

# ALGEBRAIC CHARACTERIZATION OF APPROXIMATE CONTROLLABILITY OF BEHAVIOURS OF SPATIALLY INVARIANT SYSTEMS

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ABSTRACT. An algebraic characterization of the property of approximate controllability is given, for behaviours of spatially invariant dynamical systems, consisting of distributional solutions  $w$ , that are periodic in the spatial variables, to a system of partial differential equations

$$M \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial t} \right) w = 0,$$

corresponding to a polynomial matrix  $M \in (\mathbb{C}[\xi_1, \dots, \xi_d, \tau])^{m \times n}$ . This settles an issue left open in [17].

## 1. INTRODUCTION

Consider a homogeneous, linear, constant coefficient partial differential equation, in  $\mathbb{R}^{d+1}$  described by a polynomial  $p \in \mathbb{C}[\xi_1, \dots, \xi_d, \tau]$ :

$$p \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial t} \right) w = 0. \tag{1.1}$$

That is, the differential operator

$$p \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial t} \right)$$

is obtained from the polynomial  $p \in \mathbb{C}[\xi_1, \dots, \xi_d, \tau]$  by making the replacements

$$\xi_k \rightsquigarrow \frac{\partial}{\partial x_k} \text{ for } k = 1, \dots, d, \text{ and } \tau \rightsquigarrow \frac{\partial}{\partial t}.$$

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More generally, given a polynomial *matrix*  $M \in (\mathbb{C}[\xi_1, \dots, \xi_d, \tau])^{m \times n}$ , consider the corresponding *system* of partial differential equations

$$M \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial t} \right) w := \begin{bmatrix} \sum_{j=1}^n p_{1j} \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial t} \right) w_j \\ \vdots \\ \sum_{j=1}^n p_{mj} \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial t} \right) w_j \end{bmatrix} = 0, \quad (1.2)$$

where solutions  $w$  now have the  $n$  components  $w_1, \dots, w_n$ , and  $M = [p_{ij}]$  with  $p_{ij}$  denoting the polynomial entries of  $M$  for  $1 \leq i \leq m$  and  $1 \leq j \leq n$ .

In the behavioural approach to control theory pioneered by Willems [9], the “behaviour”  $\mathfrak{B}_{\mathcal{W}}(M)$  associated with  $M$  in  $\mathcal{W}^n$  (where  $\mathcal{W}$  is an appropriate solution space, for example smooth functions  $C^\infty(\mathbb{R}^{d+1})$  or distribution spaces like  $\mathcal{D}'(\mathbb{R}^{d+1})$  or  $\mathcal{S}'(\mathbb{R}^{d+1})$  and so on), is defined to be the set of all solutions  $w \in \mathcal{W}^n$  that satisfy the above partial differential equation system (1.2). Let us recall the notion of a behaviour associated with a system of partial differential equations associated with a polynomial matrix  $M$ .

**Definition 1.1** (Solution space invariant under differentiation; Behaviour). Let  $\mathcal{W}$  be a subspace of  $(\mathcal{D}'(\mathbb{R}^{d+1}))^n$  which is *invariant under differentiation*, that is, for all  $w \in \mathcal{W}$ ,

$$\begin{aligned} \frac{\partial}{\partial x_k} w &\in \mathcal{W}, \text{ for all } k = 1, \dots, d, \text{ and} \\ \frac{\partial}{\partial t} w &\in \mathcal{W}. \end{aligned}$$

The *behaviour*  $\mathfrak{B}_{\mathcal{W}}(M)$  associated with  $M \in (\mathbb{C}[\xi_1, \dots, \xi_d, \tau])^{m \times n}$  in  $\mathcal{W}^n$  is

$$\mathfrak{B}_{\mathcal{W}}(M) := \left\{ w \in \mathcal{W}^n : M \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial t} \right) w = 0 \right\}.$$

The aim in the behavioural approach to control theory is then to obtain algebraic characterizations (in terms of algebraic properties of the polynomial matrix  $M$ ) of certain analytical properties of  $\mathfrak{B}_{\mathcal{W}}(M)$  (for example, the control theoretic properties of autonomy, controllability, stability, and so on). We refer the reader to [9] for background on the behavioural approach in the case of systems of ordinary differential equations, and to [1], [10], [16] for distinct takes on this in the context of systems described by partial differential equations (PDEs).

