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Symmetries of the Dirac operator with skew-symmetric torsion

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Abstract

In this paper, we consider the symmetries of the Dirac operator derived from a connection with skew-symmetric torsion, ∇^T. We find that the generalized conformal Killing–Yano tensors give rise to symmetry operators of the massless Dirac equation, provided an explicitly given anomaly vanishes. We show that this gives rise to symmetries of the Dirac operator in the case of strong Kähler with torsion (KT) and strong hyper-Kähler with torsion (HKT) manifolds.

1 Introduction

In the study of symmetries of pseudo-Riemannian manifolds, an important role is played by conformal Killing–Yano tensors [1, 2]. These generalize conformal Killing vectors to higher rank anti-symmetric tensors and have many elegant properties which can be exploited both in mathematical and physical contexts. We refer the reader to four recent PhD dissertations and the references therein [3].

Recently, there has been interest in the generalization of the conformal Killing–Yano tensors to pseudo-Riemannian manifolds with skew-symmetric torsion first introduced in [2], then rediscovered in [4, 5]. Such manifolds occur in superstring theories where the torsion form may be

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identified (up to a factor) with a 3-form field occurring naturally in the theory. Geometries with torsion are also of interest in the context of non-integrable special Riemannian geometries [6]. Important examples include Kähler with torsion (KT) and hyper Kähler with torsion (HKT) manifolds [7, 8], which have applications across mathematics and physics.

A key property of conformal Killing–Yano tensors in the absence of torsion is that one may construct from them symmetry operators for the massless Dirac equation [9, 10, 11]. Conversely any first-order symmetry operator of the massless Dirac equation is constructed from a conformal Killing–Yano tensor [12, 13]. In this paper, we will consider the question of when a *generalized conformal Killing–Yano* (GCKY) tensor with respect to a connection with torsion T gives rise to a symmetry of an appropriate massless Dirac equation with torsion. For reasons which we will motivate below, the appropriate Dirac operator to consider is not that constructed from the connection on the spin bundle with torsion T, which we denote ∇^T, but that constructed from the connection with torsion $T/3$, which we denote $\mathcal{D}$ and can relate to the Dirac operator of the connection ∇^T by

$$\mathcal{D} = D^T + \frac{T}{2}\,. \tag{1.1}$$

This 'modified' Dirac operator is that introduced by Bismut in [14]. One may alternatively choose to view our result as relating the symmetries of the Dirac operator with torsion T to the GCKY tensors of the connection with torsion $3T$.

We now state the main theorem of the paper, see subsequent sections for conventions.

Theorem 1.1. *Let ω be a generalized conformal Killing-Yano (GCKY) p–form obeying*

$$\nabla^T_X\omega - \frac{1}{p+1}X\lrcorner\, d^T\omega + \frac{1}{n-p+1}X^\flat\wedge\delta^T\omega = 0, \tag{1.2}$$

then the operator

$$L_\omega = e^a\omega\nabla^T_{X_a} + \frac{p}{p+1}d^T\omega - \frac{n-p}{n-p+1}\delta^T\omega + \frac{1}{2}T\omega \tag{1.3}$$

satisfies

$$\mathcal{D}L_\omega = \omega\mathcal{D}^2 + \frac{(-1)^p}{p+1}d^T\omega\mathcal{D} + \frac{(-1)^p}{n-p+1}\delta^T\omega\mathcal{D} - A\,. \tag{1.4}$$

The anomaly A, given explicitly below, contains no term where a derivative acts on a spinor and depends on T and ω. In the case where A vanishes, L_ω is a symmetry operator for the massless Dirac equation.

We also exhibit this operator in terms of γ-matrices, a formalism perhaps more familiar to physicists. In the cases where either $d^T\omega = 0$ or $\delta^T\omega = 0$, we can exhibit operators which (anti)-commute with the Dirac operator.

