varepsilon \in D} \ \bigvee_{\substack{0 \neq b' \leqslant b, \\ d(b') \leqslant \varepsilon}} \ \bigwedge_{0 \neq a' \leqslant a} d(a' \vee b')$$

defines a distance function on $\underline{L}$ *which is antitone in the first and isotone in the second component and induces* d *via*

$$d(c) = 0 \vee \bigvee_{0 \neq a \leqslant c} e(a, c).$$

This will be proved using two lemmata.

LEMMA 3.20. *For* $\alpha \lll \zeta$ *in a completely distributive co-quantale* $\underline{M}$ *with idempotent zero-filter* D*,*

$$\exists\, \alpha' \lll \zeta\ \exists\, \delta \in D\ \forall\, \vartheta \in M : (\vartheta + \delta \geqslant \alpha' \Longrightarrow \vartheta \geqslant \alpha).$$

Proof. Assume that, for all $\alpha' \lll \zeta$ and $\delta \in D$, there is $\vartheta \in M$ with $\vartheta + \delta \geqslant \alpha'$ and $\vartheta \ngeqslant \alpha$, so that

$$\bigwedge_{\delta \in D} \left(\bigvee_{\vartheta \ngeqslant \alpha} \vartheta + \delta \right) \geqslant \alpha'.$$

Then $\bigvee_{\vartheta \ngeqslant \alpha} \vartheta \geqslant \zeta$ by lower and complete distributivity. This contradicts $\alpha \lll \zeta$. □

---

[1] Note that for lattices L, condition (I) follows from distributivity but not from modularity ⟦the five-element lattice M_3 with three atoms⟧ or pseudo-complementedness ⟦the other non-distributive five-element lattice, call it D_5⟧. On the other hand, for finite L, it implies pseudo-complementedness ⟦straightforward⟧ but not upper or lower semi-modularity ⟦D_5 with an adjoined new bottom element⟧.

LEMMA 3.21. *If $c \in L$ and (I) holds,*

$$\forall \alpha \lll d(c), \delta \in D\; \exists a \leqslant c, d(a) \leqslant \delta$$
$$\qquad \forall \beta \lll \alpha, \varepsilon \in D\; \exists b \leqslant c, d(b) \leqslant \varepsilon \;:\; \delta \geqslant \beta \;\textit{ or }\; d(a \vee b) \geqslant \beta.$$

Proof. Let α and δ be given. If $\alpha \leqslant 0$, $a := b := 0$ works, so assume $\alpha \nleqslant 0$, choose some $\zeta \in D$ with $\alpha \nleqslant 3\zeta \leqslant \delta$, and, according to metricity, some $a, b_0 \leqslant c$ with $d(a), d(b_0) \leqslant \zeta$ and $d(a \vee b_0) \geqslant \alpha$. Now, for all $\beta \lll \alpha$ and $\varepsilon \in D$, there is $\vartheta \in D$ with $\zeta, \varepsilon \geqslant \vartheta$, and there are $c', c'' \leqslant a \vee b_0$ with $d(c'), d(c'') \leqslant \vartheta$ and $d(c' \vee c'') \geqslant \beta$. Neither c' nor c'' intersects both a and b_0 $[\![$otherwise $\alpha \leqslant d(a \vee b_0) \leqslant 3\zeta]\!]$. By (I), each of c', c'' must therefore be below a or b_0. If either a or b_0 is above both c' and c'', we have $\delta \geqslant d(c' \vee c'') \geqslant \beta$; otherwise we may assume that $c' \leqslant a$ and $c'' \leqslant b_0$ and put $b := c''$, so that $d(a \vee b) \geqslant d(c' \vee c'') \geqslant \beta$. ☐

Proof of Theorem 3.19. Because of metricity, the function

$$e_d(a,b) := \bigwedge_{\varepsilon \in D} \bigvee_{\substack{0 \neq b' \leqslant b,\\ d(b') \leqslant \varepsilon}} \bigwedge_{0 \neq a' \leqslant a} d(a' \vee b')$$

fulfils $e_d(a,b) = 0$ whenever $a \geqslant b$ $[\![$given ε and b', put $a' := b']\!]$. It is a distance function:

$$\begin{aligned} & e_d(a,b) + e_d(b,c) \\ & \geqslant \bigwedge_{\varepsilon \in D} \bigvee_{\substack{0 \neq c' \leqslant c,\\ d(c') \leqslant \varepsilon}} \left(\bigvee_{\substack{0 \neq b' \leqslant b,\\ d(b') \leqslant \varepsilon}} \bigwedge_{0 \neq a' \leqslant a} d(a' \vee b') + \bigwedge_{0 \neq b'' \leqslant b} d(b'' \vee c') \right) \\ & \geqslant \bigwedge_{\varepsilon \in D} \bigvee_{\substack{0 \neq c' \leqslant c,\\ d(c') \leqslant \varepsilon}} \bigwedge_{0 \neq a' \leqslant a} d(a' \vee c'), \end{aligned}$$

where the first inequality holds because of lower distributivity of $\underline{M}$ and directedness of D, and the second one can be proved like this: let $\varepsilon \in D$,
$0 \neq c' \leqslant c$, $d(c') \leqslant \varepsilon$, and $\beta \ggg \bigwedge_{0 \neq b'' \leqslant b} d(b'' \vee c')$. Because of metricity, ε, c', β
we can choose $0 \neq b''_\beta \leqslant b$ with $d(b''_\beta) \leqslant \varepsilon$ and $d(b''_\beta \vee c') \leqslant \beta$. Let $\alpha \ggg$ b''_β, α
$\bigwedge_{0 \neq a' \leqslant a} d(a' \vee b''_\beta)$ and choose $0 \neq a'_\alpha \leqslant a$ with $d(a'_\alpha \vee b''_\beta) \leqslant \alpha$. Then a'_α

$$\bigwedge_{0 \neq a' \leqslant a} d(a' \vee c') \leqslant d(a'_\alpha \vee c') \leqslant d(a'_\alpha \vee b''_\beta) + d(b''_\beta \vee c') \leqslant \alpha + \beta,$$

thus

$$\begin{aligned}\bigwedge_{0\neq a'\leqslant a} d(a'\vee c') &\leqslant \bigwedge_{\beta}\bigwedge_{\alpha}(\alpha+\beta) = \bigwedge_{\beta}\bigwedge_{\alpha}\alpha + \bigwedge_{\beta}\beta \\ &= \bigwedge_{\beta}\ \bigwedge_{0\neq a'\leqslant a} d(a'\vee b''_{\beta}) + \bigwedge_{\beta}\beta \\ &\leqslant \bigvee_{\substack{0\neq b'\leqslant b,\\ d(b')\leqslant\varepsilon}}\ \bigwedge_{0\neq a'\leqslant a} d(a'\vee b') + \bigwedge_{0\neq b''\leqslant b} d(b''\vee c')\end{aligned}$$

because of lower and complete distributivity (note that the latter implies that there is at least one β).

Finally, we can prove that

$$0\vee\bigvee_{0\neq a\leqslant c}\ \bigwedge_{\varepsilon\in D}\ \bigvee_{\substack{0\neq b\leqslant c,\\ d(b)\leqslant\varepsilon}}\ \bigwedge_{0\neq a'\leqslant a} d(a'\vee b) \geqslant d(c)$$

and thus $d(c) = \bigvee_{0\neq a\leqslant c} e_d(a,c)$. For $c=0$, we have $0\geqslant d(c)$, hence assume that $c>0$. Let $\alpha \lll d(c)$, then choose $\alpha' \lll d(c)$ and $\delta\in D$ according to Lemma 3.20. For this α', choose $0\neq a\leqslant c$ with $d(a)\leqslant\delta$ according to Lemma 3.21. Now let $\beta\lll\alpha'$, $\varepsilon\in D$, and choose b according to Lemma 3.21. Then, for all $0\neq a'\leqslant a$, $\gamma := d(a'\vee b)+\delta\geqslant\delta$ and $\gamma\geqslant d(a\vee b)$, hence $\gamma\geqslant d(a\vee b)\vee\delta\geqslant\beta$ because of Lemma 3.21. Therefore,

$$\bigwedge_{\varepsilon\in D}\ \bigvee_{\substack{0\neq b\leqslant c,\\ d(b)\leqslant\varepsilon}}\ \bigwedge_{0\neq a'\leqslant a} d(a'\vee b)+\delta\geqslant\beta,$$

so that complete distributivity gives $\bigwedge\cdots+\delta\geqslant\alpha'$, hence $\bigwedge\cdots\geqslant\alpha$ because of Lemma 3.20. Again by complete distributivity, the latter implies the proposition of the theorem. $\boxed{\text{AC}}$

COROLLARY 3.22. *A pair* $(d_+,\underline{M},D)$, $(d_-,\underline{M},D)$ *of diameter structures on* $\underline{L}$ *comes from the distance function*

$$e(a,b) := e_{d_+}(a,b)\vee e_{d_-}(b,a)$$

in the sense that

$$d_+(b) = 0\vee\bigvee_{0\neq a\leqslant b} e(a,b) \quad \textit{and} \quad d_-(b) = 0\vee\bigvee_{0\neq a\leqslant b} e(b,a).$$

Proof. For $a\leqslant b$, both $e_{d_-}(a,b)$ and $e_{d_+}(b,a)$ are zero. $\boxed{\text{AC}}$

Note that no distributivity of $\underline{L}$ is needed for all this. However, condition (I) is implied by distributivity and may thus be considered a very weak form of distributivity. On the other hand, the proof of the theorem relies heavily on the fact that $\underline{M}$ has the strongest possible form of distributivity.

PROBLEM 3.23. Is there a similar result that does not require complete distributivity of $\underline{M}$?

Pultr [Pul84] considers the relationship between *frame uniformities* and diameters and shows that every frame uniformity on a frame $\underline{L}$ can be induced by a family $\mathscr{D}$ of metric pre-diameters $\delta \in \mathscr{D}$ on $\underline{L}$. Just like a family of pseudometrics defines a multi-pseudometric, this family defines a *multi-real pre-diameter* $d : \underline{L} \to \underline{M} := (\mathbb{R}^\top)^{\mathscr{D}}$, $d \mapsto d(a)$, with $d(a) : \mathscr{D} \to \underline{\mathbb{R}}^\top$, $\delta \mapsto \delta(a)$. It is easy to see that this function d fulfils the requirements of the theorem, hence

multi-real pre-diameter

COROLLARY 3.24. *Every pair of frame uniformities comes from a single distance function on the frame.*

Concluding this chapter, the following example shows that on hyperspaces there are also useful *symmetric* distances.

EXAMPLE 3.25. On the hyperspace of a bounded metric space (X,d), the *Wijsman topology* (cf. [DMM98]) can be characterized as the coarsest topology on $C(\underline{X})$ under which the maps $f_x : C(\underline{X}) \to \underline{\mathbb{E}}_1$, $A \mapsto \bigvee_{a\in A} d(a,x)$ are continuous for all $x \in X$.

Wijsman topology

This can be used to generalize the concept to distance spaces $(X,d,\underline{M},D)$ for which $\underline{M}$ is a complete p. o. m. with some topology $\mathscr{T}$ on it. The *generalized Wijsman topology* $\mathscr{W}_{\mathscr{T}}(\underline{X})$ on $C(\underline{X})$ is then the coarsest under which all maps $A \mapsto \bigvee_{a\in A} d(a,x)$ for $x \in X$ become continuous w. r. t. $\mathscr{T}$.

generalized Wijsman topology

$\mathscr{W}_{\mathscr{T}}(\underline{X})$

If $\mathscr{T}$ comes from a distance structure $(e,\underline{N},E)$ on M, as in the original case of $\mathbb{E}_1$, then $\mathscr{W}_{\mathscr{T}}(\underline{X})$ is induced by the following multi-$\underline{N}$-distance structure $(d_{\mathscr{T}},D)$ on $C(\underline{X})$:

$$d_{\mathscr{T}} : \left\{ \begin{array}{lcl} C(\underline{X})^2 & \to & (\underline{N}^\top)^X \\ (A,B) & \mapsto & d_{\mathscr{T}}(A,B) : \left\{ \begin{array}{lcl} X & \to & \underline{N}^\top \\ x & \mapsto & e\left(\bigvee_{a\in A} d(a,x), \bigvee_{b\in B} d(b,x)\right), \end{array} \right. \end{array} \right.$$

and $D := \coprod_{x\in X} E^\top \subseteq (\underline{N}^\top)^X$. Since $(d_{\mathscr{T}},D)$ is a categorical supremum, this is easily seen from Lemma 3.12. It also follows from the more general result on initial structures, Lemma 4.21.

When $\underline{M}$ is a co-quantale with $\bigwedge \twoheaduparrow 0 = 0$, (e,E) would most naturally be the internal distance structure $(d^0_{\leftrightarrow}, \twoheaduparrow 0)$ on $\underline{M}$, so that $C(\underline{X})$ would then become a symmetric multi-$\underline{M}$-distance space. In particular, for $\underline{M} = \underline{\mathbb{R}}^\top$, $d^0_{\leftrightarrow}$ is just Euclidean distance, hence the Wijsman topology of a qp-metric space comes from a multi-pseudometric.

4.

MORE ON MAPPINGS

> *Nothing awakens in the traveller a livelier remembrance of the immense distance by which he is separated from his country, than the aspect of an unknown firmament.*
>
> *Longfellow (1867)*

Topological properties of maps

FORMS OF CONTINUITY

In Part A, several "preservation" properties of maps between distance sets have been studied that were of a mere algebraic and order-theoretic nature. Now, the additional structure of a distance *space* allows us to define as well "topological" properties of maps f between distance spaces $(X,d,\underline{M},D)$ and $(Y,e,\underline{N},E)$. The most familiar of them is of course *continuity* with a whole number of equivalent characterizations (cf. [Čec66]): images of convergent filters [or nets] converge to the images of the limits, or: for all $x \in X$ and $F \in \mathscr{C}_{e,E}(fx)$ we have $f^{-1}[F] \in \mathscr{C}_{d,D}x$, or: for all $\varepsilon \in E$ and $y \in X$, there is $\delta \in D$ such that, for all $x \in X$, $d(x,y) \leqslant \delta$ implies $ef(x,y) \leqslant \varepsilon$. Note that since E is a zero-filter, this implies that continuous maps are specialization preserving.

continuity

Also, continuity implies that pre-images of open sets are open, which is equivalent to all pre-images of closed sets being closed, but these conditions are properly *weaker* than continuity in general. They imply continuity only when E is idempotent. Indeed, the *interior*

interior

A°

$$A^\circ := \{a \in A \,|\, A \in \mathscr{C}_{e,E}a\}$$

of any set $A \subseteq Y$ is then an open set ⟦if $B_{e,\varepsilon}a \subseteq A$, there is $\delta \in E$ with $2\delta \leqslant \varepsilon$, so that $B_{e,\delta}b \subseteq A$ for all $b \in B_{e,\delta}a$, that is, $B_{e,\delta}a \subseteq A^\circ$ ⟧. Under the premise, $f^{-1}[F]$ is then an open set containing x for all $F \in \mathscr{C}_{e,E}(fx)$, that is, an element of $\mathscr{C}_{d,D}x$. To put it very clearly: continuity between distance spaces is properly stronger than continuity w. r. t. the two induced topologies.

uniform continuity

As in case of metric spaces, one gets the definition of *uniform continuity* of a map $f : \underline{X} \to \underline{Y}$ by interchanging two quantifications: for all $\varepsilon \in E$, there must be $\delta \in D$ such that

$$d(x,y) \leqslant \delta \Longrightarrow ef(x,y) \leqslant \varepsilon \quad \text{for all } x,y \in X.$$

Note that this condition shows a similarity to the definitions on page 30 in that it states a kind of "preservation of smallness" of distances instead of a preservation of inequalities. Like the step from distance inequality preserving maps to set homometries, this can be very naturally strengthened by requiring the same for sums of distances: f is *strongly uniformly continuous* if for all $\varepsilon \in E$, there is $\delta \in D$ such that

strongly uniformly continuous

$$d(s) \leqslant \delta \Longrightarrow ef(s) \leqslant \varepsilon \quad \text{for all } s \in X^{2\star}.$$

Although this seems to be a property not yet studied in the literature, it might prove very useful since it can be considered as a substitute for the still stronger Lipschitz-continuity. In fact, we will see below that in many cases a strongly uniformly continuous map between metric spaces is already Lipschitz-continuous.

Moreover, the new property fits nicely between two other properties known from real analysis. On the one hand, it is weaker than Hölder-continuity for exponents at least one ⟦Assume that $ef(x,y) \leqslant L\big(d(x,y)\big)^\alpha$ holds for all $x,y \in X$, with $L \geqslant 0$ and $\alpha \geqslant 1$. Then also $ef(s) \leqslant L\big(d(s)\big)^\alpha$ for all $s \in X^{2\star}$, hence we might put $\delta := (\varepsilon/L)^{1/\alpha}$ for any given $\varepsilon > 0$⟧. On the other hand, it is stronger than absolute continuity. A real-valued function f on a real interval $[a,b]$ is absolutely continuous if and only if it is the indefinite integral of some Lebesgue-integrable function, which is equivalent to the following: for all $\varepsilon > 0$, there is $\delta > 0$ such that $d(s) \leqslant \delta \Longrightarrow ef(s) \leqslant \varepsilon$ for all $s = x_1y_1 \cdots x_ny_n \in [a,b]^{2\star}$ with $x_1 \leqslant y_1 < x_2 \leqslant y_2 < \cdots < x_n \leqslant y_n$ (cf. [GZZ79]).

absolute continuity

This leads to the definition of *absolute continuity* for mappings f between arbitrary distance spaces: for all $\varepsilon \in E$, there is $\delta \in D$ such that, for all $s = x_1y_1 \cdots x_ny_n \in X^{2\star}$,

$$d(s) \leqslant \delta \Longrightarrow \big(ef(s) \leqslant \varepsilon, \text{ or } \overline{x_iy_i}^{d} \cap \overline{x_jy_j}^{d} \neq \emptyset \text{ for some } i < j \leqslant n\big).$$

From continuity to uniform continuity. It is a well-known fact that a continuous map from a compact metric space into another metric space is already uniformly continuous. Although the following generalization might in part follow from known facts about semi-uniform spaces (cf. [Huš64, Čec66]), I give a simple proof here that does not require the Axiom of Choice. Recall that a topological space is called *compact* if each of its open covers contains a finite subcover. A distance space $\underline{X}$ will be called *locally dwindling* if each neighbourhood filter contains balls of arbitrarily small diameter,[1] that is, if

compact

locally dwindling

$$\forall x \in X\ \forall \varepsilon \in D\ \exists \delta \in D\ \forall y,z \in X:\ d(y,x),d(z,x) \leqslant \delta \Longrightarrow d(y,z) \leqslant \varepsilon.$$

local symmetry

This is not to be confused with *local symmetry*, meaning

$$\forall y \in X\ \forall \varepsilon \in D\ \exists \delta \in D\ \forall x,z \in X:\ d(y,x),d(z,x) \leqslant \delta \Longrightarrow d(y,z) \leqslant \varepsilon.$$

point-symmetric

However, a locally dwindling distance space is also *point-symmetric*, that is,

$$\forall x \in X\ \forall \varepsilon \in D\ \exists \delta \in D\ \forall z \in X:\ d(z,x) \leqslant \delta \Longrightarrow d(x,z) \leqslant \varepsilon.$$

If D is idempotent, the converse is also true ⟦Take $\varepsilon' \in D$ with $2\varepsilon' \leqslant \varepsilon$, and $\delta \in D$ with $\delta \leqslant \varepsilon'$ and $B_{d,\delta}x \subseteq xB_{d,\varepsilon'}$. Then $y,z \in B_{d,\delta}x$ implies $d(y,z) \leqslant d(y,x) + d(x,z) \leqslant \delta + \varepsilon' \leqslant \varepsilon$⟧.

PROPOSITION 4.1. *A continuous map from a compact distance space with idempotent zero-filter into a locally dwindling distance space is uniformly continuous.*

Proof. Let $f : \underline{X} \to \underline{Y}$ be the map, and $\varepsilon \in E$. For all $z \in X$, the set

$$E'_z := \{\varepsilon' \in E \,|\, \forall x,y \in X:\ ef(x,z),ef(y,z) \leqslant \varepsilon' \Longrightarrow ef(x,y) \leqslant \varepsilon\}$$

in nonempty since $\underline{Y}$ is locally dwindling, the set

$$D'_z := \{\delta' \in D \,|\, \exists \varepsilon' \in E'_z\ \forall x \in X:\ d(x,z) \leqslant \delta' \Longrightarrow ef(x,z) \leqslant \varepsilon'\}$$

in nonempty by continuity of f, and the set

$$D_z := \{\delta \in D \,|\, \exists \delta' \in D'_z:\ 2\delta \leqslant \delta'\}$$

is nonempty because D is idempotent. Therefore, $\{B^\circ_{d,\delta}z \,|\, z \in X, \delta \in D_z\}$ is an open cover of X having a finite subcover $\{B^\circ_{d,\delta_i}z_i \,|\, i = 1,\dots,n\}$ by compactness of $\underline{X}$. Since D is directed, we can choose $\delta \in D$ with $\delta \leqslant \delta_i$ for all i. Assuming that $d(x,y) \leqslant \delta$, there is some i with $d(y,z_i) \leqslant \delta_i$, some $\delta' \in D'_{z_i}$ with $2\delta_i \leqslant \delta'$, and some corresponding $\varepsilon' \in E'_{z_i}$. Now $d(x,z_i),d(y,z_i) \leqslant \delta'$ implies that $ef(x,z_i),ef(y,z_i) \leqslant \varepsilon'$ and therefore $ef(x,y) \leqslant \varepsilon$ by choice of ε'. □

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[1] This could also have been called ‘locally small’, but that term has already a different meaning.

Note that this proof neither requires a stronger form of symmetry of one of the spaces, nor an idempotent zero-filter for the co-domain, nor the Axiom of Choice.

From uniform to strong uniform continuity. (See also the next section)

LEMMA 4.2. *A uniformly continuous map from a positive distance space into a distance space whose zero-filter has a base of idempotents is already strongly uniformly continuous.*

Proof. Let $f : \underline{X} \to \underline{Y}$ be the map. For an idempotent $\varepsilon \in E$, choose $\delta \in D$ so that $d(x,y) \leqslant \delta$ implies $ef(x,y) \leqslant \varepsilon$. Because $d \geqslant 0$, each inequality $d(x_1 y_1 \cdots x_n y_n) \leqslant \delta$ implies $d(x_i, y_i) \leqslant \delta$ for all i, hence $ef(x_i, y_i) \leqslant \varepsilon$, and finally $ef(x_1 y_1 \cdots x_n y_n) \leqslant n\varepsilon = \varepsilon$. ☐

For example, the premise of this lemma is fulfilled whenever $\underline{N}$ is a semilattice. Another special case is when $\underline{N}$ is the p. o. m. $(\uparrow\Delta_Y, \circ, \Delta_Y, \subseteq)$ of reflexive relations on Y under composition and set inclusion. The positive filters [zero-filters] of $\underline{N}$ are then exactly the [T$_1$] *semi-uniformities* (in the sense of [Huš64]) on Y, those that are idempotent are exactly the [T$_1$] *quasi-uniformities* on $\underline{Y}$, and those that have a base of idempotent elements are just the *transitive* ones among them (cf. [FL82]).

semi-uniformities

quasi-uniformities

transitive

From strong uniform to Lipschitz-continuity. For quasi-metric spaces, strong uniform continuity is much closer to Lipschitz-continuity than to uniform continuity.

PROPOSITION 4.3. *Any bounded strongly uniformly continuous map from a positive* T_1 *qp-metric space into a real distance space is already Lipschitz-continuous.*

Proof. Assume $f : (X,d) \to (Y,e)$ is not Lipschitz-continuous but bounded, say $ef(x,y) \leqslant 2\varepsilon < \infty$ for some $\varepsilon > 0$ and all $x,y \in X$. Then, for each $\delta > 0$, one can choose $a,b \in X$ with $ef(a,b) > \frac{2\varepsilon}{\delta} d(a,b) \geqslant 0$. In particular, $a \neq b$ and $0 < d(a,b) \leqslant \delta$.

Choose a positive integer n with $\delta/2n \leqslant d(a,b) \leqslant \delta/n$, and let $s := ab \circ \cdots \circ ab \in X^{2\star}$ be the word made of n syllables ab. Then $d(s) \leqslant \delta$ and $ef(s) = n \cdot ef(a,b) > n \cdot \frac{2\varepsilon}{\delta} \cdot \delta/2n = \varepsilon$, hence f is not strongly uniformly continuous. ☐

While boundedness is a condition on the range of f, also certain conditions on the domain assure Lipschitz-continuity of strongly uniformly continuous maps, for instance:

THEOREM 4.4. *Let* $\lambda > 0$ *and* $\underline{X}$ *be a positive* T_1 *qp-metric space such that, for all* $x, y \in X$ *and all* $\zeta > 0$*, there are finitely many "intermediate" points* $x = x_0, x_1, \ldots, x_{n-1}, x_n = y \in X$ *with (i)* $d(x_i, x_{i+1}) \leqslant \zeta$ *for all* $i \in n$ *and (ii)* $\sum_{i \in n} d(x_i, x_{i+1}) \leqslant \lambda(d(x,y) + \zeta)$*. Then strong uniform continuity and Lipschitz-continuity coincide for all maps from* $\underline{X}$ *into real distance spaces.*

Proof. Assume that $f : \underline{X} \to (Y, e)$ is not Lipschitz-continuous, choose $\varepsilon > 0$ arbitrarily, let $\delta > 0$, and put $\gamma := 2\varepsilon/\delta$. Choose $x, y \in X$ such that $ef(x,y) > \gamma\lambda d(x,y)$, and $\zeta > 0$ such that $\zeta \leqslant \delta$ and $ef(x,y) > \gamma\lambda(d(x,y) + \zeta)$. Finally, choose pairwise distinct points $x = x_0, x_1, \ldots, x_{n-1}, x_n = y \in X$ with $\sum_{i \in n} d(x_i, x_{i+1}) \leqslant \lambda(d(x,y) + \zeta)$ and $d(x_i, x_{i+1}) \leqslant \zeta$ for all $i \in n$. Then $ef(x_j, x_{j+1}) > \gamma d(x_j, x_{j+1})$ for some $j \in n$, since otherwise $ef(x,y) \leqslant \sum_{i \in n} ef(x_i, x_{i+1}) \leqslant \gamma \sum_{i \in n} d(x_i, x_{i+1}) \leqslant \gamma\lambda(d(x,y) + \zeta)$, a contradiction.

Now put $(a, b) := (x_j, x_{j+1})$ and proceed as in the second paragraph of the preceding proof. ☐

The above condition might be interpreted as a strong kind of "chainedness" and a weak kind of convexity.

COROLLARY 4.5. *If* X *is a subset of a Banach space whose closure is convex, strong uniform continuity and Lipschitz-continuity coincide for all maps from* X *into real distance spaces.*

Maps with both topological and non-topological properties

Being the strongest form of continuity expressible between general distance spaces, strong uniform continuity is the natural topological supplement for a set homometry, turning it into a *space homometry.* Likewise, let us call a uniformly continuous [dually] weak homometry a *[dually] weak space homometry.*

space homometry

[dually] weak space homometry

EXAMPLES 4.6. The following counter-examples, some of which are well known, show that the diagram in Figure 4 is *correct,* that is, no further implications hold for mappings between general distance spaces. It also remains correct when both occurrences of 'weak' are replaced by 'dually weak'.

a) *A contraction which is not distance equivalence preserving:* $x \mapsto |\frac{x}{2}|$ on $\underline{\mathbb{E}}_1$.

b) *Distance equivalence preserving but not specialization preserving:* the identity map from $(\mathbb{R}, x - y)$ to $\underline{\mathbb{E}}_1$.

c) *Continuous but not uniformly:* $x \mapsto e^x$ on $\underline{\mathbb{E}}_1$.

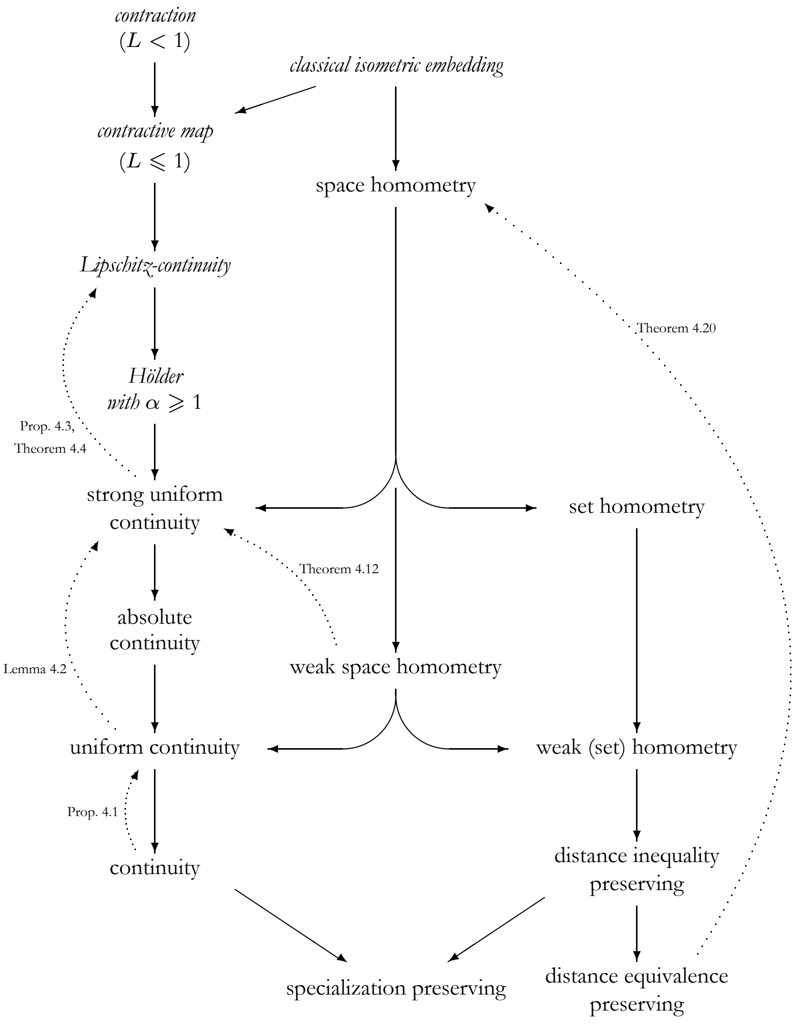

Figure 1. A correct diagram of mapping properties. Italics mark notions that are only available for real distance spaces. Dotted arrows refer to results about special cases.

d) *Uniformly but not absolutely continuous.* On the Euclidean unit interval $[0,1]$, define $f(x) = \sum_{i=0}^{\infty} g(2^i x)/2^i \in [0,1]$, where $g(x) := \min\{x - \lfloor x \rfloor, \lceil x \rceil - x\}$ is the smallest distance from x to an integer. Then f is a uniform limit of continuous functions, hence continuous, hence uniformly continuous since $[0,1]$ is compact. Moreover, f is known to be nowhere differentiable hence not absolutely continuous.

e) *Absolutely continuous but not strongly uniformly.*[1] Again on $[0,1]$, the square root map $f(x) := \sqrt{x}$ is Hölder-continuous with $L = 1$ and $\alpha = 1/2 < 1$. It has a continuous derivative almost everywhere and is thus absolutely continuous. But not strongly uniformly continuous: for any $\delta > 0$ and all $j \geqslant 1$, put $x_j := 0$ and $y_j := 6\min\{\delta,1\}/(\pi j)^2$. Now $\sum_{j=1}^{\infty} j^{-2} = \frac{\pi^2}{6}$ implies that $\sum_{i=1}^{n} |y_i - x_i| \leqslant \delta$ for all n, but $\sum_{j=1}^{n} |\sqrt{y_j} - \sqrt{x_j}|$ diverges for $n \to \infty$.

f) *Strongly uniformly but not Hölder-continuous.* On the two rays $X := \{-\frac{1}{2}, \frac{1}{2}\} \times [0,\infty)$, put $f(x,y) := y \operatorname{sgn}(x)$. Then f is not Hölder-continuous since the images of two points with unit distance can have arbitrarily large distance, but it fulfils the strong uniformity condition. For $\varepsilon > 0$, put $\delta := \min\{\varepsilon, \frac{1}{2}\}$. Then $\sum_i d(a_i,b_i) \leqslant \delta$ implies $d(a_i,b_i) < 1$ for all i, hence each pair a_i,b_i is in one of the two rays and thus $\sum_i dh(a_i,b_i) = \sum_i d(a_i,b_i) \leqslant \delta \leqslant \varepsilon$.

connected
star-shaped

g) There is also such an example on a *connected* set (that is, one that is not the disjoint union of two nonempty open subsets) which is even *star-shaped* (that is, a union of segments with a common endpoint). For all $n \in \omega$, let X_n be the "slice" of all $a \in \underline{\mathbb{E}}_2$ with $\varphi(a) \in [2^{-2n-1}\pi, 2^{-2n}\pi]$, where $(r(a), \varphi(a))$ are the standard polar coordinates of a. Then $X := \bigcup_{n\in\omega} X_n$ is a star-shaped set. Moreover, let

$$r_n := \frac{1}{5\sqrt{2 - 2\cos(2^{-2n-2}\pi)}} = \frac{1}{10\sin(2^{-2n-3}\pi)},$$

so that $d(a,b) > 2r_n \sin(2^{-2n-3}\pi) \geqslant 1/5$ whenever $a \in X_n$, $b \in X_m$, $n < m$, and $r(a), r(b) > r_n$. On X_n, define $f(a) := 0$ whenever $r(a) \leqslant r_n$, otherwise $f(a) := r(a) - r_n$ if n is even and $f(a) := r_n - r(a)$ if n is odd.