In the PDE context, while the initial works (see for example [10], [11]) treated the spatial variables on the same footing as the temporal one, this symmetry was broken in the works of the author (see for instance [15], [16], [2]). Also in the context of periodicity, while there has been work in the area of behaviours that are periodic *both* temporally and spatially (see for example [8], [20]), the case of PDE behaviours that are purely periodic in

the spatial variables does not seem to have received much attention. Tempered distributional solutions were considered right from the outset in the behavioural approach to PDEs (see for instance [10], [12], [18]), while tempered behaviour only in the spatial directions was considered in [14].

The goal of this article is to give algebraic characterizations of the properties of approximate controllability of behaviours of spatially invariant dynamical systems, consisting of distributional solutions  $w$ , that are periodic in the spatial variables, to a system of partial differential equations

$$M \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial t} \right) w = 0,$$

corresponding to a polynomial matrix

$$M \in (\mathbb{C}[\xi_1, \dots, \xi_d, \tau])^{m \times n}.$$

This settles a question left open in [17].

We remark that there has been recent interest in “spatially invariant systems”, see for example [4], [5], where one considers solutions to partial differential equations that are periodic along the spatial direction.

We give the relevant definitions below, and also state our main results in Theorem 1.3 (characterizing approximate controllability).

**1.1. Controllability and approximate controllability.** Let us first recall the property of “controllability”, which means the following.

**Definition 1.2** (Controllability; Approximate controllability). Let  $\mathcal{W}$  be a subspace of  $(\mathcal{D}'(\mathbb{R}^{d+1}))^n$  which is invariant under differentiation, and suppose that  $M \in (\mathbb{C}[\xi_1, \dots, \xi_d, \tau])^{m \times n}$ .

- (1) The behaviour  $\mathfrak{B}_{\mathcal{W}}(M)$  in  $\mathcal{W}$  is called *controllable in time*  $T > 0$  if for every  $w_1, w_2 \in \mathfrak{B}_{\mathcal{W}}(M)$ , there is a  $w \in \mathfrak{B}_{\mathcal{W}}(M)$  such that

$$\begin{aligned} w|_{(-\infty, 0)} &= w_1|_{(-\infty, 0)} \text{ and} \\ w|_{(T, +\infty)} &= w_2|_{(T, +\infty)} \end{aligned}$$

- (2) The behaviour  $\mathfrak{B}_{\mathcal{W}}(M)$  in  $\mathcal{W}$  is called *approximately controllable in time*  $T > 0$  if for every  $\epsilon > 0$  and for all  $w_1, w_2 \in \mathfrak{B}_{\mathcal{W}}(M)$ , there is a  $w \in \mathfrak{B}_{\mathcal{W}}(M)$  such that

$$w|_{(-\infty, 0)} = w_1|_{(-\infty, 0)},$$

and  $(w - w_2)|_{(T, +\infty)}$  is a regular distribution on  $(T, +\infty) \times \mathbb{R}^d$  with

$$\sup_{(t, \mathbf{x}) \in (T, +\infty) \times \mathbb{R}^d} \|(w - w_2)|_{(T, +\infty)}(t, \mathbf{x})\|_2 < \epsilon.$$

See Figure 1.

Our main result is the following.

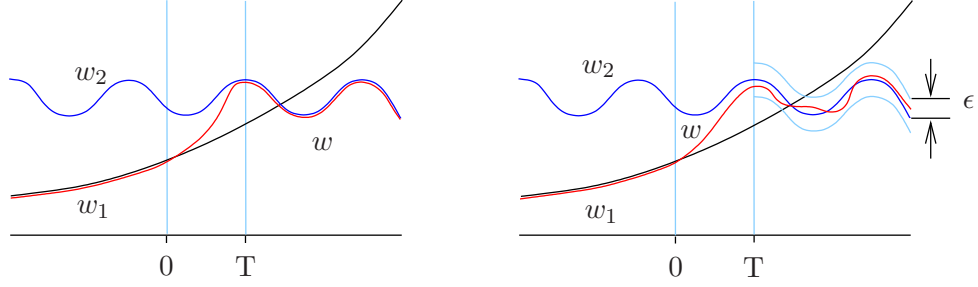


FIGURE 1. Controllability versus approximate controllability.

