The complexity of recognizing minimally tough graphs

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Abstract: Let t be a real number. A graph is called t-tough if the removal of any vertex set S that disconnects the graph leaves at most $|S|/t$ components. The toughness of a graph is the largest t for which the graph is t-tough. A graph is minimally t-tough if the toughness of the graph is t and the deletion of any edge from the graph decreases the toughness. The complexity class DP is the set of all languages that can be expressed as the intersection of a language in NP and a language in coNP. We prove that recognizing minimally t-tough graphs is DP-complete for any positive rational number t. We introduce a new notion called weighted toughness, which has a key role in our proof.

Keywords: minimally toughness, complexity, DP-completeness

1 Introduction

All graphs considered in this paper are finite, simple and undirected. Let $\omega(G)$ denote the number of components and $\alpha(G)$ denote the independence number of a graph G. For a graph G and a vertex set $V' \subseteq V(G)$ let $G[V']$ denote the subgraph of G induced by V'.

Definition 1 *Let t be a real number. A graph G is called* t-tough *if*

$$\omega(G-S) \le \frac{|S|}{t}$$

for any vertex set $S \subseteq V(G)$ that disconnects the graph (i.e. for any $S \subseteq V(G)$ with $\omega(G-S) > 1$). The toughness *of G, denoted by $\tau(G)$, is the largest t for which G is t-tough, taking $\tau(K_n) = \infty$ for all $n \ge 1$.*

We say that a cutset $S \subseteq V(G)$ is a tough set *if $\omega(G-S) = |S|/\tau(G)$.*

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Definition 2 *A graph G is* minimally t-tough *if $\tau(G) = t$ and $\tau(G - e) < t$ for all $e \in E(G)$.*

Let t be an arbitrary positive rational number and consider the following problem.

t-Tough
Instance: a graph G.
Question: is it true that $\tau(G) \geq t$?

Bauer et al. [1] proved that for any positive rational number t the problem t-Tough is coNP-complete. However, in some graph classes the toughness can be computed in polynomial time, for instance, in the class of split graphs [7].

The focus of our investigation is on the critical version of the problem t-Tough. Let t be an arbitrary positive rational number and consider the following problem.

Min-t-Tough
Instance: a graph G.
Question: is it true that G is minimally t-tough?

Extremal problems usually seem not to belong to NP $\cup$ coNP; therefore, the complexity class called DP was introduced by Papadimitriou and Yannakakis [5].

Definition 3 *A language L is in the class* DP *if there exist two languages $L_1 \in NP$ and $L_2 \in coNP$ such that $L = L_1 \cap L_2$.*

A language is called DP-hard *if all problems in DP can be reduced to it in polynomial time. A language is* DP-complete *if it is in DP and it is DP-hard.*

In our proofs we use the following problem for reduction.

α-Critical
Instance: a graph G and a positive integer k.
Question: is it true that $\alpha(G) < k$, but $\alpha(G - e) \geq k$ for any edge $e \in E(G)$?

Theorem 4 ([6]) *The problem α-Critical is DP-complete.*

Definition 5 *A graph G is called* α-critical *if $\alpha(G - e) > \alpha(G)$ for all $e \in E(G)$.*

Our main result is the following.

Theorem 6 *The problem* Min-t-Tough *is DP-complete for any positive rational number t.*

The paper is organized as follows. In Section 2 we prove some useful lemmas, including that the problem Min-t-Tough belongs to DP for any positive rational number t. In Section 3 we prove Theorem 6 for any positive rational number $1/2 < t < 1$, then we prove the theorem for any positive rational number $t \geq 1$ in Section 4. Finally, in Section 5 we prove the theorem for any positive rational number $t \leq 1/2$.

2 Preliminaries

In this section we cite some results.