As above, $f : X \to \underline{\mathbb{E}}_1$ fulfils the strong uniformity condition. For $\varepsilon > 0$, put $\delta := \min\{\varepsilon, 1/5\}$. Then $\sum_i d(a_i,b_i) \leqslant \delta$ implies $d(a_i,b_i) \leqslant 1/5$ for all i. If $a_i,b_i \in X_n$ for some n then $df(a_i,b_i) = |r(a_i) - r(b_i)| \leqslant d(a_i,b_i)$. If $a_i \in X_n$, $b_i \in X_m$, and $n \neq m$, say $n < m$, then either $r(a_i) \leqslant r_n$ and thus $f(a_i) = 0$, or $r(b_i) \leqslant r_n < r_m$ and thus $f(b_i) = 0$. In any case, $dh(a_i,b_i) \leqslant |r(a_i) - r(b_i)| \leqslant d(a_i,b_i)$ $[\![$either $f(a_i) = f(b_i) = 0$; or $f(a_i) = 0$, $r(b_i) > r_m > r(a_i)$, and $df(a_i,b_i) = |r(b_i) -$

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[1] This example is due to Marcel Erné (personal communication).

$r_m|$; or $f(b_i) = 0$, $r(a_i) > r_n \geqslant r(b_i)$, and $df(a_i,b_i) = |r(a_i) - r_n|$]]. Consequently $\sum_i df(a_i,b_i) \leqslant \sum_i d(a_i,b_i) \leqslant \delta \leqslant \varepsilon$.

And again, f is not Hölder-continuous. Observe that (i) $\lim_{n\to\infty}(r_n - r_{n-1}) = \infty$ and (ii) $\lim_{n\to\infty} r_n/r_{n-1} = 4$. Hence, for all $\alpha > 0$ and arbitrarily large $L > 0$, we can choose an odd $n \in \omega$ so that (i) $r_n - r_{n-1} \geqslant L$ and (ii) $r_n \leqslant 5r_{n-1}$. Then the border points $a \in X_{n-1}$ and $b \in X_n$ with $r(a) = r(b) = r_n$, $\varphi(a) = 2^{-2(n-1)-1}\pi$, and $\varphi(b) = 2^{-2n}\pi$ have distance $d(a,b) \leqslant 1$. On the other hand,

$$df(a,b) = |(r(a) - r_{n-1}) - (r_n - r(b))| = r_n - r_{n-1} \geqslant L \geqslant Ld(a,b)^\alpha.$$

h) *Hölder with exponent $\alpha > 1$, but not Lipschitz.* Let $\alpha > 1$ and $L < \infty$. On ω, $d(x,y) := \max\{x,y\}$ for $x \neq y$ and $e(x,y) := Ld(x,y)^\alpha$ define two metrics. By definition, $id : (\omega,d) \to (\omega,e)$ is Hölder-continuous with constant L and exponent α, but not Lipschitz-continuous since $e(x,y)/d(x,y)$ is not bounded.

i) *A space homometry which is not contractive:* $x \mapsto 2x$ on $\underline{\mathbb{E}}_1$.

j) *A set homometry which is not continuous:* the identity map from $\underline{\mathbb{E}}_1$ to the discrete space $(\mathbb{R}, e, [0,\infty))$.

k) *A [dually] weak space homometry which is not a set homometry.* On the unit circle $X := \{x \in \mathbb{C} \,|\, |x| = 1\}$, let e be Euclidean distance, and d be the *geodesic distance* (that is, the shortest path length in X) inherited from e. Then $h := id : (X,d) \to (X,e)$ is contractive but not a contraction. Since $d \leqslant \frac{\pi}{2} \cdot e$, also h^{-1} is Lipschitz-continuous. In particular, both h and h^{-1} are strongly uniformly continuous. Moreover, h is a weak homometry but not a set homometry, since its translation map $t(\alpha) = 2\sin\frac{\alpha}{2}$ is only sub-additive but not additive. For the same reason, h^{-1} is a dually weak space homometry but not a set homometry.

geodesic distance

l) *Distance inequality preserving but not a [dually] weak set homometry:* see the previous example.

m) *A uniformly continuous set homometry which is not absolutely continuous.* On the set $X := \omega \times \mathbb{R}$, define a multi-pseudometric $d : X^2 \to (\underline{\mathbb{R}}^{+\top})^\omega$ by putting $d(x,y) := \{(i, |r-s|)\} \cup \{(j,0) \,|\, j \neq i\}$ for $x = (i,r)$ and $y = (i,s)$, and $d(x,y) := \omega \times \{\infty\}$ in all other cases. Then $e(x,y) := \sum_{i=0}^\infty d(x,y)(i)$ is a metric on X, and $h := id : (X,d) \to (X,e)$ is a set homometry with calibration $c(\alpha) = \sum_{i=0}^\infty \alpha(i)$. Moreover, h is uniformly continuous w. r. t. the idempotent zero-filters $D := \uparrow\{\omega \times \{r\} \,|\, r > 0\}$ and $E := (0,\infty]$ (which both have a countable base) [[for given $\varepsilon \in E$, put $\delta := \omega \times \{\varepsilon\}$]].

> But h is not absolutely continuous. For an arbitrary $\varepsilon < \infty$ and every $\delta \in D$, there is some $r > 0$ with $\omega \times \{r\} \leqslant \delta$, and some $n \geqslant \varepsilon/r$. Now put $(x_i,y_i) := \big((i,0),(i,r)\big)$ for all $i \in n$, and $s := x_0y_0 \cdots x_ny_n$. Then $d(s) \leqslant \delta$ but $eh(s) = (n+1)r > \varepsilon$. Finally, all the segments $\overline{x_iy_i}^d = \{i\} \times [0,r]$ are pairwise disjoint.

Since all of these examples involve only idempotent zero-filters, there are also no further implications for the case of idempotent zero-filters. Also, all but the last example use only real distance functions.

UNIFORM CONTINUITY IMPLIED BY NON-TOPOLOGICAL PROPERTIES

LEMMA 4.7. *Let* $\underline{X} = (X,d,D)$ *be a distance space such that, for all* $y,z,w \in X$ *and* $\delta \in D$ *with* $d(z,w) \leqslant \delta$*, there is some* $x \in X$ *with* $d(z,w) \leqslant d(x,y) \leqslant \delta$*. Then each continuous distance inequality preserving map* $f : \underline{X} \to \underline{Y}$ *is uniformly continuous.*

If for all $y,z,w \in X$ *there is even some* $x \in X$ *with* $d(z,w) = d(x,y)$ *then also each continuous distance equivalence preserving map* $f : \underline{X} \to \underline{Y}$ *is uniformly continuous.*

The *proof* is straightforward. ☐

LEMMA 4.8. *Assume that* $\underline{X} = (X,d,D)$ *and* $\underline{Y} = (Y,e,E)$ *are distance spaces, and for all* $\varepsilon \in E$*, there is some* $A \subseteq X$ *such that (i) for all distinct points* $x,y \in A$*, there is* $\delta \in D$ *with* $\delta < d(x,y)$*, and (ii) for all* $B \subseteq Y$ *with* $|B| = |A|$ *there are distinct points* $x,y \in B$ *with* $e(x,y) \leqslant \varepsilon$*.*

Then each order representation of $\underline{X}$ *is uniformly continuous.*

Proof. Let $f : \underline{X} \to \underline{Y}$ be an order representation, and assume that there is $\varepsilon \in E$ such that for all $\delta \in D$ there are points $z,w \in X$ with $d(z,w) \leqslant \delta$ and $ef(z,w) \nleqslant \varepsilon$. For this ε, choose some $A \subseteq X$ as in the premise of the lemma, and put $B := f[A]$. By choice of A, all distinct pairs $x,y \in A$ fulfil $d(x,x) < d(x,y)$, hence $0 < ef(x,y)$, that is, $|B| = |A|$. Now we can choose distinct points $x,y \in A$ with $ef(x,y) \leqslant \varepsilon$, and some $\delta \in D$ with $\delta < d(x,y)$. By choice of ε, there are z,w with $d(z,w) \leqslant \delta < d(x,y)$ and $ef(z,w) \nleqslant \varepsilon$, in contradiction to $ef(x,y) \leqslant \varepsilon$. ☐

λ-Lindelöf
λ-bounded

For any infinite cardinal λ, let us call a distance space $\underline{X} = (X,d,D)$ λ-*Lindelöf* if each open cover of $\underline{X}$ has a sub-cover of cardinality $< \lambda$, and λ-*bounded* if for each $\delta \in D$ there is a cover $\mathscr{A}$ of X with $|\mathscr{A}| < \lambda$ whose members all have diameter $\leqslant \delta$. In particular, the ω-Lindelöf property is just compactness, and the ω^+-Lindelöf property is just the (ordinary) Lindelöf property. Moreover, ω-boundedness equals total boundedness.

Note that a point-symmetric, λ-Lindelöf distance space $\underline{X}$ with idempotent zero-filter D is also λ-bounded since then the system $\{A^\circ \,|\, x \in X,\ A \in \mathscr{C}_{d,D}x,\ A \text{ has diameter} \leqslant \delta\}$ is an open cover of $\underline{X}$ for all $\delta \in D$.

COROLLARY 4.9. *Any order representation of a* T_1 *real distance space* (X,d) *into an* $|X|$*-bounded distance space is uniformly continuous.*

Proof. Let $\underline{Y}$ be the second space. For all $\varepsilon \in E$, the set $A := X$ fulfils the premise of the previous lemma. Condition (i) follows from separatedness. As for (ii), let $B \subseteq Y$ with $|B| = |X|$, and let $\mathscr{A}$ be a cover of Y with $|\mathscr{A}| < |X| = |B|$ whose members have diameter $\leqslant \varepsilon$. Then some $A \in \mathscr{A}$ contains two distinct points x,y of B, in particular, $e(x,y) \leqslant \varepsilon$. □

A SMALL STEP:
DUALLY WEAK HOMOMETRIES WITH STRONG UNIFORM CONTINUITY

A surprising implication is true in case of non-positive real distances:

PROPOSITION 4.10. *A dually weak set homometry* $h : (X,d,\underline{\mathbb{R}}) \to (Y,e,\underline{\mathbb{R}})$ *with* $d(a,b) < 0$ *for some* $a,b \in X$ *is already Lipschitz-continuous with* $L = \frac{eh(a,b)}{d(a,b)}$.

Proof. Put $\gamma_0 := d(a,b) < 0$. The translation map $t : \mathbb{R} \to [-\infty,\infty)$, $\alpha \mapsto \bigvee\{eh(s) \,|\, s \in X^{2\star},\, d(s) \leqslant \alpha\}$ is super-additive. Since it is also isotone, we know that $t(\gamma_0) \leqslant 0$, and it suffices to show that $t(\alpha) \leqslant \alpha\frac{t(\gamma_0)}{\gamma_0}$ for all $\alpha > 0$ with $\alpha/\gamma_0 \in \mathbb{Q}$. For such an α, assume that $t(\alpha) > \alpha\frac{t(\gamma_0)}{\gamma_0}$. Inductively, define

$$n_i := \lceil -\alpha/\gamma_i \rceil = \min\{n \in \omega \,|\, \alpha + n\gamma_i \leqslant 0\}$$

and $\gamma_{i+1} := \alpha + n_i\gamma_i$ for all $i \geqslant 0$ for which $\gamma_i \neq 0$. Since $\gamma_{i+1} > \gamma_i$ and γ_i/γ_0 has the same denominator as α/γ_0, this sequence must stop, that is, $\gamma_{k+1} = 0$ for some k. In other words, $n_k = -\alpha/\gamma_k$.

For $i = 1 \dots k$, it now follows inductively that $t(\gamma_i)/\gamma_i \leqslant t(\gamma_0)/\gamma_0$ and therefore $t(\alpha) > \alpha\frac{t(\gamma_i)}{\gamma_i}$ $[\![0 = t(0) \geqslant t(\gamma_{i+1}) \geqslant t(\alpha) + n_i t(\gamma_i) > \alpha\frac{t(\gamma_i)}{\gamma_i} + n_i t(\gamma_i) = \frac{\gamma_{i+1}}{\gamma_i}t(\gamma_i)]\!]$. On the other hand, $t(\alpha) \leqslant t(\gamma_{k+1}) - n_k t(\gamma_k) = \alpha\frac{t(\gamma_k)}{\gamma_k}$, a contradiction. □

Still, such a map need not be a homometry $[\![$put $X := Y := \{0,1\}$, $d(0,1) := e(0,1) := -1$, $d(1,0) := 2$, $e(1,0) := 1$, and $h := id_X]\!]$. On the other hand, positivity of one of the distance functions leads to stronger forms of continuity as well:

PROPOSITION 4.11. *Let* $h : (X,d,\underline{M},D) \to (Y,e,E)$ *be a dually weak space homometry which is not strongly uniformly continuous, with a totally ordered monoid* $\underline{M}$ *and* $D = \{\delta \in M \,|\, \delta > 0\}$.

Then d *is not positive, and there is* $\delta \in D$ *with* $d[X^2] \cap (0,\delta] = \emptyset$.

If, additionally, $\underline{M}$ *is an archimedean group,* e *is not positive, too.*

Proof. Choose some $\varepsilon \in E$ for which strong uniform continuity is violated, then some $\delta \in D$ so that $eh(x,y) \leqslant \varepsilon$ for all x,y with $d(x,y) \leqslant \delta$, and some $s \in X^{2\star}$ with $d(s) \leqslant \delta$ and $eh(s) \nleqslant \varepsilon$. Assume that $d(x,y) \in (0,\delta] \subseteq D$ for some x,y. Then $d(t) \leqslant d(x,y)$ would imply $eh(t) \leqslant eh(x,y) \leqslant \varepsilon$ for all words t, in contradiction to the choice of ε. Hence $d[X^2] \cap (0,\delta] = \emptyset$.

Because of $eh(s) \nleqslant 0$, some syllable xy of s must fulfil $d(x,y) \nleqslant 0$, that is, $d(x,y) > \delta$. Since $d(s) \leqslant \delta$, some other syllable zw of s must fulfil $d(z,w) < 0$, thus d is not positive.

Finally, assume that $\underline{M}$ is an archimedean group and e is positive. Choose $n \in \omega$ with $nd(z,w) + d(s) \leqslant 0$. Then $eh(s) \leqslant neh(z,w) + eh(s) \leqslant 0 \leqslant \varepsilon$ in contradiction to the choice of s. ☐

Together, the last two propositions yield

THEOREM 4.12. *Every dually weak space homometry between distance spaces* $(X,d,\underline{\mathbb{R}})$ *and* $(Y,e,\underline{\mathbb{R}})$ *is strongly uniformly continuous.*

Proof. If not, d would not be positive and the proposition would imply that h is even Lipschitz-continuous. ☐

QUESTION 4.13. For which monoids other than $\underline{\mathbb{R}}$ is this also true?

A LARGER STEP: DISTANCE EQUIVALENCE PRESERVING MAPS THAT ARE HOMOMETRIES

similarity maps

$\mathrm{Aut}(\underline{\mathbb{E}}_n)$

In this section, the homometries between Euclidean spaces $\underline{\mathbb{E}}_n$ (that is, the *similarity maps*) are characterized by properties which are far weaker in general. The group of motions $\mathrm{Aut}(\underline{\mathbb{E}}_n)$ is always supposed to be endowed with the topology of pointwise convergence.

LEMMA 4.14. *Let* $(X,\cdot,1)$ *be a monoid,* $n \geqslant 0$*, and* d *a left translation-invariant distance function on* X*, that is, with* $d(zx,zy) = d(x,y)$*. If* $h : (X,d) \to \underline{\mathbb{E}}_n$ *is a [continuous] distance equivalence preserving map, there is a [continuous] monoid homomorphism* $f : (X,\cdot,1) \to \mathrm{Aut}(\underline{\mathbb{E}}_n)$ *such that* $h(x) = f(x)\big(h(1)\big)$ *for all* $x \in X$.

If, moreover, $h[X]$ *is not contained in any affine hyperplane of* $\mathbb{R}^n$*, this* f *is unique.*

Proof. Let us first consider the "non-degenerate" case where $h[X]$ is not contained in a hyperplane. Then there are $n+1$ points $x_0,\dots,x_n \in X$ such that $\{h(x_i) \,|\, i \in n+1\}$ is not contained in any hyperplane. For each $x \in X$, there is a unique motion f_x of $\underline{\mathbb{E}}_n$ with $f_x h(x_i) = h(xx_i)$ for all i ⟦Since d is left translation invariant and h is distance equivalence preserving, $eh(xx_i,xx_j) =$ x_i f_x

$eh(x_i,x_j)$, hence also $\{h(xx_i)\,|\,i \in n+1\}$ is not contained in a hyperplane. Thus, for all $a \in \mathbb{R}^n$, there is a unique $b \in \mathbb{R}^n$ such that $e\big(b,h(xx_i)\big) = e\big(a,h(x_i)\big)$ for all i. Put $f_x(a) := b$. In particular, $f_x h(x_i) = h(xx_i)$ for all i. Since for all $a,a' \in \mathbb{R}^n$, the distance $e(a,a')$ is a function of the two $(n+1)$-tuples of distances $\big(e(a,hx_i)\big)_i$ and $\big(e(a',hx_i)\big)_i$, f_x is a motion$\rrbracket$. If h is continuous, also $x \mapsto f_x$ must be continuous $\llbracket$ since d is translation-invariant, $x \mapsto h(xx_i)$ is then continuous for all i, hence $x \mapsto f_x(a)$ is continuous for all $a \in Y\rrbracket$. Now $h(xy) = f_x h(y)$ for all $x \in X$ $\llbracket f_x h(y)$ is the unique $b \in Y$ with $e\big(b,h(xx_i)\big) = e\big(h(y),h(x_i)\big)$ for all i, and $h(xy)$ is such a $b\rrbracket$, in particular $h(x) = f_x h(1)$, and the map $f : x \mapsto f_x$ is a monoid homomorphism $\llbracket f_x f_y$ is a motion with $f_x f_y h(x_i) = f_x h(yx_i) = h(xyx_i)$ for all i, hence it equals $f_{xy}\rrbracket$.

On the other hand, let $g : x \mapsto g_x$ be another homomorphism with $h(x) = g_x h(1)$ for all x. Then $g_x h(x_i) = g_x g_{x_i} h(1) = g_{xx_i} h(1) = h(xx_i)$ for all x and i, hence $g_x = f_x$ for all x, that is, $g = f$.

i, h'

For the degenerate case, let $h' := i^{-1} \circ h$, where i is an exact isometry between some $\underline{\mathbb{E}}_k$ (with $k < n$) and the affine hull of $h[X]$. Then $h' : (X,d) \to \underline{\mathbb{E}}_k$ is distance equivalence preserving [and continuous] and non-degenerate, hence there is a corresponding [continuous] monoid homomorphism $f' : (X,\cdot,1) \to \mathrm{Aut}(\underline{\mathbb{E}}_k)$. Moreover, there is a continuous embedding $g : \mathrm{Aut}(\underline{\mathbb{E}}_k) \to \mathrm{Aut}(\underline{\mathbb{E}}_n)$ such that $g(m) \circ i = i \circ m$ for all motions m of $\underline{\mathbb{E}}_k$. Then $f := g \circ f'$ is a [continuous] homomorphism such that

$$f(x)\big(h(1)\big) = \big(g(f'_x) \circ i\big)\big(h'(1)\big) = (i \circ f'_x)\big(h'(1)\big) = ih'(x) = h(x).$$

☐

LEMMA 4.15. *With the notation as in the previous lemma, assume that*

either (i) $d(x,y) \sim d(v,w)$ implies $y = xz$ and $w = vz$ for some $z \in X$,

or (ii) d is symmetric and $d(x,y) \sim d(v,w)$ implies ($y = xz$ or $x = yz$) and ($w = vz$ or $v = wz$) for some $z \in X$.

If $f : (X,\cdot,1) \to \mathrm{Aut}(\underline{\mathbb{E}}_n)$ is a [continuous] monoid homomorphism and $a \in \mathbb{R}^n$ a point, the orbit function $h(x) := f(x)(a)$ is distance equivalence preserving [and continuous].

Proof. By definition, $h(1) = a$ and $h(xy) = f_{xy}h(1) = f_x f_y h(1) = f_x h(y)$. Let $d(x,y) \sim d(v,w)$. Without loss of generality, we can assume that there is $z \in X$ with $y = xz$ and $w = vz$ (in case (ii), we probably have to exchange x with y and/or v with w first). Then

$$eh(y,x) = eh(x,y) = ef_x h(1,z) = ef_v h(1,z) = eh(v,w) = eh(w,v).$$

Moreover, if $x \mapsto f(x)$ is continuous, then so is $x \mapsto f(x)(a)$ for all a. ☐

It is well known that each continuous representation of the group $\mathbb{R}$ by motions of $\underline{\mathbb{E}}_n$ is of the following form.

EXAMPLE 4.16. *Generalized helices.* Let k be a motion of $\underline{\mathbb{E}}_n$ (then $k(x)$ is the coordinate vector of x with respect to some orthonormal coordinate system), and s a non-negative integer with $2s \leqslant n$. For each $i \in \{1,\dots,s\}$, let E_i be the plane spanned by the standard unit vectors e_{2i-1} and e_{2i}, and $\alpha_i > 0$. Moreover, let $b \in \mathbb{R}^n$ be orthogonal to all E_i. Then the map $f : x \mapsto f_x$ defined by

$$A_i \;:=\; \begin{pmatrix} \cos(x\alpha_i) & -\sin(x\alpha_i) \\ \sin(x\alpha_i) & \cos(x\alpha_i) \end{pmatrix} \quad \text{and}$$

$$k\big(f_x(a)\big) \;:=\; xb + \begin{pmatrix} \boxed{A_1} & & & & & \\ & \ddots & & & & \\ & & \boxed{A_s} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \cdot k(a)$$

is a continuous group homomorphism from $\mathbb{R}$ into $\mathrm{Aut}(\underline{\mathbb{E}}_n)$. Each f_x is a composition of rotations in the planes $k^{-1}[E_i]$ with centre $k^{-1}(0)$ and angles $x\alpha_i$, and of a translation perpendicular to all those planes.

COROLLARY 4.17. *The continuous distance equivalence preserving maps from* $\underline{\mathbb{E}}_1$ *to* $\underline{\mathbb{E}}_n$ *are exactly the generalized helices.*

We will now see that between Euclidean spaces of higher dimension than one, the continuous distance equivalence preserving maps are already similarities.

LEMMA 4.18. *For a function* $C : \mathbb{R} \longrightarrow \mathbb{R}$ *of the form*

$$C(r) = (r\lambda)^2 + \sum_{k=1}^{s} \left(\lambda_k \sqrt{2 - 2\cos(r\kappa_k)}\right)^2$$

with $0 < \kappa_1 < \cdots < \kappa_s$ *and* $\lambda_k > 0$*, all coefficients* κ_k, λ_k *are uniquely determined.*

Proof. For $n > 0$, the $(4n+1)$-st derivative of C fulfils

$$\frac{C^{(4n+1)}(r)}{2\kappa_s^{4n+1}} = \lambda_s^2 \sin(r\kappa_s) + \sum_{k=1}^{s-1} \lambda_k^2 \left(\frac{\kappa_k}{\kappa_s}\right)^{4n+1} \sin(r\kappa_k) \quad \longrightarrow \quad \lambda_s \sin(r\kappa_s)$$

for $n \longrightarrow \infty$. Therefore, κ_s is the smallest $\mu > 0$ with $C^{(4n+1)}(r) = O(\mu^{4n+1})$ for almost all r, and $\lambda_s^2 = \bigvee_r \lim C^{(4n+1)}(r)/2\kappa_s^{4n+1}$. Subtracting the s-term, one can now inductively determine all κ_k and λ_k. ☐

LEMMA 4.19. *A map* $h : \underline{\mathbb{E}}_m \longrightarrow \underline{\mathbb{E}}_n$ *(*$m > 1$*) is a similarity if and only if* $h|_E$ *is one for all affine planes* $E \subseteq \mathbb{R}^m$*.*

Proof. h is a similarity if and only if it preserves angles. Every angle lies in an affine plane. ☐

THEOREM 4.20. *The continuous distance equivalence preserving maps from* $\underline{\mathbb{E}}_m$ *to* $\underline{\mathbb{E}}_n$ *with* $m > 1$ *are exactly the similarity maps.*

Proof. By virtue of the previous lemma, we might assume that $m = 2$. Let $h : \underline{\mathbb{E}}_2 \to \underline{\mathbb{E}}_n$ be continuous and distance equivalence preserving with $h(0,0) = z$. From Lemma 4.14 we know that h is of the form

z

$$h(x,y) = f_{(x,y)}(z) = \varphi_x\psi_y(z) = \psi_y\varphi_x(z),$$

where $\varphi : x \mapsto \varphi_x$ and $\psi : y \mapsto \psi_y$ are continuous group homomorphisms from $\mathbb{R}^2$ into $\mathrm{Aut}(\underline{\mathbb{E}}_n)$. The motions φ_x and ψ_y are of the form

φ_x, ψ_y

$$\varphi_x(v) = a_x + A_x v \quad \text{and} \quad \psi_y(v) = b_y + B_y v,$$

where A_x and B_y are orthogonal matrices with $A_x B_y = B_y A_x$ $[\![$since $a_x + A_x b_y + A_x B_y v = \varphi_x\psi_y(v) = \psi_y\varphi_x(v) = b_y + B_y a_x + B_y A_x v$ for all $v \in \mathbb{R}^n$ implies $a_x + A_x b_y = b_y + B_y a_x$, hence $A_x B_y v = B_y A_x v$ for all v, that is, $A_x B_y = B_y A_x$ $]\!]$.

A_x, B_y

Therefore, there is a unitary complex matrix P such that $A'_x := P^{-1}A_x P$ and $B'_y := P^{-1}B_y P$ are (complex) diagonal matrices for all x,y $[\![$Choose $\xi,\eta \in \mathbb{R}$ so that A_ξ and B_η have a minimal number of real eigenvalues among all A_x resp. B_y. The commuting orthogonal matrices A_ξ and B_η have a common diagonalization $P^{-1}A_\xi P$, $P^{-1}B_\eta P$ with some unitary P. Since for all $x,y \in \mathbb{R}$, the planes of rotation of A_x and B_y are among those of A_ξ and B_η, respectively, every complex eigenvector of A_ξ or B_η is an eigenvector of A_x or B_y, respectively. Hence also A'_x and B'_y are diagonal$]\!]$. Now $x \mapsto A'_x$ and $y \mapsto B'_y$ are continuous homomorphisms into the group of unitary diagonal matrices, hence there are coefficients $\alpha_j,\beta_j \in \mathbb{R}$ such that

α_j, β_j

$$A'_x = \mathrm{diag}(e^{ix\alpha_1},\dots,e^{ix\alpha_n}) \text{ and } B'_y = \mathrm{diag}(e^{iy\beta_1},\dots,e^{iy\beta_n}).$$

Since h is distance equivalence preserving, its translation map c fulfils

$$c(r) = eh\big((0,0),(r\cos\gamma, r\sin\gamma)\big) = e\big(z, \varphi_{r\cos\gamma}\psi_{r\sin\gamma}(z)\big)$$

for all $r \geqslant 0$ and $\gamma \in \mathbb{R}$. Note that $\varphi_{r\cos\gamma}\psi_{r\sin\gamma}$ is a motion whose matrix $A_{r\cos\gamma}B_{r\sin\gamma}$ has the complex eigenvalues $e^{ir((\cos\gamma)\alpha_j + (\sin\gamma)\beta_j)}$, $j = 1\dots n$. Using elementary geometry, we see that c is of the form

$$c(r)^2 = \big(r\lambda(\gamma)\big)^2 + \sum_{k=1}^{s(\gamma)} \left(\lambda_k(\gamma)\sqrt{2 - 2\cos\big(r\kappa_k(\gamma)\big)}\right)^2 \quad \text{for all } \gamma \in \mathbb{R},$$

with $\lambda_k(\gamma) > 0$ and $\kappa_k(\gamma) > 0$ for all k. Indeed, $r\lambda(\gamma)$ is the length of the translational part of $\varphi_{r\cos\gamma}\psi_{r\sin\gamma}$; each $\lambda_k(\gamma)$ is a radius of rotation for some rotational part of $\varphi_{r\cos\gamma}\psi_{r\sin\gamma}$, that is, the distance from z to the affine $(n-2)$-dimensional subspace of $\mathbb{R}^n$ which is fixed under that rotation; and $r\kappa_k(\gamma)$ is the corresponding angle of that rotation.

Note that, by definition of the α_j,β_j, each $\kappa_k(\gamma)$ is of the form $|(\cos\gamma)\alpha_j + (\sin\gamma)\beta_j|$ for some j. But because of Lemma 4.18, the set $K := \{\kappa_k(\gamma) \mid k = 1\dots s(\gamma)\}$ can be determined from c and is thus the same for all $\gamma \in \mathbb{R}$. Assume that $\kappa \in K$. Then for each $\gamma \in \mathbb{R}$, there is $j \in \{1,\dots,n\}$ with $|(\cos\gamma)\alpha_j + (\sin\gamma)\beta_j| = \kappa$. This is only possible if $\kappa = \alpha_j = \beta_j = 0$ for some j, in contradiction to $\kappa > 0$. Hence K is empty and c is linear, which means that h is a homometry. $\square$

The special case of equal dimensions $n = m$ also follows from the Beckman–Quarles Theorem [BQ53] which says that a map $f : \underline{\mathbb{E}}_n \to \underline{\mathbb{E}}_n$ ($n \geqslant 2$) with $e(x,y) = 1 \implies ef(x,y) = 1$ is already a similarity. Its proof however does not generalize to the case of different dimensions.

Finest distance structures

SET-INDEXED INITIAL LIFTS

morphisms
$\mathsf{S},\mathsf{A},\mathsf{U},\mathsf{T}$
$\mathsf{W},\mathsf{D},\mathsf{H}$
$\mathsf{I},\mathsf{E},\mathsf{O}$

Let us define the following classes of *morphisms*. The classes $\mathsf{S},\mathsf{A},\mathsf{U}$, and T consist of all strongly uniformly, absolutely, uniformly, and ordinarily continuous maps, respectively. The classes $\mathsf{W},\mathsf{D},\mathsf{H};\mathsf{I},\mathsf{E},\mathsf{O}$ contain all weak, dually weak, or ordinary set homometries; distance inequality, equivalence, or specialization preserving maps, respectively. Finally, $\mathsf{SH} := \mathsf{S} \cap \mathsf{H}$ contains all space homometries, and $\mathsf{DU} := \mathsf{D} \cap \mathsf{U}$ and $\mathsf{WU} := \mathsf{W} \cap \mathsf{U}$ all [dually] weak space homometries.

$\mathsf{SH},\mathsf{DU},\mathsf{WU}$
objects
$\mathsf{DIST}_{\mathsf{M}}$
discrete
$\mathsf{DIST}^0_{\mathsf{M}}$

With all distance spaces as *objects*, these classes lead to constructs $\mathsf{DIST}_{\mathsf{M}}$, that is, concrete categories over the category of sets, where M is one of the above classes $\llbracket$it is easily seen that all the classes contain all identity maps and are closed under composition$\rrbracket$. By identifying a distance set $(X,d,\underline{M})$ with the *discrete* space $(X,d,\underline{M},\uparrow 0)$, we see that each category $\mathsf{DIST}_{\mathsf{M}}$ includes a full subcategory $\mathsf{DIST}^0_{\mathsf{M}}$ of distance sets.

Now the construction of categorical suprema on page 51 is a special case of the following general construction.

LEMMA 4.21. *For set-indexed sources* $(h_i : Y \to \underline{Y}_i)_{i\in I}$ *of maps into distance spaces* $\underline{Y}_i = (Y_i,e_i,\underline{N}_i,E_i)$, *an initial distance structure* $(e,\underline{N},E)$ *on* Y *is given by*

$$\underline{N} := \prod_{i\in I} \underline{N}_i^{\top}, \quad e(x,y)(i) := e_i h_i(x,y), \quad \text{and} \quad E := \coprod_{i\in I} E_i^{\top}.$$

M-*initial*

This structure is M-initial *for* $\mathsf{M} = \mathsf{S}\dots\mathsf{WU}$, *that is, for each distance space* $\underline{X}$, *a map* $f : \underline{X} \to (Y,e,E)$ *belongs to* M *if and only if so does each composite* $h_i f : \underline{X} \to \underline{Y}_i$.

Proof. The distance properties are inherited by e from the functions e_i. A map $f : \underline{X} \to \underline{Y}$ is strongly uniformly continuous if and only if for all $F \Subset I$ and all $\varepsilon_i \in E_i$ $(i \in F)$, there is some $\delta \in D$ such that $d(s) \leqslant \delta$ implies $eh_i f(s) \leqslant \varepsilon_i$ for all i, which, by down-directedness of D, is equivalent to strong uniform continuity of all $h_i f$. The other continuity properties only differ in that they restrict the choice of s. Furthermore, an inequality $ef(s) \leqslant ef(t)$ is equivalent to $eh_i f(s) \leqslant eh_i f(t)$ for all i, which settles the case of $\mathsf{M} = \mathsf{W}\ldots\mathsf{O}$. Now also the cases $\mathsf{M} = \mathsf{SH}, \mathsf{DU}, \mathsf{WU}$ are clear. ☐

Note that because $ef(x,y) < ef(x,z)$ does not in general imply $eh_i f(x,y) < eh_i f(x,z)$, the above construction does not work for (local) order representations as morphisms.[1]

EXAMPLE 4.22. *Čech-Stone compactification.* For a topological space $(X, \mathscr{T})$, the multi-pseudometric $d : X^2 \to [0,1]^\Phi$ with $d(x,y) : f \mapsto |f(x) - f(y)|$ is initial for the family Φ of all continuous real-valued maps f from $(X, \mathscr{T})$ to the Euclidean unit interval $[0,1]$. If $\mathscr{T}$ is completely regular, the topology induced by the T_1 bi-completion $\underline{X}'$ of $\underline{X} := (X, d)$ (to be defined in Chapter 5) is the *Čech-Stone compactification* βX of $(X, \mathscr{T})$. This is Samuel's [Sam48] construction of βX, but with distances instead of uniformities.