**Theorem 1.3.** *Suppose that  $\mathbb{A} := \{\mathbf{a}_1, \dots, \mathbf{a}_d\}$  is a linearly independent set of vectors in  $\mathbb{R}^d$ . Let  $M \in (\mathbb{C}[\xi_1, \dots, \xi_d, \tau])^{m \times n}$  and let*

$$\mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(M) := \left\{ w \in (\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1}))^n : M \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial t} \right) w = 0 \right\}.$$

*Then the following statements are equivalent:*

- (1)  $\mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(M)$  is approximately controllable in time  $T > 0$ .
- (2)  $\mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(M)$  is controllable in time  $T > 0$ .
- (3) For each  $\mathbf{v} \in A^{-1}\mathbb{Z}^d$ , there exists an  $r_{\mathbf{v}} \in \{0, 1, 2, 3, \dots\}$  satisfying  $r_{\mathbf{v}} \leq \min\{n, m\}$  and such that for all  $t \in \mathbb{C}$ ,  $\text{rank}(M(2\pi i\mathbf{v}, t)) = r_{\mathbf{v}}$ .
- (4) For each  $\mathbf{v} \in A^{-1}\mathbb{Z}^d$ , the  $\mathbb{C}[\tau]$ -module  $\mathbb{C}[\tau]^{1 \times n} / \langle M(2\pi i\mathbf{v}, \tau) \rangle$  is torsion free.

Here  $\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})$  is, roughly speaking, the set of all distributions on  $\mathbb{R}^{d+1}$  that are periodic in the spatial direction with a discrete set  $\mathbb{A}$  of periods. The precise definition of  $\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})$  is given below in Subsection 1.2.

The algebraic terminology in (4) of Theorem 1.3 is explained below. Consider the polynomial matrix

$$M = \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & & \vdots \\ p_{m1} & \cdots & p_{mn} \end{bmatrix} \in \mathbb{C}[\tau]^{m \times n}.$$

Then each row of  $M$  is an element of the free  $\mathbb{C}[\tau]$ -module  $\mathbb{C}[\tau]^{1 \times n}$ .

**Notation 1.4** ( $\langle M \rangle$ ). Given  $M \in \mathbb{C}[\tau]^{m \times n}$ , let  $\langle M \rangle$  denote the  $\mathbb{C}[\tau]$ -submodule of  $\mathbb{C}[\tau]^{1 \times n}$  generated by the rows of the polynomial matrix  $M$ .

**Definition 1.5** (Torsion element; Torsion free module). Let  $M \in \mathbb{C}[\tau]^{m \times n}$ .

- (1) An element  $[\mathbf{m}]$  in the quotient  $\mathbb{C}[\tau]$ -module  $\mathbb{C}[\tau]^{1 \times n} / \langle M(\tau) \rangle$  (corresponding to an element  $\mathbf{m} \in \langle R \rangle$ ) is called a *torsion element* if there exists a polynomial  $p \in \mathbb{C}[\tau]$  such that  $p \cdot [\mathbf{m}] = [\mathbf{0}]$ , that is,  $p \cdot \mathbf{m} \in \langle M \rangle$ .
- (2) The quotient  $\mathbb{C}[\tau]$ -module  $\mathbb{C}[\tau]^{1 \times n} / \langle M(\tau) \rangle$  is said to be *torsion free* if it has no nontrivial torsion element.

The equivalence of (2) and (3) follows from the proof of [17, Theorem 1.4].

## 1.2. The space $\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})$ .

**Definition 1.6** (Translation operator  $\mathbf{S}_{\mathbf{a}}$ ; Periodic distribution).

Let  $\mathbf{a} \in \mathbb{R}^d$ .

- (1) The *translation operation*  $\mathbf{S}_{\mathbf{a}}$  on distributions in  $\mathcal{D}'(\mathbb{R}^d)$  is defined by  $\langle \mathbf{S}_{\mathbf{a}}(T), \varphi \rangle = \langle T, \varphi(\cdot + \mathbf{a}) \rangle$  for all  $\varphi \in \mathcal{D}(\mathbb{R}^d)$ .
- (2) A distribution  $T \in \mathcal{D}'(\mathbb{R}^d)$  is said to be *periodic with a period*  $\mathbf{a} \in \mathbb{R}^d$  if  $T = \mathbf{S}_{\mathbf{a}}(T)$ .

