TEST VECTORS FOR RANKIN–SELBERG L-FUNCTIONS

ANDREW R. BOOKER, M. KRISHNAMURTHY, AND MIN LEE

Abstract. We study the local zeta integrals attached to a pair of generic representations (π,τ) of $\mathrm{GL}_n\times\mathrm{GL}_m$, $n>m$, over a p-adic field. Through a process of unipotent averaging we produce a pair of corresponding Whittaker functions whose zeta integral is non-zero, and we express this integral in terms of the Langlands parameters of π and τ. In many cases, these Whittaker functions also serve as a test vector for the associated Rankin–Selberg (local) L-function.

1. introduction

Let F be a non-archimedean local field with ring of integers $\mathfrak{o}$ and residue field of cardinality q. For $m<n$, let π and τ be irreducible admissible representations of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_m(F)$, respectively. We fix an additive character ψ of F with conductor $\mathfrak{o}$ and assume that π and τ are generic relative to ψ.

Recall that the local *zeta integral* $\Psi(s;W,W')$ is defined by

$$\Psi(s;W,W')=\int\limits_{\mathrm{U}_m(F)\backslash\mathrm{GL}_m(F)} W\begin{pmatrix} h & \\ & I_{n-m}\end{pmatrix} W'(h)\|\det h\|^{s-\frac{n-m}{2}}\,dh, \tag{1.1}$$

where $W\in\mathcal{W}(\pi,\psi)$ and $W'\in\mathcal{W}(\tau,\psi^{-1})$ are Whittaker functions in the corresponding Whittaker spaces, and U_m is the group of unipotent matrices. It converges for $\Re(s)\gg 1$, and the collection of such zeta integrals spans a fractional ideal $\mathbb{C}[q^s,q^{-s}]L(s,\pi\boxtimes\tau)$ of the ring $\mathbb{C}[q^s,q^{-s}]$. We may choose the generator to satisfy $1/L(s,\pi\boxtimes\tau)\in\mathbb{C}[q^{-s}]$ and $\lim_{s\to\infty}L(s,\pi\boxtimes\tau)=1$, and this gives the local Rankin–Selberg factor attached to the pair (π,τ) in [5, §2.7].

In particular, if we define a map

$$\mathcal{W}(\pi,\psi)\otimes\mathcal{W}(\tau,\psi^{-1})\longrightarrow\mathbb{C}(q^{-s})$$

via

$$W\otimes W'\mapsto\Psi(s;W,W'),$$

then there is an element in $\mathcal{W}(\pi,\psi)\otimes\mathcal{W}(\tau,\psi^{-1})$ that maps to $L(s,\pi\boxtimes\tau)$. However, *a priori* this element need not be a pure tensor. In this paper, we produce a pure tensor $W\otimes W'$ for which the associated zeta integral is explicitly computable and non-zero. The precise result that we prove is the following.

Theorem 1.1. *Let $\{\alpha_i\}_{i=1}^n$ and $\{\gamma_j\}_{j=1}^m$ denote the Langlands parameters of π and τ, respectively, and let $L(s,\pi\times\tau)$ be the naive Rankin–Selberg L-factor defined by*

$$L(s,\pi\times\tau)=\prod_{i=1}^n\prod_{j=1}^m\left(1-\alpha_i\gamma_jq^{-s}\right)^{-1}.$$

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Then there is a pair $(W, W') \in \mathcal{W}(\pi, \psi) \times \mathcal{W}(\tau, \psi^{-1})$*, described explicitly in §3, such that*

$$\Psi(s; W, W') = L(s, \pi \times \tau).$$

When $\Psi(s; W, W') = L(s, \pi \boxtimes \tau)$, the pair (W, W') is called a *test vector* for (π, τ). Hence the theorem produces a test vector whenever $L(s, \pi\times\tau) = L(s, \pi\boxtimes\tau)$—for instance, if either π or τ is unramified or if $L(s, \pi \boxtimes \tau) = 1$. In general, one has $L(s, \pi \times \tau) = P(q^{-s})L(s, \pi \boxtimes \tau)$ for a non-zero polynomial $P \in \mathbb{C}[X]$ (see Lemma 2.1).

The overview of our method is as follows. Let ξ^0 (resp. φ^0) denote the "essential vector" in the space of π (resp. τ), and let $W_{\xi^0} \in \mathcal{W}(\pi, \psi)$ (resp. $W_{\varphi^0} \in \mathcal{W}(\tau, \psi^{-1})$) be the associated essential Whittaker functions, as described in detail in §2. When τ is unramified, it follows from [10, Corollary 3.3] that

$$L(s, \pi \boxtimes \tau) = \int\limits_{\mathrm{U}_m(F)\backslash \mathrm{GL}_m(F)} W_{\xi^0}\begin{pmatrix} h & \\ & I_{n-m}\end{pmatrix} W_{\varphi^0}(h)\|\det h\|^{s-\frac{n-m}{2}}\, dh \tag{1.2}$$

for a suitable normalization of the measure on $\mathrm{U}_m(F)\backslash \mathrm{GL}_m(F)$. When $m = n-1$, the above equality is part of the characterization of the essential vector in [4, 6]; the fact that it holds for any $m < n$ is the result of a concrete realization of essential functions in [10]. On the other hand, if τ is ramified then the local integral in (1.2) vanishes. Through a process of unipotent averaging (see (3.1) below), we modify W_{ξ^0} to obtain a Whittaker function $W \in \mathcal{W}(\pi, \psi)$ such that the resulting zeta integral $\Psi(s; W, W_{\varphi^0})$ equals $cL(s, \pi \times \tau)$ for a non-zero number $c \in \mathbb{C}$, depending on the conductor of τ, its central character ω_τ, and ψ. Setting $W' = c^{-1}W_{\varphi^0}$, we obtain the required pair (W, W').

