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Quantum sketching protocols for Hamming distance and beyondJoão Fernando Doriguello^{1,2,*} and Ashley Montanaro^{1,†}¹*School of Mathematics, University of Bristol, Bristol BS8 1TW, United Kingdom*²*Quantum Engineering Centre for Doctoral Training, University of Bristol, Bristol BS8 1TW, United Kingdom*

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In this work we use the concept of quantum fingerprinting to develop a quantum communication protocol in the simultaneous message passing model that calculates the Hamming distance between two n -bit strings up to relative error ϵ . The number of qubits communicated by the protocol is polynomial in $\log n$ and $1/\epsilon$, while any classical protocol must communicate $\Omega(\sqrt{n})$ bits. Motivated by the relationship between Hamming distance and vertex distance in hypercubes, we apply the protocol to approximately calculate distances between vertices in graphs that can be embedded into a hypercube such that all distances are preserved up to a constant factor. Such graphs are known as ℓ_1 -graphs. This class includes all trees, median graphs, Johnson graphs, and Hamming graphs. Our protocol is efficient for ℓ_1 -graphs with low diameter, and we show that its dependence on the diameter is essentially optimal. Finally, we show that our protocol can be used to approximately compute ℓ_1 distances between vectors efficiently.

DOI: [10.1103/PhysRevA.99.062331](https://doi.org/10.1103/PhysRevA.99.062331)**I. INTRODUCTION**

Imagine that two separated parties (Alice and Bob) each have some data and would like to determine how alike their data are, using the minimal amount of communication possible. Also imagine that they are not allowed to communicate with each other, but are each only allowed to send a single message to a third party (referee), and do not share any prior information with each other. This communication model is known as the simultaneous message passing (SMP) model with private randomness [1]. It encapsulates, for example, a scenario where it is not clear in advance whose data sets are to be compared. Another motivation comes from cryptographic scenarios. For example, it could be that the inputs to the two parties are controlled by an adversary, who has access to any previously shared randomness and can choose the inputs such that the protocol fails [2]; alternatively, Alice and Bob may simply want to find an efficient protocol which hides their data from the referee.

A natural strategy for completing this task is for each of Alice and Bob to compress their data to some kind of “sketch” [3,4] and send the sketches to the referee, who uses them to determine the distance between the corresponding original data sets. Unfortunately, even for one of the simplest distance measures possible, testing equality of n -bit strings, and even if Alice and Bob are allowed a small probability of failure, this task requires $\Omega(\sqrt{n})$ bits of classical communication to the referee [5]. In comparison, if Alice and Bob are allowed access to a shared random bit string, this complexity drops to $O(1)$ [6].

Remarkably, the use of quantum information allows an exponential reduction in the complexity of equality testing. If Alice and Bob encode their n -bit strings as particular

quantum states called quantum fingerprints, then there exists a quantum protocol that communicates only $O(\log n)$ qubits [7] and succeeds with arbitrarily high constant probability.

This surprising result sparked significant interest from the perspective of computer science [8,9] and information theory [10], as well as physics. Theoretically, it has been used to shed light on the two-slit experiment [11] and detailed studies of fingerprinting schemes using few qubits have been undertaken [12,13]. Proof-of-principle quantum fingerprinting experiments have been carried out with states of one qubit realized using linear optics [14] and nuclear magnetic resonance [15]. More recently, a variant of the quantum fingerprinting protocol based on coherent states [16] has also been implemented experimentally, surpassing the best known classical protocols [17] and even the classical theoretical limit [18].

However, equality is just one distance measure, and a very special one. Here we seek other measures of distance for which quantum information can achieve a similar exponential advantage. In addition to the inherent theoretical interest of this question in terms of giving insight into the expressive power of quantum states, quantum protocols for more general distance measures could find significantly broader applications.

One example where quantum fingerprinting has been generalized is an efficient quantum communication protocol of Kumar *et al.* based on coherent states [19], which can approximately compute the Euclidean distance between unit vectors up to low additive error. This protocol is directly based on the use of the swap test to approximate ℓ_2 distances between quantum states [7]. There are many other distance measures of practical relevance where it is less clear whether similar techniques to quantum fingerprinting can be applied.

A. Our results

Our main result is a quantum protocol for approximately computing another distance measure, the Hamming distance,

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up to low *relative* error. This notion of accuracy is important when one wishes to compare objects that are similar, for example, when one of the objects is produced by a small number of errors affecting the other [20]. Approximating the Hamming distance between two n -bit strings up to additive accuracy ϵn (analogous to the accuracy achieved by the protocol of [19]) would give no useful information in this situation.

In the setting we consider, Alice and Bob are given $x, y \in \{0, 1\}^n$, respectively. Their goal is to approximately calculate the Hamming distance $d(x, y)$ between x and y , i.e., they must output $d_\epsilon(x, y)$ such that $(1 - \epsilon)d(x, y) \leq d_\epsilon(x, y) \leq (1 + \epsilon)d(x, y)$. Pang and El Gamal [21] proved a lower bound of $\Omega(n)$ for exactly calculating the Hamming distance in the multiround two-party classical communication model. Here we describe a quantum protocol that approximately computes the Hamming distance in the SMP model by communicating $\text{poly}(\log n)$ qubits.

Theorem 1. There is a quantum protocol in the SMP model with private randomness which communicates $O((\log n)^2(\log \log n)/\epsilon^5)$ qubits and computes the Hamming distance between n -bit strings up to relative error ϵ , for any $\epsilon = \Omega(1/\log n)$, with failure probability bounded above by an arbitrarily small constant.

The protocol is based on a subroutine which determines whether, for some threshold δ , $d(x, y) \leq \delta$ or $d(x, y) \geq (1 + \epsilon)\delta$. This subroutine maps x and y to n -bit strings Ax and Ay such that in the first case, $d(Ax, Ay)$ is low (less than αN , for some constant α), whereas in the second case, $d(Ax, Ay)$ is high (greater than βN , for some constant $\beta > \alpha$). Alice and Bob then encode the strings Ax and Ay as quantum superpositions, which the referee can distinguish between using the swap test [7].

Note that there exists a corresponding classical protocol in the SMP model with shared randomness, with a similar complexity. One way to see this is that the quantum protocol is ultimately based on the use of the swap test to approximately compute the inner product between unit vectors, for which there is an efficient classical protocol in this model [22].

We then generalize Theorem 1 to other distance measures, in particular those which can be interpreted as distances in graphs. A graph $G = (V, E)$ is fixed in advance and each of Alice and Bob is given a vertex of G (v and w , respectively). They aim to approximately compute $d_G(v, w)$, the length of a shortest path in G between v and w , up to relative error ϵ .

We first observe that Theorem 1 can be applied to give an efficient protocol for this problem whenever there is a distance-preserving embedding of G into the hypercube: the graph whose vertex set is $\{0, 1\}^m$, for some m , and where two vertices are connected by an edge whenever their Hamming distance is 1. In fact, this can be generalized further, to graphs which are embeddable into the hypercube such that distances are preserved up to a constant factor k . Such graphs are known as ℓ_1 -graphs, because it turns out that this criterion is equivalent to the existence of a distance-preserving embedding of the graph in ℓ_1 [23]. The class of ℓ_1 -graphs includes all trees, median graphs, Hamming graphs, and Johnson graphs [23]. (We include in the Appendix a characterization of ℓ_1 -graphs which we were not able to find in the literature.)

Distances in ℓ_1 -graphs are used in a variety of applications, a few of which we outline here. Partial cubes (ℓ_1 -graphs

with embedding constant $k = 1$) were initially introduced by Graham and Pollak [24] as a model for interconnection networks in the Bell system, with distances between vertices corresponding to the number of hops between “loops” in their network. Antimatroids (a specific subclass of ℓ_1 -graphs) are used as structures to represent the required steps to develop a student’s knowledge in a certain topic, and the distance between two points that represent concepts in these structures corresponds to the length of a student’s learning path [25]. The Barnes-Hut tree method in many-body physics [26] provides a systematic way of determining the degree of closeness between two different particles. The distance between two nodes in the tree is linked to this closeness property and can be used for various purposes, e.g., to calculate gravitational forces in star clusters and study galaxy evolution [27]. Tree structures are also used in biology, where phylogenetic trees classify organisms based on overall similarity, and the distance between vertices is related to genetic or mutation distance [28].