Proposition 7 ([2]) *For every positive rational number t the problem* Min-t-Tough *belongs to DP.*

Lemma 8 (Problem 14 of §8 in [3]) *If we replace a vertex of an α-critical graph with a clique, and connect every neighbor of the original vertex with every vertex in the clique, then the resulting graph is still α-critical.*

Lemma 9 ([4]) *Let G be an α-critical graph and w an arbitrary vertex of degree at least two. Split w into two vertices y and z, each of degree at least 1, add a new vertex x and connect it to both y and z. Then the resulting graph G' is α-critical, and $\alpha(G') = \alpha(G) + 1$.*

For one of our proofs we also need the following observation, which is a straightforward consequence of Theorem 4 and Lemmas 8 and 9.

Proposition 10 *For any positive integers l and m the following variant of the problem α-*CRITICAL *is DP-complete.*

Instance: an l-connected graph G and a positive integer k that is divisible by m.
Question: is it true that $\alpha(G) < k$, but $\alpha(G - e) \geq k$ for any edge $e \in E(G)$?

3 Minimally t-tough graphs, where $1/2 < t < 1$

Before proving Theorem 6 for any positive rational number $1/2 < t < 1$, we need some preparation: first we construct some auxilary graphs.

Let t be a rational number such that $1/2 < t < 1$. Let a, b be relatively prime positive integers such that $t = a/b$. Let k be a positive integer, and let $W = \{w_1, \dots, w_{ak}\}$ and $W' = \{w'_1, \dots, w'_{(b-1)k}\}$. Place a clique on the vertices of W and a complete bipartite graph on $(W; W')$. Obviously, the toughness of this complete split graph is $a/(b-1) > t$. Deleting an edge may decrease the toughness, and now we delete edges incident to W' until the toughness remains at least t but the deletion of any other such edge would result in a graph with toughness less than t. Let $H^*_{t,k}$ denote the obtained split graph. Now delete all the edges induced by W, and let $H^{**}_{t,k}$ denote the obtained bipartite graph.

Claim 11 *Let t be a rational number such that $1/2 < t < 1$. Let a, b be relatively prime positive integers such that $t = a/b$ and let H_t be the following graph. Let*

$$V = \{v_1, v_2, \dots, v_a\}, \quad U = \{u_1, u_2, \dots, u_b\}.$$

For any $i \in [a]$ and $j \in [b-1]$ connect v_i to u_j, and connect u_b to v_1 and v_a. (See Figure 1.) Then $\tau(H_t) = t$.

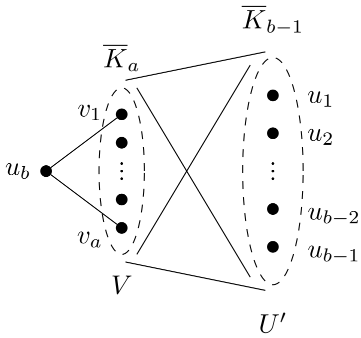

Figure 1: The graph H_t, when $1/2 < t < 1$.

By repeatedly deleting some edges of H_t, eventually we obtain a minimally t-tough graph, let us denote it with H'_t (i.e. if there exists an edge whose deletion does not decrease the toughness, then we delete it). Obviously, we could not delete the edges incident to u_b, so the vertex u_b still has degree 2. Let e denote the edge connecting v_1 and u_b and let $H''_t = H'_t - e$.

Theorem 12 *For any rational number t with $1/2 < t < 1$ the problem* MIN-t-TOUGH *is DP-complete.*

Proof: Let t be a rational number such that $1/2 < t < 1$. By Proposition 7, the problem Min-t-Tough is in DP. To show that it is DP-hard, we reduce α-Critical to it.

