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Re-thinking ‘concrete to abstract’: towards the use of symbolically structured environments

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Abstract
In this article, we question the prevalent assumption that teaching and learning mathematics should always entail movement from the concrete to the abstract. Such a view leads to reported difficulties in students moving from manipulatives and models to more symbolic work, moves that many students never make, with all the implications this entails for life chances. We propose working in “symbolically structured environments” as an alternative way of characterising students’ direct engagement with the abstract and exemplify two such environments, both of which involve early number learning. We additionally propose some roles for the teacher working in a symbolically structured environment.

Key words: manipulatives; concrete and abstract; learning mathematics; symbolically structured environment; tens chart; *TouchTimes*

Introduction

Both in most learning theories in mathematics education, and in intuitive approaches to pedagogy, there are widespread assumptions that the teaching and learning of mathematics should begin with the concrete and familiar, more abstract and symbolic knowledge arising later. The concrete to abstract assumption is supported by the large number of physical manipulatives that pervade the mathematics education landscape, in an attempt to offer entry points that are ‘concrete’ and meaningful to students. While many researchers have espoused the benefits of such physical (as well as, more recently, virtual) manipulatives, others have questioned their effectiveness. While Pimm (1995) points out that much work needs to be done in order to support students’ interactions with manipulatives, which do not in and of themselves *contain* mathematical concepts, researchers in the cognitive sciences have argued that the use of manipulatives can prolong the learning process by requiring students first to learn how to engage with the manipulative and then link the manipulative to the mathematics concepts themselves (Uttal et al., 1997). Evidently, more research is needed on what effective use of manipulatives in the mathematics classroom entails.

In this article, we propose that mathematics learning does not necessarily begin with the “concrete”. We will provide a theoretical argument for this, first by studying the acquisitionist theories on which this assumption rests and showing how they can be disrupted by a different set of philosophical assumptions that underscore the relational, embodied and material nature of thinking and knowing. We then will propose a different way of conceptualizing the concrete/abstract distinction and exemplify it by providing two examples involving the teaching and learning of arithmetic using specific symbolically structured environments (which we will described in more detail later). In discussing these examples, we draw attention not only to the design of the environments, but also, significantly, to the role of the teacher and the pedagogical choices involved in effectively using them.

Concrete and abstract

In this section, we survey some of the different ways in which the terms 'concrete' and 'abstract' have been used in the mathematics education literature, because we think that the distinctions made between them have played an important role in promoting the use of manipulatives in the teaching and learning of mathematics. A number of acquisitionist theorists posit a concrete-to-abstract developmental progression. This is evident in Piaget's (1954) work, for example, where the "concrete operational stage" precedes the formal or abstract one. While his stages do not imply that learning moves from the concrete to the abstract, his postulated notion of "reflective abstraction" takes abstraction to be a subtractive process in which one must extract elements from a lower-level structure in order to reconstruct a new, higher-level structure (Piaget, 2001). The abstract *depends on* the concrete. In building on Piaget's approach, Simon et al. (2004) characterize the "abstraction of the relationships between activity and effect" (p. 319) of the reflection phase in a reflective abstraction, which they see as the "first phase in the development of a new conception" (ibid). Here too, the activity (in this case, manipulating virtual 'sticks' in a computer-based environment in order partition one long stick into five equal parts) involves physical actions (moving sticks, observing the effect of these movements) that are then taken to be the source of the abstraction.

Physical materials have long been used in order to facilitate meaningful mathematics teaching and learning (e.g., Sowell, 1989). The impetus to do so can be traced to Bruner (1966), whose own work was based on the classic theories of cognitive development of Piaget, in which learning begins as a concrete, sensorimotor process. Bruner's (1996) three stages of enactive, iconic and symbolic privileges action on concrete objects over the more abstract manipulation of symbolic ones. Indeed, according to Bruner, new concepts should be presented in these three forms, one after another, gradually increasing the abstraction (that is, becoming symbolic). For Bruner, abstraction was specifically about decontextualizing, which is a form of removal, and involved removing the familiarity of physical, situated and non-symbolic materials. Some researchers (e.g., Fyfe, McNeil, Son & Goldstone, 2014) have empirically studied Bruner's approach, first by offering students concrete materials and then by gradually removing (or "fading out") contextual elements in order eventually to reach more abstract representations. This perspective reifies the concrete-to-abstract order and tightly associates the abstract with the symbolic. It is also associated with the ideology of 'development', which draws on a philosophical commitment that denies the materiality of the mathematical concept. In other words, concepts are seen as being abstracted away from the physical situation, so that learners are granted only enough embodiment to perform that abstraction.

Vygotsky's (1978) approach, which is sometimes contrasted with that of Piaget because of the way it posits learners as moving from abstract to concrete, may seem to counter the prevailing assumptions outlined above. However, Vygotsky is deploying the words 'concrete' and 'abstract' in slightly different ways from Piaget or Bruner, attending more to the learner-object relation. For example, the concept of 'honour' will be abstract at first for learners because they have no experience of that concept; but it will become more concrete as they gain experience related to worth, war, promises, etc[1]. This view of the concrete/abstract duo (that we see as a 'relational' one) is also put forward by Wilensky (1991), who argues that while the alphanumeric inputs of programming languages such as Logo may seem abstract, being symbolic as they are, they can also be seen as concrete for some children inasmuch as some children will have had encounters with these symbols that give them a direct and visible reference. Wilensky sidesteps the question of whether or not learning should begin with the concrete or the abstract, and instead advances the relational perspective that both sides, concrete/abstract, are context-bound and subjective (i.e., one is not necessarily harder for learners than the other) and, further, that symbolic infrastructures may provide suitable environments for learning mathematics.

