The inerter: a retrospective

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Abstract

The paper provides an introduction and overview of the inerter concept and device. Careful attention is given to the distinction between the inerter as an ideal modelling element and devices that approximate the ideal behaviour. The background is given to the formal definition of the inerter as a mechanical one-port with terminal forces proportional to the relative acceleration between them. Four major methods of construction are described and modelled. The discussion focuses particularly on: the notion of terminals; the distinction between a device and an effect; sign reversals; back-driving in geared systems; the conceptual aspects of the modelling step for inerter embodiments; the problem of reverse engineering to discover a purpose. The paper includes an analysis and discussion of the rotational inerter. A brief review of the ideas of passive network synthesis that led to the inerter concept are provided. A discussion and analysis is given on several examples of integrated mechanical devices. The article concludes with an imaginary dialogue between the author and an interlocutor on the understanding and purpose of the inerter.

1 Introduction

The "inerter" is a mechanical device with two terminals such that the equal and opposite force on the terminals is proportional to the relative acceleration between them [1, 2]. Since its introduction in 2001 the inerter has become established commercially in suspension systems for high-performance motor vehicles as the third passive element alongside the spring and damper [3]. Practical inerters may be constructed in a variety of ways with currently the main types being: ballscrew [2], rack and pinion [2, 1], gear pump [2], fluid-inerter [4, 5]. The range of potential application areas continues to expand beyond automotive suspensions [6] to include: vibration absorption [1, 7] and vibration mounts [8, 9], building suspensions [10, 11, 12, 13, 14, 15, 16, 17], vibration in cables [18, 19], control of motorcycle steering oscillations [20, 21], railway vehicle suspensions, [22, 23, 24, 25], shimmy in aircraft landing gear [26, 27], passive walking for bipedal robots [28].

Despite the increasing prominence of the inerter in theory and applications there are frequent misunderstandings of the concept. These arise principally due to the fact that the inerter is *both* a device *and* an ideal modelling element. Whilst the same is true for springs, dampers, capacitors, resistors etc, a proper understanding of the inerter does seem to require a clearer appreciation of the distinction between the two. In the following we will review the concept from first principles and provide a commentary on aspects that are sometimes misunderstood. These include:

1. the notion of terminals;
2. approximate nature of the defining law;
3. distinction between mass and inertance of the device;
4. distinction between an effect and a device;
5. reductive versus holistic understanding of the purpose of inerters.

2 Mechanical one-ports and inerter definition

Figure 1 shows a schematic of an ideal two-terminal mechanical element (a *one-port*). The two *terminals* are the connection points to other elements and have absolute displacements x_1 and x_2 in some inertial frame of reference. The forces F at the terminals are equal and opposite. Device laws are relations between the through-variable F and the across-variable $x = x_2 - x_1$.

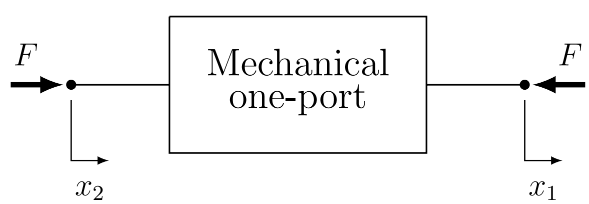

Figure 1: Mechanical one-port element with sign convention in which a positive force F is compressive and $\dot{x} = \dot{x}_2 - \dot{x}_1$ being positive corresponds to the terminals moving towards each other.

We begin with the formal definition.

Definition 1 *The ideal inerter is a mechanical one-port with the property that the equal and opposite forces at the terminals are proportional to the relative acceleration between them, in particular, in the notation of Fig. 1*

$$F(t) = b\,(\ddot{x}_2(t) - \ddot{x}_1(t)) \tag{1}$$

where $b > 0$ is the inertance *and has units of kilograms.*

The precise wording of the definition is crucial. It is essential that it concerns *relative* acceleration, which must be measured between two points—the terminals (connection points)—of the element. The terminals are also the application points for the forces. As shown in Fig. 1 it is explicit that the forces are equal and opposite and colinear with the line joining the terminals. The defining property is a proportionality between force and relative acceleration. Yet neither the force nor the relative acceleration is an input or output, just as the voltage across and current through a resistor are neither inputs nor outputs of the device.

Definition 1 is not a law of nature or mechanics such as Newton's second law which relates the force on a mass to its absolute acceleration in an inertial frame of reference. It is the property that a practical device must satisfy to count as an inerter. It must be emphasised that the definition can only ever be satisfied *approximately* by real devices. Even if we assume we can achieve zero losses through friction etc, and any components used are ideal, it is still an approximation. This can be deduced directly from Definition 1. Consider the situation that each terminal is subjected to an equal acceleration. From (1), $F(t) = 0$. More fundamentally, since the forces at the terminals are equal and opposite, there is no net force on the device. Hence there can be no net acceleration of the device, which is a contradiction, unless it has zero mass. Since any practical device will have some mass, it is evident that equation (1) will always be an approximation. A similar assumption is inherent in the ideal modelling laws for springs and dampers. Indeed Definition 1 should be viewed as the definition of an *ideal modelling element*.

It should be noted that the mass of an inerter *device* is different to, and typically much smaller than, its inertance b. We will see in Sections 4 and 5 that (1) is a useful description of a practical device if it has *small* mass compared to the objects that it is connected to.

3 The mass element in mechanical modelling

There are two standard analogies between electrical and mechanical networks which are power preserving. The older of the two is the force-voltage (velocity-current) analogy, which has its roots in the notion of electromotive force. For network analysis and synthesis the later force-current (velocity-voltage) analogy [29, 30, 31] is preferable because it preserves topological structure—series (resp. parallel) connections in one domain are of the same type in the other. Furthermore, terminals map to terminals and electrical ground maps to mechanical ground (which is a fixed point or a point with constant velocity in an inertial frame of reference). Figure 2 shows the traditional correspondences between circuit elements in the force-current analogy.

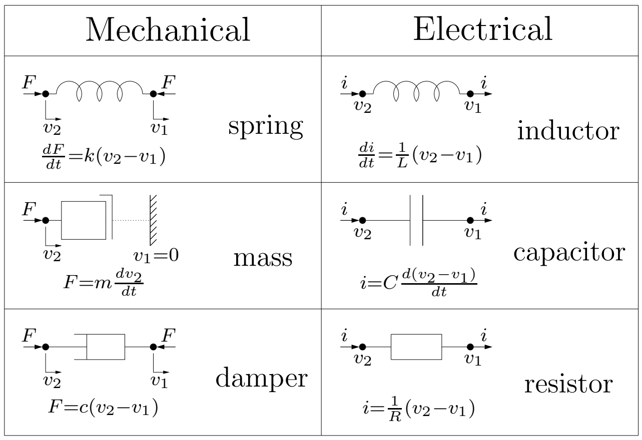

Figure 2: Standard mapping of circuit elements in the force-current analogy where stiffness k, mass m, damping c, inductance L, capacitance C, resistance R are positive constants.

Figure 2 is based on the correspondence between through-variables $\mathcal{T}$ (Force F or current i) and across-variables $\mathcal{A}$ (velocity v and voltage v). For the damper and resistor $\mathcal{T}$ and $\mathcal{A}$ are proportional, for the spring and inductor $\frac{d}{dt}\mathcal{T}$ and $\mathcal{A}$ are proportional, and for the mass and capacitor $\mathcal{T}$ and $\frac{d}{dt}\mathcal{A}$ are proportional. In this sense the correspondence seems complete. But Figure 2 highlights the fact that five of the six elements are two-terminal devices, while the sixth—the mass element—is not. From the point of view of network synthesis the mass element is exceptional in that it has only one "non-grounded" terminal. The symbol for the mass element shows a notional connection to ground through the dotted line, but this is nothing other than an indication that Newton's second law relates the applied force to the absolute acceleration in the inertial frame of reference.

Aside from the issue of the terminals there is a further sense in which the mass element is exceptional. The other five elements in Figure 2 are all "engineering devices", namely they are contrivances that must be manufactured to achieve a property. The property is only satisfied approximately because of deviations (parasitics) such as hysteresis, temperature dependence (springs), fluid compressibility, non-laminar flow, cavitation, seal friction (dampers), series inductance, parallel capacitance, temperature dependence, thermal noise (resistors), Q-factor, dielectric absorption, leakage (capacitors), Q-factor, eddy currents, hysteresis (inductors) etc. In addition there are practical upper limits on the through and across variables to prevent damage to the devices.

In contrast to the other five elements in Figure 2 mass is not an engineering device that must be manufactured. Its governing law, Newton's second law, is presumed to be exact in the classical theory and is not subject to hard limits on the magnitude of the force or acceleration. Departures from this law are not of the same nature as for the other five elements, e.g. relativisitc effects, or if a body is non-rigid. In the former case the deviation from the classical law is negligible for most engineering purposes. In the latter case, the fundamental law still holds with respect to the centre of mass.

Consideration of Figure 2 invites the question as to whether a device satisfying Definition 1 can be constructed. As emphasised in [1] any such devices would need to satisfy certain practical conditions, e.g. the device should be capable of having a small mass, independent of the required value of inertance; and the device should have a finite linear travel which is specifiable independently of the inertance. In subsequent sections we will discuss various methods of realisation which can achieve these requirements.

4 The ballscrew inerter

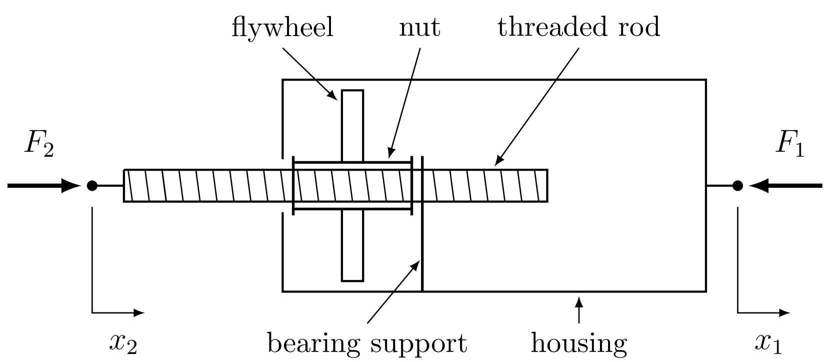

Figure 3: Ballscrew inerter.