We mention some related results in the literature. First, if π and τ are discrete series representations, then the existence of a test vector (W, W') was shown in [9], but the Whittaker function W' there is taken to be in a larger space, namely the Whittaker space associated to the standard module of τ. Second, the so-called *local Birch lemma*, arising in the context of $\mathfrak{p}$-adic interpolation of special values of twisted Rankin–Selberg (global) L-functions, is also related. It concerns evaluation of a local integral in the special case that π is unramified and τ is the twist of an unramified representation by a character with non-trivial conductor; see [8, Proposition 3.1] and [7, Theorem 2.1]. The approach in [8] is similar to ours in that it also uses a process of unipotent averaging in order to modify the Whittaker function on the larger general linear group. We can of course apply Theorem 1.1 to their setup: Since $L(s, \pi \times \tau) = 1$ in this case, the pair (W, W_{φ^0}) described above has the property that $\Psi(s; W, W_{\varphi^0})$ is an explicit constant (independent of s).

Finally, note that one can obtain a global version of Theorem 1.1 by combining the test vectors at all (finite) places. In work in progress, we study the analogous question over an archimedean local field.

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2. PRELIMINARIES

Let $\mathfrak{p}$ be the unique maximal ideal in $\mathfrak{o}$. We fix a generator ϖ of $\mathfrak{p}$ with absolute value $\|\varpi\| = q^{-1}$. Let $\mathrm{v} : F^\times \to \mathbb{Z}$ denote the valuation map, and extend it to fractional ideals in the usual way. For any $n > 1$, let $\mathrm{B}_n = \mathrm{T}_n\,\mathrm{U}_n$ be the Borel subgroup of GL_n consisting of upper triangular matrices; let $\mathrm{P}'_n \supset \mathrm{B}_n$ be the standard parabolic subgroup of type

$(n-1,1)$ with Levi decomposition $\mathrm{P}'_n = \mathrm{M}_n \mathrm{N}_n$. Then $\mathrm{M}_n \cong \mathrm{GL}_{n-1} \times \mathrm{GL}_1$ and

$$\mathrm{N}_n = \left\{ \begin{pmatrix} I_{n-1} & * \\ & 1 \end{pmatrix} \right\}.$$

Also, we write Z_n to denote the center consisting of scalar matrices and $\mathrm{A}_n \subset \mathrm{T}_n$ to denote the subtorus consisting of diagonal matrices with lower-right entry 1.

If R is any F-algebra and H is any algebraic F-group, we write $H(R)$ to denote the corresponding group of R-points. Let $\mathrm{P}_n(R) \subset \mathrm{P}'_n(R)$ denote the mirabolic subgroup consisting of matrices whose last row is of the form $(0, \dots, 0, 1)$, i.e.,

$$\mathrm{P}_n(R) = \left\{ \begin{pmatrix} h & y \\ & 1 \end{pmatrix} : h \in \mathrm{GL}_{n-1}(R), y \in R^{n-1} \right\} \cong \mathrm{GL}_{n-1}(R) \ltimes \mathrm{N}_n(R).$$

The character

$$u \mapsto \psi\left(\sum_{i=1}^{n-1} u_{i,i+1} \right) \quad \text{for } u \in \mathrm{U}_n(F)$$

defines a *generic character* of $\mathrm{U}_n(F)$, and by abuse of notation we continue to denote this character by ψ. Further, for any algebraic subgroup $V \subseteq \mathrm{U}_n$, ψ defines a character of $V(F)$ via restriction. In particular, we may consider the character $\psi|_{\mathrm{N}_n(F)}$; its stabilizer in $\mathrm{M}_n(F)$ is then $\mathrm{P}_{n-1}(F)$, where we regard P_{n-1} as a subgroup of M_n via $h \mapsto \left(\begin{smallmatrix} h & \\ & 1 \end{smallmatrix}\right)$.

An irreducible representation (π, V_π) of $\mathrm{GL}_n(F)$ is said to be *generic* if

$$\mathrm{Hom}_{\mathrm{GL}_n(F)}\big(V_\pi, \mathrm{Ind}_{\mathrm{U}_n(F)}^{\mathrm{GL}_n(F)} \psi\big) \neq 0.$$