Our protocol is efficient for ℓ_1 -graphs G whose diameter $\text{diam}(G)$ is low, where the diameter is defined as $\text{diam}(G) = \max_{v,w} d_G(v, w)$.

Theorem 2. Let $G = (V, E)$ be an ℓ_1 -graph with $|V|$ vertices and let $v, w \in V$. There is a quantum protocol in the SMP model with private randomness which communicates $O([\log \text{diam}(G)][\log \log \text{diam}(G)](\log \log |V|)/\epsilon^5)$ qubits and computes $d_G(v, w)$ up to relative error ϵ , for any $\epsilon = \Omega[1/\log \text{diam}(G)]$, with failure probability bounded above by an arbitrarily small constant.

For any graph G , even testing equality between vertices requires $\Omega(\sqrt{\log |V|})$ bits of classical communication in the SMP model without shared randomness [5], so this is an exponential separation for those ℓ_1 -graphs where, for example, $\text{diam}(G) = O(\log |V|)$. The $d_G(v, w)$ can be computed trivially using $O(\log |V|)$ bits of classical communication, by sending the labels of v and w to the referee. So for graphs G where $\text{diam}(G)$ is close to $|V|$, Theorem 2 gives little or no improvement on the classical complexity. One may wonder whether this is simply a limitation of our protocol, but we show that this is not the case.

Theorem 3. Given a graph G with diameter $\text{diam}(G)$, any one-way quantum communication protocol that computes $d_G(v, w)$ up to relative error $\epsilon < 1/4$ with failure probability $1/3$ must transmit at least $\Omega[\log \text{diam}(G)]$ qubits.

As every protocol in the SMP model implies a one-way protocol, this shows that the complexity of our protocol is nearly optimal in terms of its dependence on $\text{diam}(G)$.

Finally, we show that our protocol for approximately computing the Hamming distance can be used to give an efficient protocol for approximately computing the ℓ_1 distance between vectors in \mathbb{R}^n .

Theorem 4. Let $x, y \in [-1, 1]^n$ such that each entry of x and y is specified by a k -bit string, with $k = O(\log n)$. There is a quantum protocol in the SMP model which communicates $O((\log n)^2(\log \log n)/\epsilon^5)$ qubits and computes $\|x - y\|_1$ up to relative error ϵ , for any $\epsilon = \Omega(1/\log n)$, with failure probability bounded above by an arbitrarily small constant.

A natural special case of Theorem 4 is where x and y are probability distributions. Then our result enables Alice and

Bob to determine the distance between two distributions, one of which is a small perturbation of the other.

Two interesting questions which remain open are whether one can find a similar result to Theorem 2 which holds for all graphs, without the restriction to ℓ_1 -graphs, and if the communication complexity dependence on ϵ , currently at $1/\epsilon^5$, can be improved.

B. Related work

The Hamming distance is a fundamental distance measure and has been studied in various forms. In the context of quantum communication complexity, Liu and Zhang [29] gave a quantum sketching protocol for the related threshold problem of determining whether the Hamming distance is larger than d , for some d . Their protocol uses $O(d \log n)$ communication, improving a previous $O(d \log^2 n)$ protocol of Gavinsky *et al.* [30]. Huang *et al.* [31] had previously proven an $\Omega(d)$ lower bound for even the two-way quantum communication complexity of the threshold Hamming distance problem, together with an $O(d \log d)$ upper bound in the classical SMP model with public randomness.

A key ingredient in the upper bound of Huang *et al.* is a protocol which communicates $O(1)$ bits and distinguishes between the case that the Hamming distance is at most d and the case that the Hamming distance is at least $2d$, for arbitrary d . Their protocol can be seen as a variant of our Lemma 2 below with $N = 1$; similar analysis shows that it could be generalized to distinguish between Hamming distance d and Hamming distance $(1 + \epsilon)d$ with $O(1/\epsilon^2)$ bits of communication. Using a generic construction of Yao [9], improved by Gavinsky *et al.* [8], this implies a quantum sketching protocol for the same task which communicates $2^{O(1/\epsilon^2)} \log n$ qubits. Using a similar approach to our work, this in turn implies a protocol which solves the approximate Hamming distance problem by transmitting $2^{O(1/\epsilon^2)}$ poly $\log n$ qubits. This is the same asymptotic complexity as our protocol for constant ϵ , but in practice the $2^{O(1/\epsilon^2)}$ factor makes the protocol infeasible for even modest values of ϵ .

Classically, there has also been substantial work on approximately computing the Hamming distance between a small pattern and a larger string, both locally and in a distributed context (see [32] and references therein).

More generally, the field of communication complexity studies the amount of communication needed between two or more parties to solve a particular problem [1,33]. We now give a brief summary of this area. The simplest and most illustrative scenario is the one in which two parties, called Alice and Bob, each possesses some piece of information, often encoded into some string, so that Alice has $x \in X$ and Bob has $y \in Y$, and they want to compute some function $f(x, y)$. Since each does not know the piece of information the other has, they will need to communicate information in order to compute $f(x, y)$. The most straightforward way to solve the problem is to have Alice and Bob exchange their entire string, but sometimes more efficient protocols exist. This communication model was first introduced by Yao in [34].

An important variant of this usual general communication scenario is the model of quantum communication complexity, again introduced by Yao [35], where now Alice and Bob each

has a quantum computer and they exchange qubits instead of bits and/or make use of shared entanglement. The question is whether Alice and Bob can now compute f with less communication than in the classical case; in some cases, this is known to be possible [1].

The above communication scenarios can be narrowed down by imposing some restrictions on the communication process between Alice and Bob and by restricting or allowing resources like randomness and entanglement. The three most common communication models are the one-way, the multi-round two-party, and the SMP models. In the multi-round two-party model both Alice and Bob can communicate with the other. On the other hand, in the one-way model only one party can communicate with the other, e.g., Alice communicates with Bob. Finally, in the SMP model Alice and Bob are only allowed to send messages to a third party, called the referee, who then computes $f(x, y)$. The SMP model was also introduced by Yao [34] and is the weakest reasonable model of communication complexity. Considering the SMP model in particular, Buhrman *et al.* [7] proved that, if f is the equality function, then a communication reduction from $\Theta(\sqrt{n})$ bits to $\Theta(\log n)$ qubits is possible.

Later, Yao showed that any classical SMP protocol with shared randomness that transmits $O(1)$ bits and computes a function on n bits implies a quantum SMP protocol without shared randomness that transmits $O(\log n)$ qubits [9]. This result was generalized by Gavinsky *et al.* [8], who gave a quantum SMP protocol that simulates any two-way quantum communication protocol with shared entanglement, at communication cost exponential in the cost of the original protocol. However, Gavinsky *et al.* also proved that for most functions, quantum fingerprinting protocols, which are a subclass of quantum SMP protocols, are exponentially worse than classical deterministic SMP protocols.

Recently, more exotic communication models based on indefinite causal structures were used to demonstrate exponential quantum advantage. Wei *et al.* [36] and Guérin *et al.* [37] showed such an exponential communication advantage by using the concept of a quantum switch (a device that controls the order in which two transformations are performed) to coherently superpose the one-way communication path of information in a tripartite setting, i.e., from Alice to Bob and then to the referee or from Bob to Alice and then to the referee.

II. PROTOCOL

In this section we present our protocol for approximating the Hamming distance $d(x, y)$ between two strings $x, y \in \{0, 1\}^n$ up to relative error ϵ in the SMP model. That is, Alice and Bob seek the referee to output $d_\epsilon(x, y)$ such that $(1 - \epsilon)d(x, y) \leq d_\epsilon(x, y) \leq (1 + \epsilon)d(x, y)$. Call this problem HAM_ϵ .

We first state a lemma that is going to be useful for our protocol and which encapsulates results on quantum fingerprinting by Yao [9].