Let a, b be relatively prime positive integers such that $t = a/b$, let G be an arbitrary 2-connected graph on the vertices $v_1, \dots, v_n$ and let $G_{t,k}$ be defined as follows. For all $i \in [n]$ let

$$V_i = \{v_{i,j} \mid i \in [n], j \in [ak]\}$$

and place a clique on the vertices of V_i. For all $i_1, i_2 \in [n]$ if $v_{i_1}v_{i_2} \in E(G)$, then place a complete bipartite graph on $(V_{i_1}; V_{i_2})$. (This subgraph is denoted by $\tilde{G}$ in Figure 2.) For all $i \in [n], j \in [ak]$ "glue" the graph H''_t to the vertex $v_{i,j}$ by identifying $v_{i,j}$ with the vertex v_1 of H''_t and let $H^{i,j}$ denote the (i,j)-th copy of H''_t and let $A^{i,j}$ denote its color class which contains $v_{i,j}$, and let $v'_{i,j}$ and $u_{i,j}$ denote the (i,j)-th copies of the vertices v_a and u_b, respectively. Let

$$V = \bigcup_{i=1}^{n} V_i$$

and

$$U = \{u_{i,j} \mid i \in [n], j \in [ak]\}.$$

Add the vertex sets

$$W = \{w_j \mid j \in [ak]\}$$

and

$$W' = \{w'_1, \dots, w'_{(b-1)k}\}$$

to the graph and place the bipartite graph $H^{**}_{t,k}$ on $(W; W')$. For all $i \in [n]$ and $j \in [ak]$ connect w_j to $u_{i,j}$. See Figure 2. Now k is part of the input of the problem α-Critical, therefore the graph $H^{**}_{t,k}$ must be constructed in polynomial time, which is possible since the tougness of split graphs can be computed in polynomial time [7]. On the other hand, t is not part of the input of the problem Min-t-Tough, therefore the graph H''_t can be constructed in advance. Hence, $G_{t,k}$ can be constructed from G in polynomial time.

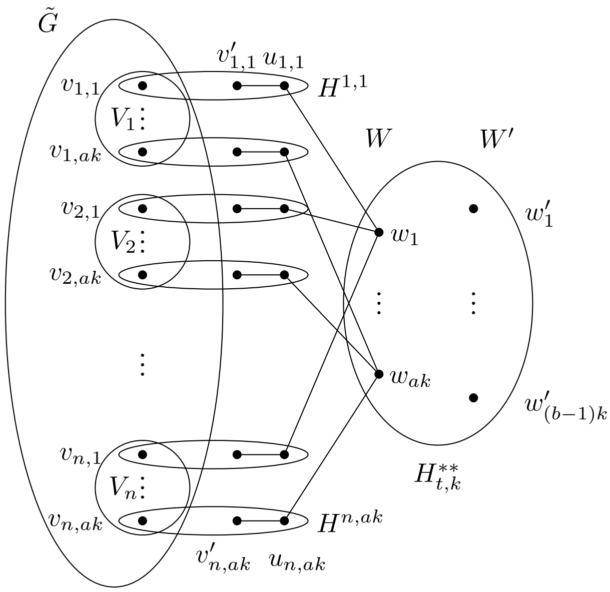

Figure 2: The graph $G_{t,k}$, when $1/2 < t < 1$.

To show that G is α-critical with $\alpha(G) = k$ if and only if $G_{t,k}$ is minimally t-tough, we need the following lemma.

Lemma 13 *Let G be a 2-connected graph with $\alpha(G) \leq k$. Then $G_{t,k}$ is t-tough.*

Let us assume that G is α-critical with $\alpha(G) = k$. By Lemma 13, $G_{t,k}$ is t-tough, i.e. $\tau(G_{t,k}) \geq t$. Let I be an independent vertex set of size $\alpha(G)$ in $G_{t,k}[V]$. Let

$$J = \{(i,j) \in [n] \times [ak] \mid v_{i,j} \in I\}$$

and

$$S = \left(\bigcup_{(i,j) \notin J} A^{i,j} \right) \cup W.$$

Then S is a cutset in $G_{t,k}$ with

$$|S| = a\big(|V| - \alpha(G)\big) + ak = a|V|$$

and

$$\omega(G_{t,k} - S) = \alpha(G) + b\big(|V| - \alpha(G)\big) + (b-1)k = b|V| = \frac{|S|}{t},$$

so $\tau(G_{t,k}) \leq t$.

Therefore, $\tau(G_{t,k}) = t$.