Consider the two-terminal device depicted in Fig. 3 consisting of a threaded rod, nut, and a flywheel rigidly connected to the nut which may rotate within the housing. The housing is attached to terminal one and the threaded rod to terminal two, and they are constrained not to rotate relative to each other. Let m be the mass of the flywheel, γ the radius of gyration of the flywheel and p the pitch of the screw. Let the rod have mass m_2 and the rest of the device including nut, bearings, housing and flywheel have mass $m_1 > m$. Let the longitudinal reaction force between the rod and nut assembly be denoted by F_0, the torque applied by the rod to the nut by T_0, and the angular rotation of the flywheel with respect to the housing by θ. For modelling purposes we assume that the rod and housing are constrained not to rotate within the frame of reference and we begin by allowing different forces F_1 and F_2 at the two terminals. We may write the dynamic equations of the device as follows:

$$-F_1 = m_1\ddot{x}_1 + F_0, \tag{2}$$
$$F_2 = m_2\ddot{x}_2 - F_0, \tag{3}$$
$$\dot{x}_2 - \dot{x}_1 = \dot{\theta}\frac{p}{2\pi}, \tag{4}$$
$$F_0 = -\frac{2\pi}{p}T_0, \tag{5}$$
$$T_0 = m\gamma^2\ddot{\theta}, \tag{6}$$

where (2)-(3) are Newton's 2nd law applied to the housing and the rod, (4)-(5) are the ideal transformer equations for the ballscrew-nut linear-to-rotational transducer, and (6) is Newton's 2nd law in rotational form applied to the flywheel. Eliminating T_0, θ and F_0 leads to the pair of equations:

$$F_1 = b(\ddot{x}_2 - \ddot{x}_1) - m_1\ddot{x}_1, \tag{7}$$
$$F_2 = b(\ddot{x}_2 - \ddot{x}_1) + m_2\ddot{x}_2 \tag{8}$$

where

$$b = \left(\frac{2\pi\gamma}{p}\right)^2 m.$$

The two terminal forces F_1 and F_2 contain a common term proportional to the relative acceleration as well as an additional term proportional to the absolute acceleration of the terminal. Clearly if these latter terms were significant then the device of Fig. 3 would not be a good approximation of an ideal inerter. On the other hand there is the scaling factor $2\pi\gamma/p$ which could be made large. For example, in a vehicle suspension application, values of $\gamma = 25\,\text{mm}$ and $p = 4\,\text{mm}$ are quite practical, which makes $(2\pi\gamma/p)^2 \approx 1542$. If m is a substantial proportion of m_1 and $m_2 < m_1$ then the second terms in (7)-(8) may reasonably be neglected in most situations. Hence, the device as depicted in Fig. 3 can be a good approximation of an ideal inerter for suitable choices of m, m_1, m_2, γ and p, namely

$$F_1 = F_2 = F = b(\ddot{x}_2 - \ddot{x}_1)$$

holds approximately for F defined in Fig. 1.

It is worth pointing out that (7)–(8) may be viewed as an ideal inerter connected between a mass m_1 at terminal 1 and a mass m_2 at terminal 2. If other masses are to be connected to the two terminals in an application then these additional terms serve just to increase these masses, probably by a negligible amount.

We remark that use of an inerter in a suspension system will generally involve many reversals of sign in the relative velocity $\dot{x}_2 - \dot{x}_1$. Although ballscrews were not conceived originally for such rapid and frequent reversals, they have proved more than capable of operating in such a manner in their now standard use as a suspension element in motorsport. (See Section 9 for further discussion.)

Penske Racing Shocks has led the commercial development and supply of inerters since 2008. Figure 4 shows one version of Penske's Formula One ballscrew inerter. The construction is similar to the schematic of Figure 3. There is no internal keying against rotation within the device. The terminal attachments for both the rod and the housing employ a clevis mount which keys the rod and housing separately against rotation. In this example the ballscrew is lubricated with grease and operates without seals.

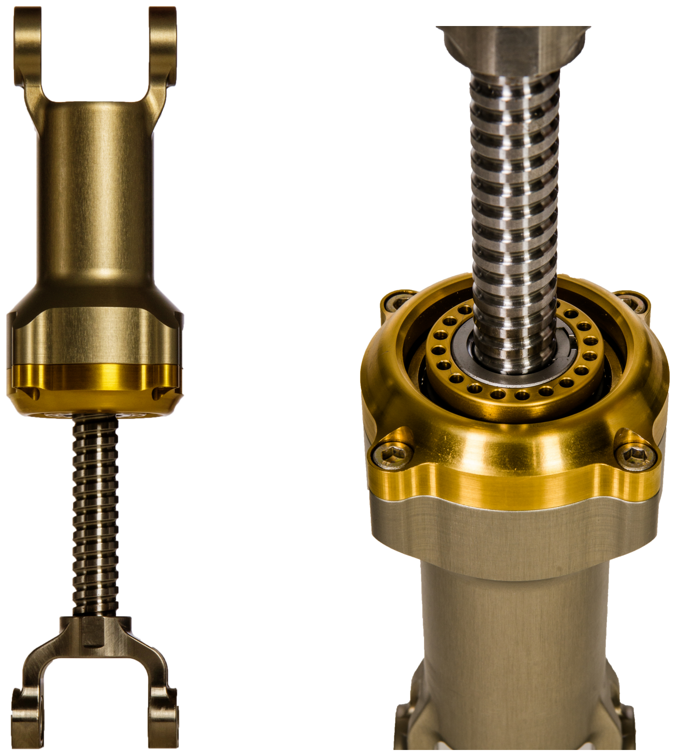

Figure 4: Penske ballscrew inerter.

5 Rack and pinion inerter

Consider the two-terminal device depicted in Fig. 5 consisting of a rack connected to terminal 2 which is constrained to slide within the housing along the line to terminal 1 which is connected to the housing. The rack pinion and gear are constrained to rotate within the housing with the angle of rotation θ_0 satisfying $r\theta_0 = x_2 - x_1$ where r is the radius of the rack pinion. The flywheel and pinion are also constrained to rotate within the housing with the angle of rotation θ satisfying $\theta = -\alpha\theta_0$ where α is the gearing ratio. Let the housing have mass m_3, the rack have mass m_2, the rack pinion and gear have mass m_0 and radius of gyration γ_0, and the flywheel and pinion have mass m and radius of gyration γ. Assuming infinitesimal displacements δx_1, δx_2, $\delta\theta_0$ and $\delta\theta$, and applying

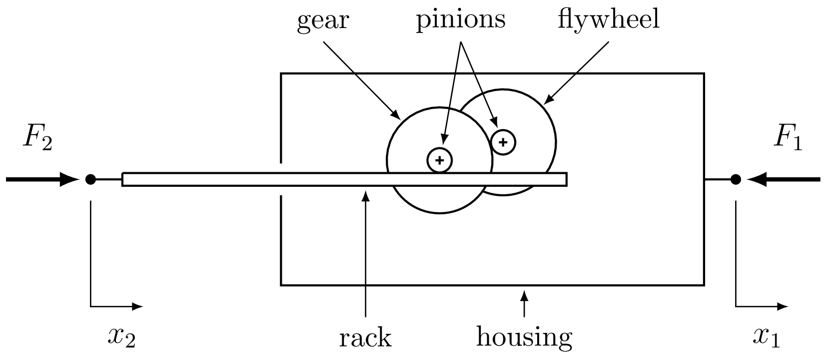

Figure 5: Rack and pinion inerter.

d'Alembert's principle [32] gives:

$$\begin{aligned} 0 &= (-F_1 - (m_0 + m_3 + m)\ddot{x}_1)\delta x_1 + (F_2 - m_2\ddot{x}_2)\delta x_2 - m_0\gamma_0^2\ddot{\theta}_0\delta\theta_0 - m\gamma^2\ddot{\theta}\delta\theta \\ &= (-F_1 - (m_0 + m_3 + m)\ddot{x}_1)\delta x_1 + (F_2 - m_2\ddot{x}_2)\delta x_2 \\ &\qquad - \frac{m_0\gamma_0^2 + m\gamma^2\alpha^2}{r^2}(\ddot{x}_2 - \ddot{x}_1)(\delta x_2 - \delta x_1). \end{aligned}$$

Since δx_1 and δx_2 are independent we obtain:

$$F_1 = b(\ddot{x}_2 - \ddot{x}_1) - m_1\ddot{x}_1, \tag{9}$$

$$F_2 = b(\ddot{x}_2 - \ddot{x}_1) + m_2\ddot{x}_2, \tag{10}$$

where $b = (m_0\gamma_0^2 + m\gamma^2\alpha^2)/r^2$ and $m_1 = m_0 + m_3 + m$. The form of (9)–(10) is similar to (7)–(8) and the same considerations apply. Values of $\gamma/r \approx 3$ and $\alpha \approx 6$ are quite practical, which provides a multiplier in excess of 300 to the flywheel mass m in the second term of b. A further gearing stage is possible, say with ratio β, which provides a third dominant term in b of the form $m(\gamma/r)^2\alpha^2\beta^2$. A simple meccano demonstrator was constructed with $m \approx 0.1\,\text{kg}$, $\gamma/r \approx 2$, $\alpha = \beta = 5$ to give an inertance $b \approx 250\,\text{kg}$. With such parameters the second terms in (9)–(10) can reasonably be neglected (or added as small contributions to any masses connected directly to terminals 1 and 2) to provide a practical implementation of an ideal inerter.

We conclude the section by recalling the circuit symbol that is commonly used to represent the inerter which is similar in proportion to the capacitor but is reminiscent of a flywheel (Figure 6).

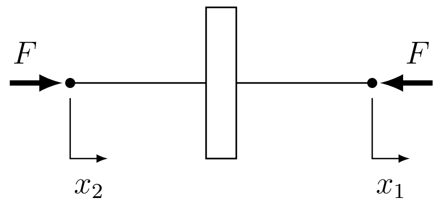

Figure 6: Inerter circuit symbol.

6 Devices and effects

Any solid body subject to a stress gives rise to a strain. If the stress is small enough the stress is proportional to the strain. This elastic behaviour is a "stiffness effect" of the material and is the basis from which most springs are made. The stiffness effect of a solid does not create a spring in full generality. Attachment points (terminals) are needed in order that a spring as described in Figure 2 can transmit forces F of either sign (both compressive and extensive). It is true that if a spring is pre-loaded so it always acts in compression the connections to other elements may be simplified. Nevertheless, for the purposes of the discussion here, we make a distinction between elasticity or compliance as an effect and a spring as a device with two attachment points.

