Internal Enriched Categories

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Dissertation Summary

Enrico Ghiorzi

This dissertation introduces and develops the theory of internal enriched categories, arising from the internalization of the theory of enriched categories. Given an internal monoidal category $\mathbf{V}$ in an ambient category $\mathscr{E}$, we define the notions of $\mathbf{V}$-enriched category, functor and natural transformation. We then develop such theory, which presents many of the good properties of standard enriched category theory. Notably, under suitable conditions, the category of internal $\mathbf{V}$-enriched categories and their functors is monoidal closed.

Internal enriched categories admit a notion of internal weighted limit, analogously to how internal categories admit internal limits. Such theory of limits constitutes a major focus point in the dissertation and yields fundamental results such as the adjoint functor theorem. It is observed that internal categories are intrinsically small and some of them are non-trivial examples of small complete categories, whereas the only standard small complete categories are complete lattices. As a consequence, the internal theory is better behaved than that of standard categories, particularly in relation with size issues, while still featuring interesting examples.

Moreover, to frame it into a wider context, the notion of internal enriched category is compared with related notions from the literature, such as those of indexed enriched category and enriched generalized multicategory. It turns out that internal enriched categories are indeed strongly connected with such other notions, thus providing a novel approach to–and, possibly, insight into–other topics in category theory.

Acknowledgments

For even the very wise cannot see all ends.

(J.R.R. Tolkien, The Fellowship of the Ring)

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Contents

Introduction

It is a remarkable feature of the Effective Topos (Hyland 1982) that it contains a small subcategory which is in some sense complete. That something of this kind should be true was originally suggested by Eugenio Moggi as a way to understand how realizability toposes give rise to models for impredicative polymorphism. Concrete versions of such models already appeared in Girard 1972. The relevant subcategory identified within the Effective Topos has been called the category of Modest Sets. The sense in which it is complete is a delicate matter. That and other aspects of the situation are considered in detail in Hyland, Robinson, and Rosolini 1990. For the purposes of this dissertation however it will be enough to know that the Modest Sets form a small complete subcategory of the Category of Assemblies (equivalently, the $\neg\neg$-separated objects in the Effective Topos, identified in Hyland 1982) in a sense which we shall make precise in chapter 2. This notion of (strong) completeness is related in a reasonably straightforward way—using the externalization of an internal category—to the established notion for indexed categories (Paré and Schumacher 1978). However we do not exploit this explicitly and keep the exposition as internal as possible. There is a rough sketch of the indexed point of view in Hyland 1982.

Aside from the applications to polymorphism, the existence of a small complete category remains underexploited. Section 3.1 of Hyland 1988 contains a little small complete category theory, inspired by the understanding of indexed categories (Paré and Schumacher 1978), but even that has never been worked through in any detail. We are not going to address that lack in this dissertation: rather, we want to propose a different perspective, that of enriched category theory.

This dissertation introduces the notion of internal enriched category theory, that is, enriched category theory internal to a suitable ambient category $\mathscr{E}$. Given a complete (and so co-complete) symmetric monoidal category $\mathscr{V}$ within $\mathscr{E}$, we consider categories enriched in $\mathscr{V}$ and internal to $\mathscr{E}$. (These themselves form a category indexed over $\mathscr{E}$.) Our aim is to develop an attractive theory of such internal enriched categories.

Our leading example of such a situation is provided by taking $\mathscr{V}$ to be the internal category of modest sets within $\mathscr{E}$ the Category of Assemblies. In that case $\mathscr{V}$ is an internal subcategory of $\mathscr{E}$ and enriched categories are essentially special categories internal to $\mathscr{E}$. Even within the Category of Assemblies, there should be many other examples of our general situation where the enriched categories are not simply special categories. In particular, if T is a commutative monad acting on $\mathscr{V}$, then the category of T-algebras should give an internal symmetric monoidal category in which one can enrich. We briefly address this at the end of the dissertation.

The dissertation is organized as follows.

Some categorical background to the dissertation is presented in chapter 1. The focus of the dissertation is on enriched category theory internal to a suitable ambient category, so we discuss internal categories and internal monoidal categories. However, when handling internal category theory, one needs to consider from time to time e.g. all such categories and give definitions which are not strictly within the internal logic. It is helpful to see such material from the perspective of indexed category theory, as that gives a way to talk about large categories over a base. While we shall not need to make this perspective explicit throughout the dissertation, we include here a treatment of indexed categories as motivating background for the main material of the dissertation.

The usual setting for enriched category theory involves a symmetric monodical closed category which is complete and cocomplete (Kelly 1982). We try to follow essentially the same approach in our internal theory, so we have to consider, in particular, the completeness of our enriching symmetric monoidal categories. There is relatively little on this topic in the existing literature and the fullest treatment (Hyland, Robinson, and Rosolini 1990) is concerned with issues irrelevant to the context in which we develop our theory. Hence, in chapter 2 we give a treatment of limits and completeness for internal categories.

In chapter 3 we introduce the main objects of study in the dissertation, that is, internal categories enriched in an internal complete symmetric monoidal closed category. We make the comparison with the indexed setting and then introduce the basic machinery of the theory of enrichment, including a crucial calculus of ends. A major topic is the construction and properties of the exponential (or enriched category of functors) between enriched categories. After we address that in some detail, we are then able to present a form of the Yoneda lemma and Yoneda embedding.

We finally come to the focal point of the dissertation in chapter 4, in which we discuss completeness and cocompleteness for internal enriched categories. Here, we find a pleasing contrast with Kelly's classical theory: there is no need to consider matters of size, as all objects under consideration are internal and can thus be thought of as small. So our results—for example the adjoint functor theorem—appear clean and uncluttered by the usual delicate set-theoretic considerations.

We have not developed the theory of internal enriched categories simply for its own sake and in our concluding chapter 5, we suggest directions for future work.

Finally, in appendix A we conduct a kind of sanity check on the notion of internal enriched category, by showing that it can be presented as an example of Leinster's notion of enriched generalised multicategory (Leinster 2002, 2004).

1. Background

In this chapter we quickly recall, without any claim of completeness, some topics in Category Theory which will be needed as background. Although these topics are standard and well-known, it is useful to spell them out anyway to set the notation. Aside from that, we assume the reader to be familiar with the basic notions of Category Theory. In this respect, we shall regard Borceux (1994) and Mac Lane (1989) as our main references.

In the context of this dissertation, let $\mathscr{E}$ be a category with finite limits, which we will regard as our ambient category. We also require $\mathscr{E}$ to have a cartesian monoidal structure, that is, a monoidal structure given by a functorial choice of binary products $\times^{\mathscr{E}} \colon \mathscr{E} \times \mathscr{E} \to \mathscr{E}$ and a chosen terminal object $\mathbb{1}_{\mathscr{E}}$.

Notice that, as a category with finite limits, $\mathscr{E}$ is a model for cartesian logic, or finite limit logic. So, we will frequently use its internal language to ease the notation. The internal language will be extended to typed lambda-calculus when we will further assume $\mathscr{E}$ to be locally cartesian closed. There are multiple accounts of the internal language of categories in the literature. In particular, we shall follow Crole (1993) and Johnstone (2002), but, since we only make a basic use of the internal language, other references would be equally adequate.

1.1. Internal Categories

We start by giving the definitions of internal category, functor and natural transformation using the internal language of $\mathscr{E}$ as described before.

Definition 1.1.1 (internal category)**.** An *internal category* (or *category object*) $\boldsymbol{A}$ in $\mathscr{E}$ is a diagram

$$A_0 \underset{\mathrm{t}_{\boldsymbol{A}}}{\overset{\mathrm{s}_{\boldsymbol{A}}}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\mathrm{id}_{\boldsymbol{A}}}}}}} A_1 \xleftarrow{\;\circ_{\boldsymbol{A}}\;} A_1 \,{}_{\mathrm{s}}{\times}_{\mathrm{t}}\, A_1$$

in $\mathscr{E}$ (where $A_1 \,{}_{\mathrm{s}}{\times}_{\mathrm{t}}\, A_1$ is the pullback of $\mathrm{s}_{\boldsymbol{A}}$ and $\mathrm{t}_{\boldsymbol{A}}$) satisfying the usual axioms for categories, which can be expressed in the internal language of $\mathscr{E}$ as follows.

$$a\colon A_0 \vdash \mathrm{s}_{\boldsymbol{A}}(\mathrm{id}_{\boldsymbol{A}}(a)) = a\colon A_0$$
$$a\colon A_0 \vdash \mathrm{t}_{\boldsymbol{A}}(\mathrm{id}_{\boldsymbol{A}}(a)) = a\colon A_0$$

$$
\begin{aligned}
(g,f) : A_1 \,{}_{\mathrm{s}}{\times}_{\mathrm{t}}\, A_1 &\vdash \mathrm{s}_{\boldsymbol{A}}(g \circ_{\boldsymbol{A}} f) = \mathrm{s}_{\boldsymbol{A}}(f) : A_0 \\
(g,f) : A_1 \,{}_{\mathrm{s}}{\times}_{\mathrm{t}}\, A_1 &\vdash \mathrm{t}_{\boldsymbol{A}}(g \circ_{\boldsymbol{A}} f) = \mathrm{t}_{\boldsymbol{A}}(g) : A_0 \\
(h,g,f) : A_1 \,{}_{\mathrm{s}}{\times}_{\mathrm{t}}\, A_1 \,{}_{\mathrm{s}}{\times}_{\mathrm{t}}\, A_1 &\vdash h \circ_{\boldsymbol{A}} (g \circ_{\boldsymbol{A}} f) = (h \circ_{\boldsymbol{A}} g) \circ_{\boldsymbol{A}} f : A_1 \\
f : A_1 &\vdash f \circ_{\boldsymbol{A}} \mathrm{id}_{\boldsymbol{A}}(\mathrm{s}_{\boldsymbol{A}}(f)) = f : A_1 \\
f : A_1 &\vdash \mathrm{id}_{\boldsymbol{A}}(\mathrm{t}_{\boldsymbol{A}}(f)) \circ_{\boldsymbol{A}} f = f : A_1
\end{aligned}
$$

We now define a notion of functor of internal categories with relative composition and identity, so that internal categories and their functors shall form a category.

Definition 1.1.2 (internal functor)**.** Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be internal categories in $\mathscr{E}$. A *functor* of internal categories $F : \boldsymbol{A} \to \boldsymbol{B}$ is given by a pair of arrows $F_0 : A_0 \to B_0$ and $F_1 : A_1 \to B_1$ such that

$$
\begin{aligned}
f : A_1 &\vdash \mathrm{s}_{\boldsymbol{B}}(F_1(f)) = F_0(\mathrm{s}_{\boldsymbol{A}}(f)) : B_0 \\
f : A_1 &\vdash \mathrm{t}_{\boldsymbol{B}}(F_1(f)) = F_0(\mathrm{t}_{\boldsymbol{A}}(f)) : B_0 \\
(g,f) : A_1 \,{}_{\mathrm{s}}{\times}_{\mathrm{t}}\, A_1 &\vdash F_1(g \circ_{\boldsymbol{A}} f) = F_1(g) \circ_{\boldsymbol{B}} F_1(f) : B_1.
\end{aligned}
$$

The composition $G \circ F$ of two consecutive functors $\boldsymbol{A} \xrightarrow{F} \boldsymbol{B} \xrightarrow{G} \boldsymbol{C}$ is defined by

$$
\begin{aligned}
a : A_0 &\vdash (G \circ F)_0(a) \coloneqq G_0(F_0(a)) : C_0 \\
f : A_1 &\vdash (G \circ F)_1(f) \coloneqq G_1(F_1(f)) : C_1.
\end{aligned}
$$

The identity functor $\mathrm{id}_{\boldsymbol{A}} : \boldsymbol{A} \to \boldsymbol{A}$ is defined by

$$
\begin{aligned}
a : A_0 &\vdash (\mathrm{id}_{\boldsymbol{A}})_0(a) \coloneqq a : A_0 \\
f : A_1 &\vdash (\mathrm{id}_{\boldsymbol{A}})_1(f) \coloneqq f : A_1.
\end{aligned}
$$

With the composition so defined, internal categories in $\mathscr{E}$ and their functors form a category $\mathrm{Cat}_{\mathscr{E}}$. In the following remark, we notice some useful properties of $\mathrm{Cat}_{\mathscr{E}}$ in relation to slicing and change of base.

Remark 1.1.3. Let $\mathscr{E}'$ be another category with finite limits, and $F : \mathscr{E} \to \mathscr{E}'$ a functor preserving finite limits. Then, there is a functor $F : \mathrm{Cat}_{\mathscr{E}} \to \mathrm{Cat}_{\mathscr{E}'}$ (with notation overload) applying F to the underlying graph of internal categories.

Remark 1.1.4. Let $i : J \to I$ be an arrow in $\mathscr{E}$. Then, there is an adjunction $i_! \dashv i^* : \mathscr{E}/I \to \mathscr{E}/J$ where the functor $i_! : \mathscr{E}/J \to \mathscr{E}/I$ is given by post-composition with i, and the functor $i^* : \mathscr{E}/I \to \mathscr{E}/J$ is given by pullback along i. This adjunction extends to internal categories, yielding $i_! \dashv i^* : \mathrm{Cat}_{\mathscr{E}/I} \to \mathrm{Cat}_{\mathscr{E}/J}$.

In particular, the unique arrow $!_I : I \to \mathbb{1}_{\mathscr{E}}$ yields an adjunction $I_! \dashv I^* : \mathrm{Cat}_{\mathscr{E}} \to \mathrm{Cat}_{\mathscr{E}/I}$.

While correct, the use we made of the internal language is quite unwieldy, especially when dealing with terms whose type is the object of arrows of an internal category. We then introduce conventions to ease the use of the internal language, by bringing it closer to the standard notation of category theory.

Notation 1.1.5. Given arrows $F\colon X \to A_1$ and $S, T\colon X \to A_0$, we shall write

$$x\colon X \vdash F(x)\colon S(x) \to T(x) : A_1 \qquad \text{or} \qquad x\colon X \vdash S(x) \xrightarrow{F(x)} T(x) : A_1$$

instead of $x\colon X \vdash \mathrm{s}_{\boldsymbol{A}} F(x) = S(x) : A_0$ and $x\colon X \vdash \mathrm{t}_{\boldsymbol{A}} F(x) = T(x) : A_0$. Moreover, given $G\colon X \to A_1$ and $U\colon X \to A_0$ such that $x\colon X \vdash G(x)\colon T(x) \to U(x) : A_1$, we shall write

$$x\colon X \vdash S(x) \xrightarrow{F(x)} T(x) \xrightarrow{G(x)} U(x) : A_1$$

instead of $x\colon X \vdash G(x) \circ_{\boldsymbol{A}} F(x) : A_1$. Finally, when we have a term $a_0, a_1 : A_0, f\colon A_1 \vdash t(a_0, a_1, f) : B$ we shall write

$$(f\colon a_0 \to a_1) : A_1 \vdash t(f, a_0, a_1) : B$$

in place of $f\colon A_1 \vdash t(\mathrm{s}_{\boldsymbol{A}}(f), \mathrm{t}_{\boldsymbol{A}}(f), f) : B$. Then, we can use the familiar notation for commuting diagrams even in the internal language.

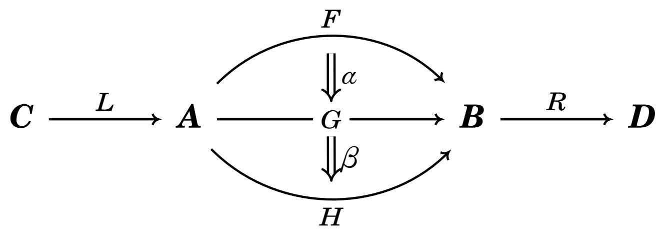
With the previous notation, it is easy to define natural transformations between functors of internal categories, with relative horizontal and vertical compositions and identity, so that internal categories with their functors and the natural transformations between them shall form a 2-category.

Definition 1.1.6 (natural transformations)**.** Let $F, G\colon \boldsymbol{A} \to \boldsymbol{B}$ be functors of internal categories in $\mathscr{E}$. A *natural transformation* $\alpha\colon F \to G\colon \boldsymbol{A} \to \boldsymbol{B}$ is given by an arrow $\alpha\colon A_0 \to B_1$ such that

$$a\colon A_0 \vdash F_0(a) \xrightarrow{\alpha(a)} G_0(a) : B_1,$$

$$(f\colon a \to a') : A_1 \vdash \begin{array}{ccc} F_0(a) & \xrightarrow{\alpha(a)} & G_0(a) \\ {\scriptstyle F_1(f)}\downarrow & & \downarrow{\scriptstyle G_1(f)} \\ F_0(a') & \xrightarrow[\alpha(a')]{} & G_0(a') \end{array} : B_1.$$

The vertical and horizontal compositions of natural transformations in



are given by

$$
\begin{aligned}
a\colon A_0 &\vdash (\alpha\beta)(a) \coloneqq F_0(a) \xrightarrow{\alpha(a)} G_0(a) \xrightarrow{\beta(a)} H_0(a) : B_1 \\
c\colon C_0 &\vdash (\alpha L)(a) \coloneqq F_0(L_0(c)) \xrightarrow{\alpha(L(c))} G_0(L_0(c)) : B_1 \\
a\colon A_0 &\vdash (R\alpha)(a) \coloneqq R_0(F_0(a)) \xrightarrow{R_1(\alpha(a))} R_0(G_0(a)) : D_1.
\end{aligned}
$$

The identity natural transformation $\mathrm{id}_F\colon F \to F\colon \boldsymbol{A} \to \boldsymbol{B}$ is defined by

$$
a\colon A_0 \vdash \mathrm{id}_F(a) \coloneqq F_0(a) \xrightarrow{\mathrm{id}_A(F_0(a))} F_0(a) : B_1.
$$

With the compositions so defined, internal categories in $\mathscr{E}$ with their functors and the natural transformations between them form a 2-category $\mathrm{Cat}_{\mathscr{E}}$. Notice that there is a notation overload, but context is usually sufficient to distinguish between occurrences of $\mathrm{Cat}_{\mathscr{E}}$ as a category or as a 2-category.

The category of internal categories is well-behaved with respect to slicing, as the following remark makes clear.

The following proposition can be proved by a completely routine application of the internal language of $\mathscr{E}$.

Proposition 1.1.7. *The category* $\mathrm{Cat}_{\mathscr{E}}$ *has finite limits induced point-wise by the corresponding limits in* $\mathscr{E}$*. In particular, there is a terminal internal category* $\mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}}$ *and a binary product* $\times^{\mathrm{Cat}_{\mathscr{E}}}$ *of internal categories making* $\mathrm{Cat}_{\mathscr{E}}$ *a cartesian monoidal category.*

There is also an obvious functor $\mathrm{Cat}_{\mathscr{E}} \to \mathscr{E}$, given in the next definition.

Definition 1.1.8. The *objects functor* is the monoidal functor $U\colon \mathrm{Cat}_{\mathscr{E}} \to \mathscr{E}$ sending an internal category $\boldsymbol{A}$ into its object of objects A, and an internal functor $F\colon \boldsymbol{A} \to \boldsymbol{B}$ into its object component $F_0\colon A \to B$.

We now present a few remarkable examples of internal categories.

Example 1.1.9. Let A be an object of $\mathscr{E}$. The *discrete category* $\mathbf{dis}(A)$ over A is given by

$$
\mathbf{dis}(A)_0 = A \;\underset{\mathrm{t}_{\mathbf{dis}(A)_1}=\mathrm{id}_A}{\overset{\mathrm{s}_{\mathbf{dis}(A)_1}=\mathrm{id}_A}{\leftleftarrows}}\;\xrightarrow[\mathrm{id}_A]{\mathrm{id}_{\mathbf{dis}(A)_1}=}\; \mathbf{dis}(A)_1 = A \xleftarrow{\circ_{\mathbf{dis}(A)_1}=\mathrm{id}_A} \mathbf{dis}(A)_1 \,{}_{\mathrm{s}}\!\times_{\mathrm{t}} \mathbf{dis}(A)_1 \cong A
$$

This extends to a monoidal functor $\mathbf{dis}\colon \mathscr{E} \to \mathrm{Cat}_{\mathscr{E}}$.

Example 1.1.10. Let A be an object of $\mathscr{E}$. The *indiscrete category* $\mathbf{ind}(A)$ over A is given by

$$\mathbf{ind}(A)_0 = A \underset{\Delta_A}{\overset{\mathrm{id}_{\mathbf{ind}(A)_1}}{=}} \mathbf{ind}(A)_1 = A \times A \underset{(\pi_1,\pi_3)}{\overset{\circ_{\mathbf{ind}(A)_1}}{=}} \mathbf{ind}(A)_1 \, {}_{\mathrm{s}}\!\times_{\mathrm{t}} \mathbf{ind}(A)_1 \cong A \times A \times A$$

with $\mathrm{s}_{\mathbf{ind}(A)_1} = \pi_1$ and $\mathrm{t}_{\mathbf{ind}(A)_1} = \pi_2$ from $\mathbf{ind}(A)_1$ to $\mathbf{ind}(A)_0$.

This extends to a monoidal functor $\mathbf{ind}\colon \mathscr{E} \to \mathrm{Cat}_{\mathscr{E}}$.

The above constructions yield the free and co-free internal categories over an object of $\mathscr{E}$, as formally stated by the following proposition, whose proof is, again, routine.

Proposition 1.1.11. *There are monoidal adjunctions* $\mathbf{dis} \dashv U \dashv \mathbf{ind}$.

As a further example, consider opposite categories.

Example 1.1.12. Let $\boldsymbol{A}$ be an internal category in $\mathscr{E}$. The opposite category $\boldsymbol{A}^{\mathbf{op}}$ is obtained by switching the source and target arrows of $\boldsymbol{A}$, so that $\mathrm{s}_{\boldsymbol{A}^{\mathbf{op}}} = \mathrm{t}_{\boldsymbol{A}}$ and $\mathrm{t}_{\boldsymbol{A}^{\mathbf{op}}} = \mathrm{s}_{\boldsymbol{A}}$. This extends to a monoidal functor $(\text{-})^{\mathbf{op}}\colon \mathrm{Cat}_{\mathscr{E}} \to \mathrm{Cat}_{\mathscr{E}}$.

Then, it is not difficult to prove the following result.

Proposition 1.1.13. *The monoidal functor* $(\text{-})^{\mathbf{op}}$ *is a self-adjoint automorphism.*

1.2. Exponentials of Internal Categories

The functors between two categories form a set, meaning that Cat is enriched over Set. Analogously, we would expect $\mathrm{Cat}_{\mathscr{E}}$ to be enriched over $\mathscr{E}$. For that, we need to assume that $\mathscr{E}$ is cartesian closed. Then, the theory of categories internal to $\mathscr{E}$ becomes clearer from the perspective of enriched category theory.

In this section, let's assume that $\mathscr{E}$ is cartesian closed, meaning that its internal language will be the simply typed lambda calculus extension of finite limit logic.

Notation 1.2.1. If A and B are objects of $\mathscr{E}$, let $\mathrm{e}_{\mathscr{E}}\colon B^A \times A \to B$ be the evaluation morphism. The notation for the evaluation arrow should also be decorated with A and B, but it is unnecessary as those are usually clear from the context.

It is also useful to introduce a convention to denote subobjects yielded by an equalizer.

Notation 1.2.2. Given two terms $a \colon A \vdash t(a), t'(a) : B$ we can form the equalizer of the two parallel arrows $A \to B$ yielded by t and t'. That is the subobject of A of those $a \colon A$ such that $t(a) = t'(a)$.

We are now ready to prove that the intuition about $\mathrm{Cat}_{\mathscr{E}}$ being enriched in $\mathscr{E}$ was correct, in the sense made precise by the statement of the following proposition.

Proposition 1.2.3. *There is a category enriched in $\mathscr{E}$ (which shall be called* $\mathrm{Cat}_{\mathscr{E}}$ *with abuse of notation) whose underlying category is isomorphic to* $\mathrm{Cat}_{\mathscr{E}}$ *itself.*

Proof. The hom-object $\mathrm{Hom}_{\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{A}, \boldsymbol{B})$ of internal categories $\boldsymbol{A}$ and $\boldsymbol{B}$ represents the functors $\boldsymbol{A} \to \boldsymbol{B}$, and is defined as the subobject of $B_0^{A_0} \times B_1^{A_1}$ of those $F = (F_0, F_1)$ satisfying the functoriality axioms, i.e. such that

$$\lambda(f \colon a_0 \to a_1) : A_1.\ \mathrm{e}_{\mathscr{E}}(F_0, a_0) \xrightarrow{\mathrm{e}_{\mathscr{E}}(F_1, f)} \mathrm{e}_{\mathscr{E}}(F_0, a_1)$$

$$\lambda(g, f) : A_1\ {}_{\mathrm{s}}{\times}_{\mathrm{t}}\ A_1.\ \mathrm{e}_{\mathscr{E}}(F_1, g \circ_{\boldsymbol{A}} f) = \lambda(g, f) : A_1\ {}_{\mathrm{s}}{\times}_{\mathrm{t}}\ A_1.\ \mathrm{e}_{\mathscr{E}}(F_1, g) \circ_{\boldsymbol{B}} \mathrm{e}_{\mathscr{E}}(F_1, f).$$

The composition of internal categories $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$ is the arrow

$$\circ_{\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}) : \mathrm{Hom}_{\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{B}, \boldsymbol{C}) \times \mathrm{Hom}_{\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{A}, \boldsymbol{B}) \to \mathrm{Hom}_{\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{A}, \boldsymbol{C})$$

defined in context $F \colon \mathrm{Hom}_{\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{A}, \boldsymbol{B}),\ G \colon \mathrm{Hom}_{\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{B}, \boldsymbol{C})$ as

$$\circ_{\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})(F, G)_0 \coloneqq \lambda a \colon A_0.\ \mathrm{e}_{\mathscr{E}}(G_0, \mathrm{e}_{\mathscr{E}}(F_0, a)) : {C_0}^{A_0}$$

$$\circ_{\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})(F, G)_1 \coloneqq \lambda f \colon A_1.\ \mathrm{e}_{\mathscr{E}}(G_1, \mathrm{e}_{\mathscr{E}}(F_1, f)) : {C_1}^{A_1}.$$

The identity of the internal category $\boldsymbol{A}$ is the arrow

$$\mathrm{id}_{\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{A}) : \mathbb{1}_{\mathscr{E}} \to \mathrm{Hom}_{\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{A}, \boldsymbol{A})$$

defined as

$$\mathrm{id}_{\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{A}) \coloneqq (\lambda a \colon A_0.\ a, \lambda f \colon A_1.\ f) : \mathrm{Hom}_{\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{A}, \boldsymbol{A}).$$

The verification that these data give an enrichment is a simple exercise in the internal language.

The points of the hom-object $\mathrm{Hom}_{\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{A}, \boldsymbol{B})$ are evidently in bijective correspondence with functors $\boldsymbol{A} \to \boldsymbol{B}$. □

The theory developed in section 1.1 extends to the context of enriched category theory. For example, consider the following proposition.

Proposition 1.2.4. *The adjunctions from proposition 1.1.11 are $\mathscr{E}$-enriched.*

Remark 1.2.5. For any object I in $\mathscr{E}$ we have that $\mathrm{Cat}_{\mathscr{E}/I}$ is enriched over $\mathscr{E}/I$. Moreover, if $u\colon I \to J$ is an arrow in $\mathscr{E}$, then $\mathrm{Cat}_{\mathscr{E}/J}$ has an enrichment on $\mathscr{E}/I$ through $u^*\colon \mathscr{E}/J \to \mathscr{E}/I$, and $(u^*)_*\colon \mathrm{Cat}_{\mathscr{E}/J} \to \mathrm{Cat}_{\mathscr{E}/I}$ is a $\mathscr{E}/I$-enriched functor.

We can finally prove that $\mathrm{Cat}_{\mathscr{E}}$ is cartesian closed. Moreover, as it is clear from the following proof, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are categories in $\mathscr{E}$, then the points of $\boldsymbol{B}^{\boldsymbol{A}}$ are in bijective correspondence with the functors $\boldsymbol{A} \to \boldsymbol{B}$.

Proposition 1.2.6. *The category* $\mathrm{Cat}_{\mathscr{E}}$ *is cartesian closed.*

Proof. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be internal categories of $\mathscr{E}$. Let's define the exponential object $\boldsymbol{B}^{\boldsymbol{A}}$, representing the internal category of functors $\boldsymbol{A} \to \boldsymbol{B}$ and natural transformations between them:

Object of objects $(\boldsymbol{B}^{\boldsymbol{A}})_0$ is $\mathrm{Hom}_{\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{A},\boldsymbol{B})$, as defined in proposition 1.2.3.

Object of arrows $(\boldsymbol{B}^{\boldsymbol{A}})_1$ is the subobject of $(\boldsymbol{B}^{\boldsymbol{A}})_0 \times (\boldsymbol{B}^{\boldsymbol{A}})_0 \times B_1^{A_0}$ given, in context $(F\colon (\boldsymbol{B}^{\boldsymbol{A}})_0, G\colon (\boldsymbol{B}^{\boldsymbol{A}})_0, \alpha\colon B_1^{A_0})$, by the axiom

$$\lambda(f\colon a \to b) : A_1.\quad \begin{array}{ccc} \mathrm{e}_{\mathscr{E}}(F_0,a) & \xrightarrow{\mathrm{e}_{\mathscr{E}}(\alpha,a)} & \mathrm{e}_{\mathscr{E}}(G_0,a) \\ {\scriptstyle \mathrm{e}_{\mathscr{E}}(F_0,f)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{e}_{\mathscr{E}}(G_0,f)} \\ \mathrm{e}_{\mathscr{E}}(F_0,b) & \xrightarrow[\mathrm{e}_{\mathscr{E}}(\alpha,b)]{} & \mathrm{e}_{\mathscr{E}}(G_0,b) \end{array} : B_1.$$

Composition is the arrow $\circ_{\boldsymbol{B}^{\boldsymbol{A}}}\colon (\boldsymbol{B}^{\boldsymbol{A}})_1 \,{}_{\mathrm{s}}{\times}_{\mathrm{t}}\, (\boldsymbol{B}^{\boldsymbol{A}})_1 \to (\boldsymbol{B}^{\boldsymbol{A}})_1$ defined, in context $((G,H,\beta),(F,G,\alpha))\colon (\boldsymbol{B}^{\boldsymbol{A}})_1 \,{}_{\mathrm{s}}{\times}_{\mathrm{t}}\, (\boldsymbol{B}^{\boldsymbol{A}})_1$, as

$$(G,H,\beta) \circ_{\boldsymbol{B}^{\boldsymbol{A}}} (F,G,\alpha) := (F,H,\lambda a\colon A_0.\ \mathrm{e}_{\mathscr{E}}(\beta,a) \circ_{\boldsymbol{B}} \mathrm{e}_{\mathscr{E}}(\alpha,a))$$

Identity is the arrow $\mathrm{id}_{\boldsymbol{B}^{\boldsymbol{A}}}\colon (\boldsymbol{B}^{\boldsymbol{A}})_0 \to (\boldsymbol{B}^{\boldsymbol{A}})_1$ defined, in context $F\colon (\boldsymbol{B}^{\boldsymbol{A}})_0$, as

$$\mathrm{id}_{\boldsymbol{B}^{\boldsymbol{A}}}(F) := (F,F,\lambda a\colon A_0.\ \mathrm{id}_{\boldsymbol{B}}(F_0(a))).$$

Then, a standard argument shows that there is an isomorphism

$$\mathrm{Hom}_{\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{A}' \times^{\mathrm{Cat}_{\mathscr{E}}} \boldsymbol{A},\boldsymbol{B}) \cong \mathrm{Hom}_{\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{A}',\boldsymbol{B}^{\boldsymbol{A}})$$

natural in $\boldsymbol{A}'$. □

Of course the fact that $\mathrm{Cat}_{\mathscr{E}}$ is cartesian closed tells us that it is enriched in itself. That extends the fact that it is enriched in $\mathscr{E}$. In fact the enrichment in $\mathscr{E}$ is essentially obtained from that in $\mathrm{Cat}_{\mathscr{E}}$ by change of base along the objects functor $U\colon \mathrm{Cat}_{\mathscr{E}} \to \mathscr{E}$.

1.3. Internal Monoidal Categories

We could not conceivably present a notion of enrichment without introducing a suitable notion of monoidal category to serve as enriching category. We introduce here the definitions, in the internal language of $\mathscr{E}$, of the notions of monoidal category, functor and natural transformation. We also give the definitions pertaining to the symmetric case, which is of most interest for the sequel.

Definition 1.3.1 (internal monoidal category)**.** An *internal monoidal category* is an internal category $\boldsymbol{V}$ in $\mathscr{E}$ equipped with functors

Monoidal product $\otimes^{\boldsymbol{V}}\colon \boldsymbol{V} \times^{\mathrm{Cat}_{\mathscr{E}}} \boldsymbol{V} \to \boldsymbol{V}$, and

Monoidal unit $\mathbb{I}_{\boldsymbol{V}}\colon \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}} \to \boldsymbol{V}$,

and natural isomorphisms

Associator $\alpha_{\boldsymbol{V}}\colon (\pi_1 \otimes^{\boldsymbol{V}} \pi_2) \otimes^{\boldsymbol{V}} \pi_3 \to \pi_1 \otimes^{\boldsymbol{V}} (\pi_2 \otimes^{\boldsymbol{V}} \pi_3) : \boldsymbol{V} \times \boldsymbol{V} \times \boldsymbol{V} \to \boldsymbol{V}$,

Left unitor $\lambda_{\boldsymbol{V}}\colon \mathbb{I}_{\boldsymbol{V}} \otimes^{\boldsymbol{V}} \mathrm{id}_{\boldsymbol{V}} \to \mathrm{id}_{\boldsymbol{V}}\colon \boldsymbol{V} \to \boldsymbol{V}$, and

Right unitor $\rho_{\boldsymbol{V}}\colon \mathrm{id}_{\boldsymbol{V}} \otimes^{\boldsymbol{V}} \mathbb{I}_{\boldsymbol{V}} \to \mathrm{id}_{\boldsymbol{V}}\colon \boldsymbol{V} \to \boldsymbol{V}$,

such that, in context $a, b, c, d\colon V_0$, the axioms

$$\begin{array}{ccc} (a \otimes_0^{\boldsymbol{V}} \mathbb{I}_{\boldsymbol{V}}) \otimes_0^{\boldsymbol{V}} b & \xrightarrow{\alpha_{\boldsymbol{V}}(a,\mathbb{I}_{\boldsymbol{V}},b)} & a \otimes_0^{\boldsymbol{V}} (\mathbb{I}_{\boldsymbol{V}} \otimes_0^{\boldsymbol{V}} b) \\ & {\scriptstyle \rho_{\boldsymbol{V}}(a) \otimes_1^{\boldsymbol{V}} \mathrm{id}_{\boldsymbol{V}}(b)} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_{\boldsymbol{V}}(a) \otimes_1^{\boldsymbol{V}} \lambda_{\boldsymbol{V}}(b)} & \\ & a \otimes_0^{\boldsymbol{V}} b & \end{array} \tag{Triangle}$$

$$\begin{array}{ccc} & ((a \otimes_0^{\boldsymbol{V}} b) \otimes_0^{\boldsymbol{V}} c) \otimes_0^{\boldsymbol{V}} d & \\ & {\scriptstyle \alpha_{\boldsymbol{V}}(a,b,c) \otimes_0^{\boldsymbol{V}} d} \swarrow \quad \searrow {\scriptstyle \alpha_{\boldsymbol{V}}(a \otimes_0^{\boldsymbol{V}} b,c,d)} & \\ (a \otimes_0^{\boldsymbol{V}} (b \otimes_0^{\boldsymbol{V}} c)) \otimes_0^{\boldsymbol{V}} d & & (a \otimes_0^{\boldsymbol{V}} b) \otimes_0^{\boldsymbol{V}} (c \otimes_0^{\boldsymbol{V}} d) \\ {\scriptstyle \alpha_{\boldsymbol{V}}(a,b \otimes_0^{\boldsymbol{V}} c,d)} \downarrow & & \downarrow {\scriptstyle \alpha_{\boldsymbol{V}}(a,b,c \otimes_0^{\boldsymbol{V}} d)} \\ a \otimes_0^{\boldsymbol{V}} ((b \otimes_0^{\boldsymbol{V}} c) \otimes_0^{\boldsymbol{V}} d) & \xrightarrow[a \otimes_0^{\boldsymbol{V}} \alpha_{\boldsymbol{V}}(b,c,d)]{} & a \otimes_0^{\boldsymbol{V}} (b \otimes_0^{\boldsymbol{V}} (c \otimes_0^{\boldsymbol{V}} d)) \end{array} \tag{Pentagon}$$

hold.

The previous definition is a direct internalization of the standard definition of monoidal category, and that alone should suffice to persuade us of its correctness. If we were still skeptical, though, it could also be argued that, since small monoidal categories are pseudomonoids in the 2-category of categories, then internal monoidal categories in $\mathscr{E}$ must be pseudomonoids in the 2-category $\mathrm{Cat}_{\mathscr{E}}$, which is what our definition amounts to.

We then proceed with the definition of monoidal functor.

Definition 1.3.2 (internal monoidal functor)**.** An *internal monoidal functor* $(F, \epsilon, \mu) : \boldsymbol{V} \to \boldsymbol{W}$ is given by an internal functor $F \colon \boldsymbol{V} \to \boldsymbol{W}$ and coherence natural isomorphisms

$$\epsilon \colon \mathbb{I}_{\boldsymbol{W}} \to F\mathbb{I}_{\boldsymbol{V}} \colon \mathbb{I}_{\mathrm{Cat}_{\mathscr{E}}} \to \boldsymbol{W}$$

and

$$\mu : F \otimes^{\boldsymbol{W}} F \to F(\text{-} \otimes^{\boldsymbol{V}} \text{-}) : \boldsymbol{V} \times \boldsymbol{V} \to \boldsymbol{W}$$

such that, in context $a, b, c \colon V_0$, the axioms

$$\begin{array}{ccc}
(F_0(a) \otimes^{\boldsymbol{W}} F_0(b)) \otimes^{\boldsymbol{W}} F_0(c) & \xrightarrow{\alpha_{\boldsymbol{W}}(F_0(a), F_0(b), F_0(c))} & F_0(a) \otimes^{\boldsymbol{W}} (F_0(b) \otimes^{\boldsymbol{W}} F_0(c)) \\
{\scriptstyle \mu(a,b) \otimes_1^{\boldsymbol{W}} \mathrm{id}_{\boldsymbol{W}}(F_0(c))} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_{\boldsymbol{W}}(F_0(a)) \otimes_1^{\boldsymbol{V}} \mu(b,c)} \\
F_0(a \otimes^{\boldsymbol{V}} b) \otimes^{\boldsymbol{W}} F_0(c) & & F_0(a) \otimes^{\boldsymbol{W}} F_0(b \otimes^{\boldsymbol{V}} c) \\
{\scriptstyle \mu(a \otimes_0^{\boldsymbol{V}} b, c)} \downarrow & & \downarrow {\scriptstyle \mu(a, b \otimes_0^{\boldsymbol{V}} c)} \\
F_0((a \otimes^{\boldsymbol{V}} b) \otimes^{\boldsymbol{V}} c) & \xrightarrow[F_1(\alpha_{\boldsymbol{V}}(a,b,c))]{} & F_0(a \otimes^{\boldsymbol{V}} (b \otimes^{\boldsymbol{V}} c))
\end{array}$$

$$\begin{array}{ccccc}
\mathbb{I}_{\boldsymbol{W}} \otimes^{\boldsymbol{W}} F_0(a) & & \xrightarrow{\lambda_{\boldsymbol{W}}(F_0(a))} & & F_0(a) \\
& {\scriptstyle \epsilon \otimes^{\boldsymbol{W}} F_0(a)} \searrow & & \nearrow {\scriptstyle F_1(\lambda_{\boldsymbol{V}}(a))} & \\
& F_0(\mathbb{I}_{\boldsymbol{V}}) \otimes^{\boldsymbol{W}} F_0(a) & \xrightarrow[\mu(\mathbb{I}_{\boldsymbol{V}}, a)]{} & F_0(\mathbb{I}_{\boldsymbol{V}} \otimes^{\boldsymbol{V}} a) &
\end{array}$$

$$\begin{array}{ccccc}
F_0(a) \otimes^{\boldsymbol{W}} \mathbb{I}_{\boldsymbol{W}} & & \xrightarrow{\rho_{\boldsymbol{W}}(F_0(a))} & & F_0(a) \\
& {\scriptstyle F_0(a) \otimes^{\boldsymbol{W}} \epsilon} \searrow & & \nearrow {\scriptstyle F_1(\rho_{\boldsymbol{V}}(a))} & \\
& F_0(a) \otimes^{\boldsymbol{W}} F_0(\mathbb{I}_{\boldsymbol{V}}) & \xrightarrow[\mu(a, \mathbb{I}_{\boldsymbol{V}})]{} & F_0(a \otimes^{\boldsymbol{V}} \mathbb{I}_{\boldsymbol{V}}) &
\end{array}$$

hold.

Then, we define natural trasformations of monoidal functors.

Definition 1.3.3 (internal monoidal natural transformation)**.** An *internal monoidal natural transformation* $\alpha \colon (F, \epsilon_F, \mu_F) \to (G, \epsilon_G, \mu_G) : \boldsymbol{V} \to \boldsymbol{W}$ is a natural transformation $\alpha\colon F \to G\colon \boldsymbol{V} \to \boldsymbol{W}$ such that, in context $a, b, : V_0$, the axioms

$$\begin{array}{ccc} F_0(a) \otimes^{\boldsymbol{W}} F_0(b) & \xrightarrow{\alpha(a)\otimes^{\boldsymbol{W}}\alpha(b)} & G_0(a)\otimes^{\boldsymbol{W}} G_0(b) \\ {\scriptstyle \mu_F(a,b)}\downarrow & & \downarrow{\scriptstyle \mu_G(a,b)} \\ F_0(a\otimes^{\boldsymbol{V}} b) & \xrightarrow[\alpha(a\otimes^{\boldsymbol{V}} b)]{} & G_0(a\otimes^{\boldsymbol{V}} b)\end{array}$$

$$\begin{array}{ccc} F_0(\mathbb{I}_{\boldsymbol{V}}) & \xrightarrow{\alpha(\mathbb{I}_{\boldsymbol{V}})} & G_0(\mathbb{I}_{\boldsymbol{V}}) \\ {\scriptstyle \epsilon_F}\nwarrow & & \nearrow{\scriptstyle \epsilon_G} \\ & \mathbb{I}_{\boldsymbol{W}} & \end{array}$$

hold.

While it is not necessary for the definition of the notion of enrichment, in many situations it is required for the monoidal product of the enriching category to be symmetric. We now define the notion of internal symmetric monoidal category and functor. Notice that there is no need for a definition of symmetric monoidal natural transformation, as the usual definition is already adequate.

Definition 1.3.4 (internal symmetric monoidal category)**.** An internal monoidal category $\boldsymbol{V}$ is a *internal symmetric monoidal category* if it comes equipped with a symmetry, that is, a natural isomorphism $\sigma_{\boldsymbol{V}}\colon \pi_1 \otimes^{\boldsymbol{V}} \pi_2 \to \pi_2 \otimes^{\boldsymbol{V}} \pi_1\colon \boldsymbol{V} \times^{\mathrm{Cat}_{\mathscr{E}}} \boldsymbol{V} \to \boldsymbol{V}$ such that, in context $a, b\colon V_0$, the following axioms

$$\begin{array}{ccc} (a\otimes^{\boldsymbol{V}} b)\otimes^{\boldsymbol{V}} c & \xrightarrow{\alpha_{\boldsymbol{V}}(a,b,c)} & a\otimes^{\boldsymbol{V}}(b\otimes^{\boldsymbol{V}} c) \\ {\scriptstyle \sigma_{\boldsymbol{V}}(a,b)\otimes^{\boldsymbol{V}} c}\downarrow & & \downarrow{\scriptstyle \sigma_{\boldsymbol{V}}(a,b\otimes^{\boldsymbol{V}} c)} \\ (b\otimes^{\boldsymbol{V}} a)\otimes^{\boldsymbol{V}} c & & (b\otimes^{\boldsymbol{V}} c)\otimes^{\boldsymbol{V}} a \\ {\scriptstyle \alpha_{\boldsymbol{V}}(b,a,c)}\downarrow & & \downarrow{\scriptstyle \alpha_{\boldsymbol{V}}(b,c,a)} \\ b\otimes^{\boldsymbol{V}}(a\otimes^{\boldsymbol{V}} c) & \xrightarrow[b\otimes^{\boldsymbol{V}}\sigma_{\boldsymbol{V}}(a,c)]{} & b\otimes^{\boldsymbol{V}}(c\otimes^{\boldsymbol{V}} a) \end{array} \tag{Hexagon}$$

$$\begin{array}{ccc} a\otimes^{\boldsymbol{V}} b & = & a\otimes^{\boldsymbol{V}} b \\ {\scriptstyle \sigma_{\boldsymbol{V}}(a,b)}\searrow & & \nearrow{\scriptstyle \sigma_{\boldsymbol{V}}(b,a)} \\ & b\otimes^{\boldsymbol{V}} a & \end{array} \tag{Involution}$$

hold.

Definition 1.3.5 (internal symmetric monoidal functor)**.** An internal monoidal functor $(F, \epsilon, \mu) : \boldsymbol{V} \to \boldsymbol{W}$ between internal symmetric monoidal categories $\boldsymbol{V}$ and $\boldsymbol{W}$ is a *internal symmetric monoidal functor* if the axiom

$$\begin{array}{ccc} F_0(a \otimes^{\boldsymbol{V}} b) & \xrightarrow{\mu(a,b)} & F_0(a) \otimes^{\boldsymbol{W}} F_0(b) \\ {\scriptstyle F_1(\sigma_{\boldsymbol{V}}(a,b))}\downarrow & & \downarrow{\scriptstyle \sigma_{\boldsymbol{W}}(F_0(a),F_0(b))} \\ F_0(b \otimes^{\boldsymbol{V}} a) & \xrightarrow[\mu(b,a)]{} & F_0(b) \otimes^{\boldsymbol{W}} F_0(a) \end{array}$$

holds.

It is routine to check in the internal language that the data above gives 2-categories, so we can give the following definitions.

Definition 1.3.6 (category of internal monoidal categories)**.** Internal monoidal categories and monoidal functors (and monoidal transformations) in $\mathscr{E}$ form a (2-)category $\mathrm{MonCat}_{\mathscr{E}}$. Internal symmetric monoidal categories and symmetric monoidal functors (and monoidal transformations) in $\mathscr{E}$ form a (2-)category $\mathrm{SymMonCat}_{\mathscr{E}}$.

Remark 1.3.7. There is an underlying-monoidal-category (2-)functor

$$U_{\mathrm{MonCat}_{\mathscr{E}}} : \mathrm{SymMonCat}_{\mathscr{E}} \to \mathrm{MonCat}_{\mathscr{E}}$$

sending symmetric monoidal categories and functors (and natural transformations) to, respectively, their underlying monoidal categories and functors (and natural transformations).

There is an underlying-internal-category (2-)functor

$$U_{\mathrm{Cat}_{\mathscr{E}}} : \mathrm{MonCat}_{\mathscr{E}} \to \mathrm{Cat}_{\mathscr{E}}$$

sending monoidal categories and functors (and natural transformations) to, respectively, their underlying internal categories and functors (and natural transformations).

The analogous result to proposition 1.2.6 does not hold for monoidal categories, as $\mathrm{MonCat}_{\mathscr{E}}$ is not generally cartesian closed even when $\mathscr{E}$ is. However, we can still extend the ideas about enrichment to the case of monoidal categories, and we have the following proposition, analogous to proposition 1.2.3 and proved in the same way.

Proposition 1.3.8. *If $\mathscr{E}$ is cartesian closed, then* $\mathrm{MonCat}_{\mathscr{E}}$ *(*$\mathrm{SymMonCat}_{\mathscr{E}}$*) has an enrichment on* $\mathrm{Cat}_{\mathscr{E}}$ *(as a cartesian monoidal category) whose underlying category is* $\mathrm{MonCat}_{\mathscr{E}}$ *(*$\mathrm{SymMonCat}_{\mathscr{E}}$*) itself.*

Finally, in this dissertation we will also need a notion of monoidal closedness for internal symmetric monoidal categories.

Definition 1.3.9 (monoidal closed internal category)**.** An internal symmetric monoidal categoy $\boldsymbol{V}$ is *monoidal closed* if the functor in $\mathscr{E}/V_0$

$$\begin{array}{ccc} \boldsymbol{V} \times V_0 & \xrightarrow{(\pi_2, \otimes^{\boldsymbol{V}})} & (V_0)^*\boldsymbol{V} \\ & {\scriptstyle \pi_2} \searrow \quad \swarrow {\scriptstyle \pi_1} & \\ & V_0 & \end{array}$$

has a right adjoint $[-,-]\colon (V_0)^*\boldsymbol{V} \to \boldsymbol{V}$. In this case, the unit and counit will be internal natural transformations

$$\eta\colon \pi_1 \to [\pi_2, \pi_1 \otimes^{\boldsymbol{V}} \pi_2]\colon \boldsymbol{V} \times V_0 \to \boldsymbol{V} \times V_0$$

and

$$\epsilon\colon [\pi_1, \pi_2] \otimes^{\boldsymbol{V}} \pi_1 \to \pi_2\colon (V_0)^*\boldsymbol{V} \to (V_0)^*\boldsymbol{V},$$

the latter providing the evaluation $\mathrm{e}_{\boldsymbol{V}}$.