**Notation 1.7** ( $\mathbb{A}, A, \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d)$ ).

Let  $\mathbb{A} := \{\mathbf{a}_1, \dots, \mathbf{a}_d\}$  be a linearly independent set vectors in  $\mathbb{R}^d$ . It will be convenient for the sequel to also introduce the following matrix:

$$A := \begin{bmatrix} \mathbf{a}_1^\top \\ \vdots \\ \mathbf{a}_d^\top \end{bmatrix}. \quad (1.3)$$

$\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d)$  is the set of all distributions  $T \in \mathcal{D}'(\mathbb{R}^d)$  that satisfy

$$\mathbf{S}_{\mathbf{a}_k}(T) = T \text{ for all } k = 1, \dots, d.$$

From [6, §34],  $T \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d)$  is a tempered distribution, and from the above it follows by taking Fourier transforms that  $(1 - e^{2\pi i \mathbf{a}_k \cdot \mathbf{y}}) \widehat{T} = 0$  for  $k = 1, \dots, d$ . It can be seen that

$$\widehat{T} = \sum_{\mathbf{v} \in A^{-1}\mathbb{Z}^d} \alpha_{\mathbf{v}}(T) \delta_{\mathbf{v}},$$

for some scalars  $\alpha_{\mathbf{v}} \in \mathbb{C}$ , and where  $A$  is the matrix given in (1.3). Also, in the above,  $\delta_{\mathbf{v}}$  denotes the usual Dirac measure with support in  $\mathbf{v}$ :

$$\langle \delta_{\mathbf{v}}, \psi \rangle = \psi(\mathbf{v}) \text{ for } \psi \in \mathcal{D}'(\mathbb{R}^d).$$

By the Schwartz Kernel Theorem (see for instance [7, p. 128, Theorem 5.2.1]),  $\mathcal{D}'(\mathbb{R}^{d+1})$  is isomorphic as a topological space to  $\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{D}'(\mathbb{R}^d))$ , the space of all continuous linear maps from  $\mathcal{D}(\mathbb{R})$  to  $\mathcal{D}'(\mathbb{R}^d)$ , thought of as vector-valued distributions. For preliminaries on vector-valued distributions, we refer the reader to [3]. We indicate this isomorphism by putting an arrow on top of elements of  $\mathcal{D}'(\mathbb{R}^{d+1})$ .

**Notation 1.8** ( $\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})$ ).

For  $w \in \mathcal{D}'(\mathbb{R}^{d+1})$ , we set  $\vec{w} \in \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{D}'(\mathbb{R}^d))$  to be the vector-valued distribution defined by  $\langle \vec{w}(\varphi), \psi \rangle = \langle w, \psi \otimes \varphi \rangle$  for  $\varphi \in \mathcal{D}(\mathbb{R})$  and  $\psi \in \mathcal{D}(\mathbb{R}^d)$ .

If  $\mathbb{A} := \{\mathbf{a}_1, \dots, \mathbf{a}_d\}$  is a linearly independent set vectors in  $\mathbb{R}^d$ , then we define

$$\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1}) := \{w \in \mathcal{D}'(\mathbb{R}^{d+1}) : \text{for all } \varphi \in \mathcal{D}(\mathbb{R}), \vec{w}(\varphi) \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d)\}.$$

For  $w \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})$ ,

$$\frac{\partial}{\partial x_k} w \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1}) \text{ for } k = 1, \dots, d, \text{ and } \frac{\partial}{\partial t} w \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1}).$$

Also, for  $w \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})$ , we define  $\widehat{w} \in \mathcal{D}'(\mathbb{R}^{d+1})$  by

$$\langle \widehat{w}, \psi \otimes \varphi \rangle = \langle \vec{w}(\varphi), \widehat{\psi} \rangle, \quad (1.4)$$

for  $\varphi \in \mathcal{D}(\mathbb{R})$  and  $\psi \in \mathcal{D}(\mathbb{R}^d)$ . In the right hand side of (1.4),  $\widehat{\cdot}$  is the usual Fourier transform  $\psi \mapsto \widehat{\psi} : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$  on the Schwartz space  $\mathcal{S}(\mathbb{R}^d)$  of test functions with rapidly decreasing derivatives. That (1.4) specifies a well-defined distribution in  $\mathcal{D}'(\mathbb{R}^{d+1})$ , can be seen using the fact that for every  $\Phi \in \mathcal{D}(\mathbb{R}^{d+1})$ , there exists a sequence of functions  $(\Psi_n)_n$  that are finite sums of direct products of test functions, that is,  $\Psi_n = \sum_k \psi_k \otimes \varphi_k$ , where  $\psi_k \in \mathcal{D}(\mathbb{R}^d)$  and  $\varphi_k \in \mathcal{D}(\mathbb{R})$ , such that  $\Psi_n$  converges to  $\Phi$  in  $\mathcal{D}(\mathbb{R}^{d+1})$ . We also have

$$\widehat{\frac{\partial}{\partial x_k} w} = 2\pi i y_k \widehat{w} \text{ for } k = 1, \dots, d, \text{ and } \widehat{\frac{\partial}{\partial t} w} = \frac{\partial}{\partial t} \widehat{w}.$$

Here  $\mathbf{y} = (y_1, \dots, y_d)$  is the Fourier transform variable.