The same considerations apply to damping. As an effect, damping is associated with the conversion of a high grade energy to heat through, for example, friction, viscosity or internal deformation. In a *damper*, such effects are exploited to produce a device law with respect to its terminals. The viscous damper, consisting of a piston moving in a cylinder which forces hydraulic fluid through an orifice, requires careful design to produce the desired relationship between force and relative velocity. The terminals of a damper allow the necessary sign reversals in the terminal forces and the relative velocity between them.

The inerter is an energy storage device, like the spring, but unlike the damper which is a dissipator. If we denote $v(t) = \dot{x}_2(t) - \dot{x}_1(t)$ then the energy delivered to the inerter between times t_1 and t_2 is given by

$$\begin{aligned}\mathcal{E} &= \int_{t_1}^{t_2} F(t)v(t)dt = \int_{t_1}^{t_2} b\dot{v}(t)v(t)dt \\ &= \frac{1}{2}b\left(v(t_2)^2 - v(t_1)^2\right)\end{aligned}$$

hence $(1/2)bv(t)^2$ is the stored energy in the inerter at time t. In the ballscrew and rack and pinion inerters the stored energy of the device is the energy stored in the rotating flywheel. The effect associated with the inerter device is "inertia". The way the effect is harnessed to provide a device according to Definition 1 is less straightforward than for the damper and spring. In both inerter types discussed so far there is, in addition to the flywheel, a linear-to-rotational transducer, some form of gearing and necessary bearing supports to connect with the terminals. The presence of a mechanical advantage is necessary to create an inertance that is much larger than the mass of the device, though this is not sufficient to indicate the presence of an inerter. For example, a car descending a hill using engine braking is not an example of an inerter (see Section 11). We will discuss a further example in Section 8 to highlight the difference between the inerter device and the effect which it exploits.

7 Simulated mass

If one terminal of an inerter is connected to a fixed point (mechanical ground) as in Figure 7 the remaining terminal behaves as a "simulated mass", namely an applied force satisfies the relationship $F = b\,\ddot{x}$ where b is the inertance. Although this is not a normal use for the inerter (the main purpose of which is to have two independently movable terminals) there are possible applications, as noted in [2, 1]. For example, if a spring-damper support or absorber is required to be tested before final installation on a mass which is extremely large (e.g. a large building) then the use of an inerter to simulate the mass as in Figure 7 may be practical.

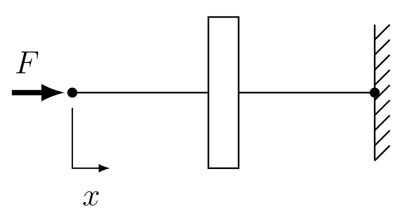

Figure 7: Inerter with one terminal grounded.

The arrangement of Figure 7 also serves to further illustrate the "effect" associated with an inerter device. If a force F is applied to the non-grounded terminal then a "resistance to acceleration" is experienced that is identical to a mass element of mass b. It is this characteristic that is instantly identifiable in any inerter embodiment. Though the effect alone doesn't prove the existence of an inerter.

8 Mousetrap, cafetière and spinning top

An instructive demonstration using household items of the three passive elements (spring, damper and inerter) can be found in a SkySports feature on racing car suspensions by Ted Kravitz and Pat Symonds [33]. A rotational spring in the form of a torsion bar is illustrated by a mousetrap. The proportionality of the torque with angle is clearly seen, though the spring is pre-loaded to act in compression, so a genuine pair of terminals is not in evidence. These are visible in the anti-roll bar graphic at the connection to the rockers shown in the first part of the feature.

The damper is represented by a cafetière filled with oil. As the oil is squeezed through the holes in the piston the resistance to motion increases with velocity in a manner that is very similar to a viscous damper. Terminals can be easily imagined attached to the handle and the bottom of the pot.

The item chosen to represent the inerter is a spinning top, which in general may be defined as a body with a pointed tip on an axis about which it may be spun. There are various methods to impart spin to a top, e.g. with the fingers or with a string or with an external device. The top shown in [33] is a type with a built-in pump-action handle. Attached to the handle is a rod with a spiral groove which may be pushed in and out of the top along its axis of rotation. When there is downward pressure on the handle the rod engages to spin the top, but disengages otherwise to allow free rotation.

In the SkySports feature Pat Symonds is careful to say that the Spinning Top "represents an inerter". The property of the inerter that the top illustrates is the inertia effect. There is a resistance to acceleration when there is downward pressure on the handle. The effect is produced in a similar manner to the ballscrew inerter. Otherwise, the top behaves very differently. There is no resistance to acceleration when the handle is pulled upwards. Also, the pointed tip is quite unlike a terminal. The purpose of the tip is to allow steady rotation about an axis which may vary, as in precession and nutation.

One can further observe, in contrast to the spinning top, that the angular momentum in the inerter flywheel is inevitably stored only for a short time since in normal use the device will reverse and give back its internal energy before the limits of travel are reached. The coriolis force that is essential in the behaviour of a spinning top is only present for short periods of time and is incidental to the function of the inerter. Counter-rotating flywheels were proposed in [2] to eliminate any gyroscopic effects, though this has not proved necessary in practical applications so far since the effect is small and transitory.

9 Sign reversals

A practical inerter has internal end stops to prevent the device falling apart. If the device is driven into the end stops then it will 'bounce' back after hitting the spring buffer with a more or less equal velocity, providing this velocity is not too large, otherwise the device may be damaged. In a suspension system, such as a spring-damper-inerter parallel arrangement, the design of the system ensures that the movement of the suspension will normally be within the limits of travel of the suspension strut. The suspension will have its own bump stops to deal with large inputs which take the suspension system out of its normal limits. These bump stops will typically engage before the inerter reaches its limit of travel. In such an application, in normal operation, an inerter undergoes repeated reversals in the sign of $v(t) = \dot{x}_2 - \dot{x}_1$ while maintaining its defining law between force and relative acceleration (1). This contrasts with many geared systems containing a clutch element which disengages a load when the drive reverses. The spinning top of Section 8 is a simple example of this type where the response is different when the sign of $v(t)$ (the velocity of the handle relative to the tip) reverses. In contrast, it is a fundamental requirement of the inerter that sign reversals do not affect its defining law. In both methods to construct inerters presented so far, the ball screw inerter (Section 4) and the rack and pinion inerter (Section 5), the flywheel angular velocity changes sign whenever $v(t)$ changes sign.

10 Back-driving

There is a further important design requirement for inerters which is related to the notion of *back-driving* in geared systems. Consider a complete cycle for an inerter device in which $v(t) = \sin(t)$ and $F(t) = b\dot{v}(t) = b\cos(t)$. The velocity $v(t)$ is positive for the first half of the cycle and negative for the second half, while the force $F(t)$ is positive only for the first and last quarter. Each of the possible combinations of sign of $v(t)$ and $F(t)$ occurs in the 4 quarter cycles. These correspond to forward-driving and back-driving within the gearing system according to whether the two signs are the same or different. This is illustrated in Figure 8 for a pair of spur gears. The primary means to transfer torque between the two shafts is through the normal reaction force R_n at tooth contact. But the frictional force R_t due to sliding must also be accounted for. The sign of R_t depends on the sense of rotation. The sense shown in the figure corresponds to anticlockwise rotation of the smaller gear (forward driving) and it is clear that the torque on the large gear can be balanced by R_n and R_t even for large pressure angles ψ. For back driving the sign of R_t reverses and this may no longer be possible, namely the resultant of R_n and R_t may

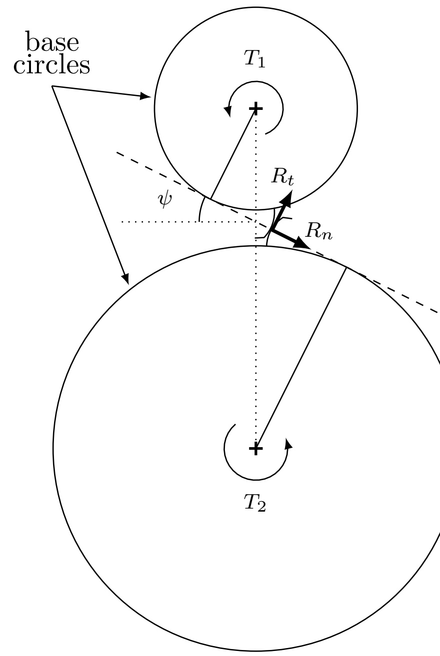

Figure 8: Normal and tangential forces, R_n and R_t, at tooth contact in a pair of spur gears with pressure angle ψ and torques T_1 and T_2 applied to the smaller and larger gear.

not be able to generate a torque on the large gear of the required sign. Thus the gears may no longer back drive if ψ is too large.

Similar considerations apply to a screw and nut. Because of the rolling contact in ball screws the frictional forces are greatly reduced so that back-driving is possible if the pitch angle of the screw is not too small. It should be noted that forward driving and back driving is determined according to the sign of $F(t)v(t)$. For both the ball screw inerter (Section 4) and the rack and pinion inerter (Section 5) back driving corresponds to $F(t)v(t) > 0$, when energy is being supplied to the device, and forward driving to $F(t)v(t) < 0$, when energy is being released.

11 When does a device count as an inerter?

A rather abstract answer to the question of the section title is: when equation (1) is satisfied to a sufficient degree of approximation by the device. The realisations discussed in Section 4 and 5 have become sufficiently well known that instances of a ball screw connected to a flywheel, or a rack and pinion connected to a flywheel through gearing, are now readily identified as inerters. But one must be careful: the mere presence of a flywheel, ballscrew etc does not guarantee the presence of an inerter. Summarising the discussion of Sections 6–10 it is useful to state the following necessary conditions.

1. The parameters (gearing ratio, pitch, masses etc) must be selected in order that deviations from (1) are small enough. Typically, the inertance is significantly larger than the mass of the device, and the latter is sufficiently small compared to other masses to which the device is to be connected.

2. There must be a pair of terminals which are capable of transmitting forces F of positive and negative sign.

3. The device must be capable of reversals of the relative velocity v while maintaining the law (1).

4. The gearing must allow back driving.

A car descending a hill using engine braking is not an example of an inerter: there is an "inertia effect" but 2. and 3. above fail. The spinning top is not an inerter since it fails 2. and 3. (at least).