Čech-Stone compactification

On the other hand, consider the *product space* $\underline{Y} := [0,1]^\Phi$, that is, the set $[0,1]^\Phi$ with the multi-real initial distance structure for the source $(\pi_f)_{f \in \Phi}$ of projections $\pi_f : [0,1]^\Phi \to [0,1]$. The distance in $\underline{Y}$ is just the pointwise $e(a,b) := |a - b|$. Now, the structure of $\underline{X}$ is initial also for the one-element source that consists of the evaluation map $h : X \to \underline{Y}, x \mapsto h_x$ with $h_x : f \mapsto f(x)$. In the completely regular case, h is injective, hence an isometric embedding, and extends to an isometry between $\underline{X}'$ and the closure of $h[X]$, the latter being Čech's [Čec37] original construction of βX.

product space

FINENESS, CANONICAL AND GENERATING STRUCTURES, AND CLASS-INDEXED INITIAL LIFTS

The finer relation in the category $\mathsf{DIST}_{\mathsf{SH}}$ is this: a distance structure (d, D) on X is *finer* than another one, (e, E), if $id_X : (X, d, D) \to (X, e, E)$ is a space homometry. The terms 'coarser' and 'equivalent' are defined in the obvious way.

finer

---

[1] Another badly behaved class of morphisms is that of *proximally continuous maps,* defined by the condition that $f[A]$ must be near $f[B]$ whenever $A \subseteq X$ is *near* $B \subseteq X$, meaning that for each $\delta \in D$ there are $x \in A$ and $y \in B$ such that $d(x,y) \leqslant \delta$. This is related to the problem that the supremum of quasi-uniformities does not induce the supremum of the induced quasi-proximities and that the supremum of quasi-proximities is not their intersection in general (cf. [FL82]).

$\check{\varepsilon}_d$

Given a space $\underline{X}$, the sets

$$\check{\varepsilon}_d := \{s \in X^{2\star} \mid d(s) \leqslant \varepsilon\} \quad (\varepsilon \in D)$$

$\check{D}_d$

are lower sets of R_d and build a base for the zero-filter

$$\check{D}_d := \uparrow\{\check{\varepsilon}_d \mid \varepsilon \in D\}$$

of $\underline{\check{M}}_d$, and $\check{D}_d$ is idempotent if so is D. Now the upper canonical modification $\check{d}$ of d leads to an *upper canonical distance structure* $(\check{d}, \check{D}_d)$ which is obviously equivalent to (d, D).

upper canonical distance structure

$(\check{d}, \check{D}_d)$

generating structures

In terms of the *generating structures* $(R_d, \check{D}_d)$, the finer relation is now simply a pair of inclusions: (d, D) is finer than (e, E) if and only if $R_d \subseteq R_e$ and $\check{D}_d \supseteq \check{E}_e$ $\llbracket$We know already that the first inclusion means that d is finer than e. Now, given that $R_d \subseteq R_e$, all $\varepsilon \in \check{E}_e$ are upper sets of R_e, hence of R_d, and the strong uniform continuity condition for $id_X : (X, \check{d}, \check{D}_d) \to (X, \check{e}, \check{E}_e)$ is equivalent to $\forall \varepsilon \in \check{E}_e\ \exists \delta \in \check{D}_d : \delta \subseteq \varepsilon$, and thus to $\check{E}_e \subseteq \check{D}_d$ since the latter is an upper set of $\underline{\check{M}}_d \rrbracket$. In particular, equivalent upper canonical distance structures coincide, and on a singleton X there is only one upper canonical structure. Because, moreover, the class of all upper canonical structures on X (called the *fibre* of X) is a set, the category canDIST$_{\mathsf{SH}}$ of *upper canonical distance spaces* is small-fibred.

fibre

canDIST$_{\mathsf{SH}}$

upper canonical distance spaces

Using generating structures, one can now construct initial structures, even for class-indexed sources, which are *unique* with respect to space homometries:

THEOREM 4.23. *Every class-indexed source* $\big(h_i : Y \to (Y_i, e_i, E_i)\big)_{i \in I}$ *has a unique upper canonical* SH*-initial lift* (e, E)*. This distance structure is also* M*-initial for* $\mathsf{M} = \mathsf{S} \ldots \mathsf{WU}$*. If all* E_i *are idempotent, so is* E*.*

Proof. For each $i \in I$, $s\, R_i\, t :\Longleftrightarrow e_i h_i(s) \leqslant e_i h_i(t)$ defines a generating quasi-order on $Y^{2\star}$. Thus, the intersection R of the set (!) $\{R_i \mid i \in I\}$ is again a generating quasi-order, and $e := d_R$ a distance function on Y. For each $i \in I$ and $\varepsilon \in E_i$, the set $\varepsilon^i := \{s \in Y^{2\star} \mid e_i h_i(s) \leqslant \varepsilon\}$ is a lower set of R_i, hence of R. Thus, the set B of all finite intersections of some ε^i is a filter base in $\underline{N} := (Y^{2\star}, \circ, \emptyset, R)_\downarrow$. Furthermore, $\bigcap B = R\{\emptyset\}$ since $s \in \bigcap B$ implies $e_i h_i(s) \leqslant \varepsilon$ for all $i \in I$ and $\varepsilon \in E_i$, hence $e_i h_i(s) \leqslant 0$ and $s\, R_i\, \emptyset$ for all $i \in I$, that is, $s\, R\, \emptyset$. In all, B is a base for a zero-filter E in $\underline{N}$.

Now $(e, \underline{N}, E)$ is an initial structure: by its definition, every h_i is a homometry from $\underline{Y}$ to (Y_i, e_i, E_i). Moreover, let $h : (X, d, D) \to \underline{Y}$ be a map such that each $h_i h$ is either (i) among $\mathsf{S} \ldots \mathsf{T}$, or (ii) among $\mathsf{W} \ldots \mathsf{O}$. We have to show that so is h.

In case of (i) there is, for each $\varepsilon_1^{i_1} \cap \cdots \cap \varepsilon_n^{i_n} \in B$, a corresponding $\delta \in D$ such that, for all words s of the correct "type", $d(s) \leqslant \delta$ implies $e_{i_j} h_{i_j} h(s) \leqslant \varepsilon_j$ for $1 \leqslant j \leqslant n$ (one δ suffices because D is a filter). Hence $eh(s) \leqslant \varepsilon_1^{i_1} \cap \cdots \cap \varepsilon_n^{i_n}$

as required. Moreover, $eh(s) \leqslant eh(t)$ is equivalent to $e_i\big(h_ih(s)\big) \leqslant e_i\big(h_ih(t)\big)$ for all i, which settles (ii).

In case of the class SH, uniqueness follows from the fact that all SH-initial structures for the given source are equivalent.

Finally, assume that all E_i are idempotent. Then, for each $\varepsilon_1^{i_1} \cap \cdots \cap \varepsilon_n^{i_n} \in B$, there are $\delta_1 \in E_{i_1}, \ldots, \delta_n \in E_{i_n}$ such that $2\delta_j \leqslant \varepsilon_j$, hence $2\delta_j^{i_j} \subseteq \varepsilon_j^{i_j}$ for all j, and therefore $2(\delta_1^{i_1} \cap \cdots \cap \delta_n^{i_n}) \subseteq 2\delta_1^{i_1} \cap \cdots \cap 2\delta_n^{i_n} \subseteq \varepsilon_1^{i_1} \cap \cdots \cap \varepsilon_n^{i_n}$. ☐

Categorically speaking, this theorem implies that $\mathsf{canDIST}_{\mathsf{SH}}$ is well-fibred, hence a topological construct (in the strong sense of [Pre88], compare [AHS90]).

SEMI- AND QUASI-UNIFORMITIES

The induced system of semi-uniformities. Let $\underline{X}$ be a distance space. In generalization of the usual definition of entourages in a metric space, let

$B_{n,d,\delta}$

$$B_{n,d,\delta} := \big\{(x,y) \in X \times X \,|\, nd(x,y) \leqslant n\delta\big\}$$

for every $\delta \in D$ and every positive integer n. As D is a positive filter, the set $\mathscr{B}_n(d,D) := \{B_{n,d,\delta} \,|\, \delta \in D\}$ is a base for a *semi-uniformity*, that is, for a filter $\mathscr{U}_n(d,D)$ of reflexive relations on X. For $n=1$, the index n might be left out in the sequel. $\mathscr{U}(d,D)$ will be called the *left semi-uniformity* of $\underline{X}$ since its induced filter convergence structure is the left convergence structure $\mathscr{C}(d^{\mathrm{op}},D)$. Likewise, the *right semi-uniformity* $\mathscr{U}^{\mathrm{op}}(d,D) = \{U^{-1} \,|\, U \in \mathscr{U}(d,D)\} = \mathscr{U}(d^{\mathrm{op}},D)$ induces the right convergence structure $\mathscr{C}(d,D)$.

$\mathscr{B}_n(d,D)$

semi-uniformity

$\mathscr{U}_n(d,D)$

left semi-uniformity

right semi-uniformity

If D is idempotent then $\mathscr{U}(d,D)$ is a quasi-uniformity, and in case of a commutative $\underline{M}$ this is also true of $\mathscr{U}_n(d,D)$:

$$nd(x,y), nd(y,z) \leqslant n\delta \text{ then implies } nd(x,z) \leqslant n\big(d(x,y)+d(y,z)\big) \leqslant 2n\delta.$$

In particular, the quasi-uniformity $\mathscr{U}_n(d,D)$ is then equal to $\mathscr{U}(nd,D)$.

Of course, there are certain relationships between the $\mathscr{U}_n(d,D)$ for different n, and in many cases most of them coincide. Obviously,

$$n = n_1 + \cdots + n_k \text{ implies } B_{n_1,d,\delta} \cap \cdots \cap B_{n_k,d,\delta} \subseteq B_{n,d,\delta}.$$

Also, $nd(x,y) \leqslant nmd(x,y) + (m-1)nd(y,x)$, so that

$$(2m-1)n\delta \leqslant n\varepsilon \text{ implies } B_{m,d,\delta} \cap B^{-1}_{n,d,\delta} \subseteq B_{n,d,\varepsilon}.$$

For a positive d,

$$n \leqslant m \text{ and } m\delta \leqslant n\varepsilon \text{ imply } B_{m,d,\delta} \subseteq B_{n,d,\varepsilon} \quad (\star).$$

On the other hand, a symmetric d fulfils $2d(x,y) = d(x,y) + d(y,x) \geqslant d(x,x) = 0$, so that here the implication $(\star)$ holds at least when $m-n$ is even. This proves the following

LEMMA 4.24.

a) $n = n_1 + \cdots + n_k$ *implies* $\mathscr{U}_n(d,D) \subseteq \mathscr{U}_{n_1}(d,D) \vee \cdots \vee \mathscr{U}_{n_k}(d,D)$*, in particular, the map* $n \mapsto \mathscr{U}_n(d,D)$ *is antitone with respect to divisibility.*

b) If D *is idempotent then* $\mathscr{U}_n(d,D) \subseteq \mathscr{U}_n^{\mathrm{op}}(d,D) \vee \mathscr{U}_m(d,D)$ *for all* n,m.

c) If D *is idempotent and* d *is positive, all* $\mathscr{U}_n(d,D)$ *coincide.*

d) If D *is idempotent and* d *is symmetric then* $\mathscr{U}_{2k}(d,D) = \mathscr{U}_2(d,D) \subseteq \mathscr{U}(d,D) = \mathscr{U}_{2k-1}(d,D)$ *and* $\mathscr{U}_k(d,D) = \mathscr{U}_k^{\mathrm{op}}(d,D)$ *for all* $k \geqslant 1$.

Note that there are indeed natural distance functions that are neither positive nor symmetric, the most important being perhaps the distance $x^{-1}y$ on groups:

EXAMPLE 4.25. Let $G := [0,2\pi)$ be the additive group of real numbers modulo 2π, $\underline{M} := (\mathscr{P}(G),+,\{0\},\subseteq)$ with element-wise addition, $D := \{(-\delta,\delta) \mid \delta \in (0,2\pi]\}$, and $d(x,y) := \{y-x\}$. Then $\mathscr{U}(d,D)$ is the usual "Euclidean" uniformity on G, while $\mathscr{U}_n(d,D)$ is this uniformity "modulo $\frac{2\pi}{n}$" since

$$x\,B_{n,d,\delta}\,y \Longleftrightarrow y-x \in \textstyle\bigcup_{k\in n}(-\delta+\frac{2k\pi}{n}, \frac{2k\pi}{n}+\delta).$$

Likewise, for $X := \mathbb{C}\setminus\{0\}$, $\underline{N} := \underline{M}\times\underline{\mathbb{R}}^+$, $E := D\times(0,\infty)$, and $e(x,y) := \big(d(\arg x,\arg y), \big||y|-|x|\big|\big)$, the semi-uniformity $\mathscr{U}_n(e,E)$ induces the Euclidean topology "modulo multiplication with nth roots of unity".

Before we construct finest distance structures for a whole class of quasi-uniformities at once, let us start with a single quasi-uniformity.

THEOREM 4.26. *Every quasi-uniformity* $\mathscr{V}$ *admits a finest distance structure* $(d_{\mathscr{V}}, D_{\mathscr{V}})$ *for which* $D_{\mathscr{V}}$ *is idempotent and* $\mathscr{V} = \mathscr{U}(d_{\mathscr{V}}, D_{\mathscr{V}})$.

V_0 R

Proof. Let $\mathscr{V}$ be some quasi-uniformity on X and $V_0 := \bigcap\mathscr{V}$ its specialization quasi-order. We will see that the essential information about $\mathscr{V}$ is contained in the idempotent zero-filter $D_{\mathscr{V}}$ which we must construct, while the generating quasi-order $R_{d_{\mathscr{V}}}$ is fully determined by the very weak condition that $xx'\,R_{d_{\mathscr{V}}}\,\emptyset$ must hold for any pair $x,x' \in X$ that fulfils $x\,V_0\,x'$ ⟦otherwise $d_{\mathscr{V}}(x,x') \not\leqslant \delta$ for some $\delta \in D_{\mathscr{V}}$, in contradiction to $V_0 \subseteq B_{d_{\mathscr{V}},\delta}$⟧. Therefore, let R be the smallest quasi-order on $X^{2\star}$ that is compatible with $\circ$ and fulfils

$$xx'\,R\,\emptyset\,R\,xx \text{ and } xz\,R\,xyyz \quad \text{for all } x,x',y,z \in X \text{ with } x\,V_0\,x'. \quad (\star\star)$$

$\underline{M}$

If we find a suitable idempotent zero-filter D such that $\mathscr{U}(\check{d}_R, D) = \mathscr{V}$ then R must obviously be the smallest relation (and thus $\check{d}_R$ a finest distance function) with this property. Let $\underline{M} := (M,+,0,\subseteq) := (X^{2\star},\circ,\emptyset,R)_\downarrow$.

Now observe that each of the resulting entourages B_{d,δ_1} has to include some entourage $V_1 \in \mathscr{V}$, hence every $\delta_1 \in D$ must include some set $\{xy \in$ V_1 $X^{2\star} \,|\, x\, V_1\, y\}$ with $V_1 \in \mathscr{V}$. Since $0 = R\{xx\}$ is a neutral element, δ_1 must even include the set

$$\begin{aligned}\{xy' \in X^{2\star} \,|\, x\, V_0 V_1 V_0\, y'\} \;&\subseteq\; R\{xx'x'yyy' \,|\, x\, V_0\, x'\, V_1\, y\, V_0\, y'\} \\ &\subseteq\; 0 + \{x'y \in X^{2\star} \,|\, x'\, V_1\, y\} + 0 \subseteq 0 + \delta_1 + 0.\end{aligned}$$

The same must be true for any $\delta_2 \in D$ that fulfils $2\delta_2 \subseteq \delta_1$, so that δ_1 must also include a set $\{xyx'y' \in X^{2\star} \,|\, x\, V_0 V_2 V_0\, y, x'\, V_0 V_2 V_0\, y'\} \subseteq 2\delta_2$ for some $V_2 \in \mathscr{V}$. V_2 This process of replacing some δ_n by some $2\delta_{n+1}$ can be continued, and in order to describe it formally, let us define W to be the smallest set of tuples of W positive integers that contains the 1-tuple (1) and fulfils

$$(n_1,\ldots,n_{i-1},n_i+1,n_i+1,n_{i+1},\ldots,n_k) \in W$$

whenever $(n_1,\ldots,n_k) \in W$ and $1 \leqslant i \leqslant k$. One can think of the elements of W as coding exactly those terms of the form '$\delta_{n_1} + \cdots + \delta_{n_k}$' which can be obtained when we start with the term 'δ_1' and then successively replace an arbitrary summand 'δ_n' by the term '$\delta_{n+1} + \delta_{n+1}$'. Accordingly, one shows by induction that for each element δ_1 of an idempotent zero-filter D there is a sequence $\delta_2, \delta_3, \ldots$ in D such that

$$(n_1,\ldots,n_k) \in W \text{ implies } \delta_{n_1} + \cdots + \delta_{n_k} \leqslant \delta_1.$$

In our situation, this observation implies that for each $\delta_1 \in D$ there must be a sequence $\mathscr{S} = (V_1, V_2, \ldots)$ in $\mathscr{V}$ with the property that δ_1 includes the set $A_{\mathscr{S}}$ of $A_{\mathscr{S}}$ all words $v_1 w_1 \cdots v_k w_k \in X^{2\star}$ for which there is some $(n_1,\ldots,n_k) \in W$ such that $v_i\, V_0 V_{n_i} V_0\, w_i$ for $i = 1,\ldots,k$. In particular, $\delta_{\mathscr{S}} := RA_{\mathscr{S}} \subseteq R\delta_1 = \delta_1$. It $\delta_{\mathscr{S}}$ turns out that this is the only restraint on the idempotent zero-filter $D_{\mathscr{V}}$. More precisely, we will see that the system B

$$B := \{\delta_{\mathscr{S}} \,|\, \mathscr{S} \in \mathscr{V}^{\omega}\}$$

of lower sets of $(X^{2\star}, R)$ is a base for an idempotent zero-filter D in $\underline{M}$, and that the distance structure $(\check{d}_R, D)$ induces the quasi-uniformity $\mathscr{V}$. It is then clear that D is the largest idempotent zero-filter with this property, so that $(d_{\mathscr{V}}, D_{\mathscr{V}}) := (\check{d}_R, D)$ is a finest distance structure inducing $\mathscr{V}$.

Since $\mathscr{V}$ is a filter and the map $\mathscr{S} \mapsto \delta_{\mathscr{S}}$ is isotone in every component of $\mathscr{S}$, B is a filter-base. In order to show that D is an idempotent zero-filter, we first observe that $(n_1,\ldots,n_k), (m_1,\ldots,m_l) \in W$ implies

$$(n_1+1,\ldots,n_k+1,m_1+1,\ldots,m_l+1) \in W$$

⟦after increasing each index by one, the replacements that produce $(n_1,\ldots,n_k)$ and $(m_1,\ldots,m_l)$ from the tuple (1) can be combined to a sequence of replacements that produce $(n_1+1,\dots,n_k+1,m_1+1,\dots,m_l+1)$ from the tuple $(2,2)$ ⟧. Hence also $v_1w_1\cdots v_kw_k, v_1'w_1'\cdots v_l'w_l' \in \delta_{(V_2,V_3,V_4,\dots)}$ implies

$$v_1w_1\cdots v_kw_kv_1'w_1'\cdots v_l'w_l' \in \delta_{(V_1,V_2,V_3,\dots)}$$

for each sequence $(V_1,V_2,\dots)$ in $\mathscr{V}$.

Secondly, we must prove that $\bigcap B = 0$, which is the harder part. Let $s = x_1z_1\cdots x_mz_m \in \bigcap B$ and $V_1 \in \mathscr{V}$. I will show that $x_j\, V_0V_1V_0\, z_j$ holds for all
$\mathscr{S}$ $j = 1,\dots,m$. Choose a sequence $\mathscr{S} = (V_1,V_2,\dots)$ in $\mathscr{V}$ such that $V_{i+1}V_0V_{i+1} \subseteq V_i$ for all $i \geqslant 1$ ⟦ such a sequence always exists in a quasi-uniformity ⟧. Note that $(n_1,\dots,n_k) \in W$ then implies $V_0V_{n_1}V_0V_{n_2}V_0\cdots V_0V_{n_k}V_0 \subseteq V_0V_1V_0$. Now $s \in RA_{\mathscr{S}}$, that is, there exists a word $t = v_1w_1\cdots v_kw_k$ and a k-tuple $(n_1,\dots,n_k) \in W$ such that $v_i\, V_0V_{n_i}V_0\, w_i$ for $i = 1,\dots,k$, and $s\,R\,t$. The latter means that, starting with $v_1w_1\cdots v_kw_k$, one gets $x_1z_1\cdots x_mz_m$ in finitely many steps in each of which some pair of letters is inserted or removed corresponding to the
ψ condition ($\star\star$). Now take the k-tuple

$$\psi := (v_1\, V_0V_{n_1}V_0\, w_1,\dots,v_k\, V_0V_{n_k}V_0\, w_k)$$

of formulae (which express true propositions about the word $v_1w_1\cdots v_kw_k$) and modify it, analogously to those finitely many steps, in the following way: (i) if (because of $xz\,R\,xyyz$) a pair yy is being removed after an odd number of letters, replace the two consecutive formulae $\dots V_0\, y, y\, V_0\cdots$ in ψ by one formula $\dots V_0\cdots$ (that is, erase the symbols '$y, y\, V_0$'); (ii) if (because of $\emptyset\, R\, xx$) a syllable xx is being removed, remove the corresponding formula $x\dots x$ from ψ; (iii) if (because of $xx'\, R\, \emptyset$) a syllable xx' is inserted, insert the formula $x\, V_0\, x'$ at the respective position in ψ. Then, by definition of R, all these modifications preserve the truth of all formulae in the tuple, and each formula in the resulting tuple $(\psi_1,\dots,\psi_m)$ expresses a true proposition of the form

$$\psi_j = x_j\, V_0V_{n_a}V_0V_{n_{a+1}}V_0\dots V_0V_{n_b}V_0\, z_j \quad \text{with } 1 \leqslant a,b \leqslant k.$$

Since all V_{n_i} are reflexive, ψ_j thus implies

$$x_j\, V_0V_{n_1}V_0V_{n_2}V_0\dots V_0V_{n_k}V_0\, z_j, \quad \text{hence} \quad x_j\, V_0V_1V_0\, z_j.$$

Because V_1 was chosen arbitrarily, we conclude that $(x_j,z_j) \in \bigcap_{V_1\in\mathscr{V}} V_0V_1V_0 = V_0$ for all j, and therefore $s = x_1z_1\cdots x_mz_m\, R\, \emptyset$, that is, $s \in 0$.

Finally, we have to show that $(\check{d}_R, D)$ induces the quasi-uniformity $\mathscr{V}$. For $V \in \mathscr{V}$, choose $V_1 \in \mathscr{V}$ such that $V_0V_1V_0 \subseteq V$, then choose a sequence $\mathscr{S}$ as in the preceding paragraph. There we have shown that, in particular,

$$\check{d}_R(x,z) \subseteq RA_{\mathscr{S}} \quad \text{implies} \quad (x,z) \in V_0V_1V_0 \subseteq V.$$

On the other hand, for each $\delta \in D$, there is some $\mathscr{S} = (V_1, \dots) \in \mathscr{V}^\omega$ such that $\delta_{\mathscr{S}} \subseteq \delta$, and

$$(x, z) \in V_1 \subseteq V_0 V_1 V_0 \quad \text{implies} \quad \check{d}_R(x, z) \subseteq \delta_{\mathscr{S}} \subseteq \delta.$$

$\boxed{\mathrm{CC}}$

A somewhat astonishing consequence of this construction is that *one* distance function is compatible to *all* T_1 quasi-uniformities on X:

COROLLARY 4.27. *The distance function $\check{d}^{\perp}_X$ is a finest distance function d on X such that for each T_1 quasi-uniformity $\mathscr{V}$ on X there is an idempotent zero-filter D such that $(\check{d}^{\perp}_X, D)$ induces $\mathscr{V}$.*

Also, almost exactly the same construction can be used to get a finest distance structure inducing a semi-uniformity $\mathscr{V}$: in that case, W is taken to be the singleton set that contains only the 1-tuple (1), and instead of sequences $\mathscr{S}$, one takes just 1-tuples $\mathscr{S} = (V_1)$. The only crucial point is that each $V \in \mathscr{V}$ must include some $V_0 V_1 V_0$ with $V_1 \in \mathscr{V}$, a condition easily seen to be true in every semi-uniformity induced by a distance structure.

THEOREM 4.28. *If $\mathscr{V}$ is a semi-uniformity on X with specialization $V_0 = \bigcap \mathscr{V}$ such that, for all $V \in \mathscr{V}$, there is $W \in \mathscr{V}$ with $V_0 W V_0 \subseteq V$, then there is a finest distance structure (d, D) with $\mathscr{V} = \mathscr{U}(d, D)$.*[1]

Without giving more details, I notice that this also leads to finest distance structures for the filter convergence structures resp. Čech-closure operators characterized in Theorem 3.15 and Corollary 3.17. Moreover, the following result can be used to prove that the assignment $(X, \mathscr{V}) \mapsto (X, d_{\mathscr{V}}, D_{\mathscr{V}})$ extends to a co-reflective, full and concrete embedding of the category QUNIF of quasi-uniform spaces with uniformly continuous maps into the category $\mathsf{iDIST_{SH}}$ of distance spaces with idempotent zero-filters and space homometries:

QUNIF

$\mathsf{iDIST_{SH}}$

THEOREM 4.29. *Let $\underline{Y} = (Y, e, \underline{N}, E)$ be a distance space with idempotent E, $\mathscr{V}$ a quasi-uniformity on X, and $h : (X, \mathscr{V}) \to \big(Y, \mathscr{U}(e, E)\big)$ a uniformly continuous map. Then h is a space homometry from $(X, d_{\mathscr{V}}, D_{\mathscr{V}})$ into $\underline{Y}$.*

Proof. Recall the construction of $d := d_{\mathscr{V}}$, and assume that $d(s) \leqslant d(t)$. Then we get t from s by a finite number of modifications of one of the following two types: (i) inserting some pair xx, (ii) removing some syllable xx' with $(x, x') \in V_0 := \bigcap \mathscr{V}$. Since one gets $eh(t)$ from $eh(s)$ by corresponding steps, it suffices to show that these are steps "down in R_e": (i) the pair $h(x)h(x)$

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[1] Here the Axiom of Countable Choice is no longer needed since it was only necessary for the choice of suitable sequences $\mathscr{S}$.

may be inserted because $h(x)h(x)\,R_e\,\emptyset$, and (ii) the syllable $h(x)h(x')$ may be removed because $x\,V_0\,x'$ implies $eh(x,x') \leqslant 0$.

Now comes the more difficult part. For $\varepsilon_1 \in E$, choose some sequence $(\varepsilon_n)_n$ in E such that $\varepsilon_{n_1} + \cdots + \varepsilon_{n_k} \leqslant \varepsilon_1$ for all tuples $(n_1,\ldots,n_k) \in W$. By uniform continuity of h, each $V_n := (h \times h)^{-1}[B_{e,\varepsilon_n}]$ is in $\mathscr{V}$. Now, put $\mathscr{S} := (V_1,V_2,\ldots) \in \mathscr{V}^\omega$ and assume $d(s) \leqslant \delta_{\mathscr{S}}$, which implies $d(s) \leqslant d(t)$ for some $t = v_1w_1 \cdots v_kw_k \in A_{\mathscr{S}}$. Then there is some $(n_1,\ldots,n_k) \in W$ such that $(v_i,w_i) \in V_0V_{n_i}V_0$ and hence $(hv_i,hw_i) \in B_{e,\varepsilon_{n_i}}$ for $i = 1,\ldots,k$. Finally, the latter implies $eh(s) \leqslant eh(t) \leqslant \varepsilon_1$. $\boxed{\text{cc}}$

Again, a similar result holds for semi-uniformities.

HOW TO INDUCE SYSTEMS OF QUASI-UNIFORMITIES

I will now extend the result of Theorem 4.26 to certain systems of quasi-uniformities and show that, in particular, every finite system and every descending sequence of T$_1$ uniformities is part of some system $\big(\mathscr{U}_n(d,D)\big)_{n\in\omega}$. Unfortunately, the proof requires four quite technical lemmata about the structure of the relation $R_X^\perp$.

$[a,b]$
$\tilde{s}$
$\ell(s)$
s_a
$s_{a,b}$
$\lambda(x,s)$

Some additional notation: Intervals of integers will here be designated by $[a,b]$. For a word $s \in X^{2\star}$, let $\tilde{s} \in X^{2\star}$ be the word s after deletion of all null syllables, that is, without those of the form xx. The length of $\tilde{s}$ in letters is designated by $\ell(s)$, and s_a is the ath letter of $\tilde{s}$ for any position $a \in [1,\ell(s)]$. The subword of $\tilde{s}$ from position a to b is $s_{a,b}$. Moreover, let $\lambda(x,s)$ and $\sigma(xy,s)$ denote the number of occurrences of the letter x resp. the syllable xy in $\tilde{s}$. Finally, $(xy)^r = xyxy\cdots xy$ is a word consisting of r equal syllables.

$\sigma(xy,s)$
$(xy)^r$
s,t,r_i

For the moment, let us fix some words $s,t \in X^{2\star}$ with $s\,R_X^\perp\,t$, where

$$\tilde{t} = (v_1w_1)^{r_1} \cdots (v_\varrho w_\varrho)^{r_\varrho}, \quad v_i \neq w_i, \qquad \text{and all } r_i \text{ are even.}$$

Then $\tilde{s}$ can be derived from $\tilde{t}$ by a finite number of successive deletions of pairs of identical letters which are neighbours at the time of deletion. A guiding example: for $s = yy\,xy\,zz\,xy\,uz\,uz\,R_X^\perp\,xy\,xy\,zz\,zu\,uz\,uz\,xx\,uz = t$, the deletion steps could be this: in $\tilde{t} = xy\,xy\,zu\,uz\,uz\,uz$, first delete uu, giving $xy\,xy\,zz\,uz\,uz$, then delete zz, giving $xy\,xy\,uz\,uz = \tilde{s}$.

D

We now also fix such a sequence of deletions and let $D \subseteq [1,\ell(t)]$ be the set of positions in $\tilde{t}$ whose corresponding letters are deleted in one of these steps (in the example: $D = [5,8]$). For $a \in D$, let $\pi(a) \in [1,\ell(t)]$ be that position in $\tilde{t}$ such that t_a and $t_{\pi(a)}$ build a deleted pair, called the *partner position* of a (in the example: $\pi(5) = 8$ and $\pi(6) = 7$). Finally, write $a \curvearrowright b$ if and only if a and b are even numbers in D such that $a < \pi(a) = b - 1$ (in the example: $6 \curvearrowright 8$).

partner position

Note that because t_c and $t_{\pi(c)}$ must first become neighbours before they can be deleted, $a \curvearrowright \cdots \curvearrowright b$ implies that (i) $[a,b-1] \subseteq D$, (ii) $\pi(c) \in [a,b-1]$ for all $c \in [a,b-1]$, and thus (iii) $\lambda(x,t_{a,b-1})$ is even for all $x \in X$.