By Frobenius reciprocity, this means that there is a non-zero linear form $\lambda : V_\pi \to \mathbb{C}$ satisfying $\lambda(\pi(u)v) = \psi(u)\lambda(v)$ for $v \in V_\pi$, $u \in \mathrm{U}_n(F)$. It is known (see [3]) that for a generic π the space of such linear functionals, or equivalently the space $\mathrm{Hom}_{\mathrm{GL}_n(F)}(V_\pi, \mathrm{Ind}_{\mathrm{U}_n(F)}^{\mathrm{GL}_n(F)} \psi)$, is of dimension 1. Let $\mathcal{W}(\pi, \psi)$ denote the Whittaker model of π, viz. the space of functions W_v on $\mathrm{GL}_n(F)$ defined by $W_v(g) = \lambda(\pi(g)v)$ for $v \in V_\pi$. Then $\mathcal{W}(\pi, \psi)$ is independent of the choice of λ, and for $u \in \mathrm{U}_n(F)$, $g \in \mathrm{GL}_n(F)$,

$$W_v(ug) = \psi(u) W(g),$$
$$W_v(g) = W_{\pi(g)v}(I_n).$$

We will consider certain compact open subgroups of $\mathrm{GL}_n(F)$; namely, for any integer $f \geq 0$, set

$$K_1(\mathfrak{p}^f) = \left\{ g \in \mathrm{GL}_n(\mathfrak{o}) : g \equiv \begin{pmatrix} & & & * \\ & * & & \vdots \\ & & & * \\ 0 & \cdots & 0 & 1 \end{pmatrix} \pmod{\mathfrak{p}^f} \right\},$$
$$K_0(\mathfrak{p}^f) = \left\{ g \in \mathrm{GL}_n(\mathfrak{o}) : g \equiv \begin{pmatrix} & & & * \\ & * & & \vdots \\ & & & * \\ 0 & \cdots & 0 & * \end{pmatrix} \pmod{\mathfrak{p}^f} \right\},$$

so that $K_1(\mathfrak{p}^f)$ is a normal subgroup of $K_0(\mathfrak{p}^f)$, with quotient $K_0(\mathfrak{p}^f)/K_1(\mathfrak{p}^f) \cong (\mathfrak{o}/\mathfrak{p}^f)^\times$.

Next we introduce our choice of measures. For $n \geq 1$ we normalize the Haar measure on $\mathrm{GL}_n(F)$ and $\mathrm{GL}_n(\mathfrak{o})$ so that $\mathrm{vol}(\mathrm{GL}_n(\mathfrak{o})) = 1$, and we fix the Haar measure on $\mathrm{U}_n(F)$ for which $\mathrm{vol}(\mathrm{U}_n(F) \cap \mathrm{GL}_n(\mathfrak{o})) = 1$. From these, we obtain a right-invariant measure on $\mathrm{U}_n(F) \backslash \mathrm{GL}_n(F)$. We may make this explicit using the Iwasawa decomposition. For instance, let dx be the Haar measure on F such that $\mathfrak{o}$ has unit volume, and let $d^\times x$ be the multiplicative measure on $F^\times$ such that $\mathrm{vol}(\mathfrak{o}^\times) = 1$, i.e., $d^\times x = \frac{q}{q-1} \frac{dx}{\|x\|}$. Let dz and da be the corresponding measures on the center $\mathrm{Z}_n(F) \cong F^\times$ and the subtorus $\mathrm{A}_n(F) \cong$

$(F^\times)^{n-1}$, respectively. We fix the isomorphism $(F^\times)^{n-1} \cong \mathrm{A}_n(F)$ via $(a_1, \ldots, a_{n-1}) \mapsto a = t(a_1, \ldots, a_{n-1})$, where

$$t(a_1, \ldots, a_{n-1}) = \begin{pmatrix} a_1 a_2 \ldots a_{n-1} & & & & \\ & a_1 a_2 \ldots a_{n-2} & & & \\ & & \ddots & & \\ & & & a_1 & \\ & & & & 1 \end{pmatrix}. \tag{2.1}$$

Then $da = d^\times a_1 \, d^\times a_2 \cdots d^\times a_{n-1}$. If $f \in C_c^\infty(\mathrm{GL}_n(F))$ is $\mathrm{U}_n(F)$-invariant on the left, we then have the integration formula

$$\int_{\mathrm{U}_n(F) \backslash \mathrm{GL}_n(F)} f(g) \, dg = \int_{\mathrm{Z}_n(F) \times \mathrm{A}_n(F) \times \mathrm{GL}_n(\mathfrak{o})} f(zak) \delta_{\mathrm{B}_n}(a)^{-1} \, dz \, da \, dk, \tag{2.2}$$

where δ_{B_n} is the modulus character, defined so that

$$\delta_{\mathrm{B}_n}(a) = \prod_{i=1}^{n-1} \|a_i\|^{i(n-i)}. \tag{2.3}$$

Next we review the notion of *conductor* and the theory of the *essential vector* associated to an irreducible, admissible, generic representation π. According to [4] (see also [6]), there is a unique positive integer $m(\pi)$ such that the space of $K_1\big(\mathfrak{p}^{m(\pi)}\big)$-fixed vectors is 1-dimensional. Further, as alluded to in the introduction, by loc. cit. there is a unique vector ξ^0 in this space, called the essential vector, with the associated essential function $W_{\xi^0} \in \mathcal{W}(\pi, \psi)$ satisfying the condition $W_{\xi^0}\big(\begin{smallmatrix} gh & \\ & 1 \end{smallmatrix}\big) = W_{\xi^0}\big(\begin{smallmatrix} g & \\ & 1 \end{smallmatrix}\big)$ for all $h \in \mathrm{GL}_{n-1}(\mathfrak{o})$ and $g \in \mathrm{GL}_{n-1}(F)$. Since U_n acts via ψ on the left, it follows that

$$W_{\xi^0}\big(t(a_1, \ldots, a_{n-1})\big) \neq 0 \implies a_1, \ldots, a_{n-1} \in \mathfrak{o}. \tag{2.4}$$