Definition 1. Given an N -bit string x , define the quantum state

$$|h_x\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^N |i\rangle |x_i\rangle, \tag{1}$$

where x_i is the i th bit of x .

Lemma 1. Given the N -bit strings x and y , their Hamming distance $d(x, y)$ can be estimated up to additive accuracy $N\epsilon$ with failure probability δ using $O(\log(1/\delta)/\epsilon^4)$ copies of $|h_x\rangle$ and $|h_y\rangle$ [9].

Proof. Given the N -bit strings x and y , we encode them with the states $|h_x\rangle$ and $|h_y\rangle$, respectively. Note that

$$\langle h_y | h_x \rangle = \frac{1}{N} \sum_{i=1}^N \langle y_i | x_i \rangle = 1 - \frac{d(x, y)}{N}. \quad (2)$$

The swap test [7] is a test which outputs either 0 or 1 on input $|h_x\rangle|h_y\rangle$ and outputs 1 with probability

$$\frac{1}{2}[1 - |\langle h_y | h_x \rangle|^2]. \quad (3)$$

We apply the swap test to k copies of $|h_x\rangle|h_y\rangle$, for some k to be determined. Let X_i correspond to the outcome of the i th swap test. In [9] it was proven that

$$\Pr[|\tilde{\eta} - |\langle h_y | h_x \rangle|| \geq \epsilon] \leq 2e^{-k\epsilon^4/32}, \quad (4)$$

where $\tilde{\eta} = \sqrt{1 - \frac{2}{k} \sum_i X_i}$. We hence conclude that

$$\Pr[|\tilde{d} - d(x, y)| \geq N\epsilon] \leq 2e^{-k\epsilon^4/32}, \quad (5)$$

where $\tilde{d} = N(1 - \sqrt{1 - \frac{2}{k} \sum_i X_i})$. Setting δ as the probability of error, we see that it is sufficient to use $k = O(\log(1/\delta)/\epsilon^4)$ copies of the states to estimate $d(x, y)$ up to additive accuracy $N\epsilon$ with failure probability δ . ■

[Given that we aim to approximately compute the inner product between $|h_x\rangle$ and $|h_y\rangle$ in Lemma 1, the reader may wonder why the Hadamard test [38] was not used instead, given that this test allows direct estimation of $\langle h_y | h_x \rangle$. The reason is that the Hadamard test requires the ability to produce the coherent superposition $\frac{1}{\sqrt{2}}(|0\rangle|h_x\rangle + |1\rangle|h_y\rangle)$, which is not available to the referee.]

In the following, we use the notation $|z|$ to mean the number of entries equal to 1 in a string $z \in \{0, 1\}^n$.

Lemma 2. Consider an $N \times n$ matrix A over \mathbb{F}_2 whose entries are randomly chosen from $\{0, 1\}$ and equal to 1 with independent probability $1/2d$ for some $d \geq 1$. Fix $\epsilon > 0$. Then there exist values $\delta_1 < \delta_2$ that do not depend on N and n such that $\delta_2 - \delta_1 = \Theta(\epsilon)$ and for any $\eta > 0$ the following hold: For all $z \in \{0, 1\}^n$ such that $|z| \leq d$, $\Pr_A[|Az| \geq N\delta_1 + N\eta] \leq e^{-2N\eta^2}$, and for all $z \in \{0, 1\}^n$ such that $|z| \geq (1 + \epsilon)d$, $\Pr_A[|Az| \leq N\delta_2 - N\eta] \leq e^{-2N\eta^2}$. Hence, for sufficiently large $N = \Theta(n/\epsilon^2)$, with high probability over the choice of A , it is sufficient to determine $|Az|$ up to additive accuracy $\Theta(N\epsilon)$ to distinguish between the cases $|z| \leq d$ and $|z| \geq (1 + \epsilon)d$.

Proof. It is shown in [39] that for any z , $\Pr[(Az)_i = 1] = \frac{1}{2}[1 - (1 - 1/2d)^{|z|}]$ and that the probabilities of this event for $|z| \leq d$ and $|z| \geq (1 + \epsilon)d$ are bounded by values δ_1 and δ_2 that do not depend on N and n and are separated by $\Theta(1 - e^{-\epsilon/2}) = \Theta(\epsilon)$, that is,

$$\Pr_A[(Az)_i = 1] \leq \delta_1 = \frac{1}{2} \left[1 - \left(1 - \frac{1}{2d} \right)^d \right] \quad (6a)$$

if $|z| \leq d$,

$$\Pr_A[(Az)_i = 1] \geq \delta_2 = \frac{1}{2} \left[1 - \left(1 - \frac{1}{2d} \right)^{(1+\epsilon)d} \right] \quad (6b)$$

if $|z| \geq (1 + \epsilon)d$.

The expected value of $|Az| = \sum_i (Az)_i$ then satisfies

$$\mathbb{E}[|Az|] \leq N\delta_1 \quad \text{if } |z| \leq d, \quad (7a)$$

$$\mathbb{E}[|Az|] \geq N\delta_2 \quad \text{if } |z| \geq (1 + \epsilon)d. \quad (7b)$$

If $|z| \leq d$ so that $\mathbb{E}[|Az|] \leq N\delta_1$, by a Chernoff bound [40] we obtain

$$\Pr_A[|Az| \geq N\delta_1 + N\eta] \leq e^{-2N\eta^2}. \quad (8)$$

By the same token, if $|z| \geq (1 + \epsilon)d$ so that $\mathbb{E}[|Az|] \geq N\delta_2$, we obtain

$$\Pr_A[|Az| \leq N\delta_2 - N\eta] \leq e^{-2N\eta^2}. \quad (9)$$

Taking a union bound over all $z \in \{0, 1\}^n$ in both cases, we have

$$\Pr_A[\exists z \text{ such that } |z| \leq d \text{ and } |Az| \geq N\delta_1 + N\eta] \leq 2^n e^{-2N\eta^2} = e^{n \ln 2 - 2N\eta^2},$$

$$\Pr_A[\exists z \text{ such that } |z| \geq (1 + \epsilon)d \text{ and } |Az| \leq N\delta_2 - N\eta] \leq 2^n e^{-2N\eta^2} = e^{n \ln 2 - 2N\eta^2},$$

so it is sufficient to choose $N = \Omega(n/\eta^2)$ to bound the probability that either case occurs by an arbitrarily small constant. Choosing $\eta = c\epsilon$ for a sufficiently small constant c , we have $|Az| \leq N(\delta_1 + c\epsilon)$ if $|z| \leq d$ and $|Az| \geq N(\delta_2 - c\epsilon)$ if $|z| \geq (1 + \epsilon)d$. Therefore, it is sufficient to determine $|Az|$ up to additive accuracy $O(N\epsilon)$ to distinguish these two cases. ■

The map A in Lemma 2 can be interpreted as a linear code. Such codes are also used in quantum fingerprinting protocols [7,17], but here, unlike previous protocols, we choose the matrix A to be sparse and random. This enables us to control its behavior when acting on strings z such that $|z| \approx d$, when d is small.

We now describe our protocol based on the two previous lemmas. In this protocol, Alice and Bob have already agreed beforehand on the matrix A , guaranteed to exist by Lemma 2, to be used. We stress that this matrix is fixed in advance and does not need to be chosen using shared randomness.

Protocol 1. Consider the following subroutine for arbitrary $d \in [1, n]$ and $\delta > 0$. Alice and Bob encode their n -bit strings x and y as Ax and Ay , respectively, where A is picked according to Lemma 2 and multiplication is over \mathbb{F}_2 . They send $O((\log 1/\delta)/\epsilon^4)$ copies of the quantum states $|h_{Ax}\rangle$ and $|h_{Ay}\rangle$ to the referee, who performs swap tests and estimates the Hamming distance $d(Ax, Ay)$ up to accuracy $N\epsilon$ with failure probability δ . By Lemma 2, this is sufficient to determine whether $d(x, y) \leq d$ or $d(x, y) \geq (1 + \epsilon)d$ with failure probability δ .