LEMMA 4.30. *Assume* $a \curvearrowright \cdots \curvearrowright b \curvearrowright \cdots \curvearrowright c$*,* $t_a = t_{b-1}$*, and* $t_b = t_{c-1}$*. Then*

a) $t_{a-1} = t_b$ *or* $t_{b-1} = t_c$*.*

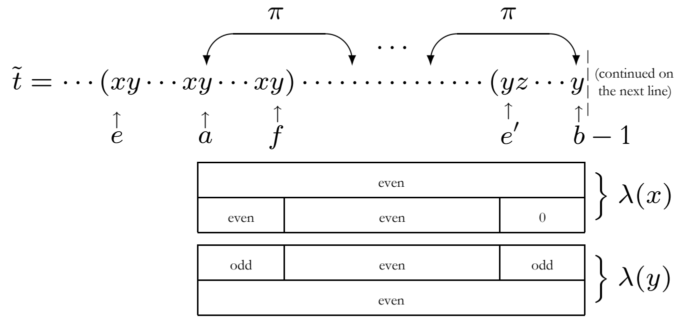
b) If $t_{a-1} \neq t_b$ *then* $\lambda(t_a,t_{c,\ell(t)})$ *is odd.*

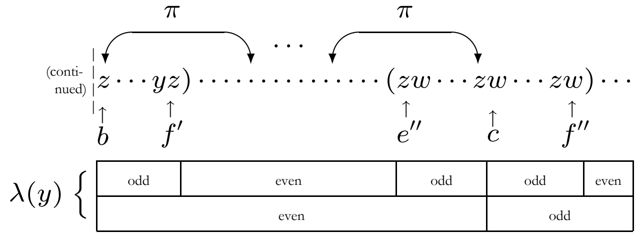
c) If $t_{b-1} \neq t_c$ *then* $\lambda(t_b,t_{1,a-1})$ *is odd.*

Proof. Let $e,f,e',f',e'',f'' \in [1,\ell(t)]$ with $e < a \leqslant f < e' < b \leqslant f' < e'' < c \leqslant f''$ such that $t_{e,f}$, $t_{e',f'}$, and $t_{e'',f''}$ are three of the defining subwords $(v_i w_i)^{r_i}$ of $\tilde{t}$. Moreover, let $x := t_{a-1}$, $y := t_a = t_{b-1}$, $z := t_b = t_{c-1}$, and $w := t_c$, and assume $x \neq z$. The situation and the parity arguments that will follow are sketched below.

e'',f''

x,y,z,w





Because of $x \neq z$, we have $\lambda(x,t_{e',b-1}) = 0$. Moreover, $\lambda(x,t_{f+1,e'-1})$ is even ⟦since all r_i are even⟧, and $\lambda(x,t_{a,b-1})$ is even because of (iii), so that also $\lambda(x,t_{a,f})$ is even and $\lambda(y,t_{a,f})$ is odd ⟦since $|[a,f]|$ is odd⟧. As before, $\lambda(y,t_{f+1,e'-1})$ and $\lambda(y,t_{a,b-1})$ are even, thus $\lambda(y,t_{e',b-1})$ is odd. Because all r_i are even, $\lambda(y,t_{b,f'})$ is also odd. Again, $\lambda(y,t_{f'+1,e''-1})$ and $\lambda(y,t_{b,c-1})$ are even, hence $\lambda(y,t_{e'',c-1})$ is odd. In particular, $y \in \{z,w\}$, that is, $y = w$ ⟦as yz is a syllable of $\tilde{t}$⟧, and $\lambda(y,t_{c,f''})$ is also odd. Finally, $\lambda(y,t_{c,\ell(t)})$ is odd because

$\lambda(y,t_{f'',\ell(t)})$ is even. This proves a) and b), whereas c) is strictly analogous to b). $\square$

LEMMA 4.31.

a) Assume that $a_0 \curvearrowright b_0 \curvearrowright a_1 \curvearrowright b_1 \cdots a_k \curvearrowright b_k \curvearrowright c$, *with* $t_{a_0} = \cdots = t_{a_k} = y$ *and* $t_{b_0} = \cdots = t_{b_k} = z$. *Then* $t_{a_0-1} = z$ *or* $y = t_c$.

b) Assume that $a \curvearrowright \cdots \curvearrowright b$, *with* $t_a = t_{b-1}$ *and* $t_{a-1} \neq t_b$. *Then both* $\lambda(t_a,t_{1,a-1})$ *and* $\lambda(t_a,t_{b,\ell(t)})$ *are odd.*

Proof. a) Define e'',f'' as above. Similarly, for each $i \in [0,k]$, find positions $e_i,f_i,e'_i,f'_i \in [1,\ell(t)]$ with $e_i < a_i \leqslant f_i < e'_i < b_i \leqslant f'_i$ such that t_{e_i,f_i} and $t_{e'_i,f'_i}$ are two of the defining subwords of $\tilde{t}$. Assuming $t_{a_0-1} = x \neq z$, one proves that $\lambda(y,t_{b_0,f'_0})$ is odd exactly as before. Since, for $i \in [1,k]$, all of $\lambda(y,t_{b_{i-1},a_i-1})$, $\lambda(y,t_{a_i,b_i-1})$, $\lambda(y,t_{f'_{i-1}+1,e_i-1})$, $\lambda(y,t_{e_i,f_i})$, $\lambda(y,t_{f_i+1,e'_i-1})$, and $\lambda(y,t_{e'_i,f'_i})$ are even, and since also $\lambda(y,t_{b_k,c-1})$ and $\lambda(y,t_{f'_k+1,e''-1})$ are even, we conclude that $\lambda(y,t_{e'',c-1})$ is odd, hence $y = t_c$.

b) Again as in the previous lemma, one proves that, for $y := t_a$, the number $\lambda(y,t_{b,f'})$ is odd, so that the first claim follows because $\lambda(y,t_{f'+1,\ell(t)})$ is even. The second claim is just the dual. $\square$

LEMMA 4.32. *Assume that* $s_{e-1}s_e = xz$ *is the syllable of* $\tilde{s}$ *that remains after all the deletions in a subword* $t_{a-1,c}$ *of* $\tilde{t}$, *with* $t_{a-1} = x$, $t_c = z$, *and* $a < c$. *Then there is some* $y \in X$ *such that* $\lambda(y,s) > 0$, $\sigma(xy,t_{a-1,c}) > 0$, *and* $\sigma(yz,t_{a-1,c}) > 0$.

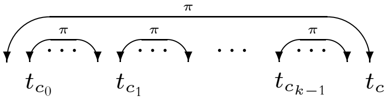
Proof. Although t_a and t_{c-1} may be different, we find $k \geqslant 2, b_1,\ldots,b_k \in [1,\ell(t)]$, and $y_0,y_1,\ldots,y_k \in X$ such that

$$a = b_1 \curvearrowright \cdots \curvearrowright b_2 \curvearrowright \cdots \curvearrowright b_3 \;\cdots\; b_{k-1} \curvearrowright \cdots \curvearrowright b_k \leqslant c,$$

$t_{b_i} = t_{b_{i+1}-1} = y_i$ for $i \in [1,k-1]$, $y_0 = x$, $y_k = z$, and $y_i \neq y_j$ for $i \neq j$ ⟦Start with $a =: b'_1 \curvearrowright b'_2 \curvearrowright \cdots \curvearrowright b'_l := c$ and $y'_i := t_{b'_i}$. As long as there are indices $j > i > 1$ with $y'_i = y'_j$, remove all the indices $i+1,\ldots,j$, so that finally all the remaining y'_i are different. Since $y'_1 = t_a \neq z = y'_l$, at least $k \geqslant 2$ of the original indices are not removed, including the index 1, and the corresponding b'_i build the required positions $b_1,\ldots,b_k$⟧.

Then $k = 2$ since otherwise Lemma 4.30 a) would imply that either $y_0 = y_2$ or $y_1 = y_3$. With $y := y_1$ and $b := b_2$, Lemma 4.31 b) implies that $\lambda(y,t_{1,a-1})$ is odd. Now, also $\lambda(y,s_{1,e-1})$ is odd because $b \in [1,a-1] \cap D$ implies $\pi(b) \in [1,a-1]$ ⟦since the letter x at position $a-1$ is not deleted⟧. In particular, $\lambda(y,s_{1,e-1}) > 0$. $\square$

LEMMA 4.33. *Assume that* $k \geqslant 2$, $c_0 \curvearrowright c_1 \cdots c_{k-1} \curvearrowright c_k$, $c_k \in D$, *and* $\pi(c_k) = c_0 - 1$, *representing a number of deletions of the form*



Let $t' := t_{c_0-1}t_{c_0}t_{c_1-1}t_{c_1}\cdots t_{c_k-1}t_{c_k}$ *be the word consisting only of the "boundary letters", and* $i \in [0,k]$. *Then* $\sigma(t_{c_i-1}t_{c_i},t') = \sigma(t_{c_i}t_{c_i-1},t')$.

Proof. Put $c_{-1} := c_k$. Obviously, $t_{c_{i-1}} = t_{c_i-1}$ for all $i \in [1,k]$, and $t_{c_k} = t_{c_0-1}$. If also $t_{c_{i-1}-1} = t_{c_i}$ for all $i \in [0,k]$ then k must be odd $[\![$ since $t_{c_k} \neq t_{c_0}$ $]\!]$, and $\sigma(t_{c_i-1}t_{c_i},t') = \sigma(t_{c_i}t_{c_i-1},t') = (k+1)/2$.

Otherwise, there are $r \geqslant 1$ positions $i(1) < \cdots < i(r)$ in $[0,k]$ with $t_{c_{i(j)-1}-1} \neq t_{c_{i(j)}}$. Then $i(j+1) - i(j)$ is even for all j $[\![$otherwise, put $a_0 := c_{i(j)-1}$, $b_0 := c_{i(j)}, \ldots, c := c_{i(j+1)-1}$ and apply Lemma 4.31 a)$]\!]$. In case that all $i(j)$ are even, we have

$$t_{c_{k-1}} \neq t_{c_k} = t_{c_0-1} = t_{c_1} = t_{c_i}$$

for all odd i, so that k must be odd. On the other hand, if all $i(j)$ are odd, we have

$$t_{c_k} = t_{c_0} - 1 \neq t_{c_0} = t_{c_i}$$

for all even i, so that again k must be odd. This shows that t' is of one of the following two forms:

$$\begin{aligned} t' &= \quad (yxxy)^{m_0}(yz_1z_1y)^{m_1}\cdots(yz_{r-1}z_{r-1}y)^{m_{r-1}}(yxxy)^{m_r} \\ \text{or} \quad t' &= xy(yxxy)^{m_0}(yz_1z_1y)^{m_1}\cdots(yz_{r-1}z_{r-1}y)^{m_{r-1}}(yxxy)^{m_r}yx, \end{aligned}$$

from which the claim follows immediately. ☐

Now we are ready for the construction. Let p_i be the ith odd prime number, and $S(A) := \{a_1 + \cdots + a_k \mid k \geqslant 1,\ a_i \in A\}$ for any set A of integers. In the theorem, we need the following sets of even numbers: for any positive integer u, let $q_{uj} = \frac{2}{p_j}\prod_{i=1}^{u} p_i$ for all $j \in [1,u]$, $Q_u := \{q_{u1},\ldots,q_{uu}\}$, and $Q_{uj} := Q_u \setminus \{q_{uj}\}$. It is easy to see that then, for each $j \in [1,u]$ and $k \in S(Q_{uj})$, $k - q_{uj} \notin S(Q_{uj})$ $[\![$since p_j divides k but not $q_{uj}$$]\!]$.

p_i
$S(A)$
q_{uj}
Q_u, Q_{uj}

THEOREM 4.34.

a) *Let* $\mathscr{V}_1,\ldots,\mathscr{V}_u$ *be a finite system of* T_1 *quasi-uniformities such that, for all* $i,j \in [1,u]$, $\mathscr{V}_j \subseteq \mathscr{V}_j^{-1} \vee \mathscr{V}_i$. *Then there is an idempotent zero-filter* D *such that* $\mathscr{V}_j = \mathscr{U}_{q_{uj}}(\check{d}_X^{\perp}, D)$ *for all* $j \in [1,u]$.

b) Let $\mathscr{V}_1 \supseteq \mathscr{V}_2 \supseteq \dots$ *be a descending sequence of* T_1 *quasi-uniformities such that, for all* j *and all* $U \in \mathscr{V}_j$*, there are* $V_1 \in \mathscr{V}_1, V_2 \in \mathscr{V}_2, \dots$ *with* $V_j^{-1} \cap \bigcup_{i \neq j} V_i \subseteq U$*. Then there is an idempotent zero-filter* D *such that* $\mathscr{V}_j = \mathscr{U}_{2^j}(\check{d}_X^{\perp}, D)$ *for all* $j \in \omega$*.*

Proof. For part a), let $I := [1,u]$, while for part b), let $I := \omega \setminus \{0\}$. In both cases, D is defined quite analogously to the proof of Theorem 4.26: its filter-base is now the system

B

$$B := \{\varepsilon_{\mathscr{S}} \mid \mathscr{S} \text{ is a sequence in } \mathscr{V}\}$$

of lower sets $\delta_{\mathscr{S}} := R_X^{\perp} A_{\mathscr{S}}$ of $R_X^{\perp}$, where $\mathscr{V} := \prod_{i \in I} \mathscr{V}_i$, and the definition of $A_{\mathscr{S}}$ changes to this: for

$\delta_{\mathscr{S}}$ $\mathscr{V}$ $A_{\mathscr{S}}$

$$\mathscr{S} = \big((V_1^{(1)}, V_1^{(2)}, \dots), (V_2^{(1)}, V_2^{(2)}, \dots), \dots\big) \in \mathscr{V}^{\omega},$$

$A_{\mathscr{S}}$ is now the set of all words $(v_1 w_1)^{r_1} (v_2 w_2)^{r_2} \cdots (v_\varrho w_\varrho)^{r_\varrho} \in X^{2\star}$ for which there is some ϱ-tuple $(n_1, \dots, n_\varrho) \in W$ and some tuple of indices $(i_1, \dots, i_\varrho)$ such that, for all $a \in [1, \varrho]$, $v_a \, V_{n_a}^{(i_a)} \, w_a$ and either $r_a = q_{u i_a}$ (for the proof of part a) or $r_a = 2^{i_a}$ (for the proof of part b).

As before, D turns out to be an idempotent zero-filter, where the only essential change is the proof of $\bigcap B = 0$: Let $s \in \bigcap B$, xz a non-null syllable of s, that is, $\sigma(xz, s) > 0$, and let $V = (V_1^{(1)}, V_1^{(2)}, \dots) \in \mathscr{V}$. Choose $\mathscr{S} \in \mathscr{V}^{\omega}$ so that $V_{k+1}^{(i)} V_{k+1}^{(i)} \subseteq V_k^{(i)}$ for all $i \in I$ and $k \in \omega$, and some $t \in A_{\mathscr{S}}$ with $s \, R_X^{\perp} \, t$. Assume that $\tilde{t} = (v_1 w_1)^{r_1} (v_2 w_2)^{r_2} \cdots (v_\varrho w_\varrho)^{r_\varrho}$. If $\sigma(xz, t) > 0$, put $y_V := x$, otherwise choose some $y_V \in X$ with $\lambda(y_V, s) > 0$, $\sigma(x y_V, t) > 0$, and $\sigma(y_V z, t) > 0$, according to Lemma 4.32. Since $\ell(s)$ is finite and $\mathscr{V}$ is filtered, there is some y such that, for all $V \in \mathscr{V}$, there is $V' \in \mathscr{V}$ with $V' \leqslant V$ and $y_{V'} = y$, where $\leqslant$ denotes component-wise set inclusion. Consequently, $x \, U_V \, y \, U_V \, z$ for all $V \in \mathscr{V}$, where $U_V = \bigcup_i V_1^{(i)}$. This implies that $x, y \in \bigcap \mathscr{V}_i$ and $x, y \in \bigcap \mathscr{V}_{i'}$ for some $i, i' \in I$, hence $x = y = z$. Since this is a contradiction to $x \neq z$, we have shown that $\tilde{s}$ is the empty word, that is, $s \in 0$.

Finally, let us show that $\mathscr{V}_j = \mathscr{U}_{q_{uj}}(\check{d}_X^{\perp}, D)$ resp. $\mathscr{V}_j = \mathscr{U}_{2^j}(\check{d}_X^{\perp}, D)$ for each $j \in I$. Fix some $j \in I$ and let $V_0^{(j)} \in \mathscr{V}_j$. Because of the premises, the following choices can now be made. For part a), choose for all $i \in I \setminus \{j\}$ some $V_0^{(i)} \in \mathscr{V}_j$ and $V_1^{(i)} \in \mathscr{V}_i$ such that $(V_0^{(i)})^{-1} \cap V_1^{(i)} \subseteq V_0^{(j)}$. Then choose $V_1^{(j)} \in \mathscr{V}_j$ such that $V_1^{(j)} \subseteq V_0^{(i)}$ for all of the finitely many $i \in I \setminus \{j\}$. For part b), choose instead some $(V_1^{(1)}, V_1^{(2)}, \dots) \in \mathscr{V}$ with $V_1^{(h)} = V_1^{(j)} \subseteq V_0^{(j)}$ for all $h \leqslant j$ and $(V_1^{(j)})^{-1} \cap \bigcup_{i \neq j} V_1^{(i)} \subseteq V_0^{(j)}$.

After that, choose the remaining components of

$$\mathscr{S} = \big((V_1^{(1)}, V_1^{(2)}, \dots), (V_2^{(1)}, V_2^{(2)}, \dots), \dots\big) \in \mathscr{V}^{\omega}$$

so that $V^{(i)}_{k+1}V^{(i)}_{k+1} \subseteq V^{(i)}_{k}$ for all $i \in I$ and $k \in \omega$, and assume that $r\check{d}^{\perp}_{\mathrm{X}}(x,y) \leqslant \delta_{\mathscr{S}}$, that is, $s := (xz)^{r}\, R^{\perp}_{\mathrm{X}}\, t \in A_{\mathscr{S}}$ with a) $r = q_{uj}$ resp. b) $r = 2^{j}$. We have to show that $x\, V^{(j)}_{0}\, z$.

By definition of $A_{\mathscr{S}}$, we have $\tilde{t} = (v_1w_1)^{r_1}(v_2w_2)^{r_2}\cdots(v_{\varrho}w_{\varrho})^{r_{\varrho}}$, and there is some corresponding tuple $(i_1,\ldots,i_{\varrho})$. Since the only letters in $\tilde{s}$ are x and z, there are exactly r occurrences of the syllable xz in $\tilde{t}$ which are not deleted ⟦because otherwise Lemma 4.32 would imply the existence of a third letter y in $\tilde{s}$⟧. All other occurrences of xz in $\tilde{t}$ are deleted as part of some set of deletions of the form represented in Lemma 4.33, that is, there are $c_0,\ldots,c_k$ with properties as in Lemma 4.33 and with $t_{c_i-1}t_{c_i} = xz$ for some $i \in [0,k]$. Then the lemma implies that $\sigma(xz,t) = r + \sigma(zx,t) =: k$.

For a): If $(v_aw_a)^{r_a} = (xz)^{q_{uj}}$ for some $a \in [1,\varrho]$, then $i_a = j$ and

$$(x,z) \in V^{(i_a)}_{n_a} \subseteq V^{(j)}_{1} \subseteq V^{(j)}_{0}.$$

Otherwise, we know that $k \in S(Q_{uj})$, that is, $\sigma(zx,t) = k - q_{uj} \in S(Q_u) \setminus S(Q_{uj})$, so that $(v_aw_a)^{r_a} = (zx)^{q_{uj}}$ and $i_a = j$ for some $a \in [1,\varrho]$. Also, $(v_bw_b)^{r_b} = (xz)^{q_{ui}}$ and $i_b = i$ for some $b \in [1,\varrho]$ and some $i \in I \setminus \{j\}$, so that $(x,z) \in (V^{(j)}_{1})^{-1} \cap V^{(i)}_{1} \subseteq V^{(j)}_{0}$.

For b) instead: If $(v_aw_a)^{r_a} = (xz)^{2^{i}}$ for some $a \in [1,\varrho]$ and $i \leqslant j$, then $i_a = i$ and $(x,z) \in V^{(i_a)}_{n_a} \subseteq V^{(i)}_{1} \subseteq V^{(j)}_{0}$. Otherwise, k is a multiple of 2^{j+1} so that $\sigma(zx,t) = k - 2^{j}$ is not such a multiple. Therefore, $(v_aw_a)^{r_a} = (zx)^{2^{i_a}}$ and $i_a \leqslant j$ for some $a \in [1,\varrho]$. Also, $(v_bw_b)^{r_b} = (xz)^{2^{i_b}}$ and $i_b \neq j$ for some $b \in [1,\varrho]$, so that again $(x,z) \in (V^{(i_a)}_{1})^{-1} \cap V^{(i_b)}_{1} \subseteq V^{(j)}_{0}$. $\boxed{\text{cc}}$

Note that this proof highly depends on the fact that the resulting value monoid has not been made commutative.

PROBLEM 4.35. Which systems of quasi-uniformities come from distance structures with a commutative monoid?

5.

FUNDAMENTAL NETS AND COMPLETENESS

> *Je kleiner das Sandkörnchen ist, desto sicherer hält es sich für den Mittelpunkt der Welt.*
>
> *Marie von Ebner-Eschenbach*

Fundamental nets and Cauchy-filters

This section is about some generalizations of the notion of a fundamental sequence.[1] There are a lot of possibilities to concretise the intuitive idea that the distances between the points of a net "become small" at the end. The same is true for generalized versions of Cauchy-filters in quasi-uniform spaces, and a lot of work was done in this area by Deák, Künzi, Romaguera, and others. When translating their definitions into the language of nets and distances instead of filters and entourages, many logical interrelations become very clear. Most of these properties can be formulated using alternating quantifications of the kind '$\forall n\ \exists i \geqslant n$' and '$\exists n\ \forall i \geqslant n$'. They are strongly related to some functions

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[1] The nowadays more usual term 'Cauchy-sequence' is misleading in that it ignores the fact that instead of Cauchy, Bernard Bolzano seems to have been the first who defined the notion of fundamental sequence in 1817. He used it for his "proof" [Bol17] of the (topological) completeness of the real numbers, then transformed this into order-theoretic completeness, and finally proved the intermediate value theorem for continuous functions on a real interval in the still usual elegant way: given $f(a) < 0 < f(b)$, a point $x \in [a,b]$ for which $f(x) = 0$ is constructed as the infimum of all y with $0 \leqslant f(y)$.

For filters, however, the 'Cauchy'-terminology is so common that I, too, will use it here.

which involve limites superiores and inferiores and are interpreted as measures of various kinds of "distances" between nets. As it turns out, these functions can be used to construct a number of different types of completions for distance spaces.

It is tried here to simplify the somewhat unstructured terminology which exists in the literature on quasi-uniform spaces, and to underline the connection between Cauchy-properties and distances between nets. For this reason, I begin with a definition of ten "nearness-relations" for nets on a distance space:

Nearness-relations and fundamentality of nets. For nets x, y on a distance space $(X,d,\underline{M},D)$, define

$$\begin{aligned}
\mathsf{x}\langle\exists\exists\forall\forall\rangle\mathsf{y} \;&:\Longleftrightarrow\; \forall\delta \quad \exists n\ \exists m \quad \forall i\geqslant n\ \forall j\geqslant m \;:\; d(x_i,y_j)\leqslant\delta,\\[4pt]
\mathsf{x}\langle\ell\exists\forall\exists\forall\rangle\mathsf{y} \;&:\Longleftrightarrow\; \forall\delta \quad \exists n\ \forall i\geqslant n \quad \exists m\ \forall j\geqslant m \;:\; d(x_i,y_j)\leqslant\delta,\\
\mathsf{x}\langle\ell\forall\exists\exists\forall\rangle\mathsf{y} \;&:\Longleftrightarrow\; \forall\delta \quad \forall n\ \exists i\geqslant n \quad \exists m\ \forall j\geqslant m \;:\; d(x_i,y_j)\leqslant\delta,\\
\mathsf{x}\langle\ell\exists\forall\forall\exists\rangle\mathsf{y} \;&:\Longleftrightarrow\; \forall\delta \quad \exists n\ \forall i\geqslant n \quad \forall m\ \exists j\geqslant m \;:\; d(x_i,y_j)\leqslant\delta,\\
\mathsf{x}\langle\ell\forall\exists\forall\exists\rangle\mathsf{y} \;&:\Longleftrightarrow\; \forall\delta \quad \forall n\ \exists i\geqslant n \quad \forall m\ \exists j\geqslant m \;:\; d(x_i,y_j)\leqslant\delta,\\[4pt]
\mathsf{x}\langle r\exists\forall\exists\forall\rangle\mathsf{y} \;&:\Longleftrightarrow\; \forall\delta \quad \exists m\ \forall j\geqslant m \quad \exists n\ \forall i\geqslant n \;:\; d(x_i,y_j)\leqslant\delta,\\
\mathsf{x}\langle r\forall\exists\exists\forall\rangle\mathsf{y} \;&:\Longleftrightarrow\; \forall\delta \quad \forall m\ \exists j\geqslant m \quad \exists n\ \forall i\geqslant n \;:\; d(x_i,y_j)\leqslant\delta,\\
\mathsf{x}\langle r\exists\forall\forall\exists\rangle\mathsf{y} \;&:\Longleftrightarrow\; \forall\delta \quad \exists m\ \forall j\geqslant m \quad \forall n\ \exists i\geqslant n \;:\; d(x_i,y_j)\leqslant\delta,\\
\mathsf{x}\langle r\forall\exists\forall\exists\rangle\mathsf{y} \;&:\Longleftrightarrow\; \forall\delta \quad \forall m\ \exists j\geqslant m \quad \forall n\ \exists i\geqslant n \;:\; d(x_i,y_j)\leqslant\delta,\\[4pt]
\mathsf{x}\langle\forall\forall\exists\exists\rangle\mathsf{y} \;&:\Longleftrightarrow\; \forall\delta \quad \forall n\ \forall m \quad \exists i\geqslant n\ \exists j\geqslant m \;:\; d(x_i,y_j)\leqslant\delta,
\end{aligned}$$

where δ, n, i, m, and j are variables of the sort D, I_x, I_x, I_y, and I_y, respectively (that is, they run only over the respective sets). If Q is one of the quantifications $\exists\exists\forall\forall$, $\ell\exists\forall\exists\forall$, …, $\forall\forall\exists\exists$, and if $\mathsf{x}\langle\mathsf{Q}\rangle\mathsf{y}$ holds, we say that x is Q-*near* y or that (x,y) is a Q-*pair*. If $\mathsf{x}\langle\mathsf{Q}\rangle\mathsf{x}$ then x is said to be Q-*fundamental*. In case of the quantification $\exists\exists\forall\forall$, I will also use the terms *bi-near* and *bi-fundamental net*.

Q-*near*

Q-*pair*

Q-*fundamental*

bi-near

bi-fundamental net

The notation is so that for a quantification whose label begins with an 'ℓ', the first two quantifiers affect the indices of the *left* net, while if the label begins with an 'r', the first two quantifiers affect the indices of the *right* net.

Moreover, call x an $\ell\exists\forall{\cdot}\forall$ resp. $r\exists\forall{\cdot}\forall$-*fundamental net*[1] if and only if

$\ell\exists\forall{\cdot}\forall$-*fundamental net*

$r\exists\forall{\cdot}\forall$-*fundamental net*

$$\begin{aligned}
&\forall\delta \quad \exists n\ \forall i\geqslant n\ \forall j\geqslant i \;:\; d(x_i,x_j)\leqslant\delta\\
\text{resp.}\quad &\forall\delta \quad \exists n\ \forall j\geqslant n\ \forall i\geqslant j \;:\; d(x_i,x_j)\leqslant\delta,
\end{aligned}$$

where of course now j is also of sort I_x.

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[1] These properties have also been called 'left' and 'right K-Cauchy' (cf. [Kün01]) but are not equivalent to the left or right K-Cauchy property of the end filter ⟦a trivial example is the sequence $(1,0,3,2,5,4,\dots)$ in $(\mathbb{Z},d)$ with $d(x,y):=0$ for $x\leqslant y$ and $d(x,y):=\infty$ for $x>y$⟧. Instead, the latter are equivalent to the weaker properties of $\ell\exists\forall\exists\forall$- and $r\exists\forall\exists\forall$-fundamentality.

LEMMA 5.1. *If D is idempotent and $\mathscr{U}$ is the quasi-uniformity induced by (d, D), then:*

a) (x,y) *is an* $\exists\exists\forall\forall$*-pair if and only if* $(\mathscr{E}\mathsf{x},\mathscr{E}\mathsf{y})$ *is a Cauchy-pair in the sense of Deák [Deá96], that is, if and only if each entourage includes some* $F \times G$*, where* $F \in \mathscr{E}\mathsf{x}$ *and* $G \in \mathscr{E}\mathsf{y}$. *(In this case* $\mathscr{E}\mathsf{y}$ *is called* D-Cauchy *[Rom96].)*

D-Cauchy

bi-Cauchy-filter [FL82],

b) x *is bi-fundamental if and only if* $\mathscr{E}\mathsf{x}$ *is a* bi-Cauchy-filter [FL82], *that is, if and only if each entourage includes some* F^2 *with* $F \in \mathscr{E}\mathsf{x}$.

weakly hereditarily Cauchy

c) x *is* $\ell\forall\exists\exists\forall$*-fundamental if and only if* $\mathscr{E}\mathsf{x}$ *is* weakly hereditarily Cauchy *[Rom96], that is, iff for all* $U \in \mathscr{U}$ *and* $F \in \mathscr{E}\mathsf{x}$ *there is* $y \in F$ *such that* $yU \in \mathscr{E}\mathsf{x}$.

left K-Cauchy

d) x *is* $\ell\exists\forall\exists\forall$*-fundamental if and only if* $\mathscr{E}\mathsf{x}$ *is* left K-Cauchy *[Rom96], that is, if and only if for all* $U \in \mathscr{U}$ *there is* $F \in \mathscr{E}\mathsf{x}$ *such that* $yU \in \mathscr{E}\mathsf{x}$ *for all* $y \in F$.

right K-Cauchy

e) Likewise, x *is* $r\exists\forall\exists\forall$*-fundamental if and only if* $\mathscr{E}\mathsf{x}$ *is* right K-Cauchy *[Rom96].*

stable

f) x *is* $r\exists\forall\forall\exists$*-fundamental if and only if* $\mathscr{E}\mathsf{x}$ *is* stable *[Deá96], that is, if and only if* $\bigcap\{FU \mid F \in \mathscr{E}\mathsf{x}\} \in \mathscr{E}\mathsf{x}$ *for each* $U \in \mathscr{U}$.

g) Every net is $\forall\forall\exists\exists$*-fundamental.*

LEMMA 5.2. *For all* $\underline{X}$,

$$\langle\exists\exists\forall\forall\rangle \subseteq \langle\ell\exists\forall\exists\forall\rangle \subseteq \langle\ell\forall\exists\exists\forall\rangle \subseteq \langle\ell\forall\exists\forall\exists\rangle \subseteq \langle\forall\forall\exists\exists\rangle,$$
$$\langle\ell\exists\forall\exists\forall\rangle \subseteq \langle\ell\exists\forall\forall\exists\rangle \subseteq \langle\ell\forall\exists\forall\exists\rangle,$$

every bi-fundamental net is $\ell\exists\forall\cdot\forall$*-fundamental, and every* $\ell\exists\forall\cdot\forall$*-fundamental net is* $\ell\exists\forall\exists\forall$*-fundamental. The same is true with '*r*' instead of '*ℓ*'.*

Both *proofs* are immediate from the definitions. ☐

LEMMA 5.3.

For (X, d, D) *with idempotent* D*:* *Proof.* (notation explained below)

1. $\langle\ell\exists\forall\forall\exists\rangle\langle r\exists\forall\exists\forall\rangle \subseteq \langle\exists\exists\forall\forall\rangle$ $[\![\forall\varepsilon\ \exists\delta\ \underline{\exists n}\ \underline{\exists k}\ \underline{\forall i}\ \underline{\forall l}\ \exists m\ [\forall]m\ \exists j\ [\forall]j\ \ \varphi]\!]$
2. $\langle\ell\exists\forall\exists\forall\rangle\langle r\exists\forall\forall\exists\rangle \subseteq \langle\exists\exists\forall\forall\rangle$ $[\![\forall\varepsilon\ \exists\delta\ \underline{\exists n}\ \underline{\exists k}\ \underline{\forall i}\ \underline{\forall l}\ \exists m\ [\forall]m\ \exists j\ [\forall]j\ \ \varphi]\!]$
3. $\langle\ell\forall\exists\exists\forall\rangle\langle\ell\forall\exists\exists\forall\rangle \subseteq \langle\ell\forall\exists\exists\forall\rangle$ $[\![\forall\varepsilon\ \exists\delta\ \underline{\forall n}\ \underline{\exists i}\ \exists m\ [\forall]m\ \exists j\ [\forall]j\ \underline{\exists k}\ \underline{\forall l}\ \ \varphi]\!]$
4. $\langle\ell\forall\exists\exists\forall\rangle\langle r\exists\forall\forall\exists\rangle \subseteq \langle\ell\forall\exists\exists\forall\rangle$ $[\![\forall\varepsilon\ \exists\delta\ \underline{\forall n}\ \underline{\exists i}\ \underline{\exists k}\ \underline{\forall l}\ \exists m\ [\forall]m\ \exists j\ [\forall]j\ \ \varphi]\!]$
5. $\langle r\exists\forall\forall\exists\rangle\langle\ell\forall\exists\exists\forall\rangle \subseteq \langle\ell\forall\exists\exists\forall\rangle$ $[\![\forall\varepsilon\ \exists\delta\ \exists m\ [\forall]m\ \exists j\ [\forall]j\ \underline{\forall n}\ \underline{\exists i}\ \underline{\exists k}\ \underline{\forall l}\ \ \varphi]\!]$

The *proofs* above are given in a short notation: the quantified variables run over all $\delta,\varepsilon \in D$, $i,n \in I_{\mathsf{x}}$, $j,m \in I_{\mathsf{y}}$, and $l,k \in I_{\mathsf{z}}$ which fulfil

$$2\delta \leqslant \varepsilon,\ \ i \geqslant n,\ \ j \geqslant m,\ \text{ and }\ l \geqslant k,$$

and φ is short for

$$d(x_i,z_l) \leqslant d(x_i,y_j) + d(y_j,z_l) \leqslant 2\delta \leqslant \varepsilon.$$

Whenever an $\forall$-quantified part of a definition is applied to some variable which is already evaluated in an earlier quantification, then the $\forall$ is surrounded by square brackets. Whether a quantifier "originates" from the first or the second relation is visualized by a slightly raised or lowered position, and those that appear in the conclusion are underlined. For example, the proof of the first line reads as follows:

Let $\mathsf{x}\langle \ell\exists\forall\forall\exists\rangle\mathsf{y}\langle r\exists\forall\exists\forall\rangle\mathsf{z}$. Then for each $\varepsilon \in D$ there is $\delta \in D$ such that $2\delta \leqslant \varepsilon$, and there are $n \in I_{\mathsf{x}}$ and $k \in I_{\mathsf{z}}$ such that for all $i \geqslant n$ and $l \geqslant k$ there is some $m \in I_{\mathsf{y}}$ and, for all and particularly for this choice of m, there is some $j \geqslant m$ such that, for all and particularly for this choice of $j \geqslant m$, $d(x_i,y_j) \leqslant \delta$ and $d(y_j,z_l) \leqslant \delta$ and thus $d(x_i,z_l) \leqslant 2\delta \leqslant \varepsilon$. ☐

Notions of completeness

Given a distance structure (d,D), let us write $\mathsf{x} \longrightarrow x$, $\mathsf{x} \succ x$, $x \longleftarrow \mathsf{x}$, $x \prec \mathsf{x}$, or $x \longleftarrow \mathsf{x} \longrightarrow x$ as shorthands for $\mathsf{x} \longrightarrow_{d,D} x$, $\mathsf{x} \succ_{d,D} x$, $\mathsf{x} \longrightarrow_{d^{\mathrm{op}},D} x$, $\mathsf{x} \succ_{d^{\mathrm{op}},D} x$, or ($\mathsf{x} \longrightarrow_{d,D} x$ and $\mathsf{x} \longrightarrow_{d^{\mathrm{op}},D} x$), respectively. In case of $x \longleftarrow \mathsf{x}$, $x \prec \mathsf{x}$, or $x \longleftarrow \mathsf{x} \longrightarrow x$, x will be called a *dual limit*, *dual cluster point*, or *bi-limit* of x, respectively, and x will be said to be *dually convergent*, *dually clustering*, or *bi-convergent*.

dual limit

dual cluster point

bi-limit

dually convergent

dually clustering

bi-convergent

Net selections. In the following, it will be convenient to call a property of nets (such as bi-fundamentality) a *net selection*. Formally, a net selection C is just a class of nets, and a C-net is an element of C. Special net selections are the class of all bi-fundamental nets, called '*bi*' for short, that of all $\ell\exists\forall\exists\forall$-fundamental nets, designated by '$\ell\exists\forall\exists\forall$', etc.

net selection

Completeness properties. For a net selection C, let us call $\underline{X}$ $(C,\prec)$-, $(C,\longleftarrow)$-, $(C,\longleftrightarrow)$-, $(C,\longrightarrow)$-, or $(C,\succ)$-*complete* if and only if each C-net on $\underline{X}$ has a dual cluster point, dual limit, bi-limit, limit, or cluster point, respectively. For *sequential* $(C,\dots)$-*completeness*, this is only required for C-sequences instead of all C-nets.

complete

sequential $(C,\dots)$-*completeness*

Sequential $(\ell\exists\forall{\cdot}\forall,\longrightarrow)$-, $(r\exists\forall{\cdot}\forall,\longrightarrow)$-, or $(bi,\longrightarrow)$-completeness, for instance, are suitable forms of completeness for the generalized version of Banach's fixed point theorem in the next chapter. The following facts generalize a result by Romaguera [Rom92].