Corollary 1.2. *Let ω be a generalized Killing-Yano (GKY) p-form such that A vanishes. Then there exists an operator K_ω such that*

$$\mathcal{D}K_\omega + (-1)^pK_\omega\mathcal{D} = 0. \tag{1.5}$$

Corollary 1.3. *Let ω be a generalized closed conformal Killing-Yano (GCCKY) p-form such that A vanishes. Then there exists an operator M_ω such that*

$$\mathcal{D}M_\omega - (-1)^p M_\omega \mathcal{D} = 0. \tag{1.6}$$

The anomaly term of Theorem 1.1 obviously vanishes for vanishing torsion, by the result of Benn and Charlton [10]. It is also shown to vanish in [15] for the case of the GCKY tensors exhibited by the charged Kerr-NUT metrics [16] (including the Kerr-Sen black hole spacetime [17]). The anomaly also vanishes in the case of strong Kähler with torsion (KT) and strong hyper-Kähler with torsion (HKT) metrics, giving the result:

Corollary 1.4. *A strong KT metric admits one operator and a strong HKT metric three operators which commute with the modified Dirac operator $\mathcal{D}$.*

In section 2 we introduce our notation and conventions and derive some basic results which we require in the proof of Theorem 1.1. Section 3 introduces the GCKY tensors, together with some of their properties. The proof of the main theorem takes up section 4.

2 Preliminaries

We assume that we work on (M^n, g), a pseudo-Riemannian spin manifold. It is convenient to denote by $\{X_a\}$ a local orthonormal basis for TM and by $\{e^a\}$ the dual basis for T^*M. We also define

$$\eta_{ab} = g(X_a, X_b), \qquad \eta^{ab} = \left(\eta^{-1}\right)^{ab}, \qquad X^a = \eta^{ab} X_b, \qquad e_a = \eta_{ab} e^b. \tag{2.1}$$

The matrix with entries η_{ab} will of course be diagonal, with entries ± 1. Throughout we sum over repeated indices.

2.1 The Clifford algebra

We will work with the conventions of Benn and Tucker [19], to whom we refer the reader for more details of this formalism. We recall that differential forms may be identified with elements of the Clifford algebra. Denoting the Clifford product by juxtaposition, our convention[1] is that for a 1-form α and p-form ω

$$\begin{aligned} \alpha\omega &= \alpha \wedge \omega + \alpha^\sharp \lrcorner\, \omega\,, \\ \omega\alpha &= (-1)^p \left(\alpha \wedge \omega - \alpha^\sharp \lrcorner\, \omega\right) . \end{aligned} \tag{2.2}$$

Repeated application of this rule allows us to construct the Clifford product between forms of arbitrary degree. Appendix A.1 contains, for convenience, expressions for the products and some (anti)-commutators. In order to state these products compactly, we introduce the contracted wedge product, defined inductively by

$$\alpha \underset{0}{\wedge} \beta = \alpha \wedge \beta, \qquad \alpha \underset{n}{\wedge} \beta = (X_a \lrcorner\, \alpha) \underset{n-1}{\wedge} (X^a \lrcorner\, \beta)\,. \tag{2.3}$$

[1] note that in the mathematical literature, a different convention with the signs of the interior derivatives reversed is often used

For those more familiar with the Clifford algebra in terms of γ-matrices, we note that the identification between differential forms and elements of the Clifford algebra can be expressed as

$$\frac{1}{p!}\omega_{a_1\dots a_p}e^{a_1}\wedge\cdots\wedge e^{a_p}\longrightarrow\frac{1}{p!}\omega_{a_1\dots a_p}\gamma^{[a_1}\cdots\gamma^{a_p]}. \tag{2.4}$$

The relations (2.2) are equivalent to the usual anti-commutator for the γ-matrices:

$$\gamma^a\gamma^b+\gamma^b\gamma^a=2\eta^{ab}. \tag{2.5}$$

We will use the short-hand $e^{a_1\dots a_p}=e^{[a_1}\cdots e^{a_p]}=e^{a_1}\wedge\cdots\wedge e^{a_p}$.

2.2 The connection with torsion

We wish to consider a connection with totally anti-symmetric torsion $T\in\Omega^3(M)$ which is defined, for arbitrary vectors X and Y, in terms of the Levi-Civita connection ∇ to be

$$\nabla^T_X Y=\nabla_X Y+\frac{1}{2}T(X,Y,X_a)X^a\,. \tag{2.6}$$

Acting on a form, we find that

$$\nabla^T_X\omega=\nabla_X\omega+\frac{1}{2}(X\lrcorner T)\underset{1}{\wedge}\omega\,. \tag{2.7}$$

This connection is metric, so obeys $X(g(Y,Z))=g(\nabla^T_X Y,Z)+g(Y,\nabla^T_X Z)$ and in addition has the same geodesics as the Levi-Civita connection since $\nabla^T_X X=\nabla_X X$.