Let $e \in E(G_{t,k})$ be an arbitrary edge. If e has an endpoint in U, then this endpoint has degree 2, so $\tau(G_{t,k} - e) \leq 1/2 < t$. If e has an endpoint in W', then by the properties of $H^*_{t,k}$, it can be shown that $\tau(G_{t,k} - e) < t$. If e is induced by H^{i_0,j_0} for some $i_0 \in [n], j_0 \in [ak]$, then by the properties of H'_t, it can be shown that $\tau(G_{t,k} - e) < t$. If e connects two vertices of V, then using the fact that $G_{t,k}[V]$ is α-critical by Lemma 8, it can be shown that $\tau(G_{t,k} - e) < t$.

Now let us assume that G is not α-critical with $\alpha(G) = k$, i.e. either $\alpha(G) \neq k$ or even though $\alpha(G) = k$, the graph G is not α-critical.

Case 1: $\alpha(G) > k$.

Let I be an independent vertex set of size $\alpha(G)$ in $G_{t,k}[V]$ and let

$$J = \{(i,j) \in [n] \times [ak] \mid v_{i,j} \in I\}$$

and

$$S = \left(\bigcup_{(i,j) \notin J} A^{i,j} \right) \cup W.$$

Then S is a cutset in $G_{t,k} - e$ with

$$|S| = a\big(|V| - \alpha(G)\big) + ak = a|V| - a\big(\alpha(G) - k\big)$$

and

$$\begin{aligned} \omega(G_{t,k} - S) = \alpha(G) + b\big(|V| - \alpha(G)\big) + (b-1)k &= b|V| - (b-1)\big(\alpha(G) - k\big) \\ &> b|V| - b\big(\alpha(G) - k\big) = |S|/t, \end{aligned}$$

so $\tau(G_{t,k}) < t$, which means that $G_{t,k}$ is not minimally t-tough.

Case 2: $\alpha(G) \leq k$.

Since G is not α-critical with $\alpha(G) = k$, there exists an edge $e \in E(G)$ such that $\alpha(G - e) \leq k$. By Lemma 13, the graph $(G-e)_{t,k}$ is t-tough, but we can obtain $(G-e)_{t,k}$ from $G_{t,k}$ by edge-deletion, which means that $G_{t,k}$ is not minimally t-tough. $\square$

4 Minimally t-tough graphs, where $t \geq 1$

This section resembles the previous one in structure. However, it requires some additional ideas that make the proofs more complicated.

Let $t \geq 1$ be a rational number. It is easy to see that either $\lceil 2t \rceil = 2\lceil t \rceil$ or $\lceil 2t \rceil = 2\lceil t \rceil - 1$. Let $T = \lceil t \rceil$, and $T' = \lceil 2t \rceil - \lceil t \rceil$ and $M = \lceil 2\lceil t \rceil / \lceil 2t \rceil \rceil$. Let a, b be the smallest positive integers such that $b \geq 3$ and $t = a/b$.

Let k be a positive integer that is divisible by a, and let

$$W = \{w_{j,l,m} \mid j \in [k], l \in [T'], m \in M\}$$

and

$$W' = \{w'_1, \ldots, w'_{(MT'/t-1)k}\}.$$

Place a clique on the vertices of W and a complete bipartite graph on $(W; W')$. Obviously, the toughness of this complete split graph is

$$\frac{kMT'}{(MT'/t - 1)k} = \frac{1}{\frac{1}{t} - \frac{1}{MT'}} > t.$$

Deleting an edge may decrease the toughness, and now we delete edges incident to W' until the toughness remains at least t but the deletion of any other such edge would result in a graph with toughness less than t. Let $H^*_{t,k}$ denote the obtained split graph. Now delete all the edges induced by W, and let $H^{**}_{t,k}$ denote the obtained bipartite graph.