LEMMA 5.4. *Let* $(C,\dots)$ *be one of the pairs* $(\ell\exists\forall\cdot\forall,\longleftarrow)$*,* $(r\exists\forall\cdot\forall,\longrightarrow)$*,* $(bi,\longleftarrow)$*,* $(bi,\longleftrightarrow)$*, or* $(bi,\longrightarrow)$*.*

Then $(C,\dots)$*-completeness is equivalent to sequential* $(C,\dots)$*-completeness for all distance spaces* $\underline{X}$ *whose zero-filter* D *is idempotent and has a countable base* $\{\delta_n \mid n \in \omega\}$*.*

Proof. Let us start with the pair $(r\exists\forall\cdot\forall,\longrightarrow)$. Let $\underline{X}$ be sequentially $(r\exists\forall\cdot\forall,\longrightarrow)$-complete, and x an $r\exists\forall\cdot\forall$-net on $\underline{X}$. For all $n \in \omega$, choose some $k_n \in I_{\mathsf{x}}$ such that $d(x_i,x_j) \leqslant \delta_n$ for all $i \geqslant j \geqslant k_n$. Since I_{x} is directed, we can also choose an increasing sequence $(k'_n)_{n\in\omega}$ with $k'_n \geqslant k_n$ for all n. Now $y_n := x_{k'_n}$ defines an $r\exists\forall\cdot\forall$-sequence y on $\underline{X}$ $\llbracket$For $\varepsilon \in D$, choose n with $\delta_n \leqslant \varepsilon$. Then $d(y_m,y'_m) \leqslant \delta_n \leqslant \varepsilon$ for all $m \geqslant m' \geqslant n\rrbracket$. Let z be some limit of y. Then also $\mathsf{x} \longrightarrow z$ $\llbracket$For $\varepsilon \in D$, choose $\delta \in D$ with $2\delta \leqslant \varepsilon$, and $n \in \omega$ such that $\delta_n \leqslant \delta$ and $d(y_m,z) \leqslant \delta$ for all $m \geqslant n$. Then, for all $i \geqslant k'_n$, $d(x_i,z) \leqslant d(x_i,y_n) + d(y_n,z) = d(x_i,x_{k'_n}) + d(y_n,z) \leqslant \delta_n + \delta \leqslant \varepsilon.\rrbracket$

The proof for the pair $(bi,\longrightarrow)$ differs only in that the requirements $i \geqslant j \geqslant k_n$ and $m \geqslant m' \geqslant n$ are replaced by $i,j \geqslant k_n$ and $m,m' \geqslant n$. Those for $(\ell\exists\forall\cdot\forall,\longleftarrow)$ and $(bi,\longleftarrow)$ are just dual, and in the proof for $(bi,\longleftrightarrow)$-completeness, z is a bi-limit of y and hence of x. $\boxed{\text{cc}}$

I will not try to compare here all of the many interesting completeness properties which arise from the different notions of fundamental net defined in the previous section. In the case of quasi-uniform spaces, many of these properties have already been investigated thoroughly (cf. [Kün01]), and those results can easily be translated to the situation of distance spaces with idempotent zero-filter.

There are, however, some differences when D is not idempotent. In a quasi-uniform space, any bi-Cauchy-filter converges to all of its cluster points. For distance spaces with non-idempotent zero-filter, it is not even the case that $(\ell\forall\exists\forall\exists,\succ)$-completeness implies sequential $(bi,\longrightarrow)$-completeness:

EXAMPLE 5.5. Let $\underline{M} = ([0,\infty],\star,0,\leqslant)$ be the p. o. m. of extended non-negative real numbers with the mutated addition defined by $\alpha \star \beta := \infty$ for all $\alpha,\beta > 0$. Then $D := (0,\infty]$ is a non-idempotent zero-filter for $\underline{M}$. On $X := [-1,1] \setminus \{0\} \cup \{a,b\}$, define a symmetric $\underline{M}$-distance function d by $d(a,r) := d(-r,b) := d(a,b) := \infty$ and $d(-r,a) := d(b,r) := r$ for all $r > 0$, and $d(x,y) := |x-y|$ for all $x,y \in [-1,1]$. Then $\underline{X} := (X,d,\underline{M},D)$ is $(\ell\exists\forall\exists\forall,\succ)$- and $(\ell\exists\forall\exists\forall,\prec)$-complete since every net is either frequently in $X_- := [-1,0) \cup \{a\}$ or eventually in $X_+ := \{b\} \cup (0,1]$, and both these subspaces are uniformly isomorphic to the Euclidean unit interval. But the sequence $\big((-2)^{-n}\big)_n \in \omega$ is bi-fundamental without having a limit.

EXAMPLE 5.6. The modified quasi-pseudometric $d(0,x) := 0$, $d(x,0) := \infty$, and $d(x,y) := |x-y|$ for $x,y > 0$ turns the interval $[0,1]$ into an $(all,\longleftarrow)$-

complete space in which the bi-fundamental sequence $(2^{-n})_{n\in\omega}$ has no cluster point.

EXAMPLE 5.7. Even with an idempotent zero-filter, $(\ell\exists\forall{\cdot}\forall,\succ)$- is not implied by $(bi,\leftrightarrow)$-completeness. Since the bi-fundamental nets on $\mathbb{R}$ with the skew-symmetric distance $d(x,y):=y-x$ are the same as those on $\mathbb{E}_1$, this space is surely bi-complete. However, $(-n)_{n\in\omega}$ is an $\ell\exists\forall{\cdot}\forall$-fundamental sequence without a cluster point.

Let us consider the 55 completeness properties $(C,\dots)$ where C is one of $\ell\forall\exists\forall\exists$, $\ell\forall\exists\exists\forall$, $\ell\exists\forall\forall\exists$, $\ell\exists\forall\exists\forall$, $\ell\exists\forall{\cdot}\forall$, $\exists\exists\forall\forall$, and their duals with '$r$' instead of '$\ell$'.

PROBLEM 5.8. Which of the following implications are true for general distance spaces:

$$\begin{array}{rcl@{\qquad}rcl}
(\ell\forall\exists\forall\exists,\leftrightarrow) & \overset{?}{\Longrightarrow} & (r\exists\forall{\cdot}\forall,\prec), & (\ell\forall\exists\forall\exists,\leftrightarrow) & \overset{?}{\Longrightarrow} & (r\exists\forall{\cdot}\forall,\succ),\\
(\ell\forall\exists\exists\forall,\leftrightarrow) & \overset{?}{\Longrightarrow} & (\ell\exists\forall\forall\exists,\prec), & (\ell\forall\exists\exists\forall,\leftrightarrow) & \overset{?}{\Longrightarrow} & (\ell\exists\forall\forall\exists,\succ),\\
(\ell\exists\forall\forall\exists,\leftrightarrow) & \overset{?}{\Longrightarrow} & (\ell\forall\exists\exists\forall,\prec), & (\ell\exists\forall\forall\exists,\leftrightarrow) & \overset{?}{\Longrightarrow} & (\ell\forall\exists\exists\forall,\succ),\\
(\ell\exists\forall{\cdot}\forall,\leftrightarrow) & \overset{?}{\Longrightarrow} & (\ell\exists\forall\exists\forall,\prec), & (\ell\exists\forall{\cdot}\forall,\leftrightarrow) & \overset{?}{\Longrightarrow} & (\ell\exists\forall\exists\forall,\succ),\\
\text{and }(bi,\leftrightarrow) & \overset{?}{\Longrightarrow} & (\ell\exists\forall{\cdot}\forall,\prec). & & &
\end{array}$$

This is a sensible choice of questions because these are exactly the weakest implications not yet decided. If all of them would turn out false, the implication partial order between the 55 properties would just be the product of the implication order among the five types of convergence and the dual of the implication order among the eleven types of fundamentality. However, this is not the case since the two implications in the second to the last line are in fact true. More precisely,

OBSERVATION 5.9. *$(\ell\exists\forall{\cdot}\forall,\rightarrow)$-completeness implies $(\ell\exists\forall\exists\forall,\succ)$-completeness, and $(\ell\exists\forall{\cdot}\forall,\leftarrow)$-completeness implies $(\ell\exists\forall\exists\forall,\prec)$-completeness.*

The *proof* is a variant of that of [KR97], Lemma 1, modified so as to avoid the need of a choice principle. Assume that $\underline{X}$ is $(\ell\exists\forall{\cdot}\forall,\rightarrow)$ or $(\ell\exists\forall{\cdot}\forall,\rightarrow)$-complete, and let x be some $\ell\exists\forall\exists\forall$-fundamental net. On

$$J:=\bigl\{(\delta,n,x_i)\,|\,\delta\in D,\ i\geqslant n\in I_{\mathsf{x}},\ \exists m\,\forall j{\geqslant}m: d(x_i,x_j)\leqslant\delta\bigr\},$$

define a directed quasi-order $\leqslant$ by putting $(\delta,n,x)\leqslant(\varepsilon,m,y)$ if and only if either (i) $(\delta,n)=(\varepsilon,m)$ or (ii) $\delta\geqslant\varepsilon$, $n\leqslant m$, and $d(x,x_j)\leqslant\delta$ for all $j\geqslant m$. Then $\mathsf{y}:=(x)_{(\delta,n,x)\in J}$ is an $\ell\exists\forall{\cdot}\forall$-fundamental net $[\![$ Let $\delta\in D$, choose some $n\in I_{\mathsf{x}}$ and $x\in X$ with $(\delta,n,x)\in J$, and assume that $(\zeta,r,z)\geqslant(\varepsilon,m,y)\geqslant(\delta,n,x)$. Then $y=x_j$ for some $j\geqslant m$ with $d(y,x_k)\leqslant\varepsilon$ for all $k\geqslant r$, and $z=x_k$ for some $k \geqslant r$, hence $d(y,z) \leqslant \varepsilon \leqslant \delta$ as required⟧. Now let a be a [dual] limit of y. In order to show that a is a [dual] cluster point of x, let $\delta \in D$ and $n \in I_{\mathsf{x}}$, and choose some $(\varepsilon,m,y) \in J$ with $d^{[\mathrm{op}]}(z,a) \leqslant \delta$ for all $(\zeta,r,z) \geqslant (\varepsilon,m,y)$. Finally, choose some $s \in I_{\mathsf{x}}$ with $d(y,x_k) \leqslant \varepsilon$ for all $k \geqslant s$, and some (ζ,r,z) with $\zeta = \varepsilon$ and $r \geqslant n,m,s$. Then $(\zeta,r,z) \geqslant (\varepsilon,m,y)$ since $d(y,x_k) \leqslant \varepsilon$ for all $k \geqslant r$. Therefore, $d^{[\mathrm{op}]}(z,a) \leqslant \delta$, which proves that a is a [dual] cluster point of x since $z = x_k$ for some $k \geqslant r \geqslant n$. ☐

Updating the set of weakest yet unproved implications, we get:

QUESTION 5.10. Which of these are true for general distance spaces:

$$\begin{array}{rclrcl}
(\ell\forall\exists\forall\exists,\leftrightarrow) & \overset{?}{\Longrightarrow} & (r\exists\forall\cdot\forall,\prec), & (\ell\forall\exists\forall\exists,\leftrightarrow) & \overset{?}{\Longrightarrow} & (r\exists\forall\cdot\forall,\succ),\\
(\ell\forall\exists\exists\forall,\leftrightarrow) & \overset{?}{\Longrightarrow} & (\ell\exists\forall\forall\exists,\prec), & (\ell\forall\exists\exists\forall,\leftrightarrow) & \overset{?}{\Longrightarrow} & (\ell\exists\forall\forall\exists,\succ),\\
(\ell\exists\forall\forall\exists,\leftrightarrow) & \overset{?}{\Longrightarrow} & (\ell\forall\exists\exists\forall,\prec), & (\ell\exists\forall\forall\exists,\leftrightarrow) & \overset{?}{\Longrightarrow} & (\ell\forall\exists\exists\forall,\succ),\\
(\ell\exists\forall\cdot\forall,\rightarrow) & \overset{?}{\Longrightarrow} & (\ell\exists\forall\exists\forall,\prec), & (\ell\exists\forall\cdot\forall,\leftarrow) & \overset{?}{\Longrightarrow} & (\ell\exists\forall\exists\forall,\succ),\\
(\ell\exists\forall\cdot\forall,\leftrightarrow) & \overset{?}{\Longrightarrow} & (\ell\exists\forall\exists\forall,\leftarrow), & (\ell\exists\forall\cdot\forall,\leftrightarrow) & \overset{?}{\Longrightarrow} & (\ell\exists\forall\exists\forall,\rightarrow),\\
(\ell\exists\forall\cdot\forall,\prec) & \overset{?}{\Longrightarrow} & (\ell\exists\forall\exists\forall,\prec), & (\ell\exists\forall\cdot\forall,\succ) & \overset{?}{\Longrightarrow} & (\ell\exists\forall\exists\forall,\succ),\\
\text{and }(bi,\leftrightarrow) & \overset{?}{\Longrightarrow} & (\ell\exists\forall\cdot\forall,\prec). & & &
\end{array}$$

Distances between nets

The idea behind the completions to be presented in the next section is to define distances between nets as certain kinds of limits of the distances between the points which constitute the nets.

limes superior
limes inferior
lim sup
lim inf

Limes superior and inferior. The *limes superior* [*limes inferior*] of a net $\mathsf{a} = (\alpha_i)_{i\in I_{\mathsf{a}}}$ on a conditionally complete lattice is the infimum [supremum] of the set of suprema [infima] of its principal ends:

$$\begin{aligned}
\limsup_{i\in I_\alpha}\alpha_i &:= \bigwedge\Big\{\bigvee\{\alpha_i\,|\,i\geqslant n\}\,\Big|\,n\in I_{\mathsf{a}}\Big\},\\
\liminf_{i\in I_\alpha}\alpha_i &:= \bigvee\Big\{\bigwedge\{\alpha_i\,|\,i\geqslant n\}\,\Big|\,n\in I_{\mathsf{a}}\Big\}.
\end{aligned}$$

If the index set is a product $I \times J$ of directed sets, one can also write $\limsup_{i\in I,j\in J}$ instead of $\limsup_{(i,j)\in I\times J}$.

LEMMA 5.11. *If α and β are nets on a co-quantale then*

$$\begin{aligned}
\limsup_{i\in I_\alpha,j\in I_\beta}(\alpha_i+\beta_j) &\leqslant \limsup_{i\in I_\alpha}\alpha_i + \limsup_{j\in I_\beta}\beta_j,\\
\liminf_{i\in I_\alpha,j\in I_\beta}(\alpha_i+\beta_j) &\leqslant \liminf_{i\in I_\alpha}\alpha_i + \liminf_{j\in I_\beta}\beta_j.
\end{aligned}$$

The *proof* is straightforward. ☐

Now define

$n_{\vee\vee}, n_{\vee\wedge}$

$n_{\wedge\vee}, n_{\wedge\wedge}$

$$\begin{aligned} n_{\vee\vee}(\mathsf{x},\mathsf{y}) &:= \limsup\nolimits_{(i,j)\in I_{\mathsf{x}}\times I_{\mathsf{y}}} d(x_i,y_j),\\ n_{\vee\wedge}(\mathsf{x},\mathsf{y}) &:= \limsup\nolimits_{i\in I_{\mathsf{x}}} \liminf\nolimits_{j\in I_{\mathsf{y}}} d(x_i,y_j),\\ n_{\wedge\vee}(\mathsf{x},\mathsf{y}) &:= \liminf\nolimits_{i\in I_{\mathsf{x}}} \limsup\nolimits_{j\in I_{\mathsf{y}}} d(x_i,y_j),\\ \text{and}\quad n_{\wedge\wedge}(\mathsf{x},\mathsf{y}) &:= \liminf\nolimits_{(i,j)\in I_{\mathsf{x}}\times I_{\mathsf{y}}} d(x_i,y_j). \end{aligned}$$

Whether these functions satisfy the requirements for a distance function mainly depends on the fundamentality properties of the nets one applies them to. This is not very surprising because the definitions of most of the fundamentality properties involve $\exists$- resp. $\forall$-quantifications over ends of nets, and such quantifications are strongly related to inequalities of the kind $\alpha \leqslant \bigvee B$ and $\alpha \geqslant \bigwedge B$, respectively $\alpha \leqslant \bigwedge B$ and $\alpha \geqslant \bigvee B$.

LEMMA 5.12. *Let $\underline{M}$ be a co-quantale and Y a set of nets on some $\underline{M}$-distance space $\underline{X}$. Then, for $a,b,x,y \in X$ and $\mathsf{x},\mathsf{y} \in Y$:*

a) $n_{\wedge\wedge} \leqslant n_{\wedge\vee} \leqslant n_{\vee\vee}$, $n_{\wedge\wedge}(\mathsf{x},\dot{y}) = n_{\wedge\vee}(\mathsf{x},\dot{y})$, $n_{\vee\vee}(\mathsf{x},\dot{y}) = n_{\vee\wedge}(\mathsf{x},\dot{y})$,
$n_{\wedge\wedge} \leqslant n_{\vee\wedge} \leqslant n_{\vee\vee}$, $n_{\wedge\wedge}(\dot{x},\mathsf{y}) = n_{\vee\wedge}(\dot{x},\mathsf{y})$, *and* $n_{\vee\vee}(\dot{x},\mathsf{y}) = n_{\wedge\vee}(\dot{x},\mathsf{y})$.

b) $n_{\vee\vee}$ *always fulfils the triangle inequality, while* $n_{\vee\wedge}$, $n_{\wedge\vee}$, *and* $n_{\wedge\wedge}$ *at least satisfy it for each triple whose second element is* $\ell\forall\exists\exists\forall$-, $\ell\exists\forall\forall\exists$-, *or bi-fundamental, respectively.*

c) Always $n_{\vee\vee}(x,x) \geqslant 0$. *If* x *is bi-near* y *then* $n_{\vee\vee}(\mathsf{x},\mathsf{y}) \leqslant 0$.
Thus, if x *is bi-fundamental,* $n_{\vee\vee}(\mathsf{x},\mathsf{x}) = 0$.

d) Always $n_{\wedge\wedge}(\mathsf{x},\mathsf{x}) \leqslant 0$. $n_{\vee\wedge}(\mathsf{x},\mathsf{x}) \leqslant 0$ *[resp.* $n_{\wedge\vee} \leqslant 0$*] if* x *is* $\ell\exists\forall\forall\exists$- *[resp.* $\ell\forall\exists\exists\forall$-*] fundamental.*

e) If $D = \uparrow 0$ *then* $n_{\vee\vee}(\mathsf{x},\mathsf{y}) \leqslant 0$ *implies that* x *is bi-near* y *and* $n_{\vee\vee}(\dot{x},\mathsf{y}) \leqslant 0$ *implies that* $x \leftarrow \mathsf{y}$.

f) If $D = \twoheaduparrow 0$ *then* $n_{\wedge\wedge}(\dot{x},\mathsf{y}) \leqslant 0$ *implies that* $x \prec \mathsf{y}$.

g)
$$\begin{aligned} \mathsf{x} \longrightarrow a \textit{ and } b \longleftarrow \mathsf{y} &\implies n_{\vee\vee}(\mathsf{x},\mathsf{y}) \leqslant d(a,b),\\ \mathsf{x} \succ a \textit{ and } b \prec \mathsf{y} &\implies n_{\wedge\wedge}(\mathsf{x},\mathsf{y}) \leqslant d(a,b). \end{aligned}$$

h)
$$\begin{aligned} a \prec \mathsf{x} \textit{ and } \mathsf{y} \succ b &\implies n_{\vee\vee}(\mathsf{x},\mathsf{y}) \geqslant d(a,b),\\ a \longleftarrow \mathsf{x} \textit{ and } \mathsf{y} \longrightarrow b &\implies n_{\wedge\wedge}(\mathsf{x},\mathsf{y}) \geqslant d(a,b). \end{aligned}$$

Proof. a), c), and d) are straightforward. As for b),

$$\begin{aligned}
&n_{\vee\vee}(\mathsf{x},\mathsf{y})+n_{\vee\vee}(\mathsf{y},\mathsf{z})\\
&\quad= \bigwedge\bigvee\cdots+\bigwedge\bigvee\cdots \overset{\substack{\text{lower}\\\text{distrib.}}}{=} \bigwedge\Big(\bigvee\cdots+\bigvee\cdots\Big)\\
&\quad\geqslant \bigwedge\Big\{\bigvee\{d(x_i,y_j)+d(y_{j'},z_l)\mid i\geqslant n,\, j\geqslant m,\, j'\geqslant m',\, l\geqslant k\}\\
&\qquad\qquad\qquad\qquad\mid n\in I_{\mathsf{x}},\, m,m'\in I_{\mathsf{y}},\, k\in I_{\mathsf{z}}\Big\}\\
&\quad\geqslant \bigwedge\Big\{\bigvee\{d(x_i,y_j)+d(y_j,z_l)\mid i\geqslant n,\, j\geqslant m,\, j\geqslant m',\, l\geqslant k\}\\
&\qquad\qquad\qquad\qquad\mid n\in I_{\mathsf{x}},\, m,m'\in I_{\mathsf{y}},\, k\in I_{\mathsf{z}}\Big\}\\
&\quad\overset{\substack{I_{\mathsf{y}}\text{ is}\\\text{directed}}}{=} \bigwedge\Big\{\bigvee\{d(x_i,y_j)+d(y_j,z_l)\mid i\geqslant n,\, j\geqslant m,\, l\geqslant k\}\\
&\qquad\qquad\qquad\qquad\mid n\in I_{\mathsf{x}},\, m\in I_{\mathsf{y}},\, k\in I_{\mathsf{z}}\Big\}\\
&\quad\geqslant \bigwedge\Big\{\bigvee\{d(x_i,z_l)\mid i\geqslant n,\, l\geqslant k\}\mid n\in I_{\mathsf{x}},\, k\in I_{\mathsf{z}}\Big\} = n_{\vee\vee}(\mathsf{y},\mathsf{z}).
\end{aligned}$$

Moreover, if y is bi-fundamental, then for all $\varepsilon\in D$ there is $m_\varepsilon\in I_{\mathsf{y}}$ such that

$$\begin{aligned}
&n_{\wedge\wedge}(\mathsf{x},\mathsf{y})+\varepsilon+n_{\wedge\wedge}(\mathsf{y},\mathsf{z})\\
&\quad= \bigvee\bigwedge\cdots+\varepsilon+\bigvee\bigwedge\cdots \geqslant \bigvee\Big(\bigwedge\cdots+\varepsilon+\bigwedge\cdots\Big)\\
&\quad\overset{\substack{\text{lower}\\\text{distrib.}}}{=} \bigvee\Big\{\bigwedge\{d(x_i,y_j)+\varepsilon+d(y_{j'},z_l)\mid i\geqslant n,\, j\geqslant m,\, j'\geqslant m',\, l\geqslant k\}\\
&\qquad\qquad\qquad\qquad\mid n\in I_{\mathsf{x}},\, m,m'\in I_{\mathsf{y}},\, k\in I_{\mathsf{z}}\Big\}\\
&\quad\geqslant \bigvee\Big\{\bigwedge\{d(x_i,y_j)+d(y_j,y_{j'})+d(y_{j'},z_l)\mid i\geqslant n,\, j,j'\geqslant m_\varepsilon,\, l\geqslant k\}\\
&\qquad\qquad\qquad\qquad\mid n\in I_{\mathsf{x}},\, k\in I_{\mathsf{z}}\Big\}\\
&\quad\geqslant \bigvee\Big\{\bigwedge\{d(x_i,z_l)\mid i\geqslant n,\, l\geqslant k\}\mid n\in I_{\mathsf{x}},\, k\in I_{\mathsf{z}}\Big\} = n_{\wedge\wedge}(\mathsf{y},\mathsf{z}),
\end{aligned}$$

hence $n_{\wedge\wedge}(\mathsf{x},\mathsf{y})+n_{\wedge\wedge}(\mathsf{y},\mathsf{z})\geqslant n_{\wedge\wedge}(\mathsf{y},\mathsf{z})$ because of lower distributivity. The remaining two cases are similar.

e) If x is not bi-near y, there is some $\varepsilon\in D$ such that none of the suprema in $\limsup_{(i,j)\in I_{\mathsf{x}}\times I_{\mathsf{y}}} d(x_i,y_j)$ is at most ε. Since $I_{\mathsf{x}}\times I_{\mathsf{y}}$ is directed, the infimum of all those suprema is a filtered infimum, hence $n_{\vee\vee}(\mathsf{x},\mathsf{y})\not\leqslant 0$ whenever $\varepsilon\gg 0$. The other implication follows with a).

f) Similarly, $n_{\wedge\wedge}(\mathsf{x},\dot{y})\leqslant 0$ resp. $n_{\wedge\wedge}(\dot{x},\mathsf{y})\leqslant 0$ implies that for all $\varepsilon\ggg 0$ and all n there is $i\geqslant n$ such that $d(x_i,y)\leqslant\varepsilon$ resp. $d(x,y_i)\leqslant\varepsilon$.

g) If for each $\varepsilon\in D$ there is $(n,m)\in I_{\mathsf{x}}\times I_{\mathsf{y}}$ such that

$$d(x_i,y_j)\leqslant d(x_i,a)+d(a,b)+d(b,y_j)\leqslant\varepsilon+d(a,b)+\varepsilon\qquad(\star)$$

for all $(i,j)\geqslant(n,m)$, then $n_{\vee\vee}(\mathsf{x},\mathsf{y})\leqslant\bigwedge(D+d(a,b)+D)\overset{\substack{\text{lower}\\\text{distrib.}}}{=}d(a,b)$. If, on the other hand, for all $\varepsilon\in D$ and $(n,m)\in I_{\mathsf{x}}\times I_{\mathsf{y}}$ there is $(i,j)\geqslant(n,m)$ such that $(\star)$ holds, then $n_{\wedge\wedge}(\mathsf{x},\mathsf{y})\leqslant\bigwedge(D+d(a,b)+D)=d(a,b)$.

h) If for all $\varepsilon \in D$ and $(n,m) \in I_\mathsf{x} \times I_\mathsf{y}$ there is $(i,j) \geqslant (n,m)$ such that $d(a,b) \leqslant d(a,x_i) + d(x_i,y_j) + d(y_j,b) \leqslant \varepsilon + d(x_i,y_j) + \varepsilon$ (†), then

$$d(a,b) \leqslant \limsup_{i \in I_\mathsf{x},\, j \in I_\mathsf{y}} (\varepsilon + d(x_i,y_j) + \varepsilon) \leqslant \varepsilon + n_{\vee\vee}(\mathsf{x},\mathsf{y}) + \varepsilon$$

for all $\varepsilon \in D$, that is, $d(a,b) \leqslant n_{\vee\vee}(\mathsf{x},\mathsf{y})$. If, on the other hand, for each $\varepsilon \in D$ there is $(n,m) \in I_\mathsf{x} \times I_\mathsf{y}$ such that (†) holds for all $(i,j) \geqslant (n,m)$, then

$$d(a,b) \leqslant \liminf_{i \in I_\mathsf{x},\, j \in I_\mathsf{y}} (\varepsilon + d(x_i,y_j) + \varepsilon) \leqslant \varepsilon + n_{\wedge\wedge}(\mathsf{x},\mathsf{y}) + \varepsilon$$

for all $\varepsilon \in D$, that is, $d(a,b) \leqslant n_{\wedge\wedge}(\mathsf{x},\mathsf{y})$. □

Some completions

A first application of the above is a construction of [T$_0$] bi-completions for [T$_0$] distance spaces. In contrast to Flagg's [Fla92] bi-completion for continuity spaces, the present construction seems far more straightforward and does neither require complete distributivity nor a zero-filter which equals the long-way-above set of 0. The intuitive idea is to take all bi-fundamental nets as the points of the completion space, but since this system of nets always is a proper class, we restrict our choice to canonical nets.

Canonical nets. A net x on a set X will be called *canonical* if and only if each index is a pair (A,a) with $a \in A \subseteq X$, and I_x is quasi-ordered by $(A,a) \leqslant (B,b) :\Longleftrightarrow A \supseteq B$. *canonical*

If we are not interested in the index set itself, but only in the end filter of the net, we can always assume that a net is canonical:

LEMMA 5.13. *For any net* x *on* X*, the set* $\tilde I_\mathsf{x} := \big\{(\{x_i \,|\, i \geqslant n\}, x_n) \,|\, n \in I_\mathsf{x}\big\}$ *is up-directed by* $(A,a) \leqslant (B,b) :\Longleftrightarrow A \supseteq B$*, and the canonical net* $\tilde{\mathsf{x}} := (a)_{(A,a) \in \tilde I_\mathsf{x}}$ *fulfils* $\mathscr{E}\tilde{\mathsf{x}} = \mathscr{E}\mathsf{x}$. $\tilde I_\mathsf{x}$ $\tilde{\mathsf{x}}$

The *proof* is immediate. □

In this sense, the system of all canonical nets on X is a set of representatives for the class of all nets on X.[1]

THEOREM 5.14. (Bi-completion). *Let* $\underline{X} = (X, d, \underline{M}, D)$ *be a distance space such that* $\underline{M}$ *is a co-quantale and* D *is an idempotent zero-filter with a base of elements* ε *for which* $\bigwedge_{\alpha \gg \varepsilon} = \varepsilon$*. Let* B *be the set of all canonical bi-fundamental nets on* $\underline{X}$*. Then* $\underline{B} := (B, n_{\vee\vee}, \underline{M}, D)$ *is a bi-complete distance space, and the map* $h : \underline{X} \to \underline{B}$*,* $x \mapsto \dot{x}$ *is a bi-dense exact isometric embedding.* B

---

[1] Although the index sets $\tilde I_\mathsf{x}$ are not partially ordered in general, there is a similar but slightly more complicated construction with partially ordered index sets.

Proof. By Lemma 5.12 b) and c), $n_{\vee\vee}$ is a distance function on B. The map h is well-defined since constant nets are always bi-fundamental and principal nets are canonical, and h is an exact isometric embedding since $n_{\vee\vee}(\dot{x},\dot{y}) = d(x,y)$.

$h\mathsf{x}$ If $\mathsf{x} \in B$ is a bi-fundamental net on $\underline{X}$, the net

$$h\mathsf{x} := (\dot{x_i})_{i\in I_{\mathsf{x}}}$$

on $\underline{B}$ bi-converges to the element x of B ⟦For $\varepsilon \in D$ choose $n \in I_{\mathsf{x}}$ such that $d(x_i,x_j) \leqslant \varepsilon$ for all $i,j \geqslant n$. Then, taking the supremum over j resp. i, we find that $n_{\vee\vee}(\dot{x_i},\mathsf{x}) \leqslant \varepsilon$ and $n_{\vee\vee}(\mathsf{x},\dot{x_j}) \leqslant \varepsilon$ for all $i,j \geqslant n$⟧. Hence h is bi-dense.