[1] This resonates with how Hadamard is said to have opined, the concrete is the abstract made familiar by time.

We take a similar perspective, but one that is rooted in our respective philosophical commitments to enactivism (Maturana & Varela, 1987) and inclusive materialism (de Freitas and Sinclair, 2014). With enactivism, we view meaning as a feature of relationships, that reside neither in an individual, nor group, nor an object or tool (Thompson & Stapleton, 2009). In every interaction between two organisms, both are changed, both leave traces; our relationship with the world we experience is in constant flux, hence so is every meaning (Varela, Thompson & Rosch, 1991). As humans, how we come to experience the world is a result of our history of co-evolution with everything around us; meaning is an achievement of joint activity and we view knowing and doing as synonymous (Maturana & Varela, 1987). In the same way that every action is an interaction, everything we might interpret as an individual construction of knowledge can only be separated, at the cost of insight, from the wider relationships in which it arises (Bateson, 1972).

With inclusive materialism, we further challenge the premise of ways in which the abstract-concrete distinction is drawn, arguing that abstract thought and materiality are entwined. Instead of seeing concepts as abstract, that is, as being abstracted from perceptions of and actions with manipulatives, which entails a binary relationship between mind and body, inclusive materialism insists on the materiality of concepts. This position challenges the socio-cultural assumptions prevalent in the current mathematics education community whereby mathematical meanings depend exclusively on human language and culture. From this perspective, the image of doing mathematics is neither dualist nor extractive, that is, about abstracting away from sensorimotor experiences or from a passive environment through language and other cultural tools. Rather, what is abstract is reconceived as being about the virtual, or the potential, that which is latent in matter but not yet actualised. Abstraction does not produce an escape from the real; it enlarges it.

In this text, we will be mobilising the terms abstract and concrete but using them to distinguish different forms of relationship, or modes of engaging, with the world (Coles, 2017). In broad terms, we find it helpful to draw the distinction that we can attend to *things* (e.g., objects, concepts) and we can attend to *relations between things* (e.g., differences, changes). We will therefore speak of attending to things as a concrete mode of mathematical activity and attending to relations as an abstract one. We underscore that in our characterization, concrete and abstract are no longer opposites of each other. Moreover, unlike in the everyday sense of concrete and abstract, in which the latter is seen as more mathematical or more advanced than the former, we see both concretising[2] and abstracting as productive shifts in attention: the former moves from relations to objects while the latter moves from objects to relations.

It is not things themselves that are concrete or abstract but rather how they come into meaningful relationship with (in our case) a person. For example, we can attend to a set of multi-link cubes as objects (e.g., noting that there is a line of 4 cubes and a line of 2 cubes) or we can attend to relations between the cubes (e.g., noting that one line is half the length of the other). The first mode would be concrete, the second abstract. *Crucially, one does not necessarily, or naturally, precede the other.* Or, to take a different example, one approach to early number might entail children being encouraged to count different sets of objects. Attention here is initially on the collection of objects being named. Perhaps by counting different sets of objects, there can then be a move to considering the fact that the same system of counting can be used in each case (i.e., moving to a more abstract consideration of the relations between counting different kinds of objects). A different approach to number might entail children counting in 1s from 1 to 9 (as a song) and then counting in 100s from

[2] It is interesting that while “abstracting” is a frequently used verb in mathematics education, the word “concretising” is seldom used.

100 to 900 and then counting in 1000s from 1000 to 9000. With no objects as reference except the number names themselves, the focus here is on the structure of the number system and the relations between different kinds of count, i.e., abstract from the start—attending to relations. Curriculum design in mathematics has tended to prioritize the counting of objects, attending to things, emphasizing cardinality as the starting point. We do not see any reason why such work has to come before more relational, and therefore abstract, work on number naming (Coles & Sinclair, 2019).

We see our characterisation of the abstract and the concrete as being relational, while leaving it an open question as to which focus might come first in any learning context. Additionally, we see them as avoiding other dichotomies that underpin most theories of learning in mathematics, including the doing-knowing dichotomy that ascribes conceptual understanding to reflections on doing; the doing-thinking dichotomy that associates doing with mere bodily actions (see also Coles & Sinclair, 2018); and, the material-symbolic dichotomy that subordinates matter to meaning.

In the next section, we trace problems that are reported around how students come to operate with abstract concepts and consider different historical perspectives on the use of manipulatives and their links to symbols. These differences suggest the move from concrete to abstract is not a developmental universal but an outcome of curriculum and pedagogical choices.

Manipulatives and symbols

One result of the assumption that learning proceeds from the concrete to the abstract has been the historical drive, for example as seen in the 1980s in the USA, to use manipulatives in learning mathematics (Sowell, 1989). The assumption here has been that manipulatives provide a mechanism for students to engage in giving meaning to mathematical objects, which can then be mapped onto more standard, abstract notation. The idea that meaning is created through an individual, or group, grappling with manipulatives and metaphors of mathematical concepts, and then generalising from particular experiences, is linked to the constructivist views of learning (Piaget, 1954) discussed above.

Rittle-Johnson et al. (2015) have suggested that there has been an unquestioned (and un-tested) assumption, which we see driving manipulative use, that instruction in mathematics should proceed from the conceptual to the procedural. Manipulatives are then used with the aim of beginning instruction with conceptual understanding of processes, e.g., base-10 blocks to understand a subtraction algorithm (Fuson & Briars, 1990). We acknowledge that positive benefits have been reported in relation to manipulative-based approaches to learning arithmetic (Mix et al., 2017). However, as far back as 1997, Uttal et al. reported on a series of experiments that suggest the use of manipulatives may set up a dual-representation system, one system being the manipulatives and the other being the symbols or operations intended to be represented. Students frequently became confident working within the manipulative system, but did not see the connection to the symbolic system. Furthermore, learners who are able to translate between systems are doing more and harder work than those able to operate purely within the symbol system (as mathematicians do). These critiques raise the question, whether the reported advantages of using manipulatives may be dependent on curriculum and pedagogical choices that do not benefit all learners.