If π is unramified, let $W_\pi^{0,\psi} \in \mathcal{W}(\pi, \psi)$ denote the normalized spherical function [6, p. 2]. If $m(\pi) = 0$ then by uniqueness of essential functions, one has the equality $W_{\xi^0} = W_\pi^{0,\psi}$. The integral ideal $\mathfrak{p}^{m(\pi)}$ is called the *conductor* of π. In passing, we mention that the integer $m(\pi)$ can also be characterized as the degree of the monomial in the local ϵ-factor $\epsilon(s, \pi, \psi)$ [4], i.e. so that

$$\epsilon(s, \pi, \psi) = \epsilon(\pi, \psi) q^{m(\pi)(\frac{1}{2} - s)}$$

for some $\epsilon(\pi, \psi) \in \mathbb{C}^\times$.

A crucial property of the conductor is that $K_0(\mathfrak{p}^m(\pi))$ acts on the space of $K_1(\mathfrak{p}^{m(\pi)})$-fixed vectors via the central character ω_π (cf. [2, Section 8]). Precisely, for $g = (g_{i,j}) \in K_0\big(\mathfrak{p}^{m(\pi)}\big)$, define

$$\chi_\pi(g) = \begin{cases} 1 & \text{if } m(\pi) = 0, \\ \omega_\pi(g_{n,n}) & \text{if } m(\pi) > 0. \end{cases}$$

It is shown in loc. cit. that χ_π is a character of $K_0(\mathfrak{p}^{m(\pi)})$ trivial on $K_1(\mathfrak{p}^{m(\pi)})$, and

$$\pi(g)\xi^0 = \chi_\pi(g)\xi^0 \quad \text{for all } g \in K_0\big(\mathfrak{p}^{m(\pi)}\big).$$

We end this section by recalling the definition of conductor of a multiplicative character χ of $F^\times$. If χ is trivial on $\mathfrak{o}^\times$ then the conductor of χ is $\mathfrak{o}$; otherwise, the conductor is $\mathfrak{p}^n$, where $n \geq 1$ is the least integer such that χ is trivial on $1 + \mathfrak{p}^n$.

2.1. **Rankin–Selberg L-functions.** In this subsection alone we drop the assumption that $m < n$ and allow (m, n) to be an arbitrary pair of positive integers. For π and τ irreducible, admissible, generic representations of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_m(F)$, respectively, let $L(s, \pi \boxtimes \tau)$ be as defined in [5]. When $m < n$, $L(s, \pi \boxtimes \tau)$ is defined as in the introduction. For $m > n$, one defines $L(s, \pi \boxtimes \tau) = L(s, \tau \boxtimes \pi)$. For $m = n$, the defining local integrals are different and involve a Schwartz function on F^n; see loc. cit.

Next we elaborate on the definition of the naive Rankin–Selberg L-factor, $L(s, \pi \times \tau)$, introduced in Theorem 1.1. By definition, the L-function $L(s, \pi)$ is of the form $P_\pi(q^{-s})^{-1}$, where $P_\pi \in \mathbb{C}[X]$ has degree at most n and satisfies $P_\pi(0) = 1$. We may then find n complex numbers $\{\alpha_i\}_{i=1}^n$ (allowing some of them to be zero) satisfying

$$L(s, \pi) = \prod_{i=1}^{n} (1 - \alpha_i q^{-s})^{-1}.$$

We call the set $\{\alpha_i\}$ the *Langlands parameters* of π; if π is spherical, they agree with the usual Satake parameters. Let $\{\gamma_j\}_{j=1}^m$ be the Langlands parameters of τ, and set

$$L(s, \pi \times \tau) = \prod_{i=1}^{n} \prod_{j=1}^{m} (1 - \alpha_i \gamma_j q^{-s})^{-1}. \tag{2.5}$$

$L(s, \pi \times \tau) = L(s, \pi \boxtimes \tau)$ if π or τ is spherical.

In the following lemma we describe the connection between $L(s, \pi \times \tau)$ and $L(s, \pi \boxtimes \tau)$. To that end, we first recall the classification of irreducible admissible representations of $\mathrm{GL}_n(F)$. Let $\mathcal{A}_n$ denote the set of equivalence classes of such representations, and put $\mathcal{A} = \bigcup \mathcal{A}_n$. The essentially square-integrable representations of $\mathrm{GL}_n(F)$ have been classified by Bernstein and Zelevinsky, and they are as follows. If σ is an essentially square-integrable representation of $\mathrm{GL}_n(F)$, then there is a divisor $a \mid n$ and a supercuspidal representation η of $\mathrm{GL}_a(F)$ such that if $b = \frac{n}{a}$ and Q is the standard (upper) parabolic subgroup of $\mathrm{GL}_n(F)$ of type $(a, \ldots, a)$, then σ can be realized as the unique quotient of the (normalized) induced representation

$$\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_n(F)}(\eta, \eta\|\cdot\|, \ldots, \eta\|\cdot\|^{b-1}).$$

The integer a and the class of η are uniquely determined by σ. In short, σ is parametrized by b and η, and we denote this by $\sigma = \sigma_b(\eta)$; further, σ is square-integrable (also called "discrete series") if and only if the representation $\eta\|\cdot\|^{\frac{b-1}{2}}$ of $\mathrm{GL}_a(F)$ is unitary.