Let H_t be the following graph. Let

$$V'_1 = \{v'_1, \ldots, v'_T\}, \qquad V'_2 = \{v'_{T+1}, \ldots, v'_{2T}\}, \qquad V'_3 = \{v'_{2T+1}, \ldots, v'_{aT}\},$$
$$V'' = \{v''_1, \ldots, v''_T\},$$
$$U'_1 = \{u'_1, \ldots, u'_T\}, \qquad U'_2 = \{u'_{T+1}, \ldots, u'_{2T}\}, \qquad U'_3 = \{u'_{2T+1}, \ldots, u'_{bT-1}\},$$
$$U'' = \{u''_1, \ldots, u''_{T'}\},$$

and

$$U''_1 = \{u''_1, \ldots, u''_T\}.$$

Place a clique on the vertices of V'_1, V'_2, V'_3, and U''. For all $l \in [T]$ connect v''_l to v'_l and to u'_l, and connect v'_{T+l} to u'_{T+l}. Connect all the vertices of V'_3 to all the vertices of $V'_1 \cup V'' \cup U'_1 \cup U'_2$, and connect all the vertices of V'_2 to all the vertices of U''. Finally, add a new vertex x to the graph and connect it to all the vertices of $V'_1 \cup U''$. See Figure 3.

Let t be a real number. Given a graph G and a positive weight function w on its vertices, we say that the graph G is weighted t-tough with respect to the weight function w if

$$\omega(G - S) \leq \frac{w(S)}{t}$$

holds for any vertex set $S \subseteq V(G)$ whose removal disconnects the graph; where

$$w(S) = \sum_{v \in S} w(v).$$

We define the weighted toughness of a noncomplete graph (with respect to the weight function w) to be the largest t for which the graph is weighted t-tough, and we define the weighted toughness of complete graphs (with respect to w) to be infinity.

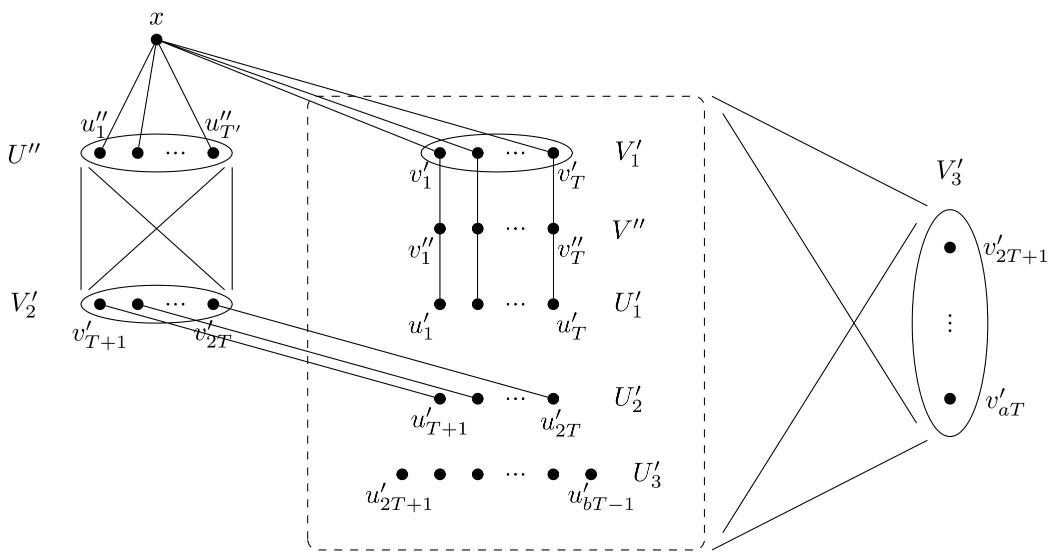

Figure 3: The graph H_t, when $t \geq 1$.

Claim 14 *For any rational number $t \geq 1$ the graph H_t has weighted toughness t with respect to the weight function w that assigns weight 1 to all the vertices of H_t except for the vertex x, to which it assigns weight t.*

Deleting an edge may decrease the weighted toughness, and now we delete edges not induced by U'' until the weighted toughness with respect to the weight function w remains at least t but the deletion of any other edge not induced by U'' would result in a graph with weighted toughness less than t. Let H'_t denote the obtained graph.