1.4. Indexed Categories

Indexed categories have been treated extensively in the literature, and the main ideas are long established. However, we shall refer to the recent exposition given in Shulman (2008, 2013), since these sources are also needed in regard to the notion of enriched indexed category.

To begin with, we shall state the definition of indexed category.

Definition 1.4.1 (indexed category)**.** An *$\mathscr{E}$-indexed category* is a pseudofunctor $\mathscr{C} : \mathscr{E}^{\mathbf{op}} \to \mathrm{Cat}$, where Cat is the 2-category of categories, functors, and natural transformations.

Consider the following, notable example, which will turn out to be useful later on.

Example 1.4.2. The self-indexing of $\mathscr{E}$ is the $\mathscr{E}$-indexed category whose fiber over an object X is the slice category $\mathscr{E}/X$ and where the reindexing along $f\colon X \to Y$ is given by pullback along f.

There is a strict relation between the theory of indexed categories and that of fibration, as established by the following, classic result.

Theorem 1.4.3. *An $\mathscr{E}$-indexed category $\mathscr{C}$ is, via the* Grothendieck construction*, equivalent to a cloven fibration $\int \mathscr{C} \to \mathscr{E}$.*

Now we want to extend the previous ideas to the context of monoidal categories.

We begin by giving the notion of indexed monoidal categories.

Definition 1.4.4 (indexed monoidal category)**.** An $\mathscr{E}$*-indexed monoidal category* is a pseudofunctor $\mathscr{W} : \mathscr{E}^{\mathbf{op}} \to$ MonCat, where MonCat is the 2-category of monoidal categories, strong monoidal functors, and monoidal transformations.

A suitable notion of monoidal fibration is required to establish a relation with indexed monoidal categories, so we recall that in the following definition.

Definition 1.4.5 (monoidal fibration)**.** Let $\mathscr{V}$ be a monoidal category. A monoidal fibration is a cloven fibration $\mathscr{V} \to \mathscr{E}$ such that the underlying functor is strict monoidal (with $\mathscr{E}$ cartesian monoidal) and the tensor product in $\mathscr{V}$ preserves the choice of cartesian arrows.

For a general monoidal base category $\mathscr{E}'$ the notions of $\mathscr{E}'$-indexed monoidal category and of monoidal fibration do not correspond under the Grothendieck construction. Indeed, if $\mathscr{W} : \mathscr{E}'^{\mathbf{op}} \to$ MonCat is an $\mathscr{E}'$-indexed monoidal category, then, in the cloven fibration $\int \mathscr{W} \to \mathscr{E}'$, it is evident that $\int \mathscr{W}$ has tensor products only for objects in the same fiber, and the result is still an object in that fiber. On the other hand, if $\mathscr{V} \to \mathscr{E}'$ is a monoidal fibration, and A and B are objects of $\mathscr{V}$ lying over the objects X and Y of $\mathscr{E}'$ respectively, then the tensor product $A \otimes^{\mathscr{V}} B$ lies over $X \otimes^{\mathscr{E}'} Y$. However, it is folklore that, in case the monodial structure on $\mathscr{E}'$ is given by the product, i.e. $\mathscr{E}'$ is cartesian monoidal, such as our ambient category $\mathscr{E}$ is, then there is a correspondence.

We now introduce some convenient notation for use in the next result.

Notation 1.4.6. Let $F\colon \mathscr{C} \to \mathscr{E}$ be a cloven fibration, and $f\colon X \to Y$ an arrow in $\mathscr{E}$. Then, call $f^* : F^{-1}(Y) \to F^{-1}(X)$ the lifting functor from the fiber along F over Y to the fiber over X.

We then state and give a sketch proof of the theorem analogous to theorem 1.4.3, in the context of monoidal categories.

Theorem 1.4.7. *An* $\mathscr{E}$*-indexed monoidal category* $\mathscr{W}$ *is, via the* Grothendieck construction*, equivalent to a monoidal fibration* $\int \mathscr{W} \to \mathscr{E}$*.*

Proof (sketch). Let the pseudo-functor $\mathscr{W} : \mathscr{E} \to$ MonCat be an index monoidal category and $\int \mathscr{W} \to \mathscr{E}$ the fibration yielded by the Grothendieck construction. Then, $\int \mathscr{W}$ has a monoidal structure. Indeed, $\mathbb{I}_{\int \mathscr{W}} = \mathbb{I}_{\mathscr{W}(1_{\mathscr{E}})}$. Let X and Y be objects of $\mathscr{E}$. Let A be an object of $\mathscr{W}(X)$ and B one of $\mathscr{W}(Y)$. Then,

$$A \otimes^{\int \mathscr{W}} B = (\pi_Y)^* A \otimes^{\mathscr{W}(X \times Y)} (\pi_X)^* B$$

where $\pi_Y\colon X \times Y \to X$ and $\pi_X\colon X \times Y \to Y$. With this monoidal structure on $\int \mathscr{W}$, the fibration $\int \mathscr{W} \to \mathscr{E}$ is strict monoidal.

Let $V\colon \mathscr{V} \to \mathscr{E}$ be a strict monoidal fibration and $\mathscr{W}: \mathscr{E} \to \mathrm{Cat}$ the pseudofunctor defined by the fibers of V. Then, $\mathscr{W}$ is an $\mathscr{E}$-indexed monoidal category, that is, it restricts to $\mathscr{W}: \mathscr{E} \to \mathrm{MonCat}$. Let X be an object of $\mathscr{E}$, and $!_X\colon X \to \mathbb{1}_{\mathscr{E}}$ the unique such arrow. Then, $\mathbb{I}_{\mathscr{W}(X)} = (!_X)^*\mathbb{I}_{\mathscr{V}}$. Let A and B be a objects of $\mathscr{W}(X)$. Then,

$$A \otimes^{\mathscr{W}(X)} B = \Delta^*(A \otimes^{\mathscr{V}} B)$$

where $\Delta\colon X \to X \times X$. □

Notice that the proof makes essential use of the assumption that $\mathscr{E}$ has (at least) finite products.

1.5. Externalization of Internal Categories

The next piece of background material that we present concerns the relationship between internal and indexed categories, and makes an essential use of the theory of indexed categories from section 1.4.

Let $\boldsymbol{A}$ be a category in $\mathscr{E}$ and X an object of $\mathscr{E}$. We regard an arrow $X \to A_0$ as representing an indexed family of objects of $\boldsymbol{A}$ over the indexing object X. Given two such indexed families $x_0, x_1 : X \to A_0$, consider the pullback

$$\begin{array}{ccc} (x_0,x_1)^*A_1 & \longrightarrow & A_1 \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{s}_{\boldsymbol{A}},\mathrm{t}_{\boldsymbol{A}})} \\ X & \xrightarrow[(x_0,x_1)]{} & A_0 \times A_0. \end{array}$$

Then, the sections of p represent indexed families of arrows of $\boldsymbol{A}$ with domain x_0 and codomain x_1. Given another family $x_2 : X \to A_0$, the composition in $\boldsymbol{A}$ restricts to an indexed composition

$$\circ_{\boldsymbol{A}|x_0,x_1,x_2} : (x_1,x_2)^*A_1 \times (x_0,x_1)^*A_1 \to (x_0,x_2)^*A_1$$

inducing a composition of indexed families of arrows: given two families of arrows $s_0 : X \to (x_0,x_1)^*A_1$ and $s_1 : X \to (x_1,x_2)^*A_1$, their composition is defined as

$$s_1 \circ_{[\boldsymbol{A}]^X} s_0 \coloneqq X \xrightarrow{(s_1,s_0)} (x_1,x_2)^*A_1 \times (x_0,x_1)^*A_1 \xrightarrow{\circ_{\boldsymbol{A}|x_0,x_1,x_2}} (x_0,x_2)^*A_1.$$

Moreover, a family of objects $x\colon X \to A_0$ induces a family of identity arrows $\mathrm{id}_{[\boldsymbol{A}]^X}(x) : X \to (x,x)^*A_1$. These data form the category $[\boldsymbol{A}]^X$ of indexed families of objects and morphisms of $\boldsymbol{A}$ over X.

Given a reindexing $u\colon X' \to X$, precomposition reindexes a family of objects $x\colon X \to A_0$ over X the family xu over X'; a family of arrows $s\colon X \to (x_0, x_1)^* A_1$ is reindexed to $u^* s\colon X' \to (ux_0, ux_1)^* A_1$ by pulling back the section s along (x_0, x_1). That gives a functor $u^*\colon [\boldsymbol{A}]^X \to [\boldsymbol{A}]^{X'}$.

The above discussion leads to the following result.

Proposition 1.5.1. *For* $\boldsymbol{A}$ *an internal category in* $\mathscr{E}$*, there is an indexed category* $[\boldsymbol{A}]$ *given by* $[\boldsymbol{A}](X) := [\boldsymbol{A}]^X$ *and* $[\boldsymbol{A}](u) := u^*$.

Remark 1.5.2. Notice that the indexed category arising from an internal one is given by a strict (rather than merely a pseudo) functor $\mathscr{E}^{\mathbf{op}} \to \mathrm{Cat}$. Then, evidently, internal categories yield rather special indexed categories, and not all indexed categories can be obtained from an internal one.

Moreover, the construction extends to the monoidal context, as shown in the following proposition.

Proposition 1.5.3. *Let* $\boldsymbol{V}$ *be an internal (symmetric) monoidal category in* $\mathscr{E}$*. Then,* $[\boldsymbol{V}]$ *is an indexed (symmetric) monoidal category on* $\mathscr{E}$.

Proof. Let X be an object in $\mathscr{E}$. Then, $[\boldsymbol{V}]^X$ has a monoidal structure induced by that of $\boldsymbol{V}$.

The monoidal product on objects is defined as

$$(X \xrightarrow{x} V_0) \otimes^{[\boldsymbol{V}]^X} (X \xrightarrow{x'} V_0) := X \xrightarrow{(x,x')} V_0 \times V_0 \xrightarrow{\otimes^{\boldsymbol{V}}} V_0.$$

Let

$$(X \xrightarrow{x_0} V_0) \xrightarrow{f\colon X \to (x_0,x_1)^* V_1} (X \xrightarrow{x_1} V_0)$$

and

$$(X \xrightarrow{x'_0} V_0) \xrightarrow{f'\colon X \to (x'_0,x'_1)^* V_1} (X \xrightarrow{x'_1} V_0)$$

be arrows of $[\boldsymbol{V}]^X$, and notice that $\otimes^{\boldsymbol{V}}$ restricts to

$$(x_0, x_1)^* V_1 \times (x'_0, x'_1)^* V_1 \to (x_0 \otimes^{[\boldsymbol{V}]^X} x'_0, x_1 \otimes^{[\boldsymbol{V}]^X} x'_1)^* V_1.$$

Then, the monoidal product of arrows $f \otimes^{[\boldsymbol{V}]^X} f'$ is given by the arrow

$$X \xrightarrow{(f,f')} (x_0, x_1)^* V_1 \times (x'_0, x'_1)^* V_1 \xrightarrow{\otimes^{\boldsymbol{V}}} (x_0 \otimes^{\boldsymbol{V}} x'_0, x_1 \otimes^{\boldsymbol{V}} x'_1)^* V_1.$$

The monoidal unit $\mathbb{I}_{[\boldsymbol{V}]^X}$ is defined by the constant family indexed by X on the monoidal unit of $\boldsymbol{V}$.

The structural isomorphisms, associator and unitors (and, in case $\boldsymbol{V}$ is symmetric, the symmetry), are defined point-wise.

Moreover, reindexing preserves the monoidal product. □

Remark 1.5.4. The strictness of the monoidal products of the fibers of the indexed monoidal category $[\boldsymbol{V}]$ obtained from an internal monoidal category $\boldsymbol{V}$ is the same as that of the original monoidal product of $\boldsymbol{V}$, so it will generally not be strict monoidal. Still, the reindexing functors for $[\boldsymbol{V}]$ strictly preserve the monoidal structure, regardless of how strict the monoidal product of $\boldsymbol{V}$ is. That means that the (actually strict) functor $[\boldsymbol{V}] : \mathscr{E} \to \mathrm{MonCat}$ factorizes through the 2-category of (non-necessarily-strict) monoidal categories, strict monoidal functors and monoidal natural transformations. Such a category is quite uncommon, since normally there is little use for strict monoidal functors, especially between non-strict monoidal categories. Nonetheless, this shows that the indexed monoidal categories arising from internal monoidal categories are rather special ones.

Remark 1.5.5. The fiber $[\boldsymbol{A}]^X$ over an object X is enriched over $\mathscr{E}/X$:

Homset: $\mathrm{Hom}_{[\boldsymbol{A}]^X}(x_0, x_1) := (x_0, x_1)^* A_1 \xrightarrow{p} X.$

Composition: $\circ_{[\boldsymbol{A}]^X}(x_0, x_1, x_2) := \circ_{\boldsymbol{A}|x_0,x_1,x_2}.$

Identity: $\mathrm{id}_{[\boldsymbol{A}]^X}(x).$

Reindexing is compatible with this structure, in that the reindexing of the homset is the same as the homset of the reindexing. More explicitly, given a reindexing $u : X' \to X$, by pullback-pasting we have that

$$u^*(x_0, x_1)^* A_1 \cong (x_0 u, x_1 u)^* A_1.$$

In fact, the reindexing functor $u^* : [\boldsymbol{A}]^X \to [\boldsymbol{A}]^{X'}$ is a fully-faithful functor of enriched categories.

Then, $[\boldsymbol{A}]$ is an indexed enriched category over the self-indexing of $\mathscr{E}$ (see example 1.4.2). Equivalently, it is the locally internal category over $\mathscr{E}$ whose underlying indexed category is (up to natural isomorphism) $[\boldsymbol{A}]$ as an indexed category (Johnstone 2002; Shulman 2013).

As shown in section 1.4, indexed (monoidal) categories are equivalent to cloven (monoidal) fibration.

Definition 1.5.6 (externalization)**.** The *externalization* of an internal category $\boldsymbol{A}$ is the total category for the fibration associated to the indexed category $[\boldsymbol{A}]$. With abuse of notation, we denote the externalization of $\boldsymbol{A}$ with $[\boldsymbol{A}]$, and context will usually suffice to distinguish between the use of the notation as a fibration or as an indexed category.

Explicitly, the externalization of $\boldsymbol{A}$ is the category given by the data

Objects: families of objects of $\boldsymbol{A}$ indexed over objects of $\mathscr{E}$, that is, arrows $X \to A_0$ with X in $\mathscr{E}$.

Morphisms: an arrow $(x\colon X \to A_0) \to (y\colon Y \to A_0)$ is given by a reindexing $u\colon X \to Y$ and a family of arrows $x \to yu$, that is, a section of the projection $p\colon (x,yu)^*A_1 \to X$.

Composition: the composition is given by

$$(X \xrightarrow{x} A_0) \xrightarrow{(X\xrightarrow{u}Y, X\xrightarrow{f}(x,yu)^*A_1)} (Y \xrightarrow{y} A_0) \xrightarrow{(Y\xrightarrow{v}Z, Y\xrightarrow{g}(y,zv)^*A_1)} (Z \xrightarrow{z} A_0)$$
$$:= (X \xrightarrow{x} A_0) \xrightarrow{(X\xrightarrow{u}Y\xrightarrow{v}Z,\, u^*g\circ_{[A]^X} f\colon X\to(x,zvu)^*A_1)} (Z \xrightarrow{z} A_0).$$

Identity: the family of identity arrows.

Let $F\colon \boldsymbol{A} \to \boldsymbol{B}$ be a functor of internal categories. Then, there is a functor of fibered categories $[F]\colon [\boldsymbol{A}] \to [\boldsymbol{B}]$ defined on objects as

$$[F]\,(X \xrightarrow{x} A_0) := X \xrightarrow{x} A_0 \xrightarrow{F_0} B_0$$

and on morphisms as

$$[F]\Big((X \xrightarrow{x} A_0) \xrightarrow{(X\xrightarrow{u}Y, X\xrightarrow{f}(x,yu)^*A_1)} (Y \xrightarrow{y} A_0)\Big)$$
$$:= (X \xrightarrow{x} A_0 \xrightarrow{F_0} B_0) \xrightarrow{\left(X\xrightarrow{u}Y, X\xrightarrow{f}(x,yu)^*A_1\xrightarrow{F_1}(F_0x,F_0yu)^*B_1\right)} (Y \xrightarrow{y} A_0 \xrightarrow{F_0} B_0)$$

which restricts to a functor on the fibers $[F]^X\colon [\boldsymbol{A}]^X \to [\boldsymbol{B}]^X$.

Let $\alpha\colon F \to G\colon \boldsymbol{A} \to \boldsymbol{B}$ be a natural transformation.Then, there is a natural transformation of fibered categories $[\alpha]\colon [F] \to [G]\colon [\boldsymbol{A}] \to [\boldsymbol{B}]$, defined as

$$[\alpha]\,(X \xrightarrow{x} A_0) := (X \xrightarrow{x} A_0 \xrightarrow{F_0} B_0) \xrightarrow{(\mathrm{id}_X, X\xrightarrow{x}A_0\xrightarrow{\alpha}(F_0x,G_0x)^*B_1)} (X \xrightarrow{x} A_0 \xrightarrow{G_0} B_0)$$

which restricts to a natural transformation on the fibers $[\alpha]^X\colon [F]^X \to [G]^X$.

Remark 1.5.7. If $\boldsymbol{V}$ is a (symmetric) monoidal category, then its externalization $[\boldsymbol{V}]$ has a (symmetric) monoidal structure induced by that of the indexed (symmetric) monoidal category $[\boldsymbol{V}]$. The monoidal product on objects is given by

$$(X \xrightarrow{x} V_0) \otimes^{[\boldsymbol{V}]} (Y \xrightarrow{y} V_0) := X \times Y \xrightarrow{x\times y} V_0 \times V_0 \xrightarrow{\otimes^{\boldsymbol{V}}} V_0.$$

The monoidal product on arrows

$$(X \xrightarrow{x} V_0) \xrightarrow{(X\xrightarrow{u}Y, X\xrightarrow{f}(x,yu)^*V_1)} (Y \xrightarrow{y} V_0)$$

and

$$(Z \xrightarrow{z} V_0) \xrightarrow{(Z \xrightarrow{v} W, Z \xrightarrow{g} (z,wv)^* V_1)} (W \xrightarrow{w} V_0)$$

is indexed by $X \times Z \xrightarrow{u \times v} Y \times W$ and given by

$$X \times Z \xrightarrow{f \otimes^{\boldsymbol{V}} g} (x \otimes^{\boldsymbol{V}} z, yu \otimes^{\boldsymbol{V}} wv)^* V_1.$$

The monoidal unit is $\mathbb{I}_{\boldsymbol{V}} \colon \mathbb{1}_{\mathscr{E}} \to V_0$. Finally, the structural isomorphisms are induced by those of $\boldsymbol{V}$.

1.6. Indexed Enriched Categories

Now that we have—at least for $\mathscr{E}$ with products—a notion of indexed monoidal category, we give, as the last piece of background material, the notion of enrichment in such a category. This comes in two versions: a general indexed version and a version which we call “small” (Shulman 2013). The latter is in a sense a hybrid notion, having an internal as well as an indexed aspect. The force of the definition will be clear in chapter 3, after we have (in the next section) treated the relationship between internal and indexed categories.

In this section, let $\mathscr{W}$ be an $\mathscr{E}$-indexed monoidal category.

Let’s introduce a convenient notation for use in the next definition.

Notation 1.6.1. If $f \colon B \to A$ is a morphism in $\mathscr{E}$ and H is an object in $\mathscr{W}(A)$, let’s write $H(f)$ for the object $\mathscr{W}(f)(H)$ of $\mathscr{W}(B)$.

First we give an outline of the definitions of the notions of small $\mathscr{W}$-category, of functors between such categories, and of natural transformations between such functors. For brevity we will omit some diagrammatic axioms. We refer to Shulman (ibid.) for those.

Definition 1.6.2 (small $\mathscr{W}$-category)**.** A *small $\mathscr{W}$-category* $\mathscr{A}$ consists of

- an object A of $\mathscr{E}$;
- an object $\mathrm{Hom}_{\mathscr{A}}$ of $\mathscr{W}(A \times A)$;
- a morphism $\mathrm{id}_{\mathscr{A}} \colon \mathbb{I}_{\mathscr{W}(A)} \to \mathrm{Hom}_{\mathscr{A}}(\Delta)$ where $\Delta \colon A \to A \times A$ is the diagonal;
- A morphism of $\mathscr{W}(A \times A \times A)$

  $$\circ_{\mathscr{A}} \colon \mathrm{Hom}_{\mathscr{A}}(\pi_2, \pi_3) \otimes^{\mathscr{W}(A \times A \times A)} \mathrm{Hom}_{\mathscr{A}}(\pi_1, \pi_2) \to \mathrm{Hom}_{\mathscr{A}}(\pi_1, \pi_2)$$

  where $\pi_1, \pi_2, \pi_3 \colon A \times A \times A \to A$ are projections.

satisfying the associativity and unitarity axioms (ibid.).

Definition 1.6.3 (functor of small $\mathscr{W}$-categories)**.** A *functor of small $\mathscr{W}$-categories* $F\colon \mathscr{A} \to \mathscr{B}$ consists of

- a morphism $F_0\colon A \to B$ of $\mathscr{E}$;
- a morphism $F_1\colon \mathrm{Hom}_{\mathscr{A}} \to \mathrm{Hom}_{\mathscr{B}}(F_0, F_0)$ of $\mathscr{W}(A \times A)$;

satisfying the functoriality axioms (ibid.).

Definition 1.6.4 (natural transformation of small $\mathscr{W}$-categories)**.** A *natural transformation of small $\mathscr{W}$-categories* $\alpha\colon F \to G\colon \mathscr{A} \to \mathscr{B}$ consists of a morphism

$$\alpha\colon \mathbb{I}_{\mathscr{W}(A)} \to \mathrm{Hom}_{\mathscr{B}}((F_0, G_0)\Delta)$$

satisfying a naturality axiom.

Let's introduce some convenient notations for use in the next definition.

Notation 1.6.5. Let $\mathscr{V}$ and $\mathscr{V}'$ be monoidal categories. If $F\colon \mathscr{V} \to \mathscr{V}'$ is a lax monoidal functor and $\mathscr{A}$ is a $\mathscr{V}$-enriched category, then there is an induced $\mathscr{V}'$-enriched category $F_\bullet(\mathscr{A})$. Moreover, if F is a closed monoidal functor (with $\mathscr{V}$ and $\mathscr{V}'$ closed monoidal), then in particular there is a fully faithful $\mathscr{V}'$-functor $F_\bullet\colon F_\bullet(\mathscr{V}) \to \mathscr{V}'$.

Now we present the hopefully more familiar definition of a general indexed category enriched in an indexed monoidal category (ibid.).

Definition 1.6.6 (indexed $\mathscr{W}$-category)**.** An *indexed $\mathscr{W}$-category* $\mathscr{B}$ consists of

- for each X object of $\mathscr{E}$, a $\mathscr{W}(X)$-category $\mathscr{B}^X$;
- for each $f\colon X \to Y$ in $\mathscr{E}$, a fully faithful $\mathscr{W}(X)$-functor $f^*\colon (f^*)_\bullet(\mathscr{B}^Y) \to \mathscr{B}^X$;
- for each $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathscr{E}$, a $\mathscr{W}(X)$-natural isomorphism $(gf)^* \cong f \circ (f^*)_\bullet(g)$ (where we implicitly identify $(f^*)_\bullet(g^*)_\bullet\mathscr{B}^Z$ with $(gf^*)_\bullet\mathscr{B}^Z$ in the domains of these functors);
- for each X object of $\mathscr{E}$, a $\mathscr{W}(X)$-natural isomorphism $(\mathrm{id}_X)^* \cong \mathrm{id}_{\mathscr{B}^X}$;

satisfying, for every $f\colon X \to Y$, $g\colon Y \to Z$ and $h\colon Z \to K$ in $\mathscr{E}$, the axioms for associativity and unitarity, analogous to those for ordinary indexed categories, by making the following diagrams of isomorphisms commute.

$$\begin{array}{ccc}
(hgf)^* & \longrightarrow & f^* \circ (f^*)_\bullet((hg)^*) \\
\downarrow & & \downarrow \\
(gf)^* \circ ((gf)^*)_\bullet(h^*) & & f^* \circ (f^*)_\bullet(g^* \circ (g^*)_\bullet(h^*)) \\
\downarrow & & \downarrow \\
f^* \circ (f^*)_\bullet(g^*) \circ ((gf)^*)_\bullet(h^*) & \longrightarrow & f^* \circ (f^*)_\bullet(g^*) \circ (f^*)_\bullet(g^*)_\bullet(h^*)
\end{array}$$

$$
\begin{array}{ccc}
 & (\mathrm{id}_{\mathscr{X}})^* \circ ((\mathrm{id}_{\mathscr{X}})^*)_\bullet(f^*) & \\
\nearrow & & \searrow \\
(f\mathrm{id}_{\mathscr{X}})^* & = & f^*
\end{array}
\qquad
\begin{array}{ccc}
 & f^* \circ (f^*)_\bullet((\mathrm{id}_{\mathscr{Y}})^*) & \\
\nearrow & & \searrow \\
(\mathrm{id}_{\mathscr{Y}}f)^* & = & f^*
\end{array}
$$

Definition 1.6.7 (functor of indexed $\mathscr{W}$-categories)**.** An *indexed $\mathscr{W}$-functor* $\mathscr{F} : \mathscr{B} \to \mathscr{B}'$ consists, for every object X of $\mathscr{E}$, of a $\mathscr{W}(X)$-enriched functor $\mathscr{F}^X : \mathscr{B}^X \to \mathscr{B}'^X$ together with, for every $f : X \to Y$, an isomorphism $\mathscr{F}^X \circ f^* \cong f^* \circ (f^*)_\bullet(\mathscr{F}^Y)$. Such data have to satisfy the functoriality axioms by making the following diagrams of isomorphisms commute, for every $f : X \to Y$ and $g : Y \to Z$ in $\mathscr{E}$.

$$
\begin{array}{ccc}
\mathscr{F}^I \circ (gf)^* & \longrightarrow & \mathscr{F}^X \circ f^* \circ (f^*)_\bullet(g^*) \\
\downarrow & & \downarrow \\
(gf)^* \circ ((gf)^*)_\bullet(\mathscr{F}^Z) & & f^* \circ (f^*)_\bullet(\mathscr{F}^Y) \circ (f^*)_\bullet(g^*) \\
\downarrow & & \downarrow \\
f^* \circ (f^*)_\bullet(g^*) \circ (f^*)_\bullet((g^*)_\bullet(\mathscr{F}^Z)) & & f^* \circ (f^*)_\bullet(\mathscr{F}^Y \circ g^*) \\
 & \searrow & \downarrow \\
 & & f^* \circ (f^*)_\bullet(g^* \circ (g^*)_\bullet(\mathscr{F}^Z))
\end{array}
$$

$$
\begin{array}{ccc}
\mathscr{F}^X \circ (\mathrm{id}_{\mathscr{X}})^* & \longrightarrow & (\mathrm{id}_{\mathscr{X}})^* \circ ((\mathrm{id}_{\mathscr{X}})^*)_\bullet(\mathscr{F}^X) \\
 & \searrow & \downarrow \\
 & & \mathscr{F}^X
\end{array}
$$

Definition 1.6.8 (natural transformation of indexed $\mathscr{W}$-categories)**.** An indexed $\mathscr{W}$-natural transformation $\alpha : \mathscr{F} \to \mathscr{G} : \mathscr{B} \to \mathscr{B}'$ consists, for every object X of $\mathscr{E}$, of a $\mathscr{W}(X)$-natural transformation $\alpha^X : \mathscr{F}^X \to \mathscr{G}^X : \mathscr{B}^X \to \mathscr{B}'^X$, satisfying naturality axioms by making the following diagram commute, for every $f : X \to Y$.

$$
\begin{array}{ccc}
\mathscr{F}^X \circ f^* & \longrightarrow & f^* \circ (f^*)_\bullet(\mathscr{F}^Y) \\
\downarrow & & \downarrow \\
\mathscr{F}^X \circ f^* & \longrightarrow & f^* \circ (f^*)_\bullet(\mathscr{F}^Y)
\end{array}
$$

With the data thus defined (plus the obvious notions of compositions and identities) we can define a 2-category of indexed enriched categories.

Definition 1.6.9 (category of indexed $\mathscr{W}$-categories)**.** We denote with $\mathscr{W}\mathrm{ICat}_{\mathscr{E}}$ the 2-category of indexed $\mathscr{W}$-categories, their functors and the natural transformations between them.

By abuse of notation, we shall denote with $\mathscr{W}\mathrm{ICat}_{\mathscr{E}}$ also the mere 1-category of indexed $\mathscr{W}$-categories and their functors. Usually, the context is sufficient to distinguish when the notation is being used referring to the 1-category or the 2-category.

Finally, we briefly mention how to change the indexed monoidal category in which we are enriching. Such operation, fundamental for every flavor of enriched category theory, is called *change of cosmos*. For more details, we refer to Shulman 2013.

First, we define the transformation between indexed monoidal categories.

Definition 1.6.10 (lax monoidal morphism)**.** If $\mathscr{W}$ is an $\mathscr{E}$-indexed monoidal category, and $\mathscr{W}'$ is an $\mathscr{E}'$-indexed monoidal category, then a *lax monoidal morphism* $\phi\colon \mathscr{W} \to \mathscr{W}'$ is a commutative square

$$\begin{array}{ccc} \int \mathscr{W} & \xrightarrow{\phi} & \int \mathscr{W}' \\ \downarrow & & \downarrow \\ \mathscr{E} & \xrightarrow[\phi]{} & \mathscr{E}' \end{array}$$

such that $\phi\colon \mathscr{E} \to \mathscr{E}'$ preserves finite products and $\phi\colon \int \mathscr{W} \to \int \mathscr{W}'$ is lax monoidal and preserves cartesian arrows.

If $\phi\colon \mathscr{W} \to \mathscr{W}'$ is a lax monoidal morphism, there are induced operations $\phi_\bullet$ from small and indexed $\mathscr{W}$-categories to the corresponding sort of $\mathscr{W}'$-categories, and similarly for functors, and natural transformations.

2. Completeness of Internal Categories

The aim of this dissertation is to develop a form of enriched category theory internal to a suitable ambient category $\mathscr{E}$. What is perhaps unexpected in our theory is that we seek enriching categories within $\mathscr{E}$ that are complete. Of course, when $\mathscr{E}$ is the category of sets, there are relatively few small complete categories in which we can usefully enrich: essentially, they are all quantales. However, there are small complete internal categories within categories provided by realizability, such as the category of modest sets within the category of assemblies (Hyland 1988).

As in the usual theory of enriched categories (Kelly 1982), two kinds of completeness play a role in the theory: first, there is the completeness of the enriching category; secondly, the completeness of the enriched categories. The main focus of the dissertation is on the second notion, which is part of the theory of internal enriched categories. However, to set the stage for that, we have to consider the first notion. So, we need some preliminaries on complete internal categories.

Sadly, the existing literature on internal complete categories is sparse. The paper Hyland 1988, which presents a leading example, gives a sketch of how the theory might develop. The more or less contemporaneous paper Hyland, Robinson, and Rosolini 1990 discusses the definitions in light of a perspective suggested by Freyd, but its main focus is on weak notions of completeness which play no role in this dissertation. The basic idea is that one has a weak limit when it is internally true that there exists a limit cone for the given diagram, while the limit is strong when the choice of limit cone is given as part of the structure. In this dissertation, though, we are not assuming that the internal logic of our ambient category features existential quantification, so the notion of weak limit is not even applicable.

As completeness of the enriching category is fundamental to the theory we develop, we have decided to adopt and clarify the definition of strong completeness given in ibid. Evidently, the use of the internal logic in ibid. is intended to ensure that the property of being a limit cone is stable under pullback. Although that makes the definition of completeness very concise, it also obfuscates its content. Instead, we will present the notion in a more explicit way, by avoiding the use of the internal language.

In this chapter $\mathscr{E}$ shall be a finitely complete, locally cartesian closed category.

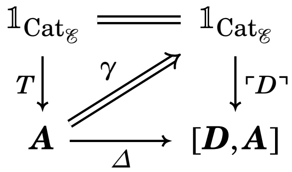
As a consequence, the internal language is now a dependent type theory (Seely 1984). In particular, these assumptions are necessary to ensure that the notions we are going to define will be stable under pullback, which in turn is necessary if we want them to be meaningful in the internal language.

2.1. Diagrams and Cones

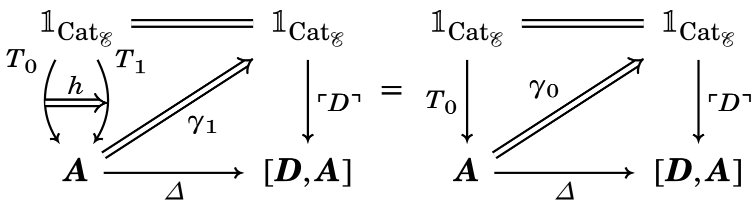
As obvious as they might seem, we shall spell out the definitions of diagrams and cones over diagrams in the internal context.

Definition 2.1.1 (diagram)**.** Let $\boldsymbol{A}$ and $\boldsymbol{D}$ be internal categories in $\mathscr{E}$. A *diagram* of shape $\boldsymbol{D}$ in $\boldsymbol{A}$ is an internal functor $D\colon \boldsymbol{D} \to \boldsymbol{A}$.

Definition 2.1.2 (cone)**.** A *cone* over a diagram $D\colon \boldsymbol{D} \to \boldsymbol{A}$ with *vertex* $T\colon \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}} \to \boldsymbol{A}$ is a natural transformation $\gamma\colon \Delta T \to \ulcorner D \urcorner\colon \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}} \to [\boldsymbol{D},\boldsymbol{A}]$, where $\Delta\colon \boldsymbol{A} \to [\boldsymbol{D},\boldsymbol{A}]$ is the diagonal functor. The situation is shown in the following diagram.



Let $\gamma_i\colon T_i\Delta \to \ulcorner D \urcorner\colon \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}} \to [\boldsymbol{D},\boldsymbol{A}]$ for $i = 0,1$ be two cones over the diagram $D\colon \boldsymbol{D} \to \boldsymbol{A}$ with tips $T_i\colon \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}} \to \boldsymbol{A}$ respectively. A morphism of cones $h\colon \gamma_0 \to \gamma_1$ is given by a natural transformation $h\colon T_0 \to T_1$ such that $\gamma_1 \circ (\Delta * h) = \gamma_0$, as represented in the following diagram.



Dually, we can give the definition of cocone (which we sum up, but all the considerations from the definition of cones apply).

Definition 2.1.3 (cocone)**.** A *cocone* over a diagram $D\colon \boldsymbol{D} \to \boldsymbol{A}$ with *vertex* $T\colon \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}} \to \boldsymbol{A}$ is a natural transformation $\gamma\colon \ulcorner D \urcorner \to \Delta T\colon \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}} \to [\boldsymbol{D},\boldsymbol{A}]$.

Let I be an object of $\mathscr{E}$, to be intended as an indexing object. We know from remark 1.1.4 that there are two adjunctions, $I_! \dashv I^*\colon \mathscr{E} \to \mathscr{E}/I$ and $I_! \dashv I^*\colon \mathrm{Cat}_{\mathscr{E}} \to \mathrm{Cat}_{\mathscr{E}/I}$. Moreover, the canonical morphism $i\colon I^*[\boldsymbol{D},\boldsymbol{A}] \to [I^*\boldsymbol{D},I^*\boldsymbol{A}]$ is an isomorphism because $\mathscr{E}$ is locally cartesian closed. Then, given a diagram $D\colon \boldsymbol{D} \to \boldsymbol{A}$, we get a diagram $I^*D\colon I^*\boldsymbol{D} \to I^*\boldsymbol{A}$, and, given a cone

$$\begin{array}{ccc} \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}} & = & \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}} \\ {\scriptstyle T}\big\downarrow & \overset{\gamma}{\Nearrow} & \big\downarrow{\scriptstyle \ulcorner D \urcorner} \\ \boldsymbol{A} & \xrightarrow[\Delta]{} & [\boldsymbol{D},\boldsymbol{A}] \end{array}$$

over D, we get a cone

$$\begin{array}{ccccc} \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}/I}} & = & \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}/I}} & \xrightarrow{\ulcorner I^*D \urcorner} & \\ {\scriptstyle I^*T}\big\downarrow & \overset{i*I^*\gamma}{\Nearrow} & \big\downarrow{\scriptstyle I^*\ulcorner D \urcorner} & & \\ I^*\boldsymbol{A} & \xrightarrow[I^*\Delta]{} & I^*[\boldsymbol{D},\boldsymbol{A}] & \xrightarrow[i]{} & [I^*\boldsymbol{D},I^*\boldsymbol{A}] \end{array}$$
$$I^*\boldsymbol{A} \xrightarrow[\Delta]{} [I^*\boldsymbol{D},I^*\boldsymbol{A}]$$

over I^*D. Notice that I^* preserves the terminal object, $i \circ I^*\ulcorner D \urcorner = \ulcorner I^*D \urcorner$ and $i \circ I^*\Delta = \Delta$. Then, we have a functor $I^*\colon \mathrm{Cone}_D \to \mathrm{Cone}_{I^*D}$.

It is worth noting that the previous notions are all internal, in the sense that they can be expressed by the internal language of $\mathscr{E}$. The internal category of cones over a diagram $D\colon \boldsymbol{D} \to \boldsymbol{A}$ is given by the lax pullback

$$\begin{array}{ccc} \mathrm{Cone}_D & \longrightarrow & \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}} \\ \big\downarrow & \lrcorner\ \Nearrow & \big\downarrow{\scriptstyle \ulcorner D \urcorner} \\ \boldsymbol{A} & \xrightarrow[\Delta]{} & [\boldsymbol{D},\boldsymbol{A}] \end{array} .$$

Then, the category of points of Cone_D is the external category of cones over D and their transformtions; in particular, any cone (T,γ) over D corresponds uniquely to a global sections $(T,\gamma)\colon \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}} \to \mathrm{Cone}_D$. Moreover, if I is an object of $\mathscr{E}$, an I-indexed family of cones over D is given by a 2-cell

$$\begin{array}{ccc} \boldsymbol{I} & \longrightarrow & \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}} \\ {\scriptstyle T}\big\downarrow & \overset{\gamma}{\Nearrow} & \big\downarrow{\scriptstyle \ulcorner D \urcorner} \\ \boldsymbol{A} & \xrightarrow[\Delta]{} & [\boldsymbol{D},\boldsymbol{A}] \end{array} \tag{2.1}$$

where $\boldsymbol{I}$ is $I_!I^*\mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}}$ (intuitively, the discrete category over the object I). Analogously, the category of all cones of shape $\boldsymbol{D}$ over $\boldsymbol{A}$ is given by the lax pullback

$$\begin{array}{ccc} \mathrm{Cone}_{\boldsymbol{D}} & \longrightarrow & [\boldsymbol{D},\boldsymbol{A}] \\ \big\downarrow & \lrcorner\ \Nearrow & \big\| \\ \boldsymbol{A} & \xrightarrow[\Delta]{} & [\boldsymbol{D},\boldsymbol{A}] \end{array} .$$

Dually, the internal category of cocones over a diagram $D\colon \boldsymbol{D} \to \boldsymbol{A}$ is given by the lax pullback

$$\begin{array}{ccc} \mathrm{CoCone}_D & \longrightarrow & \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}} \\ \downarrow & \swarrow\!\!\!\!\swarrow & \downarrow{\scriptstyle \ulcorner D \urcorner} \\ \boldsymbol{A} & \xrightarrow[\Delta]{} & [\boldsymbol{D}, \boldsymbol{A}] \end{array} .$$

2.2. Limits

As usual, we call universal a cone that is a terminal object in the category of cones over the given diagrams. Though, we have to be careful about which category of cones we are talking about, and what it means to be terminal in it. Since our definition aims to be internal in essence, what we want is a terminal object, as defined by the internal language, in the internal category of cones. That is more than a terminal object in the external category of cones: there, stability under pullback has to be enforced, thus realizing the intuition that, if $\lim_D$ is a limit for a diagram $D\colon \boldsymbol{D} \to \boldsymbol{A}$ and I is an object of $\mathscr{E}$, then $I^* \lim_D$ should be a limit for I^*D.

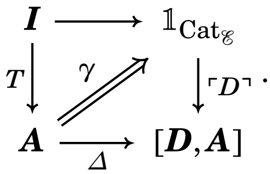
Definition 2.2.1 (universal cone/limit)**.** A cone (T, γ) over a diagram $D\colon \boldsymbol{D} \to \boldsymbol{A}$ is *universal* if it is internally a terminal object for Cone_D; that means that, for every object I of $\mathscr{E}$, the cone $I^*(T, \gamma)$ over I^*D is a terminal object in the external category of cones over I^*D. The vertex of a universal cone over D is also called the *limit* of D.

Dually, we give the definitions of universal cocone and colimit.

Definition 2.2.2 (universal cocone/colimit)**.** A cocone (T, γ) over a diagram $D\colon \boldsymbol{D} \to \boldsymbol{A}$ is *universal* if it is internally an initial object for CoCone_D; that means that, for every object I of $\mathscr{E}$, the cocone $I^*(T, \gamma)$ over I^*D is an initial object in the external category of cocones over I^*D. The vertex of a universal cocone over D is also called the *colimit* of D.

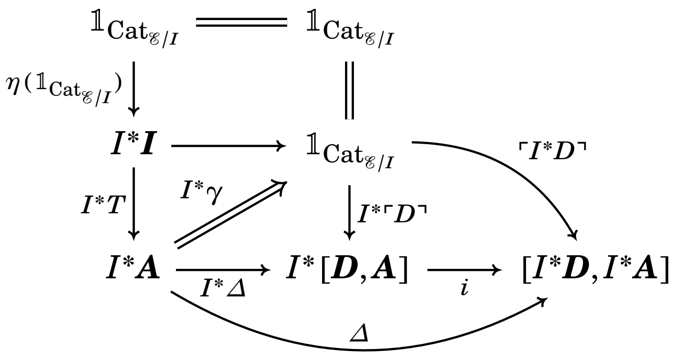
It is inconvenient that stability needs to be imposed, but such is the price of doing everything concretely, without relying on the internal logic. However, once the definition has been stabilized, we get all the desired consequences of an internal definition. For example, the universal property does not merely hold for cones, but for indexed families of cones as well: that is to say the universal quantifier for all cones has the expected property.

Indeed, let $D\colon \boldsymbol{D} \to \boldsymbol{A}$ be a diagram admitting a limit $(\lim_D, \pi)$. Then, if I is an object of $\mathscr{E}$, consider an I-indexed family of cones

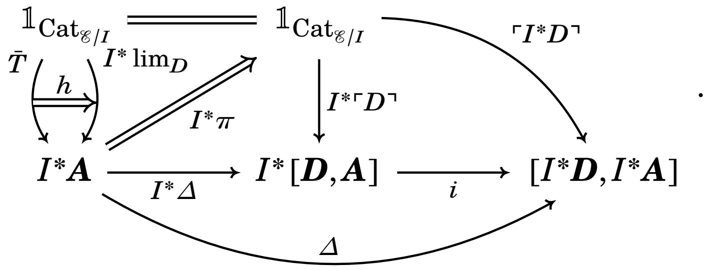


(2.2)

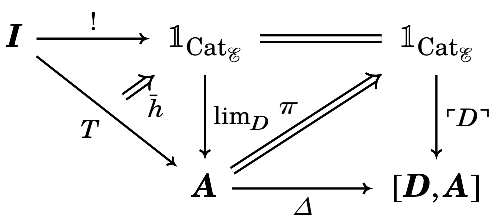
By applying I^* and the properties of the adjunction, we get



which is a cone over I^*D. Let's call $\bar{T} = I^*T \circ \eta(\mathbb{1})$ and $\bar{\gamma} = i * \gamma * \eta(\mathbb{1})$. Then, by the universal property, there exists a unique $h\colon \bar{T} \to I^* \lim_D$ such that the above diagram is equal to



On both squares, cancel the whiskering with i, which is possible because i is an isomorphism as $\mathscr{E}$ is locally cartesian closed. Apply $I_!$ to the two squares and then compose with the counit ϵ; let $\bar{h}$ be $\epsilon(\boldsymbol{A}) * I_! h$. Then, we get that the square (2.2) is equal to



thus showing that even families of cones factorize through the limit.

Moreover, any cone (T,γ) over D corresponds uniquely to a global sections $(T,\gamma)\colon \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}} \to \mathrm{Cone}_D$, and a cone is a limit if it is understood in the internal logic as a terminal object for Cone_D. Again, it is essential that the notion is stable under pullback, or it could not be expressed in the internal language. Then, that of limit is a notion in the internal logic, and that allows us to use the internal logic to argue about it.

2.3. Completeness

While completeness generally means to “just” have limits for all diagrams, in the internal setting the treatment of the notion requires special care. The category of diagrams of shape $\boldsymbol{D}$ over $\boldsymbol{A}$ is evidently the functor category $[\boldsymbol{D},\boldsymbol{A}]$, and the notion of functor category is stable under pullback. Then, we may wish to say, in the internal language, that every such diagram has a limit cone. However, limits are determined only up to (unique) isomorphism, and we have no non-unique existential quantifier in the logic of a cartesian closed category (this issue could be partially circumvented by treating completeness as a structure rather than a property, and requiring a functorial choice of limits on the internal category of diagrams). Even more limiting, while we have a category of diagrams of a certain shape, there is no category of diagrams of any shape. Thus, completeness cannot be a purely internal notion.

From an external perspective, it would look natural to call an (internal) category complete if every diagram over it admits a limit. Again, it is necessary to reconsider this intuition as the resulting notion would not be stable under pullback. Indeed, even if every diagram over a category $\boldsymbol{A}$ admits a limit, it is still possible that not every diagram over $I^*\boldsymbol{A}$ does, for some indexing object I.

Definition 2.3.1 (completeness)**.** An internal category $\boldsymbol{A}$ in $\mathscr{E}$ is *(strongly) complete* if, for every object I of $\mathscr{E}$, every diagram over $I^*\boldsymbol{A}$ admits a limit.

At this point, it is necessary to find out whether such definition of completeness admits non-trivial examples. The first place to look into is the category of sets.

Example 2.3.2. The internal complete categories in Set coincide with (the categories associate to) complete lattices.

Thus, the internal complete categories in Set are precisely the small complete categories. That means that the notion of internal completeness is compatible with the standard one, as it should be.

While the examples in the category of sets are legitimate, they are a bit disappointing: there are no small complete categories other than lattices. Fortunately, and somewhat surprisingly, there are remarkable examples in other ambient categories. Maybe the most famous of these is the internal category of the Modest Sets in the Category of Assemblies (Hyland 1988; Hyland, Robinson, and Rosolini 1990). Such example is presented in section 5.2.

The dual notion of cocompleteness is as one would expect.

Definition 2.3.3 (cocompleteness)**.** An internal category $\boldsymbol{A}$ in $\mathscr{E}$ is *(strongly) cocomplete* if, for every object I of $\mathscr{E}$, every diagram over $I^*\boldsymbol{A}$ admits a colimit.

A complete category has all indexed limits, and the object of diagrams of a certain shape over it is a suitable indexing object. Following this intuition, we can produce a functorial choice of limits in the form of a right adjoint to the diagonal functor. This shows some interesting facts. Firstly, that there exists an internal, uniform (as in functorial) choice of limits, as opposed to the external choice required by the definition of internal completeness. Secondly, that our definition of completeness is consistent with the well-known characterization of completeness via adjunctions. Finally, that the limit-choice functor comes for free from the definition of completeness, whereas, for standard categories, that would generally require the use of the axiom of choice. It is worth remarking that the choice functor can only be defined over the object of diagrams of a given shape, as there is no internal object of all diagrams of any shape.

Proposition 2.3.4. *If $\boldsymbol{A}$ is a complete internal category in $\mathscr{E}$, then for any internal category $\boldsymbol{D}$ in $\mathscr{E}$ there is an internal functor* $\lim_{\boldsymbol{D}}\colon [\boldsymbol{D},\boldsymbol{A}] \to \boldsymbol{A}$ *which is right adjoint to the diagonal functor* $\Delta\colon \boldsymbol{A} \to [\boldsymbol{D},\boldsymbol{A}]$.

Proof. The first step is to define the functor $\lim_{\boldsymbol{D}}$. We shall define the object and arrow components separately, each by using the definition of limit for a suitable diagram, in a suitable slice of E.