Now, let P be an upper parabolic subgroup of $\mathrm{GL}_n(F)$ of type $(n_1, \ldots, n_r)$. For each $i = 1, \ldots, r$, let τ_i^0 be a discrete series representation of $\mathrm{GL}_{n_i}(F)$. Let $(s_1, \ldots, s_r)$ be a sequence of real numbers satisfying $s_1 \geq \cdots \geq s_r$, and put $\tau_i = \tau_i^0 \otimes \|\cdot\|^{s_i}$ (an essentially square-integrable representation). Then the induced representation

$$\xi = \mathrm{Ind}_{\mathrm{P}}^{\mathrm{GL}_n(F)}(\tau_1 \otimes \cdots \otimes \tau_r)$$

is said to be a representation of $\mathrm{GL}_n(F)$ of Langlands type. If $\tau \in \mathcal{A}_n$, then it is well known that it is uniquely representable as the quotient of an induced representation of Langlands type. We write $\tau = \tau_1 \boxplus \cdots \boxplus \tau_r$ to denote this realization of τ. Thus one obtains a sum operation on the set $\mathcal{A}$ [5, §9.5]. It follows easily from the definition that $L(s, \pi \times \tau)$ is bi-additive, i.e.

$$L(s, \pi \times (\tau \boxplus \tau')) = L(s, \pi \times \tau) L(s, \pi \times \tau')$$
$$L(s, (\pi \boxplus \pi') \times \tau) = L(s, \pi \times \tau) L(s, \pi' \times \tau)$$

for all $\pi, \pi', \tau, \tau' \in \mathcal{A}$. The local factor $L(s, \pi \boxtimes \tau)$ is also bi-additive in the above sense, by [5, §9.5, Theorem].

Lemma 2.1. *Let m and n be positive integers, and consider $\pi \in \mathcal{A}_n$, $\tau \in \mathcal{A}_m$. Then*

$$L(s, \pi \times \tau) = P(q^{-s}) L(s, \pi \boxtimes \tau)$$

for a polynomial $P \in \mathbb{C}[X]$ (depending on π and τ) satisfying $P(0) = 1$.

Proof. Since π and τ are sums of essentially square-integrable representations and $L(s, \pi \times \tau)$ and $L(s, \pi \boxtimes \tau)$ are both additive with respect to $\boxplus$, it suffices to prove the lemma for a pair (π, τ) of essentially square-integrable representations. In particular, assume $\pi = \sigma_b(\eta)$ as above.

We proceed by induction on m. If $m = 1$ then $\tau = \chi$ is a quasi-character of $F^\times$ and $L(s, \pi \boxtimes \tau) = L(s, \pi \otimes \chi)$, where $\pi \otimes \chi$ is the representation of $\mathrm{GL}_n(F)$ defined by $g \mapsto \pi(g)\chi(\det g)$. If χ is unramified, then

$$L(s, \pi \otimes \chi) = L(s, \pi \times \chi),$$

and consequently $P = 1$. On the other hand, if χ is ramified then $L(s, \pi \times \chi) = 1$, and the assertion follows since $L(s, \pi \otimes \chi)^{-1}$ is a polynomial in q^{-s}.

We now assume $m > 1$ and τ is an essentially square-integrable representation of $\mathrm{GL}_m(F)$, say $\tau = \sigma_{b'}(\eta')$, where $\eta' \in \mathcal{A}_{a'}$ is supercuspidal and $a'b' = m$. Then the standard L-factor $L(s, \tau)$ is given by $L(s, \tau) = L(s + b' - 1, \eta')$ [5]. Therefore, $L(s, \tau) = 1$ unless $a' = 1$ and $\eta' = \chi$ is an unramified quasi-character of $F^\times$. On the other hand, if $L(s, \tau) = 1$, then $L(s, \pi \times \tau) = 1$ and the assertion of the lemma follows. Hence we may assume $\tau = \sigma_m(\chi)$ for an unramified quasi-character χ of $F^\times$, in which case

$$\begin{aligned} L(s, \pi \times \tau) &= L(s, \pi \otimes \chi \| \cdot \|^{m-1}) = L(s, \sigma_b(\eta) \otimes \chi \| \cdot \|^{m-1}) \\ &= L(s + m - 1 + b - 1, \eta \otimes \chi). \end{aligned} \tag{2.6}$$

On the other hand, it follows from [5, §8.2, Theorem] that

$$L(s, \pi \boxtimes \tau) = \begin{cases} \prod_{j=0}^{m-1} L(s + j + b - 1, \eta \otimes \chi) & \text{if } m \leq n, \\ \prod_{i=0}^{b-1} L(s + m - 1 + i, \eta \otimes \chi) & \text{if } m > n. \end{cases} \tag{2.7}$$

From (2.6) and (2.7), one sees that the ratio $\frac{L(s, \pi \times \tau)}{L(s, \pi \boxtimes \tau)}$ is a polynomial in q^{-s}, thus proving the lemma. $\square$