12 Modelling and reverse-engineering

Assuming that the conditions of Section 11 are all satisfied for a ballscrew or rack and pinion device, it is important to note that the use of (1) to describe the device's terminal behaviour is a *modelling step*. Without background knowledge on the inerter concept it would be perfectly natural to model each component separately, e.g. to include each of equations (2)–(6) individually in the system. A resulting simulation of the device, or a system incorporating the device, would generally be indistinguishable from one that used the modelling approximation. For the purpose of evaluating the performance of a system that is *given* the modelling step provides perhaps only a small advantage of simplification. It is in the design and system integration that the modelling step is decisive in allowing the manner in which the systems is controlled or compensated to be put in a general synthesis context, and with background theory to explain what is possible to build and what is not.

Prior to wide dissemination of [1] the modelling step referred to in the previous paragraph was not generally understood. There is a famous illustration of this fact from the world of Formula One racing when a drawing of an inerter came into the hands of the Renault Formula One team. A ruling from the FIA World Motor Sport Council on a possible contravention of the sporting code suggested that the “drawing did not reveal to Renault enough about the system for the championship to have been affected” [34]. In truth reverse-engineering to discover a purpose is not straightforward.

13 Mass dampers versus J-dampers

The inerter was raced in Formula One for the first time at the Spanish Grand Prix in 2005 by Kimi Räikkönen after a secret development programme at McLaren Racing Ltd in collaboration with the University of Cambridge. McLaren’s use of the inerter (code-named the J-damper) eventually became public knowledge after a scoop by the *Autosport* magazine (see [3], [34]). From an engineering perspective the use of the inerter in Formula One racing provided a perfect illustration of the contrast between the 2-terminal nature of the inerter and other more classical methods of vibration absorption.

At a similar time the Renault Formula One team was developing the mass damper approach to improving “mechanical grip” in a racing car.[3] In essence the approach is to mount a “mass on a spring” to the sprung mass of the vehicle. The idea is a classical one which dates to the early part of the 20th Century, even though the use is somewhat non-standard in the Formula One context. Classically the mass damper (or tuned vibration absorber) acts as a mechanical “notch filter” to prevent vibrations being transmitted at a single frequency. Its purpose in Formula One was to reduce the mean tyre load variations in a stochastic uneven road profile. The inerter achieves a similar improvement in mechanical grip spread over a range of frequencies. Figure 9 shows the contrasting circuit arrangements in which the mass has a single attachment point on the sprung mass, whereas the inerter acts between the sprung and unsprung masses providing an equal and opposite force in proportion to the relative acceleration.

The distinct nature of the two approaches is further illustrated by the fact that Formula Technical specifications have outlawed mass dampers, whereas the inerter in an arrangement such as that shown in Figure 9 has remained legal.

It is useful to mention that the inerter also provides an alternative to the tuned mass damper in its classical use as a “notch filter”. This is described in [1, Section III] (vibration absorption). To insulate a body from a sinusoidal oscillation of the support at a fixed frequency, a zero of transmission is needed in the transfer function from the support displacement to the displacement of the body. In contrast to the classical solution of a mass on a spring mounted on the body, it is shown that an inerter in parallel with a spring incorporated into the support may achieve the desired effect with certain advantages.

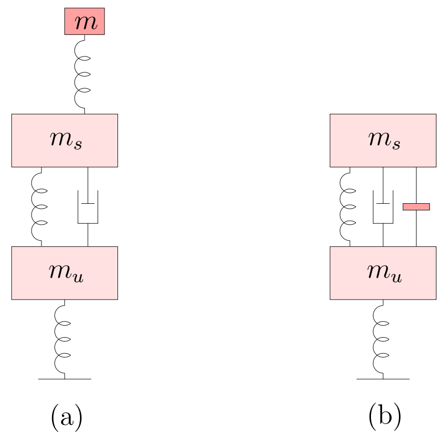

Figure 9: (a) Mass damper (tuned vibration absorber) attached to the sprung mass of a quarter car vehicle model in contrast to (b) an inerter acting in parallel with the conventional spring-damper suspension elements.

14 Gear pump inerter

A further example of a method to construct an inerter using a gear pump/motor, described in [2, 35], is illustrated in Figure 10. The device consists of a piston with through-rod and a cylinder with hydraulic fluid on both sides of the piston. Each end of the cylinder is connected by pipes to the inlet and outlet of the gear pump. Either of the gears in the motor may be connected to a flywheel (not shown), either directly or after further gearing. Alternatively, each of the gear wheels may be connected to flywheels which then would be counter-rotating. The device is provided with two spring buffers, which are effective when the limit of travel is reached but otherwise play no role in the modelling below. The device operates by means of the piston displacing hydraulic fluid from the main cylinder along the connecting pipes to cause a rotation of the gear pump, and vice versa.

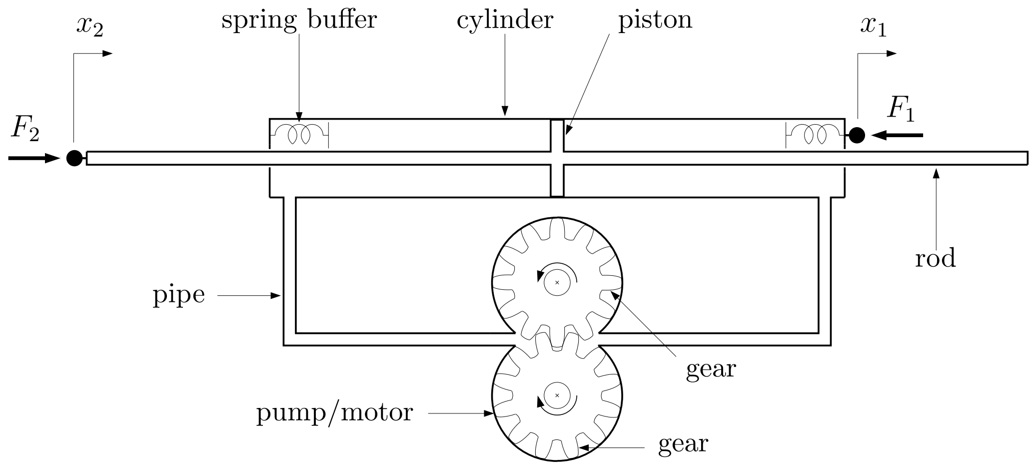

Figure 10: Gear pump inerter.

Let A be the cross-sectional area of the main cylinder minus the rod and let V be the volumetric displacement of the gear pump per radian. Suppose that a flywheel of mass m and radius of gyration γ is attached to the gear pump. Let p_1 (resp. p_2) be the pressure in the hydraulic fluid on the terminal 1 (resp. terminal 2) side of the piston. Then it can be seen from Newton's second law that

$$-F_1 + (p_1 - p_2)A = m_1\ddot{x}_1, \quad (11)$$

$$F_2 + (p_2 - p_1)A = m_2\ddot{x}_2, \quad (12)$$

where m_2 is the mass of the piston and m_1 is the mass of the device without the piston. Equating the flow rate of fluid displaced by the piston with the fluid flow rate into the gear pump gives the equation

$$V\dot{\theta} = A(\dot{x}_2 - \dot{x}_1) \quad (13)$$

where θ is the angular rotation of the upper gear. If the volumetric efficiency of the gear pump is 100%, namely there are no energy losses, we find by equating the power to drive the piston with the power to rotate the pump that:

$$-m\gamma^2\ddot{\theta}\,\dot{\theta} = A(p_2 - p_1)\,(\dot{x}_2 - \dot{x}_1). \quad (14)$$

Using (13) to eliminate θ in (14) and substituting for $p_2 - p_1$ in (11), (12) gives

$$F_1 = b(\ddot{x}_2 - \ddot{x}_1) - m_1\ddot{x}_1, \quad (15)$$

$$F_2 = b(\ddot{x}_2 - \ddot{x}_1) + m_2\ddot{x}_2, \quad (16)$$

where $b = m\gamma^2(A/V)^2$. These equations take the same form as for the ballscrew and rack and pinion inerter. Evidently the device as modelled above can be a good approximation of an ideal inerter if b is sufficiently large compared to m_1 and m_2. For practical devices, additional effects may be included in the modelling to take account of other possible departures from an ideal modelling law, e.g. viscous damping, and series compliance due to compression of the fluid. (See [35] for a detailed discussion of these issues.)

15 Fluid inerter

A method to construct inerters which is entirely fluid based (the so-called "fluid inerter") is described in [4, 5, 36, 37]. The basic concept is illustrated in Figure 11. Again there is a piston with through-rod within a cylinder with hydraulic fluid on both sides of the piston and a (helical) tube connecting the two sides of the piston. The movement of the piston relative to the cylinder causes fluid to flow along the tube.

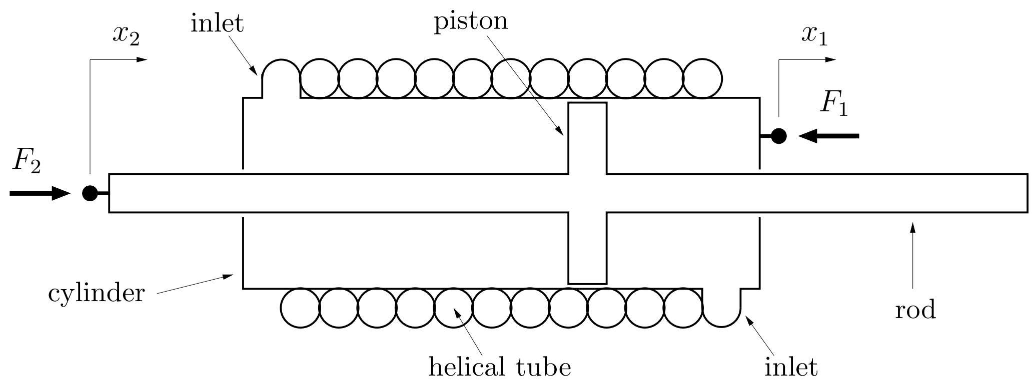

Figure 11: Fluid inerter.