Alice and Bob then apply this subroutine to the sequence S of values d ,

$$0, 1, 1 + \epsilon, (1 + \epsilon)^2, \dots, \quad (10)$$

where the last element in S corresponds to the minimal k such that $(1 + \epsilon)^{k+1} > n$; there are $O(\log n / \log(1 + \epsilon)) =$

$O((\log n)/\epsilon)$ elements in the sequence. (In the case $d = 0$, they use the standard quantum fingerprinting protocol instead.) Given the $O((\log n)/\epsilon)$ results, the referee outputs the minimal \tilde{d} such that the subroutine returned $d(x, y) \leq \tilde{d}$. ■

We first show that, if each use of the subroutine succeeds, the overall algorithm achieves the required level of accuracy. By the definition of S , there exist consecutive elements $d_0, d_1, d_2 \in S$ such that $d_0 \leq d(x, y)/(1 + \epsilon)$, $d(x, y)/(1 + \epsilon) \leq d_1 \leq d(x, y)$, and $d(x, y) \leq d_2 \leq (1 + \epsilon)d(x, y)$. Then on input d_2 the subroutine must return $d(x, y) \leq d_2$, while for input d_0 it must return $d(x, y) \geq (1 + \epsilon)d_0$, so the output \tilde{d} is either d_1 or d_2 and hence

$$(1 - \epsilon)d(x, y) \leq \frac{d(x, y)}{1 + \epsilon} \leq \tilde{d} \leq (1 + \epsilon)d(x, y).$$

Setting $\delta = O(\epsilon/\log n)$ and using a union bound over the $O((\log n)/\epsilon)$ uses of the subroutine, the probability that any of the subroutines fails can be upper bounded by an arbitrarily small positive constant. The overall communication complexity is

$$O([\log n/\epsilon] \cdot [\log(1/\delta)/\epsilon^4] \cdot (\log n + \log 1/\epsilon)) = O((\log n)^2 (\log \log n)/\epsilon^5), \tag{11}$$

assuming that $\epsilon \geq 1/\log n$. This completes the proof of Theorem 1.

III. MEASURING DISTANCES IN GRAPHS

In the following, for an arbitrary graph G and vertices v and w , let $d_G(v, w)$ denote the distance between v and w in G , i.e., the length of the shortest path between v and w . Also, the hypercube graph Q_n is defined as the graph with vertex set $\{0, 1\}^n$, where the distance between vertices is the Hamming distance.

The algorithm from the preceding section for approximately measuring the Hamming distance between two strings in the SMP model can be slightly modified to approximately compute the distance between two vertices in specific graphs in the SMP model. That is, to solve the following problem, for some graph $G = (V, E)$ and given vertices v and w as input, output \tilde{d} such that $(1 - \epsilon)d_G(v, w) \leq \tilde{d} \leq (1 + \epsilon)d_G(v, w)$. Call this problem $\text{DIS}_\epsilon[G]$. The idea is to embed a given graph G into a hypercube graph such that all the distances between vertices are preserved or rescaled by a constant factor. Once this embedding is achieved, the hypercube structure allows the equivalence between vertex distance in the graph and Hamming distance, so a binary string can be associated with each vertex and the algorithm can be applied to these binary strings.

The downside of the above approach is that it cannot be applied to any given graph, since most graphs are not isometrically embeddable into a hypercube. The graphs which can be isometrically embedded into hypercubes are known as partial cubes [41,42].

The identification of which graphs are partial cubes is an interesting question by itself. The class of partial cubes is relatively broad. The most important examples are hypercubes, trees [43], and median graphs [44]. It also includes other significant classes, e.g., tope graphs of oriented

matroids (special graphs of regions of hyperplane arrangements) [45,46], antimatroids [46,47], weak orderings [46], bipartite $(6, 3)$ -graphs [23], tiled partial cubes [48], and netlike partial cubes [49].

Partial cubes can be fully characterized via Djoković’s characterization [50,51], introduced by Djoković. It connects the property of isometric embedding to bipartiteness and convexity of some specific subgraphs of the original graph. Here a set is said to be convex if it is closed under taking shortest paths, i.e., if the shortest paths between any two points from the set are also contained in the set. Djoković’s characterization states, more specifically, that a connected graph G can be isometrically embedded into a hypercube if and only if G is bipartite and $G(a|b)$ is convex for each edge (a, b) of G , where $G(a|b) := \{x \in V(G) \mid d_G(x, a) < d_G(x, b)\}$ is the set of the vertices closer to a than b . In other words, to check if a graph is a partial cube, one needs to check first if the graph is bipartite. Apart from that, one chooses an edge and constructs the set of all vertices that are closer to one vertex of the chosen edge than the other vertex. Then one needs to check if all shortest paths connecting any two vertices from this set only pass through the vertices of the set. If yes, the set is said to be convex and the same procedure is repeated for another edge of the original graph. If all sets constructed in this way are convex, then the graph is a partial cube.

Since the original protocol is unaffected if all distances are rescaled by a constant factor, the idea of partial cubes can be expanded by the following definitions.

Definition 2. Given two connected and unweighted graphs G and H , we write $G \xrightarrow{k} H$ and say that G is a scale k embedding of H if there exists a mapping $\sigma : V(G) \rightarrow V(H)$ such that $d_H(\sigma(a), \sigma(b)) = k \cdot d_G(a, b)$ for all nodes $a, b \in V(G)$ [23,52].

It is clear that partial cubes are just graphs which can be embedded in a hypercube with a scale 1 embedding. An example of a graph which is not a partial cube, but can be embedded in a hypercube with a scale k embedding for $k > 1$, is a triangle, which embeds into Q_3 with $k = 2$.

Definition 3. A graph G is said to be an ℓ_1 -graph if its path metric d_G is ℓ_1 -embeddable, i.e., there is a map f between $V(G)$ and \mathbb{R}^m , for some m , such that $d_G(v, w) = \|f(v) - f(w)\|_1$ [50].

Theorem 5. A graph G is an ℓ_1 -graph if and only if it admits a scale embedding into a hypercube [23].

This means that the graphs we are interested in are ℓ_1 -graphs. This class of ℓ_1 -graphs includes new graphs that are not partial cubes, e.g., Hamming graphs, half cubes, and Johnson graphs are 2-embeddable into a hypercube [23]. In the Appendix we develop a similar characterization for ℓ_1 -graphs and the final result is the following theorem, which is Djoković’s characterization without the bipartite requirement.

Theorem 6. A graph G is an ℓ_1 -graph if and only if $G(a|b)$ is convex for each edge (a, b) of G .

By allowing the rescaling of all the distances by an even factor we can relax the bipartite requirement, but not the convexity of the $G(a|b)$ subgraphs. As an example of a direct consequence of the above result, it is known that graphs of the form C_{2n} and $C_{2n} \square K_2$ for $n \geq 2$ are partial cubes, where C_n is a cycle on n vertices, K_n is the complete graph with n vertices,

and \square denotes the Cartesian product [41]; therefore all graphs of the form C_n and $C_n \square K_2$, for $n \geq 2$, are ℓ_1 -graphs.

Before stating the communication protocol in the SMP model to approximately measure the distance between two vertices in an ℓ_1 -graph, we state the Johnson-Lindenstrauss lemma [8,53,54], which is going to be useful to reduce the protocol complexity. Note that we use Dirac notation for vectors which are not necessarily normalized.

Lemma 3. Consider $0 < \epsilon < 1/2$ and a positive integer n . Then for any set U of k vectors in \mathbb{R}^n , there is a linear map $f : \mathbb{R}^n \rightarrow \mathbb{R}^{O((\log k)/\epsilon^2)}$ such that for all $|u\rangle, |v\rangle \in U$,

$$(1 - \epsilon)\| |u\rangle - |v\rangle \|^2 \leq \|f|u\rangle - f|v\rangle\|^2 \leq (1 + \epsilon)\| |u\rangle - |v\rangle \|^2.$$

To find a map f achieving the bounds of Lemma 3, one can choose it at random from an appropriate distribution. A number of different constructions of such random maps are known; one simple example is a suitably normalized projection onto a random subspace of \mathbb{R}^n .

As mentioned, e.g., in [8], if the set U includes the 0-vector, then the map f also approximately preserves the inner product between all the pairs of vectors in U . This implies the following lemma.