According to the following claim we could not delete the edges induced by V'_1 or incident to any of the vertices of $\{x\} \cup V'_2 \cup U''$.

Claim 15 *Let $t \geq 1$ be a rational number. For any edge $e \in E(H_t)$ induced by V'_1 or incident to any of the vertices of $\{x\} \cup V'_2 \cup U''$ there exists a cutset $S = S(e) \subseteq V(H_t)$ in $H_t - e$ for which $\omega\big((H_t - e) - S\big) > w(S)/t$.*

Claim 16 *Let $t \geq 1$ be a rational number and $H''_t = H'_t - \{x\}$. Then the following hold.*

(i) The graph H''_t is connected.

(ii) For any cutset S of H''_t,

$$\omega(H''_t - S) \leq \frac{|S|}{t} + 1.$$

(iii) If $V'_1 \subseteq S$ holds for a cutset S of H''_t, then

$$\omega(H''_t - S) \leq \frac{|S|}{t}.$$

(iv) For any edge $e \in E(H''_t)$ not induced by U'' there exists a vertex set $S = S(e)$ whose removal from $H''_t - e$ disconnects the graph and

$$\omega\big((H''_t - e) - S\big) > \frac{|S|}{t}.$$

Theorem 17 *For any rational number $t \geq 1$ the problem* MIN-t-TOUGH *is DP-complete.*

PROOF: Let $t \geq 1$ be a rational number. By Proposition 7, the problem MIN-t-TOUGH is in DP. To show that it is DP-hard, we reduce the variant of α-CRITICAL mentioned in Proposition 10 to it.

Let $T = \lceil t \rceil$, and $T' = \lceil 2t \rceil - \lceil t \rceil$, and $M = \lceil 2\lceil t \rceil / \lceil 2t \rceil \rceil$. Let a, b be the smallest positive integers such that $b \geq 3$ and $t = a/b$, let G be an arbitrary 3-connected graph on the vertices $v_1, \ldots, v_n$ with $n \geq t + 1$, let k be a positive integer that is divisible by a and let $G_{t,k}$ be defined as follows. For all $i \in [n], j \in [k], m \in [M]$ let

$$V_{i,j,m} = \{v_{i,j,l,m} \mid l \in [T]\}.$$

For all $i \in [n]$ let

$$V_i = \bigcup_{\substack{j \in [k],\\ m \in [M]}} V_{i,j,m}$$

and place a clique on the vertices of V_i. For all $i_1, i_2 \in [n]$ if $v_{i_1}v_{i_2} \in E(G)$, then place a complete bipartite graph on $(V_{i_1}; V_{i_2})$. (This subgraph is denoted by $\tilde{G}$ in Figure 4.) For all $i \in [n], j \in [k], m \in [M]$ "glue" the graph H''_t to the vertex set $V_{i,j,m}$ by identifying $v_{i,j,l,m}$ with the vertex v'_l of H''_t for all $l \in [T]$. For all $i \in [n], j \in [k], l \in [T'], m \in [M]$ let $u''_{i,j,l,m}$ denote the (i, j, m)-th copy of u''_l. For all $j \in [k], m \in [M]$ add the vertex set

$$W_{j,m} = \{w_{j,l,m} \mid l \in [T']\}$$

to the graph and for all $i \in [n], j \in [k], l \in [T'], m \in [M]$ connect $w_{j,l,m}$ to $u''_{i,j,l,m}$. Let

$$W = \bigcup_{\substack{j \in [k],\\ m \in [M]}} W_{j,m}.$$

Add the vertex set

$$W' = \{w'_1, \ldots, w'_{(MT'/t-1)k}\}$$

to the graph and place the bipartite graph $H^{**}_{t,k}$ on $(W; W')$. See Figure 4. Similarly as in the previous case, $G_{t,k}$ can be constructed from G in polynomial time.