$\mathsf{x},\mathsf{x}_i,x_{ij}$ As for bi-completeness, let $\mathsf{x} = (\mathsf{x}_i)_{i\in I}$ be a bi-fundamental net on $\underline{B}$ which consists of canonical bi-fundamental nets $\mathsf{x}_i = (x_{ij})_{j\in I_j}$ on $\underline{X}$. The $(K,\leqslant)$ set $K := D \times I$ becomes a directed set by putting $(\zeta,i) \geqslant (\delta,n) :\Longleftrightarrow \zeta \leqslant \delta$ $m(\zeta,i)$ and $i \geqslant n$. For each index $(\zeta,i) \in K$, choose some $m(\zeta,i) \in J_i$ such that $y_{\zeta i},\mathsf{y}$ $d(x_{ij},x_{ij'}) \leqslant \zeta$ for all $j,j' \in J_i$ with $j,j' \geqslant m(\zeta,i)$, and put $y_{\zeta i} := x_{i,m(\zeta,i)}$. Then $\mathsf{y} := (y_{\zeta i})_{(\zeta,i)\in K}$ is a bi-fundamental net on $\underline{X}$ ⟦For $\varepsilon \in D$, choose $\delta \in D$ with $3\delta \leqslant \varepsilon$ and $\bigwedge_{\alpha\gg\delta} = \delta$, and choose $n \in I$ with $n_{\vee\vee}(\mathsf{x}_i,\mathsf{x}_{i'}) \leqslant \delta$ for all $i,i' \geqslant n$. Let $(\zeta,i),(\zeta',i') \in K$ with $(\zeta,i),(\zeta',i') \geqslant (\delta,n)$. We will see that $d(y_{\zeta i},y_{\zeta' i'}) \leqslant \varepsilon$. Since in particular $i,i' \geqslant n$, the directed infimum $n_{\vee\vee}(\mathsf{x}_i,\mathsf{x}_{i'})$ is at most δ. Hence, for each $\alpha \gg \delta$, there are $k \in J_i$ and $k' \in J_{i'}$ so that $d(x_{ij},x_{i'j'}) \leqslant \alpha$ for all $j \geqslant k$ and $j' \geqslant k'$. Because of directedness, there are indices $j \in J_i$ and $j' \in J_{i'}$ with $k,m(\zeta,i) \leqslant j$ and $k',m(\zeta',i') \leqslant j'$, hence

$$\begin{aligned} d(y_{\zeta i},y_{\zeta' i'}) &= d(x_{i,m(\zeta,i)},x_{i',m(\zeta',i')}) \\ &\leqslant d(x_{i,m(\zeta,i)},x_{ij}) + d(x_{ij},x_{i'j'}) + d(x_{i'j'},x_{i',m(\zeta',i')}) \\ &\leqslant \zeta + \alpha + \zeta \leqslant \delta + \alpha + \delta \end{aligned}$$

for all $\alpha \gg \delta$. Consequently, $d(y_{\zeta i},y_{\zeta' i'}) \leqslant \bigwedge_{\alpha\gg\delta}(\delta+\alpha+\delta) = 3\delta \leqslant \varepsilon$ because of lower distributivity⟧.

By Lemmata 5.1 and 5.13, also $\tilde{\mathsf{y}}$ is bi-fundamental, in particular, $\tilde{\mathsf{y}} \in B$. Moreover, the net x on $\underline{B}$ bi-converges to the element $\tilde{\mathsf{y}}$ of $\underline{B}$ ⟦For $\varepsilon \in D$, choose $\delta \in D$ and $n \in I$ as above, let $i \geqslant n$, and choose $s \in J_i$ with $d(x_{ir},x_{ij}) \leqslant \delta$ for all $r,j \geqslant s$. Then $n_{\vee\vee}(\mathsf{x}_i,\tilde{\mathsf{y}}) \leqslant \bigvee_{r\geqslant s}\bigvee_{(\zeta,i')\geqslant(\delta,n)} d(x_{ir},x_{i',m(\zeta,i')})$, and we will see that each of the latter distances is at most ε. Let $r \geqslant s$ and $(\zeta,i') \geqslant (\delta,n)$. Since in particular $i,i' \geqslant n$, the directed infimum $n_{\vee\vee}(\mathsf{x}_i,\mathsf{x}_{i'})$ is at most δ. Hence for each $\alpha \gg \delta$, there are again $k \in J_i$ and $k' \in J_{i'}$ as above. Now choose $j \in J_i$ and $j' \in J_{i'}$ with $k,s \leqslant j$ and $k',m(\zeta,i') \leqslant j'$, so that

$$\begin{aligned} &d(x_{ir},x_{i',m(\zeta,i')}) \\ &\quad\leqslant d(x_{ir},x_{ij}) + d(x_{ij},x_{i'j'}) + d(x_{i'j'},x_{i',m(\zeta,i')}) \\ &\quad\leqslant \delta + \alpha + \zeta \leqslant \delta + \alpha + \delta \end{aligned}$$

and finally $d(x_{ir}, x_{i',m(\zeta,i')}) \leqslant 3\delta \leqslant \varepsilon$ by lower distributivity$]\!]$. Therefore, $\mathsf{x} \longrightarrow \tilde{\mathsf{y}}$, and the proof of $\tilde{\mathsf{y}} \longleftarrow \mathsf{x}$ works dually. $\boxed{\text{AC}}$

Note that the above bi-completion is only T_0 when $\underline{X}$ is a singleton or empty $[\![$since the canonical bi-fundamental nets $\dot{x}$ and $\{(\{x,y\},y),(\{x\},x)\}$ are different for $x \neq y$ but have zero distance in $\underline{B}]\!]$. Since for a bi-completion of a T_0 space, one usually again requires the T_0 property, one needs to supplement the above construction with a T_0 *reflection* in that case. In this standard construction, *T_0 reflection* the set Y of T_0 classes of a distance space $(B,e,\underline{M},D)$ with partially ordered $\underline{M}$ is endowed with the inherited distance $e'(A,B) := e(a,b)$ for $(a,b) \in A \times B$, and thus becomes a T_0 distance space $(Y,e',\underline{M},D)$.

The following example shows that, in contrast, a T_1 distance space need not in general possess a T_1 bi-completion.

EXAMPLE 5.15. Let $X := (0,1] \cup \{a,b\}$ be the extension of the half-open real interval $(0,1]$ by two distinct points a,b, and define a real distance function d on X by $d(x,y) := |x-y|$, $d(x,a) := d(x,b) := x$, and $d(a,y) := d(b,y) := d(a,b) := d(b,a) := \infty$ for all $x,y \in (0,1]$. Then $\underline{X}$ is T_1 but no T_1 extension of X is (bi, $\longleftarrow$)-complete $[\![$The bi-fundamental net $(1/n)_{n\in\mathbb{N}}$ which converges to a and b cannot have a dual limit c in some T_1 extension of X, because otherwise a, b, and c would have had to coincide$]\!]$.

Non-symmetric completions. We now turn to the case of weaker fundamentality properties and see whether we can use a similar completion procedure here, too. Since any bi-convergent net is bi-fundamental, we will not be able to produce bi-convergence for weaker kinds of fundamental nets, but only convergence or dual convergence. Here the situation is as for quasi-uniform spaces: if one adjoins a new point a to $\underline{X}$ such that $d(x,a) := 0$ and $d(a,x) := \top$ for all $x \in X$, one always gets a dense embedding of $\underline{X}$ into a $\underline{M}^{\top}$-distance space $\underline{X}'$ in which *all* nets are convergent to a. Moreover, if $\underline{X}$ is T_0, then so is $\underline{X}'$ (while it is never T_1).

Therefore, only the case of T_1 completions is interesting. We concentrate our considerations to (C, $\longrightarrow$)- and (C, $\succ\!\!-$)-completeness, where C is some net selection included in a certain class of fundamental nets. Now we can build a “universal” completion space from all principal nets and all canonical $\ell\forall\exists\exists\forall$- resp. $\ell\forall\exists\forall\exists$-fundamental nets that have no dual limit resp. no dual cluster point, and use a limes superior or inferior to define the distances.

If $\underline{X}$ can be isometrically embedded into $\underline{X}'$, the latter is called an *extension* *extension* of $\underline{X}$.

LEMMA 5.16. *Assume that* $\underline{X} = (X, d, \underline{M}, D)$ *is a* T_1 *distance space such that* $\underline{M}$ *is a co-quantale, and put* $h(x) := \dot{x}$ *for all* $x \in X$.

Let Z *be the set of all canonical* $\ell\forall\exists\exists\forall$*-fundamental nets on* $\underline{X}$ *that have no dual limit, and put* $Y := Z \cup h[X]$.

Similarly, let V *be the set of all canonical* $\ell\forall\exists\forall\exists$*-fundamental nets on* $\underline{X}$ *that have no dual cluster point, and put* $W := V \cup h[X]$.

Moreover, define two distance functions

$$d_Y(\mathsf{x},\mathsf{y}) := \begin{cases} 0 & \textit{if } \mathsf{x} \in Z \textit{ and } \mathsf{y} = \mathsf{x} \\ \top & \textit{if } \mathsf{x} \in Z \textit{ and } \mathsf{y} \neq \mathsf{x} \\ \limsup_{j \in I_{\mathsf{y}}} d(x, y_j) & \textit{if } \mathsf{x} = \dot{x} \textit{ and } \mathsf{y} \in Z \\ d(x,y) & \textit{if } \mathsf{x} = \dot{x} \textit{ and } \mathsf{y} = \dot{y} \end{cases}$$

$$\textit{and} \quad d_W(\mathsf{x},\mathsf{y}) := \begin{cases} 0 & \textit{if } \mathsf{x} \in V \textit{ and } \mathsf{y} = \mathsf{x} \\ \top & \textit{if } \mathsf{x} \in V \textit{ and } \mathsf{y} \neq \mathsf{x} \\ \liminf_{j \in I_{\mathsf{y}}} d(x, y_j) & \textit{if } \mathsf{x} = \dot{x} \textit{ and } \mathsf{y} \in V \\ d(x,y) & \textit{if } \mathsf{x} = \dot{x} \textit{ and } \mathsf{y} = \dot{y}. \end{cases}$$

$\underline{Y}$ *If* $D = \uparrow 0$ *then* $\underline{Y} := (Y, d_Y, \underline{M}, D)$ *is a* T_1 *distance space into which* $\underline{X}$ *is densely isometrically embedded by* h.

$\underline{W}$ *If* $D = ↟ 0$ *then* $\underline{W} := (W, d_W, \underline{M}, D)$ *is a* T_1 *distance space into which* $\underline{X}$ *is densely isometrically embedded by* h.

Proof. It is easily verified that d_Y and d_W are distance functions since $\underline{M}$ is lower distributive. Under the given conditions, they are also T_1 $[\![$Let $\mathsf{x} \neq \mathsf{y}$. If both of them are in $h[X]$ then $d_{Y[W]}(\mathsf{x},\mathsf{y}) \nleqslant 0$ because $\underline{X}$ is T_1; if $\mathsf{x} \notin h[X]$ then $d_{Y[W]}(\mathsf{x},\mathsf{y}) = \top \nleqslant 0$; if $\mathsf{y} \notin h[X]$ and $\mathsf{x} = \dot{x}$ then $d_Y(\mathsf{x},\mathsf{y}) = n_{\vee\vee}(\dot{x}, y) \nleqslant 0$ [resp. $d_W(\mathsf{x},\mathsf{y}) = n_{\wedge\wedge}(\dot{x}, y) \nleqslant 0$] since $x \not\leftarrow \mathsf{y}$ [resp. $x \not\prec \mathsf{y}$] by Lemma 5.12 e) [f)]$]\!]$.

Finally, h is dense $[\![$Let $\mathsf{y} \in Y$ [resp. W]. For each $\varepsilon \in D$, choose some arbitrary $n \in I_{\mathsf{y}}$, and some $i(\varepsilon) \geqslant n$ corresponding to the definition of $\langle \ell\forall\exists\exists\forall \rangle$ [resp. $\langle \ell\forall\exists\forall\exists \rangle$]. Then define a net $\mathsf{x} := (x_\varepsilon)_{\varepsilon \in D}$ on $\underline{X}$, where D is up-directed by the dual order of $\underline{M}$, by setting $x_\varepsilon := y_{i(\varepsilon)}$. Then for all $\varepsilon \in D$ there is $m \in I_{\mathsf{y}}$ such that $d(y_{i(\varepsilon)}, y_j) \leqslant \varepsilon$ for all $j \geqslant m$ [resp. $\forall \varepsilon\, \forall m\, \exists j \geqslant m$: $d(y_{i(\varepsilon)}, y_j) \leqslant \varepsilon$]. The latter implies $d_Y(\dot{x}_\varepsilon, \mathsf{y}) \leqslant \varepsilon$ [resp. $d_W(\dot{x}_\varepsilon, \mathsf{y}) \leqslant \varepsilon$], hence $h\mathsf{x} \to \mathsf{y}]\!]$. $\boxed{\text{AC}}$

LEMMA 5.17. *Let* $\underline{X}$, $\underline{Y}$, *and* $\underline{W}$ *be as in Lemma 5.16, and* x *a canonical net on* $\underline{X}$.

If x *is* $\ell\forall\exists\exists\forall$*-fundamental but has no dual limit then* $h\mathsf{x} \succ \mathsf{x}$ *in* $\underline{Y}$.

If x *is* $\ell\exists\forall\exists\forall$*-fundamental but has no dual limit then* $h\mathsf{x} \to \mathsf{x}$ *in* $\underline{Y}$.

If x *is* $\ell\forall\exists\forall\exists$*-fundamental but has no dual cluster point then* $h\mathsf{x} \succ \mathsf{x}$ *in* $\underline{W}$.

If x *is* $\ell\exists\forall\forall\exists$*-fundamental but has no dual cluster point then* $h\mathsf{x} \to \mathsf{x}$ *in* $\underline{W}$.

The *proof* is straightforward. ☐

LEMMA 5.18. *Let* $\underline{X} = (X, d, D)$ *be a distance space with idempotent* D *and some* T_1 *extension* $\underline{X}' = (X', e, E)$.

A net on $\underline{X}$ *with a limit in* $\underline{X}'$ *and a dual cluster point in* $\underline{X}$ *has also a limit in* $\underline{X}$.

A net on $\underline{X}$ *with a cluster point in* $\underline{X}'$ *and a dual limit in* $\underline{X}$ *has also a cluster point in* $\underline{X}$.

Proof. Let $h : \underline{X} \to \underline{X}'$ be the embedding. If $z \prec \mathsf{x} \to y$ or $z \leftarrow \mathsf{x} \succ y$ for some $z \in X$ and $y \in X'$, then $d(z,y) \leqslant 0$, hence $eh(z,y) \leqslant 0$. Now $h(y) = h(z) \in X$ since $\underline{X}'$ is T_1, and $y = z$ since h is injective. □

T_1 $(C,\dots)$-completion

By a *T_1 $(C,\dots)$-completion* of $\underline{X}$ I mean a T_1 $(C,\dots)$-complete distance space into which $\underline{X}$ can be densely isometrically embedded. Now we are ready to prove a whole class of completion results at once, since whenever $\underline{X}$ has a T_1 completion of the desired kind at all, one of the spaces $\underline{Y}$ and $\underline{W}$ is such a completion.

THEOREM 5.19. (Non-symmetric T_1 completions). *Let* $\underline{X} = (X, d, \underline{M}, D)$ *be a* T_1 *distance space such that* $\underline{M}$ *is a co-quantale and* D *is idempotent. Define* $\underline{Y}$ *and* $\underline{W}$ *as in 5.16, and let* C *be some net selection.*

If C *is included in '*$\ell\forall\exists\exists\forall$*-fundamental',* $D = \uparrow 0$*, and* $\underline{X}$ *has a* T_1 *(*C*,* $\succ$*)-complete extension then* $\underline{Y}$ *is a* T_1 *(*C*,* $\succ$*)-completion of* $\underline{X}$.

If C *is included in '*$\ell\exists\forall\exists\forall$*-fundamental',* $D = \uparrow 0$*, and* $\underline{X}$ *has a* T_1 *(*C*,* $\to$*)-complete extension then* $\underline{Y}$ *is a* T_1 *(*C*,* $\to$*)-completion of* $\underline{X}$.

If C *is included in '*$\ell\exists\forall\forall\exists$*-fundamental',* $D = ↟0$*, and* $\underline{X}$ *has a* T_1 *(*C*,* $\to$*)-complete extension then* $\underline{W}$ *is a* T_1 *(*C*,* $\to$*)-completion of* $\underline{X}$.

Proof. Let y be a C-net on $\underline{Y}$ which is not eventually constant.

In the first case, for all $\varepsilon \in D \setminus \{\top\}$ and $n \in I_{\mathsf{y}}$, the index $i \geqslant n$ whose existence is stated in the definition of $\langle \ell\forall\exists\exists\forall \rangle$ is such that $y_i \in h[X]$ $\llbracket$ otherwise $y_i \in Z$, and there is $m \in I_{\mathsf{y}}$ with $d_Y(y_i, y_j) \leqslant \varepsilon < \top$ and thus $y_j = y_i$ for all $j \geqslant m \rrbracket$. The co-restriction y' of the net y to the set $h[X]$ is thus again a net, say $\mathsf{y}' = h\mathsf{x}$ for some net x on $\underline{X}$, and both x and $\tilde{\mathsf{x}}$ are again $\ell\forall\exists\exists\forall$-fundamental. If x has no dual limit in $\underline{X}$, $\tilde{\mathsf{x}}$ is a cluster point of $h\mathsf{x}$ by Lemma 5.17. If, on the other hand, x has a dual limit in $\underline{X}$, Lemma 5.18 implies that it has also a cluster point in $\underline{X}$ since it has one in some T_1 $(C, \succ)$-complete extension of $\underline{X}$. Anyway, $\mathsf{y}' = h\mathsf{x}$ has a cluster point in $\underline{Y}$. Since $I_{\mathsf{y}'}$ is co-final in I_{y}, also y must have one.

In the second case, for all $\varepsilon \in D \setminus \{\top\}$, the index $n \in I_{\mathsf{y}}$ whose existence is stated in the definition of $\langle \ell\exists\forall\exists\forall \rangle$ is such that $y_i \in h[X]$ for all $i \geqslant n$ $\llbracket$ for the same reason as above $\rrbracket$. Defined as before, x and $\tilde{\mathsf{x}}$ are even $\ell\exists\forall\forall\exists$-fundamental this time, and $\mathsf{y}' := h\mathsf{x}$ now has a limit in $\underline{Y}$. Since $\mathscr{E}\mathsf{y}' = \mathscr{E}\mathsf{y}$, also y has a limit in $\underline{Y}$.

The third case can be proved with completely analogous arguments. □

6.

FIXED POINTS

> *While his parents are alive, the son may not go abroad to a distance. If he does go abroad, he must have a fixed point to which he goes.*
>
> *Confucius, Analects*

Banach: Fixed points of Lipschitz-continuous maps

Using the following generalization of Lipschitz-continuity, Banach's important but easily proved fixed point theorem can be reformulated for general distance spaces. For a filtered monoid $(\underline{M}, D)$, a map $L : \underline{M} \to \underline{M}$ will be called *left* or *right contractive* if for all $\alpha \in M$ and $\delta \in D$, there is $n \in \omega$ such that

right contractive

$$L^i\alpha + \cdots + L^{j-1}\alpha \leqslant \delta \quad \text{or} \quad L^{j-1}\alpha + \cdots + L^i\alpha \leqslant \delta \quad \text{for all } j \geqslant i \geqslant n,$$

respectively, where the sums are considered zero in case of $i = j$.

PROPOSITION 6.1. *Let* $\underline{X} = (X, d, \underline{M}, D)$ *be a nonempty* T_1 *distance space,* $L : \underline{M} \to \underline{M}$*, and* $f : \underline{X} \to \underline{X}^{\mathrm{op}} = (X, d^{\mathrm{op}}, D)$ *a continuous map with* $df \leqslant Ld$. *Assume that either*

(i) $\underline{X}$ *is sequentially* $(\ell\exists\forall\cdot\forall, \to)$*-complete and* L *is left contractive, or*

(ii) $\underline{X}$ *is sequentially* $(r\exists\forall\cdot\forall, \to)$*-complete and* L *is right contractive, or*

(iii) $\underline{X}$ *is sequentially* $(bi, \to)$*-complete and* L *is both left and right contractive.*

Then f *has a unique fixed point* $x \in X$*, and* $x \leftarrow (f^n x_0)_n \to x$ *for every* $x_0 \in X$.

The *proof* is essentially the standard one. (i) For $x_0 \in X$, put $x_n := f^n(x_0)$ and $\alpha := d(x_0, x_1)$. Since for all $\delta \in D$, there is $n \in \omega$ with $\delta \geqslant L^i\alpha +$

$\cdots + L^{j-1}\alpha \geqslant d(x_i,x_j)$ for all $j \geqslant i \geqslant n$, the sequence $(x_n)_n$ is $\ell\exists\forall{\cdot}\forall$-fundamental. By sequential completeness, $(x_n)_n \to x$ for some $x \in X$, so that $f(x) \leftarrow \big(f(x_n)\big)_n = (x_n)_{n\geqslant 2}$ by continuity. Now $f(x) \leftarrow (x_n)_n \to x$, hence $d(f(x),x) \leqslant 0$ and $f(x) = x$ because of T_1. Finally, for $f(y) = y$, we have $d(x,y) = df^n(x,y) \leqslant L^n d(x,y)$ for all $n \in \omega$, hence $d(x,y) \leqslant \bigwedge D = 0$ and $x = y$ again because of T_1. Parts (ii) and (iii) are strictly analogous. ☐

Because its proof uses a similar argument, I include here the following observation:

EXAMPLE 6.2. *Possible worlds semantics with completeness.* When defined by means of the betweenness quasi-orders

$$y \leqslant_x z \iff xyyz\, R_d\, xz,$$

the counterfactual operator (!)

$$A \mathrel{\Box\!\!\rightarrow} B = \{x \in X \mid \forall w \in A\ \exists z \in A \cap B : z \leqslant_x\}$$

(compare to Example 2.27) fulfils the rule

$$A \mathrel{\Box\!\!\rightarrow} (B \cap C) \subseteq (A \mathrel{\Box\!\!\rightarrow} B) \cap (A \mathrel{\Box\!\!\rightarrow} C)$$

but *not* in general the intuitively also justified inference rule

$$(A \mathrel{\Box\!\!\rightarrow} B) \cap (A \mathrel{\Box\!\!\rightarrow} C) \subseteq A \mathrel{\Box\!\!\rightarrow} (B \cap C).$$

In the following case, however, it does: $d \geqslant 0$, $\underline{X}$ is sequentially $(r\exists\forall{\cdot}\forall, \leftrightarrow)$-complete, $\underline{M}$ is a commutative and *order-cancellative* (that is, $\alpha + \beta \leqslant \alpha + \gamma \Longrightarrow \beta \leqslant \gamma$) co-quantale, D is idempotent, $\alpha + \delta \gg \alpha$ holds for all $\alpha \in M$ and $\delta \in D$, and A, B, and C are sequentially $\mathscr{T}(d^s, D)$-closed.

order-cancellative

For the proof, assume that $x \in (A \mathrel{\Box\!\!\rightarrow} B) \cap (A \mathrel{\Box\!\!\rightarrow} C)$ and $w = z_0 \in A$, and define $(z_n)_n$ inductively: for odd or even $n \geqslant 1$, choose $z_n \in \overline{xz_{n-1}}^d$ so that $z \in B$ or $z \in C$, respectively. Now $(z_n)_n$ is $r\exists\forall{\cdot}\forall$-fundamental $\llbracket$Put $\alpha := \bigwedge_{n\in\omega} d(x,z_n)$. For $\delta \in D$, choose $n \in \omega$ with $d(x,z_j) \leqslant d(x,z_n) \leqslant \alpha + \delta$ for all $j \geqslant n$. Then $\alpha + d(z_i,z_j) \leqslant d(xz_iz_iz_j) = d(xz_j) \leqslant \alpha + \delta$ and thus $d(z_i,z_j) \leqslant \delta$ for all $i \geqslant j \geqslant n\rrbracket$. Then $z \leftarrow (z_n)_n \to z$ for some $z \in A \cap B \cap C$, and $z \in \overline{xz_0}^d$ $\llbracket$for all $\delta \in D$, there is $j \geqslant 0$ with $d(xzzz_0) \leqslant d(xz_j\, z_jz\, zz_j\, z_jz_0) \leqslant d(x,z_j) + 2\delta + d(z_j,z_0) = d(x,z_0) + 2\delta$, thus $d(xzzz_0) \leqslant d(x,z_0)\rrbracket$.

POINT-FREE GENERALIZATIONS

Let $\underline{M}$ be a q. o. m. with zero-filter D, and $K : M \to M$ a map. An $\alpha \in M$ will be called K-*contractible* if for all $\varepsilon \in D$ there is $n \in \omega$ such that $K^n\alpha \leqslant \varepsilon$.

contractible

For the following point-free "fixed point" results, let us adopt a very general standpoint. Assume that Q is a quoset with a unique smallest element 0, and $d : Q \to \underline{M}$ is an isotone function with $d(0) = 0$. For example, Q could be a frame and d a diameter on Q in the sense of Pultr [Pul84].

K-Lipschitz

A map $h : Q \to Q$ will be called *K-Lipschitz* if it preserves $\leqslant$ and 0, and for each $a \in Q$ there is $b \in Q$ with $h(b) \geqslant a$ and $d(b) \leqslant Kd(a)$. A filter $F \subseteq Q^\star$ is called a *Cauchy-filter* if and only if, for each $\varepsilon \in D$, it contains some element a with $d(a) \leqslant \varepsilon$.

Cauchy-filter

preserve down-directedness

Finally, a map between quosets is said to *preserve down-directedness* if and only if the image of each down-directed set is again down-directed.

LEMMA 6.3. *Let h be K-Lipschitz such that h^{-1} preserves down-directedness, and $a \in Q^\star$ such that $h(a) \geqslant a$. Then there is a descending sequence (a_n) in $Q^\star$ with $a_0 = a$, $h(a_{n+1}) \geqslant a_n \geqslant a_{n+1}$, and $d(a_n) \leqslant K^n d(a)$. If $d(a)$ is K-contractible, this sequence generates a Cauchy-filter F_a in $Q^\star$.*

Proof. Define (a_n) inductively: put $a_0 := a$ and, when a_n is already defined, choose b such that $d(b) \leqslant Kd(a_n)$ and $h(b) \geqslant a_n$. Since also $h(a_n) \geqslant a_n$, there is $a_{n+1} \leqslant a_n, b$ such that $h(a_{n+1}) \geqslant a_n$, in particular $a_{n+1} \neq 0$. Because d is isotone, $d(a_{n+1}) \leqslant d(b) \leqslant Kd(a_n)$. The Cauchy-property is obvious. $\boxed{\mathrm{cc}}$

Now assume that $\underline{M}$ is complete, and Q is a complete lattice L. Define K^ν inductively for all ordinals ν by $K^{\nu+1}\alpha := KK^\nu\alpha$, and $K^\lambda\alpha := \bigwedge_{\nu\in\lambda} K^\nu\alpha$ for limit ordinals λ. Now an $\alpha \in M$ is called *transfinitely K-contractible* if $K^\nu\alpha \leqslant 0$ for some ordinal ν. Note that simple K-contractibility implies $K^\omega\alpha \leqslant 0$. For a cardinal γ, an $\alpha \in L$ is [(γ,ω)-]co-compact if each down-directed set [of cardinality at most γ] whose infimum is $\leqslant \alpha$ already contains an element $\leqslant \alpha$. Usually, (ω,ω)-co-compactness is also called *countable co-compactness.*

transfinitely contractible

countably compact

LEMMA 6.4. *Let h be K-Lipchitz, and γ an infinite cardinal.*

a) Assume that $h(a) \geqslant a$ for some $a \in Q^\star$, and h preserves arbitrary infima. Then there is a descending transfinite sequence $(a_\nu)_{\nu\leqslant\gamma}$ in Q with $a_0 = a$, $h(a_\nu) \geqslant h(a_{\nu+1}) \geqslant a_\nu \geqslant a_{\nu+1}$, and $d(a_\nu) \leqslant K^\nu d(a)$.

b) If, additionally, $0 \in L$ is (γ,ω)-co-compact and $K^\gamma d(a) \leqslant 0$ then $a_\gamma \in Q^\star$ and $d(a_\gamma) = 0$.

Proof. a) For successors $\nu + 1$, choose b as in Lemma 6.3 and put $a_{\nu+1} := a_\nu \wedge b$. Note that $a_{\nu+1} \neq 0$ whenever $a_\nu \neq 0$. For limit ordinals, put $a_\lambda := \bigwedge_{\nu\in\lambda} a_\nu$. Since h preserves infima, $h(a_\lambda) = \bigwedge_{\nu\in\lambda} h(a_\nu) \geqslant a_\lambda$. Moreover, $d(a_\lambda) \leqslant \bigwedge_{\nu\in\lambda} d(a_\nu) \leqslant \bigwedge_{\nu\in\lambda} K^\nu d(a) = K^\lambda d(a)$.

b) In this case, also each d_λ with $\lambda \leqslant \gamma$ is non-zero $\llbracket$Inductively, it is an infimum of a chain of cardinality at most γ of non-zero elements and is therefore also non-zero because 0 is (γ,ω)-co-compact$]\!]$. Moreover, $0 \leqslant d(a_\gamma) \leqslant K^\gamma d(a) \leqslant 0$. $\boxed{\text{AC}}$

The above lemmata become fixed-point theorems when Q, d, and D are sufficiently well-behaved. Following [BP89], define

$\lhd$

$$b \lhd a :\Longleftrightarrow \exists \varepsilon \in D\, \forall c \in Q\ (c \text{ intersects } b \text{ and } d(c) \leqslant \varepsilon \Longrightarrow c \leqslant a),$$

and call a filter $F \subseteq Q^\star$ *regular* if and only if for each $a \in F$, there is $b \in F$ with $b \lhd a$.

regular

Now suppose that Q is a complete lattice L. Then d is a *star-prediameter* if and only if $d(a \vee b) \leqslant d(a) + d(b)$ whenever a, b intersect, and

star-prediameter

$$d(a \vee \bigvee A) \leqslant d(a) + \bigvee\{d(b) + d(c) \,|\, b, c \in A,\ b \neq c\}$$

whenever a intersects all $b \in A$. Finally, L has the *intersection property* if and only if a intersects some $b \in A$ whenever a intersects $\bigvee A$. Note that, in particular, frames have this property.

intersection property

LEMMA 6.5. *Assume that* L *is a complete lattice with the intersection property,* d *is a star-prediameter,* D *is idempotent, and* $F \subseteq L^\star$ *is a Cauchy-filter. Then*

$$\tilde{F} := \{a \in L \,|\, b \lhd a \textit{ for some } b \in F\} \subseteq F$$

is a regular Cauchy-filter.

Proof. $\tilde{F}$ is a filter $[\![$It is a non-void upper set since $b \lhd \bigvee L$, and $b' \leqslant b \lhd a \leqslant a' \Longrightarrow b' \lhd a'$ for all a, a', b, b'. It is also closed under infima: For $a, a' \in \tilde{F}$, choose $b, b' \in F$ with $b \lhd a$, $b' \lhd a'$. Choose $c \in F$ with $c \leqslant b, b'$. Since $c \lhd a, a'$, there is $\varepsilon \in D$ with $x \leqslant a \wedge a'$ for all x which intersect c and have $d(x) \leqslant \varepsilon$. Hence $a \wedge a' \in \tilde{F}]\!]$.

$\tilde{F}$ is Cauchy $[\![$For $\varepsilon \in D$, choose $\delta \in D$ with $3\delta \leqslant \varepsilon$, and $b \in F$ with $d(b) \leqslant \delta$. Put

$$a := \bigvee\{x \in L \,|\, x \text{ intersects } b,\ d(x) \leqslant \delta\}.$$

Then $b \lhd a$ and thus $a \in \tilde{F}$, and $d(a) \leqslant d(b) + 2\delta \leqslant \varepsilon$ because of the star property$]\!]$.

$\tilde{F}$ is regular $[\![$For $c \in \tilde{F}$, choose $b \in F$ with $b \lhd c$, and choose $\varepsilon \in D$ such that $x \leqslant c$ whenever x intersects b and $d(x) \leqslant \varepsilon$. Choose $\delta \in D$ with $2\delta \leqslant \varepsilon$ and define $a \in \tilde{F}$ with $b \lhd a$ as above. We finally see that $a \lhd c$: Whenever y intersects a with $d(y) \leqslant \delta$, then, because of the intersection property, y must intersect some x which intersects b with $d(x) \leqslant \delta$. Since then $d(x \vee y) \leqslant \delta + \delta \leqslant \varepsilon$ (because d is a prediameter), we have $x \vee y \leqslant c$, in particular $y \leqslant c]\!]$. $\square$

Note that still no distributivity of L is needed. When we follow Banaschewski and Pultr [BP96] in considering regular Cauchy-filters as generalized points, both Lemma 6.3 and Lemma 6.4 can now be interpreted as fixed point theorems: the above lemma implies that when Q, d, and D are well-behaved, the Cauchy-filter F_a of Lemma 6.3 resp. the filter $F_a := \uparrow a_\gamma$ of Lemma 6.4 leads to a regular Cauchy-filter $\tilde{F}_a$ in which every element intersects its image $[\![$for $b \in \tilde{F}_a$, there is $a_\nu \lhd b$, hence $h(b) \geqslant h(a_\nu) \geqslant a_\nu$, that is, b and $h(b)$ intersect$]\!]$.

Sine–Soardi: Fixed points of contractive maps

hyperconvex

In [Sin79] and [Soa79], R. Sine and P. M. Soardi independently showed that when a metric space has finite diameter and is *hyperconvex* (that is, an intersection of a family of closed balls $x_i B_{d,\alpha_i}$ is nonempty whenever $d(x_i,x_j) \leqslant \alpha_i + \alpha_j$ for all i,j), every contractive map has a fixed point. In [JMP86], Jahwari, Misane, and Pouzet generalized this result to positive distance sets in which $d(y,x) = \varphi d(x,y)$ for an isotone and dually additive function φ. In this section, I will weaken the requirement of hyperconvexity and drop the conditions on φ to obtain a slightly more general result.

weakly φ-hyperconvex

Let $\underline{X} = (X, d, \underline{M}, D)$ be a distance space so that $\underline{M}$ has a largest element $\top$, and $\varphi : M \to M$ an arbitrary map. Then $\underline{X}$ will be called *weakly φ-hyperconvex* if and only if the following conditions hold: (i) $d(y,x) = \varphi d(x,y)$ for all $x,y \in X$. (ii) For all $\varepsilon \in D$, all families $(x_i)_{i \in I}$ of elements of X, and all families $(\alpha_i)_{i \in I}$ of elements of $\underline{M}$ that fulfil $d(x_i,x_j) \leqslant \alpha_i + \varphi\alpha_j$ for all $i \in I$, there is some $x \in X$ with $d(x_i,x) \leqslant \varepsilon + \alpha_i + \varepsilon$ for all $i \in I$. In case of a symmetric distance function, φ could be the identity, for example.