It is useful to define two operators on forms related to the exterior derivative and its adjoint as

$$\begin{aligned} d^T\omega &= e^a\wedge\nabla^T_{X_a}\omega=d\omega-T\underset{1}{\wedge}\omega\,,\\ \delta^T\omega &= -X^a\lrcorner\nabla^T_{X_a}=\delta\omega-\frac{1}{2}T\underset{2}{\wedge}\omega. \end{aligned} \tag{2.8}$$

These respectively raise and lower the degree of a form by one. Unlike the standard d,δ they are not nilpotent in general.

In calculations it is often convenient to consider a basis which is parallel at a point p. For a connection with non-zero torsion, such a basis does not necessarily exist. It is always possible to arrange that at p:

$$\begin{aligned} \nabla^T_{X_a}X_b &= \frac{1}{2}T(X_a,X_b,X_c)X^c\,,\\ \nabla^T_{X_a}e^b &= \frac{1}{2}T(X_a,X^b,X_c)e^c\,,\\ [X_a,X_b] &= 0\,. \end{aligned} \tag{2.9}$$

We will assume from now on that we work with such a basis.

2.2.1 Curvature

The curvature operator is defined as usual by

$$R(X,Y)\omega = \left(\nabla^T_X\nabla^T_Y - \nabla^T_Y\nabla^T_X - \nabla^T_{[X,Y]}\right)\omega. \tag{2.10}$$

In Appendix A.3 we prove versions of the first Bianchi identity for the curvature operator.

For any pair of vectors X, Y, we introduce the following 2-form, following [19]

$$\mathcal{R}_{X,Y} = \frac{1}{4}(X_a \lrcorner R(X,Y)e_b)\, e^{ab} = -\frac{1}{4} X \lrcorner Y \lrcorner R_{ab} e^{ab}. \tag{2.11}$$

Here R_{ab} are the usual curvature 2-forms. The reason for introducing this 2-form is as follows. Using the results of Appendix A.1 it can be readily demonstrated that $[\mathcal{R}_{X,Y}, e^a] = R(X,Y)e^a$, where the commutator is taken in the Clifford sense. One may extend this result by linearity to show that for any form ω, the curvature operator is related to the 2-form $\mathcal{R}_{X,Y}$ by

$$[\mathcal{R}_{X,Y}, \omega] = R(X,Y)\omega. \tag{2.12}$$

In Appendix A.3 we establish the following Bianchi identity for $\mathcal{R}_{X,Y}$:

$$e^{ab}\mathcal{R}_{X_a,X_b} = -\frac{3}{2} d^T T - T \underset{1}{\wedge} T + \delta^T T - \frac{1}{2} s\,. \tag{2.13}$$

Here s is the scalar curvature of the connection with torsion, which we take to be defined by $s = -X^a \lrcorner R(X_a, X_b)e^b$.

2.2.2 The lift to the spinor bundle

We will consider the natural lift of the connection (2.6) to the spinor bundle, given by

$$\nabla^T_X \psi = \nabla_X \psi - \frac{1}{4} X \lrcorner T\psi, \tag{2.14}$$

where ψ is a spinor field (i.e. a section of the spinor bundle) and ∇_X is the usual lift of the Levi-Civita connection. This connection is a derivation over the Clifford product between forms and spinors,

$$\nabla^T_X(\omega\psi) = (\nabla^T_X\omega)\psi + \omega\nabla^T_X\psi, \tag{2.15}$$

for any form ω and spinor field ψ. It is shown in [19] that the curvature operator acting on a spinor is given simply by

$$R(X,Y)\psi = \left(\nabla^T_X\nabla^T_Y - \nabla^T_Y\nabla^T_X - \nabla^T_{[X,Y]}\right)\psi = \mathcal{R}_{X,Y}\psi\,. \tag{2.16}$$

2.3 The Dirac operator

The 'naïve' or 'bare' Dirac operator defined by the spinor covariant derivative given above is

$$D^T = e^a \nabla^T_{X_a} = D - \frac{3}{4} T, \tag{2.17}$$

where D is the Dirac operator of the Levi-Civita connection. For the purposes of the physics of spinning particles, among other reasons, it is convenient to consider a modified operator of the form $D^T + \alpha T$ which satisfies a Schrödinger-Lichnerowicz type formula [20].