We note that researchers have argued that effective virtual manipulatives should "link[ing] the concrete and the symbolic with feedback" (Sarama & Clements, 2009, p. 147)—as in the case of the virtual balance scale reported on by Suh and Moyer (2007), which links algebra symbols to the movement of the scale. While this emphasis on multiple, linked representations can offer benefits (Moyer-Packenham and Westenskow 2013), it often continues to rely on the use of metaphoric representations (such as the balance scale) that will still require the kind of translation Uttal et al. describe, where the action is on the representation rather than on the mathematics (*on* the numbers, *on* the shapes, *on* the relations).

A relational perspective on the concrete/abstract rarely, if ever, features in official curriculum or standards documents. Consider. for example, the well-known and influential counting principle of Gelman and colleagues (see Gelman & Meck, 1983), who propose that the earliest experiences around number should centrally involve counting objects, that is, transitive counting, and culminate in the ability to determine the number of objects in a given set, that is, in successfully answering the "How many?" question. The assumption about the concrete may not at first be obvious, but that is because so much of early number learning is conceptualised as being cardinal in nature, in which the purpose of number is precisely to determine the numerosity of a set. In such a conception, it is natural to assume children should begin their number experiences by counting things, that is, by taking 'things' as metaphors for quantities. In this line of thinking, the very meaning of a number such as seventeen, for early number learners, is the existence of a set of seventeen objects. Such an approach begins with bringing learners into contact with numbers as objects, in other words, in a concrete mode.

In our own work on learning number, we argue that offering low-attaining students concrete models of number may re-enforce the very way of thinking about number (as solely linked to objects) that students need to move away from to become more successful at arithmetic (Sinclair & Coles, 2017). It appears a short step from assuming learning begins with the concrete to concluding some children are not 'ready' for the abstract. A different conceptualisation of number can call this 'natural' assumption into question. For example, in a more ordinal conception of number alluded to above, the meaning of 'seventeen' derives from the fact that it follows 'sixteen' and precedes 'eighteen', in which case its meaning is a relational one and hence abstract (from the start). One may also attend to the symbolic patterning of the numeral 17, in which case it is the metonymic nature of number that is being stressed (e.g., its link, as a symbol, to other number symbols), rather than the metaphoric one (e.g., its link to objects). We use the word metonym in the sense of a part standing for a whole (Coles & Sinclair, 2017). In other words, a metonymic approach to number might make use of number names, initially, as the "part" of the number concept that stands for number. Within a relational conception of number, intransitive counting (reciting the number sequence) may be seen as a proper starting point. Such an approach dispenses with physical objects to be counted, and thus may strike educators as being rather abstract. But, in the Wilensky sense, its concreteness might be found in its connection to experiences like singing the number song. While Wilensky's motive was to point to the potential for so-called abstract mathematical environments being motivating for children, through the connection that these children can make with such environments, we are also interested in how working within the 'abstract' may also be conceptually advantageous, a view that will challenge deep-seated assumptions about what becomes meaningful and basic in mathematics learning.

There have been two important strands of work (Davydov, 1990; Gattegno, 1974), in the case of number learning, that examine an alternative to starting instruction with a concrete phase before moving to the abstract. In Davydov's curriculum (Dougherty, 2008) students' first experiences with number are as a comparison of measurements (of length, area or volume). The natural numbers appear as a symbolisation for the number of times a unit measure fits into a second measure. Similarly, Gattegno's curriculum for early number begins with an exploration of relationships of lengths (greater than, less than) leading to the first numerals appearing to symbolise the situation when one length fits an exact number of times into a second length[3].

In both the Davydov and Gattegno curricula, although there are differences in background philosophical assumptions, there is a similarity in the manner in which physical materials play an important part. In contrast to the kind of use of manipulatives critiqued by Uttal et al. (1997), there is a significant and, we feel, previously unmarked difference in how the manipulatives and symbols are related. A typical example of using manipulatives (Fuson & Briars, 1990) introduces a direct and absolute relation between a symbol and object, e.g., in base-10 equipment, 'units' are single cubes, 'tens' are lines of 10 cubes, 'hundreds' are squares of 100 cubes. For Davydov and Gattegno, their manipulatives also have direct and absolute symbol references (Davydov gets students to create their own notation; Gattegno uses colour names for wooden rod lengths) but these are not the symbols that are important. The key mathematical symbols are the numerals. Numerals are introduced, in both curricula, as relations between objects (Coles, 2017). Hence the concrete objects are used as a context in which to make meaningful a set of symbols, but where the key symbols are abstract from the start, denoting actions on, or relations between objects. An immediate consequence is that the numeral symbols develop connections to each other, independent of objects, because a relationship (which is what the symbols symbolise) can be viewed from two sides, e.g., if one length is double another, then the second length is also half the first, so students come to see that if, e.g., 2w=r, then w=½r. Gattegno introduces fraction notation in the first year of his curriculum; conceptual work (giving meaning to symbols) and procedural work (using symbol-manipulation rules) proceed together.