Corollary 2.2. *If $L(s, \pi \boxtimes \tau) = 1$ then either $L(s, \pi) = 1$ or $L(s, \tau) = 1$.*

Proof. If $L(s, \pi \boxtimes \tau) = 1$ then Lemma 2.1 implies that $L(s, \pi \times \tau)$ is a polynomial in q^{-s}, and hence must be 1. This in turn implies the conclusion. $\square$

3. The main calculation

Recall that ξ^0 and φ^0 are the essential vectors of π and τ, respectively. Here we construct a pair $(W, W') \in \mathcal{W}(\pi, \psi) \times \mathcal{W}(\tau, \psi^{-1})$ as in Theorem 1.1. Let $\mathfrak{n}$, $\mathfrak{q}$ and $\mathfrak{c}$ denote the conductors of π, τ and ω_τ, respectively. If τ is an unramified representation of $\mathrm{GL}_m(F)$, then by (1.2) we have

$$\Psi(s; W_{\xi^0}, W_{\varphi^0}) = L(s, \pi \boxtimes \tau) = L(s, \pi \times \tau).$$

Thus, in this case we can take $(W, W') = (W_{\xi^0}, W_{\varphi^0})$.

Let us assume from now on that τ is ramified, meaning $\mathrm{v}(\mathfrak{q}) > 0$. Since $\mathfrak{c} \supseteq \mathfrak{q}$, we have $\mathrm{v}(\mathfrak{c}) \leq \mathrm{v}(\mathfrak{q})$. Consider $\beta = (\beta_1, \ldots, \beta_m) \in F^m$, with $\beta_i \in \mathfrak{q}^{-1}$ for $i = 1, \ldots, m$, and let $u(\beta)$ denote the $n \times n$ matrix with 1s on the diagonal and β^t embedded above the diagonal in the $(m+1)^{\text{st}}$ column. Let ξ^0_β denote the vector $\xi^0_\beta = \pi(u(\beta))\xi^0$, and define

$$\overline{\xi} = \frac{1}{[\mathfrak{o}:\mathfrak{q}]^{m-1}} \sum_{(\beta_1,\ldots,\beta_{m-1}) \in (\mathfrak{q}^{-1}/\mathfrak{o})^{m-1}} \xi^0_{(\beta_1,\ldots,\beta_{m-1},\varpi^{-\mathrm{v}(\mathfrak{c})})}. \tag{3.1}$$

(When $m = 1$ we understand there to be one summand, so that $\overline{\xi} = \xi^0_{(\varpi^{-\mathrm{v}(\mathfrak{c})})}$.) We will now calculate $\Psi(s; W_{\overline{\xi}}, W_{\varphi^0})$, which by linearity equals

$$\frac{1}{[\mathfrak{o}:\mathfrak{q}]^{m-1}} \sum_{(\beta_1,\ldots,\beta_{m-1}) \in (\mathfrak{q}^{-1}/\mathfrak{o})^{m-1}} \Psi\big(s; W_{\xi^0_{(\beta_1,\ldots,\beta_{m-1},\varpi^{-\mathrm{v}(\mathfrak{c})})}}, W_{\varphi^0}\big).$$

Put $K = \mathrm{GL}_m(\mathfrak{o})$. By (2.2), for fixed $\beta = (\beta_1, \ldots, \beta_m)$, we have

$$\begin{aligned}\Psi(s; W_{\xi^0_\beta}, W_{\varphi^0}) &= \int\limits_{\mathrm{Z}_m(F)\times \mathrm{A}_m(F)\times K} \omega_\tau(z) W_{\xi^0}\left(\begin{pmatrix} zak & \\ & I_{n-m}\end{pmatrix} u(\beta)\right) W_{\varphi^0}(ak) \\ &\qquad \cdot \delta_{\mathrm{B}_m}(a)^{-1} \|\det(za)\|^{s-\frac{n-m}{2}}\, dz\, da\, dk \\ &= \int\limits_{F^\times \times \mathrm{A}_m(F)\times K} \omega_\tau(z) \left(\sum_{j=1}^m \psi(zk_j\beta_j)\right) W_{\xi^0}\begin{pmatrix} za & \\ & I_{n-m}\end{pmatrix} W_{\varphi^0}(ak) \\ &\qquad \cdot \delta_{\mathrm{B}_m}(a)^{-1} \|z^m \det a\|^{s-\frac{n-m}{2}}\, d^\times z\, da\, dk,\end{aligned}$$

where $(k_1, \ldots, k_m)$ is the bottom row of the matrix k. Here we have used the fact that the function $h \mapsto W_{\xi^0}\left(\begin{smallmatrix} h & \\ & I_{n-m}\end{smallmatrix}\right)$, $h \in \mathrm{GL}_m(F)$, is right K-invariant. Now, performing the average over $\beta_j \in \mathfrak{q}^{-1}/\mathfrak{o}$ for each $j < m$, we see that $\Psi(s; W_{\overline{\xi}}, W_{\varphi^0})$ equals

$$\begin{aligned}\int\limits_{\left\{\substack{F^\times \times \mathrm{A}_m(F)\times K \\ zk_j \in \mathfrak{q}\ \forall j<m}\right\}} &\omega_\tau(z)\psi\big(zk_m\varpi^{-\mathrm{v}(\mathfrak{c})}\big) W_{\xi^0}\begin{pmatrix} za & \\ & I_{n-m}\end{pmatrix} W_{\varphi^0}(ak) \\ &\cdot \delta_{\mathrm{B}_m}(a)^{-1} \|z^m \det a\|^{s-\frac{n-m}{2}}\, d^\times z\, da\, dk.\end{aligned}$$

By (2.4) we have $W_{\xi^0}(t(a_1, \ldots, a_{n-1})) = 0$ unless $a_1, \ldots, a_{n-1} \in \mathfrak{o}$. In view of (2.1), it follows that the integrand vanishes unless z is integral.