Lemma 4. Let $0 < \epsilon < 1/2$. Let U be a set of unit vectors in \mathbb{R}^n and let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map such that, for all $|u\rangle, |v\rangle \in U \cup \{\vec{0}\}$,

$$(1 - \epsilon)\| |u\rangle - |v\rangle \|^2 \leq \|f|u\rangle - f|v\rangle\|^2 \leq (1 + \epsilon)\| |u\rangle - |v\rangle \|^2.$$

Define the unit vectors $|\tilde{u}\rangle = f|u\rangle/\|f|u\rangle\|$ for all $|u\rangle \in U$. Then

$$|\langle \tilde{u} | \tilde{v} \rangle| - |\langle u | v \rangle| \leq 4\epsilon$$

for all $|u\rangle, |v\rangle \in U$.

Proof. For clear notation, define $|u'\rangle := f|u\rangle$. By the conditions on f , we have that

$$1 - \epsilon \leq \langle u' | u' \rangle \leq 1 + \epsilon,$$

$$(1 - \epsilon)\| |u\rangle - |v\rangle \|^2 \leq \| |u'\rangle - |v'\rangle \|^2 \leq (1 + \epsilon)\| |u\rangle - |v\rangle \|^2$$

for all $|u\rangle, |v\rangle \in U$, where the first line was obtained by taking the 0-vector as one of the vectors and using linearity of f . From the above inequalities it follows that

$$(1 + \epsilon)\langle u | v \rangle - 2\epsilon \leq \langle u' | v' \rangle \leq (1 - \epsilon)\langle u | v \rangle + 2\epsilon.$$

These inequalities in turn lead to

$$\langle \tilde{u} | \tilde{v} \rangle \geq \frac{(1 + \epsilon)\langle u | v \rangle - 2\epsilon}{1 + \epsilon} \geq \langle u | v \rangle - 2\epsilon$$

and

$$\langle \tilde{u} | \tilde{v} \rangle \leq \frac{(1 - \epsilon)\langle u | v \rangle + 2\epsilon}{1 - \epsilon} \leq \langle u | v \rangle + 4\epsilon,$$

using that $0 < \epsilon < 1/2$. Therefore,

$$|\langle \tilde{u} | \tilde{v} \rangle| - |\langle u | v \rangle| \leq |\langle \tilde{u} | \tilde{v} \rangle - \langle u | v \rangle| \leq 4\epsilon. \quad \blacksquare$$

Consider applying Lemma 1 to the normalized quantum states $|\tilde{h}_x\rangle$ and $|\tilde{h}_y\rangle$ that are produced by using the Johnson-Lindenstrauss lemma (Lemma 3), in the sense that the original states $|h_x\rangle$ and $|h_y\rangle$ in Lemma 1 are replaced with the states $|\tilde{h}_x\rangle$ and $|\tilde{h}_y\rangle$. We argue that this does not change the

parameters of the lemma substantially. To see that, we note that $|\tilde{\eta} - |\langle h_y | h_x \rangle|| + ||\langle \tilde{h}_y | \tilde{h}_x \rangle| - |\langle h_y | h_x \rangle|| \geq |\tilde{\eta} - |\langle \tilde{h}_y | \tilde{h}_x \rangle||$ and hence $|\tilde{\eta} - |\langle \tilde{h}_y | \tilde{h}_x \rangle|| \geq 5\epsilon \Rightarrow |\tilde{\eta} - |\langle h_y | h_x \rangle|| \geq \epsilon$, which means

$$\Pr[|\tilde{\eta} - |\langle \tilde{h}_y | \tilde{h}_x \rangle|| \geq 5\epsilon] \leq \Pr[|\tilde{\eta} - |\langle h_y | h_x \rangle|| \geq \epsilon], \quad (12)$$

where $\tilde{\eta}$ is as defined in Lemma 1. With this in mind, and recalling that $\text{diam}(G)$ is defined to be the diameter of the graph G , i.e., the greatest distance between any pair of vertices, we present the communication protocol.

Protocol 2. Alice and Bob each hold vertices $u, v \in V(G)$, respectively, from a graph G which admits a scale k embedding into a hypercube Q_n , for some n . Their vertex images are the n -bit strings $x, y \in Q_n$, respectively. The communication protocol to measure $(1 \pm \epsilon)d_G(u, v)$ can be divided into three parts.

First, given $d \in [1, \text{diam}(G)]$ and a matrix A picked according to Lemma 2, Alice and Bob encode their n -bit strings x and y as Ax and Ay , respectively, where multiplication is over \mathbb{F}_2 . Differently from the original protocol, Alice and Bob apply the Johnson-Lindenstrauss lemma to their data Ax and Ay , which are then encoded into the quantum states $|\tilde{h}_{Ax}\rangle$ and $|\tilde{h}_{Ay}\rangle$. There are $|V|$ possible vectors to encode, so the number of qubits to be used is reduced from $O(\log n + \log(1/\epsilon))$ to $O(\log \log |V| + \log(1/\epsilon))$.

Second, Alice and Bob send $O((\log 1/\delta)/\epsilon^4)$ copies of their quantum states $|\tilde{h}_{Ax}\rangle$ and $|\tilde{h}_{Ay}\rangle$ to the referee, who performs swap tests and estimates the Hamming distance $d(Ax, Ay)$ up to accuracy $N\epsilon$ with failure probability δ , and from this decides if $d(x, y) \leq d$ or $d(x, y) \geq (1 + \epsilon)d$.

The third and final part is to apply the first and second parts to the sequence S of values d ,

$$0, 1, 1 + \epsilon, (1 + \epsilon)^2, \dots, \quad (13)$$

where the last element in S corresponds to the minimal k such that $(1 + \epsilon)^{k+1} > \text{diam}(G)$; there are $O([\log \text{diam}(G)]/\epsilon)$ elements in the sequence. Based on the results from the swap tests, the referee outputs \tilde{d} such that $(1 - \epsilon)d(x, y) \leq \tilde{d} \leq (1 + \epsilon)d(x, y)$, in the same way as in Protocol 1. \blacksquare

Setting $\delta = O(\epsilon/\log \text{diam}(G))$, the overall communication complexity is then

$$O([\log \text{diam}(G)][\log \log \text{diam}(G)](\log \log |V|)/\epsilon^5), \quad (14)$$

assuming that $\epsilon \geq 1/[\log \text{diam}(G)]$.

The performance of the protocol is limited by the diameter of the graph. It is known that $\log_{\Delta-1} |V| - \frac{2}{\Delta} \leq \text{diam}(G) < |V|$, where Δ is the maximum vertex degree [55]. If $\text{diam}(G) = O(\log |V|)$, the overall complexity is poly log log in $|V|$. On the other hand, if $\text{diam}(G) = \Theta(|V|)$, the overall complexity is poly log in $|V|$, which is no better than the trivial protocol where Alice and Bob send their entire inputs to the referee.

Lower bound

One can ask if there could exist other protocols substantially more efficient than ours. In order to answer this, we prove a lower bound on the quantum communication complexity for the problem of approximately calculating the graph distance between two vertices on a graph, which demonstrates

that our protocol is essentially optimal in terms of the dependence of its complexity on the graph diameter. We do not know whether the fifth-power dependence on ϵ is optimal and suspect that it may not be.

The idea behind our proof is to transform the approximate graph distance problem into the problem of approximating the modulus of the difference between two integers. We then show that two uses of a protocol for this approximate modulus problem can compute the greater than function in the one-way communication model. It was shown by Zhang [56] that the one-way quantum communication complexity of this problem is maximal, improving a previous lower bound of Klauck [57] by a logarithmic term. The bound of [56] is used to obtain the lower bound for the approximate modulus problem, and hence for the approximate graph distance problem.