Making use of the Clifford algebra, together with (2.13) one may show that

$$(D^T)^2 = -X^a \lrcorner T \nabla^T_{X_a} - \Delta^T + \frac{1}{2} e^{ab} \mathcal{R}_{X_a, X_b}, \tag{2.18}$$

where we have introduced the spinor Laplacian

$$\Delta^T = -\nabla^T_{X_a} \nabla^T_{X^a} + \nabla^T_{\nabla^T_{X_a} X^a}. \tag{2.19}$$

This suggests that a field obeying the massless Dirac equation will not obey the natural spinor Klein-Gordon equation associated with the connection ∇^T as there is an additional term linear in derivatives of the field. In order to remove this unwanted term, we note that

$$T D^T + D^T T = 2 X_a \lrcorner T \nabla^T_{X_a} + d^T T - \delta^T T, \tag{2.20}$$

so that

$$\begin{aligned} -(D^T + \alpha T)^2 \;\; = \;\; & \Delta^T + (1 - 2\alpha) X_a \lrcorner T \nabla^T_{X_a} + (\alpha - \frac{1}{2}) \delta^T T + (\frac{3}{4} - \alpha) d^T T \\ & + (\frac{1}{2} - \alpha^2) T \underset{1}{\wedge} T + \frac{\alpha^2}{6} \|T\|^2 + \frac{s}{4}, \end{aligned} \tag{2.21}$$

where we have made use of the results of Appendix A.1 to express the Clifford product of T with itself.

We may think of (2.21) as a formula relating the square of a Dirac operator with torsion $(1 - 4\alpha/3)T$ to derivatives involving the connection T. Clearly the term linear in derivatives will vanish if we take $\alpha = 1/2$. It is therefore natural when we have a connection with torsion T to consider the modified Dirac operator [14, 6, 28]

$$\mathcal{D} \equiv D^T + \frac{T}{2} = D^{T/3}. \tag{2.22}$$

In terms of the Levi-Civita connection and with the help of correspondence (2.4) this reads

$$\mathcal{D} = D - \frac{1}{4} T = \gamma^a \nabla_a - \frac{1}{24} T_{abc} \gamma^{abc}. \tag{2.23}$$

Such an operator satisfies the elegant relation

$$\mathcal{D}^2 = -\Delta^T - \frac{dT}{4} - \frac{s}{4} - \frac{\|T\|^2}{24}. \tag{2.24}$$

Up to quantum corrections this gives the Hamiltonian considered at the spinning particle level in [21, 22, 4]. It will be convenient for us to have the alternative form

$$\mathcal{D}^2 = -\Delta^T + \frac{1}{2} e^{ab} \mathcal{R}_{X_a X_b} + \frac{1}{2}(d^T T - \delta^T T) + \frac{1}{4} T^2. \tag{2.25}$$

3 Generalized conformal Killing–Yano tensors

A p-form ω is called a *generalized conformal Killing-Yano* (GCKY) tensor [2, 4] if it obeys the conformal Killing–Yano equation with torsion:

$$\nabla^T_X \omega - \frac{1}{p+1} X \lrcorner\, d^T \omega + \frac{1}{n-p+1} X^\flat \wedge \delta^T \omega = 0 \tag{3.1}$$

for any $X \in TM$. If $\delta^T\omega = 0$, ω is called a generalized Killing–Yano (GKY) tensor, while if $d^T\omega = 0$ it is a generalized closed conformal Killing–Yano (GCCKY) tensor. Eq. (3.1) is conformally invariant and also invariant under $\omega \to *\omega$. The Hodge dual interchanges GKY and GCCKY tensors.