## 2. PROOF OF THEOREM 1.3

Before we prove our main result, we illustrate the key idea behind the proof of our algebraic condition. For a trajectory in the behaviour, by taking Fourier transform with respect to the spatial variables, the partial derivatives with respect to the spatial variables are converted into the polynomial coefficients  $c_{ij}(2\pi i \mathbf{y})$ , where  $\mathbf{y}$  is the vector of Fourier transform variables  $y_1, \dots, y_d$ . But the support of  $\widehat{w}$  is carried on a family of lines, indexed by  $\mathbf{n} \in \mathbb{Z}^d$ , in  $\mathbb{R}^{d+1}$ , parallel to the time axis. So we obtain a family of ordinary differential equations, parameterized by  $\mathbf{n} \in \mathbb{Z}^d$ , and by ‘‘freezing’’ an  $\mathbf{n} \in \mathbb{Z}^d$ , we get an ordinary differential equation. So essentially the proof is completed by looking at the ordinary differential equation characterizations of controllability and approximate controllability, and it turns out that the two notions actually coincide there.

*Proof of Theorem 1.3.* We will show that (1) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

**(1)  $\Rightarrow$  (4):** Suppose that  $\mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(M)$  is approximately controllable. Let  $\mathbf{v} \in A^{-1}\mathbb{Z}^d$ . Suppose that  $\Theta \in \mathfrak{B}_{\mathcal{D}'(\mathbb{R})}(M(2\pi i \mathbf{v}, \tau))$ . Set

$$\begin{aligned} w_1 &:= 0, \\ w_2 &:= e^{2\pi i \mathbf{v} \cdot \mathbf{x}} \otimes \Theta. \end{aligned}$$

Then  $w_1, w_2 \in (\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1}))^n$ , since for all  $k \in \{1, \dots, d\}$ , we have

$$\mathbf{S}_{\mathbf{a}_k} w_2 = e^{2\pi i \mathbf{v} \cdot (\mathbf{x} + \mathbf{a}_k)} \otimes \Theta = e^{2\pi i \mathbf{v} \cdot \mathbf{a}_k} e^{2\pi i \mathbf{v} \cdot \mathbf{x}} \otimes \Theta = 1 \cdot e^{2\pi i \mathbf{v} \cdot \mathbf{x}} \otimes \Theta = w_2.$$

Also,  $w_1, w_2 \in \mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(M)$ , because

$$M\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial t}\right) w_2 = e^{2\pi i \mathbf{v} \cdot \mathbf{x}} M\left(2\pi i \mathbf{v}, \frac{d}{dt}\right) \Theta = 0 \cdot e^{2\pi i \mathbf{v} \cdot \mathbf{x}} = 0,$$

thanks to the fact that  $\Theta \in \mathfrak{B}_{\mathcal{D}'(\mathbb{R})}(M(2\pi i \mathbf{v}, \tau))$  giving

$$M\left(2\pi i \mathbf{v}, \frac{d}{dt}\right) \Theta = 0.$$

As  $\mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(M)$  is approximately controllable in time  $T$ , there exists a sequence  $(w_k)_{k \in \mathbb{N}}$  in  $\mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(M)$  such that  $(w_k - w_2)|_{(T, \infty)}$  converges to 0 in  $(L^\infty((T, \infty) \times \mathbb{R}^d))^n$ . But this implies that  $(w_k - w_2)|_{(T, \infty)}$  converges to 0 in  $(\mathcal{D}'_{\mathbb{A}}((T, \infty) \times \mathbb{R}^d))^n$ , and owing to the continuity of the Fourier transform  $\hat{\cdot} : \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1}) \rightarrow \mathcal{D}'(\mathbb{R}^{d+1})$  with respect to the spatial variables, it follows that  $(\widehat{w}_k - \widehat{w}_2)|_{(T, \infty)}$  converges to 0 in  $(\mathcal{D}'((T, \infty) \times \mathbb{R}^d))^n$ . We can write