The first step of our proof is to show that two uses of a protocol for the approximate modulus problem can solve the greater than function in the one-way communication model. Consider the greater than problem (GT) defined by the Boolean function $GT : \{0, 1\}^m \times \{0, 1\}^m \rightarrow \{0, 1\}$ as

$$GT(x, y) = \begin{cases} 1 & \text{if } x \geq y \\ 0 & \text{if } x < y, \end{cases}$$

where x and y are interpreted as m -bit integers. Given $0 \leq \epsilon < 1$, consider the approximate modulus problem where Alice and Bob are each given an integer x and y , respectively, each expressed as an m -bit string, and seek to output \tilde{d} such that $(1 - \epsilon)|x - y| \leq \tilde{d} \leq (1 + \epsilon)|x - y|$. Call this problem MOD_ϵ .

In the following we prove that two uses of this protocol suffice to solve the GT problem. Let \mathcal{P} be a quantum communication protocol in the one-way communication model which solves a problem f with failure probability δ . Denote by $Q^1(\mathcal{P})$ the communication cost of the protocol \mathcal{P} (in qubits) and define by $Q^1(f) = \min_{\mathcal{P}} Q^1(\mathcal{P})$ the minimum communication cost over all protocols \mathcal{P} that solve f with failure probability $1/3$.

Lemma 5. For any $\epsilon < 1/4$, $Q^1(GT) = O(Q^1(MOD_\epsilon))$.

Proof. Let \mathcal{P}_{MOD} be a communication protocol for MOD_ϵ in the one-way communication model with failure probability $1/6$. [We can obtain a protocol which achieves this failure probability and communicates $O(Q^1(MOD_\epsilon))$ qubits using $O(1)$ repetitions of the protocol which achieves failure probability $1/3$ and communicates $Q^1(MOD_\epsilon)$ qubits.]

Two uses of \mathcal{P}_{MOD} suffice to obtain a communication protocol for GT in the one-way communication model with failure probability $1/3$ as follows. Alice and Bob apply the protocol \mathcal{P}_{MOD} using x and y as inputs and Bob obtains $z_0 \in [(1 - \epsilon)|x - y|, (1 + \epsilon)|x - y|]$. They both apply the same protocol again, but now Bob inputs $y + z_0$ (Alice still inputs x). Bob obtains z_1 . If $z_0 < z_1$, then $x < y$ and he outputs 0. Otherwise, $x \geq y$ and he outputs 1.

To see why this protocol works (assuming that each use of \mathcal{P}_{MOD} succeeds), note that if $x < y$, then $(2 - \epsilon)|x - y| \leq |x - y - z_0| \leq (2 + \epsilon)|x - y|$, and hence

$$(2 - \epsilon)(1 - \epsilon)|x - y| \leq z_1 \leq (2 + \epsilon)(1 + \epsilon)|x - y|.$$

If $x \geq y$, then $0 \leq |x - y - z_0| \leq \epsilon|x - y|$, and hence

$$0 \leq z_1 \leq \epsilon(1 + \epsilon)|x - y|.$$

For $x < y$ we want to have $z_0 < z_1$, i.e., $1 + \epsilon < (2 - \epsilon)(1 - \epsilon)$, which holds if $\epsilon < 2 - \sqrt{3}$. For $x \geq y$ we need $z_0 \geq z_1$, i.e., $\epsilon(1 + \epsilon) \leq 1 - \epsilon$, which holds if $\epsilon \leq \sqrt{2} - 1$. Therefore, by taking $\epsilon < 1/4$, for example, one can distinguish the cases $x < y$ and $x \geq y$ through a comparison between z_0 and z_1 .

Given that every protocol for MOD_ϵ in the one-way communication model implies a protocol for GT, we conclude that $Q^1(GT) = O(Q^1(MOD_\epsilon))$. ■

The next step is to reduce the approximate modulus problem to the approximate graph distance problem, which we achieve as follows. Let G be a graph with diameter $\text{diam}(G)$. By the definition of diameter, there exists a path graph $P_n \subseteq G$ with $n = \text{diam}(G)$. Therefore, a lower bound for the approximate graph distance problem on P_n implies a lower bound for the same problem on G .

The vertices of P_n can be listed in the order v_1, v_2, \dots, v_n such that the edges are (v_i, v_{i+1}) , where $i = 1, 2, \dots, n - 1$. A given vertex v_i can then be labeled by a binary string $x_i \in \{0, 1\}^m$, with $m = \Theta(\log n)$, and hence, given $v_i, v_j \in G$, $d_G(v_i, v_j) = |x_i - x_j|$. Therefore, a communication protocol which outputs \tilde{d} such that $(1 - \epsilon)d_G(v_i, v_j) \leq \tilde{d} \leq (1 + \epsilon)d_G(v_i, v_j)$ is equivalent to a communication protocol which solves MOD_ϵ on inputs x_i and x_j . So computing an approximate modulus reduces to computing an approximate graph distance. With this in mind, we can state our lower bound.

Theorem 7. Given a graph G with diameter $\text{diam}(G)$, the quantum communication complexity for the problem $DIS_\epsilon[G]$ in the one-way communication model with $\epsilon < 1/4$ and failure probability $1/3$ is $Q^1(DIS_\epsilon[G]) = \Omega[\log \text{diam}(G)]$.

Proof. As mentioned before, the approximate graph distance problem on a path graph $P_n \subseteq G$ with $n = \text{diam}(G)$ should be at least as hard as the same problem on G , i.e., $Q^1(DIS_\epsilon[G]) \geq Q^1(DIS_\epsilon[P_n])$. Moreover, $DIS_\epsilon[P_n]$ is equivalent to MOD_ϵ on inputs of size $m = \Theta(\log[\text{diam}(G)])$, hence $Q^1(DIS_\epsilon[G]) \geq Q^1(MOD_\epsilon)$. According to Lemma 5, $Q^1(MOD_\epsilon) = \Omega(Q^1(GT))$, but $Q^1(GT) = \Theta(m)$ [56], and therefore $Q^1(DIS_\epsilon[G]) = \Omega[\log \text{diam}(G)]$. ■

The above result for the one-way communication model also holds for the SMP model. It then states that our communication protocol is optimal in terms of its dependence on $\text{diam}(G)$.

IV. MEASURING ℓ_1 DISTANCES

As seen in the previous sections, our communication protocol for approximating the Hamming distance can be adapted to ℓ_1 -graphs. A graph G is said to be an ℓ_1 -graph if there exist vectors $u_1, \dots, u_n \in \mathbb{R}^m$ for some m , and with $n = |V(G)|$, such that $d_G(v_i, v_j) = \|u_i - u_j\|_1$ for all $v_i, v_j \in V(G)$. This connection between graphs and ℓ_1 -norm suggests an application of our approximate Hamming distance protocol to ℓ_1 distances. More specifically, consider the following problem. Alice and Bob are each given a vector x and y , respectively, from $[-1, 1]^d$. Each entry of each vector is specified by k bits, for some k (1 bit to specify the sign and $k - 1$ bits z_1, \dots, z_{k-1} to specify a binary fraction $z_1 2^{-1} + z_2 2^{-2} + \dots + z_{k-1} 2^{1-k}$). Alice and Bob's task is to approximate the ℓ_1 distance between x and y up to relative error ϵ in the SMP model.

A natural special case of this problem is where Alice and Bob are each given a probability distribution $x, y \in \mathbb{R}^d$,

respectively, and are asked to approximately compute the ℓ_1 distance between them (equivalently, the total variation distance, which is defined as half the ℓ_1 distance). This corresponds to the special case where $x_i, y_i \geq 0$ for all i , and $\sum_i x_i = \sum_i y_i = 1$.

Alice and Bob can use our approximate Hamming distance protocol to approximately compute $\|x - y\|_1$: The idea is to map these vectors into a Hamming metric via a unary representation [58]. Each entry $z \in [-1, 1]$ of each vector is mapped to a 2^k -bit string $s(z)$ such that the first $2^{k-1}(z + 1)$ bits of $s(z)$ are set to 1 and the remaining bits are set to 0. Then, for any z and w , $|z - w| = d(s(z), s(w))/2^{k-1}$. Letting $s(x)$ denote the result of applying this map to each entry of x and concatenating the results, we have $\|x - y\|_1 = d(s(x), s(y))/2^{k-1}$ for bit strings $s(x)$ and $s(y)$ of length $m = 2^k d$. So we can use our usual communication protocol (Protocol 1) to deliver an estimate of $\|x - y\|_1$ up to relative error ϵ using $O((\log^2 m)(\log \log m)/\epsilon^5)$ qubits of communication, which is $O((\log^2 d)(\log \log d)/\epsilon^5)$ when $k \leq \log d$.