φ-inaccessible

Moreover, an element $\alpha \in M$ is *φ-inaccessible* if and only if for all $\beta \in M$, $\alpha \leqslant \beta + \varphi\beta$ implies $\alpha \leqslant \beta$. In case that $\underline{M} = \underline{\mathbb{R}}^\top$ and $\varphi = id$, for instance, the φ-inaccessible elements are exactly all non-positive numbers and ∞.

Finally, let us call $\underline{X}$ *φ-bounded*, if and only if any φ-inaccessible element below the diameter of $\underline{X}$ is already below 0.

THEOREM 6.6. *Let $\underline{X}$ be nonempty, weakly φ-hyperconvex, and φ-bounded. Then for each contractive $f : \underline{X} \to \underline{X}$, there is a nonempty, weakly φ-hyperconvex, and closed subset $S \subseteq X$ with zero diameter and $f[S] \subseteq S$.*

The *proof* is almost verbatim as in [JMP86]. Let $\mathscr{B}$ be the system of all nonempty intersections S of balls $xB_{d,\alpha}$ with $x \in X$ and $\alpha \in M$, containing $X = xB_{d,\top}$. $\mathscr{B}$ contains the intersection S of every chain $\mathscr{C} \subseteq \mathscr{B}$ $[\![$Assume that $\mathscr{C} = \{S_i = \bigcap_{j \in J_i} x_{ij} B_{d,\alpha_{ij}} \mid i \in I\}$. Then for $i,i' \in I$, either every $x_{ij}B_{d,\alpha_{ij}}$ ($j \in J_i$) includes $S_{i'}$ or every $x_{i'j'}B_{d,\alpha_{i'j'}}$ ($j' \in J_{i'}$) includes

S_i. Thus $x_{ij}B_{d,\alpha_{ij}}$ and $x_{i'j'}B_{d,\alpha_{i'j'}}$ intersect for all i,i',j,j', in particular, $d(x_{ij},x_{i'j'}) \leqslant \alpha_{ij} + \varphi\alpha_{i'j'}$. Then $S \neq \emptyset$ by weak φ-hyperconvexity⟧.

Now Zorn's Lemma gives a minimal element $S = \bigcap_{k\in K} x_k B_{d,\alpha_k}$ of $\mathscr{B}$. In order to show that the diameter δ of S is φ-inaccessible, assume that $\delta \leqslant \beta + \varphi\beta$. Since for $a,b \in S$ and $k \in K$, we have $d(a,b) \leqslant \beta + \varphi\beta$, $d(a,x_k) \leqslant \varphi\alpha_k$, and $d(x_k,b) \leqslant \alpha_k$, weak φ-hyperconvexity implies that $T := S \cap \bigcap_{a\in S} aB_{d,\beta}$ is nonempty.

For all $y \in T$, we have $S \subseteq yB_{d,\varphi\beta}$, hence $f[S] \subseteq f(y)B_{d,\varphi\beta}$ since f is contractive. Putting $\overline{A} := \bigcap\{B \in \mathscr{B} \,|\, A \subseteq B\}$, this implies that $\overline{f[S]} \subseteq f(y)B_{d,\varphi\beta}$. On the other hand, $\overline{f[S]} = S$ ⟦$f[S] \subseteq S$ by definition, hence $\overline{f[S]} \subseteq S$. Since this implies $f[\overline{f[S]}] \subseteq \overline{f[S]}$, the latter is in $\mathscr{B}$ and must therefore equal S by minimality of S⟧. Now $S \subseteq f(y)B_{d,\varphi\beta}$, hence $f(y) \in T$. This shows that $f[T] \subseteq T$.

Since then $T \in \mathscr{B}$, it must equal S, too. Now, for $x,y \in S = T$, we have $d(x,y) \leqslant \beta$, hence $\delta \leqslant \beta$, which shows that δ is φ-inaccessible and thus zero by φ-boundedness. AC

Brouwer: Fixed points of continuous maps

One of the most famous results of 20th century mathematics is Brouwer's fixed point theorem: *every continuous self-map of* $[0,1]^n$ *has a fixed point.* It is somewhat surprising that Brouwer, who should become one of the most important representatives of constructivism, gave only a non-constructive proof of this result. Only later, a combinatorial lemma of Sperner led to a constructive method to approximate these fixed points.

The question of whether all continuous self-maps of a topological space X have a fixed point is of course a purely *topological* one, and it is therefore not surprising that the existence of a very rich additional structure like that of a normed vector space is not essential to it. Indeed, there are large classes of quite different topological spaces which also have this *continuous fixed point property (CFPP),* for example many lower set topologies of posets: *if* P *is a poset with least element* 0 *in which each chain has a supremum, then every isotone self-map of* P *has a fixed point.* The proof is very easy and in a sense constructive. The recursively defined ascending transfinite sequence $a_0 := 0$, $a_\nu := \bigvee_{\mu\in\nu} f(a_\mu)$ eventually becomes stationary at a fixed point of f ⟦By transfinite induction, $a_{\nu+1} = f(a_\nu) \geqslant \bigvee_{\mu\in\nu} f(a_\mu) \geqslant a_\nu$ since $a_1 = f(a_0) \geqslant 0 = a_0$. In particular, $(a_\nu)_\nu$ is increasing. Since $a_{\nu+1} > a_\nu$ cannot hold for more than $|P|$ many ordinals ν, the sequence becomes stationary at the latest for $\nu > |P|$. Thus eventually $a_\nu = a_{\nu+1} = f(a_\nu)$⟧.

continuous fixed point property

The special cases where P is a complete lattice or f preserves suprema of chains are known as Tarski's and Scott's fixed point theorems. It would be quite promising to know that these and Brouwer's theorem are instances of a much more general result on fixed points of continuous maps, and indeed there seems to be a step in the right direction: using a discrete algorithm by Scarf [Sca73] and Tuy [Tuy79], van Maaren [vM87, vM91] was able to replace the vector space structure by a finite set of total quasi-orders on X.

Not much is known about the continuous fixed point property in general. Only very few of the more familiar properties are implied by it: if X has the CFPP, it must be T_0 $[\![$otherwise take two indistinguishable points $x \neq y$, and map x to y and $X \setminus \{x\}$ to $x]\!]$ and connected $[\![$otherwise map a whole component C to a single point in another component and the rest to a single point in $C]\!]$, but not T_1 (as Tarski's and Scott's theorems show). Moreover, the following example shows that CFPP does not imply path-connectedness (or even convexity), metrical boundedness, completeness, absolute closedness, countable compactness, or pseudocompactness (see e. g. [Wil70] for definitions).

EXAMPLE 6.7. Define $x_0 := (0,0)$ and $x_r := (r, r^{-1} \sin \frac{1}{r})$ for $r > 0$, and consider the subspace $\underline{X}$ defined by $X := \{x_r \mid r \in [0,1]\}$ of the Euclidean space $\underline{\mathbb{E}}_2$. Then $\underline{X}$ has the CFPP but none of the before-mentioned properties.

For the *proof,* order X by putting $x_r \leqslant x_s :\Longleftrightarrow r \leqslant s$. Assume that f is a continuous self-map without a fixed point. Since then $f(x_0) > x_0$, continuity implies that there is also some $r > 0$ with $f(x_r) > x_r$. Hence $s := \bigvee\{r \in [0,1] \mid f(x_r) > x_r\} = \bigvee\{r \in (0,1] \mid f(x_r) > x_r\} \in (0,1]$. Note that $x_r \mapsto -1/r$ is an order-preserving homeomorphism between $\underline{X} \setminus \{x_0\}$ and $\underline{\mathbb{E}}_1|_{(-\infty,-1]}$. By continuity, $f(x_s) \geqslant x_s$. But then $f(x_s) > x_s$ so that, again by continuity, there must be some $r > s$ with $f(x_r) \geqslant x_r$—in contradiction to the choice of s.

On the other hand, one can easily see that $\underline{X}$ is neither path-connected, metrically bounded, closed, complete, countably compact, or pseudocompact.

The next theorem generalizes van Maaren's result to the non-separable case. Some additional notation will be useful:

$$\begin{aligned} C_K(x) &:= \{y \in X \mid y \leqslant_i x \text{ for all } i \in K\}, \\ C_K^0(x) &:= \{y \in X \mid y <_i x \text{ for all } i \in K\}. \end{aligned}$$

completely labelled primitive set

LEMMA 6.8. *[Tuy79] For each finite family $(\leqslant_i)_{i \in I}$ of total orders on a finite set F, and each map $\ell : F \longrightarrow I$, there is a* completely labelled primitive *set, that is, a subset $U \subseteq F$ with $U = \{\bigvee_i U \mid i \in \ell[U]\}$ for which there is no $x \in F$ with $x <_i \bigvee_i U$ for all $i \in \ell[U]$.*

THEOREM 6.9. *Let* (X,τ) *be a compact topological space, and* $(\leqslant_i)_{i\in I}$ *a finite family of total quasi-orders on* X *such that, for all* $i \in I$, $K \subseteq I$, *and* $x,y \in X$, *the following three conditions hold:*

(i) $x \leqslant_i y$ *for some* $i \in I$,

(ii) The principal strictly upper and lower sets $x{<_i}$ *and* ${<_i}x$ *are* τ*-open,*

(iii) If $C_K(x) \neq \{x\}$ *then* $C^0_K(x)$ *intersects each* $A \subseteq X$ *which contains* x *and which is either a principal strictly lower set or a finite intersection of principal strictly upper sets.*

Then each continuous self-map of (X,τ) *has a fixed point.*

Proof. Let f be the continuous self-map. Because of (i), we find for all $x \in X$ some $\ell(x) \in I$ with $f(x) \leqslant_{\ell(x)} x$. The system $\mathscr{P}_f X$ of all finite subsets of X is an up-directed poset under set inclusion. For each $i \in I$, choose some total order $\leqslant'_i$ on X with $x <'_i y \iff x <_i y$ for all $x,y \in X$. Then Lemma 6.8 provides us with a completely labelled primitive set $U_F \subseteq F$ for each $F \in \mathscr{P}_f X$. Let $J_F := \ell[U_F]$ and define $\alpha_F : J_F \to X$ by $\alpha_F(i) := F_i := \bigvee'_i U_F$. Now we have to pass from combinatorics to topology:

In a *first step,* we use compactness to find a nice "limit" $\alpha : J \to X$ of the α_F. Since I has only finitely many subsets and $\mathscr{P}_f X$ is up-directed, there is some $J \subseteq I$ such that, for all $E \in \mathscr{P}_f X$, there is $F \in \mathscr{P}_f X$ with $F \supseteq E$ and $J_F = J$. For these F, $\sigma_F := \ell \circ \alpha_F$ is a permutation of J, of which there are also just finitely many. Therefore, there is some permutation σ such that, for all $E \in \mathscr{P}_f X$, there is $F \in \mathscr{P}'$ with $F \supseteq E$, where

$$\mathscr{P}' := \{F \in \mathscr{P}_f X \mid J_F = J \text{ and } \sigma_F = \sigma\}.$$

In particular, $\mathscr{P}'$ is up-directed, hence $(\alpha_F)_{F\in\mathscr{P}'}$ is a net. This net has a cluster "point" $\alpha : J \to X$, since with (X,τ) also its finite power $(X,\tau)^J$ is compact. Let $x_i := \alpha(i)$. Because of the "continuity" conditions (ii) and (iii), $\alpha[J]$ behaves like a primitive set as well:

(iv) $x_i \leqslant_j x_j$ holds for all $i,j \in J$ $[\![$Assume $x_j <_j x_i$. Put $x := x_i$, $K := \{j\}$, and $A := x_j{<_j} \in \tau$. Then $x_j \neq x$ and $x_j \in C_K(x)$, so that (iii) implies that $\alpha(j) = x_j <_j y <_j x = \alpha(i)$ for some $y \in X$. Since both ${<_j}y$ and $y{<_j}$ are open and α is a cluster point, there is $F \in \mathscr{P}'$ such that $F_j = \alpha_F(j) <_j y <_j \alpha_F(i) = F_i$. But $F_j <_j F_i$ contradicts U_F's being a completely labelled primitive set$]\!]$.

(v) There is no $x \in X$ with $x <_j x_j$ for all $j \in J$ $[\![$since by (ii) there would then be $F \in \mathscr{P}'$ with $x <_j F_j$ for all $j \in J]\!]$.

In a *second step,* (ii) and (iii) imply that in fact all x_i are equal to one and the same fixed point of f. Let $i \in J$, $L := \{j \in J \mid x_i <_j x_j\}$, $K := J \setminus L$, and $A := \bigcap\{{<_j}x_j \mid j \in L\}$. Note that $x_i \in A \in \tau$ by (ii), $K = \{k \in J \mid x_i \sim_k x_k\}$

by (iv), and $C^0_K(x_i) \cap A = \emptyset$ by (v) $[\![$otherwise $x <_j x_j$ and $x <_k x_i \sim_k x_k$ for all $j \in L$, $k \in K$, and some $x \in X]\!]$. Thus $C_K(x_i) = \{x_i\}$ by (iii). But, for all $j \in J$, $x_j \leqslant_k x_k \sim_k x_i$ implies $x_j \in C_K(x_i)$ and thus $x_j = x_i =: x$. Again by (v), $C^0_J(x) \cap X = \emptyset$, and again by (iii), $C_J(x) = \{x\}$. Moreover, $f(x) = f(x_i) \leqslant_{\sigma(i)} x_i = x$ for all $i \in J$ $[\![$as in (iv), assuming $f(x_i) >_{\sigma(i)} x_i$ leads to $f(x_i) >_{\sigma(i)} y >_{\sigma(i)} x_i$ for some $y \in X$, so that, by continuity of f, $f(F_i) >_{\sigma(i)} y >_{\sigma(i)} F_i$ for some $F \in \mathscr{P}'$, in contradiction to $\sigma(i) = \ell(F_i)]\!]$, that is, $f(x) \in C_J(x)$, so that finally $f(x) = x$. $\boxed{\text{AC}}$

Compared with van Maaren's original proof, there is one main difference: in order to be able to work with sequences in X rather than a net in X^J, he required separability of (X,τ).

In a distance space, a natural way to get quasi-orders is to compare points by their distances to some reference points z_i—just put

$$x \leqslant_i y :\Longleftrightarrow d(x,z_i) \leqslant d(y,z_i).$$

The set $C_K(x)$ is then just the intersection of the balls $N_{d(x,z_i)} z_i$ with $i \in K$. Moreover, if $\underline{M}$ is a totally ordered co-quantale, $C^0_K(x)$ is an intersection of open balls.

COROLLARY 6.10. *Let $(X,d,\underline{M},D)$ be a distance space for which $\underline{M}$ is a totally ordered co-quantale and $\tau := \mathscr{T}(d^S,D)$ is compact, Moreover, let $(z_i)_{i\in I}$ be a finite family of points in X such that, for all $i \in I$, $K \subseteq I$, and $x,y \in X$, the following conditions hold:*

(i) $d(x,z_i) \leqslant d(y,z_i)$ for some $i \in I$,

(ii) If $C_K(x) \neq \{x\}$ then $C^0_K(x)$ intersects all τ-neighbourhoods of x.

Then each τ-continuous self-map of X has a fixed point.

Proof. Conditions (i) and (ii) imply conditions (i) and (iii) of the above theorem by definition of the $\leqslant_i$, while condition (ii) of the theorem follows easily from lower distributivity of $\underline{M}$. $\boxed{\text{AC}}$

EXAMPLE 6.11. Let $T = (V,E)$ be a finite tree, and $\varphi : V \to \mathbb{R}^2$ a straight embedding of T into $\mathbb{R}^2$, that is, φ is injective, and for each two distinct edges $\{v,w\},\{x,y\} \in E$, the segments $\overline{\varphi(v)\varphi(w)}$ and $\overline{\varphi(x)\varphi(y)}$ are disjoint up to common endpoints. Then the subspace $\underline{X}$ of $\mathbb{E}_2$ defined by $X := \bigcup\{\overline{\varphi(v)\varphi(w)} \mid \{v,w\} \in E\}$ has the CFPP. Indeed, with $I := V$, $z_v := \varphi(v)$, and d the geodesic distance, it fulfils the requirements of the corollary.

A different proof of this fact is [LT89], but it follows already from Ward's fixed point theorem for generalized trees [War57].

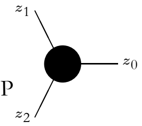

Figure 2. Example of a space with the CFPP

EXAMPLE 6.12. Let $\underline{X}$ be the subspace of the Euclidean complex plane defined by

$$X := \{z \in \mathbb{C} \mid |z| \leqslant 1\} \cup \{re^{2k\pi i/3} \mid r \in [0,3],\ k \in 3\},$$

that is, a closed unit disk with three symmetrically distributed radial arms of length two. Since their ends $z_k := 3e^{2k\pi i/3}$ ($k \in I := 3$) fulfil the requirements of the corollary, $\underline{X}$ has the CFPP (see Figure 2). This is a special case of an arcwise connected non-separating plane continuum for which class of spaces the CFPP was proved in general by Hagopian [Hag71].

However, the obvious generalization of $\underline{X}$ to an n-dimensional unit ball with $n + 1$ symmetrically distributed radial arms of length two still fulfils the requirements of the theorem and does not appear to belong to a class of continua for which the CFPP has been proved in general yet.

Since most of the distance functions presented so far do not have a totally ordered monoid, it would be very desirable to get rid of that requirement.

QUESTION 6.13.
Is there a version of Theorem 6.9 for more general quasi-orders?

APPENDIX

Visualization of distances
Additional proofs
References
Indices

VISUALIZATION OF DISTANCES

Schnell wachsende Keime
Welken geschwinde;
Zu lange Bäume
Brechen im Winde.
Schätz nach der Länge
Nicht das Entsprungne!
Fest im Gedränge
Steht das Gedrungne.

Wilhelm Busch, Schein und Sein

Introduction

The question of how distance information might be visualized is of importance for many sciences including physics, medicine, sociology, and others. Mathematicians have early studied the possibility of *embedding* a finite metric space $\underline{X}$ into other, in some sense better spaces like the Euclidean plane or 3-space. Beginning with Menger [Men28], who gave the precise criteria for $\underline{X}$ to be *isometrically* embeddable (that is, under exact preservation of the distances) into some Euclidean space, most of them have focused on mappings which map $\underline{X}$ into some standard space in a "quantitative" manner. The goal in this field of research, known under the name *metric scaling,* is to preserve the *values* of the distances as good as possible, that is, to minimize a certain error, known as "stress" (cf. [She62]).

The aim of this paper[1] is to study more "qualitative" kinds of visualization of distance data. In contrast to metric scaling, we will not be interested in the actual values of distances but rather in their comparison. Considering only the linear *order* among the distances instead of their value, a measure of *order accuracy* of a representation is introduced. Unlike stress, order accuracy has an easy interpretation as a certain probability of correctness. After an experimental exploration of different types of representations, a lower bound on the possible accuracy of plane representations will be proved using some clustering method and a result on maximal cuts in graphs. The experimental methods include random generation, optimization of accuracy by a rubber-band algorithm, and automatic proof generation. All results are summarized in Table II.

Order accuracy

Throughout this paper, $\underline{X} = (X,d)$ is a *finite metric space,* that is, X is finite, and $d : X^2 \to [0,\infty]$ fulfils $d(x,y) = d(y,x)$, $d(x,y) + d(y,z) \geqslant d(x,z)$, and $d(x,y) = 0$ if and only if $x = y$. However, one advantage of the following approach is that it also applies to any *finite, symmetric distance set* in the sense of [Hei98] and [Hei02], which is a far more general type of object than a metric space. For the sake of simplicity, we will also assume that X equals the set $n = \{0,\dots,n-1\}$ of non-negative integers, and that the pairwise distances between the points of $\underline{X}$ are all different, that is, $d(x,y) = d(x',y') > 0$ implies $\{x,y\} = \{x',y'\}$. In particular, each $x \in X$ has exactly one *nearest neighbour* $\mathrm{nn}(x) \in X$ and one *farthest neighbour* $\mathrm{fn}(x)$ which fulfil $d\big(x,\mathrm{nn}(x)\big) < d(x,y) < d\big(x,\mathrm{fn}(x)\big)$ for all $y \in X \setminus \{x,\mathrm{nn}(x),\mathrm{fn}(x)\}$.

We will be mostly interested in representing the points of X by points of either some Euclidean space $\underline{\mathbb{E}}_m$, that is, the real vector space $\mathbb{R}^m$ with Euclidean distance, or the L_1-plane $\underline{\mathbb{M}}_2$, that is, the set $\mathbb{R}^2$ with the "Manhattan"-distance $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$.

The *order accuracy* $\alpha(f)$ of a map f from $\underline{X}$ into some metric space $\underline{Y} = (Y,e)$ is defined as the probability that, of two randomly chosen pairs $\{x,y\}$ and $\{z,w\}$ of distinct elements of X, the one with the larger distance in the "representation" f also has the larger "original" distance. More formally,

$$\alpha(f) \;=\; \binom{\binom{n}{2}}{2}^{-1} \cdot \Big|\Big\{\{\{x,y\},\{z,w\}\} \subseteq \mathscr{P}(X) : \\ x \neq y,\ z \neq w,\ \{x,y\} \neq \{z,w\},\ \text{and} \\ d(x,y) < d(z,w) \iff e(fx,fy) < e(fz,fw)\Big\}\Big|.$$

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[1] The text of this additional chapter consists of an article submitted to *Experimental Mathematics.*

Table II. Representable fraction of linear orders on $\mathscr{B}(n)$ for different kinds of representations and different spaces (open intervals show exact bounds, $\gtrsim$ and $\lesssim$ denote estimated bounds, the question mark denotes a conjecture)

kind of representation	space	n (no. of points) 4	5	6	7	8	9	13	15	any
order	$\underline{\mathbb{E}}_2$	**1**	$(.538, 1)$	$\gtrsim .020$						
	$\underline{\mathbb{M}}_2$	1	$(.652, 1)$							
	$\underline{\mathbb{E}}_3$		**1**	$\gtrsim .60$	$\gtrsim .095$	< 1				
local order	$\underline{\mathbb{E}}_2$		(**.667**, 1), $\lesssim .928$	$\lesssim .60$	$\lesssim .21$	$\lesssim .030$	$\lesssim .0022$			
	$\underline{\mathbb{M}}_2$		$(.677, 1)$							
extremal neighbours	$\underline{\mathbb{E}}_2$		(**.883**, 1)							
1st & 2nd nearest nbs.	$\underline{\mathbb{E}}_2$		(**.933**, 1)							
two nearest neighbours	$\underline{\mathbb{E}}_2$		(**.963**, 1)							
nearest neighbour	$\underline{\mathbb{E}}_2$			**1**	**.999**...	**.998**...	**.997**...			
nearest neighbour	$\underline{\mathbb{E}}_3$							**1?**	< 1	
farthest neighbour	$\underline{\mathbb{E}}_2$									**1**
cluster	$\underline{\mathbb{E}}_2$									**1**

Note that $2\alpha(f) - 1$ is just Kendall's rank correlation coefficient ϱ between the two linear orders on the $\binom{n}{2}$ pairs $\{x,y\}$ that result when these pairs are compared with respect to either their original or their image distance. Using a variant of the merge-sort algorithm, ϱ can be computed in linear-times-logarithmic time, hence $\alpha(f)$ can be computed in $O(n^2 \log n)$ time.

Order and weaker representations

An *order representation* of $\underline{X}$ in $\underline{Y}$ is some map $f : \underline{X} \to \underline{Y}$ with $\alpha(f) = 1$, that is, with $d(x,y) < d(z,w) \iff e(fx,fy) < e(fz,fw)$. Likewise, an *order representation* of a (strict) linear order $<$ on the set $\mathscr{B}(X)$ of two-element subsets of X is a map $f : X \to \underline{Y}$ with $\{x,y\} < \{z,w\} \iff e(fx,fy) < e(fz,fw)$. It will be convenient to identify the metric space $\underline{X}$ with its associated linear order $<$ which is given by $\{x,y\} < \{z,w\} :\iff d(x,y) < d(z,w)$ here.

For $\underline{Y} = \underline{\mathbb{E}}_{n-2}$, there is always an order representation of $\underline{X}$—there is even a map f for which $e(fx,fy) = d(x,y) + C$ for some constant $C \geqslant 0$. This was proved by Cailliez [Cai83]. A random generation of five-element subsets of $\underline{\mathbb{E}}_3$ confirmed this result for $n = 5$, and a similar experiment showed that all four-element metric spaces not only have an order representation in $\underline{\mathbb{E}}_2$ but also in $\underline{\mathbb{M}}_2$.

To get a feeling how probable a *plane* order representation is for a five-element metric space, I also repeatedly drew five-element samples from the uniform distribution on the unit square and determined the resulting order among the ten pairwise distances. In this way, of the $10! = 3\,628\,800$ linear orders on $\mathscr{B}(5)$, at least 53.8% [resp. 65.2%] were found to have an order representation in $\mathbb{R}^2$ with the Euclidean [resp. "Manhattan"] metric. Moreover, at least 66.7% [resp. 67.7%] had a *local order representation,* that is, a map $f : \underline{X} \to \mathbb{R}^2$ such that $\{x,y\} < \{x,z\} \iff e(fx,fy) < e(fx,fz)$ for all x,y,z, where again e was the Euclidean [resp. "Manhattan"] metric. Judging from these empirical numbers, order representability seems to be considerably stronger than local order representability in the Euclidean case, but not in the "Manhattan" case.

Considering only the information coded in the functions nn and fn, it was also found that at least 88.3% of the 10! orders had a plane *extremal neighbours representation,* that is, a map $f : \underline{X} \to \underline{\mathbb{E}}_2$ such that $\mathrm{nn}(fx) = f\big(\mathrm{nn}(x)\big)$ and $\mathrm{fn}(fx) = f\big(\mathrm{fn}(x)\big)$ for all $x \in X$. Likewise, at least 93.3% allowed for a map under which both the nearest and second-nearest neighbours were represented accurately, and another 3% allowed for a map under which at least the information about which points were the two nearest to x was represented accurately for all x (see Table II).

In view of the quickly growing number $\binom{n}{2}!$ of orders on $\mathscr{B}(n)$ and the limited space for storing the list of orders already found, such a random generation did not make much sense for $n > 5$. It is, however, possible to estimate some similar lower bounds at least for $n \in \{6,7\}$ from the following experiment.

Representation by accuracy optimization

Starting with a randomly generated $f : X \to \mathbb{E}_m$, an order representation of a linear order $<$ on $\mathscr{B}(X)$ can often be produced by a stepwise maximization of order accuracy. The following optimization step proved useful: for each pair $\{x,y\}$, $\{z,w\}$ with $\{x,y\} < \{z,w\}$ and $e(fx,fy) \geqslant e(fz,fw)$, move x,y towards each other by some fixed fraction of $e(fx,fy)$, and move z,w away from each other by the same fixed fraction of $e(fz,fw)$. I have tested this kind of rubber-band algorithm in several ways:

(i) When $<$ was taken to be the order that corresponded to 8 or 25 independently uniformly distributed random points in the unit square, the algorithm found an order representation of $<$ in $\mathbb{E}_2$ in about 96% of all cases, no matter whether 8 or 25 points were taken. For 25 points, the resulting representations were almost similar to the original sets. More precisely, for each edge the quotient between its original length and its length in the representation was determined, and on average the relative difference between maximal and minimal quotient was less than 5% (compared to 12% for 15 points and over 60% for 8 points).

(ii) When $<$ was taken from a uniform distribution of all linear orders on $\mathscr{B}(5)$, the algorithm succeeded in only 45% of the cases. Since, as mentioned before, more than 53% of the orders actually have an order representation, this indicates that the algorithm is susceptible to being caught in a local optimum.

However, in both (i) and (ii), the success of the algorithm did not seem to depend on the initial state: when a cluster representation (see below) instead of a random initial state was used, only the average number of iterations that were needed shrinked slightly.

(iii) As in (i), but for five points in a 100-dimensional cube. Here the success rate was about 79%. Such finite subspaces of high-dimensional spaces frequently occur in multivariate statistics, for example.

(iv) Generating the orders as in (ii), an order representation in $\mathbb{E}_3$ of six-point metric spaces was found in about 65% of 1000 cases, but of seven-point spaces in only 10.5% of 7000 cases.

The rubber-band algorithm has also been implemented as a Java applet which can be tested at `http://www-ifm.math.uni-hannover.de/~heitzig/distance`.

Despite the algorithm's lack of optimality, we can use these results to estimate lower bounds for the fraction of representable orders. As the samples were large enough, one can use the approximate confidence bound that arises from the approximation of the actual binomial distribution by a normal distribution (see [Kre91]). For a sample of size N, $s + 1/2$ successes, and confidence level β, it has the form

$$\frac{s + \frac{c^2}{2} - c\sqrt{s - \frac{s^2}{N} + \frac{c^2}{4}}}{N + c^2} \quad \text{with} \quad c = \Phi^{-1}(\beta).$$

Taking $\beta = 0.995$, this leads to the following conjectured bounds:

CONJECTURE 6.14. *A six- [seven-] element metric space has an order representation in* $\underline{\mathbb{E}}_3$ *with probability at least* 60% *[*9.5%*].*

For six points in $\underline{\mathbb{E}}_2$, the same method gives a conjectured lower bound of only 2% (see Table II).

Disproving local order representability

A local order representation can also be characterized as a map that preserves the order among the three sides of any triangle. More precisely, $f : \underline{X} \to \underline{Y}$ is a local order representation if and only if for each three distinct points $x, y, z \in X$ with $d(x,y) < d(y,z) < d(z,x)$, also $e(fx,fy) < e(fy,fz) < e(fz,fx)$. Using elementary geometry, one sees that, in the Euclidean plane, the latter is equivalent to $\angle fx\,fz\,fy < \angle fy\,fx\,fz < \angle fz\,fy\,fx$ $(\star)$.

Therefore, the existence of a plane local order representation for some order $<$ can be disproved by showing that a certain set of inequalities between angles in the plane has no solution. The advantage of using angles instead of distances is that the additional equations and inequalities which every n-point subset of the plane must fulfil are all linear in the angles:

(i) $\angle abc \in [0, \pi]$,
(ii) $\angle abc + \angle bca + \angle cab = \pi$,
(iii) $\angle azc \leqslant \angle azb + \angle bzc$,
(iv) $\angle azb + \angle bzc + \angle cza = 2\pi$ if z is in the convex hull of a, b, c,
(v) $\angle azc = \angle azb + \angle bzc$ if b is "between" a and c as seen from z.

In search of a local order representation for $\underline{X}$, these linear relations together with those of type $(\star)$ enable us, starting with the largest interval $[0, \pi]$, to successively narrow down the interval of possible values of each angle. If some angle's interval becomes empty, there can be no local order representation of

```
TEST OF EDGE ORDER de < ad < ac < ab < ce < be < bc < cd < ae < bd
USING ONLY EXTREMAL NEIGHBOURS INFORMATION

legend: points are labeled a,b,c,d,e
        xy is a segment, xyz is a triangle, x:yz is the angle in xyz at vertex x
        x:ywz means that x:yz=x:yw+x:wz
                                                                        follows
 line type      proposition                                             from
---------------------------------------------------------------------------------
   1. smallest  a:de,b:ad,b:de,c:ad,c:de,d:bc,e:ab,e:ac < 60
   2. dominated a:be,a:ce,b:ac,b:cd,c:ab,d:ab,d:ac,d:be,d:ce,e:ad < 90
   3. largest   a:bc,a:bd,a:cd,b:ae,c:ae,c:bd,d:ae,e:bd,e:cd > 60
   4. on bndry  a,b,d,e since in fn[X]
   5. tripod    a:bd <=a:be+a:de < 90+60= 150                           2.1.
   6. tripod    a:cd <=a:ce+a:de < 90+60= 150                           2.1.
   7. tripod    b:ae <=b:ad+b:de < 60+60= 120                           1.1.
   8.       not c in abd since c:ad+c:bd+c:ab<360                       1.0.2.
   9.       not c:abd since c:ad<c:ab+c:bd                              1.0.3.
  10. tripod    c:ae <=c:ad+c:de < 60+60= 120                           1.1.
  11. larger    a:be > (180-b:ae)/2>(180-120)/2= 30                     7.
  12. larger    a:ce > (180-c:ae)/2>(180-120)/2= 30                     10.
  13.       not a:cbe since a:ce<a:bc+a:be                              2.3.11.
  14.       not a:bce since a:be<a:ce+a:bc                              2.12.3.
  15.     hence a:bec                                                   4.13.14.

CASE ANALYSIS using points a,bcd:

  16. (i) ASSUMING a:bcd...
  17. sum       a:bd =a:bc+a:cd > 60+60= 120                            16.3.3.
  18. sum       a:bc =a:bd-a:cd < 150-60= 90                            16.5.3.
  19. tripod    a:be >=a:bd-a:de > 120-60= 60                           17.1.
  20.       not a:bec since a:bc<a:ce+a:be                              18.12.19.
  21.     hence a in bce                                                14.13.20.
  22. contradiction!                                                    21.4.
  23. (ii) ASSUMING a:cbd...
  24. sum       a:cd =a:bc+a:bd > 60+60= 120                            23.3.3.
  25. sum       a:bc =a:cd-a:bd < 150-60= 90                            23.6.3.
  26. tripod    a:ce >=a:cd-a:de > 120-60= 60                           24.1.
  27.       not a:bec since a:bc<a:be+a:ce                              25.11.26.
  28.     hence a in bce                                                13.14.27.
  29. contradiction!                                                    28.4.
  30. (iii) ASSUMING a:bdc...
  31.       not d:acb since a:bdc                                       30.
  32.       not d:abc since a:bdc                                       30.
  33.     hence d:bac                                                   31.4.32.
  34.       not c:bad since a:bdc                                       30.
  35.     hence c:adb                                                   8.9.34.
  36. new sum   c:abd since ad diag in cabd                             30.33.
  37. new circ  d in abc since a:bdc and c:adb                          30.35.
  38. contradiction!                                                    37.4.
  39. (iv) ASSUMING a in bcd...
  40. contradiction!                                                    39.4.

CONTRADICTION in all four cases!
```

Figure 3. A computer generated non-representability proof.

this order $<$. This method can also be used to disprove the existence of even weaker kinds of representations such as extremal neighbours representations.