Let A_1 (resp. A_2) be the cross-sectional area of the main cylinder minus the rod (resp. the helical tube). Suppose the tube has length ℓ and the density of the fluid is ρ. Again let p_1 (resp. p_2) be the pressure in the hydraulic fluid on the terminal 1 (resp. terminal 2) side of the piston. Then from Newton's second law

$$-F_1 + (p_1 - p_2)A_1 = m_1\ddot{x}_1, \quad (17)$$

$$F_2 + (p_2 - p_1)A_1 = m_2\ddot{x}_2, \quad (18)$$

where m_2 is the mass of the piston and m_1 is the mass of the device without the piston. In the next modelling steps we will assume the fluid is inviscid and incompressible, and denote by u the mean velocity of the fluid in the helical tube. Equating flow rates we find $A_1(\dot{x}_2 - \dot{x}_1) = A_2 u$. Considering the bulk motion of the fluid in the helical tube and applying Newton's second law we obtain

$$(p_1 - p_2)A_2 = \rho \ell A_2 \dot{u}. \tag{19}$$

After substituting in (17), (18) we find

$$F_1 = b(\ddot{x}_2 - \ddot{x}_1) - m_1 \ddot{x}_1, \tag{20}$$

$$F_2 = b(\ddot{x}_2 - \ddot{x}_1) + m_2 \ddot{x}_2, \tag{21}$$

where $b = \rho \ell A_1^2 / A_2$. Once again, the device as modelled by (20), (21) can be a good approximation of an ideal inerter if b is sufficiently large compared to m_1 and m_2.

Of course, real fluids are neither incompressible nor inviscid, and it cannot be assumed that the fluid flow is properly modelled by its bulk motion. A study of these additional effects is given in [5, 37]. It should be noted that such effects may sometimes be exploited to provide an integrated device with a circuit model comprising inerters, springs and nonlinear dampers [37]. (See Section 18 for a more general discussion of integrated devices.)

16 Rotational inerter

Rotational mechanical networks are analogous both to electrical networks and translational mechanical network through the notion of through and across variables ([38], [29]). The terminals of a rotational network or element are a pair of colinear shafts which may be freely and independently rotated, and the laws of network elements express a relationship between the equal and opposite torque T applied to the shafts and their relative displacement $\theta_2 - \theta_1$, velocity or acceleration (see Figure 12). In [2] a definition was proposed of an ideal rotational inerter as follows.

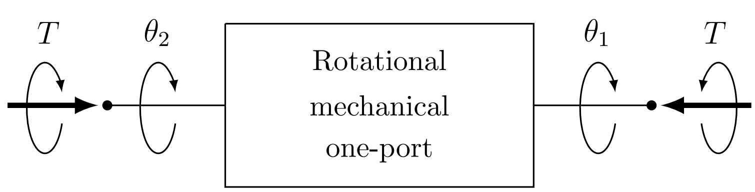

Figure 12: Rotational mechanical one-port element with equal and opposite torques T and rotations θ_1 and θ_2 about two colinear, rotational terminals.

Definition 2 *The ideal (rotational) inerter is a rotational mechanical one-port with the property that the equal and opposite torques $T(t)$ at the terminals are proportional to the relative angular acceleration between them, in particular,*

$$T(t) = b\left(\ddot{\theta}_2(t) - \ddot{\theta}_1(t)\right) \tag{22}$$

where $b > 0$ is the inertance *and has units of* kg m^2.

To be useful in applications the inertance b should greatly exceed the moment of inertia of the device. A method to realise a rotational inerter using epicyclic gears was described in [2]. A similar construction is shown in Figure 13. Let θ denote the absolute rotation of the flywheel. Treating the gear box as ideal with ratio n and neglecting all inertias except that of the flywheel with inertia J gives

$$\begin{aligned} T_1 = T_2 &= nJ\ddot{\theta}, \\ \theta &= n(\theta_2 - \theta_1) \end{aligned}$$

from which (22) is satisfied with $T = T_1 = T_2$ and $b = n^2 J$.

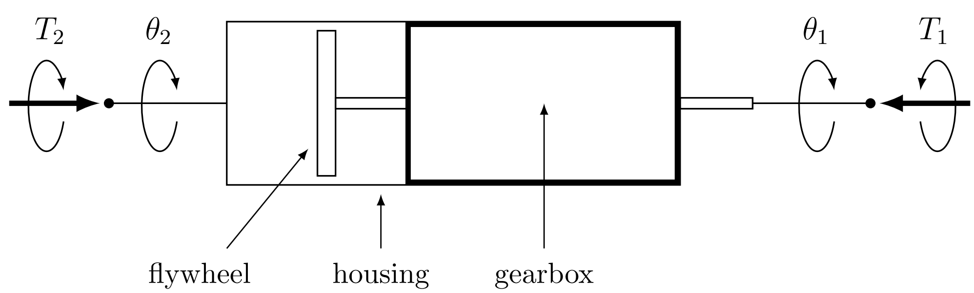

Figure 13: Rotational inerter embodiment

It is instructive to carry out a detailed modelling analysis of the construction in [2] which makes use of a compound epicyclic gear train. An example is shown in Figure 14 in which there are two coaxial shafts (the terminals) with angular velocities and applied torques ω_2, T_2 and ω_1, $-T_1$ respectively (note the opposite sign convention for the torque at terminal one), gears of radii r_1 and r_4, and an intermediate gear wheel with gears of radii r_3 and r_2 fixed to a spindle A-B rotating with angular velocity ω_4 which is attached to a planet carrier which rotates about the main axis with angular velocity ω_3. The gearing imposes the following constraints on the angular velocities:

$$\begin{aligned} \omega_2 r_1 &= \omega_3 R - \omega_4 r_3, \\ \omega_1 r_4 &= \omega_3 R - \omega_4 r_2, \end{aligned}$$

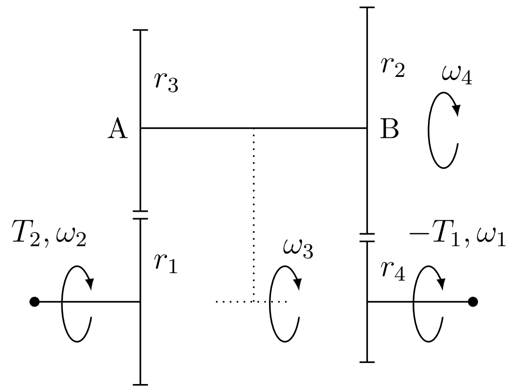

Figure 14: Compound epicyclic gear train

where $R = r_1 + r_3 = r_2 + r_4$. Eliminating ω_4, noting that $R(r_2 - r_3) = r_1 r_2 - r_3 r_4$, leads to the identity

$$\omega_2 = \gamma\omega_1 + (1 - \gamma)\omega_3 \tag{23}$$

where $\gamma = r_3 r_4 / r_1 r_2$. We can also derive the following expression for ω_4 in terms of ω_1 and ω_2:

$$\omega_4 = -\frac{\gamma R}{r_3(1-\gamma)}\omega_1 + \frac{R}{r_2(1-\gamma)}\omega_2. \tag{24}$$

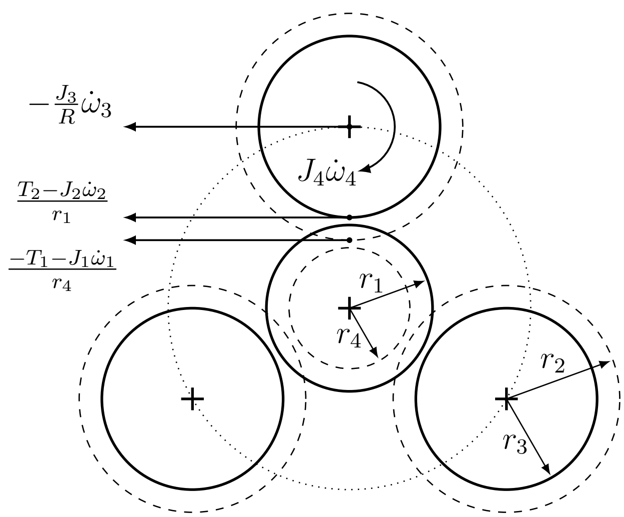

Figure 15: Compound epicyclic gear train (axial view)

Suppose that the moments of inertia of the gears attached to terminals 1 and 2 are J_1 and J_2 respectively, that the moment of inertia of the planet carrier about the principal axis (including the intermediate gears) is J_3, and that the total moment of inertia of the intermediate gears about their own axis is J_4. The axial view in Figure 15 shows the real and inertial forces and torques on the intermediate gear from tooth contact and the planet carrier assembly (all forces summed and shown as acting at one intermediate gear). Equilibrium conditions give the following two equations:

$$\frac{-T_1 - J_1\dot{\omega}_1}{r_4} + \frac{T_2 - J_2\dot{\omega}_2}{r_1} - \frac{J_3}{R}\dot{\omega}_3 = 0, \tag{25}$$

$$\frac{(-T_1 - J_1\dot{\omega}_1)r_2}{r_4} + \frac{(T_2 - J_2\dot{\omega}_2)r_3}{r_1} + J_4\dot{\omega}_4 = 0. \tag{26}$$

We can verify that

$$\begin{pmatrix} \frac{1}{r_4} & \frac{1}{r_1} \\ \frac{r_2}{r_4} & \frac{r_3}{r_1} \end{pmatrix}^{-1} = \frac{R}{1-\gamma}\begin{pmatrix} -\gamma & \frac{\gamma}{r_3} \\ 1 & -\frac{1}{r_2} \end{pmatrix}$$

and hence

$$\begin{aligned} \begin{pmatrix} -T_1 - J_1\dot{\omega}_1 \\ T_2 - J_2\dot{\omega}_2 \end{pmatrix} &= \frac{J_3}{(1-\gamma)^2}\begin{pmatrix} -\gamma \\ 1 \end{pmatrix}(-\gamma\dot{\omega}_1 + \dot{\omega}_2) \\ &\quad - \frac{J_4R^2}{(1-\gamma)^2}\begin{pmatrix} \frac{\gamma}{r_3} \\ -\frac{1}{r_2} \end{pmatrix}\left(-\frac{\gamma}{r_3}\dot{\omega}_1 + \frac{1}{r_2}\dot{\omega}_2\right) \\ &= (M_1 + M_2)\begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{pmatrix} \end{aligned} \tag{27}$$

where

$$M_1 = \frac{J_3}{(1-\gamma)^2}\begin{pmatrix} \gamma^2 & -\gamma \\ -\gamma & 1 \end{pmatrix}, \quad M_2 = \frac{J_4}{(1-\gamma)^2}\begin{pmatrix} \gamma^2\alpha^2 & -\gamma\alpha\beta \\ -\gamma\alpha\beta & \beta^2 \end{pmatrix},$$

and $\alpha = R/r_3$, $\beta = R/r_2$.