The use of a unary representation may seem wasteful, but a straightforward binary representation would not preserve distances correctly for all inputs. There is also a lower bound that the communication complexity of this problem must have at least a linear dependence on k : By the lower bound on the complexity of the MOD_ϵ problem that follows from Lemma 5, $\Omega(k)$ bits of communication are required to approximately compute $\|x - y\|_1$ even for $d = 1$. Finally, the protocol can easily be extended to the setting where $x, y \in [-M, M]^d$, for some $M \geq 1$, by rescaling the vectors appropriately.

V. CONCLUSION

We developed an efficient quantum communication protocol to approximately compute the Hamming distance between two n -bit strings in the SMP model up to relative error ϵ , which uses $\tilde{O}((\log n)^2/\epsilon^5)$ qubits of communication, whereas any classical communication protocol needs to transmit at least $\Omega(\sqrt{n})$ bits. We stress that the protocol approximates the Hamming distance up to a relative, and not additive, error, so small Hamming distances are approximated accurately.

The Hamming distance protocol was modified to approximate the distance between any two vertices in a graph. This modification was based on embedding the graph into a subgraph of the Hamming cube such that all distances are preserved up to a constant factor. This requirement restricts the class of graphs to which the original Hamming distance protocol can be applied. Graphs with this property are known as ℓ_1 -graphs. The modified quantum communication protocol to approximate the vertex distance in ℓ_1 -graphs in the SMP model up to relative error ϵ transmits $\tilde{O}(\log[\text{diam}(G)]/\epsilon^5)$ qubits, where $\text{diam}(G)$ is the diameter of the graph, so the protocol is only efficient for low diameter graphs. A lower bound on the number of communicated qubits needed to approximate the vertex distance shows that this limitation of our protocol is due to the problem itself. More specifically, we proved that any one-way quantum protocol to approximate the distance between any two vertices in a graph needs to communicate at least $\Omega[\log \text{diam}(G)]$ qubits. Finally, the original Hamming distance protocol was also modified to approximate

the ℓ_1 -norm between two vectors $x, y \in [-1, 1]^d$ specified by $k \leq \log d$ bits.

Two interesting questions remain open: Can one find a similar result to Theorem 2 which holds for all graphs, without the restriction to ℓ_1 -graphs, and can one improve the communication complexity dependence on ϵ , currently at $1/\epsilon^5$?

No new data were created during this study.

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APPENDIX: ℓ_1 -GRAPH CHARACTERIZATION

In this Appendix we will prove Theorem 6. Remember that $G(a|b) := \{x \in V(G) \mid d_G(x, a) < d_G(x, b)\}$.

This theorem is a generalization of Djoković's characterization [50,51] for partial cubes by introducing the concept of scale k embedding, which is linked to the concept of ℓ_1 -graphs. A partial cube is then just a special case of ℓ_1 -graphs.

While the idea of scale k embedding and some of its properties related to partial cubes were already studied, we could not find a clear and direct characterization for ℓ_1 -graphs as it is stated in Theorem 6, similar to Djoković's. For example, in [52] it is proved that a graph is embeddable into a hypercube with an odd scale if and only if it is 1-embeddable into a hypercube, meaning that odd scale embeddings do not add anything new. This makes sense since an odd scale embedding cannot alter the bipartiteness requirement.

The proof of the theorem is sketched as follows. The direction (i) \Rightarrow (ii) is a direct generalization of Djoković's proof (see [50]). On the other hand, the direction (ii) \Rightarrow (i) does not follow Djoković's proof, but instead introduces the idea of a k -rescaling map which transforms a given connected and unweighted graph into a new graph by adding $k - 1$ new vertices on each original edge. In this way, the original distances are rescaled by a factor of k . We show in Lemma 7 that if k is even, then this new graph is bipartite. Also, we show in Lemma 9 that this map preserves the convexity of subgraphs. This means that, if the sets $G(a|b)$ are convex for each edge (a, b) , then the new rescaled graph will fulfill the requirements from Djoković's characterization for k even and is therefore a partial cube. Since the original vertices are a subset of the new ones, the original graph is an ℓ_1 -graph.

In all the following, let $G = (V, E)$ be a connected and unweighted graph. We start by proving (i) \Rightarrow (ii).

Lemma 6. If G is an ℓ_1 -graph, then $G(a|b)$ is convex for each edge (a, b) of G .

Proof. Let (a, b) be an edge of G and let $x, y \in G(a|b)$ and $z \in V(G)$ lie on the shortest path from x to y . Consider a hypercube scale k embedding $\sigma_k : V \rightarrow \mathcal{Q}_n$ in which node a is labeled by $\sigma_k(a) = 0^n$ (where $c^j = ccc \dots c$ means c repeated

j times), node b is labeled by $\sigma_k(b) = 1^k 0^{n-k}$, and nodes x, y, z are labeled by the strings X, Y, Z . Given an n -bit string A , we define its i th bit as A_i .

We first prove that $v \in G(a|b)$ if and only if $[\sigma_k(v)]_i \neq 1$ for $i \in [1, k]$. Consider that $[\sigma_k(v)]_i \neq 1$ for $i \in [1, k]$. Therefore, $d_{Q_n}(\sigma_k(v), \sigma_k(b)) = k + d_{Q_n}(\sigma_k(v), \sigma_k(a))$ and hence v is closer to a than b , i.e., $v \in G(a|b)$. Now consider that $v \in G(a|b)$. This means $d_{Q_n}(\sigma_k(v), \sigma_k(a)) = lk$ and $d_{Q_n}(\sigma_k(v), \sigma_k(b)) = (l+1)k$ for some $l \in \mathbb{N}$. Suppose that $[\sigma_k(v)]_i = 1$ for m indices i in $[1, k]$. Therefore, $d_{Q_n}(\sigma_k(v), \sigma_k(b)) - k + m = d_{Q_n}(\sigma_k(v), \sigma_k(a)) - m$, which gives $(l+1)k - k + m = lk - m \Rightarrow m = 0$, i.e., $[\sigma_k(v)]_i \neq 1$ for $i \in [1, k]$.

Given this, then $X_i, Y_i \neq 1$ for $i \in [1, k]$, and $d_{Q_n}(X, Y) = d_{Q_n}(X, Z) + d_{Q_n}(Z, Y)$ since $d_G(x, y) = d_G(x, z) + d_G(z, y)$. This implies that $Z_i \neq 1$ for $i \in [1, k]$, i.e., $z \in G(a|b)$. This shows that the set $G(a|b)$ is convex. ■

To prove (ii) \Rightarrow (i), we first make the following definitions.

Definition 4. Let $\mathbb{G}_k : G \rightarrow G^{(k)}$ be the k -rescaling map which adds $k - 1$ new nodes on every edge $e \in E$. The resulting graph $G^{(k)} = (V^{(k)}, E^{(k)})$ is called the k -rescaled image of G . Also, $G^{(1)} = G$. It is straightforward that $|E^{(k)}| = k|E|$ and $|V^{(k)}| = |V| + (k - 1)|E|$.

Definition 5. Let $v \in V$. We define $G \oplus (v, v')$ as the graph $G' = (V', E')$ obtained by connecting an extra node v' to the node v so that $V' = V \cup \{v'\}$ and $E' = E \cup (v, v')$. If $v' = v$, we define $G \oplus (v, v) = G$.

Definition 6. Let $\mathbb{G}_k : G \rightarrow G^{(k)}$. Given $(u, v) \in E$, we define the set $V^{(k)}(u, v) := \{w \in V^{(k)} \mid d_{G^{(k)}}(u, w) < k; d_{G^{(k)}}(v, w) < k\}$.

The set $V^{(k)}(u, v)$ is just the nodes added between the nodes $u, v \in V$. With the above definition, $V^{(k)} = V \cup [\bigcup_{e \in E} V^{(k)}(e)]$.

We now state the following auxiliary lemmas.