The reported successes of uses of the Davydov and Gattegno approaches to learning number provide evidence that there is no universal pattern of learners reaching the abstract as a culmination of experiences that begin with concrete examples, and/or in which mathematical symbols start off having direct and absolute concrete referents. We propose that a common feature of these approaches is captured by the concept of a "symbolically structured environment", which we explore in the next section. Our definition of this concept comes towards the end of the article.

Symbolically structured environments

The notion of a "symbolically structure environment" (SSE) first arose for us in exploring the role of ritualisation in early mathematics learning (Coles & Sinclair, 2019). In the work of anthropologist Catherine Bell (1991), we found an approach to ritualisation that eschewed the dichotomy between thought and action that characterises much of the work on ritual in the mathematics education research, and instead proposed that ritualisation "is embedded within the dynamics of the body defined within a symbolically structured environment. […] It is designed to do what it does without bringing what it is doing across the threshold of discourse or systematic thinking" (p. 93). Although Bell was speaking of environments that one might find in religious ceremonies, we see a resonance to the mathematics classroom in the notion of a way "of acting that is designed and orchestrated to distinguish and privilege what is being done in comparison to other, usually more quotidian, activities" (p. 74). In a mathematics classroom context, design and orchestration of ways of acting are an aspect of the role of the teacher.

We see the symbolically structured environments as having features in common with the notion of a microworld, first introduced by Papert to describe self-contained worlds where students can "learn to transfer habits of exploration from their personal lives to the formal domain of scientific

[3] Thus using what Fowler (1979) calls "ratio numbers".

construction” (1980, p. 177). For Papert, the microworld was an immersive environment in which students could develop mathematically fluency. It was not a manipulative that made mathematics concrete; it was mathematics, but restricted to an accessible and generative subdomain that was both formal and body-syntonic. These latter qualities connect closely with Bell’s emphasis on SSEs that enable certain structured acts of moving. In the case of Turtle Geometry, which was an early LOGO-based programming language that was used to explore shape, the propositional commands provided structured ways of body-syntonically moving a turtle on the screen, that is, of experiencing movements by having the Turtle make them.

Another SSE that is not computer based in nature is found in the Gattegno tens chart (Figure 1) which makes available the structure of our written number system, not via a cardinal linking of numerals and objects, but via a more ordinal, relational sense of making links among numerals themselves, for example, how the spoken number names form patterns, and how you can get from one number to the next (see Coles, 2014, for further possibilities for working with the chart).

1	2	3	4	5	6	7	8	9
10	20	30	40	50	60	70	80	90
100	200	300	400	500	600	700	800	900
1,000	2,000	3,000	4,000	5,000	6,000	7,000	8,000	9,000
10,000	20,000	30,000	40,000	50,000	60,000	70,000	80,000	90,000
100,000	200,000	300,000	400,000	500,000	600,000	700,000	800,000	900,000

Figure 1: An example of Gattegno’s whole number tens chart

The symbols used in both Turtle Geometry (such as “fd 10” or “rt 90”) and the Gattegno chart can be meaningful from the start if they are introduced (likely by a teacher) and used to perform actions or mark distinctions. Entry into an SSE requires a constrained beginning, for a teacher to establish the “rules” of the game (some of which are arbitrary (Hewitt, 1999)). Later on, students can have opportunities to instigate their own use of the symbols so that their activity is not merely repetitive or procedural. In these SSEs, student work is metonymical since they act on things or objects which are not taken to represent mathematics, but to be mathematics itself, as we illustrate in the next section, drawing on empirical work we have carried out using two different symbolically structured environments. This data is offered in an illustrative manner, to point to possibilities and ground the theoretical ideas above.

Learning in a non-digital symbolically structured environment

We have reported elsewhere (e.g., Sinclair & Coles, 2017) on student work using the tens chart and other SSEs. The project for which we have the most student data was one run by the first author (AC), with 7- to 8-year-old children in which the tens chart was used as an introduction to multiplication and division (see Coles, 2014). Work with the class occurred in a fairly typical (in terms of prior attainment) rural primary school in the UK, as part of a project linked to the charity “5x5x5=creativity” which place artists (in this case, a ‘mathematician’) into schools to run projects. Having set up how to multiply and divide by 10 and 100 on the chart, emphasising the visual and gestural way this can be done, AC proposed a challenge. The students had to choose a number on the chart, go on a journey multiplying or dividing by 10, 100 and to get back to where they started. There was a constrained beginning, in which the students had to learn how to ‘play the game’ and use the symbols ($\times$ and $\div$) in the way that AC modelled. The idea of a teacher offering a constrained entry into a complex mathematical environment was written about by Gattegno (1965) in a description of his technique of teaching and is picked up by Coles and Brown (2016) in discussion of task design.

Linked to our discussion of Davydov and Gattegno, the symbols we cared about in this activity (× and ÷) arose as standing for actions or relations on the chart (i.e., how you go ‘down’ and ‘up’ a row) and hence were abstract from the start. The availability to students, of the symbolic structuring of the chart, then allowed scope for innovation. One student, on her fourth ‘journey’, wrote what we have copied in Figure 2.