To define the object component of $\lim_{\boldsymbol{D}}$, consider $\mathrm{Fun}(\boldsymbol{D},\boldsymbol{A})$ as the indexing object. There is a diagram

$$\epsilon\colon \mathrm{Fun}(\boldsymbol{D},\boldsymbol{A})^*\boldsymbol{D} \to \mathrm{Fun}(\boldsymbol{D},\boldsymbol{A})^*\boldsymbol{A}$$

in $\mathrm{Cat}_{\mathscr{E}/\mathrm{Fun}(\boldsymbol{D},\boldsymbol{A})}$, defined as

$$F\colon \mathrm{Fun}(\boldsymbol{D},\boldsymbol{A}), d\colon D_0 \vdash \epsilon_0(F,d) \coloneqq (F, \mathrm{e}_{\mathscr{E}}(F_0,d))$$
$$F\colon \mathrm{Fun}(\boldsymbol{D},\boldsymbol{A}), f\colon D_1 \vdash \epsilon_1(F,f) \coloneqq (F, \mathrm{e}_{\mathscr{E}}(F_1,f)).$$

Then, the diagram has a limit given by

$$(\lim_{\boldsymbol{D}})_0\colon \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}/\mathrm{Fun}(\boldsymbol{D},\boldsymbol{A})}} \to \mathrm{Fun}(\boldsymbol{D},\boldsymbol{A})^*\boldsymbol{A},$$

corresponding to an arrow $(\lim_{\boldsymbol{D}})_0\colon \mathrm{Fun}(\boldsymbol{D},\boldsymbol{A}) \to A_0$ yielding the object component of $\lim_{\boldsymbol{D}}$. We also get the universal cone, let it be the natural transformation

$$\pi\colon (\lim_{\boldsymbol{D}})_0! \to \epsilon\colon \mathrm{Fun}(\boldsymbol{D},\boldsymbol{A})^*\boldsymbol{D} \to \mathrm{Fun}(\boldsymbol{D},\boldsymbol{A})^*\boldsymbol{A}.$$

To define the arrow component of $\lim_{\boldsymbol{D}}$, consider $\mathrm{Nat}(\boldsymbol{D},\boldsymbol{A})$ as the indexing object. There are diagrams $\epsilon_{\mathrm{s}}, \epsilon_{\mathrm{t}} \colon \mathrm{Nat}(\boldsymbol{D},\boldsymbol{A})^*\boldsymbol{D} \to \mathrm{Nat}(\boldsymbol{D},\boldsymbol{A})^*\boldsymbol{A}$ in $\mathrm{Cat}_{\mathscr{E}/\mathrm{Nat}(\boldsymbol{D},\boldsymbol{A})}$, defined as

$$(\alpha \colon F \to G) : \mathrm{Nat}(\boldsymbol{D},\boldsymbol{A}), d \colon D_0 \vdash (\epsilon_{\mathrm{s}})_0(\alpha, d) := (\alpha, \mathrm{e}_{\mathscr{E}}(F_0, d))$$
$$(\alpha \colon F \to G) : \mathrm{Nat}(\boldsymbol{D},\boldsymbol{A}), f \colon D_1 \vdash (\epsilon_{\mathrm{s}})_1(\alpha, f) := (\alpha, \mathrm{e}_{\mathscr{E}}(F_1, f))$$

and

$$(\alpha \colon F \to G) : \mathrm{Nat}(\boldsymbol{D},\boldsymbol{A}), d \colon D_0 \vdash (\epsilon_{\mathrm{t}})_0(\alpha, d) := (\alpha, \mathrm{e}_{\mathscr{E}}(G_0, d))$$
$$(\alpha \colon F \to G) : \mathrm{Nat}(\boldsymbol{D},\boldsymbol{A}), f \colon D_1 \vdash (\epsilon_{\mathrm{t}})_1(\alpha, f) := (\alpha, \mathrm{e}_{\mathscr{E}}(G_1, f)),$$

and there is also a natural transformation $\bar{\epsilon} \colon \epsilon_{\mathrm{s}} \to \epsilon_{\mathrm{t}}$ defined as

$$(\alpha \colon F \to G) : \mathrm{Nat}(\boldsymbol{D},\boldsymbol{A}), d \colon D_0 \vdash \bar{\epsilon}(\alpha, d) := (\alpha, \mathrm{e}_{\mathscr{E}}(\alpha, d)).$$

Then, the diagram ϵ_{t} has a limit given by

$$L \colon \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}/\mathrm{Nat}(\boldsymbol{D},\boldsymbol{A})}} \to \mathrm{Nat}(\boldsymbol{D},\boldsymbol{A})^*\boldsymbol{A},$$

corresponding to an arrow $L \colon \mathrm{Nat}(\boldsymbol{D},\boldsymbol{A}) \to A_0$, and a universal cone

$$p : L! \to \epsilon_{\mathrm{t}} : \mathrm{Nat}(\boldsymbol{D},\boldsymbol{A})^*\boldsymbol{D} \to \mathrm{Nat}(\boldsymbol{D},\boldsymbol{A})^*\boldsymbol{A}.$$

Observe that, in context $d \colon D_0, (\alpha \colon F \to G) : \mathrm{Nat}(\boldsymbol{D},\boldsymbol{A})$, we have $L(\alpha) = (\lim_{\boldsymbol{D}})_0(G)$ and $p(\alpha, d) = \pi(G, d)$. Consider $L' \colon \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}/\mathrm{Nat}(\boldsymbol{D},\boldsymbol{A})}} \to \mathrm{Nat}(\boldsymbol{D},\boldsymbol{A})^*\boldsymbol{A}$ given by $L'(\alpha) := (\lim_{\boldsymbol{D}})_0(F)$, and the natural transformations $p' : L'! \to \epsilon_{\mathrm{s}}$ defined as

$$p'(\alpha, d) := (\alpha, (\lim_{\boldsymbol{D}})_0(F) \xrightarrow{\pi(F,d)} \epsilon(F, d)).$$

Let $p'' := \bar{\epsilon}p' : L! \to \epsilon_{\mathrm{t}}$. Then, by the universal property of the limit, there is a unique natural transformation

$$(\lim_{\boldsymbol{D}})_1 : L' \to L \colon \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}/\mathrm{Nat}(\boldsymbol{D},\boldsymbol{A})}} \to \mathrm{Nat}(\boldsymbol{D},\boldsymbol{A})^*\boldsymbol{A}$$

such that $p'' = p(\lim_{\boldsymbol{D}})_1$ and whose underlying arrow in $\mathscr{E}$ is a morphism $(\lim_{\boldsymbol{D}})_1 : \mathrm{Nat}(\boldsymbol{D},\boldsymbol{A}) \to A_1$. That yields the arrow component of $\lim_{\boldsymbol{D}}$.

We shall now give the unit and counit for the adjunction $\Delta \dashv \lim_{\boldsymbol{D}}$. Define the unit $\eta \colon \mathrm{id}(\boldsymbol{A}) \to \lim_{\boldsymbol{D}} \Delta \colon \boldsymbol{A} \to \boldsymbol{A}$ as the unique natural isomorphism yielded by the universal property of the limit. Define the counit $\mu \colon \Delta \lim_{\boldsymbol{D}} \to \mathrm{id}([\boldsymbol{D},\boldsymbol{A}])$ as the natural transformation induced by the universal cone π, which can be regarded as an arrow $\mathrm{Fun}(\boldsymbol{D},\boldsymbol{A}) \to \mathrm{Nat}(\boldsymbol{D},\boldsymbol{A})$. That is natural because, in context $(\alpha \colon F \to G) : \mathrm{Nat}(\boldsymbol{D},\boldsymbol{A})$ and $d \colon D_0$, the equation $p'' = p(\lim_{\boldsymbol{D}})_1$ implies that the square

$$\begin{array}{ccc} (\lim_{\boldsymbol{D}})_0(F) & \xrightarrow{\pi(F,d)} & \epsilon(F,d) = \mathrm{e}_{\mathscr{E}}(F_0, d) \\ {\scriptstyle (\lim_{\boldsymbol{D}})_1(\alpha)}\downarrow & & \downarrow{\scriptstyle \mathrm{e}_{\mathscr{E}}(\alpha,d)} \\ (\lim_{\boldsymbol{D}})_0(G) & \xrightarrow{\pi(G,d)} & \epsilon(G,d) = \mathrm{e}_{\mathscr{E}}(G_0, d) \end{array}$$

commutes. These data yield the required adjunction. □

Together, the functor $\lim_{\boldsymbol{D}}$ and the counit of the adjunction $\Delta \lim_{\boldsymbol{D}} \to \mathrm{id}_{[\boldsymbol{D},\boldsymbol{A}]}$ induce an internal functor $[\boldsymbol{D},\boldsymbol{A}] \to \mathrm{Cone}_{\boldsymbol{D}}$ which is a right adjoint of the projection $p\colon \mathrm{Cone}_{\boldsymbol{D}} \to [\boldsymbol{D},\boldsymbol{A}]$. Likewise, we could produce the functor $\lim_{\boldsymbol{D}}$ from the right adjoint of p. In other words, the two formulations of a limit functor in terms of a right adjoint to $p\colon \mathrm{Cone}_{\boldsymbol{D}} \to [\boldsymbol{D},\boldsymbol{A}]$ and in terms of a right adjoint to $\Delta\colon \boldsymbol{A} \to [\boldsymbol{D},\boldsymbol{A}]$ are equivalent. Such equivalence is just a matter of routine 2-category-theory calculations.

Notice that the limit functor is stable under pullback, as it is the right adjoint of the diagonal functor, which is a stable notion in a locally cartesian closed category. More precisely, $I^* \lim_{\boldsymbol{D}} \cong \lim_{I^*\boldsymbol{D}}\colon \boldsymbol{A} \to [\boldsymbol{D},\boldsymbol{A}]$ for every indexing object I of $\mathscr{E}$.

The strength of the notion of internal completeness is constrained by the shapes of the diagrams we can build in $\mathscr{E}$. For example, if $\mathscr{E}$ doesn't have a coproduct $\mathbb{1} + \mathbb{1}$, we are unable to build a diagram with a pair of objects. That means that an internal category may in principle be complete, but not have binary products. Even more strikingly, if $\mathscr{E}$ does not have an initial object, we are unable to build the degenerate diagram having the terminal object as its limit. Fortunately, it is possible to express the concepts of terminal object, internal binary product and equalizer independently of the notion of internal limits.

Firstly, we treat the terminal object.

Definition 2.3.5 (internal terminal object)**.** An internal category $\boldsymbol{A}$ in $\mathscr{E}$ has terminal object if the functor $!\colon \boldsymbol{A} \to \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}}$ has a right adjoint.

Proposition 2.3.6. *If $\mathscr{E}$ has initial object and $\boldsymbol{A}$ is a complete internal category in $\mathscr{E}$, then $\boldsymbol{A}$ has terminal object.*

Proof. Let $\mathbb{0}_{\mathrm{Cat}_{\mathscr{E}}}$ be the initial object in $\mathrm{Cat}_{\mathscr{E}}$, built from the initial object $\mathbb{0}_{\mathscr{E}}$ of $\mathscr{E}$. Then, $\Delta\colon \boldsymbol{A} \to [\mathbb{0}_{\mathrm{Cat}_{\mathscr{E}}},\boldsymbol{A}]$ is $!\colon \boldsymbol{A} \to \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}}$, and the limit functor of $\boldsymbol{A}$ from proposition 2.3.4 yields the right adjoint. □

Secondly, we turn our attention to binary products.

Definition 2.3.7 (internal binary product)**.** An internal category $\boldsymbol{A}$ in $\mathscr{E}$ has binary products if the functor $\Delta\colon \boldsymbol{A} \to \boldsymbol{A} \times^{\mathrm{Cat}_{\mathscr{E}}} \boldsymbol{A}$ has a right adjoint.

It turns out that when one can give a natural definition in terms of diagrams, then it coincides with the general definition.

Proposition 2.3.8. *If $\mathscr{E}$ has binary coproducts and $\boldsymbol{A}$ is a complete internal category in $\mathscr{E}$, then $\boldsymbol{A}$ has binary products.*

Proof. Let $\mathbb{D} = \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}} +^{\mathrm{Cat}_{\mathscr{E}}} \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}}$ be the internal category with just two objects (let them be called 0 and 1) and their identity arrows. Consider the diagram in the slice $\mathscr{E}/A_0 \times A_0$

$$D\colon (A_0 \times A_0)^*\mathbb{D} \to (A_0 \times A_0)^*\boldsymbol{A}$$

defined, in context $a_0, a_1 : A_0$, as $D_0(a_0, a_1, i) := (a_0, a_1, a_i)$ for $i = 0, 1$. Then, the limit of D yields the right adjoint to Δ. □

Finally, we treat the case of equalizers. Consider the object $(\boldsymbol{A}_\mathrm{p})_0$ of pairs of parallel arrows in $\boldsymbol{A}$, defined by means of the internal language as the subobject of $A_1 \times A_1$ given, in context $(f, g) : A_1 \times A_1$, by $\mathrm{s}_{\boldsymbol{A}}(f) = \mathrm{s}_{\boldsymbol{A}}(g)$ and $\mathrm{t}_{\boldsymbol{A}}(f) = \mathrm{t}_{\boldsymbol{A}}(g)$. Analogously, there is an object of commutative squares between pairs of parallel arrows in $\boldsymbol{A}$, which is the subobject of $(\boldsymbol{A}_\mathrm{p})_0 \times (\boldsymbol{A}_\mathrm{p})_0 \times A_1 \times A_1$ given, in context $((f,g),(f',g'),h_0,h_1)$, by

$$\begin{array}{ccc} X_0 & \overset{f}{\underset{g}{\rightrightarrows}} & X_1 \\ {\scriptstyle h_0}\downarrow & & \downarrow{\scriptstyle h_1} \\ Y_0 & \overset{f'}{\underset{g'}{\rightrightarrows}} & Y_1 \end{array}$$

where $h_1 \circ_{\boldsymbol{A}} f = f' \circ_{\boldsymbol{A}} h_0$ and $h_1 \circ_{\boldsymbol{A}} g = g' \circ_{\boldsymbol{A}} h_0$. Thus we get the category $\boldsymbol{A}_\mathrm{p}$ of parallel arrows of $\boldsymbol{A}$. There is also a delta functor $\Delta\colon \boldsymbol{A} \to \boldsymbol{A}_\mathrm{p}$ sending an object of $\boldsymbol{A}$ into the pair given by its identity arrow (twice).

Definition 2.3.9 (internal equalizer)**.** An internal category $\boldsymbol{A}$ in $\mathscr{E}$ has equalizers if the functor $\Delta\colon \boldsymbol{A} \to \boldsymbol{A}_\mathrm{p}$ has a right adjoint.

Again, when we can give a definition in terms of diagrams it coincides with the general definition.

Proposition 2.3.10. *If $\mathscr{E}$ has binary coproducts and $\boldsymbol{A}$ is a complete internal category in $\mathscr{E}$, then $\boldsymbol{A}$ has equalizers.*

Proof. Let $\mathbb{P}$ be the internal category in $\mathscr{E}$ with two objects (let them be called 0 and 1) and two parallel arrows between them (let them be called $\vec{0}$ and $\vec{1}$). Consider the diagram in the slice $\mathscr{E}/(\boldsymbol{A}_\mathrm{p})_0$

$$D\colon (\boldsymbol{A}_\mathrm{p})_0^*\mathbb{P} \to (\boldsymbol{A}_\mathrm{p})_0^*\boldsymbol{A}$$

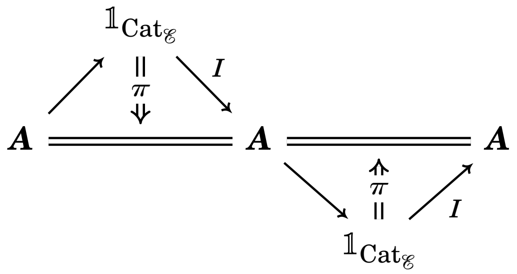
defined, in context $P = (f_0, f_1 : a_0 \to a_1) : (\boldsymbol{A}_\mathrm{p})_0$ as $D_0(P, i) := a_i$ for $i = 0, 1$, and $D_1(P, \vec{\imath}) := f_i$ for $i = 0, 1$. sending pairs of parallel arrows of $\boldsymbol{A}$ into the corresponding diagram, yields the right adjoint to Δ. □

Notice that the previous notions are all stable under pullback, as they are defined as the right adjoint of the diagonal functor, which is a stable notion in a locally cartesian closed category. Moreover, we could similarly define the dual special colimits: initial object, binary sum and coequalizer.

It is part of the folklore of category theory that, were it not for issues of size, completeness and cocompleteness would be equivalent notions. For example, the thought plays a manifest role in the proof of the General Adjoint Functor Theorem. In the context of internal categories the size issues disappear and there is an equivalence. This important fact is alluded to in Hyland (1988), but there is no written account of which we are aware, so we think it useful to indicate full details of a proof in a modern categorical style. The proof of the equivalence is a consequence of the following remarkable propositions.

Proposition 2.3.11. *Let $\boldsymbol{A}$ be a category in $\mathscr{E}$. If the identity* $\mathrm{id}_{\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{A}) : \boldsymbol{A} \to \boldsymbol{A}$ *has a limit, then that limit is an initial object in $\boldsymbol{A}$.*

Proof. Let the functor $I\colon \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}} \to \boldsymbol{A}$ be the limit of $\mathrm{id}_{\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{A})$, and the natural transformation $\pi\colon I!_{\boldsymbol{A}} \to \mathrm{id}_{\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{A}) : \boldsymbol{A} \to \boldsymbol{A}$ be its universal cone. We want to prove that I is the left adjoint of $!_{\boldsymbol{A}}$, with π as the counit of the adjunction (the unit being the obvious degenerate natural transformation). Then, the only non-trivial triangular equation (after canceling the identity terms) requires us to prove that $\pi * I = \mathrm{id}(I)$. Observe that the interchange law of composition applied to the natural transformations

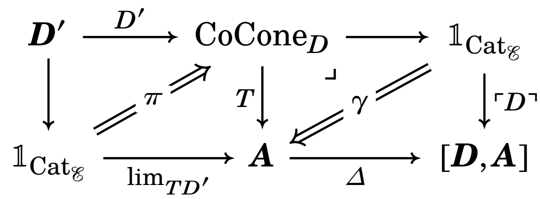


says that $\pi = \pi(\pi * I!)$ (given that $I! * \pi$ is an identity as it factors through $\mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}}$). Then, by the universal property of the limit, there is a unique $h : I \to I$ such that $\pi(h*!) = \pi$, which is obviously $\mathrm{id}(I)$. But, by the previous observation, also $\pi * I$ features such property, and then it must be that $\pi * I = \mathrm{id}(I)$, which is the required triangular equation. □

Proposition 2.3.12. *Let $\boldsymbol{A}$ be a complete category in $\mathscr{E}$ and $D\colon \boldsymbol{D} \to \boldsymbol{A}$ a diagram over it. Then,* CoCone_D *is complete too.*

Proof. Let $D' : \boldsymbol{D}' \to \mathrm{CoCone}_D$ be a diagram over CoCone_D. Observe that, by composition with the projection $T\colon \mathrm{CoCone}_D \to \boldsymbol{A}$, we have a diagram TD'

over $\boldsymbol{A}$. By hypothesis, TD' has a limit $\lim_{TD'}\colon \mathbb{1}_{\mathrm{Cat}_{\mathscr{C}}} \to \boldsymbol{A}$ with universal cone $\pi\colon \Delta \lim_{TD'} \to \ulcorner TD' \urcorner$. The situation is as follows.



Because Δ preserves limits, $\lim_{\Delta TD'}$ is (isomorphic to) $\Delta \lim_{TD'}$ and its universal cone is (isomorphic to) $\Delta * \pi$. But $\gamma * D'\colon \ulcorner D \urcorner ! \to \Delta TD'$ is a cone over $\Delta TD'$ with vertex $\ulcorner D \urcorner$, so, by the universal property of the limit, there is a unique natural transformation $h\colon \ulcorner D \urcorner \to \Delta \lim_{TD'}$ such that $(\Delta * \pi)(h*!) = \gamma * D'$. Then, by the universal property of the lax pullback, there is a unique $L\colon \mathbb{1}_{\mathrm{Cat}_{\mathscr{C}}} \to \mathrm{CoCone}_D$ such that $TL \cong \lim_{TD'}$ and (up to this isomorphism) $\gamma * L = h$. Moreover, observe that by the defining properties of h and L it holds $\gamma * D' = (\Delta * \pi)(\gamma * L!)$. Then, by the 2-dimensional universal property of the lax pullback, there is a unique $p\colon L! \to D'$ such that $T * p = \pi$.

We claim (L, p) is a limit for D'. We need to show that, for any other cone (L', p') on D', there is a unique $\beta\colon L' \to L$ such that $p(\beta*!) = p'$.

Let's produce β. To begin with, notice that $T * p'\colon TL'! \to TD'$ is a cone on TD' and thus, by the universal property of the limit, there exists a unique $k\colon TL' \to \lim_{TD'}$ such that $\pi(k*!) = T * p'$. Then, observe that $(\Delta * k)(\gamma * L')$ is h, as it has the same unique property:

$$
\begin{aligned}
&(\Delta * \pi)\big((\Delta * k)(\gamma * L')*!\big) \\
&\quad = (\Delta * \pi)(\Delta * k*!)(\gamma * L'!) \\
&\quad = \big(\Delta * (\pi(k*!))\big)(\gamma * L'!) \\
\text{(by the unique property of } k\text{)} &\quad = (\Delta T * p')(\gamma * L'!) \\
\text{(by the interchange law of composition)} &\quad = (\gamma * D')(\ulcorner D \urcorner ! * p') \\
\text{(because } \ulcorner D \urcorner ! * p' \text{ factors through } \mathbb{1}_{\mathrm{Cat}_{\mathscr{C}}}\text{)} &\quad = \gamma * D'.
\end{aligned}
$$

Finally, by the 2-dimensional universal property of the lax pullback, there is a unique $\beta\colon L' \to L$ such that $T * \beta = k$.

Now we need to show that p' factorizes through p, i.e. that $p(\beta*!) = p'$. Notice that $\gamma * D' = (\Delta * (\pi(k*!)))(\gamma * L'!)$ (it is the same calculation as before). Then, there is a unique $q\colon L'! \to D'$ such that $T * q = \pi(k*!)$. But both $p(\beta*!)$ and p' have this unique property. Indeed, $T * (p(\beta*!)) = (T * p)(T * \beta*!)$ and that, because of the unique properties of p and β, is $\pi(k*!)$; and for p' it holds because the unique property of k. But then, $p(\beta*!) = p'$.

To conclude, let's prove that β is unique with its property. Let $\beta'\colon L' \to L$ be such that $p(\beta'*!) = p'$. To prove that β' is β, we need to show that it has the same unique property, i.e. that $T*\beta' = k$. Likewise, to prove that $T*\beta'$ is k, we need show that it has the same unique property, i.e. that $\pi(T*\beta'*!) = T*p'$. But that follows from $(T*p)(T*\beta'*!) = T*p'$, by applying the unique property of p. □

We can now finally prove the result.

Theorem 2.3.13. *Let* $\boldsymbol{A}$ *be a category in* $\mathscr{E}$*. Then,* $\boldsymbol{A}$ *is complete if and only if it is cocomplete.*

Proof. By duality, it is enough to show only one direction of the equivalence.

Assume $\boldsymbol{A}$ is complete, and let's prove that it is also cocomplete. Consider the diagram $D\colon \boldsymbol{D} \to \boldsymbol{A}$. A colimit of D is by definition an initial object in CoCone_D, which is, by proposition 2.3.11, the identity on CoCone_D. Such limit exists because, by proposition 2.3.12, CoCone_D is complete. □

The equivalence of completeness and cocompleteness is, essentially, a generalization of the well-known fact that complete lattices are also cocomplete. We can now retrieve such fact by recalling that complete lattices are complete internal categories in Set (example 2.3.2) and applying theorem 2.3.13.

Another immediate consequence is that the category of Modest Sets in the Category of Assemblies is cocomplete.

3. Internal Enriched Categories

In this chapter we introduce the notion of internal enriched category, abstracting both the notions of internal and enriched category. After giving the main definitions, we develop the foundations of such theory by presenting a selection of standard categorical topics in the new setting. That shows that our theory parallels standard (enriched) category theory and it has many of its features. Moreover, we compare the new notion to that of enriched indexed category, showing that the two are closely related.

It is worth remarking that standard category theory has an intrinsic bias towards sets, in that (small) categories are implicitly enriched over the (large) category of sets. In the internal context, there is no immediate notion of large category: the usual substitute is via a theory of indexed categories, but that has the disadvantage that the notion of large is not intrinsic to the situation. As a result, a considerable portion of standard category theory is problematic from the internal point of view. After all one cannot even state the Yoneda Lemma internally. However having an internal enriching category obviates these difficulties. Thus, the theory of internal enriched categories has many of the features of the standard theory of categories, while retaining the generality of the theory of internal categories.

In this chapter, let $\mathscr{E}$ be a category with finite limits, and $\boldsymbol{V}$ be an internal monoidal category in $\mathscr{E}$.

3.1. Internal Enriched Categories

Let's define a notion of internal enrichment in $\boldsymbol{V}$, substantially by translating the standard notion of enriched category in the internal language of $\mathscr{E}$.

Definition 3.1.1 (internal enriched category)**.** An *internal $\boldsymbol{V}$-enriched category* $\boldsymbol{X}$ in $\mathscr{E}$, or $\boldsymbol{V}$-category, is given by

Underlying object: an object X of $\mathscr{E}$;

Internal hom: a morphism $\mathrm{Hom}_{\boldsymbol{X}} \colon X \times X \to V_0$;

Composition: a morphism $\circ_{\boldsymbol{X}} \colon X \times X \times X \to V_1$ such that, in context $x_0, x_1, x_2 \colon X$,

$$\circ_{\boldsymbol{X}}(x_0, x_1, x_2) \colon \mathrm{Hom}_{\boldsymbol{X}}(x_1, x_2) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_0, x_1) \to \mathrm{Hom}_{\boldsymbol{X}}(x_0, x_2);$$

Identity: a morphism $\mathrm{id}_{\boldsymbol{X}} \colon X \to V_1$ such that, in context $x \colon X$,

$$\mathrm{id}_{\boldsymbol{X}} \colon \mathbb{I}_{\boldsymbol{V}} \to \mathrm{Hom}_{\boldsymbol{X}}(x, x);$$

satisfying, in context $x_0, x_1, x_2, x_3 \colon X$, the axioms of 3.1, 3.2, and 3.3.

$$\begin{array}{ccc}
(\mathrm{Hom}_{\boldsymbol{X}}(x_2,x_3) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_1,x_2)) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) & \xrightarrow{\circ_{\boldsymbol{X}}(x_1,x_2,x_3) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1)} & \mathrm{Hom}_{\boldsymbol{X}}(x_1,x_3) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \\
\Big\downarrow {\scriptstyle \alpha_{\boldsymbol{V}}(\mathrm{Hom}_{\boldsymbol{X}}(x_2,x_3), \mathrm{Hom}_{\boldsymbol{X}}(x_1,x_2), \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1))} & & \\
\mathrm{Hom}_{\boldsymbol{X}}(x_2,x_3) \otimes^{\boldsymbol{V}} (\mathrm{Hom}_{\boldsymbol{X}}(x_1,x_2) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1)) & & \Big\downarrow {\scriptstyle \circ_{\boldsymbol{X}}(x_0,x_1,x_3)} \\
\Big\downarrow {\scriptstyle \mathrm{Hom}_{\boldsymbol{X}}(x_2,x_3) \otimes^{\boldsymbol{V}} \circ_{\boldsymbol{X}}(x_0,x_1,x_2)} & & \\
\mathrm{Hom}_{\boldsymbol{X}}(x_2,x_3) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_2) & \xrightarrow[\circ_{\boldsymbol{X}}(x_0,x_2,x_3)]{} & \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_3)
\end{array} \tag{3.1}$$

$$\begin{array}{ccc}
\mathbb{I}_{\boldsymbol{V}} \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) & \xrightarrow{\lambda_{\boldsymbol{V}}(\mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1))} & \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \\
{\scriptstyle \mathrm{id}_{\boldsymbol{X}}(x_1) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1)} \searrow & & \nearrow {\scriptstyle \circ_{\boldsymbol{X}}(x_0,x_1,x_1)} \\
& \mathrm{Hom}_{\boldsymbol{X}}(x_1,x_1) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) &
\end{array} \tag{3.2}$$

$$\begin{array}{ccc}
\mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \otimes^{\boldsymbol{V}} \mathbb{I}_{\boldsymbol{V}} & \xrightarrow{\rho_{\boldsymbol{V}}(\mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1))} & \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \\
{\scriptstyle \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \otimes^{\boldsymbol{V}} \mathrm{id}_{\boldsymbol{X}}(x_0)} \searrow & & \nearrow {\scriptstyle \circ_{\boldsymbol{X}}(x_0,x_0,x_1)} \\
& \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_0) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) &
\end{array} \tag{3.3}$$

Notice how the conventions on the internal language of $\mathscr{E}$ allow one to express those axioms in a form very close to that used to define standard enriched categories.

Example 3.1.2. Let $\mathscr{V}$ be a small monoidal category. Then, $\mathscr{V}$ is an internal category in Set, and internal $\mathscr{V}$-enriched categories in Set are standard (small) $\mathscr{V}$-enriched categories.

Continuing in the style of the previous definition, we give a notion of internal enriched functor, by translating the standard definition into the internal language.

Definition 3.1.3 (internal enriched functor)**.** Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be $\boldsymbol{V}$-enriched categories. A *$\boldsymbol{V}$-enriched functor*, or $\boldsymbol{V}$-functor, $F\colon \boldsymbol{X} \to \boldsymbol{Y}$ is given by

Objects component: an arrow $F_0\colon X \to Y$;

Morphisms component: an arrow $F_1\colon X \times X \to V_1$ of shape

$$x_0, x_1 \colon X \vdash F_1(x_0, x_1) \colon \mathrm{Hom}_{\boldsymbol{X}}(x_0, x_1) \to \mathrm{Hom}_{\boldsymbol{Y}}(F_0(x_0), F_0(x_1)) \colon V_1;$$

satisfying, in context $x_0, x_1, x_2 \colon X$, the axiom 3.4.

$$\begin{array}{ccc}
\mathrm{Hom}_{\boldsymbol{X}}(x_1,x_2) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) & \xrightarrow{\circ_{\boldsymbol{X}}(x_0,x_1,x_2)} & \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_2) \\
\Big\downarrow{\scriptstyle F_1(x_1,x_2) \otimes^{\boldsymbol{V}}_1 F_1(x_0,x_1)} & & \Big\downarrow{\scriptstyle F_1(x_0,x_2)} \\
\mathrm{Hom}_{\boldsymbol{Y}}(F_0(x_1),F_0(x_2)) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(F_0(x_0),F_0(x_1)) & \xrightarrow[\circ_{\boldsymbol{Y}}(F_0(x_0),F_0(x_1),F_0(x_2))]{} & \mathrm{Hom}_{\boldsymbol{Y}}(F_0(x_0),F_0(x_2))
\end{array} \tag{3.4}$$

We would expect the definition above to provide a category of internal $\boldsymbol{V}$-enriched categories and functors. We present the relevant data for that.

The composition of $\boldsymbol{V}$-functors $G \circ F\colon \boldsymbol{X} \xrightarrow{F} \boldsymbol{Y} \xrightarrow{G} \boldsymbol{Z}$ is defined by

$$(G \circ F)_0 \coloneqq G_0 F_0 \colon X \to Z$$

and

$$x_0, x_1 \colon X \vdash (G \circ F)_1(x_0, x_1) \coloneqq \begin{array}{c} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \\ \Big\downarrow{\scriptstyle F_1(x_0,x_1)} \\ \mathrm{Hom}_{\boldsymbol{Y}}(F_0(x_0),F_0(x_1)) \\ \Big\downarrow{\scriptstyle G_1(F_0(x_0),F_0(x_1))} \\ \mathrm{Hom}_{\boldsymbol{Z}}(G_0F_0(x_0),G_0F_0(x_1)) \end{array} \colon V_1.$$

The identity $\boldsymbol{V}$-functor $\mathrm{id}_{\boldsymbol{X}}\colon \boldsymbol{X} \to \boldsymbol{X}$ on $\boldsymbol{X}$ is defined by

$$(\mathrm{id}_{\boldsymbol{X}})_0 \coloneqq \mathrm{id}_{\mathscr{E}}(X) \colon X \to X$$

and

$$(\mathrm{id}_{\boldsymbol{X}})_1 \coloneqq X \xrightarrow{\mathrm{Hom}_{\boldsymbol{X}}} V_0 \xrightarrow{\mathrm{id}_{\boldsymbol{V}}} V_1.$$

It is just an exercise in the internal language to prove that the data so defined yield a category, as stated in the following proposition.

Proposition 3.1.4. *Composition and identity of internal enriched functors strictly satisfy associativity and unitarity. Then, $\boldsymbol{V}$-enriched categories and functors form a category $\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}$.*

Example 3.1.5. There is an underlying-object functor $U\colon \boldsymbol{V}\mathrm{Cat}_{\mathscr{E}} \to \mathscr{E}$ sending $\boldsymbol{V}$-enriched categories to their underlying object, and $\boldsymbol{V}$-enriched functors to their object-component.

Example 3.1.6. Let X be an object of $\mathscr{E}$. The indiscrete $\boldsymbol{V}$-enriched category $\mathbf{ind}(X)$ on X is given by

$$\mathrm{Hom}_{\mathbf{ind}(X)} \coloneqq X \times X \xrightarrow{!} \mathbb{1}_{\mathscr{E}} \xrightarrow{\mathbb{I}_{\boldsymbol{V}}} V_0.$$

The rest of the structure follows from that. Analogously, a morphism $f\colon X \to Y$ induces an indiscrete $\boldsymbol{V}$-enriched functor $\mathbf{ind}(f) : \mathbf{ind}(X) \to \mathbf{ind}(Y)$. Then, there is a functor $\mathbf{ind}\colon \mathscr{E} \to \boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}$.

Remark 3.1.7. To define the discrete $\boldsymbol{V}$-enriched category over an object of $\mathscr{E}$, we would need to assume some extra hypothesis. Firstly, we would need to be able to tell whether two elements of the underlying object of the $\boldsymbol{V}$-enriched category are equal. Secondly, we would need an initial object in $\boldsymbol{V}$ to be the homset of non-equal elements of the underlying object. Both hypothesis do not hold in general. For example, the first one does not hold in the Effective Topos.

Finally, again by translating the standard definition into the internal language, we give the definition of internal enriched natural transformation.

Definition 3.1.8 (internal enriched natural transformation)**.** Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be $\boldsymbol{V}$-enriched categories, and F and G be $\boldsymbol{V}$-enriched functors $\boldsymbol{X} \to \boldsymbol{Y}$. A *$\boldsymbol{V}$-enriched natural transformation*, or $\boldsymbol{V}$-natural transformation, $\alpha : F \to G : \boldsymbol{X} \to \boldsymbol{Y}$ is given by an arrow $\alpha : X \to V_1$ of shape

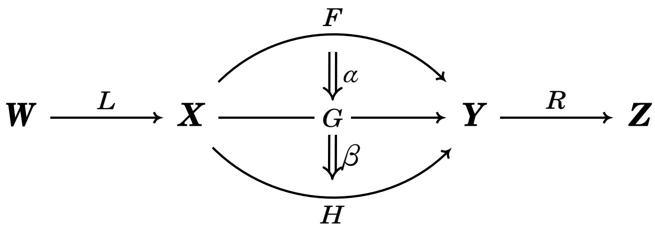
$$x : X \vdash \alpha(x) : \mathbb{I}_{\boldsymbol{V}} \to \mathrm{Hom}_{\boldsymbol{Y}}(F_0(x), G_0(x)) : V_1$$

satisfying, in context $x_0, x_1 : X$, the axiom 3.5.

$$\begin{array}{ccc}
\mathbb{I}_{\boldsymbol{V}} \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) & \xrightarrow{\alpha(x_1)\otimes^{\boldsymbol{V}} F_1(x_0,x_1)} & \mathrm{Hom}_{\boldsymbol{Y}}(F_0(x_1),G_0(x_1)) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(F_0(x_0),F_0(x_1)) \\
\uparrow{\scriptstyle \lambda_{\boldsymbol{V}}^{-1}(\mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1))} & & \downarrow{\scriptstyle \circ_{\boldsymbol{Y}}(F_0(x_0),F_0(x_1),G_0(x_1))} \\
\mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) & & \mathrm{Hom}_{\boldsymbol{Y}}(F_0(x_0),G_0(x_1)) \\
\downarrow{\scriptstyle \rho_{\boldsymbol{V}}^{-1}(\mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1))} & & \uparrow{\scriptstyle \circ_{\boldsymbol{Y}}(F_0(x_0),G_0(x_0),G_0(x_1))} \\
\mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \otimes^{\boldsymbol{V}} \mathbb{I}_{\boldsymbol{V}} & \xrightarrow[G_1(x_0,x_1)\otimes^{\boldsymbol{V}}\alpha(x_0)]{} & \mathrm{Hom}_{\boldsymbol{Y}}(G_0(x_0),G_0(x_1)) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(F_0(x_0),G_0(x_0))
\end{array} \tag{3.5}$$

We would expect the definition above to provide a 2-category of internal $\boldsymbol{V}$-enriched categories, functors and natural transformations. We present the relevant data for that.

Consider $\boldsymbol{V}$-categories, $\boldsymbol{V}$-functors and $\boldsymbol{V}$-natural transformations as shown in the diagram:



Vertical composition of $\boldsymbol{V}$-natural transformations $\beta \circ \alpha \colon F \to H \colon X \to Y$ is defined by

$$x \colon X \vdash (\beta \circ \alpha)(x) \coloneqq \begin{array}{c} \mathbb{I}_{\boldsymbol{V}} \cong \mathbb{I}_{\boldsymbol{V}} \otimes^{\boldsymbol{V}} \mathbb{I}_{\boldsymbol{V}} \\ \Big\downarrow {\scriptstyle \beta(x) \otimes^{\boldsymbol{V}} \alpha(x)} \\ \mathrm{Hom}_{\boldsymbol{X}}(G_0(x), H_0(x)) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(F_0(x), G_0(x)) \\ \Big\downarrow {\scriptstyle \circ_{\boldsymbol{Y}}(F_0(x), G_0(x), H_0(x))} \\ \mathrm{Hom}_{\boldsymbol{X}}(F_0(x), H_0(x)) \end{array} : V_1.$$

The left whiskering $\alpha * L \colon F \circ L \to G \circ L \colon \boldsymbol{W} \to \boldsymbol{Y}$ is defined by

$$w \colon W \vdash (\alpha * L)(w) \coloneqq \alpha(L_0(w)) : V_1.$$

The right whiskering $R * \beta \colon R \circ G \to R \circ H \colon \boldsymbol{X} \to \boldsymbol{Z}$ is defined by

$$x \colon X \vdash (R * \beta)(x) \coloneqq \begin{array}{c} \mathbb{I}_{\boldsymbol{V}} \\ \Big\downarrow {\scriptstyle \beta(x)} \\ \mathrm{Hom}_{\boldsymbol{Y}}(G_0(x), H_0(x)) \\ \Big\downarrow {\scriptstyle R_1(G_0(x), H_0(x))} \\ \mathrm{Hom}_{\boldsymbol{Z}}(R_0 G_0(x), R_0 H_0(x)) \end{array} : V_1.$$

The identity $\boldsymbol{V}$-natural transformation $\mathrm{id}_F \colon F \to F \colon \boldsymbol{X} \to \boldsymbol{Y}$ is defined as

$$\mathrm{id}_F \coloneqq X \xrightarrow{F_0} Y \xrightarrow{\mathrm{id}_{\boldsymbol{Y}}} V_1.$$

It is just an exercise in the internal language to prove that a pair of $\boldsymbol{V}$-enriched categories yield a category of functors and natural transformations, as stated in the following proposition.

Proposition 3.1.9. *Vertical composition and identity of internal enriched natural transformation strictly satisfy associativity and unitarity. Then, given* $\boldsymbol{V}$*-enriched categories* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$*, the* $\boldsymbol{V}$*-enriched functors* $\boldsymbol{X} \to \boldsymbol{Y}$ *and natural transformations between them form a category* $\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}(\boldsymbol{X},\boldsymbol{Y})$.

Moreover, with the horizontal composition defined before, the above assignment turns out to yield an enrichment in Cat. Equivalently, internal enriched categories, functors and natural transformations form a 2-category, as stated in the following result, whose proof is again an exercise in the internal language.

Proposition 3.1.10. *Horizontal and vertical composition of* $\boldsymbol{V}$*-enriched natural transformations satisfy the interchange laws. So,* $\boldsymbol{V}$*-enriched categories, functors, and natural transformations form a strict 2-category* $\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}$.

By abuse of notation, we called by $\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}$ both the category of $\boldsymbol{V}$-enriched categories and their functors, and the 2-category of $\boldsymbol{V}$-enriched categories, their functors and their natural transformations. As a consequence, given two $\boldsymbol{V}$-enriched categories $\boldsymbol{X}$ and $\boldsymbol{Y}$, we will denote by $\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}(\boldsymbol{X},\boldsymbol{Y})$ both the hom-set of $\boldsymbol{V}$-enriched functors $\boldsymbol{X} \to \boldsymbol{Y}$ and the hom-category of $\boldsymbol{V}$-enriched functors $\boldsymbol{X} \to \boldsymbol{Y}$ and their natural transformations. Context will usually suffice to determine in which sense the notation is being used.

Remark 3.1.11. Let $\boldsymbol{X}$ be a $\boldsymbol{V}$-enriched category. There is an underlying $\mathscr{E}$-category $U(\boldsymbol{X})$, such that $U(\boldsymbol{X})_0 \coloneqq X$ and $U(\boldsymbol{X})_1$ is the subobject of $X \times X \times V_1$ given by

$$(x_0,x_1,f) : U(\boldsymbol{X})_1 \vdash f \colon \mathbb{I}_{\boldsymbol{V}} \to \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) : V_1$$

with the first and second projections as source and target. The composition is defined, in context $(x_1,x_2,g),(x_0,x_1,f) : U(\boldsymbol{X})_1 {}_{\mathrm{s}}\!\times_{\mathrm{t}} U(\boldsymbol{X})_1$, as

$$(x_1,x_2,g) \circ_{U(\boldsymbol{X})} (x_0,x_1,f) \coloneqq (x_0,x_2,\circ_{\boldsymbol{X}}(x_0,x_1,x_2) \circ_{\boldsymbol{V}} (g \otimes^{\boldsymbol{V}} f)) : U(\boldsymbol{X})_1.$$

Let $F \colon \boldsymbol{X} \to \boldsymbol{Y}$ be a $\boldsymbol{V}$-enriched functor. There is an underlying functor $U(F) : U(\boldsymbol{X}) \to U(\boldsymbol{Y})$ in $\mathscr{E}$, with $U(F)_0$ defined as F_0 and $U(F)_1(x_0,x_1,f)$, in context $(x_0,x_1,f) : U(\boldsymbol{X})_1$, as the tuple

$$\left(F_0(x_0),F_0(x_1),\mathbb{I}_{\boldsymbol{V}} \xrightarrow{f} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \xrightarrow{F_1(x_0,x_1)} \mathrm{Hom}_{\boldsymbol{X}}(F_0(x_0),F_0(x_1))\right)$$

in $U(\boldsymbol{Y})_1$.

Let $\alpha : F \to G \colon \boldsymbol{X} \to \boldsymbol{Y}$ be a $\boldsymbol{V}$-enriched natural transformation. There is an underlying natural transformation $U(\alpha) : U(F) \to U(G) : U(\boldsymbol{X}) \to U(\boldsymbol{Y})$ in $\mathscr{E}$, defined as

$$x \colon U(\boldsymbol{X})_0 \vdash U(\alpha)(x) \coloneqq (F_0(x),G_0(x),\alpha(x)) : U(\boldsymbol{Y})_1.$$

Those data yield the underlying-category-in-$\mathscr{E}$ 2-functor $U \colon \boldsymbol{V}\mathrm{Cat}_{\mathscr{E}} \to \mathrm{Cat}_{\mathscr{E}}$.

It is an established practice in enriched category theory to focus on enrichment in symmetric monoidal closed category. *Inter alia* this provides a self-enrichment. We check that this happens in our internal setting.

Proposition 3.1.12. *If $\boldsymbol{V}$ is symmetric and monoidal closed, then it has an internal $\boldsymbol{V}$-category structure such that (with abuse of notation) $U(\boldsymbol{V}) = \boldsymbol{V}$.*

Proof. Let V_0 be the underlying object. Define

$$a, b \colon V_0 \vdash \mathrm{Hom}_{\boldsymbol{V}}(a,b) \coloneqq [a,b] \colon V_0.$$

Let the composition

$$a,b,c \colon V_0 \vdash \circ_{\boldsymbol{V}}(a,b,c) \colon \mathrm{Hom}_{\boldsymbol{V}}(b,c) \otimes \mathrm{Hom}_{\boldsymbol{V}}(a,b) \to \mathrm{Hom}_{\boldsymbol{V}}(a,c) \colon V_1$$

be the exponential transpose of

$$\mathrm{Hom}_{\boldsymbol{V}}(b,c) \otimes \mathrm{Hom}_{\boldsymbol{V}}(a,b) \otimes a \xrightarrow{\mathrm{Hom}_{\boldsymbol{V}}(b,c) \otimes \mathrm{e}(a,b)} \mathrm{Hom}_{\boldsymbol{V}}(b,c) \otimes b \xrightarrow{\mathrm{e}(b,c)} c$$

and the identity

$$a \colon V_0 \vdash \mathrm{id}_{\boldsymbol{V}}(a) \colon \mathbb{I} \to \mathrm{Hom}_{\boldsymbol{V}}(a,a) \colon V_1$$

be the exponential transpose of the identity $\mathrm{id}_{\boldsymbol{V}}(a) \colon a \to a$. (Notice again that there is some notation overload.)

The $\boldsymbol{V}$-category axioms hold because of the defining property of monoidal closure. □

We now consider the issue of the change of base. In this context, though, there are two sensible such notions, one coming from internal category theory and one from enriched category theory. Indeed, we can change both the ambient category and the enriching category.

To begin with, let's state the internal version of the standard result changing the enriching category.

Proposition 3.1.13. *Let $\boldsymbol{V}'$ be another monoidal category in $\mathscr{E}$, and $F \colon \boldsymbol{V} \to \boldsymbol{V}'$ a monoidal functor. Then there is an induced 2-functor $F_* \colon \boldsymbol{V}\mathrm{Cat}_{\mathscr{E}} \to \boldsymbol{V}'\mathrm{Cat}_{\mathscr{E}}$.*

Proof. Let $\boldsymbol{X}$ be a $\boldsymbol{V}$-category. Define a $\boldsymbol{V}'$-category $F_*(\boldsymbol{X})$ on X given by

Internal hom: $\mathrm{Hom}_{F_*(\boldsymbol{X})} \coloneqq X \times X \xrightarrow{\mathrm{Hom}_{\boldsymbol{X}}} V_0 \xrightarrow{F_0} V_0'$;

Composition: $\circ_{F_*(\boldsymbol{X})} \coloneqq X \times X \times X \xrightarrow{\circ_{\boldsymbol{X}}} V_1 \xrightarrow{F_1} V_1'$;

Identity: $\mathrm{id}_{F_*(\boldsymbol{X})} \coloneqq X \xrightarrow{\mathrm{Hom}_{\boldsymbol{X}}} V_1 \xrightarrow{F_1} V_1'$.

Let $G\colon \boldsymbol{X} \to \boldsymbol{Y}$ be a $\boldsymbol{V}$-functor. Define a $\boldsymbol{V}'$-functor $F_*(G): F_*(\boldsymbol{X}) \to F_*(\boldsymbol{Y})$, with the same object component as G and arrow component given by

$$(F_*(G))_1 := X \times X \xrightarrow{G_1} V_1 \xrightarrow{F_1} V_1'.$$

Let $\alpha\colon G \to G' : \boldsymbol{X} \to \boldsymbol{Y}$ be a $\boldsymbol{V}$-natural transformation. Define a $\boldsymbol{V}'$-natural transformation $F_*(\alpha) : F_*(G) \to F_*(G') : F_*(\boldsymbol{X}) \to F_*(\boldsymbol{Y})$ as

$$F_*(\alpha) := X \xrightarrow{\alpha} V_1 \xrightarrow{F_1} V_1'.$$

The axioms for the above definitions hold because of the functoriality of F. □

Then, let's check that changing the ambient category induces a 2-functorial operation on internal enriched categories, just as it does on internal categories.

Proposition 3.1.14. *Let $\mathscr{E}'$ be another finitely complete category and $F\colon \mathscr{E} \to \mathscr{E}'$ a functor preserving finite limits. Then there is an induced monoidal category $F(\boldsymbol{V})$ in $\mathscr{E}'$ and a 2-functor $F_*\colon \boldsymbol{V}\mathrm{Cat}_{\mathscr{E}} \to F(\boldsymbol{V})\mathrm{Cat}_{\mathscr{E}'}$.*

Proof. Let $\boldsymbol{X}$ be a $\boldsymbol{V}$-category. Define a $F(\boldsymbol{V})$-category $F_*(\boldsymbol{X})$ on $F(X)$ by applying the functor F to the structural arrows $\mathrm{Hom}_{\boldsymbol{X}}$, $\circ_{\boldsymbol{X}}$ and $\mathrm{id}_{\boldsymbol{X}}$ of $\boldsymbol{X}$. That gives a $F(\boldsymbol{V})$-enriched category because F preserves finite-limit logic, in terms of which internal enriched categories are defined. Analogously, define F_* on $\boldsymbol{V}$-enriched functors and natural transformations. □

3.2. Comparison with Indexed Enriched Categories

We now establish the connection between internal enriched categories and the notions from section 1.4. Recall that the externalization of $\boldsymbol{V}$ is a monoidal indexed category $[\boldsymbol{V}]$ over $\mathscr{E}$. Thus, we will investigate which relationship subsists between $\boldsymbol{V}$-enriched categories, small $[\boldsymbol{V}]$-categories and indexed $[\boldsymbol{V}]$-categories.