EXAMPLE 6.15. Figure 3 shows a computer generated proof that the order $\{d,e\} < \{a,d\} < \cdots < \{b,d\}$ (listed on top) cannot occur among the distances between five points in the plane. Lines 1, 2, and 3 state that certain angles are smaller than 60°, smaller than 90°, or larger than 60° because they are the smallest, second smallest, or largest in their corresponding triangle, respectively. Line 4 states that only c can be in the convex interior of the five points, since each of the remaining four is the farthest neighbour of some other. Lines 5–7 apply the "tripod" inequality (iii), using bounds already known from lines 1 and 2, this dependence being logged at the end of the lines. Line 8 notices a violation of (iv) so that c cannot be in the convex hull of a, b, d. Similarly, line 9 states that also b cannot be between a and d as seen from c. In line 11, (ii) is used to derive a lower bound for a second smallest angle from an upper bound for a largest angle. This is the only kind of argument the algorithm can use to derive bounds which are not just multiples of 30°. The rest of the proof shall be clear now.

Note that the premises in lines 1–4 already follow from the information coded in the maps nn and fn alone, hence the order under consideration does not even have an extremal neighbours representation.

There is a similar example which shows that it may also be impossible in the plane to accurately represent the set of two nearest neighbours of five points. Since for disjoint five-element subsets of some metric space $\underline{X}$, the distribution of the orders that correspond to these subsets are independent, we have:

COROLLARY 6.16. *For an n-element metric space, the probability of a plane extremal neighbours representation shrinks exponentially for $n \longrightarrow \infty$.*

To get explicit upper bounds for local representability, I tested several thousand randomly generated orders with this algorithm. For five points, 795 out of 10 000 orders could be shown to have no plane local order representation in this way. Using estimated confidence bounds with $\beta = .995$ again, this results in an estimated upper bound of .928 for the fraction of plane locally order representable orders on $\mathscr{B}(5)$. For $n = 6$, 7, 8, and 9, the corresponding numbers were 4156 out of 10 000, 3627 out of 4500, 11 690 out of 12 000, and 9990 out of 10 000, respectively, resulting in the upper bounds shown in Table II.

CONJECTURE 6.17. *In $\underline{\mathbb{E}}_2$, a six-element metric space has a local order representation with probability at most* 60%.

This fast vanishing of the probability of plane local order representability on the one hand shows that the above algorithm is quite successful, and on the other hand motivates the study of even weaker kinds of plane representation.

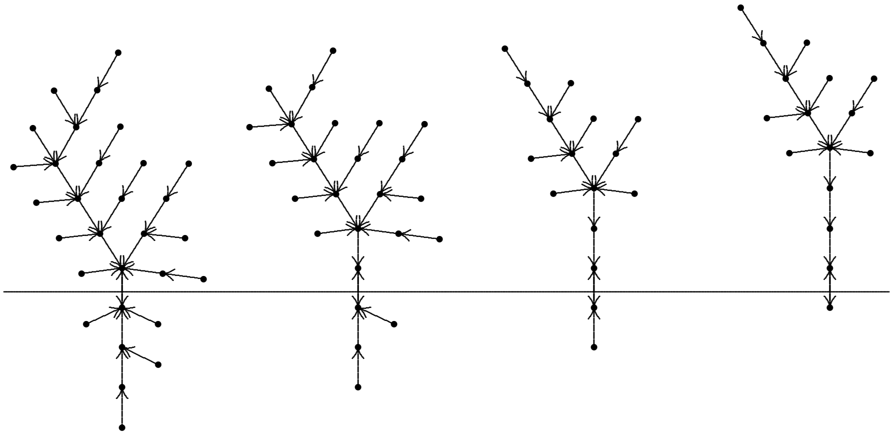

Figure 4. A "universal" nearest neighbour graph of nine points in the plane

Nearest and farthest neighbour representations

The directed graph $G_{\rm nn}(X)$ with vertex set $V(G) = X$ and edge set $E(G) = \{(x,{\rm nn}(x)) : x \in X\}$ is known as the *nearest neighbour graph* of $\underline{X}$. Asymptotic properties of nearest neighbour graphs of subsets of the plane have been studied in [EPY97]. The *farthest neighbour graph* of $\underline{X}$ is defined similarly. By a *down-tree* I mean a finite connected digraph all of whose vertices have out-degree one, except for a *root* vertex with out-degree zero.

PROPOSITION 6.18. *A finite digraph G is a nearest [farthest] neighbour graph of a metric space if and only if each of its components is a disjoint union of two down-trees whose roots are joined by a double edge.*

Since the *proof* is easy but quite technical, it is omitted here.

The digraphs characterized by this result will be called *bi-rooted forests* in the sequel, and a pair of roots will be called a *bi-root* for short. A *proper child* of a vertex x in a digraph is a vertex y for which there is an edge (y,x) but no edge (x,y).

PROPOSITION 6.19. *A bi-rooted forest of size at most nine occurs as a nearest neighbour graph in the plane if and only if no vertex has more than four proper children.*

Proof. Let G be a bi-rooted forest with $|V(G)| \leqslant 9$. If some vertex x has five proper children $x_1,\ldots,x_5$, there is no nearest neighbour representation in $\underline{\mathbb{E}}_2$. Otherwise, for $i \neq j$, the longest side of the triangle $x_i x_j x$ would be $x_i x_j$, hence the angle between the segments $x_i x$ and $x_j x$ would be larger than $\pi/3$.

Likewise, the longest side of the triangle $x_i x \operatorname{nn}(x)$ is $x_i \operatorname{nn}(x)$, hence the angle $\angle x_i x \operatorname{nn}(x)$ would also be larger than $\pi/3$ which is impossible in the plane.

On the other hand, one can verify that all bi-rooted forests with at most nine vertices and without vertices that have more than four proper children fit into the "universal" forest sketched in Figure 4. Each of its four components is constructed from its two roots (joined by a double edge of length 100) by successively adding children, where the edges originating from children of order n have length $100 + n$ and share a mutual angle of $(65 + i - n)^\circ$ if they are neighboured. Since in that figure, each edge points towards the nearest neighbour, the proposition is proved. ☐

Using this result, it was possible to calculate the fractions of linear orders on $\mathscr{B}(n)$ with a plane nearest neighbour representation shown in Table II. Note that for $n = 10$, the analogue of the above proposition is false, a counter-example being the bi-rooted forest consisting of two connected roots with four children each.

As for nearest neighbour representations in $\underline{\mathbb{E}}_3$, it was proved by Fejes Tóth [FT43] that of n points on a unit sphere in $\underline{\mathbb{E}}_3$, at least two must have a distance of at most

$$\delta_n := \sqrt{4 - \operatorname{cosec}^2 \frac{n}{n-2} \frac{\pi}{6}}.$$

In particular, $\delta_{14} \approx 0.98$, hence there exist no fourteen points on the unit sphere with pairwise distance larger than one. In other words, of fourteen rays in $\underline{\mathbb{E}}_3$ with a common source, at least two have an angle of at most 60°. Therefore, a bi-rooted forest with a root that has thirteen children cannot have a representation in $\underline{\mathbb{E}}_3$. In particular, not all linear orders on $\mathscr{B}(15)$ have a nearest neighbour representation in $\underline{\mathbb{E}}_3$. However, one may hope that at least all linear orders on $\mathscr{B}(13)$ have a representation since there exist twelve such points on the sphere.

CONJECTURE 6.20. *Every metric space of up to thirteen elements has a nearest neighbour representation in* $\underline{\mathbb{E}}_3$.

Note that $\delta_{13} \approx 1.014 > 1$, and the empirically supported conjecture that there are no thirteen such points is still unproved—this shows that questions of representability of larger sets might also be quite difficult.

Surprisingly, a small degree at all vertices of the nearest neighbour graph does not assure plane nearest neighbour representability: Eppstein, Paterson, and Yao [EPY97] could show that for a subset X of $\underline{\mathbb{E}}_2$, $|X| = O\big(D\big(G_{\mathrm{nn}}(X)\big)^5\big)$, where $D(G)$ is the *depth* of G, that is, the maximal length of a path from a vertex to the nearest root. Using their exact bounds, one can show that for instance the complete binary bi-rooted tree with $2^{66} - 2 \approx 10^{20}$ vertices does not have a nearest neighbour representation in $\mathbb{E}_2$. However, it seems likely that already far smaller binary trees fail to have one.

Eppstein et al. also showed that the expected number of components of $G_{\text{nn}}(X)$ is asymptotic to approximately $0.31|X|$ if the points of X are independently uniformly distributed in the unit square. More precisely, the probability for a vertex to belong to a bi-root is $6\pi/(8\pi+3\sqrt{3}) \approx 0.6215$ in that case. From this it is also clear that the expected fraction of elements of X which are not the nearest neighbour of some other element is at most 0.2785. However, the smallest exact upper bound on this fraction is far larger:

PROPOSITION 6.21. *In any finite subset of* $\mathbb{E}_2$*, at most* $7/9$ *of its elements are not a nearest neighbour of some other element, and this bound is sharp.*

Proof. It is quite easy to see that the bi-rooted forest consisting of a root with four and another with three children has a nearest neighbour representation in $\mathbb{E}_2$, hence $7/9$ is possible.

On the other hand, let C be a component of the nearest neighbour graph of a finite subset of the plane. Then its roots r and q together have $k \leqslant 7$ children, and C can be constructed from these $k+2$ vertices by subsequently adding $k_i \leqslant 4$ children to some end vertex, thereby increasing the number of end vertices by $k_i - 1$ in step i. Thus, the final fraction of end vertices in C is

$$\frac{k+\sum_i (k_i - 1)}{(k+2)+\sum_i k_i} \leqslant \frac{7}{9}$$

since $7(k+2+\sum_i k_i) - 9(k+\sum_i (k_i-1)) = 14-2k+9s-2\sum_i k_i \geqslant 9s - 2\cdot 4s \geqslant 0$, where s is the number of steps needed. $\square$

In view of these facts about nearest neighbour graphs, the following might be a bit surprising:

THEOREM 6.22. *Every finite metric space has a farthest neighbour representation in* $\mathbb{E}_2$.

Proof. Let $G := G_{\text{fn}}(X)$ be the corresponding farthest neighbour graph, D its depth, and define an infinite bi-rooted forest H as follows. The vertices of H are labelled a_{jt} and b_{jt}, where j is a non-negative integer and t runs over all tuples of at most D non-negative integers, including the empty tuple $\emptyset$. The bi-roots are the pairs $\{a_{j\emptyset}, b_{j\emptyset}\}$ with non-negative integer j, each vertex $a_{j(\dots,k,m)}$ is a child of $a_{j(\dots,k)}$, and each vertex $b_{j(\dots,k,m)}$ is a child of $b_{j(\dots,k)}$. In other words, H has countably many isomorphic components (numbered by j), and each vertex has countably many children, up to depth D. This digraph H contains an isomorphic copy of G, hence it suffices to give a representation of H. To address points of the plane, it will be convenient to identify $\mathbb{R}^2$ with the set $\mathbb{C}$ of complex numbers in the usual way.

For each non-negative integer j, let C_{j0} and C_{j1} be the circles of radius 2 with centres $c_{j0} := e^{2^{-j-1}\pi i}$ and $c_{j1} := e^{(1+2^{-j-1})\pi i}$, respectively. These curves can be parametrized using the following functions, where the coefficients $\lambda_j > 0$ will be determined later:

$$f_{j0}(\xi) := c_{j0} + 2e^{(2^{-j-1}+\lambda_j\xi)\pi i} \quad \text{and} \quad f_{j1}(\xi) := c_{j1} + 2e^{(1+2^{-j-1}+\lambda_j\xi)\pi i}.$$

In particular, $f_{j0}(0) = 3c_{j0}$, $f_{j1}(0) = 3c_{j1}$, $F_{j0} := f_{j0}[I] \subseteq C_{j0}$, and $F_{j1} := f_{j1}[I] \subseteq C_{j1}$, where $I = [-2^D, 2^D] \subseteq \mathbb{R}$. Now the coefficients λ_j are chosen small enough so that $2^D\lambda_j < \pi/2$ and so that the smallest distance between the sets F_{j0} and F_{j1} is still larger than the largest distance between a point in $F_{j0} \cup F_{j1}$ and a point in $F_{k0} \cup F_{k1}$ for any $k \neq j$. This ensures that, for $q \in \{0,1\}$ and all $\xi \in I$, the unique point in $\bigcup_k F_{k0} \cup F_{k1}$ which is farthest away from the point $f_{jq}(\xi)$ is the point $f_{j,1-q}(\xi/2)$. More generally, given $q \in \{0,1\}$ and $\xi, \beta, \gamma \in I$, we have

$$|f_{jq}(\xi) - f_{j,1-q}(\beta)| > |f_{jq}(\xi) - f_{j,1-q}(\gamma)| \iff |\beta - \xi/2| < |\gamma - \xi/2| \quad (\star).$$

Using this equivalence, one sees that the following recursive definition results in a farthest neighbour representation f of H:

$$f(a_{jt}) := f_{j,q(t)}\big(\xi(t)\big) \quad \text{and} \quad f(b_{jt}) := f_{j,1-q(t)}\big(-\xi(t)\big),$$

where the bi-roots have $q(\emptyset) := 0$ and $\xi(\emptyset) := 0$, their children have $q\big((m)\big) := 1$ and $\xi\big((m)\big) := 1 + 2^{-m}$, and all others have $q\big((\ldots,k,m)\big) := 1 - q\big((\ldots,k)\big)$ and

$$\begin{aligned} \xi\big((\ldots,k,m)\big) &:= 2\xi\big((\ldots,k)\big) - (1-2^{-m})\Big(\xi\big((\ldots,k)\big) - \xi\big((\ldots,k+1)\big)\Big) \\ &= (1+2^{-m})\xi\big((\ldots,k)\big) + (1-2^{-m})\xi\big((\ldots,k+1)\big). \end{aligned}$$

Because of $(\star)$, we need only verify that (i) $\big|0 - \xi\big((m)\big)/2\big| < \big|\xi\big((k,\ell)\big) - \xi\big((m)\big)/2\big|$, which is true because of $\xi\big((m)\big) < 2 < \xi\big((k,\ell)\big)$, and that (ii)

$$\big|2\xi\big((\ldots,k)\big) - \xi\big((\ldots,k,m)\big)\big| < \big|2\xi\big((\ldots,k\pm 1)\big) - \xi\big((\ldots,k,m)\big)\big|,$$

where the left hand side equals $(1-2^{-m})c$ with $c = \big(\xi\big((\ldots,k)\big) - \xi\big((\ldots,k+1)\big)\big)$, and the right hand side is the absolute value of $c + 2\big(\xi\big((\ldots,k\pm 1)\big) - \xi\big((\ldots,k,m)\big)\big)$ which is larger than c in the '$-$' case and smaller than $-c$ in the '$+$' case. $\square$

Cluster representations, and lower bounds for accuracy

A important question in applications of finite metric spaces is that of clustering the elements into homogeneous, mutually heterogeneous groups. Formally, a hierarchical clustering of $\underline{X}$ produces what I will call a *cluster tree* here, which can be formalized as a chain of partitions $\mathscr{P}_1, \dots, \mathscr{P}_n$ on X, where $\mathscr{P}_1 = \big\{\{x\} : x \in X\big\}$ is the discrete and $\mathscr{P}_n = \{X\}$ the indiscrete partition, and each $\mathscr{P}_{k+1}$ with $k < n$ arises from $\mathscr{P}_k$ by joining two clusters, that is, replacing some $A, B \in \mathscr{P}_k$ by their union $A \cup B$. Most common clustering methods fulfil the following property ($\star$): if $k < n$, $A, B \in \mathscr{P}_k$, $A \neq B$, and for all $a \in A$, $b \in B$, and $x, y \in X$, either $x, y \in A \cup B$, or $x, y \in C$ for some $C \in \mathscr{P}_k$, or $d(a,b) < d(x,y)$, then $A \cup B \in \mathscr{P}_{k+1}$. In other words, when all distances between members of A and B are smaller than all distances between points of other clusters, then A and B are joined next. Now, a cluster tree for X is said to have a *cluster representation* $f : X \longrightarrow \underline{Y}$ when all clustering methods that fulfil ($\star$) reproduce this cluster tree when they are applied to the metric space $\underline{X}' := (X, d')$ with $d'(x,y) := e(fx, fy)$.

PROPOSITION 6.23. *Every cluster tree* $\mathscr{P}_1, \dots, \mathscr{P}_n$ *for a finite set* X *has a cluster representation in* $\{0, \dots, \lfloor (1+\sqrt{2})^n/4 \rfloor\}$ *with Euclidean distance.*

Proof. Inductively, we construct maps $f_i : X \longrightarrow \mathbb{Z}$ and integers δ_i such that f_n is a cluster representation, and each f_i is already "correct" for all $C \in \mathscr{P}_i$. For $C \in \mathscr{P}_i$, the convex hull of $f_i[C]$ will be the interval $[0, w_i(C)]$. For $A, B \in \mathscr{P}_i$ and $A \cup B \in \mathscr{P}_{i+1}$, the image $f_{i+1}[A \cup B]$ will be constructed by placing $f_i[A]$ and $f_i[B]$ besides each other at a distance δ_i which is larger than the diameter of any $C \in \mathscr{P}_i$, that is, with $\delta_i > w_i(C)$.

We start with $f_1(a) := 0$ for all $a \in X$, so that $w_1(A) = 0$ for all $A \in \mathscr{P}_1$, and put $\delta_1 := 1$. For $i \geqslant 1$, let $A_i, B_i \in \mathscr{P}_i$ be those elements with $C_i := A_i \cup B_i \in \mathscr{P}_{i+1}$ and $\min A_i < \min B_i$. Now put

$$\begin{aligned} f_{i+1}(a) &:= f_i(a) \quad \text{for all } a \in A_i, \\ f_{i+1}(b) &:= f_i(b) + \delta_i + w_i(A) \quad \text{for all } b \in B_i, \\ f_{i+1}(x) &:= f_i(x) \quad \text{for all } x \notin C_i, \end{aligned}$$

and $\delta_{i+1} := w_{i+1}(C_i) + 1$, where, by construction, $w_{i+1}(C_i) = \delta_i + w_i(A_i) + w_i(B_i)$. Then the convex hull of $f_{i+1}[C_i]$ is $[0, w_{i+1}(C_i)]$ as proposed. For all $C \in \mathscr{P}_{i+1}$ different from C_i, we have $C \in \mathscr{P}_i$ and thus $\delta_{i+1} > \delta_i > w_i(C) = w_{i+1}(C)$ as required. In case that $i \geqslant 2$, one of A_i, B_i is in $\mathscr{P}_{i-1}$, hence either $w_i(A_i) = w_{i-1}(A_i)$ or $w_i(B_i) = w_{i-1}(B_i)$. Putting $m_i := \max\{w_i(A) : A \in \mathscr{P}_i\}$, this gives $m_{i+1} \leqslant 2m_i + m_{i-1} + 1$. It is easy to verify that the corresponding recursive upper bound b_i with $b_{i+1} = 2b_i + b_{i-1} + 1$ and initial conditions $b_1 = 0$ and $b_2 = 1$ is $b_i = \big((1+\sqrt{2})^i + (1-\sqrt{2})^i\big)/4 - 1/2 = \lfloor (1+\sqrt{2})^i/4 \rfloor$. In particular, $w_n(X) = m_n \leqslant b_n = \lfloor (1+\sqrt{2})^n/4 \rfloor$.

Finally, f_n is a cluster representation: Let $i \leqslant n$, $a \in A_i$, $b \in B_i$, $A' \neq B' \in \mathscr{P}_i$ with $\{A', B'\} \neq \{A_i, B_i\}$, and $a' \in A'$, $b' \in B'$. Then the smallest index j for which there is $C \in \mathscr{P}_j$ with $a',b' \in C$ is at least $i+1$, hence $df_n(a,b) = df_i(a,b) < \delta_i \leqslant \delta_{j-1} \leqslant df_j(a',b') = df_n(a',b')$. □

Finally, this construction can be used to show the following lower bound on order accuracy for maps into the real line:

THEOREM 6.24. *For every n-element metric space $\underline{X}$ with $n = 2^p$ for some integer p, there is a map $f : \underline{X} \longrightarrow \mathbb{E}_1$ with order accuracy at least $3/7 - r(n)$, where $r(n) = O(1/n)$.*

Proof. We iteratively define a binary cluster tree. For $k < n$, $\mathscr{P}_k$ is constructed from $\mathscr{P}_{k+1}$ as follows: Choose some $C \in \mathscr{P}_{k+1}$ of maximal size, and let $w_C(\{x,y\})$ be the number of pairs $\{z,w\} \subseteq C$ with $0 < d(z,w) < d(x,y)$. In [PT86] it was proved that there is a partition of C into two sets A and B of equal size such that

$$\sum_{x \in A,\ y \in B} w_C(\{x,y\}) \geqslant \frac{1}{2} \cdot \sum_{\{x,y\} \subseteq C} w_C(\{x,y\}) = \frac{1}{2} \cdot \binom{\binom{|C|}{2}}{2}.$$

Let $\mathscr{P}_k := \mathscr{P}_{k+1} \setminus \{C\} \cup \{A,B\}$. Note that $w_C(\{x,y\})$ is now the sum of $w_{A,B}(\{x,y\})$, the number of pairs $\{z,w\} \subseteq C$ with $0 < d(z,w) < d(x,y)$, $z \in A$, and $w \in B$, and of $w'_{A,B}(\{x,y\})$, the number of pairs $\{z,w\} \subseteq C$ with $0 < d(z,w) < d(x,y)$ and either $z,w \in A$ or $z,w \in B$.

Now we construct a representation as in the previous proposition, except that we might sometimes use $f'_i(a) := w_i(A_i) - f_i(a)$ and $f'_i(b) := w_i(B_i) - f_i(b)$ instead of $f_i(a)$ and $f_i(b)$ for the definition of $f_{i+1}|_{C_i}$. More precisely, when f_i has already been defined and A_i, B_i, C_i are as in the proposition, let γ be the number of quadruples $(x,y,z,w) \in A_i \times B_i \times A_i \times B_i$ with $0 < d(z,w) < d(x,y)$ and $f_i(w) - f_i(z) < f_i(y) - f_i(x)$, and let γ' be the number of quadruples $(x,y,z,w) \in A_i \times B_i \times A_i \times B_i$ with $0 < d(z,w) < d(x,y)$ and $f_i(z) - f_i(w) < f_i(x) - f_i(y)$. These numbers tell how many pairs of edges between A_i and B_i will be represented with the correct order of lengths when either f_i or f'_i is used for the definition of $f_{i+1}|_{C_i}$. Now put $f_{i+1}(x) := f_i(x)$ for all $x \notin C_i$, and either

$$\begin{aligned} f_{i+1}(a) &:= f_i(a) \quad \text{for all } a \in A_i, \text{ and} \\ f_{i+1}(b) &:= f_i(b) + \delta_i + w_i(A) \quad \text{for all } b \in B_i \end{aligned}$$

if $\gamma \geqslant \gamma'$, or otherwise

$$\begin{aligned} f_{i+1}(a) &:= f'_i(a) \quad \text{for all } a \in A_i, \text{ and} \\ f_{i+1}(b) &:= f'_i(b) + \delta_i + w_i(A) \quad \text{for all } b \in B_i. \end{aligned}$$

This assures that $|f_{i+1}(x) - f_{i+1}(y)| > |f_{i+1}(z) - f_{i+1}(w)|$ whenever $x \in A_i$, $y \in B_i$, and either $z,w \in A_i$ or $z,w \in B_i$. Moreover, since the sum of γ and γ' is $\binom{|A_i||B_i|}{2}$, their maximum is at least $|A_i||B_i|\big(|A_i||B_i| - 1\big)/4$. Hence, this step i of the construction contributes to the overall accuracy α a summand α_i with

$$\begin{aligned}
\alpha_i \cdot \binom{\binom{n}{2}}{2} &\geqslant \sum_{x\in A_i,\ y\in B_i} w'_{A_i,B_i}(\{x,y\}) + \frac{|A_i||B_i|\big(|A_i||B_i|-1\big)}{4} \\
&= \sum_{x\in A_i,\ y\in B_i} \big(w_{C_i}(\{x,y\}) - w_{A_i,B_i}(\{x,y\})\big) + \frac{|A_i||B_i|\big(|A_i||B_i|-1\big)}{4} \\
&= \sum_{x\in A_i,\ y\in B_i} w_{C_i}(\{x,y\}) - \binom{|A_i||B_i|}{2} + \frac{|A_i||B_i|\big(|A_i||B_i|-1\big)}{4} \\
&\geqslant \frac{1}{2}\cdot\binom{\binom{|C_i|}{2}}{2} - \frac{|A_i||B_i|\big(|A_i||B_i|-1\big)}{4} = \frac{3}{64}|C_i|^4 + O(|C_i|^3).
\end{aligned}$$

Finally, all C_i are of size $n/2^q$ for some q with $0 \leqslant q < p$, and there are exactly 2^q many of this size. Hence the overall accuracy is

$$\alpha = \sum_{i=1}^{n} \alpha_i \geqslant \sum_{q=0}^{p-1} 2^q \cdot \frac{3}{8}(1/2^q)^4 + O(1/n) = \frac{3}{7} - O(1/n).$$

□

However, this lower bound is very likely not the best possible. The rank correlation ϱ between two independently chosen linear orders on m elements is nearly normally distributed with expected value 0 and standard deviation $O(1/\sqrt{m})$ (cf. [KG90]). Hence $(\varrho+1)/2$ has expected value $1/2$, which motivates the following conjecture.

CONJECTURE 6.25. *Every finite metric space can be mapped into* $\mathbb{E}_1$ *with accuracy* $\geqslant 1/2$.

ADDITIONAL PROOFS

Nicht des Weges Länge
noch des Pfades Enge
schreckt mich davon.
Paul Fleming, Tugend ist mein Leben

GRAPHS

Proof of Proposition 1.3 on page 11.

1. is trivial.

2. Assume that C is of smallest cardinality among all odd circles, let x,y be neighbours on C, and let z be the vertex on C opposite to the edge xy. Then C contains a shortest path P_{xz} joining x and z ⟦ Otherwise C splits into two paths P_1 and P_2 between x and z that are both longer than some shortest path P_{xz}. Since C is odd, we may assume that also $\ell_G(P_{xz}) + \ell_G(P_1)$ is odd (otherwise exchange P_1 and P_2). But then the closed walk $P_{xz} + P_1$ would contain an odd circle shorter than C ⟧. Similarly, C contains a shortest path P_{yz} joining y and z, so that C splits into the paths P_{xz} and P_{yz} and the edge xy. Hence $d_G(x,z) + d_G(y,z) + d_G(x,y) = \ell_G(P_{xz}) + \ell_G(P_{yz}) + 1 = \ell_G(C)$ is odd.

On the other hand, assume that x,y,z minimize $d_G(x,y) + d_G(y,z) + d_G(z,x)$ among all triples for which this value is odd. Choose corresponding shortest paths P_{xy}, P_{yz}, and P_{zx}. These three paths must have disjoint edges ⟦ Assume that P_{xy} and $P_{xz} := P_{zx}^{-1}$ share an edge vw. At w, P_{xy} splits into shortest paths P_{xw} and P_{wy}, and P_{xz} into shortest paths P'_{xw} and P_{wz}. Since $\ell_G(P_{xw}) = \ell_G(P'_{xw}) = d_G(x,w) \geqslant 1$, we would have an odd value $d_G(w,y) + d_G(y,z) + d_G(z,w) < d_G(x,y) + d_G(y,z) + d_G(z,x)$ ⟧. Hence, their union is an odd circle.

3. Similarly, assume that x,y,z minimize $d_G(x,y) + d_G(y,z) + d_G(z,x)$ among all triples for which the intersection of the three generalized segments $\overline{xy}^{d_G}, \overline{yz}^{d_G}$, and $\overline{zx}^{d_G}$ is empty. In particular, x,y,z must be distinct. Again, each triple of corresponding shortest paths P_{xy}, P_{yz}, and P_{zx} must have disjoint edges ⟦Assume that P_{xy} and $P_{xz} := P_{zx}^{-1}$ share an edge vw, and replace x by w as above, so that $d_G(w,y) + d_G(y,z) + d_G(z,w) < d_G(x,y) + d_G(y,z) + d_G(z,x)$. Since then $\overline{wy}^{d_G} \subseteq \overline{xy}^{d_G}$ and $\overline{zw}^{d_G} \subseteq \overline{zx}^{d_G}$, the corresponding intersection of segments would still be empty⟧. Since x,y,z are distinct, the three paths unite to a proper circle.

On the other hand, let C be a smallest circle. (i) If C is odd, there are x,y,z for which $\alpha := d_G(x,y) + d_G(y,z) + d_G(z,x)$ is odd. Then there can be no $c \in \overline{xy}^{d_G} \cap \overline{yz}^{d_G} \cap \overline{zx}^{d_G}$, for otherwise $\alpha = 2\big(d_G(x,c) + d_G(y,c) + d_G(z,c)\big)$. (ii) If C is even, choose successive vertices x,v,y on C, and let z be the vertex on C opposite to v. Note that these four vertices are distinct. Again, C splits into shortest paths P_{xz}, P_{yz}, and $P_{xy} = (x,v,y)$. Also, $d_G(x,z) = \ell_G(P_{xz}) = d_G(y,z) = \ell_G(P_{yz}) \geqslant 2$ since $\ell_G(C) \geqslant 6$. In particular, $z \notin \overline{xy}^{d_G}$ ⟦since $d_G(x,y) = 2$⟧, $x \notin \overline{yz}^{d_G}$, and $x \notin \overline{yz}^{d_G}$ ⟦since $x \neq y$ but $d_G(x,z) = d_G(y,z)$⟧. Assuming that $c \in \overline{xy}^{d_G} \cap \overline{yz}^{d_G} \cap \overline{zx}^{d_G}$, we therefore know that c is distinct from x,y,z. In particular, c cannot be on both P_{xz} and P_{yz}, so we may assume that it is not on P_{xz} ⟦otherwise exchange x and y⟧. But then there are shortest paths P_{xc} and P_{cz} with $\ell_G(P_{xc}) + \ell_G(P_{cz}) = \ell_G(P_{xz}) < \ell_G(C)/2$ that are not contained in P_{xz}. Hence the union of P_{xc}, P_{cz}, and P_{xz} would contain a circle shorter than C. □

QUOTIENT VECTOR SPACES

Proof of Lemma 1.8 on page 21.

The first equivalence is trivial ⟦torsion-freeness means $rx + pM = (r + pR)(x + pM))pM \implies r + pR = pR$ or $x + pM = pM$⟧. If $pM \neq M$ then M/pM is not a singleton, hence R/pR cannot have zero-divisors when M/pM is torsion-free ⟦$rs = 0 \implies r(sx) = 0 \implies r = 0$ or $sx = 0 \implies r = 0$ or $s = 0$ or $x = 0$; now choose $x \neq 0$⟧. This is equivalent to p being prime.

On the other hand, let p be prime, R be a principal ideal domain, $rx \in pM$, and $p \notin pR$. Then $rx = py$ for some $y \in M$, and $1 = sp + tr$ for some $s,t \in R$, hence $x = spx + trx = spx + tpy = p(sx + ty) \in pM$. Finally, additivity of w_p follows from the fact that for an M-prime p, $rx \in p^n M$ is equivalent to $r \in p^k R$ and $x \in p^{n-k} M$ for some $k \in \{0,\ldots,n\}$ ⟦Assume that $r_0 x_0 = p^n y$. Inductively, define $r_i \in R$ and $x_i \in M$ with $r_i x_i \in p^{n-i} M$ like this: Whenever (i) $r_{i-1} \in pR$, choose $r_i \in R$ with $r_{i-1} = pr_i$ and put $x_i := x_{i-1}$. Whenever instead (ii) $x_{i-1} \in pM$, choose $x_i \in M$ with $x_{i-1} = px_i$ and put $r_i := r_{i-1}$. Finally, $r_0 = p^k r_n$ and $x_0 = p^{n-k} x_n$, where k is the number of times case (i) was applied⟧. □

Proof of Theorem 1.9 on page 21.

Reflexivity and symmetry of $\sim$ are trivial. Transitivity: $sx = ry$ and $ty = sz$ imply $stx = rty = srz$, hence $tx = rz$ since M is torsion-free and $s \neq 0$. Congruence: $(y,s) \sim (y',s')$ implies that $rs'(sx+ry) = rs(s'x+ry')$, hence $(x,r)+(y,s) \sim (x,r)+(y',s')$. M is not a singleton since $pM \neq M$. Since M is also torsion-free, R cannot have zero-divisors, so that its quotient field Q exists. The proof that V is a Q-vector space is straightforward. That the valuation w_p on V is well-defined is proved just as in the quotient field case since w_p is additive on both M and R and finite on R. Additivity: $w_p(\frac{sx}{tr}) = \big(w_p(s)+w_p(x)\big) - \big(w_p(t)+w_p(r)\big) = w_p(\frac{s}{t}) + w_p(\frac{x}{r})$. Finally,

$$\begin{aligned} w_p\left(\frac{x}{r}+\frac{y}{s}\right) &= w_p\left(\frac{sx+ry}{rs}\right) = w_p(sx+ry) - w_p(rs) \\ &\geqslant \big(w_p(sx) \wedge w_p(ry)\big) - w_p(rs) = w_p\left(\frac{sx}{rs}\right) \wedge w_p\left(\frac{ry}{rs}\right) \\ &= w_p\left(\frac{x}{r}\right) \wedge w_p\left(\frac{y}{s}\right), \end{aligned}$$

so that d_p is an ultra-metric. □

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>
> *Nietzsche,*
> *Thus Spake Zarathustra*

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INDICES

Numbers set in boldface refer to pages that contain a term's definition or explanation.

Symbols

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