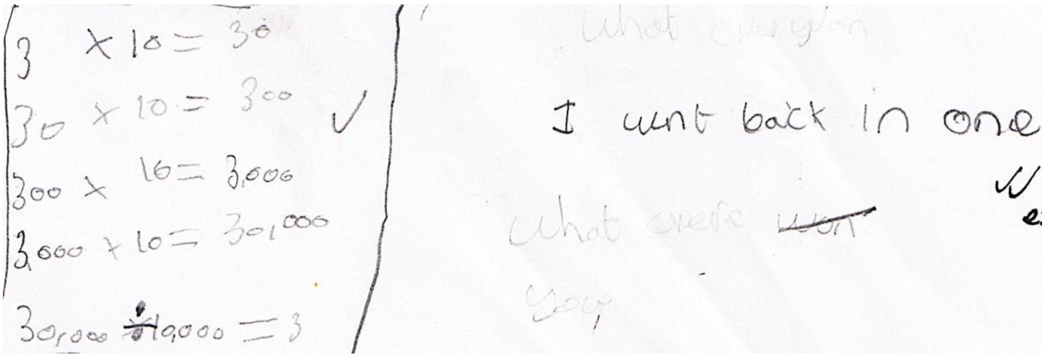


Figure 2: A student’s ‘journey’ on the tens chart, “I went back in one” (from Coles, 2017)

The student had chosen to extend the pattern of multiplication and division by powers of 10 (only 10 and 100 had been modelled) to include division by 10,000, having set herself the challenge to “get back in one”. During each lesson there were times of whole class discussion in which students were invited to share what they had done, meaning that innovations by one student were sometimes picked up by others and, perhaps as importantly, permission to innovate was implicitly given through AC’s encouragement and links he made between innovations and what mathematicians do. Many of the students showed evidence of similar creativity in setting themselves challenges and extending their symbol use, in a metonymic manner, beyond that to which they had been introduced.

The student in Figure 2 might not have been able to read “ten thousand” as a number name and we realise this kind of metonymical way of working may concern some readers. However, we would like to highlight the important role that children’s bodies are playing in their interactions within this SSE (i.e., gestures on the chart). This bodily engagement, which we see as an embodied extension with a tool, and the evident inventiveness of what children do and write, indicates that what we are advocating is far from a mechanical activity of meaningless symbol manipulation. It is also important to emphasise that it is not the symbols in the tens chart that, for us, make it “symbolic” in terms of a symbolically structured environment, but rather the fact it is the actions performed on the chart that are symbolised.

There is a visual chart with symbols that might act as objects of attention (hence, concrete). However, the task directs students’ attention onto relationships between those objects. And it is the symbolisation of these relationships (multiplication and division by powers of ten) that seems key to what allows innovation and, for us, the power the chart affords. Attention to relationships means that the work done by students, described above, includes the abstract, from the start. A symbol such as “10” or “100” was used, by all students in the class, as both a number on the chart and as part of the symbolisation of a representation of relationships between numbers on the chart. There was seemingly no difficulty for students in handling this apparent ambiguity and the use of the same symbol in both a concrete (object-based) and abstract (relational) manner simultaneously.

Learning in a digital symbolically structured environment

We now provide a second example of an SSE, this time using the touchscreen digital tool called TouchTimes (TT) (Jackiw & Sinclair, 2019), which supports children's exploration of multiplication. After a brief description of the Grasplify world of TT, we offer a transcript of two boys interacting with a researcher during pilot studies in a grade two classroom conducted in December, before multiplication was formally introduced to the children. When first opened, the Grasplify world displays a screen that is divided in half by a vertical line (Figure 3a). Whichever side of the iPad screen is first touched by the finger(s) of the user's hand, a different-coloured disc (termed a 'pip') appears beneath each one and the numeral corresponding to the number of visible pips is displayed at the top of the screen (Figure 3b). The numeral adjusts instantly when fingers are added or removed, whether temporally in sequence or simultaneously. It represents the multiplicand. In order to preserve each pip (that otherwise vanish) the finger(s) must maintain continuous screen contact. When a user taps on the other side of the line with her second hand, bundles of pips (called 'pods') appear beneath each contact finger of this second hand (Figure 3c). The number of pods is the multiplier and each pod contains a duplicate of the pip configuration, matching both the relative locations created by the first-hand placement (Figure 1d) and the colours. When pods are created, a second numeral also appears, separated from the first by the multiplication sign ('x').

When a finger is removed from above a pod, that pod remains on the screen, but becomes slightly smaller, so that more of them can be seen at the same time (Figure 3d). After pods are created, TT encircles all of them into a single unit, by surrounding them with a white line (like a lasso) and (after a short delay) displays the corresponding mathematical expression (e.g. '3x4=12') at the top of the screen. The product is also in white font (see Figure 3d). If a pip-creating finger is lifted, the contents of each of the pods adjust accordingly: that is, each pod will then contain one fewer pip (the vanished pip from each pod being the same colour as the disappearing pip on the other side).

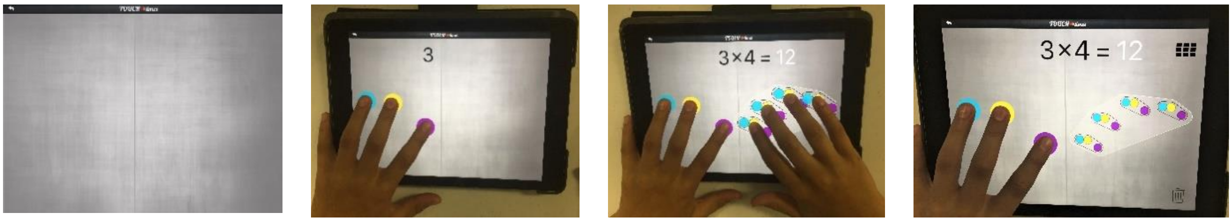
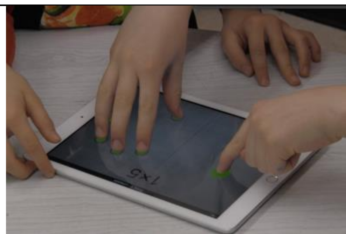
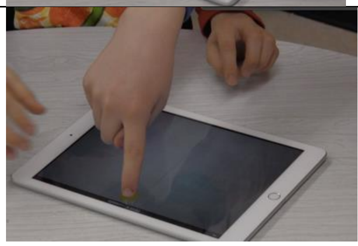