Lemma 7. The k -rescaled image $G^{(k)}$ of G is bipartite if k is even.

Proof. A graph G is bipartite if and only if it does not contain an odd cycle. If G does not have cycles, then neither does $G^{(k)}$, since the k -rescaling map \mathbb{G}_k cannot create cycles. Therefore, $G^{(k)}$ is bipartite. Now suppose G has cycles. Given a cycle $S \subseteq V$, its k -rescaled image $S^{(k)} = \mathbb{G}_k(S)$ is such that $|S^{(k)}| = k|S|$. If S is an even cycle, then so is $S^{(k)}$. If S is an odd cycle, then $S^{(k)}$ is an even cycle if k is even. Therefore, $S^{(k)}$ cannot have odd cycles for k even and hence is bipartite. ■

Lemma 8. Let $S \subseteq V$ and $v \in S$. Take a completely new vertex $v' \notin V$. Construct the new graph $G' = G \oplus (v, v')$ and consider the new subset $S' = S \cup \{v'\}$. If S is convex, then so is S' .

Proof. Let $x, y \in S'$ and $z \in V \cup \{v'\}$ be such that $d_G(x, y) = d_G(x, z) + d_G(y, z)$. We have two cases: Either $x, y \in S$ or, without loss of generality, $x = v'$ and $y \in S$. If $x, y \in S$, then it is straightforward that $z \neq v'$; otherwise the node $v \in S$ would be traversed twice. Therefore, $z \in V$ and since S is convex, $z \in S \subseteq S'$ and S' is convex. On the other hand, if $x = v'$ and $y \in S$, the fact that v' is only connected to v implies that $d_G(v', y) = d_G(v', z) + d_G(y, z) \iff$

$d_G(v, y) = d_G(v, z) + d_G(y, z)$, which, together with S being convex, means that $z \in S \subseteq S'$. ■

Lemma 9. Let $G^{(k)} = (V^{(k)}, E^{(k)})$ be the k -rescaled image of G . Then $S \subseteq V$ is convex if and only if $S^{(k)} = \mathbb{G}_k(S) \subseteq V^{(k)}$ is convex.

Proof. We start by proving S convex $\Rightarrow S^{(k)}$ convex. Given $U \subseteq V$, we write $U^{(k)} = U \cup U'$, where $U' = \bigcup_{e \in E} U^{(k)}(e)$ is the set of added nodes. Let $x, y \in S^{(k)}$ and $z \in V^{(k)}$ be such that $d_{G^{(k)}}(x, y) = d_{G^{(k)}}(x, z) + d_{G^{(k)}}(z, y)$. We will show that $z \in S^{(k)}$. Let us define the sets $A = \{a \in V \mid d_{G^{(k)}}(x, y) = d_{G^{(k)}}(x, a) + d_{G^{(k)}}(a, y)\}$ and $A' = \{a \in V' \mid d_{G^{(k)}}(x, y) = d_{G^{(k)}}(x, a) + d_{G^{(k)}}(a, y)\}$, i.e., A is the set of original nodes that lie in the shortest path between x and y , while A' is the set of added nodes that lie in the shortest path between x and y . Note that $z \in A \cup A'$. Suppose $A = \emptyset$. This means that $x, y \in S^{(k)}(e)$ for some edge $e \in E$. Therefore, we must have $z \in S^{(k)}(e) \subseteq S^{(k)}$.

Now suppose $A \neq \emptyset$. Let $a(x), a(y) \in A$ be the closest nodes to x and y , respectively. We must have $a(x) \in S$ [and $a(y) \in S$] since either $x \in S$ and then $a(x) = x$, or $x \in S^{(k)}(e)$ for some edge e , and then $a(x)$ is an end node of e . We can have two situations: Either $a(x) = a(y)$ or $a(x) \neq a(y)$.

Suppose $a(x) = a(y)$. Since $x \neq y$, this means that $x \in S^{(k)}(a(x), v_1)$ and $y \in S^{(k)}(a(x), v_2)$, for some $v_1, v_2 \in V$ such that $v_1 \neq v_2$, i.e., they are added nodes to two different edges with the common node $a(x)$. Therefore, either $z \in S^{(k)}(a(x), v_1)$ or $z \in S^{(k)}(a(x), v_2)$, or $z = a(x)$, which leads to $z \in S^{(k)}$.

Suppose then that $a(x) \neq a(y)$. Consider for now that $z \in A$. Since S is convex and $d_{G^{(k)}}(a(x), a(y)) = d_{G^{(k)}}(a(x), z) + d_{G^{(k)}}(a(y), z) \iff d_G(a(x), a(y)) = d_G(a(x), z) + d_G(a(y), z)$, we conclude that $z \in S$, i.e., $A \subseteq S$. Now consider that $z \in A'$, so $z \in V^{(k)}(v_1, v_2)$ for some nodes $v_1, v_2 \in V$. We must have $v_1, v_2 \in S$. Indeed, if $x \in V^{(k)}(v_1, v_2)$ (or y), by construction $x \in S^{(k)}$ and $x \in V^{(k)}(v_1, v_2) \Rightarrow v_1, v_2 \in S$. In addition, if $x, y \notin V^{(k)}(v_1, v_2)$, it implies that $v_1, v_2 \in A \subseteq S$. Hence $z \in S^{(k)}(v_1, v_2) \subseteq S^{(k)}$. Thus $z \in S^{(k)}$ in all cases, so $S^{(k)}$ is convex.

We now prove the other direction, $S^{(k)}$ convex $\Rightarrow S$ convex. Let $x, y \in S$ and $z \in V$ be such that $d_G(x, y) = d_G(x, z) + d_G(z, y)$. Suppose $z \notin S$. Recalling the equivalence between $d_{G^{(k)}}$ and d_G , this implies that $\exists z \in V \subset V^{(k)}$ but $z \notin S^{(k)}$ such that $d_{G^{(k)}}(x, y) = d_{G^{(k)}}(x, z) + d_{G^{(k)}}(z, y)$ for $x, y \in S \subset S^{(k)}$, which is a contradiction since $S^{(k)}$ is convex. We conclude that $z \in S$ and S is convex. ■

The above lemmas lead to the following one.

Lemma 10. Let $G^{(k)} = (V^{(k)}, E^{(k)})$ be the k -rescaled image of G . If $G(a|b)$ is convex for each $(a, b) \in E$, then $G^{(k)}(a'|b')$ is also convex for each $(a', b') \in E^{(k)}$.

Proof. Consider the edge $(a', b') \in E^{(k)}$ such that $a', b' \in \{a, b\} \cup V^{(k)}(a, b)$ for $a, b \in V$, i.e., $(a, b) \in E$ is the original edge. We note that the subgraph $G^{(k)}(a'|b')$ is just $\mathbb{G}_k(G(a|b)) \oplus (a, w_1) \oplus (w_1, w_2) \oplus \dots \oplus (w_j, a')$ for $j = d_{G^{(k)}}(a, a') - 1$ [if $d_{G^{(k)}}(a, a') = 1$, then $\mathbb{G}_k(G(a|b)) \oplus (a, a')$, and if $a = a'$, then just $\mathbb{G}_k(G(a|b))$]. Since $G(a|b)$ is convex, by Lemma 9 $\mathbb{G}_k(G(a|b))$ is also convex and by Lemma 8 $G^{(k)}(a'|b')$ is convex. ■

Finally, with the above lemmas, we can prove $(ii) \Rightarrow (i)$ in Theorem 6.

Lemma 11. If $G(a|b)$ is convex for each edge (a, b) of G , then G is an ℓ_1 -graph.

Proof. Consider the k -rescaled graph $G^{(k)} = (V^{(k)}, E^{(k)})$ corresponding to G for k even. By Lemma 7, $G^{(k)}$

is bipartite. By Lemma 10, $G^{(k)}(a'|b')$ is convex for each $(a', b') \in E^{(k)}$. Therefore, by Djokovic's characterization $G^{(k)}$ can be isometrically embedded into a hypercube [51]. Since $V \subset V^{(k)}$, we conclude that G can be k embedded into the same hypercube, i.e., it is an ℓ_1 -graph. ■

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