Writing $\lambda = r_1/r_3$ then $\alpha = 1 + \lambda$, $\beta = 1 + \gamma\lambda$ and $\gamma\alpha - \beta = \gamma - 1$. Hence from (27)

$$T_1 = b(\dot{\omega}_2 - \dot{\omega}_1) + \left(\frac{\gamma(J_3 + \alpha J_4)}{1-\gamma} - J_1\right)\dot{\omega}_1, \tag{28}$$

$$T_2 = b(\dot{\omega}_2 - \dot{\omega}_1) + \left(\frac{J_3 + \beta J_4}{1-\gamma} + J_2\right)\dot{\omega}_2, \tag{29}$$

where $b = \gamma(J_3 + \alpha\beta J_4)/(1-\gamma)^2$. By taking γ close to 1 the inertance b may assume a large value. The equations also have additional terms as in (9)–(10) and (15)-(16) but with the important difference of a term proportional to $1/(1-\gamma)$. This will not grow as quickly as the inertance as $\gamma \to 1$ but may not necessarily be assumed small as in previous cases. This term could be viewed as addition of some inertia to terminal 2 and corresponding subtraction from terminal 1, when $\gamma > 1$, and vice versa if $\gamma < 1$. It is possible that this term could be exploited to advantage in some mechanical control applications.

17 Synthesis of mechanical networks

The proposal in [1] to define a new ideal modelling element arose from the following question: what is the most general mechanical impedance that can be realised passively? The answer to the corresponding question in the electrical domain was well-known from classical circuit theory which we now describe.

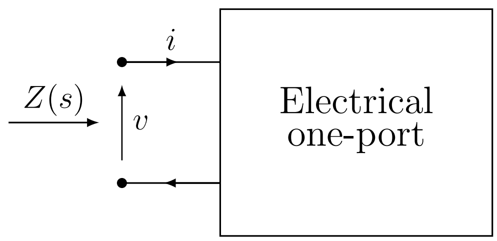

Figure 16: Electrical network with two external terminals, terminal voltage v, terminal current i and driving-point impedance $Z(s)$.

For a two-terminal electrical element or network the *impedance* (or more fully, the driving-point impedance) is defined by $Z(s) = \hat{v}(s)/\hat{i}(s)$ where $v(t)$ is the voltage across the terminals and $i(t)$ is the current through the network, as shown in Figure 16, and $\hat{}$ denotes the Laplace transform. A network is defined to be *passive* [39, p. 26],[40, p. 21] if for all admissible v, i which are square integrable on $(-\infty, T]$,

$$\int_{-\infty}^{T} v(t)i(t)\,dt \geq 0. \tag{30}$$

The quantity on the left-hand side of (30) has the interpretation of the total energy delivered to the network up to time T. Thus, a passive network cannot deliver energy to the environment.

A function $Z(s)$ is defined to be *positive-real* if $Z(s)$ is real for real s and one of the following two equivalent conditions is satisfied:

1. $Z(s)$ is analytic and $Z(s) + Z(s)^* \geq 0$ in Re(s) > 0.
2. $Z(s)$ is analytic in Re(s) > 0, $Z(j\omega) + Z(j\omega)^* \geq 0$ for all ω at which $Z(j\omega)$ is finite, and any poles of $Z(s)$ on the imaginary axis or at infinity are simple and have a positive residue.

It can then be shown [39, Chs. 4,5], [40, Theorem 2.7.1,2] that a one-port electrical network with impedance $Z(s)$ is passive if and only if $Z(s)$ is positive-real. The same statement holds for the admittance $Y(s) = Z(s)^{-1}$.

In the mechanical domain the impedance of an element or network is defined as $Z(s) = \hat{v}(s)/\hat{F}(s)$ with $F(t)$ and $v(t) = \dot{x}$ in the notation of Fig. 1, and the admittance is defined by $Y(s) = Z(s)^{-1}$. Using the force-current analogy (see Section 3) we deduce the following result.

Theorem 1 *A one-port mechanical network is passive if and only if its impedance $Z(s)$, or equivalently its admittance $Y(s)$, is positive-real.*

The above theorem sets the boundary of what is possible to build as a passive mechanical device. But it doesn't state that any such impedance *can* be physically realised. Fortunately, this question has been answered in the electrical domain by the Bott-Duffin theorem [41], which states that any positive real impedance may be realised by a circuit comprised of resistors, capacitors and inductors only. The correspondences of Figure 2 with the mass element replaced by an inerter, and using the force-current analogy, allows the result to be translated over directly, which we now state.

Theorem 2 *Any positive real function may be realised as the driving-point impedance or admittance of a mechanical network comprising springs, dampers and inerters.*

The Bott-Duffin theorem is a famous and celebrated result in circuit theory, and its importance increases with its potential use for mechanical networks. Despite it being a classical result there remained unresolved questions about the construction when research on the topic petered out in the 1960s. These questions relate to whether the construction is the simplest possible among circuits that may realise a given mechanical impedance. Recently it has been shown that the Bott-Duffin realisation is the simplest among series-parallel circuits for some positive-real functions [42] and that a similar fact holds for some classically known simplifications of the Bott-Duffin procedure to bridge circuits [43]. Simplicity of realisation is an important matter in the mechanical domain and there has been much research on this topic over the last decade. The reader is referred to [44] and the references therein for further discussion.

18 Integrated mechanical devices

Mechanical network elements are often packaged into single units for reasons of space and mechanical simplicity. For a more general one-port mechanical impedance/admittance the external terminals will be in evidence as the external attachment points, as for a stand alone network element. Internal nodes or terminals may or may not be explicitly identifiable in certain realisations, and indeed there may be advantages of mechanical simplicity to exploit an effect internally without requiring explicit internal nodes. A few examples will serve to illustrate this point.

18.1 Parallel spring-damper

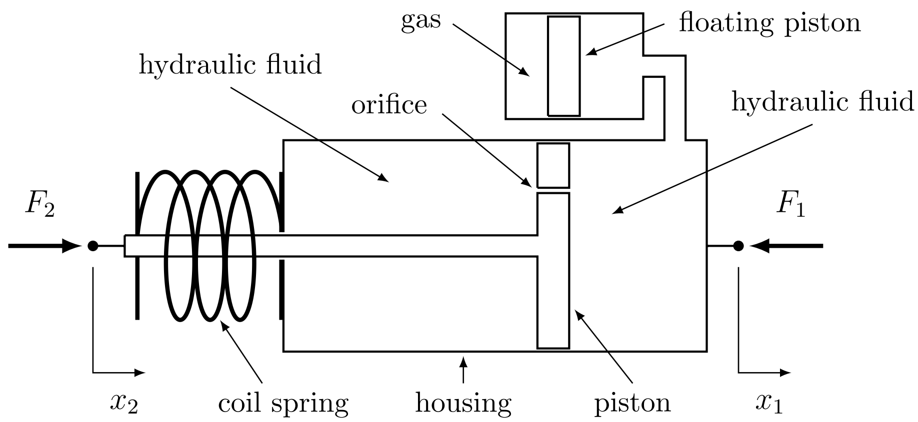

Figure 17: Telescopic hydraulic damper with floating piston and coil spring acting in parallel.

Figure 17 shows a coil spring mounted between the housing and the end of the damper rod of a telescopic hydraulic damper. Terminal 2 of the damper is rigidly connected to one of the terminals of the spring, and the other terminal is connected to the housing to which terminal 1 is rigidly connected. In this example, both terminals of the spring and damper are clearly identifiable, and the integrated device may easily be dismantled to exhibit the two devices separately.

Now consider Figure 17 in which the compressibility of the hydraulic fluid is taken account of. Let us consider the case where the coil spring is removed and the piston orifice is sealed. Movement of the piston relative to the housing causes an expansion/compression of the fluid on either side of the piston and will result in a proportionality $F_2 = F_1 = k_1 x$ to a first linear approximation. This force arises as a pressure difference across the main piston, as does the damper force, and therefore these forces are notionally equal. In the linear domain the device is consistent with the circuit diagram of Figure 18 where k is the stiffness of the coil spring and c is the (linear) damper rate, though the node/terminal between the damper and series spring is not identifiable as a physical attachment point.

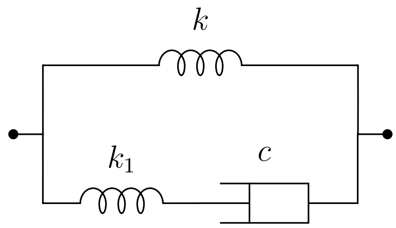

Figure 18: Equivalent network for the device of Figure 17 showing the coil spring k, damper c and series compliance k_1.

18.2 Parallel damper-inerter

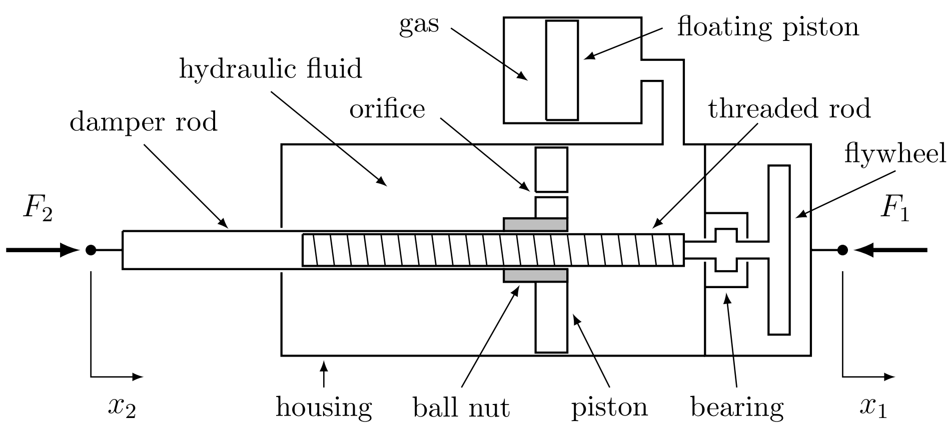

Figure 19: Telescopic hydraulic damper with floating piston and parallel ballscrew inerter.