The most immediate fact is that $\boldsymbol{V}$-enriched categories and small $[\boldsymbol{V}]$-categories are the same thing, in a very strict sense: their definitions coincide!

Proposition 3.2.1. *To give a $\boldsymbol{V}$-enriched category (functor, natural transformation) is to give a small $[\boldsymbol{V}]$-category (functor, natural transformation).*

Proof. A small $[\boldsymbol{V}]$-category $\boldsymbol{X}$ is yielded by

- an object X of $\mathscr{E}$;
- an object $\mathrm{Hom}_{\boldsymbol{X}}\colon X \times X \to V_0$ of $[\boldsymbol{V}]^{X \times X}$;

- a morphism of $[\boldsymbol{V}]$

$$(X \xrightarrow{!_X} \mathbb{1}_{\mathscr{E}} \xrightarrow{\mathbb{I}_V} V_0) \xrightarrow{(X \xrightarrow{\Delta_X} X\times X, X \xrightarrow{\mathrm{id}_X} (\mathbb{I}_V !_X, \mathrm{Hom}_X \Delta_X)^* V_1)} (X\times X \xrightarrow{\mathrm{Hom}_X} V_0);$$

- a morphism of $[\boldsymbol{V}]$

$$(X\times X\times X \xrightarrow{\mathrm{Hom}_X(\pi_2,\pi_3)\otimes^V \mathrm{Hom}_X(\pi_1,\pi_2)} V_0) \to (X\times X \xrightarrow{\mathrm{Hom}_X} V_0)$$

over $X\times X\times X \xrightarrow{(\pi_1,\pi_3)} X\times X$, given by

$$X\times X\times X \xrightarrow{\circ_X} (\mathrm{Hom}_X(\pi_2,\pi_3) \otimes^V \mathrm{Hom}_X(\pi_1,\pi_2), \mathrm{Hom}_X(\pi_1,\pi_3))^* V_1$$

satisfying associativity and unitarity axioms. But these are precisely the same data that yield an internal $\boldsymbol{V}$-enriched category.

Analogously, to give a functor or a natural transformation of small $[\boldsymbol{V}]$-categories is to give a functor or a natural transformation of internal $\boldsymbol{V}$-enriched categories. □

The relationship between $\boldsymbol{V}$-enriched categories and indexed $[\boldsymbol{V}]$-categories is more complicated. We will prove that $\boldsymbol{V}$-enriched categories are a sub-case of indexed $[\boldsymbol{V}]$-categories, in the sense precisely stated in propositions 3.2.2 and 3.2.3.

First, we give the construction yielding an indexed $[\boldsymbol{V}]$-category from a $\boldsymbol{V}$-enriched category. Let $\boldsymbol{X}$ be a $\boldsymbol{V}$-enriched category and let's define an indexed $[\boldsymbol{V}]$-category $[\boldsymbol{X}]$. Given an indexing object I of $\mathscr{E}$, define the $[\boldsymbol{V}]^I$-enriched category $[\boldsymbol{X}]^I$ as

Objects: I-indexed families $x\colon I \to X$ of elements of X;

Internal hom: $\mathrm{Hom}_{[\boldsymbol{X}]^I}(x_0\colon I \to X, x_1\colon I \to X) \coloneqq I \xrightarrow{(x_0,x_1)} X\times X \xrightarrow{\mathrm{Hom}_X} V_0$;

Composition: $\circ_{[\boldsymbol{X}]^I}(x_0,x_1,x_2) \coloneqq I \xrightarrow{(x_0,x_1,x_1)} X\times X\times X \xrightarrow{\circ_X} V_1$;

Identity: $\mathrm{id}_{[\boldsymbol{X}]^I}(x) \coloneqq I \xrightarrow{x} X \xrightarrow{\mathrm{id}_X} V_1$.

Let $f\colon I \to J$ be a re-indexing. Define the $[\boldsymbol{V}]^I$-functor $f^*\colon (f^*)_\bullet([\boldsymbol{X}]^J) \to [\boldsymbol{X}]^I$ as

$$f_0^*(J \xrightarrow{x} X) \coloneqq I \xrightarrow{f} J \xrightarrow{x} X$$
$$f_1^*(J \xrightarrow{x_0} X, J \xrightarrow{x_1} X) \coloneqq \mathrm{Hom}_{(f^*)_\bullet([\boldsymbol{X}]^J)}(x_0,x_1) \xrightarrow{\mathrm{id}} \mathrm{Hom}_{[\boldsymbol{X}]^I}(x_0 f, x_1 f).$$

Since $f_1^*(x_0, x_1)$ is the identity of $\mathrm{Hom}_{\boldsymbol{X}}(x_0 f, x_1 f)$ as an object of $[\boldsymbol{V}]^I$, then f^* is full and faithful, as required by the definition. The rest of the structure is given by canonical isomorphisms verifying the axioms.

Secondly, let $F \colon \boldsymbol{X} \to \boldsymbol{Y}$ be a $\boldsymbol{V}$-enriched functor and let's define an indexed $[\boldsymbol{V}]$-enriched functor $[F] : [\boldsymbol{X}] \to [\boldsymbol{Y}]$ induced by F. For an indexing object I, define the $[\boldsymbol{V}]^I$-enriched functor $[F]^I : [\boldsymbol{X}]^I \to [\boldsymbol{Y}]^I$ as

Objects component: $[F]^I(I \xrightarrow{x} X) := I \xrightarrow{x} X \xrightarrow{F_0} Y$;

Morphisms component: $[F]^I(I \xrightarrow{x_0} X, I \xrightarrow{x_1} X) := I \xrightarrow{(x_0, x_1)} X \times X \xrightarrow{F_1} V_1$

Notice that, for any reindexing $f \colon I \to J$, we have an equality $[F]^I \circ f^* = f^* \circ (f^*)_\bullet([F]^J)$, meaning that the axioms for indexed $[\boldsymbol{V}]$-enriched functors are automatically satisfied.

Finally, let $\alpha : F \to G \colon \boldsymbol{X} \to \boldsymbol{Y}$ be a $\boldsymbol{V}$-enriched natural transformation and let's define an indexed $[\boldsymbol{V}]$-natural transformation $[\alpha] : [F] \to [G] : [\boldsymbol{X}] \to [\boldsymbol{Y}]$ induced by α. Let I be an indexing object. Then, define the $[\boldsymbol{V}]^I$-enriched natural transformation $[\alpha]^I : [F]^I \to [G]^I : [\boldsymbol{X}]^I \to [\boldsymbol{Y}]^I$ as

$$[\alpha]^I(I \xrightarrow{x} X) := I \xrightarrow{x} X \xrightarrow{\alpha} V_1.$$

The naturality condition for indexed $[\boldsymbol{V}]$-natural transformations is trivially satisfied because the defining isomorphisms of indexed $[\boldsymbol{V}]$-functors $[F]$ and $[G]$ are identities.

With the data previously defined, we have the following proposition.

Proposition 3.2.2. *There is a 2-functor* $[\text{-}]$ *from* $\boldsymbol{V}\mathrm{Cat}_{\mathscr{C}}$ *to the 2-category of indexed* $[\boldsymbol{V}]$*-categories* $[\boldsymbol{V}]\mathrm{ICat}_{\mathscr{C}}$.

The previous result is extremely weak. Indeed, we would like to better understand the 2-functor $[\text{-}]$.

Observe that there is a construction inducing $\boldsymbol{V}$-enriched functors from indexed $[\boldsymbol{V}]$-functors. Let $\mathscr{F} : [\boldsymbol{X}] \to [\boldsymbol{Y}]$ be a $[\boldsymbol{V}]$-functor. Define the $\boldsymbol{V}$-functor $\bar{\mathscr{F}} : \boldsymbol{X} \to \boldsymbol{Y}$ as

$$\bar{\mathscr{F}}_0 := (\mathscr{F}^X)_0(\mathrm{id}(X)) : X \to Y$$
$$\bar{\mathscr{F}}_1 := \phi \circ_{\boldsymbol{Y}} (\mathscr{F}^{X \times X})_1(\pi_1, \pi_2) : X \times X \to V_1.$$

The isomorphism appearing in the definition of the morphism component requires some explanation. The source and target of $(\mathscr{F}^{X \times X})_1(\pi_1, \pi_2)$ are, respectively, $(\mathscr{F}^{X \times X})_0(\pi_1)$ and $(\mathscr{F}^{X \times X})_0(\pi_2)$, while we need an arrow from $(\mathscr{F}^X)_0(\mathrm{id}(X))\pi_1$ to $(\mathscr{F}^X)_0(\mathrm{id}(X))\pi_2$ to match the definition of $\bar{\mathscr{F}}$ on objects. We fix this issue by introducing a suitable isomorphism. By the definition of $[\boldsymbol{V}]$-functor, we have an isomorphism

$$\mathscr{F}^{X \times X} \circ {\pi_i}^* \cong {\pi_i}^* \circ ({\pi_i}^*)_\bullet(\mathscr{F}^X) : ({\pi_i}^*)_\bullet([\boldsymbol{X}]^X) \to [\boldsymbol{X}]^{X \times X}$$

which we apply to the object $\mathrm{id}(X)$ of $(\pi_i{}^*)_\bullet([\boldsymbol{X}]^X)$ to get an isomorphism

$$\phi_i\colon (\mathscr{F}^{X\times X})_0(\pi_i) \cong (\mathscr{F}^X)_0(\mathrm{id}(X))\pi_i$$

in $[\boldsymbol{X}]^{X\times X}$. From ϕ_1 and ϕ_2 we get the isomorphism ϕ that we need.

Moreover, there is also a construction inducing $\boldsymbol{V}$-enriched natural transformations from indexed $[\boldsymbol{V}]$-natural transformations. Let $\alpha\colon \mathscr{F} \to \mathscr{G}\colon [\boldsymbol{X}] \to [\boldsymbol{Y}]$ be an indexed $[\boldsymbol{V}]$-natural transformation. Define the $\boldsymbol{V}$-enriched natural transformation $\bar{\alpha}\colon \bar{\mathscr{F}} \to \bar{\mathscr{G}}\colon \boldsymbol{X} \to \boldsymbol{Y}$ as

$$\bar{\alpha} := \alpha^X(\mathrm{id}(X))\colon X \to V_1.$$

It is clear that, for a $\boldsymbol{V}$-enriched functor $F\colon \boldsymbol{X} \to \boldsymbol{Y}$, we have $F = \overline{[F]}$, and for a $\boldsymbol{V}$-enriched natural transformation $\alpha\colon F \to G\colon \boldsymbol{X} \to \boldsymbol{Y}$, we have $\alpha = \overline{[\alpha]}$. This provides a strengthening of proposition 3.2.2, in that it shows that $\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}$ is a sub-2-category of $[\boldsymbol{V}]\mathrm{ICat}_{\mathscr{E}}$. Moreover, as proved in the following proposition, $\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}$ is a full sub-2-category, meaning that there is an equivalence of hom-categories.

Proposition 3.2.3. *The 2-category* $\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}$ *is a full sub-2-category of* $[\boldsymbol{V}]\mathrm{ICat}_{\mathscr{E}}$*, and* $[\text{-}]$ *is the relative inclusion.*

Proof. Consider indexed $[\boldsymbol{V}]$-categories $[\boldsymbol{X}]$ and $[\boldsymbol{Y}]$. We need to show that there is an equivalence of categories

$$\mathrm{Cat}_{\boldsymbol{V}}(\boldsymbol{X},\boldsymbol{Y}) \equiv \mathrm{Cat}_{[\boldsymbol{V}]}([\boldsymbol{X}],[\boldsymbol{Y}]).$$

Let $\mathscr{F}\colon [\boldsymbol{X}] \to [\boldsymbol{Y}]$ be a $[\boldsymbol{V}]$-functor, and consider the indexed $[\boldsymbol{V}]$-functor $[\bar{\mathscr{F}}]\colon [\boldsymbol{X}] \to [\boldsymbol{Y}]$. We need to prove that there is a $[\boldsymbol{V}]$-natural isomorphism $\mathscr{F} \cong [\bar{\mathscr{F}}]$. Let I be an indexing object in $\mathscr{E}$. Then we need a natural isomorphism $\mathscr{F}^I \cong [\bar{\mathscr{F}}]^I$. Let $x\colon I \to X$ be an object of $[\boldsymbol{X}]^I$. By the definition of $[\boldsymbol{V}]$-functor, we have an isomorphism

$$\mathscr{F}^I \circ x^* \cong x^* \circ (x^*)_\bullet(\mathscr{F}^X)\colon (x^*)_\bullet([\boldsymbol{X}]^X) \to [\boldsymbol{X}]^I$$

which we apply to the object $\mathrm{id}(X)$ of $(x^*)_\bullet([\boldsymbol{X}]^X)$ to get an isomorphism

$$\mathscr{F}^I_0(x) \cong_x \mathscr{F}^X_0(\mathrm{id}(X)) \circ x$$

in $[\boldsymbol{X}]^I$. Then, we take that as the definition of the isomorphism $\mathscr{F}^I \cong [\bar{\mathscr{F}}]^I$ on $x\colon I \to X$.

We need to prove that the isomorphism just defined is natural. Let I be an indexing object in $\mathscr{E}$ and $x_1, x_2\colon I \to X$ objects of $[\boldsymbol{X}]^I$. By the definition of $[\boldsymbol{V}]$-functor, we have an isomorphism

$$\mathscr{F}^I \circ (x_1,x_2)^* \cong (x_1,x_2)^* \circ ((x_1,x_2)^*)_\bullet(\mathscr{F}^{X\times X})\colon ((x_1,x_2)^*)_\bullet([\boldsymbol{X}]^{X\times X}) \to [\boldsymbol{X}]^I$$

which we apply to the objects π_i of $((x_1,x_2)^*)_\bullet([\boldsymbol{X}]^{X\times X})$ to get an isomorphism

$$\mathscr{F}_0^I(x_i) \cong \mathscr{F}_0^{X\times X}(\pi_i) \circ (x_1,x_2)$$

in $[\boldsymbol{X}]^I$. Then, consider the following diagram.

$$\begin{array}{ccccc}
\mathscr{F}_0^I(x_1) & \xrightarrow{\cong} & \mathscr{F}_0^{X\times X}(\pi_1)\circ(x_1,x_2) & \xrightarrow{\cong} & \mathscr{F}_0^X(\mathrm{id}(X))\circ x_1 \\
{\scriptstyle \mathscr{F}_1^I(x_1,x_2)}\downarrow & & \downarrow{\scriptstyle \mathscr{F}_1^{X\times X}(\pi_1,\pi_2)\circ(x_1,x_2)} & & \downarrow{\scriptstyle [\bar{\mathscr{F}}]_1^I(x_1,x_2)} \\
\mathscr{F}_0^I(x_2) & \xrightarrow{\cong} & \mathscr{F}_0^{X\times X}(\pi_2)\circ(x_1,x_2) & \xrightarrow{\cong} & \mathscr{F}_0^X(\mathrm{id}(X))\circ x_1
\end{array}$$

Firstly, the left-hand-side square commutes because of the naturality of the isomorphism. Secondly, the right-hand-side square commutes because such is the definition of $[\bar{\mathscr{F}}]$. Finally, the composition of the consecutive isomorphisms $\mathscr{F}_0^I(x_i) \to \mathscr{F}_0^X(\mathrm{id}(X)) \circ x_i$ is the isomorphism $\mathscr{F}^I \cong [\bar{\mathscr{F}}]^I$ computed on x_i, because of the functoriality axiom for $[\boldsymbol{V}]$-functors applied to the functor $\mathscr{F}$ and the composition $\pi_i \circ (x_1,x_2) = x_i$. But then the outer square is the naturality diagram, and we have shown that it commutes.

Then, we need to prove that the isomorphism $\mathscr{F} \cong [\bar{\mathscr{F}}]$ satisfies the naturality condition for $[\boldsymbol{V}]$-natural transformations. Let $f\colon I \to J$ be a reindexing and $x\colon J \to X$ an object of $[\boldsymbol{X}]$. Then the naturality diagram for the reindexing f and computed on x is

$$\begin{array}{ccc}
\mathscr{F}_0^I(j\circ f) & \xrightarrow{\cong_f} & \mathscr{F}_0^I(x)\circ f \\
{\scriptstyle \cong_{x\circ f}}\searrow & & \swarrow{\scriptstyle \cong_x\circ f} \\
& [\bar{\mathscr{F}}]_0^I(x\circ f) = \mathscr{F}_0^X(\mathrm{id}(X))\circ x\circ f &
\end{array}$$

which commutes thanks to the functoriality axiom for $[\boldsymbol{V}]$-functors applied to the functor $\mathscr{F}$ and the composition of f and x.

Finally, we need to prove that, for any indexed $[\boldsymbol{V}]$-natural transformation $\alpha\colon \mathscr{F} \to \mathscr{G}\colon [\boldsymbol{X}] \to [\boldsymbol{Y}]$, the following square commute.

$$\begin{array}{ccc}
\mathscr{F} & \xrightarrow{\alpha} & \mathscr{G} \\
{\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\
[\bar{\mathscr{F}}] & \xrightarrow{[\bar{\alpha}]} & [\bar{\mathscr{G}}]
\end{array}$$

Let I be an indexing object in $\mathscr{E}$ and $x\colon I \to X$ an object of $[\boldsymbol{X}]^I$, and compute the I-th component of the above diagram on x. We get a commutative square, as it is an instance of the naturality axiom for the indexed $[\boldsymbol{V}]$-natural transformation α, relative to the reindexing x and computed on $\mathrm{id}(X)$. □

Remark 3.2.4. The converse of proposition 3.2.3 does not seem to hold, that is, indexed $[\boldsymbol{V}]$-categories don't canonically induce internal $\boldsymbol{V}$-enriched categories. In particular, the categories $[\boldsymbol{X}]^I$ are small, as their object of objects is the homset $\mathscr{E}(I,X)$, but that is not generally the case for indexed $[\boldsymbol{V}]$-categories.

We then straightforwardly get the following corollary.

Corollary 3.2.5. *The 2-category of small $[\boldsymbol{V}]$-categories, functors and natural transformations is a full sub-2-category of the 2-category of indexed $[\boldsymbol{V}]$-categories, functors and natural transformations.*

To conclude, we look at the interplay between externalization and the underlying category of points.

Proposition 3.2.6. *Let $\boldsymbol{X}$ be a $\boldsymbol{V}$-enriched category. Then, there is a natural isomorphism of indexed categories $U([\boldsymbol{X}]) \cong [U(\boldsymbol{X})]$ between the underlying indexed category of the indexed $[\boldsymbol{V}]$-enriched category $[\boldsymbol{X}]$ and the externalization of the underlying $\mathscr{E}$-category $U(\boldsymbol{X})$.*

Proof. Let I be an indexing object. We need to prove that there is an isomorphism $U([\boldsymbol{X}]^I) \cong [U(\boldsymbol{X})]^I$ between the underlying standard category of the $[\boldsymbol{V}]^I$-enriched category $[\boldsymbol{X}]^I$ and the fiber over I of the externalization of the underlying $\mathscr{E}$-category $U(\boldsymbol{X})$. Moreover, for any reindexing $f\colon J \to I$ in $\mathscr{E}$, the square

$$\begin{array}{ccc} U([\boldsymbol{X}]^I) & \xrightarrow{\cong} & [U(\boldsymbol{X})]^I \\ {\scriptstyle U(f^*([\boldsymbol{X}]))}\big\downarrow & & \big\downarrow{\scriptstyle f^*([U(\boldsymbol{X})])} \\ U([\boldsymbol{X}]^J) & \xrightarrow{\cong} & [U(\boldsymbol{X})]^J \end{array}$$

has to commute.

For both categories, the objects are I-indexed families of objects of $\boldsymbol{X}$, and the arrows $(I \xrightarrow{x_0} X) \to (I \xrightarrow{x_1} X)$ are the sections of the projection

$$(\mathbb{I}_{\boldsymbol{V}}!_X, \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1))^* V_1 \to I,$$

so that the categories are clearly isomorphic to each other, and the square commutes trivially. □

The previous result can be extended to the following proposition.

Proposition 3.2.7. *The following diagram of 2-functors commutes.*

$$\begin{array}{ccc} \boldsymbol{V}\mathrm{Cat}_{\mathscr{E}} & \xrightarrow{U} & \mathrm{Cat}_{\mathscr{E}} \\ {\scriptstyle [\text{-}]}\big\downarrow & & \big\downarrow{\scriptstyle [\text{-}]} \\ [\boldsymbol{V}]\mathrm{ICat}_{\mathscr{E}} & \xrightarrow{U} & \mathrm{ICat}_{\mathscr{E}} \end{array}$$

3.3. Categories of Functors

Analogously to the treatment of internal categories, we now look at the enrichment of $\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}$ over $\mathrm{Cat}_{\mathscr{E}}$. That is, we show how two $\boldsymbol{V}$-enriched categories give rise to a category in $\mathscr{E}$ of the functors between them and their natural transformations. That will be propaedeutic to the investigation regarding the monoidal closure of $\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}$, whereas, instead of an internal category of functors and natural transformations, we will need an internally enriched one.

In this section, we shall assume that $\mathscr{E}$ is cartesian closed.

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be $\boldsymbol{V}$-categories. Define $\mathrm{Fun}(\boldsymbol{X},\boldsymbol{Y})$ as the object representing the $\boldsymbol{V}$-enriched functors $\boldsymbol{X} \to \boldsymbol{Y}$; that is, the subobject of $Y^X \times {V_1}^{X\times X}$ of those $F = (F_0 \colon Y^X, F_1 \colon {V_1}^{X\times X})$ such that, in the context $x_0, x_1, x_2 \colon X$, the axioms

$$\mathrm{e}(F_1, (x_0, x_1)) \colon \mathrm{Hom}_{\boldsymbol{X}}(x_0, x_1) \to \mathrm{Hom}_{\boldsymbol{Y}}(\mathrm{e}(F_0, x_0), \mathrm{e}(F_0, x_1)) \colon V_1$$

and

$$\begin{array}{ccc}
\mathrm{Hom}_{\boldsymbol{X}}(x_1,x_2) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) & \xrightarrow{\circ_{\boldsymbol{X}}(x_0,x_1,x_2)} & \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_2) \\
\Big\downarrow {\scriptstyle \mathrm{e}(F_1,(x_1,x_2)) \otimes^{\boldsymbol{V}}_1 \mathrm{e}(F_1,(x_0,x_1))} & & \Big\downarrow {\scriptstyle \mathrm{e}(F_1,(x_0,x_2))} \\
\mathrm{Hom}_{\boldsymbol{Y}}(\mathrm{e}(F_0,x_1), \mathrm{e}(F_0,x_2)) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(\mathrm{e}(F_0,x_0), \mathrm{e}(F_0,x_1)) & \xrightarrow{\circ_{\boldsymbol{Y}}(\mathrm{e}(F_0,x_0),\mathrm{e}(F_0,x_1),\mathrm{e}(F_0,x_2))} & \mathrm{Hom}_{\boldsymbol{Y}}(\mathrm{e}(F_0,x_0), \mathrm{e}(F_0,x_2))
\end{array}$$

hold.

We shall now validate the intuition that $\mathrm{Fun}(\boldsymbol{X},\boldsymbol{Y})$ is indeed an internal representation of $\boldsymbol{V}$-enriched functors $\boldsymbol{X} \to \boldsymbol{Y}$, or, in other words, that $\boldsymbol{V}$-enriched functors $\boldsymbol{X} \to \boldsymbol{Y}$ correspond to the global sections of $\mathrm{Fun}(\boldsymbol{X},\boldsymbol{Y})$. That suggests that $\mathrm{Fun}(\text{-},\text{-})$ may actually yield the hom-objects for an enriched category extending $\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}$.

Proposition 3.3.1. *There is an $\mathscr{E}$-enriched category whose underlying category is isomorphic to $\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}$ (and, with abuse of notation, it shall be called in the same way). That is, for all $\boldsymbol{V}$-enriched categories $\boldsymbol{X}$ and $\boldsymbol{Y}$, there is a natural bijection between the hom-sets*

$$\mathscr{E}(\mathbb{1}_{\mathscr{E}}, \mathrm{Hom}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{X},\boldsymbol{Y})) \cong \boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}(\boldsymbol{X},\boldsymbol{Y}). \tag{3.6}$$

Proof. Let $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ be $\boldsymbol{V}$-categories. Define the enrichment of $\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}$ in $\mathscr{E}$:

Hom-object: $\mathrm{Hom}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{X},\boldsymbol{Y})$, the hom-object of $\boldsymbol{X}$ and $\boldsymbol{Y}$, is $\mathrm{Fun}(\boldsymbol{X},\boldsymbol{Y})$.

Composition: The composition of $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ is given by the arrow

$$\circ_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}) \colon \mathrm{Hom}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{X},\boldsymbol{Y}) \times \mathrm{Hom}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{X},\boldsymbol{Y}) \to \mathrm{Hom}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{X},\boldsymbol{Z})$$

where, in context $F\colon \mathrm{Hom}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{C}}}(\boldsymbol{X},\boldsymbol{Y}), G\colon \mathrm{Hom}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{C}}}(\boldsymbol{Y},\boldsymbol{Z})$, the component $\circ_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{C}}}(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z})(F,G)_0 \colon Z^X$ is defined as $\lambda x\colon X.\ \mathrm{e}_{\mathscr{C}}(G_0, \mathrm{e}_{\mathscr{C}}(F_0, x))$ and the component $\circ_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{C}}}(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z})(F,G)_1 \colon {V_1}^{X\times X}$ as

$$\lambda x_0, x_1 \colon X. \quad \begin{array}{c} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \\ \big\downarrow {\scriptstyle \mathrm{e}_{\mathscr{C}}(F_1,(x_0,x_1))} \\ \mathrm{Hom}_{\boldsymbol{Y}}(\mathrm{e}(F_0,x_0),\mathrm{e}(F_0,x_1)) \\ \big\downarrow {\scriptstyle \mathrm{e}_{\mathscr{C}}(G_1,(\mathrm{e}_{\mathscr{C}}(F_0,x_0),(\mathrm{e}_{\mathscr{C}}(F_0,x_1))))} \\ \mathrm{Hom}_{\boldsymbol{Z}}\big(\mathrm{e}(G_0,\mathrm{e}(F_0,x_0)),\mathrm{e}(G_0,\mathrm{e}(F_0,x_1))\big) \end{array} .$$

Identity: The identity of $\boldsymbol{X}$ is the arrow

$$\mathrm{id}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{C}}}(\boldsymbol{X}) \colon \mathbb{1}_{\mathscr{C}} \to \mathrm{Hom}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{C}}}(\boldsymbol{X},\boldsymbol{X})$$

where the component $\mathrm{id}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{C}}}(\boldsymbol{X})_0 \colon X^X$ is defined as $\lambda x\colon X.\ x$ and the component $\mathrm{id}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{C}}}(\boldsymbol{X})_1 \colon {V_1}^{X\times X}$ as $\lambda x_0, x_1 \colon X.\ \mathrm{id}_{\boldsymbol{V}}(\mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1))$.

Finally, the bijection (3.6) is obtained straightforwardly by applying the exponential transpose. □

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be $\boldsymbol{V}$-categories. Let's define $\mathrm{Nat}(\boldsymbol{X},\boldsymbol{Y})$ as the object representing the $\boldsymbol{V}$-enriched natural transformations $\boldsymbol{X} \to \boldsymbol{Y}$; that is, the subobject of $\mathrm{Fun}(\boldsymbol{X},\boldsymbol{Y}) \times \mathrm{Fun}(\boldsymbol{X},\boldsymbol{Y}) \times {V_1}^X$ of those

$$(F\colon \mathrm{Fun}(\boldsymbol{X},\boldsymbol{Y}), G\colon \mathrm{Fun}(\boldsymbol{X},\boldsymbol{Y}), \alpha\colon {V_1}^X)$$

such that, in context $x, x_0, x_1 \colon X$, the axioms

$$\mathrm{e}(\alpha,x) \colon \mathbb{I}_{\boldsymbol{V}} \to \mathrm{Hom}_{\boldsymbol{Y}}(\mathrm{e}(F_0,x),\mathrm{e}(F_0,x)) \colon V_1$$

and

$$\begin{array}{ccc} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) & \xrightarrow{\mathrm{e}(\alpha,x_1)\otimes^{\boldsymbol{V}}\mathrm{e}(F_1,(x_0,x_1))} & \begin{array}{c}\mathrm{Hom}_{\boldsymbol{Y}}(\mathrm{e}(F_0,x_1),\mathrm{e}(G_0,x_1)) \\ \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(\mathrm{e}(F_0,x_0),\mathrm{e}(F_0,x_1))\end{array} \\ {\scriptstyle \mathrm{e}(G_1,(x_0,x_1))\otimes^{\boldsymbol{V}}\mathrm{e}(\alpha,x_0)}\big\downarrow & & \big\downarrow {\scriptstyle \circ_{\boldsymbol{Y}}(\mathrm{e}(F_0,x_0),\mathrm{e}(F_0,x_1),\mathrm{e}(G_0,x_1))} \\ \begin{array}{c}\mathrm{Hom}_{\boldsymbol{Y}}(\mathrm{e}(G_0,x_0),\mathrm{e}(G_0,x_1)) \\ \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(\mathrm{e}(F_0,x_0),\mathrm{e}(G_0,x_0))\end{array} & \xrightarrow[\circ_{\boldsymbol{Y}}(\mathrm{e}(F_0,x_0),\mathrm{e}(G_0,x_0),\mathrm{e}(G_0,x_1))]{} & \mathrm{Hom}_{\boldsymbol{Y}}(\mathrm{e}(F_0,x_0),\mathrm{e}(G_0,x_1)) \end{array}$$

hold.

As before, we now validate the intuition that $\mathrm{Nat}(\boldsymbol{X},\boldsymbol{Y})$ represents the object of $\boldsymbol{V}$-natural transformations $\boldsymbol{X} \to \boldsymbol{Y}$. By doing that, we prove that the enrichment of $\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}$ over $\mathscr{E}$ extends to an enrichment over $\mathrm{Cat}_{\mathscr{E}}$; in other words, that two $\boldsymbol{V}$-enriched categories yield an internal category in $\mathscr{E}$ whose underlying standard category is that of $\boldsymbol{V}$-functors between them and their natural transformations.

Proposition 3.3.2. *There is a* $\mathrm{Cat}_{\mathscr{E}}$*-enriched category whose underlying category is isomorphic to* $\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}$ *(and, with abuse of notation, it shall be called in the same way). Such enrichment extends that of proposition 3.3.1. That is, for all* $\boldsymbol{V}$*-enriched categories* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$*, the natural bijection (3.6) extends to an isomorphism of categories*

$$\mathrm{Cat}_{\mathscr{E}}(\mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}}, \mathrm{Hom}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{X},\boldsymbol{Y})) \cong \boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}(\boldsymbol{X},\boldsymbol{Y}). \tag{3.7}$$

Proof. Let $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ be $\boldsymbol{V}$-categories. Let's define the category $\mathrm{Hom}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{X},\boldsymbol{Y})$ in $\mathscr{E}$ as:

Object of objects: let $\mathrm{Hom}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{X},\boldsymbol{Y})_0$ be $\mathrm{Fun}(\boldsymbol{X},\boldsymbol{Y})$.

Object of arrows: let $\mathrm{Hom}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{X},\boldsymbol{Y})_1$ be $\mathrm{Nat}(\boldsymbol{X},\boldsymbol{Y})$ with the first and second projections as source and target, respectively.

Composition: In context $(F,G,\alpha),(G,H,\beta) : \mathrm{Nat}(\boldsymbol{X},\boldsymbol{Y})$, composition is defined as

$$\beta \circ_{\mathrm{Hom}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{X},\boldsymbol{Y})} \alpha \coloneqq (F,H,\lambda x : X.\ \mathrm{e}(\beta,x) \circ_{\boldsymbol{V}} \mathrm{e}(\alpha,x) : V_1) : \mathrm{Nat}(\boldsymbol{X},\boldsymbol{Y}).$$

Identity: in context $F : \mathrm{Fun}(\boldsymbol{X},\boldsymbol{Y})$, identity is defined as

$$\mathrm{id}_{\mathrm{Hom}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{X},\boldsymbol{Y})}(F) \coloneqq (F,F,\lambda x : X.\ \mathrm{id}_{\boldsymbol{Y}}(\mathrm{e}(F_0,x)) : V_1) : \mathrm{Nat}(\boldsymbol{X},\boldsymbol{Y}).$$

The composition

$$\circ_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}) : \mathrm{Hom}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{Y},\boldsymbol{Z}) \times^{\mathrm{Cat}_{\mathscr{E}}} \mathrm{Hom}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{X},\boldsymbol{Y}) \to \mathrm{Hom}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{X},\boldsymbol{Z})$$

is the functor which, on the object component, acts as the composition of functors in proposition 3.3.1. On the morphism component,

$$(F',G',\beta)\,(\circ_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}})_1\,(F,G,\alpha) \coloneqq \begin{pmatrix} F'\,(\circ_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}})_0 F, \\ G'\,(\circ_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}})_0 G, \\ \lambda x : X.\ \mathrm{e}(\beta,\mathrm{e}(G_0,x)) \circ_{\boldsymbol{Z}} \mathrm{e}(F_0,\mathrm{e}(\alpha,x)) \end{pmatrix}$$

(which is one out of two equivalent ways to define composition of natural transformations).

The identity

$$\mathrm{id}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{X})\colon \mathbb{I}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}} \to \mathrm{Hom}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{X},\boldsymbol{X})$$

targets the identity functor in $\mathrm{Fun}(\boldsymbol{X},\boldsymbol{X})$ and the identity natural transformation on it in $\mathrm{Nat}(\boldsymbol{X},\boldsymbol{X})$.

Finally, the object and morphism components of the isomorphism (3.7) are obtained by applications of the exponential transpose. □

There is abuse of notation in that $\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}$ is enriched both in $\mathscr{E}$ and in $\mathrm{Cat}_{\mathscr{E}}$, and thus $\mathrm{Hom}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{X},\boldsymbol{Y})$ can either be an object or an internal category in $\mathscr{E}$. Moreover, $\mathrm{Hom}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{X},\boldsymbol{Y})$ as an object is the object of objects of $\mathrm{Hom}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{X},\boldsymbol{Y})$ as an internal category in $\mathscr{E}$.

3.4. Profunctors and Extranatural Transformations

In the following, we will need the notion of end. In order to properly define ends, though, it is useful to dedicate some time to the treatment of profunctors and extranatural transformations, which will be the main topic of the present section. This will allow for a very general and elegant theory of ends, requiring no unnecessary assumptions.

In standard enriched category theory, profunctor are normally defined as certain functors into the enriching category. Such a notion, though, is well-defined only if the enriching category is monoidal closed and thus enriched over itself. The same situation presents itself in the internal context. Luckily, by stating the definitions carefully, it is possible to introduce profunctors, extranatural transformations and ends without requiring extra assumptions on the enriching category.

Definition 3.4.1 (profunctor)**.** Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be $\boldsymbol{V}$-enriched categories. A $\boldsymbol{V}$-*enriched profunctor* $P\colon \boldsymbol{X} \nrightarrow \boldsymbol{Y}$ is given by

Object component: an arrow $P_0\colon Y \times X \to V_0$;

Morphism component: an arrow $P_1\colon (Y \times X) \times (Y \times X) \to V_1$ of shape

$$\begin{aligned} &x_0,x_1\colon X, y_0,y_1\colon Y \vdash P_1((y_0,x_0),(y_1,x_1))\colon \\ &\qquad \mathrm{Hom}_{\boldsymbol{Y}}(y_1,y_0) \otimes P_0(y_0,x_0) \otimes \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \to P_0(y_1,x_1)\colon V_1; \end{aligned}$$

satisfying, in context $x, x_0, x_1, x_2 \colon X, y, y_0, y_1, y_2 \colon Y$, the axioms

$$\begin{array}{ccc}
\begin{array}{c}\mathrm{Hom}_{\boldsymbol{Y}}(y_2,y_1) \otimes \mathrm{Hom}_{\boldsymbol{Y}}(y_1,y_0) \\ \otimes P_0(y_0,x_0) \otimes \\ \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \otimes \mathrm{Hom}_{\boldsymbol{X}}(x_1,x_2)\end{array} & \xrightarrow{\mathrm{Hom}_{\boldsymbol{Y}}(y_2,y_1) \otimes P_1((y_0,x_0),(y_1,x_1)) \otimes \mathrm{Hom}_{\boldsymbol{X}}(x_1,x_2)} & \begin{array}{c}\mathrm{Hom}_{\boldsymbol{Y}}(y_2,y_1) \\ \otimes P_0(y_1,x_1) \otimes \\ \mathrm{Hom}_{\boldsymbol{X}}(x_1,x_2)\end{array} \\
\Big\downarrow {\scriptstyle \circ_{\boldsymbol{Y}}(y_0,y_1,y_2) \otimes P_0(y_0,x_0) \otimes \circ_{\boldsymbol{X}}(x_0,x_1,x_2)} & & \Big\downarrow {\scriptstyle P_1((y_1,x_1),(y_1,x_1))} \\
\begin{array}{c}\mathrm{Hom}_{\boldsymbol{Y}}(y_2,y_0) \\ \otimes P_0(y_0,x_0) \otimes \\ \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_2)\end{array} & \xrightarrow[P_1((y_0,x_0),(y_2,x_2))]{} & P_0(y_2,x_2)
\end{array}$$

and

$$\begin{array}{ccc}
P_0(y,x) & \xrightarrow{\mathrm{id}} & P_0(y,x) \\
{\scriptstyle \mathrm{id}_{\boldsymbol{Y}}(y) \otimes P_0(y,x) \otimes \mathrm{id}_{\boldsymbol{X}}(x)} \searrow & & \nearrow {\scriptstyle P_1((y,x),(y,x))} \\
& \mathrm{Hom}_{\boldsymbol{Y}}(y,y) \otimes P_0(y,x) \otimes \mathrm{Hom}_{\boldsymbol{X}}(x,x) &
\end{array} .$$

Remark 3.4.2. Given a profunctor $P \colon \boldsymbol{X}' \nrightarrow \boldsymbol{Y}'$ and functors $F \colon \boldsymbol{X} \to \boldsymbol{X}'$ and $G \colon \boldsymbol{Y} \to \boldsymbol{Y}'$, there is a composite profunctor $P(G,F) : \boldsymbol{X} \nrightarrow \boldsymbol{Y}$ defined, in context $x_0, x_1 \colon X, y_0, y_1 \colon Y$, as

$$(P(G,F))_0 \coloneqq Y \times X \xrightarrow{F \times G} Y' \times X' \xrightarrow{P} V_0$$

$$(P(G,F))_1((y_0,x_0),(y_1,x_1)) \coloneqq$$

$$\begin{array}{c}
\mathrm{Hom}_{\boldsymbol{Y}}(y_1,y_0) \otimes (P(G,F))_0(y_0,x_0) \otimes \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \\
\Big\downarrow {\scriptstyle G_1(y_1,y_0) \otimes (P(G,F))_0(y_0,x_0) \otimes F_1(x_0,x_1)} \\
\mathrm{Hom}_{\boldsymbol{Y}'}(G_0(y_1),G_0(y_0)) \otimes P_0(G_0(y_0),F_0(x_0)) \otimes \mathrm{Hom}_{\boldsymbol{X}'}(F_0(x_0),F_0(x_1)) \\
\Big\downarrow {\scriptstyle P_1((G_0(y_0),F_0(x_0)),(G_0(y_1),F_0(x_1)))} \\
P_0(G_0(y_1),F_0(x_1)) = (P(G,F))_0(y_1,x_1).
\end{array}$$

Example 3.4.3. Any $\boldsymbol{V}$-enriched category $\boldsymbol{X}$ has an associated profunctor induced by the internal hom, $\mathrm{Hom}_{\boldsymbol{X}} \colon \boldsymbol{X} \nrightarrow \boldsymbol{X}$, where $(\mathrm{Hom}_{\boldsymbol{X}})_0$ is the internal hom of $\boldsymbol{X}$ and $(\mathrm{Hom}_{\boldsymbol{X}})_1$ is given by the composition in $\boldsymbol{X}$.

Example 3.4.4. Let $F \colon \boldsymbol{X} \to \boldsymbol{Y}$ be a $\boldsymbol{V}$-enriched functor. Then, there is a $\boldsymbol{V}$-enriched profunctor $\mathrm{Hom}_{\boldsymbol{Y}}(\text{-},F) : \boldsymbol{X} \nrightarrow \boldsymbol{Y}$ given by the composition of $\mathrm{Hom}_{\boldsymbol{Y}}$ and F. Analogously, there is a $\boldsymbol{V}$-enriched profunctor $\mathrm{Hom}_{\boldsymbol{Y}}(F,\text{-}) : \boldsymbol{Y} \nrightarrow \boldsymbol{X}$.

Example 3.4.5. The $\boldsymbol{V}$-enriched profunctors $\mathbb{I}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{C}}} \nrightarrow \mathbb{I}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{C}}}$ are the same thing as the internal functors $\mathbb{I}_{\mathrm{Cat}_{\mathscr{C}}} \to \boldsymbol{V}$.

We shall now give a definition of extranatural transformation in the context of internal enriched categories. That is, a transformation between profunctors that is (ordinary-)natural in some variables and extranatural in others (the meaning of which is clarified in the definition). Notice that some occurrences of identity arrows of enriched categories (such as $\mathrm{id}_{\boldsymbol{X}}(x,x)$) have been omitted in the definition's axioms to improve clarity.

Definition 3.4.6 (extranatural transformation)**.** Let $P\colon \boldsymbol{X} \otimes \boldsymbol{Y} \nrightarrow \boldsymbol{Y}$ and $Q\colon \boldsymbol{X} \otimes \boldsymbol{Z} \nrightarrow \boldsymbol{Z}$ be $\boldsymbol{V}$-enriched profunctors. A $\boldsymbol{V}$*-enriched extranatural transformation* $\epsilon\colon P \to Q$ is given by a morphism $\epsilon\colon X \times Y \times Z \to V_1$ of shape

$$x\colon X, y\colon Y, z\colon Z \vdash \epsilon(x,y,z) : P_0(x,y,y) \to Q_0(x,z,z) : V_1$$

satisfying, in context $x_0, x_1 : X, y\colon Y, z\colon Z$, the axiom

$$\begin{array}{ccc} P_0(x_0,y,y) \otimes \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) & \xrightarrow{\epsilon(x_0,y,z)\otimes \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1)} & Q_0(x_0,z,z) \otimes \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \\ \downarrow{\scriptstyle P_1((x_0,y,y),(x_1,y,y))} & & \downarrow{\scriptstyle Q_1((x_0,z,z),(x_1,z,z))} \\ P_0(x_1,y,y) & \xrightarrow{\epsilon(x_1,y,z)} & Q_0(x_1,z,z) \end{array}$$

(naturality in $\boldsymbol{X}$); in context $x\colon X, y_0, y_1 : Y, z\colon Z$, the axiom

$$\begin{array}{ccc} P_0(x,y_0,y_1) \otimes \mathrm{Hom}_{\boldsymbol{Y}}(y_0,y_1) & \xrightarrow{P_1((x,y_0,y_1),(x,y_1,y_1))} & P_0(x,y_1,y_1) \\ \downarrow{\scriptstyle P_1((x,y_0,y_1),(x,y_0,y_0))} & & \downarrow{\scriptstyle \epsilon(x,y_1,z)} \\ P_0(x,y_0,y_0) & \xrightarrow{\epsilon(x,y_0,z)} & Q_0(x,z,z) \end{array}$$

(extranaturality in $\boldsymbol{Y}$); and, in context $x\colon X, y\colon Y, z_0, z_1 : Z$, the axiom

$$\begin{array}{ccc} P_0(x,y,y) \otimes \mathrm{Hom}_{\boldsymbol{Z}}(z_0,z_1) & \xrightarrow{\epsilon(x,y,z_1)\otimes \mathrm{Hom}_{\boldsymbol{Z}}(z_0,z_1)} & Q_0(x,z_1,y_1) \otimes \mathrm{Hom}_{\boldsymbol{Z}}(z_0,z_1) \\ \downarrow{\scriptstyle \epsilon(x,y,z_0)\otimes \mathrm{Hom}_{\boldsymbol{Z}}(z_0,z_1)} & & \downarrow{\scriptstyle Q_1((x,z_1,z_1),(x,z_0,z_1))} \\ Q_0(x,z_0,z_0) \otimes \mathrm{Hom}_{\boldsymbol{Z}}(z_0,z_1) & \xrightarrow{Q_1((x,z_0,z_0),(x,z_0,z_1))} & Q_0(x,z_0,z_1) \end{array}$$

(extranaturality in $\boldsymbol{Z}$).

The definition of extranatural transformation supports the calculus of ends and coends (as defined in the next section). For applications, it is important to have definitions parametrized by an additional category representing "ordinary-natural" variables (that is $\boldsymbol{X}$ in our definition). However, we shall not draw further attention to this parametrization in our use of ends and coends, as those are defined relatively to profunctors having "extra-natural" variables only (that are $\boldsymbol{Y}$ and $\boldsymbol{Z}$ in our definition).

Notice that there are many special cases of this definition.

Example 3.4.7. An extranatural transformation $\epsilon\colon A \to P$ between the profunctors $A\colon \mathbb{I}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}} \nrightarrow \mathbb{I}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}}$ (or, equivalently, a functor $A\colon \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}} \to \boldsymbol{V}$) and $P\colon \boldsymbol{X} \nrightarrow \boldsymbol{X}$ only needs to satisfy the extranaturality axiom on $\boldsymbol{X}$.

Example 3.4.8. An extranatural transformation $\epsilon\colon P \to Q$ between the profunctors $P, Q\colon \boldsymbol{X} \nrightarrow \mathbb{I}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}}$ only needs to satisfy the naturality axiom on $\boldsymbol{X}$.

In standard category theory, profunctors and extranatural transformations form a bicategory. Defining the composition of profunctors requires the use of coends, which have not yet been introduced. Anyway, we are not going to need such general composition, but it will be useful to define just a few special cases, for use in the following sections.

Given a $\boldsymbol{V}$-enriched profunctor $P\colon \boldsymbol{X} \nrightarrow \boldsymbol{X}$, a functor $A\colon \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}} \to \boldsymbol{V}$, a $\boldsymbol{V}$-enriched functor $F\colon \boldsymbol{Y} \to \boldsymbol{X}$, and a $\boldsymbol{V}$-enriched extranatural transformation $\epsilon\colon A \to P$, there is an horizontal composition $\epsilon * F\colon A \to P(F, F)$ given by

$$y\colon Y \vdash (\epsilon F)(y) := A \xrightarrow{\epsilon(F_0(y))} P_0(F_0(y), F_0(y))\colon V_1.$$

Moreover, given a $\boldsymbol{V}$-enriched profunctor $P\colon \boldsymbol{X} \nrightarrow \boldsymbol{X}$, a functor $A\colon \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}} \to \boldsymbol{V}$, a $\boldsymbol{V}$-enriched extranatural transformation $\epsilon\colon A \to P$, and a natural transformation $h\colon A' \to A\colon \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}} \to \boldsymbol{V}$, there is an horizontal composition $\epsilon \circ h\colon A' \to P$ given by

$$y\colon Y \vdash (\epsilon \circ_{\boldsymbol{V}} h)(y) := A' \xrightarrow{h} A \xrightarrow{\epsilon(y)} P_0(y, y)\colon V_1.$$

Finally, given $\boldsymbol{V}$-enriched profunctors $P, Q\colon \boldsymbol{X} \nrightarrow \mathbb{I}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}}$, a $\boldsymbol{V}$-enriched extranatural transformation $\epsilon\colon P \to Q$ and a $\boldsymbol{V}$-enriched functor $F\colon \boldsymbol{Y} \to \boldsymbol{X}$, there is an horizontal composition $\epsilon * F\colon PF \to QF$ given by

$$y\colon Y \vdash (\epsilon F)(y) := P_0(F_0(x)) \xrightarrow{\epsilon(F_0(x))} Q_0(F_0(x))\colon V_1.$$

3.5. Ends

Equipped with the tools defined in the previous section, and informed by the discussion from chapter 2, we are now ready to introduce the notion of end, which will turn out to be essential to the treatment of exponentials of internal enriched categories and their weighted limits.

Definition 3.5.1 (universal extranatural transformation)**.** Let $P\colon \boldsymbol{X} \nrightarrow \boldsymbol{X}$ a $\boldsymbol{V}$-enriched profunctor and $E\colon \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}} \to \boldsymbol{V}$ an internal functor. An extranatural transformation $\epsilon\colon E \to P$ is *universal* if, given another functor $E'\colon \mathbb{1}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}} \to \boldsymbol{V}$ and another extranatural transformation $\epsilon'\colon E' \to P$, there exists a unique natural transformation $h\colon E' \to E$ such that $\epsilon' = \epsilon \circ h$.

By duality, we also have the definition of couniversal natural transformation (so called for lack of better terminology in the literature).

Definition 3.5.2 (couniversal extranatural transformation)**.** Let $P\colon \boldsymbol{X} \nrightarrow \boldsymbol{X}$ a $\boldsymbol{V}$-enriched profunctor and $E\colon \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}} \to \boldsymbol{V}$ an internal functor. An extranatural transformation $\epsilon\colon P \to E$ is *couniversal* if, given another functor $E'\colon \mathbb{1}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}} \to \boldsymbol{V}$ and another extranatural transformation $\epsilon'\colon P \to E'$, there exists a unique natural transformation $h\colon E \to E'$ such that $\epsilon' = h \circ \epsilon$.