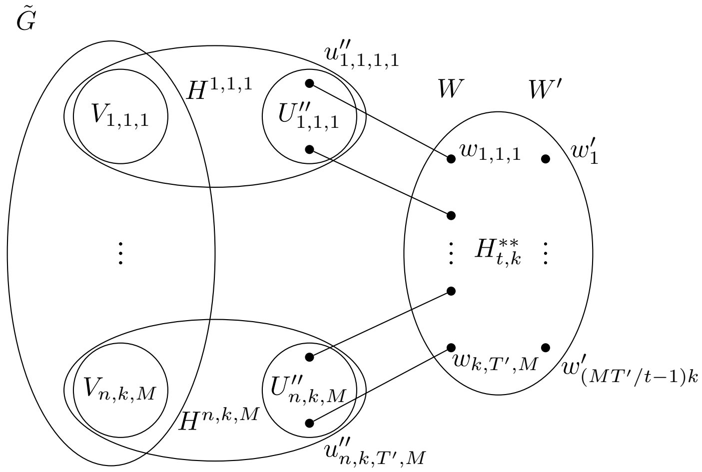

Figure 4: The graph $G_{t,k}$, when $t \geq 1$.

Using a similar but more complicated argument as in the proof of Theorem 12, it can be shown that G is α-critical with $\alpha(G) = k$ if and only if $G_{t,k}$ is minimally t-tough. $\square$

5 Minimally t-tough graphs with $t \leq 1/2$

The case when $t \leq 1/2$ is special in some sense: graphs with toughness at most $1/2$ can have cut-vertices. Unlike in the previous cases, we reduce Min-1-Tough to this problem.

Proposition 18 *Let $t \leq 1/2$ be a positive rational number. Let a, b be relatively prime positive integers such that $t = a/b$ and let H_t be the following graph. Let*

$$V = \{v_1, v_2, \dots, v_a\}, \qquad U = \{u_1, u_2, \dots, u_{b-a}\}, \qquad W = \{w_1, w_2, \dots, w_a\}.$$

Place a clique on the vertices of V, connect every vertex of V to every vertex of U, and connect v_i to w_i for all $i \in [n]$. Then $\tau(H_t) = t$.

By repeatedly deleting some edges of H_t, eventually we obtain a minimally t-tough graph; let us denote it with H_t' (i.e. if there exists an edge whose deletion does not decrease the toughness, then we delete it). Obviously, we could not delete the edges between V and W, so the vertices of W still have degree 1 in H_t'.

Definition 19 *Let H be a graph with a vertex u of degree 1, and let v be the neighbor of u. Let G be an arbitrary graph, and "glue" $H - \{u\}$ separately to all vertices of G by identifying each vertex of G with v. Let $G \oplus_v H$ denote the obtained graph.*

Theorem 20 *For any positive rational number $t \leq 1/2$ the problem* Min-t-Tough *is DP-complete.*

Proof: Let $t \leq 1/2$ be a positive rational number. By Proposition 7, the problem Min-t-Tough is in DP. To show that it is DP-hard, we reduce a variant of Min-1-Tough to it.

Let G be an arbitrary graph and $n = |V(G)|$. Consider the graph H_t' and let $u \in U$ be an arbitrary vertex of H_t' having degree 1, and let v be its neighbor.

It can be shown that $G_t = G \oplus_v H_t'$ (see Figure 5) is minimally t-tough if and only if G is minimally 1-tough or $G \simeq K_2$ or $G \simeq K_3$. $\square$

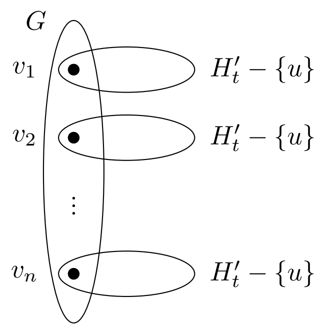

Figure 5: The graph $G_t = G \oplus_v H_t'$, when $t \leq 1/2$.

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