Next we consider an integrated damper and inerter acting in parallel realised as in Figure 19 following [45]. The damper rod and piston with orifice(s) operates as a conventional hydraulic damper which provides, in a linear approximation, an equal and opposite force at the terminals in proportion to a relative velocity between them. The damper rod and housing are keyed against any relative rotation. Within the piston is the nut of a ballscrew/nut assembly. The threaded rod is arranged to rotate within the damper rod and telescopes within it so that a relative movement between the damper rod and the housing causes the threaded rod to rotate. The equal and opposite axial force on the threaded rod is transmitted to the damper rod by the nut and to the housing by a bearing assembly. A flywheel is attached to the threaded rod to provide an inertial force when the rod rotates. The integrated device produces terminal forces which are approximately given by

$$F_1 = F_2 = b\,(\ddot{x}_2(t) - \ddot{x}_1(t)) + c\,(\dot{x}_2(t) - \dot{x}_1(t))$$

and hence the device functions as an integrated parallel damper and inerter as depicted in Figure 20. A coil spring may easily be added as in Figure 17 to realise an integrated parallel spring-damper-inerter.

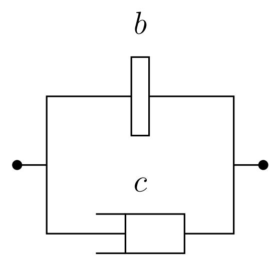

Figure 20: Equivalent network for the device of Figure 19 showing an inerter b in parallel with a damper c.

Figure 21 shows Penske's hybrid damper-inerter according to Figure 19 and [45]. Fly weights attached to the end of the threaded rod may be adjusted by removal of the end cap to vary the inertance. The damper is of standard telescopic type with an external floating piston in a pressurised chamber similar to Figure 19. The ballscrew and nut operate in the hydraulic fluid in a single sealed unit. See [53], [45] for further details on this device.

Figure 21: Penske hybrid damper-inerter.

18.3 Inerter in parallel with a mechanical admittance

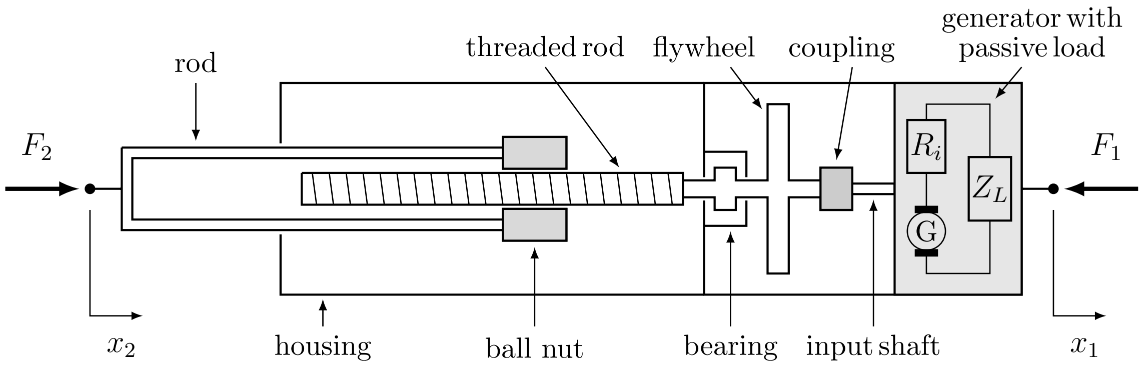

Figure 22: Ballscrew inerter with parallel network.

Following [46], [47] and [48] we consider an integrated device as shown in Figure 22 in which a general passive admittance, implemented by a motor/generator (e.g. a DC permanent magnet machine) and a passive electrical network, acts in parallel with an inerter. In Figure 22, if the coupling to the generator is removed, the device takes the form of a ballscrew inerter as in Section 4, with the only difference being that the threaded rod rotates when there is relative movement of the terminals, rather than the nut as in Figure 3. With the coupling in place, the input shaft to the generator rotates with the threaded rod and flywheel. If the rod and ball nut have mass m_2, the rest of the device has mass m_1 and p is the pitch of the screw then equations (2)–(5) hold with F_0, T_0 and θ defined as in Section 4. Let the rotational elements including the flywheel have a combined inertia of J_m, and denote by T_e the electrical torque produced by the generator. Then

$$J_m\ddot{\theta} = T_0 - T_e. \tag{31}$$

If the rotational motion of the generator induces a voltage V_s and associated armature current I then these are related to the torque and angular velocity as follows:

$$V_s = k_E\dot{\theta}, \tag{32}$$

$$T_e = k_T I, \tag{33}$$

where k_E and k_T are the voltage and torque constants, respectively, and $k_E = k_T$ in SI units. It is assumed that the generator has an armature resistance R_i and an electrical load with complex impedance $Z_L(s)$ is connected across the terminals of the generator. Thus

$$\frac{\hat{V}_s}{\hat{I}} = R_i + Z_L(s) \tag{34}$$

If the masses m_1 and m_2 are neglected then from (2)–(5) and (31)–(34) we find that $F_1 = F_2 = F$ and the admittance of the device of Figure 22 takes the form

$$Y(s) = \left(\frac{2\pi}{p}\right)^2 \left[J_m s + \frac{k_E^2}{R_i + Z_L(s)}\right].$$

Hence the device can be represented as the circuit of Figure 23 consisting of an inerter b in parallel with a mechanical admittance $Y_1(s)$ where

$$\begin{aligned} b &= \left(\frac{2\pi}{p}\right)^2 J_m, \\ Y_1(s) &= \left(\frac{2\pi}{p}\right)^2 \frac{k_E^2}{R_i + Z_L(s)}. \end{aligned}$$

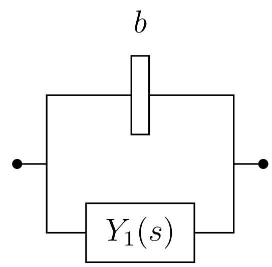

Figure 23: Equivalent network for the device of Figure 19 showing an inerter b in parallel with a mechanical admittance $Y_1(s)$.

18.4 Rotational mechanical compensator

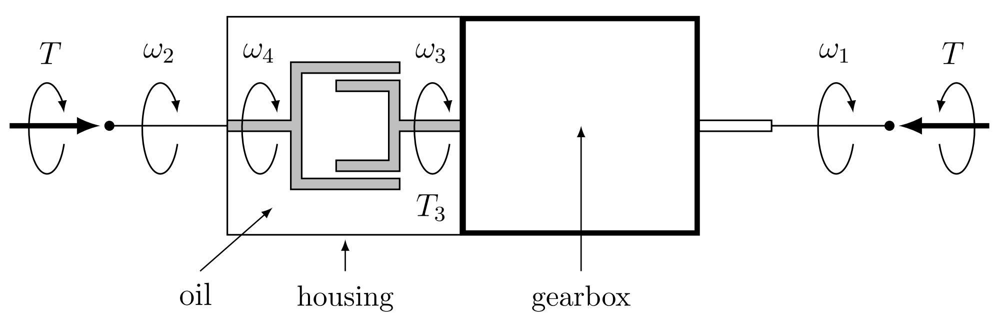

Figure 24: Schematic of the physical model of the steering compensator.

A passive mechanical network incorporating inerters was proposed in [20], [21] as a steering compensator for the control of wobble and weave instabilities in motorcycles. A schematic of a physical model of the steering compensator [49] is shown in Fig. 24. The model consists of a rotational $n\!:\!1$ gearbox, a cylinder of inertia J_1 connected rigidly to the output shaft, and a second cylinder with inertia J_2 which may rotate independently of J_1. The assembly is contained in a housing filled with oil intended to provide a shear film damper between the cylinders with linear damping with coefficient γ by means of a small clearance δ between the two cylindrical surfaces. Additionally, there is a smaller damping with coefficient γ_1, acting directly on the flywheel due to the clearance between the outer cylinder and the housing. We can write down the modelling equations of the compensator as follows, neglecting the inertia of the housing and rotating elements connected to the input shaft of the gearbox:

$$\begin{aligned} T_3 &= n^{-1}T, \\ \omega_3 &= n(\omega_2 - \omega_1), \\ T_3 &= J_1\dot{\omega}_3 + \gamma\left(\omega_3 - \omega_4\right), \\ J_2\dot{\omega}_4 &= \gamma\left(\omega_3 - \omega_4\right) - \gamma_1\omega_4. \end{aligned}$$

We can verify that the admittance function of the compensator is given by

$$\frac{\hat{T}}{\hat{\omega}} = n^2\left(sJ_1 + \gamma - \frac{\gamma^2}{sJ_2 + \gamma + \gamma_1}\right) \tag{35}$$

$$= n^2\left(sJ_1 + \frac{\gamma\gamma_1}{\gamma+\gamma_1} + \left(\left(\frac{\gamma^2}{\gamma+\gamma_1}\right)^{-1} + \left(\frac{\gamma^2 J_2 s}{(\gamma+\gamma_1)^2}\right)^{-1}\right)^{-1}\right). \tag{36}$$

where $\omega = \omega_2 - \omega_1$. The network corresponding to this linear admittance function is shown in Fig. 25 in which

$$b = n^2\left(\frac{\gamma}{\gamma+\gamma_1}\right)^2 J_2, \quad b_1 = n^2 J_1$$
$$c = n^2\frac{\gamma^2}{\gamma+\gamma_1}, \quad c_1 = n^2\frac{\gamma\gamma_1}{\gamma+\gamma_1}.$$

and an additional series spring of stiffness k is included to represent the compliance of the housing (see [49]).

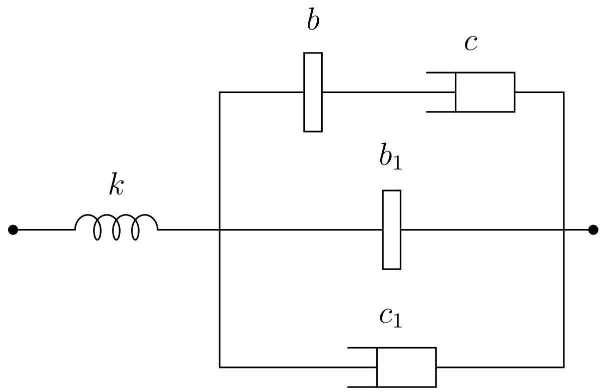

Figure 25: Equivalent network for the device of Figure 24 showing the series connection of an inerter b and damper c in parallel with an inerter b_1 and a damper c_1, all in series with a spring k.