Note that

$$zk_j \in \mathfrak{q}\ \forall j < m \iff k \in K_0(z^{-1}\mathfrak{q} \cap \mathfrak{o}).$$

For $r \in \mathbb{Z}_{\geq 0}$, put

$$\begin{aligned}\Psi_r = \int\limits_{\mathrm{A}_m(F)\times K_0(\mathfrak{q}\mathfrak{p}^{-r}\cap\mathfrak{o})} &\mathcal{G}(\omega_\tau, \psi, \varpi^{r-\mathrm{v}(\mathfrak{c})}k_m) W_{\xi^0}\begin{pmatrix} \varpi^r a & \\ & I_{n-m}\end{pmatrix} W_{\varphi^0}(ak) \\ &\cdot \delta_{\mathrm{B}_m}(a)^{-1} \|\det a\|^{s-\frac{n-m}{2}}\, da\, dk,\end{aligned}$$

where $\mathcal{G}(\omega_\tau, \psi, y)$ denotes the Gauss sum

$$\mathcal{G}(\omega_\tau, \psi, y) = \int_{\mathfrak{o}^\times} \omega_\tau(z)\psi(yz)\, d^\times z.$$

Then we have

$$\Psi(s; W_{\overline{\xi}}, W_{\varphi^0}) = \sum_{r\geq 0} \omega_\tau(\varpi^r) q^{-rm(s-\frac{n-m}{2})} \Psi_r.$$

Suppose that ω_τ is ramified, so that $\mathrm{v}(\mathfrak{c}) > 0$. Then $\mathcal{G}(\omega_\tau, \psi, \varpi^{r-\mathrm{v}(\mathfrak{c})}k_m)$ vanishes unless $\mathrm{v}(\varpi^{r-\mathrm{v}(\mathfrak{c})}k_m) = -\mathrm{v}(\mathfrak{c})$, which implies $\mathrm{v}(k_m) = -r$. Since k_m is integral, it follows that $r = 0$ is the only contributing term to $\Psi(s; W_{\bar{\xi}}, W_{\varphi^0})$, so that

$$\Psi(s; W_{\bar{\xi}}, W_{\varphi^0}) = \int\limits_{\mathrm{A}_m(F)\times K_0(\mathfrak{q})} \mathcal{G}\big(\omega_\tau, \psi, \varpi^{-\mathrm{v}(\mathfrak{c})}k_m\big) W_{\xi^0}\begin{pmatrix} a & \\ & I_{n-m}\end{pmatrix} W_{\varphi^0}(ak) \cdot \delta_{\mathrm{B}_m}(a)^{-1}\|\det a\|^{s-\frac{n-m}{2}}\, da\, dk.$$

Moreover, since $k_m \in \mathfrak{o}^\times$, we have $\mathcal{G}(\omega_\tau, \psi, \varpi^{-\mathrm{v}(\mathfrak{c})}k_m) = \omega_\tau(k_m)^{-1}\mathcal{G}(\omega_\tau, \psi, \varpi^{-\mathrm{v}(\mathfrak{c})})$, and thus

$$\Psi(s; W_{\bar{\xi}}, W_{\varphi^0}) = c\int_{\mathrm{A}_m(F)} W_{\xi^0}\begin{pmatrix} a & \\ & I_{n-m}\end{pmatrix} W_{\varphi^0}(a)\delta_{\mathrm{B}_m}(a)^{-1}\|\det a\|^{s-\frac{n-m}{2}}\, da, \tag{3.2}$$

where

$$c = \frac{\mathcal{G}\big(\omega_\tau, \psi, \varpi^{-\mathrm{v}(\mathfrak{c})}\big)}{[\mathrm{GL}_m(\mathfrak{o}) : K_0(\mathfrak{q})]} \neq 0.$$

Suppose now that ω_τ is unramified, so that $\mathfrak{c} = \mathfrak{o}$ and $m > 1$. Then $\mathcal{G}(\omega_\tau, \psi, \varpi^{r-\mathrm{v}(\mathfrak{c})}k_m) = 1$ for all r. If $r > 0$ then $K_0(\mathfrak{q}\mathfrak{p}^{-r} \cap \mathfrak{o}) \supsetneq K_0(\mathfrak{q})$; since the conductor of τ is $\mathfrak{q}$, it follows from [4, Theorem 5.1] that

$$\int\limits_{K_0(\mathfrak{q}\mathfrak{p}^{-r}\cap\mathfrak{o})} W_{\varphi^0}(ak)\, dk = 0,$$

which in turn implies that $\Psi_r = 0$. Hence only the $r = 0$ term contributes, and again we arrive at (3.2).

It remains only to identify the integral over $\mathrm{A}_m(F)$.