A key property of GCCKY tensors is that they form a (graded) algebra under the wedge product [4, 26]. We summarize this in a lemma

Lemma 3.1. *If ω_1 and ω_2 are GCCKY tensors then so is $\omega_1 \wedge \omega_2$. Further, the operator*

$$\delta^T \circ (n - \pi + 1)^{-1}, \tag{3.2}$$

where π is the linear map taking a p-form α to $p\alpha$, is a graded derivation on the exterior algebra of GCCKY tensors.

Proof. First note that $\omega \in \Omega^p M$ is GCCKY if and only if there exists an $\tilde{\omega}$ such that $\nabla^T_X \omega = X^\flat \wedge \tilde{\omega}$ and if this holds, $\tilde{\omega} = -(n-p+1)^{-1}\delta^T\omega$. Suppose ω_i are GCCKY forms of degree p_i. We calculate

$$\nabla^T_X(\omega_1 \wedge \omega_2) = \nabla^T_X \omega_1 \wedge \omega_2 + \omega_1 \wedge \nabla^T_X \omega_2 = -X^\flat \wedge \left(\frac{\delta^T \omega_1 \wedge \omega_2}{n - p_1 + 1} + (-1)^{p_1} \frac{\omega_1 \wedge \delta^T \omega_2}{n - p_2 + 1} \right).$$

□

3.1 Integrability conditions

We will require integrability conditions for the GCKY equation. By differentiating (3.1) we find the following conditions, generalizing results of Semmelmann [18].

$$\Delta^T \omega = \frac{1}{p+1} \delta^T d^T \omega + \frac{1}{n-p+1} d^T \delta^T \omega \,, \tag{3.3}$$

$$e^b \wedge X^a \lrcorner\, R(X_a, X_b)\omega = \frac{p}{p+1} \delta^T d^T \omega + \frac{n-p}{n-p+1} d^T \delta^T \omega + (X_b \lrcorner\, T) \underset{1}{\wedge} \nabla^T_{X_b} \omega \,. \tag{3.4}$$

Here $\Delta^T = -\nabla^T_{X_a} \nabla^T_{X^a} + \nabla^T_{\nabla^T_{X_a} X^a}$ is the Bochner Laplacian derived from the connection ∇^T. These are consistent with the Weitzenböck identity in the presence of torsion:

$$\delta^T d^T + d^T \delta^T = \Delta^T + e^b \wedge X^a \lrcorner\, R(X_a, X_b) - (X_b \lrcorner\, T) \underset{1}{\wedge} \nabla^T_{X_b} \,. \tag{3.5}$$

The condition (3.4) was already known to Yano and Bochner [2] in the case where $\delta^T\omega$ vanishes.

At this stage it will be convenient to deduce an identity relating the curvature form $\mathcal{R}_{XY}$ to the integrability condition. We consider

$$
\begin{aligned}
\frac{1}{2}[e^{ab},\omega]\mathcal{R}_{X_aX_b} &- \Big(\frac{p}{p+1}\delta^T d^T\omega + \frac{n-p}{n-p+1}d^T\delta^T\omega\Big)\\
&= \frac{1}{2}[e^{ab},\omega]\mathcal{R}_{X_aX_b} - e^b\wedge X^a\lrcorner R(X_a,X_b)\omega + (X_b\lrcorner T)\underset{1}{\wedge}\nabla^T_{X_b}\omega\\
&= \frac{1}{2}[e^{ab},\omega]\mathcal{R}_{X_aX_b} + \frac{1}{2}e^{ab}[\mathcal{R}_{X_aX_b},\omega] - \frac{1}{2}e^{ab}\wedge R(X_a,X_b)\omega\\
&\quad -\frac{1}{2}X^a\lrcorner X^b\lrcorner R(X_a,X_b)\omega + (X_b\lrcorner T)\underset{1}{\wedge}\nabla^T_{X_b}\omega\\
&= \frac{1}{2}[e^{ab}\mathcal{R}_{X_aX_b},\omega] - d^Td^T\omega - (X^a\lrcorner T)\wedge\nabla^T_{X_a}\omega - \delta^T\delta^T\omega + \frac{1}{2}(X^a\lrcorner T)\underset{2}{\wedge}\nabla^T_{X_a}\omega\\
&\quad +(X_b\lrcorner T)\underset{1}{\wedge}\nabla^T_{X_b}\omega\,.
\end{aligned}
$$