Analogously to what happens in chapter 2, the definition of universal extranatural transformation is not stable under pullback. So, we shall enforce stability explicitly in the definition of end.

Definition 3.5.3 (end)**.** Let $P\colon \boldsymbol{X} \nrightarrow \boldsymbol{X}$ a $\boldsymbol{V}$-enriched profunctor. An *end* for P is given by an internal functor $\int_{\boldsymbol{X}} P\colon \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}} \to \boldsymbol{V}$ and an extranatural transformation $\pi\colon \int_{\boldsymbol{X}} P \to P$ such that, for any object I of $\mathscr{E}$, the extranatural transformation $I^*\pi\colon I^* \int_{\boldsymbol{X}} P \to I^*P$ is universal.

Definition 3.5.4 (coend)**.** Let $P\colon \boldsymbol{X} \nrightarrow \boldsymbol{X}$ a $\boldsymbol{V}$-enriched profunctor. A *coend* for P is given by an internal functor $\int^{\boldsymbol{X}} P\colon \mathbb{1}_{\mathrm{Cat}_{\mathscr{E}}} \to \boldsymbol{V}$ and an extranatural transformation $\pi\colon P \to \int^{\boldsymbol{X}} P$ such that, for any object I of $\mathscr{E}$, the extranatural transformation $I^*\pi\colon I^*P \to I^* \int^{\boldsymbol{X}} P$ is universal.

Analogously to the definition of complete internal category, we say a category has all ends (for lack of better terminology from the literature) if, *in every slice*, every profunctor has an end, so that the notion is forcefully made stable under pullback.

Definition 3.5.5 (category with all (co)ends)**.** We say that $\boldsymbol{V}$ *has all (co)ends* if, for every object I in $\mathscr{E}$ and every $I^*\boldsymbol{V}$-enriched category $\boldsymbol{X}$, every $I^*\boldsymbol{V}$-enriched profunctor $P\colon \boldsymbol{X} \nrightarrow \boldsymbol{X}$ has a (co)end.

Let's note down (without proof) a few properties of ends and coends.

Proposition 3.5.6. *It is possible to swap ends (and coends) with each other (that is the so-called Fubini theorem):*

$$\int_{y:Y}\int_{x:X} P_0((x,y),(x,y)) \cong \int_{x:X}\int_{y:Y} P_0((x,y),(x,y)) \cong \int_{p:X\times Y} P_0(p,p)$$
$$\int^{y:Y}\int^{x:X} P_0((x,y),(x,y)) \cong \int^{x:X}\int^{y:Y} P_0((x,y),(x,y)) \cong \int^{p:X\times Y} P_0(p,p).$$

Moreover, if $\boldsymbol{V}$ *is monoidal closed, there is an interchange between the exponential objects and (co)ends:*

$$\mathrm{Hom}_{\boldsymbol{V}}(v, \int_{x:X} P_0(x,x)) \cong \int_{x:X} \mathrm{Hom}_{\boldsymbol{V}}(v, P_0(x,x))$$
$$\mathrm{Hom}_{\boldsymbol{V}}(\int_{x:X} P_0(x,x), v) \cong \int^{x:X} \mathrm{Hom}_{\boldsymbol{V}}(P_0(x,x), v).$$

Let's introduce a notation for products in $\boldsymbol{V}$ indexed over some object of $\mathscr{E}$ which will be helpful in the future.

Notation 3.5.7 (product)**.** Let $f\colon X \to V_0$ be a morphism. Consider the discrete category $\mathbf{dis}(X)$ (see example 1.1.9) and the diagram $\mathbf{dis}(F) : \mathbf{dis}(X) \to \boldsymbol{V}$. The limit of that diagram, if it exists, is the *product* of f, and we write it as $\prod_{x:X} f(x) : V_0$.

Proposition 3.5.8. *Assume* $\boldsymbol{V}$ *is symmetric, complete (in the internal sense) and monoidal closed, and it has equalizers. Then,* $\boldsymbol{V}$ *has all ends.*

Proof. Notice that all the hypothesis are stable under slicing. Then, the proof reduces to show that, given the above assumptions, any $\boldsymbol{V}$-enriched profunctor $P\colon \boldsymbol{X} \nrightarrow \boldsymbol{X}$ has an end.

There are two morphisms

$$\phi, \psi\colon \prod_{x:X} P_0(x,x) \to \prod_{x_0,x_1:X} \mathrm{Hom}_{\boldsymbol{V}}(\mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1), P_0(x_0,x_1))$$

which we define component-wise as

$$\phi(x_0,x_1): \quad \begin{array}{c} \prod_{x:X} P_0(x,x) \\ \downarrow \scriptstyle \mathrm{id}_{\boldsymbol{X}}(x_0) \otimes^{\boldsymbol{V}} \pi(x_0) \\ \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_0) \otimes^{\boldsymbol{V}} P_0(x_0,x_0) \\ \downarrow \scriptstyle \overline{P_1((x_0,x_0),(x_0,x_1))} \\ \mathrm{Hom}_{\boldsymbol{V}}(\mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1), P_0(x_0,x_1)) \end{array}$$

where $\overline{P_1((x_0,x_0),(x_0,x_1))}$ is the exponential transpose of $P_1((x_0,x_0),(x_0,x_1))$, and

$$\psi(x_0,x_1): \quad \begin{array}{c} \prod_{x:X} P_0(x,x) \\ \Big\downarrow \mathrm{id}_{\boldsymbol{X}}(x_1)\otimes^{\boldsymbol{V}}\pi(x_1) \\ \mathrm{Hom}_{\boldsymbol{X}}(x_1,x_1)\otimes^{\boldsymbol{V}} P_0(x_1,x_1) \\ \Big\downarrow \overline{P_1((x_1,x_1),(x_0,x_1))} \\ \mathrm{Hom}_{\boldsymbol{V}}(\mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1),P_0(x_0,x_1)) \end{array}$$

where $\overline{P_1((x_1,x_1),(x_0,x_1))}$ is the exponential transpose of $P_1((x_1,x_1),(x_0,x_1))$.

Notice that an extra-natural transformation $\epsilon\colon E \to P$ induces a morphism $e\colon E \to \prod_{x:X} P_0(x,x)$. Then, ϵ being extranatural is equivalent to e equalizing ϕ and ψ.

We claim that the equalizer E of ϕ and ψ is the end of P, and the projection is given by the extranatural transformation

$$x: X \vdash \epsilon(x) := E \xrightarrow{e} \prod_{x:X} P_0(x,x) \xrightarrow{\pi(x)} P_0(x,x).$$

Indeed, notice that the product and equalizer used to construct the above extranatural transformation are stable under pullback, so stability will not be a concern. Then, let $E'\colon V_0$ and $\epsilon'\colon E' \to P$ be another extranatural transformation. There is an induced equalizing arrow $e'\colon E \to \prod_{x:X} P_0(x,x)$. Then, define $h\colon E' \to E$ as the unique arrow making the equalizer e commute with e'. But, by construction, h is also the unique arrow making ϵ and ϵ' commute. □

We could also derive the dual result.

Proposition 3.5.9. *Assume* $\boldsymbol{V}$ *is symmetric, cocomplete (which it is, by theorem 2.3.13, if and only if it is complete) and monoidal closed, and it has coequalizers. Then,* $\boldsymbol{V}$ *has all coends.*

3.6. Monoidal Product of Enriched Categories

Standard enriched categories have a monoidal product induced by that of their enriching category, if that is symmetric monoidal. Indeed, symmetry is necessary to define the composition of the product category. The situation is analogous in the setting of internal enriched categories.

Proposition 3.6.1. *If* $\boldsymbol{V}$ *is a symmetric monoidal category, then so is* $\boldsymbol{V}\mathrm{Cat}_{\mathscr{C}}$.

Proof. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be $\boldsymbol{V}$-categories. The monoidal product $\boldsymbol{X} \otimes_0^{\boldsymbol{V}\mathrm{Cat}_{\mathscr{C}}} \boldsymbol{Y}$ is the $\boldsymbol{V}$-category given by the data:

Underlying object: $X \times Y$;

Internal hom: in context $(x_0,y_0),(x_1,y_1) : X \times Y$, define

$$\mathrm{Hom}_{\boldsymbol{X}\otimes^{\boldsymbol{V}\mathrm{Cat}_{\mathscr{C}}}\boldsymbol{Y}}((x_0,y_0),(x_1,y_1)) \coloneqq \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_0,y_1);$$

Composition: in context $(x_0,y_0),(x_1,y_1),(x_2,y_2) : X \times Y$, define

$$\circ_{\boldsymbol{X}\otimes^{\boldsymbol{V}\mathrm{Cat}_{\mathscr{C}}}\boldsymbol{Y}}((x_0,y_0),(x_1,y_1),(x_2,y_2)) \coloneqq$$

$$\begin{array}{c} \mathrm{Hom}_{\boldsymbol{X}}(x_1,x_2) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_1,y_2) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_0,y_1) \\ \downarrow \scriptstyle \mathrm{Hom}_{\boldsymbol{X}}(x_1,x_2)\otimes^{\boldsymbol{V}}\sigma_{\boldsymbol{V}}(\mathrm{Hom}_{\boldsymbol{Y}}(y_1,y_2),\mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1))\otimes^{\boldsymbol{V}}\mathrm{Hom}_{\boldsymbol{Y}}(y_0,y_1) \\ \mathrm{Hom}_{\boldsymbol{X}}(x_1,x_2) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_1,y_2) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_0,y_1) \\ \downarrow \scriptstyle \circ_{\boldsymbol{X}}(x_0,x_1,x_2)\otimes^{\boldsymbol{V}}\circ_{\boldsymbol{Y}}(y_0,y_1,y_2) \\ \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_2) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_0,y_2) \end{array} ;$$

Identity: in context $(x,y) : X \times Y$, define $\mathrm{id}_{\boldsymbol{X}\otimes^{\boldsymbol{V}\mathrm{Cat}_{\mathscr{C}}}\boldsymbol{Y}}(x,y) \coloneqq \mathrm{id}_{\boldsymbol{X}}(x) \otimes^{\boldsymbol{V}} \mathrm{id}_{\boldsymbol{Y}}(y)$.

As it can be straightforwardly checked by using the internal logic, these data satisfy the axioms 3.1 to 3.3.

Let $F : \boldsymbol{X} \to \boldsymbol{Y}$ and $G : \boldsymbol{W} \to \boldsymbol{Z}$ be $\boldsymbol{V}$-enriched functors. Define the monoidal product $F \otimes_1^{\boldsymbol{V}\mathrm{Cat}_{\mathscr{C}}} G$ as:

Object component: $(F \otimes_1^{\boldsymbol{V}\mathrm{Cat}_{\mathscr{C}}} G)_0 \coloneqq F_0 \times G_0 : X \times W \to Y \times Z$;

Morphism component: in context $(x_0,w_0),(x_1,w_1) : X \times W$, define

$$(F \otimes_1^{\boldsymbol{V}\mathrm{Cat}_{\mathscr{C}}} G)_1((x_0,w_0),(x_1,w_1)) \coloneqq F_1(x_0,x_1) \otimes^{\boldsymbol{V}} G_1(w_0,w_1).$$

which satisfy the axiom 3.4.

That gives a functor $\otimes^{\boldsymbol{V}\mathrm{Cat}_{\mathscr{C}}} : \boldsymbol{V}\mathrm{Cat}_{\mathscr{C}} \times \boldsymbol{V}\mathrm{Cat}_{\mathscr{C}} \to \boldsymbol{V}\mathrm{Cat}_{\mathscr{C}}$ satisfying the associativity axiom up to a suitable associator isomorphism.

The monoidal unit $\mathbb{I}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{C}}}$ is the $\boldsymbol{V}$-category given by the data:

Underlying object: $\mathbb{1}_{\mathscr{C}}$;

Internal hom: $\mathbb{I}_{\mathscr{M}} : \mathbb{1}_{\mathscr{C}} \times \mathbb{1}_{\mathscr{C}} \cong \mathbb{1}_{\mathscr{C}} \to V_0$;

Composition: $\rho_{\boldsymbol{V}}(\mathbb{I}_{\boldsymbol{V}}) \circ_{\boldsymbol{V}} (\rho_{\boldsymbol{V}}(\mathbb{I}_{\boldsymbol{V}}) \otimes_1^{\boldsymbol{V}} \mathrm{id}_{\boldsymbol{V}}(\mathbb{I}_{\boldsymbol{V}})) : \mathbb{1}_{\mathscr{C}} \times^{\mathscr{C}} \mathbb{1}_{\mathscr{C}} \times^{\mathscr{C}} \mathbb{1}_{\mathscr{C}} \to V_1$;

Identity: $\mathrm{id}_{\boldsymbol{V}}(\mathbb{I}_{\boldsymbol{V}}) : \mathbb{1}_{\mathscr{C}} \to V_1$.

The unit satisfies the unitarity axioms up to left/right unitor isomorphisms.

Given $\boldsymbol{V}$-categories $\boldsymbol{X}$ and $\boldsymbol{Y}$, their symmetry

$$\sigma_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{X},\boldsymbol{Y}) : \boldsymbol{X} \otimes^{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}} \boldsymbol{Y} \to \boldsymbol{Y} \otimes^{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}} \boldsymbol{X}$$

is defined as

$$(x,y) : X \times Y \vdash \sigma_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{X},\boldsymbol{Y})_0(x,y) := (y,x) : Y \times X$$

$$(x_0,y_0),(x_1,y_1) : X \times Y \vdash \sigma_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{X},\boldsymbol{Y})_1((x_0,y_0),(x_1,y_1))$$

$$:= \begin{array}{c} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_0,y_1) \\ \Big\downarrow {\scriptstyle \sigma_{\boldsymbol{V}}(\mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1),\mathrm{Hom}_{\boldsymbol{Y}}(y_0,y_1))} \\ \mathrm{Hom}_{\boldsymbol{Y}}(y_0,y_1) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1). \end{array} : V_1.$$

That satisfies the monoidal symmetry axioms. □

3.7. Exponentials of Internal Enriched Categories

Remember that in section 3.3 we showed how, given two $\boldsymbol{V}$-enriched categories $\boldsymbol{X}$ and $\boldsymbol{Y}$, there is an object $\mathrm{Fun}(\boldsymbol{X},\boldsymbol{Y})$ in $\mathscr{E}$ representing $\boldsymbol{V}$-enriched functors $\boldsymbol{X} \to \boldsymbol{Y}$, yielding an enrichment of $\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}$ in $\mathscr{E}$. Then, we extended the previous argument assigning to $\boldsymbol{X}$ and $\boldsymbol{Y}$ an object $\mathrm{Nat}(\boldsymbol{X},\boldsymbol{Y})$ in $\mathscr{E}$ of $\boldsymbol{V}$-enriched natural transformations between $\boldsymbol{V}$-enriched functors $\boldsymbol{X} \to \boldsymbol{Y}$. We thus have an internal category in $\mathscr{E}$ of $\boldsymbol{V}$-enriched functors and their $\boldsymbol{V}$-enriched natural transformations, yielding an enrichment of $\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}$ in $\mathrm{Cat}_{\mathscr{E}}$.

We now prove that the category of internal $\boldsymbol{V}$-enriched categories is monoidal closed with respect to the monoidal product defined in section 3.6. That requires associating to $\boldsymbol{X}$ and $\boldsymbol{Y}$ an internal $\boldsymbol{V}$-enriched category whose underlying object is $\mathrm{Fun}(\boldsymbol{X},\boldsymbol{Y})$, and whose hom-morphism represents $\boldsymbol{V}$-enriched natural transformations as an object of $\boldsymbol{V}$. As in standard enriched Category Theory, that is provided by an end.

In this section, assume $\mathscr{E}$ is locally cartesian closed and $\boldsymbol{V}$ is symmetric monoidal (so that $\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}$ is too, by proposition 3.6.1) and it has all ends.

Let's begin by defining a suitable candidate to be the exponential of two internal enriched categories.

Definition 3.7.1 (exponential $\boldsymbol{V}$-category)**.** Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be $\boldsymbol{V}$-categories. The *exponential of $\boldsymbol{X}$ and $\boldsymbol{Y}$* is the $\boldsymbol{V}$-enriched category $[\boldsymbol{X},\boldsymbol{Y}]$ defined as follows.

Underlying object: is given by $\mathrm{Fun}(\boldsymbol{X},\boldsymbol{Y})$, the object of $\boldsymbol{V}$-enriched functors between $\boldsymbol{X}$ and $\boldsymbol{Y}$.

Hom: is given by

$$F,G\colon \mathrm{Fun}(\boldsymbol{X},\boldsymbol{Y}) \vdash \mathrm{Hom}_{[\boldsymbol{X},\boldsymbol{Y}]}(F,G) \coloneqq \int_{x\colon X} \mathrm{Hom}_{\boldsymbol{Y}}(\mathrm{e}(F_0,x),\mathrm{e}(G_0,x)) : V_0$$

and it comes with an extranatural transformation

$$\begin{aligned}&x\colon X, F,G\colon \mathrm{Fun}(\boldsymbol{X},\boldsymbol{Y}) \vdash\\ &\quad \pi(F,G,x) \coloneqq \mathrm{Hom}_{[\boldsymbol{X},\boldsymbol{Y}]}(F,G) \to \mathrm{Hom}_{\boldsymbol{Y}}(\mathrm{e}(F_0,x),\mathrm{e}(G_0,x)) : V_1.\end{aligned}$$

Composition: is given by an arrow

$$\begin{aligned}&F,G,H\colon \mathrm{Fun}(\boldsymbol{X},\boldsymbol{Y}) \vdash\\ &\circ_{[\boldsymbol{X},\boldsymbol{Y}]}(F,G,H) : \mathrm{Hom}_{[\boldsymbol{X},\boldsymbol{Y}]}(G,H) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{[\boldsymbol{X},\boldsymbol{Y}]}(F,G) \to \mathrm{Hom}_{[\boldsymbol{X},\boldsymbol{Y}]}(F,H)\end{aligned}$$

defined as the unique arrow induced by the end on the extranatural transformation

$$F,G,H\colon \mathrm{Fun}(\boldsymbol{X},\boldsymbol{Y}), x\colon \boldsymbol{X} \vdash$$

$$\begin{array}{c}\mathrm{Hom}_{[\boldsymbol{X},\boldsymbol{Y}]}(G,H) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{[\boldsymbol{X},\boldsymbol{Y}]}(F,G)\\ \big\downarrow \pi_x \otimes^{\boldsymbol{V}} \pi_x\\ \mathrm{Hom}_{\boldsymbol{Y}}(\mathrm{e}(G_0,x),\mathrm{e}(H_0,x)) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(\mathrm{e}(F_0,x),\mathrm{e}(G_0,x))\\ \big\downarrow \circ_{\boldsymbol{Y}}(\mathrm{e}(F_0,x),\mathrm{e}(G_0,x),\mathrm{e}(H_0,x))\\ \mathrm{Hom}_{\boldsymbol{Y}}(\mathrm{e}(F_0,x),\mathrm{e}(H_0,x))\end{array} : V_1.$$

Identity: is given by an arrow

$$F\colon \mathrm{Fun}(\boldsymbol{X},\boldsymbol{Y}) \vdash \mathrm{id}_{[\boldsymbol{X},\boldsymbol{Y}]}(F) : \mathbb{I}_{\boldsymbol{V}} \to \mathrm{Hom}_{[\boldsymbol{X},\boldsymbol{Y}]}(F,F)$$

defined as the unique arrow induced by the end on the extranatural transformation

$$F\colon \mathrm{Fun}(\boldsymbol{X},\boldsymbol{Y}), x\colon X \vdash \mathbb{I}_{\boldsymbol{V}} \xrightarrow{\mathrm{id}_{\boldsymbol{Y}}(\mathrm{e}(F_0,x))} \mathrm{Hom}_{\boldsymbol{Y}}(\mathrm{e}(F_0,x),\mathrm{e}(F_0,x)) : V_1.$$

Often in this dissertation we have omitted routine calculations, but in this case it seems instructional to spell out the calculations in detail.

Let $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ be $\boldsymbol{V}$-categories. To prove that $[\boldsymbol{Y},\boldsymbol{Z}]$ is indeed the exponential object of $\boldsymbol{Y}$ and $\boldsymbol{Z}$, we need to show that $\boldsymbol{V}$-enriched functors $\boldsymbol{X} \otimes^{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}} \boldsymbol{Y} \to \boldsymbol{Z}$ correspond to $\boldsymbol{V}$-enriched functors $\boldsymbol{X} \to [\boldsymbol{Y},\boldsymbol{Z}]$. That is, that there is a natural bijection

$$\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}(\boldsymbol{X} \otimes^{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}} \boldsymbol{Y}, \boldsymbol{Z}) \cong \boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}(\boldsymbol{X}, [\boldsymbol{Y},\boldsymbol{Z}]).$$

We shall first give the two constructions yielding the correspondence, and then show that these are mutually inverse.

Let's first give the construction of

$$\bar{(\text{-})} : \boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}(\boldsymbol{X} \otimes^{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}} \boldsymbol{Y}, \boldsymbol{Z}) \to \boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}(\boldsymbol{X}, [\boldsymbol{Y}, \boldsymbol{Z}]).$$

Let $F \colon \boldsymbol{X} \otimes^{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}} \boldsymbol{Y} \to \boldsymbol{Z}$ be a $\boldsymbol{V}$-enriched functor. Define the transposed $\boldsymbol{V}$-enriched functor $\bar{F} \colon \boldsymbol{X} \to [\boldsymbol{Y}, \boldsymbol{Z}]$ as follows.

Object component: is given by

$$x \colon X \vdash \bar{F}_0(x)_0 \coloneqq \lambda y \colon Y.\, F_0(x,y) : Z$$

and

$$x \colon X \vdash \bar{F}_0(x)_1 \coloneqq \lambda y_0, y_1 \colon Y. \begin{array}{c} \mathrm{Hom}_{\boldsymbol{Y}}(y_0,y_1) \\ \big\downarrow {\scriptstyle \mathrm{id}_{\boldsymbol{X}}(x) \otimes^{\boldsymbol{V}} \mathrm{id}} \\ \mathrm{Hom}_{\boldsymbol{X}}(x,x) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_0,y_1) \\ \big\downarrow {\scriptstyle F_1((x,y_0),(x,y_1))} \\ \mathrm{Hom}_{\boldsymbol{Z}}(F_0(x,y_0), F_0(x,y_1)) \end{array} : V_1.$$

Morphism component: is an arrow of shape

$$x_0, x_1 \colon X \vdash \bar{F}_1(x_0,x_1) : \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \to \mathrm{Hom}_{[\boldsymbol{Y},\boldsymbol{Z}]}(\bar{F}_0(x_0), \bar{F}_0(x_1)) : V_1$$

defined, in context $x_0, x_1 \colon X, y \colon Y$, as the arrow yielded by the end over the extranatural transformation

$$\begin{array}{c} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \\ \big\downarrow {\scriptstyle \mathrm{id} \otimes^{\boldsymbol{V}} \mathrm{id}_{\boldsymbol{Y}}(y)} \\ \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y,y) = \mathrm{Hom}_{\boldsymbol{X} \otimes^{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}} \boldsymbol{Y}}((x_0,y), (x_1,y)) \\ \big\downarrow {\scriptstyle F_1((x_0,y),(x_1,y))} \\ \mathrm{Hom}_{\boldsymbol{Z}}(F_0(x_0,y), F_0(x_1,y)) = \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(\bar{F}_0(x_0),y), \mathrm{e}(\bar{F}_0(x_1),y)) \end{array}$$

so that the above arrow is the composition $\pi(\bar{F}_0(x_0), \bar{F}_0(x_1), y)\bar{F}_1(x_0,x_1)$. Note that here, by the definitions, we have genuine equality of objects, not mere isomorphisms.

Let's prove that $\bar{F}$ is functorial. In context $x_0, x_1, x_2 : X, y : Y$, consider the following composite.

$$
\begin{array}{c}
\mathrm{Hom}_{\boldsymbol{X}}(x_1, x_2) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_0, x_1) \\
\Big\downarrow {\scriptstyle \bar{F}_1(x_1,x_2) \otimes^{\boldsymbol{V}} \bar{F}_1(x_1,x_2)} \\
\mathrm{Hom}_{[\boldsymbol{Y},\boldsymbol{Z}]}(\bar{F}_0(x_1), \bar{F}_0(x_2)) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{[\boldsymbol{Y},\boldsymbol{Z}]}(\bar{F}_0(x_1), \bar{F}_0(x_2)) \\
\Big\downarrow {\scriptstyle \circ_{[\boldsymbol{Y},\boldsymbol{Z}]}(\bar{F}_0(x_0), \bar{F}_0(x_1), \bar{F}_0(x_2))} \\
\mathrm{Hom}_{[\boldsymbol{Y},\boldsymbol{Z}]}(\bar{F}_0(x_0), \bar{F}_0(x_2)) \\
\Big\downarrow {\scriptstyle \pi(\bar{F}_0(x_0), \bar{F}_0(x_2), y)} \\
\mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(\bar{F}_0(x_0), y), \mathrm{e}(\bar{F}_0(x_2), y))
\end{array}
$$

By definition of $\circ_{[\boldsymbol{Y},\boldsymbol{Z}]}$ and $\bar{F}_0$, the previous map is equal to the composite

$$
\begin{array}{c}
\mathrm{Hom}_{\boldsymbol{X}}(x_1, x_2) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_0, x_1) \\
\Big\downarrow {\scriptstyle \bar{F}_1(x_1,x_2) \otimes^{\boldsymbol{V}} \bar{F}_1(x_0,x_1)} \\
\mathrm{Hom}_{[\boldsymbol{Y},\boldsymbol{Z}]}(\bar{F}_0(x_1), \bar{F}_0(x_2)) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{[\boldsymbol{Y},\boldsymbol{Z}]}(\bar{F}_0(x_0), \bar{F}_0(x_1)) \\
\Big\downarrow {\scriptstyle \pi(\bar{F}_0(x_1), \bar{F}_0(x_2), y) \otimes^{\boldsymbol{V}} \pi(\bar{F}_0(x_0), \bar{F}_0(x_1), y)} \\
\mathrm{Hom}_{\boldsymbol{Z}}(F_0(x_1, y), F_0(x_2, y)) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Z}}(F_0(x_0, y), F_0(x_1, y)) \\
\Big\downarrow {\scriptstyle \circ_{\boldsymbol{Z}}(F_0(x_0,y), F_0(x_1,y), F_0(x_2,y))} \\
\mathrm{Hom}_{\boldsymbol{Z}}(F_0(x_0, y), F_0(x_2, y))
\end{array}
$$

which, by definition of $\bar{F}_1$, is equal to the composite

$$
\begin{array}{c}
\mathrm{Hom}_{\boldsymbol{X}}(x_1, x_2) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_0, x_1) \\
\Big\downarrow {\scriptstyle \mathrm{id} \otimes^{\boldsymbol{V}} \mathrm{id}_{\boldsymbol{Y}}(y) \otimes^{\boldsymbol{V}} \mathrm{id} \otimes^{\boldsymbol{V}} \mathrm{id}_{\boldsymbol{Y}}(y)} \\
\mathrm{Hom}_{\boldsymbol{X}}(x_1, x_2) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y, y) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_0, x_1) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y, y) \\
\Big\downarrow {\scriptstyle F_1((x_1,y),(x_2,y)) \otimes^{\boldsymbol{V}} F_1((x_0,y),(x_1,y))} \\
\mathrm{Hom}_{\boldsymbol{Z}}(F_0(x_1, y), F_0(x_2, y)) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Z}}(F_0(x_0, y), F_0(x_1, y)) \\
\Big\downarrow {\scriptstyle \circ_{[\boldsymbol{Y},\boldsymbol{Z}]}(F_0(x_0,y), F_0(x_1,y), F_0(x_2,y))} \\
\mathrm{Hom}_{\boldsymbol{Z}}(F_0(x_0, y), F_0(x_2, y))
\end{array}
$$

which, by functoriality of F, is equal to the composite

$$\begin{array}{c} \mathrm{Hom}_{\boldsymbol{X}}(x_1,x_2) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \\ \downarrow{\scriptstyle \circ_{\boldsymbol{X}}(x_0,x_1,x_2)} \\ \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_2) \\ \downarrow{\scriptstyle \mathrm{id}\otimes^{\boldsymbol{V}}\mathrm{id}_{\boldsymbol{Y}}(y)} \\ \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_2) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y,y) \\ \downarrow{\scriptstyle F_1((x_0,y),(x_2,y))} \\ \mathrm{Hom}_{\boldsymbol{Z}}(F_0(x_0,y),F_0(x_2,y)) \end{array}.$$

By applying the argument backwards, we get that the map we started from is equal to

$$\begin{array}{c} \mathrm{Hom}_{\boldsymbol{X}}(x_1,x_2) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \\ \downarrow{\scriptstyle \circ_{\boldsymbol{X}}(x_0,x_1,x_2)} \\ \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_2) \\ \downarrow{\scriptstyle \bar{F}_1(x_0,x_2)} \\ \mathrm{Hom}_{[\boldsymbol{Y},\boldsymbol{Z}]}(\bar{F}_0(x_0),\bar{F}_0(x_2)) \\ \downarrow{\scriptstyle \pi(\bar{F}_0(x_0),\bar{F}_0(x_2),y)} \\ \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(\bar{F}_0(x_0),y),\mathrm{e}(\bar{F}_0(x_2),y)) \end{array}$$

and with this we have proved functoriality.

Let's now give the construction of

$$\widehat{(\text{-})}\colon \boldsymbol{V}\mathrm{Cat}_{\mathscr{C}}(\boldsymbol{X},[\boldsymbol{Y},\boldsymbol{Z}]) \to \boldsymbol{V}\mathrm{Cat}_{\mathscr{C}}(\boldsymbol{X}\otimes^{\boldsymbol{V}\mathrm{Cat}_{\mathscr{C}}}\boldsymbol{Y},\boldsymbol{Z}).$$

Let $G\colon \boldsymbol{X} \to [\boldsymbol{Y},\boldsymbol{Z}]$ be a $\boldsymbol{V}$-enriched functor. Define the transposed functor $\hat{G}\colon \boldsymbol{X}\otimes^{\boldsymbol{V}\mathrm{Cat}_{\mathscr{C}}}\boldsymbol{Y} \to \boldsymbol{Z}$ as follows.

Object component: is the arrow $\hat{G}_0\colon X\times Y \to Z$ defined as

$$x\colon X, y\colon Y \vdash \hat{G}_0(x,y) \coloneqq \mathrm{e}(G_0(x)_0,y)\colon Z.$$

Morphism component: is the arrow $\hat{G}_1\colon X\times Y\times X\times Y \to V_1$ defined as

$$x_0,x_1\colon X, y_0,y_1\colon Y \vdash \hat{G}_1((x_0,y_0),(x_1,y_1)) \coloneqq$$

$$\begin{array}{c}
\mathrm{Hom}_{\boldsymbol{X}\otimes^{V\mathrm{Cat}_{\mathscr{C}}}\boldsymbol{Y}}((x_0,y_0),(x_1,y_1)) = \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_0,y_1) \\
\downarrow {\scriptstyle G_1(x_0,x_1)\otimes^{\boldsymbol{V}}\mathrm{e}(G_0(x_0)_1,(y_0,y_1))} \\
\mathrm{Hom}_{[\boldsymbol{Y},\boldsymbol{Z}]}(G_0(x_0),G_0(x_1)) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0,y_0),\mathrm{e}(G_0(x_0)_0,y_1)) \\
\downarrow {\scriptstyle \pi(G_0(x_0),G_0(x_1),y_1)\otimes^{\boldsymbol{V}}\mathrm{id}} \\
\mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0,y_1),\mathrm{e}(G_0(x_1)_0,y_1)) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0,y_0),\mathrm{e}(G_0(x_0)_0,y_1)) \\
\downarrow {\scriptstyle \circ_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0,y_0),\mathrm{e}(G_0(x_0)_0,y_1),\mathrm{e}(G_0(x_1)_0,y_1))} \\
\mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0,y_0),\mathrm{e}(G_0(x_1)_0,y_1)) = \mathrm{Hom}_{\boldsymbol{Z}}(\hat{G}_0(x_0,y_0),\hat{G}_0(x_1,y_1)).
\end{array}$$

Let's prove that $\hat{G}$ is functorial. In context $x_0,x_1,x_2 : X, y_0,y_1,y_2 : Y$, consider the composite

$$\begin{array}{c}
\mathrm{Hom}_{\boldsymbol{X}\otimes^{V\mathrm{Cat}_{\mathscr{C}}}\boldsymbol{Y}}((x_1,y_1),(x_2,y_2)) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}\otimes^{V\mathrm{Cat}_{\mathscr{C}}}\boldsymbol{Y}}((x_0,y_0),(x_1,y_1)) \\
\downarrow {\scriptstyle \hat{G}_1((x_1,y_1),(x_2,y_2))\otimes^{\boldsymbol{V}}\hat{G}_1((x_0,y_0),(x_1,y_1))} \\
\mathrm{Hom}_{\boldsymbol{Z}}(\hat{G}_0(x_1,y_1),\hat{G}_0(x_2,y_2)) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Z}}(\hat{G}_0(x_0,y_0),\hat{G}_0(x_1,y_1)) \\
\downarrow {\scriptstyle \circ_{\boldsymbol{Z}}(\hat{G}_0(x_0,y_0),\hat{G}_0(x_1,y_1),\hat{G}_0(x_2,y_2))} \\
\mathrm{Hom}_{\boldsymbol{Z}}(\hat{G}_0(x_0,y_0),\hat{G}_0(x_2,y_2)).
\end{array}$$

By definition of $\hat{G}$, that is equal to the composite arrow

$$
\begin{array}{l}
\mathrm{Hom}_{\boldsymbol{X}}(x_1,x_2) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_1,y_2) \\
\quad \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_0,y_1) \\
\quad\quad \Big\downarrow {\scriptstyle G_1(x_1,x_2)\otimes^{\boldsymbol{V}}\mathrm{e}(G_0(x_1)_1,(y_1,y_2)) \otimes^{\boldsymbol{V}} G_1(x_0,x_1)\otimes^{\boldsymbol{V}}\mathrm{e}(G_0(x_1)_1,(y_0,y_1))} \\
\mathrm{Hom}_{[\boldsymbol{Y},\boldsymbol{Z}]}(G_0(x_1),G_0(x_2)) \\
\quad \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_1)_0,y_1),\mathrm{e}(G_0(x_1)_0,y_2)) \\
\quad \otimes^{\boldsymbol{V}} \mathrm{Hom}_{[\boldsymbol{Y},\boldsymbol{Z}]}(G_0(x_0),G_0(x_1)) \\
\quad \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_1)_0,y_0),\mathrm{e}(G_0(x_1)_0,y_1)) \\
\quad\quad \Big\downarrow {\scriptstyle \pi(G_0(x_1),G_0(x_2),y_2)\otimes^{\boldsymbol{V}}\mathrm{id}\otimes^{\boldsymbol{V}}\pi(G_0(x_0),G_0(x_1),y_0)\otimes^{\boldsymbol{V}}\mathrm{id}} \\
\mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_1)_0,y_2),\mathrm{e}(G_0(x_2)_0,y_2)) \\
\quad \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_1)_0,y_1),\mathrm{e}(G_0(x_1)_0,y_2)) \\
\quad \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0,y_0),\mathrm{e}(G_0(x_1)_0,y_0)) \\
\quad \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_1)_0,y_0),\mathrm{e}(G_0(x_1)_0,y_1)) \\
\quad\quad \Big\downarrow {\scriptstyle \circ_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_1)_0,y_1),\mathrm{e}(G_0(x_1)_0,y_2),\mathrm{e}(G_0(x_2)_0,y_2)) \otimes^{\boldsymbol{V}} \circ_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0,y_0),\mathrm{e}(G_0(x_1)_0,y_0),\mathrm{e}(G_0(x_1)_0,y_1))} \\
\mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_1)_0,y_1),\mathrm{e}(G_0(x_2)_0,y_2)) \\
\quad \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0,y_0),\mathrm{e}(G_0(x_1)_0,y_1)) \\
\quad\quad \Big\downarrow {\scriptstyle \circ_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0,y_0),\mathrm{e}(G_0(x_1)_0,y_1),\mathrm{e}(G_0(x_2)_0,y_2))} \\
\mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0,y_0),\mathrm{e}(G_0(x_2)_0,y_2)).
\end{array}
$$

By shuffling the compositions, we get another composite equal to the previous one,

$$\begin{array}{c}
\mathrm{Hom}_{\boldsymbol{X}}(x_1,x_2) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_1,y_2) \\
\otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_0,y_1) \\
\Big\downarrow {\scriptstyle G_1(x_1,x_2)\otimes^{\boldsymbol{V}}\mathrm{e}(G_0(x_1)_1,(y_1,y_2)) \otimes^{\boldsymbol{V}} G_1(x_0,x_1)\otimes^{\boldsymbol{V}}\mathrm{e}(G_0(x_1)_1,(y_0,y_1))} \\
\mathrm{Hom}_{[\boldsymbol{Y},\boldsymbol{Z}]}(G_0(x_1),G_0(x_2)) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_1)_0,y_1),\mathrm{e}(G_0(x_1)_0,y_2)) \\
\otimes^{\boldsymbol{V}} \mathrm{Hom}_{[\boldsymbol{Y},\boldsymbol{Z}]}(G_0(x_0),G_0(x_1)) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_1)_0,y_0),\mathrm{e}(G_0(x_1)_0,y_1)) \\
\Big\downarrow {\scriptstyle \pi(G_0(x_1),G_0(x_2),y_2)\otimes^{\boldsymbol{V}}\pi(G_0(x_0),G_0(x_1),y_0) \otimes^{\boldsymbol{V}} \circ_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_1)_0,y_0),\mathrm{e}(G_0(x_1)_0,y_1),\mathrm{e}(G_0(x_1)_0,y_2))} \\
\mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_1)_0,y_2),\mathrm{e}(G_0(x_2)_0,y_2)) \\
\otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_1)_0,y_0),\mathrm{e}(G_0(x_1)_0,y_2)) \\
\otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0,y_0),\mathrm{e}(G_0(x_1)_0,y_0)) \\
\Big\downarrow {\scriptstyle \circ_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0,y_0),\mathrm{e}(G_0(x_1)_0,y_0),\mathrm{e}(G_0(x_1)_0,y_2),\mathrm{e}(G_0(x_2)_0,y_2))} \\
\mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0,y_0),\mathrm{e}(G_0(x_2)_0,y_2))
\end{array}$$

which, by functoriality of G, is equal to

$$\begin{array}{c}
\mathrm{Hom}_{\boldsymbol{X}}(x_1,x_2) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_1,y_2) \\
\otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_0,y_1) \\
\Big\downarrow {\scriptstyle G_1(x_1,x_2)\otimes^{\boldsymbol{V}}\circ_{\boldsymbol{Y}}(y_0,y_1,y_2)\otimes^{\boldsymbol{V}}G_1(x_0,x_1)} \\
\mathrm{Hom}_{[\boldsymbol{Y},\boldsymbol{Z}]}(G_0(x_1),G_0(x_2)) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_0,y_2) \\
\otimes^{\boldsymbol{V}} \mathrm{Hom}_{[\boldsymbol{Y},\boldsymbol{Z}]}(G_0(x_0),G_0(x_1)) \\
\Big\downarrow {\scriptstyle \pi(G_0(x_1),G_0(x_2),y_2)\otimes^{\boldsymbol{V}}\mathrm{e}(G_0(x_1)_1,(y_0,y_2)) \otimes^{\boldsymbol{V}}\pi(G_0(x_0),G_0(x_1),y_0)} \\
\mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_1)_0,y_2),\mathrm{e}(G_0(x_2)_0,y_2)) \\
\otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_1)_0,y_0),\mathrm{e}(G_0(x_1)_0,y_2)) \\
\otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0,y_0),\mathrm{e}(G_0(x_1)_0,y_0)) \\
\Big\downarrow {\scriptstyle \circ_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0,y_0),\mathrm{e}(G_0(x_1)_0,y_0),\mathrm{e}(G_0(x_1)_0,y_2),\mathrm{e}(G_0(x_2)_0,y_2))} \\
\mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0,y_0),\mathrm{e}(G_0(x_2)_0,y_2))
\end{array}$$

which, by extranaturality of π, is equal to

$$\begin{array}{c}
\mathrm{Hom}_{\boldsymbol{X}}(x_1,x_2) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_1,y_2) \\
\otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_0,y_1) \\
\Big\downarrow {\scriptstyle G_1(x_1,x_2)\otimes^{\boldsymbol{V}} G_1(x_0,x_1)\otimes^{\boldsymbol{V}} \circ_{\boldsymbol{Y}}(y_0,y_1,y_2)} \\
\mathrm{Hom}_{[\boldsymbol{Y},\boldsymbol{Z}]}(G_0(x_1),G_0(x_2)) \\
\otimes^{\boldsymbol{V}} \mathrm{Hom}_{[\boldsymbol{Y},\boldsymbol{Z}]}(G_0(x_0),G_0(x_1)) \\
\otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_0,y_2) \\
\Big\downarrow {\scriptstyle \pi(G_0(x_1),G_0(x_2),y_2)\otimes^{\boldsymbol{V}}\pi(G_0(x_0),G_0(x_1),y_2) \otimes^{\boldsymbol{V}} \mathrm{e}(G_0(x_0)_1,(y_0,y_2))} \\
\mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_1)_0,y_2),\mathrm{e}(G_0(x_2)_0,y_2)) \\
\otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0,y_2),\mathrm{e}(G_0(x_1)_0,y_2)) \\
\otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0,y_0),\mathrm{e}(G_0(x_0)_0,y_2)) \\
\Big\downarrow {\scriptstyle \circ_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0,y_0),\mathrm{e}(G_0(x_0)_0,y_2),\mathrm{e}(G_0(x_1)_0,y_2),\mathrm{e}(G_0(x_2)_0,y_2))} \\
\mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0,y_0),\mathrm{e}(G_0(x_2)_0,y_2)).
\end{array}$$

By definition of composition in $[\boldsymbol{Y},\boldsymbol{Z}]$, and again by functoriality of G, the previous composite is equal to

$$\begin{array}{c}
\mathrm{Hom}_{\boldsymbol{X}}(x_1,x_2) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_1,y_2) \\
\otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_0,y_1) \\
\Big\downarrow {\scriptstyle \circ_{\boldsymbol{X}}(x_0,x_1,x_2)\otimes^{\boldsymbol{V}} \circ_{\boldsymbol{Y}}(y_0,y_1,y_2)} \\
\mathrm{Hom}_{\boldsymbol{X}}(x_0,x_2) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_0,y_2) \\
\Big\downarrow {\scriptstyle G_1(x_0,x_2)\otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_0,y_2)} \\
\mathrm{Hom}_{[\boldsymbol{Y},\boldsymbol{Z}]}(G_0(x_0),G_0(x_2)) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_0,y_2) \\
\Big\downarrow {\scriptstyle \pi(G_0(x_0),G_0(x_2),y_2)\otimes^{\boldsymbol{V}} \mathrm{e}(G_0(x_0)_1,(y_0,y_2))} \\
\mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0,y_2),\mathrm{e}(G_0(x_2)_0,y_2)) \\
\otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0,y_0),\mathrm{e}(G_0(x_0)_0,y_2)) \\
\Big\downarrow {\scriptstyle \circ_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0,y_0),\mathrm{e}(G_0(x_0)_0,y_2),\mathrm{e}(G_0(x_2)_0,y_2))} \\
\mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0,y_0),\mathrm{e}(G_0(x_2)_0,y_2))
\end{array}$$

which is equal to

$$\begin{array}{c} \mathrm{Hom}_{\boldsymbol{X}\otimes^{V\mathrm{Cat}_{\mathscr{C}}}\boldsymbol{Y}}((x_1,y_1),(x_2,y_2)) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}\otimes^{V\mathrm{Cat}_{\mathscr{C}}}\boldsymbol{Y}}((x_0,y_0),(x_1,y_1)) \\ \downarrow \circ_{\boldsymbol{X}\otimes^{V\mathrm{Cat}_{\mathscr{C}}}\boldsymbol{Y}}((x_0,y_0),(x_1,y_1),(x_2,y_2)) \\ \mathrm{Hom}_{\boldsymbol{X}\otimes^{V\mathrm{Cat}_{\mathscr{C}}}\boldsymbol{Y}}((x_0,y_0),(x_2,y_2)) \\ \downarrow \hat{G}_1((x_0,y_0),(x_2,y_2)) \\ \mathrm{Hom}_{\boldsymbol{Z}}(\hat{G}_0(x_0,y_0),\hat{G}_0(x_2,y_2)) \end{array}$$

thus showing that $\hat{G}$ is functorial.

The final and most fundamental part of the argument is showing that the two constructions yield a bijection. That is the content of the next proposition.

Proposition 3.7.2. *The constructions* $\hat{(\text{-})}$ *and* $\bar{(\text{-})}$ *are mutually inverse.*

Proof. Let's prove that $\hat{\bar{F}} = F$. On the object component, we simply have the following chain of equalities,

$$x : X, y : Y \vdash \hat{\bar{F}}_0(x,y) = \mathrm{e}(\bar{F}_0(x)_0, y) = F_0(x,y) : Z.$$

On the morphism component, in context $x_0, x_1 : X, y_0, y_1 : Y$, we have by definition of $\hat{\bar{F}}$ that $\hat{\bar{F}}_1((x_0,y_0),(x_1,y_1))$ is

$$\begin{array}{c} \mathrm{Hom}_{\boldsymbol{X}\otimes^{V\mathrm{Cat}_{\mathscr{C}}}\boldsymbol{Y}}((x_0,y_0),(x_1,y_1)) = \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_0,y_1) \\ \downarrow \bar{F}_1(x_0,x_1)\otimes^{\boldsymbol{V}}\mathrm{e}(\bar{F}_0(x_0)_1,(y_0,y_1)) \\ \mathrm{Hom}_{[\boldsymbol{Y},\boldsymbol{Z}]}(\bar{F}_0(x_0),\bar{F}_0(x_1)) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(\bar{F}_0(x_0)_0,y_0),\mathrm{e}(\bar{F}_0(x_0)_0,y_1)) \\ \downarrow \pi(\bar{F}_0(x_0),\bar{F}_0(x_1),y_1)\otimes^{\boldsymbol{V}}\mathrm{id} \\ \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(\bar{F}_0(x_0)_0,y_1),\mathrm{e}(\bar{F}_0(x_1)_0,y_1)) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(\bar{F}_0(x_0)_0,y_0),\mathrm{e}(\bar{F}_0(x_0)_0,y_1)) \\ \downarrow \circ_{\boldsymbol{Z}}(\mathrm{e}(\bar{F}_0(x_0)_0,y_0),\mathrm{e}(\bar{F}_0(x_0)_0,y_1),\mathrm{e}(\bar{F}_0(x_1)_0,y_1)) \\ \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(\bar{F}_0(x_0)_0,y_0),\mathrm{e}(\bar{F}_0(x_1)_0,y_1)) = \mathrm{Hom}_{\boldsymbol{Z}}(\hat{\bar{F}}_0(x_0,y_0),\hat{\bar{F}}_0(x_1,y_1)) \end{array}$$

which, by definition of $\bar{F}_1$, and by definition of $\bar{F}_0(x_0)_1$, is equal to

$$\begin{array}{c}
\mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_0,y_1) \\
\downarrow {\scriptstyle \mathrm{id}\otimes^{\boldsymbol{V}}\mathrm{id}_{\boldsymbol{Y}}(y_1)\otimes^{\boldsymbol{V}}\mathrm{id}_{\boldsymbol{X}}(x_0)\otimes^{\boldsymbol{V}}\mathrm{id}} \\
\mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_1,y_1) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_0) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_0,y_1) \\
\downarrow {\scriptstyle F_1((x_0,y_1),(x_1,y_1))\otimes^{\boldsymbol{V}}F_1((x_0,y_0),(x_0,y_1))} \\
\mathrm{Hom}_{\boldsymbol{Z}}(F_0(x_0,y_1),F_0(x_1,y_1)) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Z}}(F_0(x_0,y_0),F_0(x_0,y_1)) \\
\downarrow {\scriptstyle \circ_{\boldsymbol{Z}}(F_0(x_0,y_0),F_0(x_0,y_1),F_0(x_1,y_1))} \\
\mathrm{Hom}_{\boldsymbol{Z}}(F_0(x_0,y_0),F_0(x_1y_1)).
\end{array}$$

By functoriality, swap the application of the functor F and the composition; then, apply the unit law. This way, it remains only the application of F_1.