Figure 3: (a) Initial screen of TT; (b) Creating pips; (c) Creating pods; (d) Finished expression Multiplying by 1

The excerpt presented below, which lasted 55 seconds, occurred about 7 minutes into a clinical interview, which was conducted as part of a larger project investigating the potential of teaching and learning multiplication using *TouchTimes*. The two boys, both eight years old, were new to TouchTimes. The interviewer allowed them to explore before intervening with any questions or prompts. No use of the word "multiplication" or "times" was used by the researcher, who was trying to understand how the boys made sense of the environment. We have chosen this excerpt because it illustrates the co-arising of abstract and concrete modes that we see as typical in children's engagement with SSEs. To help read the transcript, our interpretation is that Roger uses the word "minus", part way through, for what we would conventionally take to be "multiplied by".

Phrases underlined correspond to gestures shown in the images on the right. RS = right side of the screen and LS = left side of the screen.

(*Roger holds his index finger on the LS and tapping on the RS with a pod-making finger to 10 to produce 1 x 10 = 10*) Luke: Ten? T: What happens if you add a finger on this side (*pointing to the LS*)? (*Roger lets go of his index finger, which makes the screen blank. Luke puts two pip-making fingers on the LS.*)	
Roger: I think because (*pointing to the top part of the screen*) one minus (*Luke puts five fingers on the RS and produces 2 x 5 = 10*) anything equals (*Luke adds two more pod-making fingers on the RS to produce 2 x 7 = 14*) the number (*pointing to the '14'*). T: Say is again, sorry. (*Luke lets go of his right hand but then places more pod-making fingers to produce 2 x 11 = 22*)	
Roger: So like (*Luke adds three more pod-making fingers to the RS to produce 2 x 14 = 28*) if you put the one right here (*pointing to the symbol '2'*) minus this (*pointing to the symbol '14'*) it would be fourteen. If this was sixty-four (*pointing to the symbol '14'*) it would be sixty-four (*pointing to the symbol '28'*). Luke: Oh I get it (*moving his two pip-making fingers on the LS*). T: Would it work for this situation (*Luke lifts a pip-making finger to produce 1 x 14 = 14*) that [Luke] is doing (*Luke places a pip-making finger to produce 2 x 14 = 28*)?	
Roger: Well mostly what I understand (*Luke moves his two pip-making fingers on the LS and then Roger pushes his fingers away*) what I understand (*Roger places one pip-making finger and five pod-making fingers to produce 1 x 5 = 5*). Luke: Equals five see (*Rogers lifts both hands*). T: So when you say see what do I see? Say it again.	
Roger: So if you (*puts one pip-making finger on the LS*) add one here and you add (*puts five pod-making fingers on the RS*) five there Luke: Ya (*points to the equation*) it's going to make one. Roger: And if there's one here (*points to the '1' in the equation*) and if (*lets go of his hands*) there was sixty-four here (*points to the top middle of the screen*) where the five was, it would be sixty-four (*points to the top right of the screen*).	

In this episode, both children (Roger in particular) make a generalization about the relation between the three numbers that appear on the screen, namely that when the first number is a 1 (the multiplicand), the second number (the multiplier) will be the same as the last number (the product). The interviewer has a role in provoking a question "What happens if …" and in suggesting a way of testing Roger's suggestion, "Would it work for this situation". There is a subtlety to these interventions that point to the role of the teacher in a SSE. The interviewer's suggestions support a noticing (Mason, 2002) of things that have already engaged the attention of the students. Roger does not generalize[4] for any number, but chooses a large, arbitrary number (sixty-four, which acts as a generic example in the sense of Mason and Pimm (1984)) to illustrate the relation he has noticed. However, as readers will have likely noticed, Roger does not describe this verbally as a multiplication statement. Indeed, he points to the ‘x’ symbol and calls is “minus” and he speaks of “adding” numbers to each of the sides. From a certain point of view, Roger does not appear to understand what multiplication is. He certainly does not seem to see it as an operation offering repeated addition.

From a Piagetian perspective, it would be tempting to say that the children abstracted a relationship between action (changing the number of the multiplier) and effect (the resulting product), having been perhaps perturbed by the double occurrence of the same number (1 as multiplicand and 1 as product). But this dichotomises the action (placing and removing fingers) and the thinking (reflecting on the effect of such actions) and takes the relationship to be emerging from the objects involved. If instead we notice that Roger and Luke were acting on the relation (abstractly) at the same time as they were producing and modifying finger touches (conjuring pips and pods), then we can say that they were working with the concrete and the abstract at the same time, exploring, indeed, the way multiplication functions or behaves. They did not first need to use a physical manipulative to represent multiplication (as groups of objects), before extracting a symbolic meaning. In other words, the meanings do not rest solely in the visual representation given on the screen (of pips and pods or, more formally, a unit of units). Instead, they arise from the combination of the manipulating fingers and the symbolic output that changes as a result of actions on the screen.

Characterising work within a symbolically structured environment

Taking our characterization of concrete and abstract from earlier in this article, we observe that in SSEs both concrete and abstract modes of engagement seem to be around simultaneously and permanently. With the tens chart there are the numbers on the chart (objects, attention to which is concrete) and relations between them (x10, attention to which is abstract); with *TouchTimes* there are gestures that generate pips and pods (objects, attention to which is concrete) and gestures that generate a relationship between pips and pods (represented by multiplicative statements, attention to which is abstract). In the contexts reported above it seems clear that children have no difficulty working with relations, i.e., abstract modes of engagement, when these are offered in ways where the objects being related are visible or tangible. There are, of course, many features of what we are characterizing as a SSE; our focus is on symbol use and the way in which concrete and abstract modes of engagement are mobilized.