Here we have made liberal use of the results of appendices A.1 and A.3. Putting this together with (2.13) we find

$$
\begin{aligned}
\frac{1}{2}[e^{ab},\omega]\mathcal{R}_{X_aX_b} &- \Big(\frac{p}{p+1}\delta^T d^T\omega + \frac{n-p}{n-p+1}d^T\delta^T\omega\Big)\\
&= \frac{1}{2}\Big[-\frac{3}{2}d^TT - T\underset{1}{\wedge}T + \delta^TT,\omega\Big] - d^Td^T\omega - \delta^T\delta^T\omega - (X^a\lrcorner T)\nabla^T_{X_a}\omega\,. \qquad (3.6)
\end{aligned}
$$

4 Symmetry operators

4.1 Massless Dirac

We are now ready to construct a symmetry operator for the Dirac operator. Our goal is to show that (up to anomaly terms) the operator

$$
L_\omega = e^a\omega\nabla^T_{X_a} + \frac{p}{p+1}d^T\omega - \frac{n-p}{n-p+1}\delta^T\omega + \frac{1}{2}T\omega \qquad (4.1)
$$

satisfies

$$
\mathcal{D}L_\omega = \omega\mathcal{D}^2 + \frac{(-1)^p}{p+1}d^T\omega\mathcal{D} + \frac{(-1)^p}{n-p+1}\delta^T\omega\mathcal{D}, \qquad (4.2)
$$

provided ω is a GCKY form and to calculate the anomaly terms. This means that when the anomaly terms (given explicitly by (4.7) below) vanish, the operator L_ω gives an *on-shell* ($\mathcal{D}\psi = 0$) symmetry operator for $\mathcal{D}$.

In the rest of this section, we shall sketch the proof of these assertions. It essentially consists of commuting $\mathcal{D}$ through the operator L_ω. We calculate

$$
\begin{aligned}
\mathcal{D}e^a\omega\nabla^T_{X_a} &= -\omega\Delta^T + \frac{1}{2}e^{ab}\omega\mathcal{R}_{X_aX_b} + 2\nabla^T_{X_a}\omega\nabla^T_{X_a} - e^b(d^T\omega - \delta^T\omega)\nabla^T_{X_b} - X^b \lrcorner\, T\omega\nabla^T_{X_b} + \frac{1}{2}Te^b\omega\nabla^T_{X_b}\,, \\
\mathcal{D}d^T\omega &= (d^Td^T\omega - \delta^Td^T\omega) + e^a d^T\omega\nabla^T_{X_a} + \frac{1}{2}Td^T\omega\,, \\
\mathcal{D}\delta^T\omega &= (d^T\delta^T\omega - \delta^T\delta^T\omega) + e^a\delta^T\omega\nabla^T_{X_a} + \frac{1}{2}T\delta^T\omega\,, \\
\mathcal{D}T\omega &= (d^TT - \delta^TT)\omega + e^aT\nabla^T_{X_a}\omega + e^aT\omega\nabla^T_{X_a} + \frac{1}{2}T^2\omega\,,
\end{aligned}
\tag{4.3}
$$

We need to simplify the sum of these four terms (with appropriate coefficients as given by (4.1)). Let us first gather the terms with one derivative acting on a spinor

$$
\begin{aligned}
&\Big(2\nabla^T_{X_a}\omega \;-\; \frac{1}{p+1}e^ad^T\omega + \frac{1}{n-p+1}e^a\delta^T\omega\Big)\nabla^T_{X_a} + \big(-X^b\lrcorner\, T + \frac{1}{2}Te^b + \frac{1}{2}e^bT\big)\omega\nabla^T_{X_b} \\
&\quad = \; 2\Big(\nabla^T_{X_a}\omega - \frac{X^a\lrcorner\, d^T\omega}{p+1} + \frac{e^a\wedge\delta^T\omega}{n-p+1}\Big)\nabla^T_{X_a} + \frac{(-1)^p}{p+1}d^T\omega e^a\nabla^T_{X_a} + \frac{(-1)^p}{n-p+1}\delta^T\omega e^a\nabla^T_{X_a} \\
&\quad = \; \frac{(-1)^p}{p+1}d^T\omega\mathcal{D} + \frac{(-1)^p}{n-p+1}\delta^T\omega\mathcal{D} - \frac{(-1)^p}{2(p+1)}d^T\omega T - \frac{(-1)^p}{2(n-p+1)}\delta^T\omega T\,,
\end{aligned}
\tag{4.4}
$$