Let's prove that $\bar{\hat{G}} = G$. On the object component we have, in context $x\colon X$, the chain of equalities

$$\bar{\hat{G}}_0(x)_0 = \lambda y\colon Y.\, \hat{G}_0(x,y) = \lambda y\colon Y.\, \mathrm{e}(G_0(x),y) = G_0(x)\colon Z$$

whereas, in context $x\colon X, y_0, y_1\colon Y$, we have that $\mathrm{e}_{\mathscr{E}}(\bar{\hat{G}}_0(x)_1, (y_0,y_1))$ is the composite arrow

$$\begin{array}{c}
\mathrm{Hom}_{\boldsymbol{Y}}(y_0,y_1) \\
\downarrow {\scriptstyle \mathrm{id}_{\boldsymbol{X}}(x)\otimes^{\boldsymbol{V}}\mathrm{id}} \\
\mathrm{Hom}_{\boldsymbol{X}}(x,x) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y_0,y_1) \\
\downarrow {\scriptstyle G_1\otimes^{\boldsymbol{V}}\mathrm{e}(G_0(x)_1,(y_0,y_1))} \\
\mathrm{Hom}_{[\boldsymbol{Y},\boldsymbol{Z}]}(G_0(x),G_0(x)) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x)_0,y_0),\mathrm{e}(G_0(x)_0,y_1)) \\
\downarrow {\scriptstyle \pi(G_0(x),G_0(x),y_1)\otimes^{\boldsymbol{V}}\mathrm{id}} \\
\mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x)_0,y_1),\mathrm{e}(G_0(x)_0,y_1)) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x)_0,y_0),\mathrm{e}(G_0(x)_0,y_1)) \\
\downarrow {\scriptstyle \circ_{\boldsymbol{Z}}(\mathrm{e}(G_0(x)_0,y_0),\mathrm{e}(G_0(x)_0,y_1),\mathrm{e}(G_0(x)_0,y_1))} \\
\mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x)_0,y_0),\mathrm{e}(G_0(x)_0,y_1)).
\end{array}$$

But $\pi(y_1) \circ_{\boldsymbol{V}} G_1 \circ_{\boldsymbol{V}} \mathrm{id}_{\boldsymbol{X}}(x)$ is equal to $\mathrm{id}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x)_0,y_0))$, thus the above composite arrow is equal to $G_0(x)_1$ applied to (y_0,y_1). On the morphism component, in context $x_0,x_1\colon X, y\colon Y$, the y-component of $\bar{\hat{G}}_1(x_0,x_1)$, that is, the composition $\pi(\bar{\bar{G}}_0(x_0), \bar{\bar{G}}_0(x_1), y)\bar{\bar{G}}_1(x_0, x_1)$, is:

$$\begin{array}{c}
\mathrm{Hom}_{\boldsymbol{X}}(x_0, x_1) \\
\big\downarrow \scriptstyle \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1)\otimes^{\boldsymbol{V}}\mathrm{id}_{\boldsymbol{Y}}(y) \\
\mathrm{Hom}_{\boldsymbol{X}}(x_0, x_1) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Y}}(y, y) \\
\big\downarrow \scriptstyle G_1(x_0,x_1)\otimes^{\boldsymbol{V}} G_0(x_0)_1(y,y) \\
\mathrm{Hom}_{[\boldsymbol{Y},\boldsymbol{Z}]}(G_0(x_0), G_0(x_1)) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0, y), \mathrm{e}(G_0(x_0)_0, y)) \\
\big\downarrow \scriptstyle \pi(G_0(x_0),G_0(x_1),y)\otimes^{\boldsymbol{V}}\mathrm{Hom}_{\boldsymbol{Z}}(\dots) \\
\mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0, y), \mathrm{e}(G_0(x_1)_0, y)) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0, y), \mathrm{e}(G_0(x_0)_0, y)) \\
\big\downarrow \scriptstyle \circ_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0,y),\mathrm{e}(G_0(x_0)_0,y),\mathrm{e}(G_0(x_1)_0,y)) \\
\mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0, y), \mathrm{e}(G_0(x_1)_0, y))
\end{array}$$

which, by the extranaturality of π, is

$$\begin{array}{c}
\mathrm{Hom}_{\boldsymbol{X}}(x_0, x_1) \\
\big\downarrow \scriptstyle G_1(x_0,x_1) \\
\mathrm{Hom}_{[\boldsymbol{Y},\boldsymbol{Z}]}(G_0(x_0), G_0(x_1)) \\
\big\downarrow \scriptstyle \pi(G_0(x_0),G_0(x_1),y) \\
\mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{e}(G_0(x_0)_0, y), \mathrm{e}(G_0(x_1)_0, y))
\end{array}$$

that is, the y-component of $G_1(x_0, x_1)$. □

Now at last we have proved the following.

Theorem 3.7.3. *The category* $\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}$ *is monoidal closed.*

3.8. Yoneda Lemma for Internal Enriched Categories

No theory of enriched categories can do without a suitable form of the Yoneda lemma. The major concern in our context is not to prove the result, as the proof just works out by means of the usual arguments, but rather to correctly express the statement in an internal way. Indeed, while there is a tradition to view the Yoneda lemma for internal categories in terms of indexed categories, the point of the dissertation is that internal enrichment avoids all that.

In this section, let's work under the hypothesis that $\mathscr{E}$ is locally cartesian closed, $\boldsymbol{V}$ is symmetric, monoidal closed (and thus $\boldsymbol{V}$-enriched itself) and it has all ends (so that $\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}$ is monoidal closed, by theorem 3.7.3).

We begin by defining the Yoneda embedding.

Definition 3.8.1 (Yoneda embedding)**.** Let $\boldsymbol{X}$ be a $\boldsymbol{V}$-category. The *Yoneda embedding for $\boldsymbol{X}$* is the $\boldsymbol{V}$-enriched functor $\mathrm{Y}\colon \boldsymbol{X} \to [\boldsymbol{X}^{\mathbf{op}}, \boldsymbol{V}]$ which is the exponential transpose in $\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}$ of the hom-functor $\mathrm{Hom}_{\boldsymbol{X}}\colon \boldsymbol{X}^{\mathbf{op}} \otimes \boldsymbol{X} \to \boldsymbol{V}$.

We now give a version of the Yoneda Lemma for $\boldsymbol{V}$-enriched categories. Notice how the statement is different from its usual form, as it has to capture the fundamentally internal nature of the result.

Theorem 3.8.2 (Yoneda Lemma)**.** *The functor*

$$\boldsymbol{X}^{\mathbf{op}} \otimes [\boldsymbol{X}^{\mathbf{op}}, \boldsymbol{V}] \xrightarrow{\mathrm{Y}^{\mathrm{op}} \otimes [\boldsymbol{X}^{\mathbf{op}}, \boldsymbol{V}]} [\boldsymbol{X}^{\mathbf{op}}, \boldsymbol{V}]^{\mathbf{op}} \otimes [\boldsymbol{X}^{\mathbf{op}}, \boldsymbol{V}] \xrightarrow{\mathrm{Hom}_{[\boldsymbol{X}^{\mathbf{op}}, \boldsymbol{V}]}} \boldsymbol{V}$$

is naturally isomorphic to the evaluation functor.

Proof. Define the natural transformation

$$x\colon X, F\colon [\boldsymbol{X}^{\mathbf{op}}, \boldsymbol{V}] \vdash \mathrm{y}_{x,F}\colon \mathrm{e}(F_0, x) \to \mathrm{Hom}_{[\boldsymbol{X}^{\mathbf{op}}, \boldsymbol{V}]}(\mathrm{Y}_0(x), F) : V_1$$

on the projections

$$x, x' : X, F\colon [\boldsymbol{X}^{\mathbf{op}}, \boldsymbol{V}] \vdash \pi_{x'} \mathrm{y}_{x,F}\colon \mathrm{e}(F_0, x) \to \mathrm{Hom}_{\boldsymbol{V}}(\mathrm{Hom}_{\boldsymbol{X}}(x', x), \mathrm{e}(F_0, x')) : V_1$$

as the exponential transpose of $\mathrm{e}(F_1, (x, x'))$, meaning that

$$\mathrm{e}_{\boldsymbol{V}} \circ_{\boldsymbol{V}} (\pi_{x'} \mathrm{y}_{x,F} \otimes \mathrm{id}_{\boldsymbol{V}}(\mathrm{Hom}_{\boldsymbol{X}}(x', x))) = \mathrm{e}_{\boldsymbol{V}} \circ_{\boldsymbol{V}} (\mathrm{e}(F_1, (x, x')) \otimes \mathrm{id}_{\boldsymbol{V}}(\mathrm{e}(F_0, x))).$$

Define the natural transformation y^{-1} as

$$x\colon X, F\colon [\boldsymbol{X}^{\mathbf{op}}, \boldsymbol{V}] \vdash \mathrm{y}^{-1}_{x,F}\colon \mathrm{Hom}_{[\boldsymbol{X}^{\mathbf{op}}, \boldsymbol{V}]}(\mathrm{Y}_0(x), F) \to \mathrm{e}(F_0, x)$$

$$\coloneqq \begin{array}{c} \mathrm{Hom}_{[\boldsymbol{X}^{\mathbf{op}}, \boldsymbol{V}]}(\mathrm{Y}_0(x), F) \\ \downarrow{\scriptstyle \pi_x \otimes \mathrm{id}_{\boldsymbol{X}}(x)} \\ \mathrm{Hom}_{\boldsymbol{V}}(\mathrm{Hom}_{\boldsymbol{X}}(x, x), \mathrm{e}(F_0, x)) \otimes \mathrm{Hom}_{\boldsymbol{X}}(x, x) \\ \downarrow{\scriptstyle \mathrm{e}_{\boldsymbol{V}}} \\ \mathrm{e}(F_0, x) \end{array} : V_1 \qquad (3.8)$$

Let's prove that $\mathrm{y}^{-1}\mathrm{y} = \mathrm{id}$. In context $x\colon X, F\colon [\boldsymbol{X}^{\mathbf{op}}, \boldsymbol{V}]$, we have by definition that $\mathrm{y}^{-1}_{x,F}\mathrm{y}_{x,F}$ is

$$\begin{array}{c} \mathrm{e}(F_0,x) \\ \big\downarrow{\scriptstyle \pi_{x'}\mathrm{y}_{x,F}\otimes \mathrm{id}_{\boldsymbol{X}}(x)} \\ \mathrm{Hom}_{\boldsymbol{V}}(\mathrm{Hom}_{\boldsymbol{X}}(x,x), \mathrm{e}(F_0,x)) \otimes \mathrm{Hom}_{\boldsymbol{X}}(x,x) \\ \big\downarrow{\scriptstyle \mathrm{e}_{\boldsymbol{V}}} \\ \mathrm{e}(F_0,x) \end{array}$$

which, again expanding the definition, is

$$\begin{array}{c} \mathrm{e}(F_0,x) \\ \big\downarrow{\scriptstyle \mathrm{e}(F_0,x)\otimes \mathrm{id}_{\boldsymbol{X}}(x)} \\ \mathrm{e}(F_0,x) \otimes \mathrm{Hom}_{\boldsymbol{X}}(x,x) \\ \big\downarrow{\scriptstyle \mathrm{e}(F_0,x)\otimes \mathrm{e}(F_1,(x,x))} \\ \mathrm{e}(F_0,x) \otimes \mathrm{Hom}_{\boldsymbol{V}}(\mathrm{e}(F_0,x), \mathrm{e}(F_0,x)) \\ \big\downarrow{\scriptstyle \mathrm{e}_{\boldsymbol{V}}} \\ \mathrm{e}(F_0,x) \end{array}$$

which, by functoriality of F_1, is

$$\begin{array}{c} \mathrm{e}(F_0,x) \\ \big\downarrow{\scriptstyle \mathrm{e}(F_0,x)\otimes \mathrm{id}_{\boldsymbol{V}}(\mathrm{e}(F_0,x))} \\ \mathrm{e}(F_0,x) \otimes \mathrm{Hom}_{\boldsymbol{V}}(\mathrm{e}(F_0,x), \mathrm{e}(F_0,x)) \\ \big\downarrow{\scriptstyle \mathrm{e}_{\boldsymbol{V}}} \\ \mathrm{e}(F_0,x) \end{array}$$

and that amounts to $\mathrm{id}_{\boldsymbol{V}}(\mathrm{e}(F_0,x))$.

Let's prove that $\mathrm{y}\mathrm{y}^{-1} = \mathrm{id}$. In context $x, x'\colon X, F\colon [\boldsymbol{X}^{\mathbf{op}}, \boldsymbol{V}]$, the exponential transpose of $\pi_{x'}\mathrm{y}_{x,F}\mathrm{y}_{x,F}^{-1}$ is

$$\begin{array}{c}
\mathrm{Hom}_{\boldsymbol{X}}(x',x)\otimes\mathrm{Hom}_{[\boldsymbol{X}^{\mathbf{op}},\boldsymbol{V}]}(\mathrm{Y}_0(x),F)\\
\Big\downarrow{\scriptstyle \mathrm{Hom}_{\boldsymbol{X}}(x',x)\otimes\pi_x\otimes\mathrm{id}_{\boldsymbol{X}}(x)}\\
\mathrm{Hom}_{\boldsymbol{X}}(x',x)\otimes\mathrm{Hom}_{\boldsymbol{V}}(\mathrm{Hom}_{\boldsymbol{X}}(x,x),\mathrm{e}(F_0,x))\otimes\mathrm{Hom}_{\boldsymbol{X}}(x,x)\\
\Big\downarrow{\scriptstyle \mathrm{Hom}_{\boldsymbol{X}}(x',x)\otimes\mathrm{e}_{\boldsymbol{V}}}\\
\mathrm{Hom}_{\boldsymbol{X}}(x',x)\otimes\mathrm{e}(F_0,x)\\
\Big\downarrow{\scriptstyle \mathrm{Hom}_{\boldsymbol{X}}(x',x)\otimes\pi_{x'}\mathrm{y}_{x,F}}\\
\mathrm{Hom}_{\boldsymbol{X}}(x',x)\otimes\mathrm{Hom}_{\boldsymbol{V}}(\mathrm{Hom}_{\boldsymbol{X}}(x',x),\mathrm{e}(F_0,x'))\\
\Big\downarrow{\scriptstyle \mathrm{e}_{\boldsymbol{V}}}\\
\mathrm{e}(F_0,x')
\end{array}.$$

By definition of y, the above morphism is equal to

$$\begin{array}{c}
\mathrm{Hom}_{\boldsymbol{X}}(x',x)\otimes\mathrm{Hom}_{[\boldsymbol{X}^{\mathbf{op}},\boldsymbol{V}]}(\mathrm{Y}_0(x),F)\\
\Big\downarrow{\scriptstyle \mathrm{Hom}_{\boldsymbol{X}}(x',x)\otimes\pi_x\otimes\mathrm{id}_{\boldsymbol{X}}(x)}\\
\mathrm{Hom}_{\boldsymbol{X}}(x',x)\otimes\mathrm{Hom}_{\boldsymbol{V}}(\mathrm{Hom}_{\boldsymbol{X}}(x,x),\mathrm{e}(F_0,x))\otimes\mathrm{Hom}_{\boldsymbol{X}}(x,x)\\
\Big\downarrow{\scriptstyle \mathrm{e}(F_1,(x,x'))\otimes\mathrm{e}_{\boldsymbol{V}}}\\
\mathrm{Hom}_{\boldsymbol{V}}(\mathrm{e}(F_0,x),\mathrm{e}(F_0,x'))\otimes\mathrm{e}(F_0,x)\\
\Big\downarrow{\scriptstyle \mathrm{e}_{\boldsymbol{V}}}\\
\mathrm{e}(F_0,x')
\end{array}$$

which, by naturality, is

$$\begin{array}{c}
\mathrm{Hom}_{\boldsymbol{X}}(x',x)\otimes\mathrm{Hom}_{[\boldsymbol{X}^{\mathbf{op}},\boldsymbol{V}]}(\mathrm{Y}_0(x),F)\\
\Big\downarrow{\scriptstyle \mathrm{id}_{\boldsymbol{X}}(x)\otimes\mathrm{e}(\mathrm{e}(\mathrm{Y}_0,x)_1,(x,x'))\otimes\mathrm{Hom}_{[\boldsymbol{X}^{\mathbf{op}},\boldsymbol{V}]}(\mathrm{Y}_0(x),F)}\\
\mathrm{Hom}_{\boldsymbol{X}}(x,x)\otimes\mathrm{Hom}_{\boldsymbol{V}}(\mathrm{Hom}_{\boldsymbol{X}}(x,x),\mathrm{Hom}_{\boldsymbol{X}}(x',x))\otimes\mathrm{Hom}_{[\boldsymbol{X}^{\mathbf{op}},\boldsymbol{V}]}(\mathrm{Y}_0(x),F)\\
\Big\downarrow{\scriptstyle \mathrm{e}_{\boldsymbol{V}}\otimes\pi_{x'}}\\
\mathrm{Hom}_{\boldsymbol{X}}(x',x)\otimes\mathrm{Hom}_{\boldsymbol{V}}(\mathrm{Hom}_{\boldsymbol{X}}(x',x),\mathrm{e}(F_0,x'))\\
\Big\downarrow{\scriptstyle \mathrm{e}_{\boldsymbol{V}}}\\
\mathrm{e}(F_0,x')
\end{array}.$$

Then, the above composite arrow computes to

$$\begin{array}{c} \mathrm{Hom}_{\boldsymbol{X}}(x',x) \otimes \mathrm{Hom}_{[\boldsymbol{X}^{\mathbf{op}},\boldsymbol{V}]}(\mathrm{Y}_0(x),F) \\ \big\downarrow {\scriptstyle \mathrm{Hom}_{\boldsymbol{X}}(x',x)\otimes \pi_{x'}} \\ \mathrm{Hom}_{\boldsymbol{X}}(x',x) \otimes \mathrm{Hom}_{\boldsymbol{V}}(\mathrm{Hom}_{\boldsymbol{X}}(x',x), \mathrm{e}(F_0,x')) \\ \big\downarrow {\scriptstyle \mathrm{e}_{\boldsymbol{V}}} \\ \mathrm{e}(F_0,x') \end{array}$$

which is the exponential transpose of $\pi_{x'}$. □

There are some immediate consequences of the Yoneda Lemma. For a start, that the Yoneda embedding is a full and faithful functor.

Corollary 3.8.3. *Let $\boldsymbol{X}$ be a $\boldsymbol{V}$-enriched category. Then, in context $x_0, x_1 : X$,*

$$\mathrm{Y}_1(x_0,x_1) : \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) \to \mathrm{Hom}_{[\boldsymbol{X}^{\mathbf{op}},\boldsymbol{V}]}(\mathrm{Y}_0(x_0),\mathrm{Y}_0(x_1))$$

is an isomorphism.

Proof. By Yoneda lemma (theorem 3.8.2)

$$\mathrm{Hom}_{[\boldsymbol{X}^{\mathbf{op}},\boldsymbol{V}]}(\mathrm{Y}_0(x_0),\mathrm{Y}_0(x_1)) \cong \mathrm{Y}_0(x_1)(x_0) = \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1). \qquad \square$$

It follows that also post-composition by the Yoneda embedding is a full and faithful functor.

Corollary 3.8.4. *Let $\boldsymbol{X}$ be a $\boldsymbol{V}$-enriched category and $F, G : \boldsymbol{Y} \to \boldsymbol{X}$ be $\boldsymbol{V}$-enriched functors. There is a natural bijection between natural transformations $\mathrm{Y}F \to \mathrm{Y}G$ and natural transformations $F \to G$, which internalizes to an isomorphism*

$$\mathrm{Hom}_{[\boldsymbol{Y},\boldsymbol{X}]}(F,G) \cong \mathrm{Hom}_{[\boldsymbol{Y},[\boldsymbol{X}^{\mathbf{op}},\boldsymbol{V}]]}(\mathrm{Y}F,\mathrm{Y}G).$$

Proof. Any natural transformation $\mathrm{Y}F \to \mathrm{Y}G$ yields one $F \to G$ by composing each component with the isomorphism of corollary 3.8.3,

$$y : Y \vdash \mathrm{Hom}_{[\boldsymbol{X}^{\mathbf{op}},\boldsymbol{V}]}(\mathrm{Y}_0(F_0(y)),\mathrm{Y}_0(G_0(y))) \cong \mathrm{Hom}_{\boldsymbol{X}}(F_0(y),G_0(y)),$$

and vice-versa. □

In particular, that applies to natural isomorphisms, which is a fact we will need later on.

Corollary 3.8.5. *Let $F, G : \boldsymbol{Y} \to \boldsymbol{X}$ be $\boldsymbol{V}$-enriched functors. If there is a natural isomorphism $\phi : \mathrm{Y}F \cong \mathrm{Y}G$, then there is a natural isomorphism $\psi : F \cong G$ such that $\mathrm{Y}\psi = \phi$.*

Finally, we get the classic density result characterizing presheaves as colimits of representable presheaves. Proposition 3.5.9 reminds us that, for coends to exist, in addition to the hypothesis already made at the beginning of the section the enriching category also has to have coequalizers.

Corollary 3.8.6 (density of presheaves)**.** *Assume* $\boldsymbol{V}$ *has coequalizers and let* $\boldsymbol{X}$ *be a* $\boldsymbol{V}$*-category. Then, in context* $F\colon [\boldsymbol{X}^{\mathbf{op}}, \boldsymbol{V}], x\colon X$*, there is a natural isomorphism*

$$\mathrm{e}(F_0, x) \cong \int^{x'\,:\,X} \mathrm{Hom}_{\boldsymbol{X}}(x, x') \otimes^{\boldsymbol{V}} \mathrm{e}(F_0, x'). \tag{3.9}$$

Proof. Adding $v\colon V_0$ to the context, we have

$$\begin{aligned}
&\mathrm{Hom}_{\boldsymbol{V}}\Big(\int^{x'\,:\,X} \mathrm{Hom}_{\boldsymbol{X}}(x, x') \otimes^{\boldsymbol{V}} \mathrm{e}(F_0, x'), v\Big) \\
&\quad\cong \int_{x'\,:\,X} \mathrm{Hom}_{\boldsymbol{V}}(\mathrm{Hom}_{\boldsymbol{X}}(x, x') \otimes^{\boldsymbol{V}} \mathrm{e}(F_0, x'), v) \\
&\quad\cong \int_{x'\,:\,X} \mathrm{Hom}_{\boldsymbol{V}}(\mathrm{Hom}_{\boldsymbol{X}}(x, x'), \mathrm{Hom}_{\boldsymbol{V}}(\mathrm{e}(F_0, x'), v)) \\
&\quad= \mathrm{Hom}_{[\boldsymbol{X}^{\mathbf{op}}, \boldsymbol{V}]}(\mathrm{Hom}_{\boldsymbol{X}}(x, \text{-}), \mathrm{Hom}_{\boldsymbol{V}}(\mathrm{e}(F_0, \text{-}), v)) \\
&\quad\cong \mathrm{Hom}_{\boldsymbol{V}}(\mathrm{e}(F_0, x), v)
\end{aligned}$$

where the last step is an application of theorem 3.8.2. But then, by corollary 3.8.5, we obtain the isomorphism (3.9). □

4. Limits and Completeness

In this chapter, we develop the theory of weighted (co)limits and (co)completeness for internal enriched categories.

With regards to limits and completeness, the notion of size plays an important role. Particularly, one should be clear about what is the size of the diagrams in respect to which a category is said to be complete. In standard category theory, we generally mean that a category is complete with respect to small diagrams, even when such a category is large. One needs to look no further then to the category of sets to find an instance of this phenomenon. That, indeed, is a large category which we say complete, meaning with respect to small diagrams, while it is certainly not so with respect to large diagrams. For example, the product of all non-empty sets is not a set, but a proper class.

It is also worth noting that all small complete categories are, essentially, complete lattices. These feature a most remarkable behavior. For example, for lattices, completeness and cocompleteness are equivalent. That is just another way to look at the classic result for which a lattice has all meets if and only if it has all joins. Moreover, the adjoint functor theorem for complete lattices does away with size conditions, as those are trivially satisfied.

It is essential to note that the notion of internal enriched category is fundamentally "small" with respect to the given environment, in that underlying objects are objects of the ambient category. That is in sharp contrast with the standard theory of enriched categories, which are allowed to be "large" (as in having a proper class of objects). This consideration suggests that internal enriched categories should be treated as analogous to small enriched categories over a small monoidal enriching category. Moreover, for those that are complete, we should also expect to get the same kind of results holding for small complete categories.

In the internal setting there are non-trivial (as in other than lattices) examples of complete small categories. A most notable example is the internal category of modest sets in the category of assemblies (Hyland 1988). Such category is not only complete, but it also has a symmetric monoidal product (the categorical product) with respect to which it is closed (that is to say, it is a cartesian closed category). This provides a setting featuring all the good properties we need, thus guaranteeing that our theory of internal enriched categories is well-founded.

For the extent of this chapter, we shall assume that $\mathscr{E}$ is locally cartesian closed, and $\boldsymbol{V}$ is a symmetric monoidal closed category in $\mathscr{E}$ with all ends.

4.1. Complete Categories

In this section we define the notion of internal weighted limits, following that for internal limits (see definition 2.2.1). Then, we derive a few standard results which will be instrumental in the following sections.

We begin by defining internal weighted limits.

Definition 4.1.1 (weighted limits for $\boldsymbol{V}$-categories)**.** Let $\boldsymbol{X}$ and $\boldsymbol{K}$ be $\boldsymbol{V}$-enriched categories and $F\colon \boldsymbol{K} \to \boldsymbol{X}$ and $W\colon \boldsymbol{K} \to \boldsymbol{V}$ be $\boldsymbol{V}$-enriched functors. A *weighted limit* over F with respect to the weight W is given by a $\boldsymbol{V}$-enriched functor $\{W;F\}\colon \mathbb{I}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}} \to \boldsymbol{X}$ and a $\boldsymbol{V}$-enriched natural isomorphism

$$\phi\colon \mathrm{Hom}_{\boldsymbol{X}}(\text{-}, \{W;F\}) \xrightarrow{\cong} \mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W, \mathrm{Hom}_{\boldsymbol{X}}(\text{-}, F)) : \boldsymbol{X}^{\mathbf{op}} \to \boldsymbol{V}.$$

Notice that, unlike the notions of internal limit and end, that of internal weighted limit is automatically stable under pullback, being defined through a natural isomorphism between homsets. For completeness, though, we have to enforce stability again.

Definition 4.1.2 (completeness for $\boldsymbol{V}$-categories)**.** We say $\boldsymbol{X}$ is *complete* if, for every I object of $\mathscr{E}$, $I^*\boldsymbol{V}$-enriched category $\boldsymbol{K}$, $I^*\boldsymbol{V}$-enriched functors $F\colon \boldsymbol{K} \to I^*\boldsymbol{X}$ and $W\colon \boldsymbol{K} \to I^*\boldsymbol{V}$, the limit of F with respect to the weight W exists.

By duality, we have the notions of colimits and cocompleteness as well.

Definition 4.1.3 (weighted colimits for $\boldsymbol{V}$-categories)**.** Let $\boldsymbol{X}$ and $\boldsymbol{K}$ be $\boldsymbol{V}$-enriched categories and $G\colon \boldsymbol{K} \to \boldsymbol{X}$ and $W\colon \boldsymbol{K}^{\mathbf{op}} \to \boldsymbol{V}$ be $\boldsymbol{V}$-enriched functors. A *weighted colimit* over G with respect to the weight W is given by a $\boldsymbol{V}$-enriched functor $F \cdot W\colon \mathbb{I}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}} \to \boldsymbol{X}$ and a $\boldsymbol{V}$-enriched natural isomorphism

$$\phi\colon \mathrm{Hom}_{\boldsymbol{X}}(G \cdot W, \text{-}) \xrightarrow{\cong} \mathrm{Hom}_{[\boldsymbol{K}^{\mathbf{op}},\boldsymbol{V}]}(W, \mathrm{Hom}_{\boldsymbol{X}}(G, \text{-})) : \boldsymbol{X} \to \boldsymbol{V}.$$

Definition 4.1.4 (cocompleteness for $\boldsymbol{V}$-categories)**.** We say $\boldsymbol{X}$ is *cocomplete* if, for every I object of $\mathscr{E}$, $I^*\boldsymbol{V}$-enriched category $\boldsymbol{K}$, $I^*\boldsymbol{V}$-enriched functors $G\colon \boldsymbol{K} \to I^*\boldsymbol{X}$ and $W\colon \boldsymbol{K}^{\mathbf{op}} \to I^*\boldsymbol{V}$, the colimit of G with respect to the weight W exists.

As a sanity check, we make sure that the previous definitions of limit and colimit yield a pair of dual notions.

Remark 4.1.5. Colimits in a $\boldsymbol{V}$-enriched category $\boldsymbol{X}$ are, equivalently, limits in $\boldsymbol{X}^{\mathbf{op}}$. Indeed, let $F \cdot W\colon \mathbb{I}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}} \to \boldsymbol{X}$ be the colimit of $G\colon \boldsymbol{K} \to \boldsymbol{X}$ with respect to the weight $W\colon \boldsymbol{K}^{\mathbf{op}} \to \boldsymbol{V}$, and $\mathrm{Hom}_{\boldsymbol{X}}(G \cdot W, \text{-}) \cong \mathrm{Hom}_{[\boldsymbol{K}^{\mathbf{op}},\boldsymbol{V}]}(W, \mathrm{Hom}_{\boldsymbol{X}}(G, \text{-}))$ the associated isomorphism. That can, equivalently, be regarded as an isomorphism

$$\mathrm{Hom}_{\boldsymbol{X}^{\mathbf{op}}}(\text{-}, (G \cdot W)^{\mathbf{op}}) \cong \mathrm{Hom}_{[\boldsymbol{K}^{\mathbf{op}},\boldsymbol{V}]}(W, \mathrm{Hom}_{\boldsymbol{X}^{\mathbf{op}}}(\text{-}, G^{\mathbf{op}})),$$

showing that $(G \cdot W)^{\mathbf{op}} \colon \mathbb{I}_{\mathbf{VCat}_{\mathscr{C}}} \to \boldsymbol{X}^{\mathbf{op}}$ (which corresponds to the same point of X as $G \cdot W$ does) is the limit of $G^{\mathbf{op}} \colon \boldsymbol{K}^{\mathbf{op}} \to \boldsymbol{X}^{\mathbf{op}}$ with respect to the weight W.

Limits are often thought of in terms of their universal property, which we can indeed derive from the definition.

Remark 4.1.6. Let $F \colon \boldsymbol{K} \to \boldsymbol{X}$ and $W \colon \boldsymbol{K} \to \boldsymbol{V}$ be $\boldsymbol{V}$-enriched functors. Then, in context $x \colon X$, by the Yoneda Lemma (theorem 3.8.2), we have an isomorphism

$$\mathrm{Hom}_{[\boldsymbol{X}^{\mathbf{op}},\boldsymbol{V}]}\Big(\mathrm{Y}(x), \mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W, \mathrm{Hom}_{\boldsymbol{X}}(\text{-},F))\Big) \cong \mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W, \mathrm{Hom}_{\boldsymbol{X}}(x,F)).$$

The previous remark, applied to the definition of weighted limit, suggests the following proposition.

Proposition 4.1.7. *Let $F \colon \boldsymbol{K} \to \boldsymbol{X}$ and $W \colon \boldsymbol{K} \to \boldsymbol{V}$ be $\boldsymbol{V}$-enriched functors. If the weighted limit $\{W;F\}$ exists, then by remark 4.1.6 there is an associated $\boldsymbol{V}$-enriched natural transformation*

$$\pi \colon W \to \mathrm{Hom}_{\boldsymbol{X}}(\{W;F\},F) : \boldsymbol{K} \to \boldsymbol{V}$$

which is called the universal cone *for the limit $\{W;F\}$. Then, given another $\boldsymbol{V}$-enriched functor $L \colon \mathbb{I}_{\mathbf{VCat}_{\mathscr{C}}} \to \boldsymbol{X}$ and another $\boldsymbol{V}$-enriched natural transformation $p \colon W \to \mathrm{Hom}_{\boldsymbol{X}}(L,F)$, there is a unique $h \colon L \to \{W;F\}$ such that $\mathrm{Hom}_{\boldsymbol{X}}(h,F)\pi = p$.*

Proof. The natural transformation p has an associated $\boldsymbol{V}$-enriched natural transformation

$$\alpha \colon \mathrm{Hom}_{\boldsymbol{X}}(\text{-},L) \to \mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W, \mathrm{Hom}_{\boldsymbol{X}}(\text{-},F)) : \boldsymbol{X}^{\mathbf{op}} \to \boldsymbol{V}.$$

Then, h is given by

$$\begin{array}{c} \mathbb{I} \\ \big\downarrow {\scriptstyle \mathrm{id}_{\boldsymbol{X}}(L)} \\ \mathrm{Hom}_{\boldsymbol{X}}(L,L) \\ \big\downarrow {\scriptstyle \alpha(L)} \\ \mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W, \mathrm{Hom}_{\boldsymbol{X}}(L,F)) \\ \big\downarrow {\scriptstyle \phi^{-1}(L)} \\ \mathrm{Hom}_{\boldsymbol{X}}(L, \{W;F\}). \end{array}$$

Notice that $\alpha(L)\mathrm{id}_{\boldsymbol{X}}(L)$ is the exponential transpose of p, and that

$$\begin{array}{c} W \otimes \mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W, \mathrm{Hom}_{\boldsymbol{X}}(L,F)) \\ \big\downarrow{\scriptstyle \pi\otimes\phi^{-1}(L)} \\ \mathrm{Hom}_{\boldsymbol{X}}(\{W;F\},F) \otimes \mathrm{Hom}_{\boldsymbol{X}}(L,\{W;F\}) \\ \big\downarrow{\scriptstyle \circ} \\ \mathrm{Hom}_{\boldsymbol{X}}(L,F) \end{array}$$

is the evaluation on W. Then, $\mathrm{Hom}_{\boldsymbol{X}}(h,F)\pi = p$. □

Remark 4.1.8. It is not generally true that the converse of proposition 4.1.7 holds, that is, that the $\boldsymbol{V}$-enriched natural transformation associated to a universal cone is necessarily an isomorphism, and it thus yields a weighted limit. The same issue arises in standard enriched category theory and, in that context, the classic approach involves the notion of conservative category (Kelly 1982). It is likely that the same could be done in our context, but in this dissertation we shall not pursue this topic any further.

As an immediate consequence of the limit's universal property, we have their uniqueness property.

Proposition 4.1.9 (uniqueness of weighted limits)**.** *A weighted limit, if it exists, is unique up to isomorphism.*

Using proposition 2.3.4 as an analogy, we would like to have a choice-of-limits functor which, in a suitable sense, yields limits for internal functors relative to internal weights.

Notation 4.1.10. We shall now define a functor $\{\text{-};\text{-}\}\colon [\boldsymbol{K},\boldsymbol{V}]^{\mathbf{op}} \otimes [\boldsymbol{K},\boldsymbol{X}] \to \boldsymbol{X}$ for which it will be convenient to use an infix notation. We shall thus write $\{W;F\}$ for $(\{\text{-};\text{-}\})_0(W,F)$ and $\{W_0,W_1;F_0,F_1\}$ for $(\{\text{-};\text{-}\})_1((W_0,F_0),(W_1,F_1))$.

Proposition 4.1.11 (weighted limits functor)**.** *If $\boldsymbol{X}$ is a complete $\boldsymbol{V}$-enriched category, then there is a $\boldsymbol{V}$-enriched functor*

$$\{\text{-};\text{-}\}\colon [\boldsymbol{K},\boldsymbol{V}]^{\mathbf{op}} \otimes [\boldsymbol{K},\boldsymbol{X}] \to \boldsymbol{X}$$

inducing a $\boldsymbol{V}$-enriched isomorphism

$$\mathrm{Hom}_{\boldsymbol{X}}(x,\{W;F\}) \cong \mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W, \mathrm{Hom}_{\boldsymbol{X}}(x,F)) \tag{4.1}$$

natural in $x\colon X$, $W\colon [\boldsymbol{K},\boldsymbol{V}]$ and $F\colon [\boldsymbol{K},\boldsymbol{X}]$.

Proof. Consider an indexing object $I = \mathrm{Fun}(\boldsymbol{K},\boldsymbol{V}) \times \mathrm{Fun}(\boldsymbol{K},\boldsymbol{X})$. In context $W\colon [\boldsymbol{K},\boldsymbol{V}], F\colon [\boldsymbol{K},\boldsymbol{X}], k, k_0, k_1 : K$, let $\bar{W} : I^*\boldsymbol{K} \to I^*\boldsymbol{V}$ be the $I^*\boldsymbol{V}$-enriched functor defined by $\bar{W}_0(W,F,k) \coloneqq \mathrm{e}(W_0,k)$ and $\bar{W}_1(W,F,k_0,k_1) \coloneqq \mathrm{e}(W_1,(k_0,k_1))$. Let $\bar{F}\colon I^*\boldsymbol{K} \to I^*\boldsymbol{X}$ be the $I^*\boldsymbol{V}$-enriched functor defined by $\bar{F}(W,F,k) \coloneqq \mathrm{e}(F_0,k)$ and $\bar{F}_1(W,F,k_0,k_1) \coloneqq \mathrm{e}(F_1,(k_0,k_1))$.

Then, the limit of $\bar{F}$ with respect to the weight $\bar{W}$ yields an arrow $\{\text{-};\text{-}\} : I \to X$ and a family of $\boldsymbol{V}$-enriched natural isomorphisms

$$\phi_{W,F}\colon \mathrm{Hom}_{\boldsymbol{X}}(\text{-},\{W;F\}) \to \mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W,\mathrm{Hom}_{\boldsymbol{X}}(\text{-},F)) : \boldsymbol{X}^{\mathbf{op}} \to \boldsymbol{V}$$

inducing universal cones $\pi_{W,F}\colon W \to \mathrm{Hom}_{\boldsymbol{X}}(\{W;F\},F)$. Let $\{W_0,W_1;F_0,F_1\}$ be the arrow

$$\begin{array}{c}
\mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W_1,W_0) \otimes \mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{X}]}(F_0,F_1) \\
\Big\downarrow \scriptstyle \mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W_1,\pi_{W_0,F_0}) \otimes \mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{X}]}(F_0,F_1) \\
\mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W_1,\mathrm{Hom}_{\boldsymbol{X}}(\{W_0;F_0\},F_0)) \otimes \mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{X}]}(F_0,F_1) \\
\Big\downarrow \scriptstyle \circ_{[\boldsymbol{K},\boldsymbol{X}]} \\
\mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W_1,\mathrm{Hom}_{\boldsymbol{X}}(\{W_0;F_0\},F_1)) \\
\Big\downarrow \scriptstyle \phi^{-1}_{W_1,F_1}(\{W_0;F_0\}) \\
\mathrm{Hom}_{\boldsymbol{X}}(\{W_0;F_0\}_0,\{W_1;F_1\}_0).
\end{array}$$

Then, the isomorphism 4.1 holds point-wise and is natual in $x\colon X$. We have to show that it is natural in $W\colon [\boldsymbol{K},\boldsymbol{V}]$ and $F\colon [\boldsymbol{K},\boldsymbol{X}]$. That is, that the diagram

$$\begin{array}{ccc}
\begin{array}{c}\mathrm{Hom}_{\boldsymbol{X}}(x,\{W_0;F_0\}) \\ \otimes\, \mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{X}]}(F_0,F_1) \\ \otimes\, \mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W_1,W_0)\end{array} & \xrightarrow{\phi_{W_0,F_0}(x)\otimes\ldots} & \begin{array}{c}\mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W_0,\mathrm{Hom}_{\boldsymbol{X}}(x,F_0)) \\ \otimes\, \mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{X}]}(F_0,F_1) \\ \otimes\, \mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W_1,W_0)\end{array} \\
\Big\downarrow \scriptstyle \mathrm{Hom}_{\boldsymbol{X}}(x,\{W_0;F_0\})\otimes\{W_0,W_1;F_0,F_1\} & & \\
\begin{array}{c}\mathrm{Hom}_{\boldsymbol{X}}(x,\{W_0;F_0\}) \\ \otimes\, \mathrm{Hom}_{\boldsymbol{X}}(\{W_0;F_0\},\{W_1;F_1\})\end{array} & & \Big\downarrow \scriptstyle \circ \\
\Big\downarrow \scriptstyle \circ_{\boldsymbol{X}}(x,\{W_0;F_0\},\{W_1;F_1\}) & & \\
\mathrm{Hom}_{\boldsymbol{X}}(x,\{W_1;F_1\}) & \xrightarrow{\phi_{W_1,F_1}(x)} & \mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W_1,\mathrm{Hom}_{\boldsymbol{X}}(x,F_1))
\end{array}$$

commutes. But that follows from the fact that

$$\begin{array}{c}
\mathrm{Hom}_{\boldsymbol{X}}(x,\{W_0;F_0\}) \otimes \mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W_1,\mathrm{Hom}_{\boldsymbol{X}}(\{W_0;F_0\},F_1)) \\
\big\downarrow \scriptstyle \mathrm{Hom}_{\boldsymbol{X}}(x,\{W_0;F_0\})\otimes\phi^{-1}_{W_1,F_1}(\{W_0;F_0\}) \\
\mathrm{Hom}_{\boldsymbol{X}}(x,\{W_0;F_0\}) \otimes \mathrm{Hom}_{\boldsymbol{X}}(\{W_0;F_0\},\{W_1;F_1\}) \\
\big\downarrow \scriptstyle \circ_{\boldsymbol{X}}(x,\{W_0;F_0\},\{W_1;F_1\}) \\
\mathrm{Hom}_{\boldsymbol{X}}(x,\{W_1;F_1\}) \\
\big\downarrow \scriptstyle \phi_{W_1,F_1}(x) \\
\mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W_1,\mathrm{Hom}_{\boldsymbol{X}}(x,F_1))
\end{array}$$

is the composition replacing $\{W_0;F_0\}$ with x, by the naturality of ϕ. □

Observe that limits and the limit-functor behave consistently with respect to parameters, in the sense made precise by the following remark.

Remark 4.1.12. Let $\boldsymbol{X}$ be a complete category, and $F\colon \boldsymbol{K}\times\boldsymbol{L} \to \boldsymbol{X}$ and $W\colon \boldsymbol{K} \to \boldsymbol{V}$ be $\boldsymbol{V}$-enriched functors. Consider the exponential transposes $\hat{F}\colon \boldsymbol{K} \to [\boldsymbol{L},\boldsymbol{X}]$ and $\bar{F}\colon \boldsymbol{L} \to [\boldsymbol{K},\boldsymbol{X}]$. Then, the limit $\{W;\hat{F}\}$ is a $\boldsymbol{V}$-enriched functor $\mathbb{I}_{\boldsymbol{V}\mathrm{Cat}_{\otimes}} \to [\boldsymbol{L},\boldsymbol{X}]$ or, equivalently, a $\boldsymbol{V}$-enriched functor $\boldsymbol{L} \to \boldsymbol{X}$. Moreover, we can consider the composition of $\bar{F}$ and the identity of W with the limit-functor for $\boldsymbol{X}$, and get a functor $\{W;\bar{F}\}\colon \boldsymbol{L} \to \boldsymbol{X}$. Then, $\{W;\hat{F}\} \cong \{W;\bar{F}\}$, indeed, in context $G\colon [\boldsymbol{L},\boldsymbol{X}]$

$$\begin{aligned}
&\mathrm{Hom}_{[\boldsymbol{L},\boldsymbol{X}]}(G,\{W;\hat{F}\}) \\
&\quad\cong \mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W,\mathrm{Hom}_{[\boldsymbol{L},\boldsymbol{X}]}(G,\hat{F})) \\
&\quad= \int_{k:K}\int_{l:L} \mathrm{Hom}_{\boldsymbol{V}}(W_0(k),\mathrm{Hom}_{\boldsymbol{X}}(\mathrm{e}(G_0,l),\mathrm{e}(\hat{F}_0(k),l))) \\
&\quad= \int_{k:K}\int_{l:L} \mathrm{Hom}_{\boldsymbol{V}}(W_0(k),\mathrm{Hom}_{\boldsymbol{X}}(\mathrm{e}(G_0,l),\mathrm{e}(\bar{F}_0(l),k))) \\
&\quad\cong \mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W,\mathrm{Hom}_{[\boldsymbol{L},\boldsymbol{X}]}(G,\bar{F})) \\
&\quad\cong \mathrm{Hom}_{[\boldsymbol{L},\boldsymbol{X}]}(G,\{W;\bar{F}\}).
\end{aligned}$$

In particular, in context $l\colon L$, we have that $\{W;\hat{F}\}_0(l) \cong \{W;\bar{F}_0(l)\}$.

The enriching internal category $\boldsymbol{V}$ itself has a $\boldsymbol{V}$-enrichment when it is monoidal closed (see proposition 3.1.12). In that case, it is legitimate to ask whether it is complete as a $\boldsymbol{V}$-enriched category. Unintuitively, that does not (directly) depend on $\boldsymbol{V}$ being complete as an internal category, but on its having all ends (even though that is connected to completeness, see proposition 3.5.8).

Proposition 4.1.13. *The category $\boldsymbol{V}$ is complete as a $\boldsymbol{V}$-enriched category, and its limits for diagrams of shape $\boldsymbol{K}$ are given by the hom of the functor category $[\boldsymbol{K},\boldsymbol{V}]$.*

Proof. All hypothesis are stable under slicing, so we can assume $I = \mathbb{1}_{\mathscr{C}}$. Let $\boldsymbol{K}$ be a $\boldsymbol{V}$-category, and $F, W \colon \boldsymbol{K} \to \boldsymbol{V}$ be $\boldsymbol{V}$-enriched functors. Then, the limit is $\mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W,F)$. Indeed, in context $v \colon V_0$,

$$\begin{aligned}
&\mathrm{Hom}_{\boldsymbol{V}}(v, \mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W,F)) \\
&\quad = \mathrm{Hom}_{\boldsymbol{V}}(v, \textstyle\int_{k \colon K} \mathrm{Hom}_{\boldsymbol{V}}(W_0(k), F_0(k))) \\
&\quad \cong \textstyle\int_{k \colon K} \mathrm{Hom}_{\boldsymbol{V}}(v, \mathrm{Hom}_{\boldsymbol{V}}(W_0(k), F_0(k))) \\
&\quad \cong \textstyle\int_{k \colon K} \mathrm{Hom}_{\boldsymbol{V}}(W_0(k), \mathrm{Hom}_{\boldsymbol{V}}(v, F_0(k))) \\
&\quad = \mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W, \mathrm{Hom}_{\boldsymbol{V}}(v, F)).
\end{aligned}$$

□

The previous result produces a first example of complete $\boldsymbol{V}$-enriched category. We can produce further examples by means of the following proposition.

Proposition 4.1.14. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be $\boldsymbol{V}$-enriched categories. If $\boldsymbol{Y}$ is complete, then $[\boldsymbol{X},\boldsymbol{Y}]$ is complete and its weighted limits are computed point-wise.*

Proof. All hypothesis are stable under slicing, so we can assume we are slicing over the terminal object. Moreover, by proposition 4.1.11, there is a $\boldsymbol{V}$-enriched functor $\{\text{-};\text{-}\} \colon [\boldsymbol{K},\boldsymbol{V}]^{\mathbf{op}} \otimes [\boldsymbol{K},\boldsymbol{Y}] \to \boldsymbol{Y}$ yielding weighted limits in $\boldsymbol{Y}$.

Let $\boldsymbol{K}$ be a $\boldsymbol{V}$-category, and $F \colon \boldsymbol{K} \to [\boldsymbol{X},\boldsymbol{Y}]$ and $W \colon \boldsymbol{K} \to \boldsymbol{V}$ be $\boldsymbol{V}$-enriched functors. The limit of F with respect to the weight W has to be a point of $\mathrm{Fun}(\boldsymbol{X},\boldsymbol{Y})$, which is equivalent to a $\boldsymbol{V}$-enriched functor $\boldsymbol{X} \to \boldsymbol{Y}$. Moreover, consider the exponential transpose $\bar{F} \colon \boldsymbol{X} \to [\boldsymbol{K},\boldsymbol{Y}]$ of F. Then, let's define such a functor $\{W;F\}$ as

$$\begin{aligned}
x \colon X &\vdash \{W;F\}_0(x) \coloneqq \{W;\bar{F}_0(x)\} \\
x_0, x_1 \colon X &\vdash \{W;F\}_1(x_0,x_1) \coloneqq \{W;\bar{F}_0(x_0),\bar{F}_0(x_1)\}.
\end{aligned}$$

Let's prove that, in context $H \colon [\boldsymbol{X},\boldsymbol{Y}]$, there is a natural isomorphism

$$\mathrm{Hom}_{[\boldsymbol{X},\boldsymbol{Y}]}(H, \{W;F\})) \cong \mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W, \mathrm{Hom}_{[\boldsymbol{X},\boldsymbol{Y}]}(H,F)).$$

That is yielded by the following chain of equalities and natural isomorphims,

$$\begin{aligned}
&\mathrm{Hom}_{[\boldsymbol{X},\boldsymbol{Y}]}(H,\{W;F\})) \\
&\quad= \int_{x\colon X}\mathrm{Hom}_{\boldsymbol{Y}}(\mathrm{e}(H_0,x),\{W;F\}_0(x)) \\
&\quad= \int_{x\colon X}\mathrm{Hom}_{\boldsymbol{Y}}(\mathrm{e}(H_0,x),\{W;\bar{F}_0(x)\}) \\
&\quad\cong \int_{x\colon X}\mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W,\mathrm{Hom}_{\boldsymbol{Y}}(\mathrm{e}(H_0,x),\bar{F}_0(x))) \\
&\quad\cong \int_{x\colon X}\int_{k\colon K}\mathrm{Hom}_{\boldsymbol{V}}(W_0(k),\mathrm{Hom}_{\boldsymbol{Y}}(\mathrm{e}(H_0,x),\mathrm{e}(\bar{F}_0(x)_0,k))) \\
&\quad= \int_{x\colon X}\int_{k\colon K}\mathrm{Hom}_{\boldsymbol{V}}(W_0(k),\mathrm{Hom}_{\boldsymbol{Y}}(\mathrm{e}(H_0,x),\mathrm{e}(F_0(k)_0,x))) \\
&\quad\cong \int_{k\colon K}\mathrm{Hom}_{\boldsymbol{V}}(W_0(k),\int_{x\colon X}\mathrm{Hom}_{\boldsymbol{Y}}(\mathrm{e}(H_0,x),\mathrm{e}(F_0(k)_0,x))) \\
&\quad= \int_{k\colon K}\mathrm{Hom}_{\boldsymbol{V}}(W_0(k),\mathrm{Hom}_{[\boldsymbol{X},\boldsymbol{Y}]}(H,F_0(k))) \\
&\quad= \mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W,\mathrm{Hom}_{[\boldsymbol{X},\boldsymbol{Y}]}(H,F)). \qquad \square
\end{aligned}$$

As an immediate consequence of the previous results, we have that categories of presheaves are complete. That is a fundamental fact of standard category theory, and it was legitimate to expect it would hold in our context as well.