$$\widehat{w}_k = \sum_{\mathbf{v} \in A^{-1}\mathbb{Z}^d} \delta_{\mathbf{v}} \otimes T_k^{(\mathbf{v})},$$

where each  $T_k^{(\mathbf{v})} \in (\mathcal{D}'(\mathbb{R}))^n$ . Since  $\widehat{w}_k|_{(T, \infty)}$  converges to  $\widehat{w}_2|_{(T, \infty)}$  in the space  $(\mathcal{D}'((T, \infty) \times \mathbb{R}^d))^n$ , it follows that  $T_k^{(\mathbf{v})}|_{(T, \infty)}$  converges in  $\mathcal{D}'((T, \infty))$  to  $\Theta|_{(T, \infty)}$  for all  $\mathbf{v} \in A^{-1}\mathbb{Z}^d$ . Also,  $T_k^{(\mathbf{v})}|_{(-\infty, 0)} = 0$ . As

$$M\left(2\pi i \mathbf{v}, \frac{d}{dt}\right) T_k^{(\mathbf{v})} = 0,$$

so that  $T_k^{(\mathbf{v})} \in \mathfrak{B}_{\mathcal{D}'(\mathbb{R})}(M(2\pi i \mathbf{v}, \tau))$ .

Now suppose that there exists a nontrivial element  $[\mathbf{m}]$  in the  $\mathbb{C}[\tau]$ -quotient module  $\mathbb{C}[\tau]^{1 \times n} / \langle M(2\pi i \mathbf{v}, \tau) \rangle$  and a nonzero polynomial  $p \in \mathbb{C}[\tau]$  such that  $p \cdot \mathbf{m} \in \langle M(2\pi i \mathbf{v}, \tau) \rangle$ . As  $\mathbf{m} \notin \langle M(2\pi i \mathbf{v}, \tau) \rangle$ , it follows from the cogenerator property of  $\mathcal{D}'(\mathbb{R})$  (see for example Definition 3.4 on page 774 and the paragraph following the proof of Lemma 3.5 on page 775 of [19]) that  $\mathbf{m}(d/dt)$  is not identically 0 on  $\mathfrak{B}_{\mathcal{D}'(\mathbb{R})}(M(2\pi i \mathbf{v}, \tau))$ . Let the element  $w_0 \in \mathfrak{B}_{\mathcal{D}'(\mathbb{R})}(M(2\pi i \mathbf{v}, \tau))$  be such that

$$\mathbf{m}\left(\frac{d}{dt}\right) \neq 0.$$

Without loss of generality, we may assume that

$$u_0 := \mathbf{m}\left(\frac{d}{dt}\right) \Big|_{(0, \infty)} \neq 0$$

(otherwise  $w_0$  can be shifted to achieve this). As all topological vector spaces are Hausdorff ([13, Theorem 1.12]), it follows that in the topological vector space  $\mathcal{D}'((0, \infty))$ , there exists a neighbourhood  $N$  of  $u_0$  that does not contain 0. Since the map

$$\mathbf{m}\left(\frac{d}{dt}\right) : (\mathcal{D}'((0, \infty)))^n \rightarrow \mathcal{D}'((0, \infty))$$

is continuous, there exists a neighbourhood  $N_1$  of  $w_0|_{(0,\infty)}$  in  $(\mathcal{D}'((0,\infty)))^n$  such that  $\tilde{w}_0 \in N_1$  implies that

$$\mathbf{m}\left(\frac{d}{dt}\right)\tilde{w}_0 \in N.$$

Choose  $k$  large enough so that  $\mathbf{S}_{-T}T_k^{(\mathbf{v})} \in N_1$ . Set

$$u := \mathbf{m}\left(\frac{d}{dt}\right)T_k^{(\mathbf{v})}.$$

Then we have  $u \neq 0$ .

On the other hand, since  $p \cdot \mathbf{m} \in \langle M(2\pi i\mathbf{v}, \tau) \rangle$  and since the element  $T_k^{(\mathbf{v})} \in \mathfrak{B}_{\mathcal{D}'(\mathbb{R})}(M(2\pi i\mathbf{v}, \tau))$ , it follows that

$$p\left(\frac{d}{dt}\right)u = 0.$$

But we know that since  $T_k^{(\mathbf{v})}|_{(-\infty,0)} = 0$ , also

$$u|_{(-\infty,0)} = \mathbf{m}\left(\frac{d}{dt}\right)T_k^{(\mathbf{v})}|_{(-\infty,0)} = 0.$$

By the autonomy of behaviours corresponding to nonzero polynomials (see for example [17, Theorem 1.2]) we conclude that  $u = 0$ , a contradiction to the last sentence in the previous paragraph. This completes the proof of (1) $\Rightarrow$ (4).