18.5 Internal nodes

It has already been noted that the device of Figure 17 does not display an explicit internal node if the series compliance is modelled. Namely, an effect (fluid compressibility) produces a behaviour consistent with the circuit diagram of Figure 18.

In the device of Figure 19 the two terminals of the inerter are the nut and the bearing support which connect rigidly to the external terminals. Hence each of the component parts have identifiable terminals which coincide with the external terminals.

In the device of Figure 22 the two terminals of the inerter connect directly to the external terminals. The mechanical torque from the generator adds directly to the inertial load of the flywheel and hence the reactions transmit to the external terminals exactly as for a stand alone inerter.

In the device of Figure 24 the parallel inerter acts between the external terminals exactly as in the rotational inerter of Figure 13. The sheer film damper torque adds directly to the inertial load of the rotational inertia J_1 as in the device of Figure 22 and is equal to the inertial load of the rotational inertia J_2 plus an additional damping term. The latter acts between the housing and the rotational inertia J_2. The mechanical admittance takes the form of (35) and it is only after algebraic manipulation that the form (36) is obtained, which corresponds to the circuit diagram of Figure 25. Thus, there is not a direct correspondence between the internal nodes in Figure 25 and identifiable features in Figure 24.

19 Epilogue: the purpose of inerters

Stand alone mechanical components typically have an identifiable purpose: springs provide compliant support, dampers dissipate energy etc. But what about the inerter? Its origin was the completion of an analogy with the electrical capacitor. Fundamental theory (see Section 17) then shows how to construct the most general passive mechanical impedances which may be physically realised. The door is opened to improved dynamical control of mechanical systems by purely passive means. Wherever springs and dampers are employed, more general passive mechanical networks incorporating inerters could be considered.

This line of thinking is characteristic of electrical engineering (from where the fundamental theory is drawn) and control systems (which are often designed to optimise performance measures over some class of controllers). In electrical networks, components are combined together to synthesise an impedance. A given impedance may be equivalently realised by a number of different networks. Thus, it is a matter of debate whether one should seek a "purpose" in each individual element of a network, or even if it is possible to do so. The issue may usefully be illustrated by an imaginary interview of the author (though not so different from some actual interviews).

> Q. Thank you for your lecture, Professor Smith. So, please tell us, what does the inerter actually do?
>
> A. It does exactly what it says on the tin: it produces an equal and opposite force at the terminals which is proportional to the relative acceleration between them.
>
> Q. That doesn't tell me what it does. Please express it in a different way.
>
> A. Well, it's an energy storage element; energy is stored whenever there is relative movement between the terminals and it is given up when there is no relative movement.
>
> Q. But the spring is also an energy storage element and I know what a spring does.
>
> A. Yes indeed. The spring stores energy in proportion to a relative displacement of the terminals from an unloaded state, either in compression or extension. The inerter stores energy when there is a relative motion between the terminals, either moving towards or away from each other. The amount of energy stored in the spring is proportional to the square of the load. The amount of energy stored in the inerter is proportional to the square of the

relative velocity between its terminals. Force and relative velocity are dual variables. Thus, the inerter is the mechanical dual of the spring, just as the capacitor is the electrical dual of the inductor.

Q. This is not the sort of answer I am looking for. If I take a damper I know that it takes energy out of a system. It dampens things down, and I know exactly how it does it. I want you to give me a similar explanation of the inerter.

A. I think I can explain things exactly as you want. But first I must step back a bit. When electrical engineers talk about impedance they are usually describing something that is frequency dependent, i.e. the notion of a complex impedance. It is not just a proportionality between voltage and current, like Ohm's law, but there is a frequency dependence as well. It is the same for mechanical elements. The damper on its own produces a direct proportionality between force and velocity. For the spring, it is a proportionality between force and the integral of velocity. The spring lags the damper by 90 degrees.[1] Electrical engineers know that you need another element that would lead by 90 degrees. That's the inerter.

Q. You are just giving me more of the same without getting to the point. What is the use of having something that leads by 90 degrees?

A. At one level you can say that it expands the range of possibilities for control with elements that don't require a power source. You would expect to get better performance, once the system is optimised, at the expense of using an extra component. Beyond that, there is a deep theoretical result which says that *any* complex impedance that is realisable passively can be synthesised by a circuit comprising elements of the three types (spring, damper and inerter) only.[2] So it is the most you can do in the linear domain before going active.

Q. What I am still struggling with is what the inerter is actually doing when it is connected up as you describe. What effect is it having? When one has an inerter in one's hand one notices a resistance to changes in velocity. There is an inertia effect. Isn't the inerter nothing other than an inertial damper?

A. No. In the first place "inertial damper" is not a precisely defined term. It has a rough meaning that something is damped out by inertia. At its simplest level it might mean the use of a flywheel on a rotating shaft to reduce fluctuations in angular velocity. But there is no understanding that it is a two-terminal mechanical device with the defining property of the inerter.

Q. Yet your emphasis on the terminals and defining law of the inerter contrasts with how one tries to understand other circuit elements. One has a feeling that something is missing.

A. In regard to other circuit elements it is instructive to consider the analogous electrical component: the capacitor. What does the capacitor actually do? Well, it stores electrical energy. Or, current flows in and out of its terminals in proportion to the rate of change of potential difference across them. Capacitors have a multitude of applications: high-pass and low-pass filters, oscillators, rectifiers, temporary power supply, sensing, power factor correction and so on. Each of these is an example of what a capacitor can do, but it is misleading to say that any one of them explains what a capacitor does.

Q. Please give some examples of the practical use of inerters.

A. The first practical use of an inerter was to improve the "mechanical grip" in a racing car,[3] namely, it reduces the tyre normal load fluctuations to unevenness in the road surface, which for complex reasons allows the driver to go a bit faster without coming off the track. A slightly different way to deploy it in the same application is to improve the damping ratio of key suspension modes, which allows the static stiffness to be increased, which allows more aerodynamic downforce to be used, which in turn allows faster cornering speed. The interesting thing about this second use is that it improves "damping" (by working in combination with the dampers and springs) in a way that is not possible with springs and dampers alone.[4]

Q. Isn't the inerter just adding "virtual" mass to the vehicle to allow the suspension to work better without increasing the actual mass?

A. That's not correct. Firstly, one would need to be clear if the inerter is adding to the sprung mass (which would be an advantage), the unsprung mass (which would be a disadvantage), or both (for which there is no clear advantage).[5] In fact it is doing none of the above. The easiest way to see this is to consider the simple system of a pair of terminals between which an inerter is connected. Compare that to the two terminals being point masses with nothing connected between them. In the second case a force applied to one terminal has no effect on the other, in contrast to the behaviour in the first case. Thus, an inerter connected between two independently moveable terminals is fundamentally different from any effect that can be produced by point masses alone, and hence is different from any notion of "virtual" mass.

Q. Returning to the overall effect of the inerter on the vehicle. In essence, it is making the car go faster, isn't it?

A. Yes, but that's not the best way to put it. It allows, by passive means, better control. Hence it permits the driver to go round corners faster without coming off the track, if the driver has the necessary skill. That is what is required in this domain of application. The inerter can also be used for other things in a car suspension, e.g. to improve ride comfort. It is also used in drag car racing to keep the tyre in contact with the ground in transient situations.

Q. Are these the only applications of inerters?

A. By no means. We have looked at the problem of track wear for railway trains. There the goal is to reduce the static yaw stiffness in the train suspension, but that causes instability of the motion of the wheelsets and reduces the maximum speed. With inerters in the lateral suspension we have found that we can improve the stability and maximum speed. So it is the opposite of the approach in racing cars, namely we are trying to reduce stiffness while preventing instabilities from arising.

Q. So, it is a passive mechanical component to improve the behaviour of vehicles?

A. Not just vehicles in fact. There is a lot of work now to study its application to building suspensions, e.g. to make them more resistant to earthquakes. I recently heard of an application to improve the vibration suppression of an isolation system for Michelangelo's Rondanini Pietà.[6]

Q. Finally, we are making progress. At least we have established that the inerter is a suspension component.

A. No, that is false. The examples of use I have given are all suspension systems of some type. But those are not the only applications of the inerter. For example, it can be applied to the control of steering instabilities in motorcycles.

Q. Let me try one last time. I want to understand this thing you call an inerter. I want to taste it, touch it, feel it. I want to perceive its very essence. Tell me what it does in a very simple way.

A. Okay, here's my best shot. (Pause). The inerter is a mechanical network element which can be combined with other network elements to produce an overall effect.[7]

Q. (Angrily). What effect?

A. It depends what effect you want.

Q. I give up. (Aside to the audience.) Will someone tell this fellow to answer the question?

Notes

1. This treats the velocity as input and the force as output, so strictly the interpretation is based on the complex admittance rather than the impedance.

2. See Theorem 2 in Section 17.

3. The term "Mechanical grip" is used in motorsport to describe the ability of the suspension to keep the tyre in contact with the road. In terms of analytical performance measures it maps most closely to J_3 in [50] which is the rms tyre load variation for a standard Brownian motion road excitation. Mechanical grip contrasts with "aerodynamical grip" which results from the increased normal load provided by the downforce of the wings.

4. See [1, Section IV.E] (suspension strut design example).

5. Vehicle performance metrics may typically be improved if the ratio m_u/m_s of unsprung mass m_u to sprung mass m_s can be reduced. See [50] and [51].

6. See [9].

7. The reply is perhaps not as obtuse as the interlocutor thinks. It expresses the idea of classical electrical circuit synthesis that circuits are built from a standard set of components together with interconnection rules, and that for a given allowed set of components one seeks to characterise the precise set of impedances that may be realised. Classical circuit theory answered this question in a number of important cases: one-ports with RLC components with or without transformers, multiports with RLC plus transformers with or without gyrators (see [39], [40]). The inerter allows a corresponding answer in the mechanical domain for one-ports comprising springs, dampers and inerters (Section 17). The "overall effect" can be interpreted abstractly as the set of impedances generated by the class, or as the value of a performance measure optimised over that class, or as a particular outcome such as a zero of transmission at a given frequency to eliminate the effect of vibrations. The reply also hints at similar ideas in the field of control systems that place emphasis on the notion of interconnection, e.g. the behavioural theory of Willems [52].

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