We are now in a position to offer a tentative characterization of a teacher and students’ working within a SSE, as meaning: (a) symbols are offered to stand for actions or distinctions; (b) symbol use is governed by mathematical rules or constraints embedded in the structuring of environment; (c) operations can be immediately linked to their inverse; (d) complexity can be constrained, while still engaging with a mathematically integral, whole environment; (e) novel symbolic moves can be made. While in a school context it is likely to be a teacher who offers the symbols and their uses and who constrains the entry into the environment, such roles can be taken on by students, particularly if they work with a particular environment over an extended period of time.

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[4] While the terms generalization and abstraction are often used as synonyms of sorts, our charaterisation of the abstract as attending to relations clearly distinguishes these two types of activity.

We will consider each element in relation to the two examples above.

	Tens chart	**TouchTimes**
(a)	Actions of moving between rows are symbolized by $\times$ 10 and $\div$ 10, etc.	Actions of touching the screen generate symbols ($\times$ and =).
(b)	The structuring of the chart itself ensures that the actions of moving between rows have a consistent mathematical meaning.	The App ensures that what is symbolized in numerals is always a mathematically accurate statement, a description of the relationship between objects on the screen.
(c)	Children work with multiplication and division at the same time, as moving up and down rows of the chart.	Inverse arises in TouchTimes through the capacity to "do" and "un-do" actions, e.g., placing a second finger down has an effect and removing it undoes that effect on pips and pods.
(d)	The teacher can constrain the number of rows visible on the chart and e.g., in the context of multiplication and division, to constrain work initially to single columns.	In the example above, the teacher makes suggestions (adding one finger on one side) that will entail some things changing and some staying the same, constraining the potential complexity.
(e)	In the example we offered above, Figure 2 shows a child performing a division by 10,000 which was not a move that had been demonstrated by the teacher.	There is novelty in Roger invoking the number sixty-four in order to try to express the relationship he observes in how TouchTimes symbolizes the fingers on the screen.

Across the uses we have made of SSEs, including those described above, we observe students being energised by making sense of the environment they are offered. We conjecture that there may be something engaging in being offered symbols that allow learners to do things or in learners being occasioned to do things which they then symbolise. There is perhaps a sense of anticipation about the feedback from the environment which supports the development of meaning for the symbols being used. While the role of the teacher has been alluded to in our discussion so far, we are aware that there is much more to be said and that is the focus of the next section.

Roles of the teacher in a SSE

One claim we are definitely *not* wanting to make is that providing children with symbolically structured environments will somehow make learning happen by itself. The role of the teacher is vital, albeit changed. We first consider the role of the teacher in relation to the aspects of an SSE that we have identified above. When working in an SSE, the teacher must: (a) devise a way to symbolise actions; (b) incorporate inverse processes; (c) provide constraints (such as tasks); and, (d) attend to students' novel symbolic moves. We now discuss each of these in turn, exemplifying them with the two SSEs described above. We note that (b) has also been discussed by Greer (2012) who writes that "[i]nversion is a fundamental relational building block both within mathematics" (p. 429) and who argues for a much greater focus on inversion in mathematics teaching and learning. Additionally, (c) has been identified over the past decade as a central feature of teaching (e.g., Watson & Ohtani, 2015), particularly in relation to the use of digital technologies (Leung & Baccaglini-Frank, 2017). We see (a) and (d) as relatively new practices that are strongly associated with the use of SSEs. Significantly, they avoid the procedural-conceptual dichotomy that has received so much attention in research on mathematics teaching and learning, and that is related to the more traditional concrete-abstract distinction.

As mentioned earlier, our interest in symbolically structured environments arose out of our engagement with the work of Bell (1991) and consideration of the potential for ritualization practices in the mathematics classroom. We argued (Coles & Sinclair, 2019) that there is a role for the mathematics teacher in the "setting apart of distinct and privileged activity" (p.189), for instance, in proposing particular symbols and actions. These choices constrain the activity of students in ways that can be productive in terms of their doing of mathematics. And far from ritualization being a rote and meaningless practice, there can be a sense of alignment of teacher and student attention and a letting in on the rules of the game of mathematics, which we see as potentially significant in terms of access to mathematics for students.

Devise a way to symbolise actions: sometimes the teacher may need to set up the ways in which actions are symbolized. The symbolizing function is already programmed into *TouchTimes*, but with the Tens Chart, it is the teacher who needs to introduce how to express the movement between rows. In general, our sense is that symbolization is arbitrary and so can be introduced without explanation in a game-like manner. Children need to get used to symbols and they will only do this by using them. It is for this reason that the teacher does not correct Roger in the *TouchTimes* episode when he says "minus" instead of "times", since the focus is on the relation and not on the specific name of the relation. It is this aspect of an SSE—the symbolizing of actions—that is perhaps most linked to a teacher's planning of a lesson. The actions children perform need to generate some feedback from the environment and that feedback needs to ensure that the symbols their actions create have mathematical coherence. In the case of the Tens Chart, the teacher planned to use the gestures of moving down and up and to do so in a deliberate way as the words 'divide' and 'multiply' were said. In the case of *TouchTimes*, the specific gestures associated with making pips and pods have been determined to come extent by the design of the environment.