(4)  $\Rightarrow$  (3): Suppose that (3) does not hold, and let  $\mathbf{v} \in A^{-1}\mathbb{Z}^d$  be such that

$$\neg\left(\exists r_{\mathbf{v}} \in \mathbb{Z} \text{ with } 0 \leq r_{\mathbf{v}} \leq \min\{n, m\} \text{ and } \forall t \in \mathbb{C}, \text{rank}(M(2\pi i\mathbf{v}, t)) = r_{\mathbf{v}}\right). \quad (2.1)$$

From [9, Theorem B.1.4, page 404], it follows that there exist unimodular polynomial matrices  $U, V$  with entries from  $\mathbb{C}[\tau]$  such that

$$M(2\pi i\mathbf{v}, \tau) = U\Sigma V,$$

where

$$\Sigma := \left[ \begin{array}{ccc|c} d_1 & & & \mathbf{0} \\ & \ddots & & \\ & & d_r & \\ \hline & & & \mathbf{0} \end{array} \right],$$

and the  $d_k$ s polynomials such that  $d_k$  divides  $d_{k+1}$  for all  $k \in \{1, \dots, r-1\}$ . Thus any vector in  $\langle M(2\pi i\mathbf{v}, \tau) \rangle$  is of the form

$$\mathbf{u}U\Sigma V = \tilde{\mathbf{u}}\Sigma V = \tilde{u}_1 d_1 \mathbf{v}_1 + \dots + \tilde{u}_r d_r \mathbf{v}_r, \quad (2.2)$$

where  $\mathbf{u} \in \mathbb{C}[\tau]^{1 \times m}$ ,

$$\tilde{\mathbf{u}} := \mathbf{u}U = [ \tilde{u}_1 \quad \dots \quad \tilde{u}_r ],$$

and  $\mathbf{v}_1, \dots, \mathbf{v}_n$  are the rows of  $V$ . Let  $\mathbf{m} := \mathbf{v}_r$ . It follows that  $d_r$  is not constant thanks to (2.1). Clearly  $\mathbf{m} \notin \langle M(2\pi i\mathbf{v}, \tau) \rangle$ , for otherwise we would obtain  $\tilde{u}_1 d_1 \mathbf{v}_1 + \dots + \tilde{u}_r d_r \mathbf{v}_r = \mathbf{v}_r$ , and so  $\tilde{u}_r d_r = 1$ , contradicting the fact



that  $d_r$  is not a constant. Moreover, from (2.1),  $d_r \cdot \mathbf{m} \in \langle M(2\pi i\mathbf{v}, \tau) \rangle$ . Hence  $[\mathbf{m}]$  is a nontrivial torsion element in the  $\mathbb{C}[\tau]$ -module  $\mathbb{C}[\tau]^{1 \times n} / \langle M(2\pi i\mathbf{v}, \tau) \rangle$ . Consequently,  $\mathbb{C}[\tau]^{1 \times n} / \langle M(2\pi i\mathbf{v}, \tau) \rangle$  is not torsion free, that is, (4) does not hold. Hence we have shown that  $\neg(3) \Rightarrow \neg(4)$ , that is, (4)  $\Rightarrow$  (3).

**(3)  $\Rightarrow$  (2):** Suppose that  $T > 0$ . Let  $w_1, w_2 \in \mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(M)$ . Then we have

$$\widehat{w}_1 = \sum_{\mathbf{v} \in A^{-1}\mathbb{Z}^d} \delta_{\mathbf{v}} \otimes T_1^{(\mathbf{v})}, \quad \widehat{w}_2 = \sum_{\mathbf{v} \in A^{-1}\mathbb{Z}^d} \delta_{\mathbf{v}} \otimes T_2^{(\mathbf{v})},$$