Lemma 3.1. *When τ is ramified, we have*

$$\int_{\mathrm{A}_m(F)} W_{\xi^0}\begin{pmatrix} a & \\ & I_{n-m}\end{pmatrix} W_{\varphi^0}(a)\delta_{\mathrm{B}_m}(a)^{-1}\|\det a\|^{s-\frac{n-m}{2}}\, da = L(s, \pi\times\tau).$$

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $(\gamma_1, \ldots, \gamma_m)$ denote the Langlands parameters of π and τ, respectively. Since τ is ramified, we may take $\gamma_m = 0$. We set $\gamma_{m+1} = \ldots = \gamma_n = 0$ and write $\gamma = (\gamma_1, \ldots, \gamma_n)$.

Writing $a = t(a_1, \ldots, a_{m-1})$ as in (2.1), by (2.4) we see that $W_{\varphi^0}(a)$ vanishes unless each a_i is integral. Setting

$$\lambda_i = \sum_{1\le j\le m-i} \mathrm{v}(a_j) \quad \text{for } i = 1, \ldots, n,$$

the integral in question may be written as

$$\sum_{\substack{\lambda=(\lambda_1,\ldots,\lambda_n)\in\mathbb{Z}^n_{\ge 0}\\ \lambda_1\ge\cdots\ge\lambda_{m-1}\\ \lambda_m=\ldots=\lambda_n=0}} W_{\xi^0}\big(d_n(\lambda)\big) W_{\varphi^0}\big(d_m(\lambda)\big)\delta_{\mathrm{B}_m}\big(d_m(\lambda)\big)^{-1}\|\det d_m(\lambda)\|^{s-\frac{n-m}{2}},$$

where

$$d_n(\lambda) = \mathrm{diag}\big(\varpi^{\lambda_1}, \ldots, \varpi^{\lambda_n}\big) \quad \text{and} \quad d_m(\lambda) = \mathrm{diag}\big(\varpi^{\lambda_1}, \ldots, \varpi^{\lambda_m}\big).$$

On the other hand, by [11, Theorem 4.1], we have

$$W_{\xi^0}(d_n(\lambda)) = \delta_{\mathrm{B}_n}(d_n(\lambda))^{\frac{1}{2}} s_\lambda(\alpha) \quad \text{and} \quad W_{\varphi^0}(d_m(\lambda)) = \delta_{\mathrm{B}_m}(d_m(\lambda))^{\frac{1}{2}} s_\lambda(\gamma),$$

where s_λ denotes the Schur polynomial

$$s_\lambda(X_1,\dots,X_n) = \frac{\det\left(X_j^{\lambda_i+n-i}\right)_{1\le i,j\le n}}{\prod_{1\le i<j\le n}(X_i - X_j)}$$

(see [1, Theorem 38.1]). Thus the integral becomes

$$\sum_{\substack{\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{Z}^n_{\ge 0}\\ \lambda_1\ge\cdots\ge\lambda_{m-1}\\ \lambda_m=\ldots=\lambda_n=0}} s_\lambda(\alpha)s_\lambda(\gamma)\left(\frac{\delta_{\mathrm{B}_n}(d_n(\lambda))}{\delta_{\mathrm{B}_m}(d_m(\lambda))}\right)^{\frac{1}{2}} \|\det d_m(\lambda)\|^{s-\frac{n-m}{2}}$$
$$= \sum_{\substack{\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{Z}^n_{\ge 0}\\ \lambda_1\ge\cdots\ge\lambda_n}} s_\lambda(\alpha)s_\lambda(\gamma)\|\det d_m(\lambda)\|^{s},$$

where the last line follows from (2.3) and the fact that $\gamma_m = \ldots = \gamma_n = 0$.

By the Cauchy identity [1, Theorem 43.3], for sufficiently small $x\in\mathbb{C}$, we have

$$\prod_{i=1}^n\prod_{j=1}^n(1-x\alpha_i\gamma_j)^{-1} = \sum_{\substack{\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{Z}^n_{\ge 0}\\ \lambda_1\ge\cdots\ge\lambda_n}} s_\lambda(x\alpha)s_\lambda(\gamma) = \sum_{\substack{\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{Z}^n_{\ge 0}\\ \lambda_1\ge\cdots\ge\lambda_n}} s_\lambda(\alpha)s_\lambda(\gamma)x^{\lambda_1+\cdots+\lambda_n}.$$

Therefore,

$$\sum_{\substack{\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{Z}^n_{\ge 0}\\ \lambda_1\ge\cdots\ge\lambda_n}} s_\lambda(\alpha)s_\lambda(\gamma)\|\det d_m(\lambda)\|^{s} = \prod_{i=1}^n\prod_{j=1}^n(1-\alpha_i\gamma_j q^{-s})^{-1} = L(s,\pi\times\tau).$$

□

Finally, we take $W = W_{\overline{\xi}}$ and $W' = c^{-1}W_{\varphi^0}$ to conclude the proof of Theorem 1.1.

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E-mail address: andrew.booker@bristol.ac.uk

E-mail address: muthu-krishnamurthy@uiowa.edu

E-mail address: min.lee@bristol.ac.uk

School of Mathematics, University of Bristol, University Walk, Bristol, BS8 1TW, United Kingdom

Department of Mathematics, University of Iowa, 14 MacLean Hall, Iowa City, IA 52242-1419, USA