Corollary 4.1.15. *Let $\boldsymbol{X}$ be a $\boldsymbol{V}$-enriched category. Then, the category $[\boldsymbol{X}^{\mathbf{op}},\boldsymbol{V}]$ of presheaves over $\boldsymbol{X}$ is complete.*

Proof. It follows from propositions 4.1.13 and 4.1.14. □

Finally, we mention that weighted limits endow complete $\boldsymbol{V}$-enriched categories with a cotensor on $\boldsymbol{V}$.

Remark 4.1.16. If $\boldsymbol{X}$ is a complete $\boldsymbol{V}$-enriched category, there is a cotensor $\{\text{-};\text{-}\}\colon \boldsymbol{V}^{\mathbf{op}}\otimes\boldsymbol{X}\to\boldsymbol{X}$ yielding, in context $v\colon V_0, x, x'\colon X$, a natural isomorphism

$$\mathrm{Hom}_{\boldsymbol{X}}(x,\{v;x'\})\cong\mathrm{Hom}_{\boldsymbol{V}}(v,\mathrm{Hom}_{\boldsymbol{X}}(x,x')).$$

Such a cotensor and its associate isomorphism are obtained by composing the limit-yielding functor $\{\text{-};\text{-}\}\colon [\mathbb{I}_{\boldsymbol{V}\mathrm{Cat}_{\otimes}},\boldsymbol{V}]^{\mathbf{op}}\otimes[\mathbb{I}_{\boldsymbol{V}\mathrm{Cat}_{\otimes}},\boldsymbol{X}]\to\boldsymbol{X}$ with the obvious equivalences $\boldsymbol{X}\to[\mathbb{I}_{\boldsymbol{V}\mathrm{Cat}_{\otimes}},\boldsymbol{X}]$ and $\boldsymbol{V}\to[\mathbb{I}_{\boldsymbol{V}\mathrm{Cat}_{\otimes}},\boldsymbol{V}]$.

In the special case of a presheaf category $[\boldsymbol{X}^{\mathbf{op}},\boldsymbol{V}]$, which is complete because of corollary 4.1.15, we have a very concrete representation of the cotensor. Indeed, in context $v\colon V_0, F\colon [\boldsymbol{X}^{\mathbf{op}},\boldsymbol{V}]$, we have

$$\{v;F\}\cong\mathrm{Hom}_{\boldsymbol{V}}(v,\text{-})\circ F.$$

Essentially, this is a special case of proposition 4.1.14, but explicitely we see that, in context $H\colon [\boldsymbol{X}^{\mathbf{op}}, \boldsymbol{V}]$,

$$\begin{aligned}
&\mathrm{Hom}_{[\boldsymbol{X}^{\mathrm{op}},\boldsymbol{V}]}(H, \mathrm{Hom}_{\boldsymbol{V}}(v,\text{-}) \circ F) \\
&\quad = \int_{x\colon X} \mathrm{Hom}_{\boldsymbol{V}}(\mathrm{e}(H_0,x), \mathrm{Hom}_{\boldsymbol{V}}(v, \mathrm{e}_{\mathscr{E}}(F_0,x))) \\
&\quad \cong \int_{x\colon X} \mathrm{Hom}_{\boldsymbol{V}}(v, \mathrm{Hom}_{\boldsymbol{V}}(\mathrm{e}(H_0,x), \mathrm{e}_{\mathscr{E}}(F_0,x))) \\
&\quad \cong \mathrm{Hom}_{\boldsymbol{V}}(v, \int_{x\colon X} \mathrm{Hom}_{\boldsymbol{V}}(\mathrm{e}(H_0,x), \mathrm{e}_{\mathscr{E}}(F_0,x))) \\
&\quad = \mathrm{Hom}_{\boldsymbol{V}}(v, \mathrm{Hom}_{[\boldsymbol{X}^{\mathrm{op}},\boldsymbol{V}]}(H,F)).
\end{aligned}$$

4.2. Limit-Preserving Functors

The concept of a limit-preserving functor will be particularly helpful in the next section, so we shall dedicate some space to the definition and study of such a notion.

As a terminology remark, notice that, since internal categories are intrinsically small, to preserve limits is the same as to preserve small limits. Then, it would also be appropriate to call limit-preserving functors as continuous functors (which, in the standard terminology, denotes functors preserving small limits).

Definition 4.2.1 (limit-preserving functor)**.** Let $H\colon \boldsymbol{X} \to \boldsymbol{Y}$ be a $\boldsymbol{V}$-enriched functor. We say H preserves weighted limits if, for every indexing object I in $\mathscr{E}$, $I^*\boldsymbol{V}$-enriched category $\boldsymbol{K}$ and $I^*\boldsymbol{V}$-enriched functors $F\colon \boldsymbol{K} \to I^*\boldsymbol{X}$ and $W\colon \boldsymbol{K} \to I^*\boldsymbol{V}$, the evident $\boldsymbol{V}$-enriched natural transformation determined by universality $(I^*H)\{W;F\} \to \{W;(I^*H)F\}$ is an isomorphism.

In the rest of the chapter we sometimes have to prove that a functor is limit-preserving. In such situation, we will show the existence of an isomorphism, but omit the routine and tedious check that such isomorphism is indeed the correct one, i.e. the one determined by universality.

A first example of weighted limits-preserving $\boldsymbol{V}$-enriched functor is given by covariant and controvariant representable presheaves.

Proposition 4.2.2. *Representable $\boldsymbol{V}$-enriched covariant presheaves preserve weighed limits.*

Proof. Let $\boldsymbol{X}$ and $\boldsymbol{K}$ be $\boldsymbol{V}$-enriched categories, and $x\colon \mathbb{I}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}} \to \boldsymbol{X}$ an object of $\boldsymbol{X}$. Let's show that $\mathrm{Hom}_{\boldsymbol{X}}(x,\text{-})\colon \boldsymbol{X} \to \boldsymbol{V}$ preserves limits. Let $F\colon \boldsymbol{K} \to \boldsymbol{X}$ and $W\colon \boldsymbol{K} \to \boldsymbol{V}$ be $\boldsymbol{V}$-functors, and $\{W;F\}\colon \mathbb{I}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}} \to \boldsymbol{X}$ the weighted limit of F with respect to the weight W. Then, by proposition 4.1.13, we have that

$$\{W;\mathrm{Hom}_{\boldsymbol{X}}(x,\text{-}) \circ F\} \cong \mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W, \mathrm{Hom}_{\boldsymbol{X}}(x,\text{-}) \circ F) \cong \mathrm{Hom}_{\boldsymbol{X}}(x, \{W;F\}). \quad \square$$

Proposition 4.2.3. *Representable controvariant $\boldsymbol{V}$-enriched presheaves preserve weighed limits.*

Proof. Let $\boldsymbol{X}$ and $\boldsymbol{K}$ be $\boldsymbol{V}$-enriched categories, and $x\colon \mathbb{I}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}} \to \boldsymbol{X}$ an object of $\boldsymbol{X}$. Let's show that $\mathrm{Hom}_{\boldsymbol{X}}(\text{-},x) : \boldsymbol{X}^{\mathbf{op}} \to \boldsymbol{V}$ preserves limits. Let $F\colon \boldsymbol{K} \to \boldsymbol{X}^{\mathbf{op}}$ and $W\colon \boldsymbol{K} \to \boldsymbol{V}$ be $\boldsymbol{V}$-functors, and $\{W;F\}\colon \mathbb{I}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}} \to \boldsymbol{X}^{\mathbf{op}}$ the weighted limit of F with respect to the weight W. Then, by proposition 4.1.13, we have that

$$\begin{aligned}
&\{W;\mathrm{Hom}_{\boldsymbol{X}}(\text{-},x) \circ F\} \\
&\quad\cong \mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W, \mathrm{Hom}_{\boldsymbol{X}^{\mathbf{op}}}(x,\text{-}) \circ F) \\
&\quad\cong \mathrm{Hom}_{\boldsymbol{X}^{\mathbf{op}}}(x, \{W;F\}) \\
&\quad= \mathrm{Hom}_{\boldsymbol{X}}(\{W;F\}, x).
\end{aligned}$$

□

Then, by duality, we get that representable presheaves also preserve colimits. The following corollary sums up the situation.

Corollary 4.2.4. *Covariant and controvariant representable presheaves preserve both limits and colimits.*

The Yoneda embedding is another notable example of limit-preserving functor.

Proposition 4.2.5. *Let $\boldsymbol{X}$ be a $\boldsymbol{V}$-enriched category. Then, the Yoneda embedding* $\mathrm{Y}\colon \boldsymbol{X} \to [\boldsymbol{X}^{\mathbf{op}}, \boldsymbol{V}]$ *preserves weighted limits.*

Proof. All hypothesis are stable under slicing and, for an indexing object I, we have that $I^*\mathrm{Y}$ is the Yoneda embedding of $I^*\boldsymbol{X}$. So, we can assume $I = \mathbb{1}_{\mathscr{E}}$.

Let $F\colon \boldsymbol{K} \to \boldsymbol{X}$ and $W\colon \boldsymbol{K} \to \boldsymbol{V}$ be $\boldsymbol{V}$-enriched functors, such that it exists the limit of F with respect to the weight W. Then,

$$\begin{aligned}
&\mathrm{Hom}_{[\boldsymbol{X}^{\mathbf{op}},\boldsymbol{V}]}(H, \mathrm{Y}\{W;F\}) \\
&\quad= \int_{x\colon X} \mathrm{Hom}_{\boldsymbol{V}}\big(H_0(x), \mathrm{Hom}_{\boldsymbol{X}}(x, \{W;F\})\big) \\
&\quad\cong \int_{x\colon X} \mathrm{Hom}_{\boldsymbol{V}}\big(H_0(x), \mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W, \mathrm{Hom}_{\boldsymbol{X}}(x,F))\big) \\
&\quad= \int_{x\colon X} \mathrm{Hom}_{\boldsymbol{V}}\big(H_0(x), \int_{k\colon K} \mathrm{Hom}_{\boldsymbol{V}}(W_0(k), \mathrm{Hom}_{\boldsymbol{X}}(x, F_0(k)))\big) \\
&\quad\cong \int_{k\colon K} \mathrm{Hom}_{\boldsymbol{V}}\big(W_0(k), \int_{x\colon X} \mathrm{Hom}_{\boldsymbol{V}}(H_0(x), \mathrm{Hom}_{\boldsymbol{X}}(x, F_0(k)))\big) \\
&\quad= \int_{k\colon K} \mathrm{Hom}_{\boldsymbol{V}}\big(W_0(k), \mathrm{Hom}_{[\boldsymbol{X}^{\mathbf{op}},\boldsymbol{V}]}(H, \mathrm{Y}_0(F_0(k)))\big) \\
&\quad= \mathrm{Hom}_{[\boldsymbol{K},\boldsymbol{V}]}(W, \mathrm{Hom}_{[\boldsymbol{X}^{\mathbf{op}},\boldsymbol{V}]}(H, \mathrm{Y}F))
\end{aligned}$$

meaning that $\{W;\mathrm{Y}F\}$ exists and it is $\mathrm{Y}\{W;F\}$. □

It is worth noticing that we cannot internalize the notion of limit-preserving functor (meaning that we have no object of all limit-preserving functors), as the relative defining condition spans over all possible diagrams, and there is no internal representation of the object of all diagrams (just as there is no set of all sets). Still, fixed a diagram shape (other than a domain and a codomain $\boldsymbol{V}$-categories), it is possible to consider an internal representation of all functors (with the given domain and codomain) preserving limits of diagrams of the given shape.

Remark 4.2.6. Let $\boldsymbol{K}$ be a $\boldsymbol{V}$-enriched category. Then, there is in $\mathscr{E}$ an object of $\boldsymbol{K}$-limit-preserving functors $\boldsymbol{X} \to \boldsymbol{Y}$, denoted by $\mathrm{LPFun}_{\boldsymbol{K}}(\boldsymbol{X}, \boldsymbol{Y})$, which also tracks the required isomorphisms. This induces a $\boldsymbol{V}$-enriched category $[\boldsymbol{X}, \boldsymbol{Y}]_{\mathrm{LP}(\boldsymbol{K})}$.

Analogously to the case of completeness for $\boldsymbol{V}$-enriched categories, a limit-preserving functor $H \colon \boldsymbol{X} \to \boldsymbol{Y}$ induces, for each $\boldsymbol{V}$-enriched category $\boldsymbol{K}$, an internal choice of isomorphisms witnessing its property of preserving the limits of $\boldsymbol{V}$-enriched functors $\boldsymbol{K} \to \boldsymbol{X}$ with respect to weights $\boldsymbol{K} \to \boldsymbol{V}$.

4.3. Adjoint Functor Theorem

In this section, we prove some further results about internal weighted limits and completeness, notably the adjoint functor theorem, by taking advantage of the intrinsic smallness of internal enriched categories. The results in this section will then be analogous to those that can be obtained for lattices, regarded as small categories.

The next, fundamental proposition establishes a link between limits and representability of functors, and it will be the base upon which we will build all of the following results.

Proposition 4.3.1. *Let $\boldsymbol{X}$ be a $\boldsymbol{V}$-enriched category and $F \colon \boldsymbol{X} \to \boldsymbol{V}$ a limit-preserving $\boldsymbol{V}$-functor. If the limit of $\mathrm{id}_{\boldsymbol{V}\mathrm{Cat}_{\mathscr{E}}}(\boldsymbol{X})$ relative to the weight F, denoted by $\{F;\boldsymbol{X}\}$, exists, then*

$$\mathrm{Hom}_{\boldsymbol{X}}(\{F;\boldsymbol{X}\}, \text{-}) \cong F$$

(that is, F is representable).

Proof. Let $\bar{p} \colon F \to \mathrm{Hom}_{\boldsymbol{X}}(\{F;\boldsymbol{X}\}, \text{-})$ be the universal cone for the limit $\{F;\boldsymbol{X}\}$, which we shall take as one side of the isomorphism.

Since F preserves limits, $F_0(\{F;\boldsymbol{X}\})$ is (isomorphic to) the limit $\{F;F\}$. Because of that, the natural transformation

$$F_1(\{F;\boldsymbol{X}\}, \text{-}) \colon \mathrm{Hom}_{\boldsymbol{X}}(\{F;\boldsymbol{X}\}, \text{-}) \to \mathrm{Hom}_{\boldsymbol{V}}(\{F;F\}, F(\text{-})) \colon \boldsymbol{X} \to \boldsymbol{V}$$

yields the universal cone of $\{F;F\}$,

$$\bar{q} = F \xrightarrow{\bar{p}} \mathrm{Hom}_{\boldsymbol{X}}(\{F;\boldsymbol{X}\}, \text{-}) \xrightarrow{F_1(\{F;\boldsymbol{X}\}, \text{-})} \mathrm{Hom}_{\boldsymbol{V}}(\{F;F\}, F(\text{-})).$$

Consider the identical natural transformation $\mathrm{id}(F): F \to F \cong \mathrm{Hom}_{\boldsymbol{V}}(\mathbb{I}_{\boldsymbol{V}}, F)$. By the universality of the cone $\bar{q}$, there is a (unique) $\mu\colon \mathbb{I}_{\boldsymbol{V}} \to \{F;F\}$ such that

$$F \xrightarrow{\bar{q}} \mathrm{Hom}_{\boldsymbol{V}}(\{F;F\}, F) \xrightarrow{\mathrm{Hom}_{\boldsymbol{V}}(\mu,F)} \mathrm{Hom}_{\boldsymbol{V}}(\mathbb{I}_{\boldsymbol{V}}, F) \cong F \tag{4.2}$$

is the identity on F. That suggests that the natural transformation $\bar{p}^{-1}$ that we need is

$$\mathrm{Hom}_{\boldsymbol{X}}(\{F;\boldsymbol{X}\}, \text{-}) \xrightarrow{F_1(\{F;\boldsymbol{X}\},\text{-})} \mathrm{Hom}_{\boldsymbol{V}}(\{F;F\}, F(\text{-})) \xrightarrow{\mathrm{Hom}_{\boldsymbol{V}}(\mu,F)} F.$$

Let's prove that

$$\mathrm{Hom}_{\boldsymbol{X}}(\{F;\boldsymbol{X}\}, \text{-}) \xrightarrow{\bar{p}^{-1}} F \xrightarrow{\bar{p}} \mathrm{Hom}_{\boldsymbol{X}}(\{\boldsymbol{X};F\}, \text{-}) = \mathrm{id}. \tag{4.3}$$

The diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\boldsymbol{X}}(\{F;\boldsymbol{X}\}, \{F;\boldsymbol{X}\}) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(\{F;\boldsymbol{X}\}, \text{-}) & \xrightarrow{\circ_{\boldsymbol{X}}} & \mathrm{Hom}_{\boldsymbol{X}}(\{F;\boldsymbol{X}\}, \text{-}) \\
\Big\downarrow {\scriptstyle F_1(\{F;\boldsymbol{X}\},\{F;\boldsymbol{X}\})\otimes^{\boldsymbol{V}}\mathrm{id}} & & \Big\downarrow {\scriptstyle F_1(\{F;\boldsymbol{X}\},\text{-})} \\
\mathrm{Hom}_{\boldsymbol{V}}(\{F;F\}, \{F;F\}) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(\{F;\boldsymbol{X}\}, \text{-}) & \xrightarrow{\circ_{\boldsymbol{X}}(\mathrm{id}\otimes^{\boldsymbol{V}} F_1(\{F;\boldsymbol{X}\},\text{-}))} & \mathrm{Hom}_{\boldsymbol{V}}(\{F;F\}, \text{-})F \\
\Big\downarrow {\scriptstyle \mathrm{Hom}_{\boldsymbol{V}}(\mu,\{F;F\})\otimes^{\boldsymbol{V}}\mathrm{id}} & & \Big\downarrow {\scriptstyle \mathrm{Hom}_{\boldsymbol{V}}(\mu,\text{-})} \\
F_0(\{F;\boldsymbol{X}\}) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(\{F;\boldsymbol{X}\}, \text{-}) & \xrightarrow{\circ_{\boldsymbol{X}}(\mathrm{id}\otimes^{\boldsymbol{V}} F_1)} & F \\
\Big\downarrow {\scriptstyle \bar{p}_{\{F;\boldsymbol{X}\}}\otimes^{\boldsymbol{V}}\mathrm{id}} & & \Big\downarrow {\scriptstyle \bar{p}} \\
\mathrm{Hom}_{\boldsymbol{X}}(\{F;\boldsymbol{X}\}, \{F;\boldsymbol{X}\}) \otimes^{\boldsymbol{V}} \mathrm{Hom}_{\boldsymbol{X}}(\{F;\boldsymbol{X}\}, \text{-}) & \xrightarrow{\circ_{\boldsymbol{X}}} & \mathrm{Hom}_{\boldsymbol{X}}(\{F;\boldsymbol{X}\}, \text{-})
\end{array}$$

commutes because of functoriality of F, associativity of composition and naturality of $\bar{p}$. Notice that $\bar{p}_{\{F;\boldsymbol{X}\}}\mu\colon \mathbb{I}_{\boldsymbol{V}} \to \mathrm{Hom}_{\boldsymbol{V}}(\{F;\boldsymbol{X}\}, \{F;\boldsymbol{X}\})$ yields a natural transformation

$$\mathrm{Hom}_{\boldsymbol{X}}(\bar{p}_{\{F;\boldsymbol{X}\}}\mu, \text{-}) : \mathrm{Hom}_{\boldsymbol{X}}(\{F;\boldsymbol{X}\}, \text{-}) \to \mathrm{Hom}_{\boldsymbol{X}}(\{F;\boldsymbol{X}\}, \text{-}).$$

By diagram-chasing $\mathrm{id}_{\boldsymbol{X}}(\{F;\boldsymbol{X}\})$, we get that

$$\bar{p}\mathrm{Hom}_{\boldsymbol{V}}(\mu, \{F;F\})F_1(\{F;\boldsymbol{X}\}, \text{-}) = \mathrm{Hom}_{\boldsymbol{X}}(\bar{p}_{\{F;\boldsymbol{X}\}}\mu, \text{-}). \tag{4.4}$$

By pre-composing with $\bar{p}$ and using eq. (4.2), it follows that

$$\mathrm{Hom}_{\boldsymbol{X}}(\bar{p}_{\{F;\boldsymbol{X}\}}\mu, \text{-})\bar{p} = \bar{p}\mathrm{Hom}_{\boldsymbol{V}}(\mu, \{F;F\})F_1(\{F;\boldsymbol{X}\}, \text{-})\bar{p} = \bar{p}.$$

But, by the universal property of the limit, the only such arrow $\{F;\boldsymbol{X}\} \to \{F;\boldsymbol{X}\}$ is the identity, which means that eq. (4.4) yields eq. (4.3). $\square$

We would like to internalize proposition 4.3.1 to get an isomorphism that is natural over the object of limit-preserving functors. That could be achieved by internalizing the whole proof by using, in particular, the internal limit-functor for a complete category from proposition 4.1.11 and the internal choice of isomorphisms from remark 4.2.6. Still, the result would be hard to read and provide little insight. Instead, we provide a proof via indexing over the object of limit-preserving functors, and then applying the external version of the result.

Notice how, in the proof of proposition 4.3.1, the only limits the functor needs to truly preserve are those of diagrams with shape $\boldsymbol{X}$. Of course, since the functor turns out to be (isomorphic to) a presheaf, it indeed preserves all limits. This observation, though, is useful for internalizing the result, since internally we can represent the object of functors preserving limits of diagrams with shape $\boldsymbol{X}$, but not general limit-preserving functors.

Corollary 4.3.2. *Let $\boldsymbol{X}$ be a complete $\boldsymbol{V}$-enriched category. Then there is a $\boldsymbol{V}$-enriched isomorphism* $\mathrm{Hom}_{\boldsymbol{X}}(\{F;\boldsymbol{X}\},\text{-}) \cong F$ *natural in* $F\colon \mathrm{LPFun}_{\boldsymbol{X}}(\boldsymbol{X},\boldsymbol{V})$ *(see remark 4.2.6).*

Proof. In the slice category of $\mathscr{E}$ over $\mathrm{LPFun}_{\boldsymbol{X}}(\boldsymbol{X},\boldsymbol{V})$, consider the diagram

$$\mathrm{e}\colon \mathrm{LPFun}_{\boldsymbol{X}}(\boldsymbol{X},\boldsymbol{V})^{*}\boldsymbol{X} \to \mathrm{LPFun}_{\boldsymbol{X}}(\boldsymbol{X},\boldsymbol{V})^{*}\boldsymbol{V}$$

defined, in context $F\colon \mathrm{LPFun}_{\boldsymbol{X}}(\boldsymbol{X},\boldsymbol{V}), x\colon X$, as $\mathrm{e}(F,x) = (F, \mathrm{e}_{\mathscr{E}}(F_0,x))$. Such functor preserves limits because it is indexed over the object of limit-preserving functors, so we can apply proposition 4.3.1 to get an isomorphism

$$\mathrm{Hom}_{\mathrm{LPFun}_{\boldsymbol{X}}(\boldsymbol{X},\boldsymbol{V})^{*}\boldsymbol{X}}(\{\mathrm{LPFun}_{\boldsymbol{X}}(\boldsymbol{X},\boldsymbol{V})^{*}\boldsymbol{X};\mathrm{e}\}(F),(F,x)) \cong (F,\mathrm{e}_{\mathscr{E}}(F_0,x)).$$

Notice that the weighted limit $\{\mathrm{LPFun}_{\boldsymbol{X}}(\boldsymbol{X},\boldsymbol{V})^{*}\boldsymbol{X};\mathrm{e}\}$ with parameter F will be a pair (F,l) where $l\colon X$. Then, the chain of isomorphisms

$$\begin{aligned}
&(F,\mathrm{Hom}_{\boldsymbol{X}}(x',l))\\
&\quad= \mathrm{Hom}_{\mathrm{LPFun}_{\boldsymbol{X}}(\boldsymbol{X},\boldsymbol{V})^{*}\boldsymbol{X}}((F,x'),(F,l))\\
&\quad= \mathrm{Hom}_{\mathrm{LPFun}_{\boldsymbol{X}}(\boldsymbol{X},\boldsymbol{V})^{*}\boldsymbol{X}}((F,x'),\{\mathrm{LPFun}_{\boldsymbol{X}}(\boldsymbol{X},\boldsymbol{V})^{*}\boldsymbol{X};\mathrm{e}\}(F))\\
&\quad\cong \mathrm{Hom}_{\mathrm{LPFun}_{\boldsymbol{X}}(\boldsymbol{X},\boldsymbol{V})^{*}[\boldsymbol{X},\boldsymbol{X}]}(\mathrm{LPFun}_{\boldsymbol{X}}(\boldsymbol{X},\boldsymbol{V})^{*}\boldsymbol{X},\mathrm{Hom}_{\mathrm{LPFun}_{\boldsymbol{X}}(\boldsymbol{X},\boldsymbol{V})^{*}\boldsymbol{X}}((F,x'),\mathrm{e}(F)))\\
&\quad= (F,\mathrm{Hom}_{[\boldsymbol{X},\boldsymbol{X}]}(\boldsymbol{X},\mathrm{Hom}_{\mathrm{LPFun}_{\boldsymbol{X}}(\boldsymbol{X},\boldsymbol{V})^{*}\boldsymbol{X}}(x',F)))
\end{aligned}$$

shows that l is $\{\boldsymbol{X};F\}$. But then, $\mathrm{e}_{\mathscr{E}}(F_0,x) \cong \mathrm{Hom}_{\boldsymbol{X}}(\text{-},\{\boldsymbol{X};F\})$. □

By corollary 4.3.2, a presheaf on a $\boldsymbol{V}$-enriched category $\boldsymbol{X}$ that preserves limits of shape $\boldsymbol{X}$ is representable. We know that representable presheaves preserves all limits, so it may be tempting to say that $\mathrm{LPFun}_{\boldsymbol{X}}(\boldsymbol{X},\boldsymbol{V})$ is the object of limit-preserving presheaves over $\boldsymbol{X}$. There is, though, a philosophical issue: $\mathrm{LPFun}_{\boldsymbol{X}}(\boldsymbol{X}, \boldsymbol{V})$ tracks, for each presheaf it contains and each diagram of shape $\boldsymbol{X}$, the isomorphism witnessing that the limit of such diagram is preserved by such presheaf, but it does not do the same for diagrams of other shapes, as an object representing limit-preserving functors should.

Remarkably, corollary 4.3.2 provides a pseudo-inverse of the Yoneda embedding. Unexpectedly, it is then possible to establish an equivalence between a $\boldsymbol{V}$-enriched category $\boldsymbol{X}$ and the $\boldsymbol{V}$-enriched category of limit-preserving presheaves over it.

Corollary 4.3.3. *Let $\boldsymbol{X}$ be a complete $\boldsymbol{V}$-enriched category. Then there is an equivalence of $\boldsymbol{V}$-enriched categories*

$$\boldsymbol{X} \equiv [\boldsymbol{X}, \boldsymbol{Y}]_{\mathrm{LP}(\boldsymbol{K})}.$$

Proof. It follows from corollaries 3.8.5 and 4.3.2 and proposition 4.2.3. □

From the previous results, it follows that, in the context of internal enriched categories, completeness implies cocompleteness.

Corollary 4.3.4. *Complete $\boldsymbol{V}$-enriched categories are cocomplete.*

Proof. Let $\boldsymbol{X}$ be a complete $\boldsymbol{V}$-enriched category, and $W\colon \boldsymbol{K}^{\mathbf{op}} \to \boldsymbol{V}$ and $G\colon \boldsymbol{K} \to \boldsymbol{X}$ be $\boldsymbol{V}$-enriched functors. Define

$$F := \mathrm{Hom}_{[\boldsymbol{K}^{\mathbf{op}},\boldsymbol{V}]}(W, \mathrm{Hom}_{\boldsymbol{X}}(G, \text{-})) : \boldsymbol{X} \to \boldsymbol{V}.$$

By propositions 4.1.14 and 4.2.5, F preserves limits. Then, the weighted colimit $W \cdot G$ is given by the weighted limit $\{F; \boldsymbol{X}\}$. Indeed, from proposition 4.3.1, we get

$$\mathrm{Hom}_{\boldsymbol{X}}(\{F; \boldsymbol{X}\}, \text{-}) \cong F = \mathrm{Hom}_{[\boldsymbol{K}^{\mathbf{op}},\boldsymbol{V}]}(W, \mathrm{Hom}_{\boldsymbol{X}}(G, \text{-})).$$ □

By duality we also get the converse, so that completeness and cocompleteness are equivalent in the context of $\boldsymbol{V}$-enriched categories.

Corollary 4.3.5. *Cocomplete $\boldsymbol{V}$-enriched categories are complete.*

We have an immediate consequence by applying the previous result to categories of presheaves.

Corollary 4.3.6. *All $\boldsymbol{V}$-enriched categories of presheaves are co-complete.*

Proof. It follows from corollaries 4.1.15 and 4.3.4. □

We also have a version of the adjoint functor theorem. What is remarkable about it is that it doesn't require any size-related hypothesis, such as the solution set condition.

Corollary 4.3.7 (adjoint functor theorem)**.** *Let* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *be* $\boldsymbol{V}$*-enriched categories, of which* $\boldsymbol{X}$ *is complete, and* $F\colon \boldsymbol{X} \to \boldsymbol{Y}$ *a limit-preserving* $\boldsymbol{V}$*-enriched functor. Then,* F *has a left adjoint.*

Proof. Consider the functor $\tilde{F} \coloneqq \mathrm{Hom}_{\boldsymbol{Y}}(\text{-}, F) : \boldsymbol{Y}^{\mathbf{op}} \to [\boldsymbol{X}, \boldsymbol{V}]$. Notice that, in context $y\colon Y$, the functor $\tilde{F}_0(y) : \mathrm{LPFun}_{\boldsymbol{X}}(\boldsymbol{X}, \boldsymbol{V})$ preserves limits. Then, the functor

$$G \coloneqq \boldsymbol{Y} \xrightarrow{\tilde{F}^{\mathbf{op}} \otimes \mathrm{id}_{\mathrm{VCat}_{\mathscr{C}}}(\boldsymbol{X})} [\boldsymbol{X}, \boldsymbol{V}]^{\mathbf{op}} \otimes^{\boldsymbol{V}} [\boldsymbol{X}, \boldsymbol{X}] \xrightarrow{\{\text{-};\text{-}\}} \boldsymbol{X}$$

is left adjoint to F. Indeed, in context $x\colon X$ and $y\colon Y$,

$$\mathrm{Hom}_{\boldsymbol{X}}(G_0(y), x) = \mathrm{Hom}_{\boldsymbol{X}}(\{\tilde{F}_0(y); \boldsymbol{X}\}, x) \cong \mathrm{e}(\tilde{F}_0(y), x) = \mathrm{Hom}_{\boldsymbol{Y}}(y, F_0(x))$$

by corollary 4.3.2. □

By duality, we also get a version of the adjoint functor theorem for colimits.

Corollary 4.3.8. *Let* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *be* $\boldsymbol{V}$*-enriched categories, of which* $\boldsymbol{X}$ *is cocomplete, and* $F\colon \boldsymbol{X} \to \boldsymbol{Y}$ *a colimit-preserving* $\boldsymbol{V}$*-enriched functor. Then,* F *has a right adjoint.*

Finally, the previous results can be applied to categories of presheaves.

Corollary 4.3.9. *The Yoneda embedding for a complete* $\boldsymbol{V}$*-enriched category has a left adjoint.*

Proof. It follows from proposition 4.2.5 and corollary 4.3.7. □

Remark 4.3.10. It would be way too good if, analogously to corollary 4.3.5, a $\boldsymbol{V}$-enriched functor preserved limits if and only if it preserved colimits. Sadly, counter-examples are easy to find: frames are complete lattices (and thus they are complete Set-enriched categories) satisfying an infinite distributive law, whose homomorphisms preserve arbitrary joins (so preserve limits) and finite meets, but do not necessarily preserve arbitrary meets (and so do not necessarily preserve colimits). The lattice of open sets of topologies provides examples of such frames, and the inverse-image of continuous functions yields frame homomorphisms, which notoriously preserve arbitrary joins (unions of opens sets) and finite meets (intersections of open sets), but not arbitrary meets (interiors of intersections of open sets).

5. Future Work

In this dissertation, we developed the foundations of the theory of internal enriched categories. This merging of the separate subjects of internal and enriched category theory raises many issues. In particular one would like to investigate how the theory plays out in examples and exploit the theory in application areas. In this chapter we suggest what seem to be promising areas for future research.

5.1. Sets

Within the category of sets, our theory corresponds to the standard theory of enriched categories (with the restriction that both the enriched categories and the enriching monoidal categories are small).

With regards to completeness of the enriching category, the small complete symmetric monoidal closed categories are—up to equivalence—what are known as commutative unital quantales. There is some existing theory of quantale enriched categories which has been studied in particular by Dirk Hofmann, and it would be of interest to consider this subject from the point of view of this dissertation. On the one hand, the theorems which we give can be read as the elementary results of that theory. On the other hand, one might try to generalize developments of that theory to our context, such as the duality theory given in Hofmann and Waszkiewicz (2012).

5.2. Modest Sets

The internal category of modest sets in the effective topos (or better, its subcategory of assemblies) represents the leading example for the theory of internal enriched categories, as it fits all the requirements we need to develop the theory presented in this dissertation.

Let's recall the definition of the effective topos, Eff (Hyland 1982).

Definition 5.2.1 (effective topos)**.** An object $(X, [\text{-} =_X \text{-}])$ of the effective topos is a set X with a non-standard equality predicate $[\text{-} =_X \text{-}]$ on $X \times X$, that is, a function $X \times X \to \mathscr{P}(\mathbb{N})$, such that the axioms

Symmetry: $[x =_X x'] \implies [x' =_X x]$, and

Transitivity: $[x =_X x'] \wedge [x' =_X x''] \implies [x =_X x'']$

are valid.

Let's denote the term $[x =_X x]$ with $[x \in X]$, since, in a way, it represents the set of witnesses to the belonging of x to X.

An arrow $(X, [\text{-} =_X \text{-}]) \to (Y, [\text{-} =_Y \text{-}])$ of the effective topos is a class of equivalence of functional relations $X \to Y$, that is, functions $f\colon X \times Y \to \mathscr{P}(\mathbb{N})$ such that the axioms

Relational: $f(x,y) \wedge [x =_X x'] \wedge [y =_Y y'] \implies f(x',y')$,

Strict: $f(x,y) \implies [x \in X] \wedge [y \in Y]$,

Single-valued: $f(x,y) \wedge f(x,y') \implies [y =_Y y']$, and

Total: $[x \in X] \implies \exists y : Y.\, f(x,y)$

are valid, and where who such functional relations f and g are equivalent if $f(x,y) \iff g(x,y)$ is valid.

Given two morphisms

$$(X, [\text{-} =_X \text{-}]) \xrightarrow{[f]} (Y, [\text{-} =_Y \text{-}]) \xrightarrow{[g]} (Z, [\text{-} =_Z \text{-}]),$$

the composition is given by the class of equivalence of the predicate

$$(g \circ f)(x,z) \coloneqq \exists y : Y.\, f(x,y) \wedge g(y,z).$$

The identity for an object is given by the non-standard equality predicate of the object.

There is a functor $\Delta\colon \mathrm{Set} \to \mathrm{Eff}$ sending a set X to the object (X, δ_X), where $\delta_X(x,x') = \mathbb{N}$ if $x = x'$ and $\emptyset$ otherwise.

Let's now define the internal category of modest sets Mod in the effective topos (Hyland 1988).

Definition 5.2.2 (modest sets)**.** A modest set is an object $(X, [\text{-} =_X \text{-}])$ in Eff such that $[x =_X x'] = [x \in X] \cap [x' \in X]$ and $[x =_X x'] \neq \emptyset$ implies $x = x'$ for all $x, x' \in X$.

Modest sets are, equivalently, partial equivalence relations on $\mathbb{N}$. Let PER be the category of partial equivalence relations on $\mathbb{N}$.

Definition 5.2.3 (category of modest sets)**.** The object of objects of the internal category Mod in Eff is $\mathrm{Mod}_0 = \Delta(\mathrm{PER}_0)$. The object of arrows of Mod is the subobject of $\Delta(\mathrm{PER}_1)$ with existence predicate given by $[f \in \mathrm{Mod}_1] = \{\, n \mid n \text{ is an index for } f \}$.

In the effective topos, Mod is weakly complete, in the sense of Hyland, Robinson, and Rosolini 1990. Sadly that is not much use for the purposes of this dissertation. However we can restrict attention to the quasi-topos of separated objects for the double-negation topology, which is, equivalently, the category of assemblies.

Definition 5.2.4 (assemblies)**.** An assembly is an object $(X, [\text{-} =_X \text{-}])$ of Eff such that $[x =_X x'] \neq \emptyset$ implies $x = x'$ for all $x, x' \in X$. Let's call Asm the full subcategory of assemblies in the effective topos.

Then, Mod is an internal strongly complete category in Asm (ibid.). Moreover, the cartesian product in Mod makes it a symmetric monoidal closed category. Then, we can then consider Mod-enriched categories in Asm, and the theory developed in this dissertation will hold in that context.

5.3. Groupoids

The original suggestion that there should be an internal enrichment over the category of Modest Sets came from a discussion with Rosolini on the topic of categorical models of type theory, in an attempt to formalize the intuition of a groupoid-like object whose hom-sets look like modest sets.

In the early stages of the research, I abandoned the attempt to develop a theory for groupoids, focusing instead on the mere enrichment over an internal monoidal category. The reason is that, to define an enriched groupoid, the enriching category has to be cartesian monoidal, an that would have much reduced the generality of the argument. Indeed, the internal logic of a monoidal category does not even allow to express a formula of the kind $f \circ f^{-1} = \mathrm{id}$!

Still, the leading example of the category of Modest Sets is cartesian monoidal (other then cartesian closed and complete), so it would support a notion of groupoid enriched over it. It would then be interesting to specialize the present theory to groupoids enriched over a cartesian monoidal category, and then apply that theory to the category of Modest Sets.

5.4. Monads

Monads are an essential tool in category theory, and have found wide application in the area of theoretical computer science (Moggi 1991). It is then opportune to develop a theory of monads in the context of internal enriched categories. There two aspects of monads which seem worth following up from the point of view of this dissertation.

First note that the category of modest sets just discussed is cartesian closed and we might wish to have examples of more general complete symmetric monoidal closed internal categories. One way to do that is by using the theory of commutative monads as developed by Anders Kock (Kock 1970, 1971a,b, 1972a,b). Suppose we have an internal complete cartesian closed or even symmetric monoidal closed category and an internal commutative monad T on it. Then we would expect the category of T-algebras to be complete symmetric monoidal closed. Typically algebraic theories where there is a natural tensor product on algebras are commutative. There are many mathematical examples and if anything more examples have been studied in connection with Moggi's approach to computational effects. There will surely be versions of such monads acting on modest sets and that will give a large collection of examples of the theory presented in this dissertation.

Secondly we might develop a theory of internal enriched monads. We sketch the basic framework. Assume $\boldsymbol{V}$ is an internal symmetric monoidal category in $\mathscr{E}$ with equalizer.

Definition 5.4.1 (internal enriched monad)**.** Let $\boldsymbol{X}$ be a $\boldsymbol{V}$-enriched category. A *monad* T on $\boldsymbol{X}$ is a triple

$$(T\colon \boldsymbol{X} \to \boldsymbol{X}, \eta\colon \mathrm{id}(\boldsymbol{X}) \to T, \mu\colon T^2 \to T)$$

such that $\mu(\eta T) = \mathrm{id}(T) : T \to T$ and $\mu(\mu T) = \mu(T\mu) : T^3 \to T$.

The object of algebras on T is the subobject of $X \times V_1$ of those $(x\colon X, h\colon V_1)$ such that

$$h\colon \mathbb{I}_{\boldsymbol{V}} \to \mathrm{Hom}_{\boldsymbol{X}}(T_0(x), x),$$

$$\begin{array}{ccc}
 & \mathrm{Hom}_{\boldsymbol{X}}(T_0(x), x) \otimes \mathrm{Hom}_{\boldsymbol{X}}(x, T_0(x)) & \\
{}^{h\otimes\eta(x)}\nearrow & & \searrow^{\circ_{\boldsymbol{X}}(x,T_0(x),x)} \\
\mathbb{I}_{\boldsymbol{V}} & \xrightarrow{\quad\mathrm{id}_{\boldsymbol{X}}(x)\quad} & \mathrm{Hom}_{\boldsymbol{X}}(x, x)
\end{array}$$

and

$$\begin{array}{ccc}
\mathbb{I}_{\boldsymbol{V}} & \xrightarrow{\quad h\otimes h\quad} & \mathrm{Hom}_{\boldsymbol{X}}(T_0(x), x) \otimes \mathrm{Hom}_{\boldsymbol{X}}(T_0(x), x) \\
 & & \downarrow{\scriptstyle T_1(T_0(x),x)\otimes\mathrm{id}} \\
\downarrow{\scriptstyle \mu(x)\otimes h} & & \mathrm{Hom}_{\boldsymbol{X}}(T_0(T_0(x)), T_0(x)) \otimes \mathrm{Hom}_{\boldsymbol{X}}(T_0(x), x) \\
 & & \downarrow{\scriptstyle \circ_{\boldsymbol{X}}(T_0(T_0(x)),T_0(x),x)} \\
\mathrm{Hom}_{\boldsymbol{X}}(T_0(T_0(x)), T_0(x)) \otimes \mathrm{Hom}_{\boldsymbol{X}}(T_0(x), x) & \xrightarrow[\circ_{\boldsymbol{X}}(T_0(T_0(x)),T_0(x),x)]{} & \mathrm{Hom}_{\boldsymbol{X}}(T_0(T_0(x)), x).
\end{array}$$

The homset in the context of algebras (x_0, h_0) and (x_1, h_1) is the equalizer of

$$\begin{array}{c} \mathrm{Hom}_{\boldsymbol{X}}(x_0, x_1) \\ \big\downarrow {\scriptstyle \mathrm{id} \otimes h_0} \\ \mathrm{Hom}_{\boldsymbol{X}}(x_0, x_1) \otimes \mathrm{Hom}_{\boldsymbol{X}}(T_0(x_0), x_0) \\ \big\downarrow {\scriptstyle \circ_{\boldsymbol{X}}(T_0(x_0), x_0, x_1)} \\ \mathrm{Hom}_{\boldsymbol{X}}(T_0(x_0), x_1) \end{array} \quad \text{and} \quad \begin{array}{c} \mathrm{Hom}_{\boldsymbol{X}}(x_0, x_1) \\ \big\downarrow {\scriptstyle h_0 \otimes T_1(x_0, x_1)} \\ \mathrm{Hom}_{\boldsymbol{X}}(T_0(x_1), x_1) \otimes \mathrm{Hom}_{\boldsymbol{X}}(T_0(x_0), T_0(x_1)) \\ \big\downarrow {\scriptstyle \circ_{\boldsymbol{X}}(T_0(x_0), T_0(x_1), x_1)} \\ \mathrm{Hom}_{\boldsymbol{X}}(T_0(x_0), x_1) \end{array}$$

and the identity in the context of algebra (x, h) is $\mathrm{id}_{\boldsymbol{X}}(x)$. With this data, we have an internal $\boldsymbol{V}$-enriched category of algebras Alg_T.

There are some standard definitions and results which would now have to be proved, such as the characterization of monadic functors. Of particular interest would be to prove that categories of algebras Alg_T are complete. This would imply that categories of algebras are cocomplete too. In standard category theory, it is not generally true that if a category is cocomplete then so is a category of algebras over it. Although there are various circumstances in which such categories are cocomplete, these require individual attention. Even the case of algebras over sets is non-trivial.

A theory of monads in the enriched setting is one notable omission in Kelly's treaties on enriched categories and rather little seems to be have appeared in the classical literature. There may be good reason for this. For example, in his consideration of the related topics of adjoint lifting in the enriched setting, John Power explicitly remarks (Power 1988) that he conducts his arguments essentially in the context of Street's Formal Theory of Monads (Street 1972). Perhaps because of this intellectual history it would be instructive to develop the theory of enriched monads on an internal enriched category. One would expect initial problems e.g. as to whether the category of algebras is cocomplete to evaporate. But would one now be able to refer to Street's theory as a suitable background?

5.5. Cubical Sets

There is currently much interest in models for Type Theory (especially for Homotopy Type Theory) in categories of presheaves. Because of its seeming simplicity, internal enriched category theory should provide an ideal setting for this kind of investigations.

We shall comment on the possibility of developing a theory of cubical sets (M. Grandis and Mauri 2003) in the context of modest sets. Indeed, cubical sets have been successfully used to yield univalent models for homotopy type theory (Bezem, Coquand, and Huber 2014), and a theory of "modest" cubical sets would provide a potentially interesting variation of those models. In particular, it would be interesting to investigate how the properties of the effective topos, especially those regarding representability, affect such models.

To begin with, it would be necessary to define a suitable cube category $\square$ in Asm. It could seem tempting to just use the set-theoretic definition, via the embedding Δ, but likely that would not be the best choice: the set of object of the cube category is $\mathbb{N}$, but $\Delta(\mathbb{N})$ is not the natural number object of Eff. A better idea would probably be to consider the natural number object of Eff as the object of objects, and a suitable subobject of $\mathbb{N}^{\mathbb{N}}$ (whose underlying set is, notably, the set of recursive functions $\mathbb{N} \to \mathbb{N}$, see Hyland 1982, Lemma 10.1) as the object of arrows. Then, the internal category of "modest" cubical sets, $[\square^{\mathbf{op}}, \mathrm{Mod}]$, would be a complete symmetric monoidal closed category.

Moreover, we can consider $[\square^{\mathbf{op}}, \mathrm{Mod}]$-enriched categories, which provide an interesting take on higher category theory and have potential applications as models for type theory. It is an under-developed idea that categories enriched over cubical sets may be, essentially, types.

It is also worth noticing that there is a category of cubical categories (Marco Grandis 2009) which is (according to preliminary investigations) monadic on the category of cubical sets. This again should translate to the language of internal categories, so that we may get an internal category of "modest" cubical categories which is still complete and cartesian closed.

Concluding Remarks

This dissertation introduces the notion of internal enriched category, whose definition is a straightforward internalization of that of standard enriched category. This turns out to be a special case of other well-known notions, notably those of indexed enriched category (Shulman 2013) and enriched generalized multicategory (Leinster 2004).

The methodical study of internal enriched categories in the spirit of Kelly (1982) shows that they feature many of the good properties of standard enriched categories. In particular, under suitable conditions, the category of internal enriched categories and their functors is monoidal closed, in close analogy to what happens in the external setting. That makes it possible to derive an internalized version of the Yoneda lemma.

Moreover, a notion of completeness, compatible with that of strong internal completeness as given in Hyland, Robinson, and Rosolini (1990), applies to internal enriched categories. This is remarkable because there are interesting examples of internal complete categories, such as, notably, the internal category of Modest Sets inside the category of Assemblies (a subcategory of the Effective Topos, see Hyland (1982)).

We argue that the the theory of internal enriched categories is particularly nice to work with and suitable for applications in areas where the use of standard category theory turns out to be problematic, thanks to its remarkable combination of features. Indeed, as shown in the course of this dissertation, the theory follows the familiar patterns from standard enriched category theory, while offering the generality and wide array of examples of internal category theory. Moreover, internal enriched categories are especially well-behaved in regard to size issues, as they are always small with respect to the ambient category; as a result, many size-related issues and inconveniences that traditionally plague the treatment of completeness, especially in relation to presheaves categories, disappear.

In particular, the theory developed herein is meant to find application in the setting of categorical models of type theory. For example, internal enrichment over the category of Modest Sets in the category of Assemblies could provide a natural environment for the development of models with attractive computational features. This is a broad area of research well outside the scope of this dissertation, though nevertheless a worthy investigation for future work.

More generally, the approach taken in this thesis is designed to facilitate applications. In particular, note that many type theorists start with the idea of models in locally cartesian closed categories, as in Seely (1984). Even though in our leading example of the Modest Sets in the category of Assemblies we do in fact have some existential quantification (after all, as they are a category of separated objects, Assemblies form a quasitopos—see for example Johnstone (2002)), we have been careful to present the whole theory without reference to that. Hence traditional type theorists can exploit the theory here just as it stands.