Incorporate inverse processes: allowing work to take place simultaneously on inverse processes is potentially one of the keys to make engaging in an SSE powerful in terms of learning mathematics. There is an important role for the teacher here in terms of focusing attention on the potential for inverse processes. This can be seen in the Tens Chart example, where the task for students inevitably entails them working with both multiplication and division in order to "get back where they started" on the chart. With *TouchTimes,* although only multiplication is represented, there are powerful inverse processes involved in the movement between a "1x …" screen and a "2x …" screen. The teacher has a subtle role here in terms of drawing attention to these changes.

Provide constraints (such as tasks): the constraints on working within an SSE are likely to need to come from a teacher. We recognize there is often an urge in children to explore the limits of a new environment and, while this might be a useful initial task, there soon becomes a need to focus on something more constrained in order to build awareness of relations within the environment. Neither the Tens Chart nor *TouchTimes* comes with ready-made questions or exercises. However, well-designed tasks can elicit rich activity. Designing a good task can be challenging in the examples we have provided, since they do not benefit from a singular focus on objects, which is the case for many mathematical manipulatives (and certainly for most worksheets). The journey task in the Tens Chart cannot be done outside of this environment and depends on the feedback of the environment, which students can evaluate (have they returned to the starting position) without the teacher's help. In the *TouchTimes* example, the students came up with their own task, but effective tasks offered by the teacher must rely on the feedback of the environment (see Sinclair & Zazkis, 2017 for a discussion of task design for a similar touchscreen number app).

Attend to students' novel symbolic moves: such moves, by definition, need to come from the children. However, there is again a key role for the teacher both in terms of setting up tasks that allow for novelty and also in terms of noticing those novel moves. It can be powerful to point them out to other students, both in order to occasion the expansion of the space of possible moves for everyone and also to highlight that the making of novel moves is a possibility within the environment.

Given the importance of the teacher in the effective use of SSEs, we have chosen a third example, better to articulate the way in which the teacher's role differs from typical reform-based teaching in which, for example, children work in groups to solve a rich problem using manipulatives. The example comes from a National Film Board film produced in 1961, focusing on the teaching method of Caleb Gattegno (see https://www.youtube.com/watch?v=Kw94gmzRrOY). In the film, we see a lesson with kindergarten children (5 years old) using small, wooden Cuisinaire rods to work with fractions, a concept that would be deemed too abstract for children of that age. In the lesson (which is in French), Gattegno has chosen to *symbolize the actions* by associating the putting on and taking off of rods, which he does with a deliberate, large gesture, with doubling and halving—using those words explicitly to accompany the actions (see Figure 4). While holding two blocks, for example, he places another on top of his pile and says "and what happens if I double?". Then he takes the block off and asks, "what happens if I halve?" thus *incorporating inverse relations*.

Figure 4: Gattegno taking a rod off of the pile of rods while saying "divide by two"

The children answer, in chorus, but they also have a stack of rods on their desks and make the same action as Gattegno. Later, after using the word/action for halving, he begins to substitute it with "divide by two". These conventional words, which are arbitrary, are associated with particular actions, which establish the meanings of the words. Students learn not just a way to move the manipulatives, but also how to move them in a certain way that has a certain meaning. He thus offers a range of possible actions, which *provide constraints* around what can be done with the rods. In the film, evidence of attending to students' novel symbolizing is not evident. However, there is novelty in the iteration of operations as Gattegno gets children working on challenges such as half of a half of a half of a half of a half of 128.

In each of our examples, the work with a SSE has involved the concept of number. We do not see any barrier to developing SSEs across the curriculum. We alluded earlier to Papert's Logo microworld, in which tasks can afford work within geometry, algebra and programming. Dynamic geometry software, such as *The Geometer's Sketchpad* (Jackiw, 1991), are also environments in which the focus can be on objects (such as a static circle or parallelogram) or on relationships produced through dragging (as the parallelogram's four angles become right) or even on relationships with other objects (such as diagonals). In non-digital examples, tasks on geoboards or on square dotted paper can provide entries into SSEs linked to geometry. While classroom research in the environments above has taken place, the perspective provided by a SSE points to particular ways of working and brings a focus on symbol use. Such new research is needed.

Conclusions

In the space of school mathematics, the focus on manipulatives as concrete representations seems to have eclipsed a more fundamental reason to incorporate tools into mathematics learning. As articulated by Piaget (1954), and since elaborated by many mathematics education researchers with interests in the bodily basis of understanding (see de Freitas & Sinclair, 2014), the goal of using manipulatives is neither to make a concept concrete nor to recover its Platonic meaning, but to move. That is, one's senses of number, shape, and so on "have more to do with structured acts of moving than with acts of moving structures" (Ng et al., 2018). From our perspective, the main purpose of a manipulative is not to re-present mathematical concepts, but to mould the learner's motions, and in the process occasioning opportunities for them to expand and weave their repertoires of mathematically relevant structures.

With the notion of an SSE, we are attempting to provide an alternative kind of tool for mathematics learning that is both abstract and metonymical, but also accessible and meaningful. To re-iterate, we characterise an SSE as one in which: (a) symbols are offered to stand for actions or distinctions; (b) symbol use is governed by mathematical rules or constraints embedded in the structuring of environment; (c) symbols can be immediately linked to their inverse; (d) complexity can be constrained, while still engaging with a mathematically integral, whole environment; (e) novel symbolic moves can be made.

We argue that such tools—along with careful teacher support—may provide students with less cumbersome and more direct ways of developing mathematical fluency than through the kinds of manipulatives that insist on offering direct, concrete representations. This argument has been informed by theoretical insight and our experiences working with the SSEs described above. More empirical research will be required to appreciate better the ways in which our approach can invoke significant changes in current teaching practices and curricular progressions.

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