for some  $T_1^{(\mathbf{v})}, T_2^{(\mathbf{v})} \in (\mathcal{D}'(\mathbb{R}))^n$ . Moreover, owing to the correspondence between  $\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d)$  and the space of sequences  $s'(\mathbb{Z}^d)$  of at most polynomial growth, it follows that for each  $\varphi \in \mathcal{D}(\mathbb{R})$ , there exist  $M_\varphi > 0$  and a positive integer  $k_\varphi$ , such that we have the estimates

$$\|\langle T_1^{(\mathbf{v})}, \varphi \rangle\|_2 \leq M_\varphi (1 + \|\mathbf{n}\|_2)^{k_\varphi}, \quad \|\langle T_2^{(\mathbf{v})}, \varphi \rangle\|_2 \leq M_\varphi (1 + \|\mathbf{n}\|_2)^{k_\varphi},$$

for all  $\mathbf{n} := A\mathbf{v} \in \mathbb{Z}^d$ , and where  $\|\cdot\|_2$  is the usual Euclidean norm. Let  $\theta \in C^\infty(\mathbb{R})$  be such that  $\theta(t) = 1$  for all  $t \leq 0$ ,  $\theta(t) = 0$  for all  $t > T/4$  and  $0 \leq \theta(t) \leq 1$  for all  $t \in \mathbb{R}$ . Define  $T^{(\mathbf{v})} \in (\mathcal{D}'(\mathbb{R}))^n$  by

$$T^{(\mathbf{v})} := \theta T_1^{(\mathbf{v})} + \theta(T - \cdot) T_2^{(\mathbf{v})}.$$

Set  $\widehat{w} \in \mathcal{D}'(\mathbb{R}^{d+1})$  to be

$$\widehat{w} = \sum_{\mathbf{v} \in A^{-1}\mathbb{Z}^d} \delta_{\mathbf{v}} \otimes T^{(\mathbf{v})}.$$

Then for every  $\varphi \in \mathcal{D}(\mathbb{R})$ , we have

$$\begin{aligned} \|\langle T^{(\mathbf{v})}, \varphi \rangle\|_2 &\leq \|\langle \theta T_1^{(\mathbf{v})}, \varphi \rangle\|_2 + \|\langle \theta(T - \cdot) T_2^{(\mathbf{v})}, \varphi \rangle\|_2 \\ &\leq M_{\theta\varphi} (1 + \|\mathbf{n}\|_2)^{k_{\theta\varphi}} + M_{\theta(T-\cdot)\varphi} (1 + \|\mathbf{n}\|_2)^{k_{\theta(T-\cdot)\varphi}} \\ &\leq \max\{M_{\theta\varphi}, M_{\theta(T-\cdot)\varphi}\} (1 + \|\mathbf{n}\|_2)^{\max\{k_{\theta\varphi}, k_{\theta(T-\cdot)\varphi}\}}, \end{aligned}$$

and so  $\vec{w}(\varphi) \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d)$ . Thus  $w \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})$ . Also,  $w \in \mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(M)$  because

$$M \left( 2\pi i\mathbf{y}, \frac{\partial}{\partial t} \right) (\delta_{\mathbf{v}} \otimes T^{(\mathbf{v})}) = M \left( 2\pi i\mathbf{v}, \frac{d}{dt} \right) (\delta_{\mathbf{v}} \otimes T^{(\mathbf{v})}) = 0,$$

for each  $\mathbf{v} \in A^{-1}\mathbb{Z}^d$ , and so

$$M \left( 2\pi i\mathbf{y}, \frac{\partial}{\partial t} \right) \widehat{w} = 0.$$

Consequently,

$$M \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial t} \right) w = 0,$$

that is,  $w \in \mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(M)$ .

Finally, because  $T^{(\mathbf{v})}|_{(-\infty,0)} = T_1^{(\mathbf{v})}|_{(-\infty,0)}$  and  $T^{(\mathbf{v})}|_{(T,+\infty)} = T_2^{(\mathbf{v})}|_{(T,+\infty)}$ , it follows that  $\widehat{w}|_{(-\infty,0)} = \widehat{w}_1|_{(-\infty,0)}$  and  $\widehat{w}|_{(T,+\infty)} = \widehat{w}_2|_{(T,+\infty)}$ . Consequently,  $w|_{(-\infty,0)} = w_1|_{(-\infty,0)}$  and  $w|_{(T,+\infty)} = w_2|_{(T,+\infty)}$ , showing that the behaviour is controllable in time  $T$ . This completes the proof of (3) $\Rightarrow$ (2).

(2)  $\Rightarrow$  (1): This follows trivially from the definitions.  $\square$

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