A. Enriched Generalized Multicategories

The notion of internally enriched category is an occurrence of Leinster's more general notion of enriched generalized multicategory (Leinster 2002, 2004). This section will follow Leinster 2004's notation.

A.1. Generalized Multicategories

Let $\mathscr{E}$ be a finitely complete category. In addition to the usual assumption, we shall need to also assume that the forgetful functor $U\colon \mathrm{Cat}_{\mathscr{E}} \to \mathrm{Graph}_{\mathscr{E}}$ has a left adjoint, the free-category-on-a-graph functor $\mathscr{F} : \mathrm{Graph}_{\mathscr{E}} \to \mathrm{Cat}_{\mathscr{E}}$. In particular, that is true if $\mathscr{E}$ has countable products and finite coproducts.

Definition A.1.1 (cartesian monad, ibid., definition 4.1.1)**.** Call *cartesian*

- a category, if it has pullbacks;
- a functor, if it preserves pullbacks;
- a natural transformation, if all of its naturality squares are pullbacks;
- a monad (T, η, μ) on $\mathscr{E}$, if the category $\mathscr{E}$, the functor T, and the natural transformations μ and η are all cartesian.

Let $\mathrm{id}_{\mathscr{E}}$ be the identity monad on $\mathscr{E}$. This is a cartesian monad, since $\mathscr{E}$ is finitely complete (and thus, in particular, cartesian).

Definition A.1.2 (generalized multicategory, ibid., definition 4.2.2)**.** Let T be a cartesian monad on $\mathscr{E}$. A *T-graph* is a diagram

$$T(C_0) \xleftarrow{\sigma_{\mathscr{C}}} C_1 \xrightarrow{\tau_{\mathscr{C}}} C_0$$

in $\mathscr{E}$. A *(generalized) T-multicategory* is a monad in the category of T-algebras $\mathscr{E}_{(T)}$. Explicitely, a T-multicategory $\mathscr{C}$ consists of a T-graph (as above) together with arrows

$$\cdot_{\mathscr{C}} : C_1 \times_{T(C_0)} T(C_1) \to C_1 \tag{A.1}$$

and

$$\mathrm{i}_{\mathscr{E}} \colon C_0 \to C_1 \tag{A.2}$$

satisfying associativity and unitarity axioms.

Remark A.1.3. Observe that $\mathrm{Multicat}_{\mathrm{id}_{\mathscr{E}}} = \mathrm{Cat}_{\mathscr{E}}$ and $\mathrm{Graph}_{\mathrm{id}_{\mathscr{E}}} = \mathrm{Graph}_{\mathscr{E}}$. Consider the forgetful functor $U\colon \mathrm{Multicat}_{\mathrm{id}_{\mathscr{E}}} \to \mathrm{Graph}_{\mathrm{id}_{\mathscr{E}}}$, that is, $U\colon \mathrm{Cat}_{\mathscr{E}} \to \mathrm{Graph}_{\mathscr{E}}$, and its left adjoint $\mathscr{F}$. The adjunction $(\mathscr{F}, U, \eta, \epsilon)$ is monadic, and it induces the monad $(\mathscr{F} = U\mathscr{F}, \eta, \bullet = U\epsilon\mathscr{F})$ over $\mathrm{Graph}_{\mathscr{E}}$ (Leinster 2004, theorem 6.5.2).

Given an object X of $\mathscr{E}$, there exists a unique $\mathscr{E}$-category structure on the $\mathscr{E}$-graph

$$\begin{array}{ccc} & X \times X & \\ {\scriptstyle \pi_1}\swarrow & & \searrow{\scriptstyle \pi_2} \\ X & & X \end{array}$$

which is called $\mathbf{ind}(X)$, the indiscrete category over X. Composition is given by

$$(X \times X) \;{}_{\pi_1}\!\times_{\pi_2}\; (X \times X) \xrightarrow{(\pi_1 p_2, \pi_2 p_1)} (X \times X)$$

(where p_1 and p_2 are the pullback's projections), and identity is given by the diagonal $\Delta\colon X \to X \times X$.

The (underlying $\mathscr{E}$-graph of the) indiscrete $\mathscr{E}$-category $\mathbf{ind}(X)$ has an $\mathscr{F}$-algebra structure

$$\circ_{\mathbf{ind}(X)} = U\epsilon_{\mathbf{ind}(X)} \colon \mathscr{F}\mathbf{ind}(X) \to \mathbf{ind}(X)$$

given by the counit of the adjunction. Then, there is a (unique) $\mathscr{F}$-multicategory structure $\mathbf{ind}(X)^+$ (ibid., example 4.2.22) given by a $\mathscr{F}$-graph

$$\begin{array}{ccc} & \mathscr{F}\mathbf{ind}(X) & \\ {\scriptstyle \mathrm{s}=\mathrm{id}}\swarrow & & \searrow{\scriptstyle \mathrm{t}=\circ_{\mathbf{ind}(X)}} \\ \mathscr{F}\mathbf{ind}(X) & & \mathbf{ind}(X) \end{array}$$

and operations

$$\cdot_{\mathbf{ind}(X)^+} = \mathscr{F}\mathbf{ind}(X) \times_{\mathscr{F}\mathbf{ind}(X)} \mathscr{F}\mathscr{F}\mathbf{ind}(X) \xrightarrow{\pi_2} \mathscr{F}\mathscr{F}\mathbf{ind}(X) \xrightarrow{\bullet_{\mathbf{ind}(X)}} \mathscr{F}\mathbf{ind}(X)$$

and

$$\mathrm{i}_{\mathbf{ind}(X)^+} = \mathbf{ind}(X) \xrightarrow{\eta_{\mathbf{ind}(X)}} \mathscr{F}\mathbf{ind}(X)$$

satisfying associativity and unitarity.

Intuitively, the composition of $\mathbf{ind}(X)^+$ collapses a list of lists of elements of X into a single list:

$$(x_0, \dots, x_n) \cdot_{\mathbf{ind}(X)^+} ((x_0, \dots, x_1), \dots, (x_{n-1}, \dots, x_n)) = (x_0, \dots, x_1, \dots, x_{n-1}, \dots, x_n).$$

Definition A.1.4 (enriched generalized multicategory, ibid., definition 6.8.1)**.** Let $\mathscr{V}$ be an $\mathscr{F}$-multicategory. A *$\mathscr{V}$-enriched* $\mathrm{id}_{\mathscr{E}}$*-multicategory* is an object X of $\mathscr{E}$ together with a map $\mathbf{ind}(X)^+ \to \mathscr{V}$ of $\mathscr{F}$-multicategories.

A.2. The $\mathscr{F}$-multicategory $\mathscr{V}$

Given the monoidal $\mathscr{E}$-category $\boldsymbol{V}$, there is a $\mathscr{F}$-multicategory (a $\mathscr{F}$-operad, to be more specific) $\mathscr{V}$, constructed as follows.

Let $\boldsymbol{V}_0$ be the $\mathscr{E}$-graph $\mathbb{1}_{\mathscr{E}} \leftleftarrows V_0$. Consider the $\mathscr{E}$-category $(\boldsymbol{V}_0, \otimes^{\boldsymbol{V}}, \mathbb{I}_{\boldsymbol{V}})$ given by the monoidal structure on $\boldsymbol{V}$. Then, by adjunction, there is a morphism of $\mathscr{E}$-categories

$$\otimes^0 \colon \mathscr{F}(\boldsymbol{V}_0) \to (\boldsymbol{V}_0, \otimes^{\boldsymbol{V}}, \mathbb{I}_{\boldsymbol{V}}),$$

induced by the morphism of $\mathscr{E}$-graphs $\mathrm{id}_{\boldsymbol{V}_0}$. On the underlying graph, that is a morphism $\otimes^0 \colon \mathscr{F}(\boldsymbol{V}_0) \to \boldsymbol{V}_0$, which is non-trivial only on the arrow's component.

Let $\boldsymbol{V}_1 = V_0^+ \leftleftarrows V_1^+$ be the pullback of $\mathscr{E}$-graphs

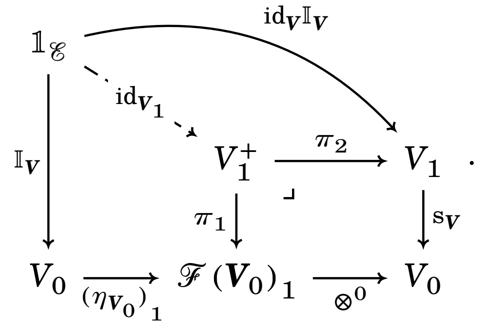
$$\begin{array}{ccc} \boldsymbol{V}_1 & \xrightarrow{\pi_2} & \mathbb{1}_{\mathscr{E}} \leftleftarrows V_1 \\ {\scriptstyle \pi_1}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle (!,\mathrm{s}_{\boldsymbol{V}})} \\ \mathscr{F}(\boldsymbol{V}_0) & \xrightarrow[\otimes^0]{} & \boldsymbol{V}_0 \end{array} = \left(\begin{array}{ccc} V_0^+ & \xrightarrow{\pi_2} & \mathbb{1}_{\mathscr{E}} \\ {\scriptstyle \pi_1}\big\downarrow & \lrcorner & \big\downarrow \\ \mathbb{1}_{\mathscr{E}} & \longrightarrow & \mathbb{1}_{\mathscr{E}} \end{array} \leftleftarrows \begin{array}{ccc} V_1^+ & \xrightarrow{\pi_2} & V_1 \\ {\scriptstyle \pi_1}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \mathrm{s}_{\mathscr{M}}} \\ \mathscr{F}(\boldsymbol{V}_0)_1 & \xrightarrow[\otimes^0]{} & V_0 \end{array} \right).$$

Notice that $V_0^+ = \mathbb{1}_{\mathscr{E}}$. In the internal language of $\mathscr{E}$, the pullback V_1^+ has elements those pairs $((A_i) : \mathscr{F}(\boldsymbol{V}_0)_1, f \colon V_1)$ such that $\mathrm{s}_{\boldsymbol{V}}(f) = \otimes^0(A_i)$. Thus, informally speaking, $\boldsymbol{V}_1$ is the graph with one vertex and arrows $f \colon A_0 \otimes^{\boldsymbol{V}} \ldots \otimes^{\boldsymbol{V}} A_n \to A$ of $\boldsymbol{V}$ as edges, remembering the sequence $A_0 \ldots A_n$ yielding the domain of f.

There is an $\mathscr{E}$-category structure $(\boldsymbol{V}_1, \circ_{\boldsymbol{V}_1}, \mathrm{id}_{\boldsymbol{V}_1})$ where composition is defined as

$$\begin{array}{ccccc} V_1^+ \times V_1^+ & & \xrightarrow{\pi_2 \times \pi_2} & & V_1 \times V_1 \\ & \searrow^{\circ_{\boldsymbol{V}_1}} & & & \big\downarrow{\scriptstyle \otimes^{\boldsymbol{V}}} \\ {\scriptstyle \pi_1 \times \pi_1}\big\downarrow & & V_1^+ & \xrightarrow{\pi_2} & V_1 \\ & & {\scriptstyle \pi_1}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \mathrm{s}_{\mathscr{M}}} \\ \mathscr{F}(\boldsymbol{V}_0)_1 \times \mathscr{F}(\boldsymbol{V}_0)_1 & \xrightarrow[\circ_{\mathscr{F}(\boldsymbol{V}_0)}]{} & \mathscr{F}(\boldsymbol{V}_0)_1 & \xrightarrow[\otimes^0]{} & V_0 \end{array}$$

and identity as



In the internal language,

$$(A_i,f),(B_j,g): V_1^+ \vdash (B_j,g) \circ_{\boldsymbol{V}_1} (A_i,f) = (A_i \circ_{\mathscr{F}(\boldsymbol{V}_0)} B_j, f \otimes^{\boldsymbol{V}} g).$$

Informally speaking,

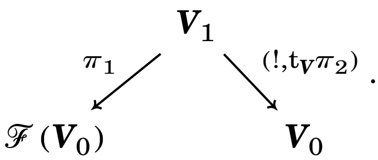
$$(A_0 \otimes \ldots \otimes A_n \xrightarrow{f} A) \circ_{\boldsymbol{V}_1} (B_0 \otimes \ldots \otimes B_m \xrightarrow{g} B) = (A_0 \otimes \ldots \otimes A_n \otimes B_0 \otimes \ldots \otimes B_m \xrightarrow{f \otimes g} A \otimes B).$$

By adjunction, there is a morphism of $\boldsymbol{V}$-categories

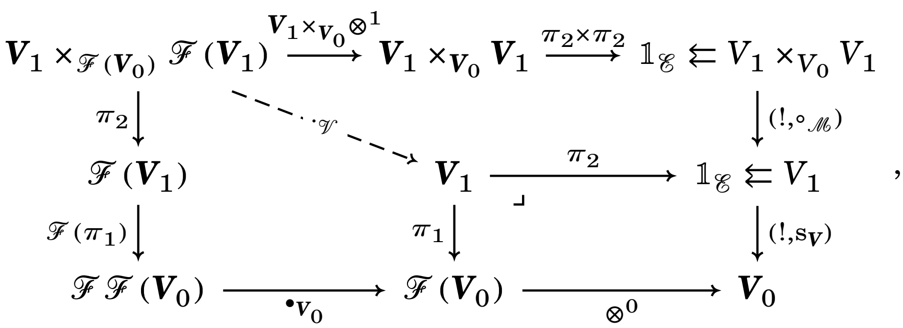
$$\otimes^1 : \mathscr{F}(\boldsymbol{V}_1) \to (\boldsymbol{V}_1, \circ_{\boldsymbol{V}_1}, \mathrm{id}_{\boldsymbol{V}_1})$$

induced by the morphism of $\mathscr{E}$-graphs $\mathrm{id}_{\boldsymbol{V}_1}$.

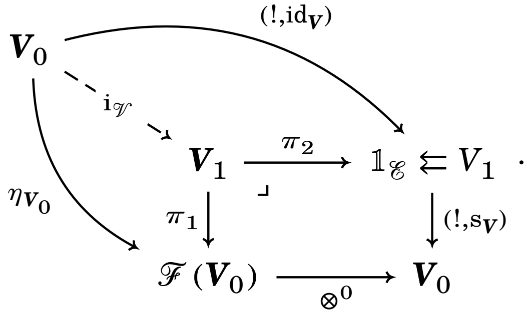
There is a $\mathscr{F}$-multicategory $\mathscr{V}$ with structure given by



The composition $\cdot_{\mathscr{V}} : \boldsymbol{V}_1 \times_{\mathscr{F}(\boldsymbol{V}_0)} \mathscr{F}(\boldsymbol{V}_1) \to \boldsymbol{V}_1$ is given by



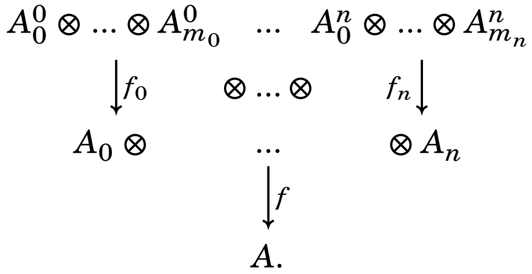
and the identity $i_{\mathscr{V}} : \boldsymbol{V}_0 \to \boldsymbol{V}_1$ given by



Observe that only the edges' part is non-trivial. In the internal language, given $(A_i, f) : V_1^+$ and $((A_j)_i, f_i) : \mathscr{F}(\boldsymbol{V}_1)_1$ such that $A_i = t_{\boldsymbol{V}}(f_i) : \mathscr{F}(\boldsymbol{V}_0)_1$, multi-category composition on the arrows' component is defined as

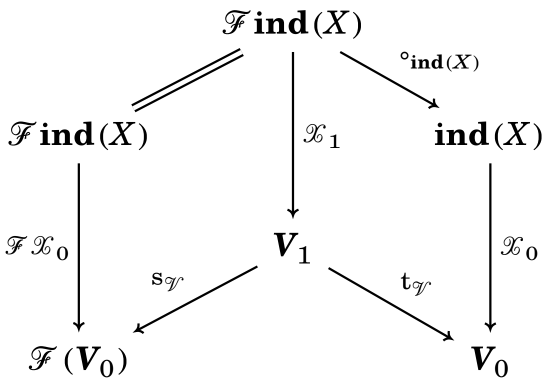
$$(A_i, f)(\cdot_{\mathscr{V}})_1((A_j)_i, f_i) \coloneqq (A_{(i,j)}, f \circ_{\boldsymbol{V}} \otimes^1(f_i)) : V_1^+$$

where $A_{(i,j)} = (\bullet_{\boldsymbol{V}_0})_1((A_j)_i) : \mathscr{F}(\boldsymbol{V}_0)_1$ (and it is trivial on the object component). Informally, $f \cdot_{\mathscr{V}} (f_0, \ldots, f_n)$ is



A.3. Internal Enrichment to Enriched Generalized Multicategories

Let $\boldsymbol{X}$ be a $\boldsymbol{V}$-enriched category. We shall build a $\mathscr{V}$-enriched id-multicategory $\mathscr{X}$ out of it. That is given by an object of $\mathscr{E}$, let's choose X, and a morphism $\mathbf{ind}(X)^+ \to \mathscr{V}$ of $\mathscr{F}$-categories; that means a commutative diagram in $\mathrm{Graph}_{\mathscr{E}}$



(A.3)

such that the diagrams

$$\begin{array}{ccc} \mathbf{ind}(X) & \xrightarrow{\mathrm{i}_{\mathbf{ind}(X)^+}} & \mathscr{F}\,\mathbf{ind}(X) \\ {\scriptstyle \mathscr{X}_0}\downarrow & & \downarrow{\scriptstyle \mathscr{X}_1} \\ \boldsymbol{V}_0 & \xrightarrow[\mathrm{i}_{\mathscr{V}}]{} & \boldsymbol{V}_1 \end{array} \tag{A.4}$$

and

$$\begin{array}{ccc} \mathscr{F}\,\mathbf{ind}(X) \times_{\mathscr{F}\mathbf{ind}(X)} \mathscr{F}\mathscr{F}\,\mathbf{ind}(X) & \xrightarrow{\cdot_{\mathbf{ind}(X)^+}} & \mathscr{F}\,\mathbf{ind}(X) \\ {\scriptstyle \mathscr{X}_1 \times \mathscr{F}\mathscr{X}_1}\downarrow & & \downarrow{\scriptstyle \mathscr{X}_1} \\ \boldsymbol{V}_1 \times_{\mathscr{F}(\boldsymbol{V}_0)} \mathscr{F}(\boldsymbol{V}_1) & \xrightarrow[\cdot_{\mathscr{V}}]{} & \boldsymbol{V}_1 \end{array} \tag{A.5}$$

commute.

Let $\mathscr{X}_0$ be $\mathbf{ind}(X) \xrightarrow{(!,\mathrm{Hom}_{\boldsymbol{X}})} \boldsymbol{V}_0$.

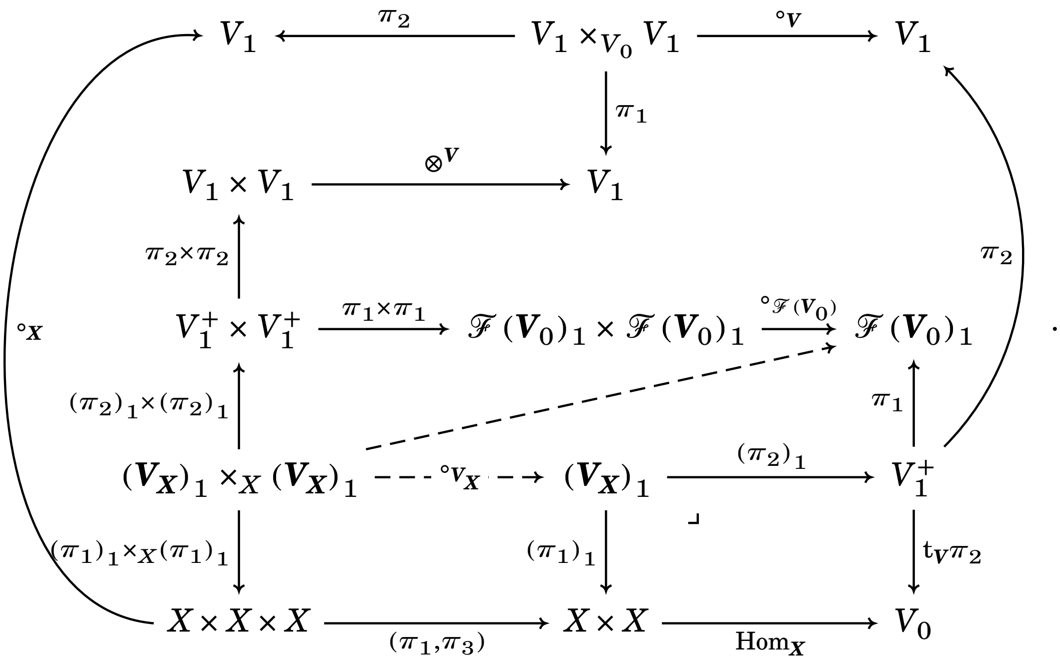
The arrow $\mathscr{X}_1 \colon \mathscr{F}\,\mathbf{ind}(X) \to \boldsymbol{V}_1$ requires a two-step construction. Consider the $\mathscr{E}$-graph $\boldsymbol{V_X} \colon (\boldsymbol{V_X})_0 \leftleftarrows (\boldsymbol{V_X})_1$, given by

$$\begin{array}{ccc} \boldsymbol{V_X} & \xrightarrow{\pi_2} & \boldsymbol{V}_1 \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle (!,\mathrm{t}_{\boldsymbol{V}}\pi_2)} \\ \mathbf{ind}(X) & \xrightarrow[\mathscr{X}_0]{} & \boldsymbol{V}_0 \end{array} = \left(\begin{array}{ccc} (\boldsymbol{V_X})_0 & \xrightarrow{\pi_2} & \mathbb{1}_{\mathscr{C}} \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & \mathbb{1}_{\mathscr{C}} \end{array} \;\leftleftarrows\; \begin{array}{ccc} (\boldsymbol{V_X})_1 & \xrightarrow{\pi_2} & V_1^+ \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{t}_{\boldsymbol{V}}\pi_2} \\ X \times X & \xrightarrow[\mathrm{Hom}_{\boldsymbol{X}}]{} & V_0 \end{array} \right)$$

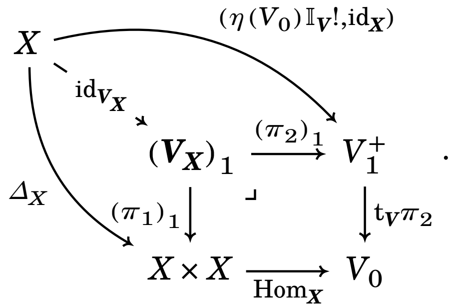
and notice that $(\boldsymbol{V_X})_0 = X$, while, in the internal language, $(\boldsymbol{V_X})_1$ has elements those tuples $(x_0 \colon X, x_1 \colon X, (A_i) \colon \mathscr{F}(\boldsymbol{V}_0)_1, f \colon V_1)$ such that $\mathrm{s}_{\boldsymbol{V}}(f) = \otimes^0(A_i)$ and $\mathrm{t}_{\boldsymbol{V}}(f) = \mathrm{Hom}_{\boldsymbol{X}}(x_0, x_1)$. Informally, $\boldsymbol{V_X}$ is the graph with elements of X as vertices, and arrows $f \colon A_0 \otimes \ldots \otimes A_n \to \mathrm{Hom}_{\boldsymbol{X}}(x_0, x_1)$ as edges from the vertex x_0 to the vertex x_1.

There is a $\mathscr{E}$-category structure $(\boldsymbol{V_X}, \circ_{\boldsymbol{V_X}}, \mathrm{id}_{\boldsymbol{V_X}})$. Composition $\circ_{\boldsymbol{V_X}} \colon (\boldsymbol{V_X})_1 \times_X$

$(\boldsymbol{V_X})_1 \to (\boldsymbol{V_X})_1$ is given by



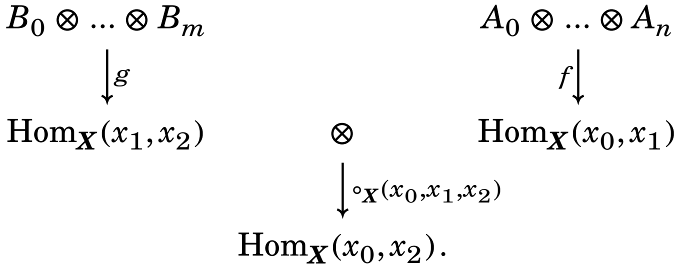
Identity $\mathrm{id}_{\boldsymbol{V_X}} \colon X \to (\boldsymbol{V_X})_1$ is given by



In the internal language, the composition is given by

$$(x_1, x_2, B_j, g) \circ_{\boldsymbol{V_X}} (x_0, x_1, A_i, f) \coloneqq (x_0, x_2, A_i \circ_{\mathscr{F}(\boldsymbol{V}_0)} B_j, \circ_{\boldsymbol{X}}(x_0, x_1, x_2)(f \otimes^{\boldsymbol{V}} g)).$$

Informally, the composition is given by



$\boldsymbol{V}$-functors $\mathscr{F}\,\mathbf{ind}(X) \to \boldsymbol{V}_{\boldsymbol{X}}$ correspond bijectively, by adjointness, to morphisms $\mathbf{ind}(X) \to \boldsymbol{V}_{\boldsymbol{X}}$ of $\mathscr{E}$-graphs (by abuse of notation, the same symbol is being used for both a $\mathscr{E}$-category and its underlying $\mathscr{E}$-graph). Let $\circ_{\boldsymbol{X}}\colon \mathscr{F}\,\mathbf{ind}(X) \to \boldsymbol{V}_{\boldsymbol{X}}$ extend the morphism of $\mathscr{E}$-categories

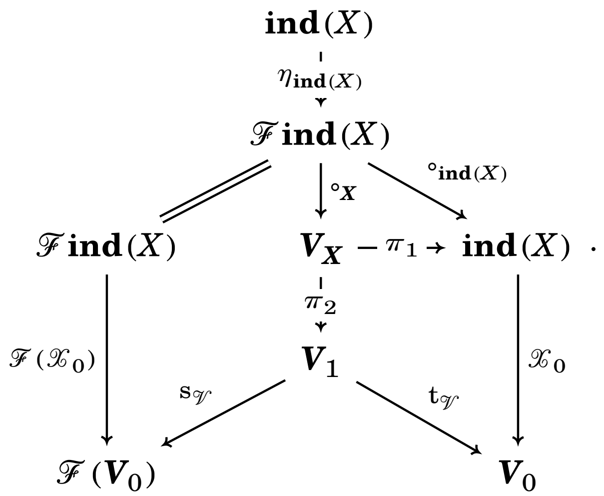
$$\mathbf{ind}(X) \xrightarrow{(\mathrm{id}_{\mathbf{ind}(X)},\mathrm{Hom}_{\boldsymbol{V}})} \mathbf{ind}(X) \times_{\boldsymbol{V}_0} \boldsymbol{V}_0 \xrightarrow{\mathrm{id}_{\mathbf{ind}(X)} \times (\eta_{\boldsymbol{V}_0},\mathrm{id}_{\boldsymbol{V}})} \mathbf{ind}(X) \times_{\boldsymbol{V}_0} \boldsymbol{V}_1 \cong \boldsymbol{V}_{\boldsymbol{X}}$$

that is, intuitively, the functor sending x to $\mathrm{id}_{\boldsymbol{X}}(x)$ and (x_0, x_1) to $\mathrm{id}_{\boldsymbol{V}}(\mathrm{Hom}_{\boldsymbol{X}}(x_0, x_1))$. Then,

$$\circ_{\boldsymbol{X}}(x_0, \dots, x_n) = \mathrm{Hom}_{\boldsymbol{X}}(x_{n-1}, x_n) \otimes \dots \otimes \mathrm{Hom}_{\boldsymbol{X}}(x_0, x_1) \xrightarrow{\circ_{\boldsymbol{X}}} \mathrm{Hom}_{\boldsymbol{X}}(x_0, x_n).$$

By applying the forgetful functor, let's consider $\circ_{\boldsymbol{X}}$ as a mere morphism of $\mathscr{E}$-graphs. Let then $\mathscr{X}_1\colon \mathscr{F}\,\mathbf{ind}(X) \to \boldsymbol{V}_1$ be the composition of $\circ_{\boldsymbol{X}}$ with the morphism of $\mathscr{E}$-graphs $\pi_2\colon \boldsymbol{V}_{\boldsymbol{X}} \to \boldsymbol{V}_1$.

Diagram (A.3) commutes. To prove this, consider the diagram



Consider the two morphisms $\mathscr{F}\,\mathbf{ind}(X) \to \boldsymbol{V}_0$ of $\mathscr{E}$-graphs forming the right square. Direct calculation shows that the square at the bottom commutes. The triangle on top is the underlying diagram of $\mathscr{E}$-graphs of a diagram of $\mathscr{E}$-categories. Restricted by precomposition on $\mathbf{ind}(X)$, the two $\mathscr{E}$-graph paths commute. Both the paths $\mathscr{F}\,\mathbf{ind}(X) \to \mathbf{ind}(X)$ extend those to $\mathscr{E}$-functors, and as such they must coincide by adjunction.

Consider the two morphisms $\mathscr{F}\,\mathbf{ind}(X) \to \mathscr{F}(\boldsymbol{V}_0)$ of $\mathscr{E}$-graphs forming the right square. Direct calculation shows that the two paths, restricted on $\mathbf{ind}(X)$, commute. Moreover, they are functors of $\mathscr{E}$-categories. Both morphisms of $\mathscr{E}$-graphs extend that to $\mathscr{E}$-functors on $\mathscr{F}\,\mathbf{ind}(X)$, and as such they must coincide by adjunction.

Diagram (A.4) commutes, because

$$
\begin{array}{ccccc}
\mathbf{ind}(X)\times_{\boldsymbol{V}_0}\boldsymbol{V}_0 & & \xrightarrow{\mathrm{id}_{\mathbf{ind}(X)}\times \mathrm{i}_{\mathscr{V}}} & & \boldsymbol{V}_{\boldsymbol{X}} \\
\big\uparrow{\scriptstyle (\mathrm{id}_{\mathbf{ind}(X)},\mathscr{X}_0)} & & & & \big\downarrow{\scriptstyle \pi_2} \\
\mathbf{ind}(X) & \xrightarrow[\eta_{\mathbf{ind}(X)}]{\mathrm{i}_{\mathbf{ind}(X)}{}^{+} =} & \mathscr{F}\,\mathbf{ind}(X) & \xrightarrow{\mathscr{X}_1} & \boldsymbol{V}_1 \\
 & \searrow{\scriptstyle \mathscr{X}_0} & \boldsymbol{V}_0 & \nearrow{\scriptstyle \mathrm{i}_{\mathscr{V}}} &
\end{array}
\quad .
$$

Diagram (A.5) commutes. To begin with, define $\cdot_{\boldsymbol{V}_{\boldsymbol{X}}} : \boldsymbol{V}_{\boldsymbol{X}} \times_{\mathscr{F}(\boldsymbol{V}_0)} \mathscr{F}(\boldsymbol{V}_{\boldsymbol{X}}) \to \boldsymbol{V}_{\boldsymbol{X}}$ as

$$
\begin{array}{ccccc}
\boldsymbol{V}_{\boldsymbol{X}} \times_{\mathscr{F}(\boldsymbol{V}_0)} \mathscr{F}(\boldsymbol{V}_{\boldsymbol{X}}) & & \xrightarrow{\pi_2 \times \mathscr{F}(\pi_2)} & & \boldsymbol{V}_1 \times_{\mathscr{F}(\boldsymbol{V}_0)} \mathscr{F}(\boldsymbol{V}_1) \\
 & \searrow{\scriptstyle \cdot_{\boldsymbol{V}_{\boldsymbol{X}}}} & & & \big\downarrow{\scriptstyle \cdot_{\mathscr{V}}} \\
\big\downarrow{\scriptstyle \pi_1} & & \boldsymbol{V}_{\boldsymbol{X}} & \xrightarrow{\pi_2} & \boldsymbol{V}_1 \\
 & & \big\downarrow{\scriptstyle \pi_1} & \lrcorner & \big\downarrow{\scriptstyle (!,\mathrm{t}_{\boldsymbol{V}}\pi_2)} \\
\boldsymbol{V}_{\boldsymbol{X}} & \xrightarrow[\pi_1]{} & \mathbf{ind}(X) & \xrightarrow[\mathscr{X}_0]{} & \boldsymbol{V}_0
\end{array}
\quad .
$$

In the internal language, given $(x_0,x_1,A_i,f) : (\boldsymbol{V}_{\boldsymbol{X}})_1$ and $(x_0^i,x_1^i,(A_j)_i,f_i) : \mathscr{F}(\boldsymbol{V}_{\boldsymbol{X}})_1$ such that $A_i = \mathrm{Hom}_{\boldsymbol{X}}(x_0^i,x_1^i) : \mathscr{F}(\boldsymbol{V}_0)_1$, multi-category composition on the arrows' component is defined as

$$
(x_0,x_1,A_i,f)\,(\cdot_{\boldsymbol{V}_{\boldsymbol{X}}})_1\,(x_0^i,x_1^i,(A_j)_i,f_i) \coloneqq (x_0,x_1,A_{(i,j)},f \circ_{\boldsymbol{V}} \otimes^1(f_i)) : (\boldsymbol{V}_{\boldsymbol{X}})_1.
$$

Intuitively, the composition $f \cdot_{\boldsymbol{V}_{\boldsymbol{X}}} (f_0,\dots,f_n)$ is given by

$$
\begin{array}{ccc}
A_0^0 \otimes \dots \otimes A_{m_0}^0 & \dots & A_0^n \otimes \dots \otimes A_{m_n}^n \\
\big\downarrow{\scriptstyle f_0} & \otimes \dots \otimes & {\scriptstyle f_n}\big\downarrow \\
\mathrm{Hom}_{\boldsymbol{X}}(x_0^0,x_1^0) \otimes & \dots & \otimes\, \mathrm{Hom}_{\boldsymbol{X}}(x_0^n,x_1^n) \\
 & \big\downarrow{\scriptstyle f} & \\
 & \mathrm{Hom}_{\boldsymbol{X}}(x_0,x_1) &
\end{array}
$$

so that $\cdot_{\boldsymbol{V}_{\boldsymbol{X}}}$ is an extension of $\cdot_{\mathscr{V}}$.

The morphism of $\mathscr{E}$-graphs $\cdot_{\boldsymbol{V}_{\boldsymbol{X}}}$ is a functor of $\mathscr{E}$-categories, where $\boldsymbol{V}_{\boldsymbol{X}}$ is given the $\mathscr{E}$-category structure $(\boldsymbol{V_X}, \circ_{\boldsymbol{V_X}}, \mathrm{id}_{\boldsymbol{V_X}})$. This means that the diagrams

$$\begin{array}{ccc} (\boldsymbol{V_X} \times_{\mathscr{F}(\boldsymbol{V}_0)} \mathscr{F}(\boldsymbol{V_X}))_0 & \xrightarrow{\mathrm{id}_{\boldsymbol{V_X} \times \mathscr{F}(\boldsymbol{V_X})}} & (\boldsymbol{V_X} \times_{\mathscr{F}(\boldsymbol{V}_0)} \mathscr{F}(\boldsymbol{V_X}))_1 \\ {\scriptstyle (\cdot_{\boldsymbol{V_X}})_0}\downarrow & & \downarrow{\scriptstyle (\cdot_{\boldsymbol{V_X}})_1} \\ (\boldsymbol{V_X})_0 & \xrightarrow[\mathrm{id}_{\boldsymbol{V_X}}]{} & (\boldsymbol{V_X})_1 \end{array} \tag{A.6}$$

and

$$\begin{array}{ccc} (\boldsymbol{V_X} \times_{\mathscr{F}(\boldsymbol{V}_0)} \mathscr{F}(\boldsymbol{V_X}))_1 \times_X (\boldsymbol{V_X} \times_{\mathscr{F}(\boldsymbol{V}_0)} \mathscr{F}(\boldsymbol{V_X}))_1 & \xrightarrow{\circ_{(\boldsymbol{V_X} \times \mathscr{F}(\boldsymbol{V_X}))}} & (\boldsymbol{V_X} \times_{\mathscr{F}(\boldsymbol{V}_0)} \mathscr{F}(\boldsymbol{V_X}))_1 \\ {\scriptstyle (\cdot_{\boldsymbol{V_X}})_1 \times (\cdot_{\boldsymbol{V_X}})_1}\downarrow & & \downarrow{\scriptstyle (\cdot_{\boldsymbol{V_X}})_1} \\ (\boldsymbol{V_X})_1 \times_X (\boldsymbol{V_X})_1 & \xrightarrow[\circ_{\boldsymbol{V_X}}]{} & (\boldsymbol{V_X})_1 \end{array} \tag{A.7}$$

commute.

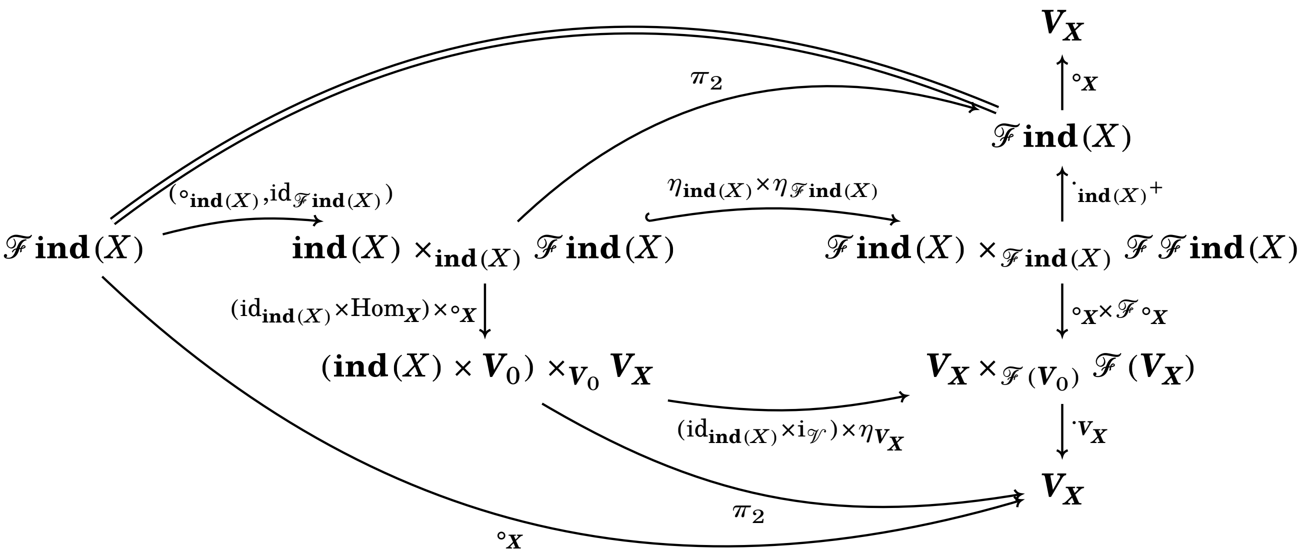
Then, to prove that (A.5) commutes is to prove that

$$\begin{array}{ccc} \mathscr{F}\mathbf{ind}(X) \times_{\mathscr{F}\mathbf{ind}(X)} \mathscr{F}\mathscr{F}\mathbf{ind}(X) & \xrightarrow{\cdot_{\mathbf{ind}(X)^+}} & \mathscr{F}\mathbf{ind}(X) \\ {\scriptstyle \circ_X \times \mathscr{F}\circ_X}\downarrow & & \downarrow{\scriptstyle \circ_X} \\ \boldsymbol{V_X} \times_{\mathscr{F}(\boldsymbol{V}_0)} \mathscr{F}(\boldsymbol{V_X}) & \xrightarrow[\cdot_{\boldsymbol{V_X}}]{} & \boldsymbol{V_X} \end{array} \tag{A.8}$$

commutes. All morphisms in (A.8) are functors of $\mathscr{E}$-categories. Thus, it suffices to show that the paths commute on the generators of

$$\mathscr{F}\mathbf{ind}(X) \times_{\mathscr{F}\mathbf{ind}(X)} \mathscr{F}\mathscr{F}\mathbf{ind}(X) \cong \mathscr{F}(\mathbf{ind}(X) \times_{\mathbf{ind}(X)} \mathscr{F}\mathbf{ind}(X)).$$

Also, notice that $\mathbf{ind}(X) \times_{\mathbf{ind}(X)} \mathscr{F}\mathbf{ind}(X) \cong \mathscr{F}\mathbf{ind}(X)$. With this in mind, the diagram



proves that (A.8) commutes.

Remark A.3.1. For the notion of natural transformation of functors of T-multicategory, and of $\mathscr{V}$-enriched functor, see Leinster 2002.

Given a functor $\boldsymbol{F}\colon \boldsymbol{X} \to \boldsymbol{Y}$ of $\boldsymbol{V}$-enriched categories, there is a $\mathscr{V}$-enriched $\mathscr{F}$-functor $\mathscr{X} \to \mathscr{Y}$ given by $F_0\colon X_0 \to Y_0$, and $F_1\colon \mathbf{ind}(X) \to \boldsymbol{V}_1$ induced by $F_1\colon X \times X \to V_1$.

A.4. Enriched Generalized Multicategories to Internal Enrichment

Let $\mathscr{X}\colon \mathbf{ind}(X)^+ \to \mathscr{V}$ be a $\mathscr{V}$-enriched $\mathscr{F}$-multicategory as in eq. (A.3). Then there is a $\boldsymbol{V}$-category $\boldsymbol{X}$ on X whose hom is $\mathrm{Hom}_{\boldsymbol{X}} = (\mathscr{X}_0)_1\colon X \times X =$

$\mathbf{ind}(X)_1 \to V_0$, whose composition is

$$\begin{array}{c} X \times X \times X \cong \mathbf{ind}(X)_1 \; {}_{s_{\mathbf{ind}(X)}}\!\times_{t_{\mathbf{ind}(X)}} \mathbf{ind}(X)_1 \\ \big\downarrow (\eta_{\mathbf{ind}(X)})_1 \times (\eta_{\mathbf{ind}(X)})_1 \\ (\mathscr{F}\mathbf{ind}(X))_1 \; {}_{s_{\mathscr{F}\mathbf{ind}(X)}}\!\times_{t_{\mathscr{F}\mathbf{ind}(X)}} (\mathscr{F}\mathbf{ind}(X))_1 \\ \big\downarrow \circ_{\mathscr{F}\mathbf{ind}(X)} \\ (\mathscr{F}\mathbf{ind}(X))_1 \\ \big\downarrow (\mathscr{X}_1)_1 \\ (\boldsymbol{V}_1)_1 \\ \big\downarrow (\pi_2)_1 \\ V_1 \end{array},$$

and whose identity $\mathrm{id}_{\boldsymbol{X}}$ is

$$X = \mathbf{ind}(X)_0 \xrightarrow{(\eta)_0} (\mathscr{F}\mathbf{ind}(X))_0 \xrightarrow{i_{\mathscr{F}\mathbf{ind}(X)}} (\mathscr{F}\mathbf{ind}(X))_1 \xrightarrow{(\mathscr{X}_1)_1} (\boldsymbol{V}_1)_1 \xrightarrow{(\pi_2)_1} V_1.$$

Given a $\mathscr{F}$-enriched functor $\boldsymbol{F}\colon \boldsymbol{X} \to \boldsymbol{Y}$ of $\mathscr{F}$-enriched categories, there is a functor of $\boldsymbol{V}$-categories given by $F_0\colon X_0 \to Y_0$, and $F_1\colon X \times X \to V_1$ induced by $F_1\colon \mathbf{ind}(X) \to \boldsymbol{V}_1$.

A.5. Equivalence

Finally, let's show that the constructions defined in the previous sections are mutually inverse (up to isomorphism), meaning that $\mathscr{V}$-enriched id-multicategories and $\boldsymbol{V}$-enriched categories are equivalent notions.

Proposition A.5.1. *Let $\boldsymbol{X}$ be a $\boldsymbol{V}$-enriched category, $\mathscr{X}$ the $\mathscr{V}$-enriched* id-*multicategory associated to $\boldsymbol{X}$, and $\bar{\boldsymbol{X}}$ the $\boldsymbol{V}$-enriched category associated to $\mathscr{X}$. Then, $\boldsymbol{X} \cong \bar{\boldsymbol{X}}$.*

Proof. Observe that $\mathrm{Hom}_{\boldsymbol{X}} = \mathrm{Hom}_{\bar{\boldsymbol{X}}}$. The composition

$$(\mathscr{F}\mathbf{ind}(X))_1 \; {}_s\!\times_t (\mathscr{F}\mathbf{ind}(X))_1 \xrightarrow{\circ_{\mathscr{F}\mathbf{ind}(X)}} (\mathscr{F}\mathbf{ind}(X))_1 \xrightarrow{(\mathscr{X}_1)_1} (\boldsymbol{V}_1)_1$$

is equal to

$$(\mathscr{F}\mathbf{ind}(X))_1 \; {}_s\!\times_t (\mathscr{F}\mathbf{ind}(X))_1 \xrightarrow{(\mathscr{X}_1)_1 \times (\mathscr{X}_1)_1} (\boldsymbol{V}_1)_1 \times (\boldsymbol{V}_1)_1 \xrightarrow{\circ_{\boldsymbol{V}_1}} (\boldsymbol{V}_1)_1.$$

Then, the composition $\circ_{\bar{\boldsymbol{X}}}$ of $\bar{\boldsymbol{X}}$ is

$$\begin{array}{ccc}
\mathbf{ind}(X)_1\ {}_{\mathrm{s}}\!\times_{\mathrm{t}}\ \mathbf{ind}(X)_1 & & \mathbf{ind}(X)_1\ {}_{\mathrm{s}}\!\times_{\mathrm{t}}\ \mathbf{ind}(X)_1 \\
\downarrow{\scriptstyle (\eta_{\mathbf{ind}(X)})_1 \times (\eta_{\mathbf{ind}(X)})_1} & & \downarrow{\scriptstyle (\eta_{\mathbf{ind}(X)})_1 \times (\eta_{\mathbf{ind}(X)})_1} \\
(\mathscr{F}\mathbf{ind}(X))_1\ {}_{\mathrm{s}}\!\times_{\mathrm{t}}\ (\mathscr{F}\mathbf{ind}(X))_1 & & (\mathscr{F}\mathbf{ind}(X))_1\ {}_{\mathrm{s}}\!\times_{\mathrm{t}}\ (\mathscr{F}\mathbf{ind}(X))_1 \\
\downarrow{\scriptstyle (\mathscr{X}_1)_1 \times (\mathscr{X}_1)_1} & & \downarrow{\scriptstyle (\mathscr{X}_1)_1 \times (\mathscr{X}_1)_1} \\
(\boldsymbol{V}_1)_1 \times (\boldsymbol{V}_1)_1 & = & (\boldsymbol{V}_1)_1 \times (\boldsymbol{V}_1)_1 \\
\downarrow{\scriptstyle \circ_{\boldsymbol{V}_1}} & & \downarrow{\scriptstyle (\pi_2)_1 \times (\pi_2)_1} \\
(\boldsymbol{V}_1)_1 & & V_1 \times_X V_1 \\
\downarrow{\scriptstyle (\pi_2)_1} & & \downarrow{\scriptstyle \circ_{\boldsymbol{X}}(\pi_1 \otimes \pi_2)} \\
V_1 & & V_1
\end{array}$$

which is $\circ_{\boldsymbol{X}}$. □

Proposition A.5.2. *Let $\mathscr{X}$ be a $\mathscr{V}$-enriched* id*-multicategory, $\boldsymbol{X}$ the $\boldsymbol{V}$-enriched category associated to $\mathscr{X}$, and $\bar{\mathscr{X}}$ the $\mathscr{V}$-enriched* id*-multicategory associated to $\boldsymbol{X}$. Then, $\mathscr{X} \cong \bar{\mathscr{X}}$.*

Proof. Observe that $\mathscr{X}_0 = \bar{\mathscr{X}}_0$. Then, it's enough to check $(\mathscr{X}_1)_1 = (\bar{\mathscr{X}}_1)_1$ (up to canonical isomorphism) on the generators of the free category $\mathscr{F}\mathbf{ind}(X)$, that is, on

$$\begin{array}{c}
\mathbf{ind}(X)_1\ {}_{\mathrm{s}_{\mathbf{ind}(X)}}\!\times_{\mathrm{t}_{\mathbf{ind}(X)}}\ \mathbf{ind}(X)_1 \\
\downarrow{\scriptstyle (\eta_{\mathbf{ind}(X)})_1 \times (\eta_{\mathbf{ind}(X)})_1} \\
(\mathscr{F}\mathbf{ind}(X))_1\ {}_{\mathrm{s}_{\mathscr{F}\mathbf{ind}(X)}}\!\times_{\mathrm{t}_{\mathscr{F}\mathbf{ind}(X)}}\ (\mathscr{F}\mathbf{ind}(X))_1 \\
\downarrow{\scriptstyle \circ_{\mathscr{F}\mathbf{ind}(X)}} \\
(\mathscr{F}\mathbf{ind}(X))_1
\end{array}$$

but that means to check that $\circ_{\boldsymbol{X}}$ is equal to the composition of the $\boldsymbol{V}$-enriched category associated to $\bar{\mathscr{X}}$, which has already been proven in the previous proposition. □

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