where in order to remove differential terms not proportional to $\mathcal{D}$ we impose the GCKY equation (3.1). This condition on ω arises also at the spinning particle level [4]. So we have

$$
\begin{aligned}
\mathcal{D}L_\omega \;=\; &\Big(2\nabla^T_{X_a}\omega - \frac{1}{p+1}e^ad^T\omega + \frac{1}{n-p+1}e^a\delta^T\omega\Big)\nabla^T_{X_a} + \big(-X^b\lrcorner\, T + \frac{1}{2}Te^b + \frac{1}{2}e^bT\big)\omega\nabla^T_{X_b} \\
&-\omega\Delta^T + \frac{1}{2}e^{ab}\omega\mathcal{R}_{X_aX_b} + \frac{1}{2}(d^TT - \delta^TT)\omega + \frac{1}{4}T^2\omega \\
&+\frac{1}{2}e^aT\nabla^T_{X_a}\omega + \frac{p}{2(p+1)}Td^T\omega - \frac{n-p}{2(n-p+1)}T\delta^T\omega \\
&+\frac{p}{p+1}d^Td^T\omega + \frac{n-p}{n-p+1}\delta^T\delta^T\omega - \Big(\frac{p}{p+1}\delta^Td^T\omega + \frac{n-p}{n-p+1}d^T\delta^T\omega\Big) \\
= \; &\frac{(-1)^p}{p+1}d^T\omega\mathcal{D} + \frac{(-1)^p}{n-p+1}\delta^T\omega\mathcal{D} - \frac{(-1)^p}{2(p+1)}d^T\omega T - \frac{(-1)^p}{2(n-p+1)}\delta^T\omega T \\
&+\omega\mathcal{D}^2 + \frac{1}{2}[e^{ab},\omega]\mathcal{R}_{X_aX_b} + \frac{1}{2}\big[d^TT - \delta^TT + \frac{1}{2}T^2,\omega\big] \\
&+X^a\lrcorner\, T\nabla^T_{X_a}\omega - \frac{Td^T\omega}{2(p+1)} + \frac{T\delta^T\omega}{2(n-p+1)} \\
&+\frac{p}{p+1}d^Td^T\omega + \frac{n-p}{n-p+1}\delta^T\delta^T\omega - \Big(\frac{p}{p+1}\delta^Td^T\omega + \frac{n-p}{n-p+1}d^T\delta^T\omega\Big)\,,
\end{aligned}
$$

where we have used (2.25). Consolidating, we obtain

$$
\mathcal{D}L_\omega = \omega\mathcal{D}^2 + \frac{(-1)^p}{p+1}d^T\omega\mathcal{D} + \frac{(-1)^p}{n-p+1}\delta^T\omega\mathcal{D} - A\,,
\tag{4.5}
$$

where we define A to be the anomaly, i.e., the obstruction to L_ω giving a symmetry operator of the massless Dirac equation. This completes the proof of Theorem 1.1.

In order to find a compact expression for the anomaly A, we make use of the integrability condition (3.6) to find

$$A = \frac{1}{2}\left(\frac{[T, d^T\omega]_p}{p+1} - \frac{[T, \delta^T\omega]_{p+1}}{n-p+1}\right) + \frac{1}{4}[dT, \omega] + \left(\frac{d^Td^T\omega}{p+1} + \frac{\delta^T\delta^T\omega}{n-p+1}\right), \tag{4.6}$$

using the bracket notation of appendix A.1. Expanding the commutators gives

$$A = \frac{1}{p+1}\left[d(d^T\omega) - \frac{1}{6}T \underset{3}{\wedge} d^T\omega\right] + \frac{\delta(\delta^T\omega) - T \wedge \delta^T\omega}{n-p+1} - \frac{1}{2}dT \underset{1}{\wedge} \omega + \frac{1}{12}dT \underset{3}{\wedge} \omega\,. \tag{4.7}$$

The anomaly A splits as $A^{(cl)}$ and $A^{(q)}$ a $(p+2)$-form and a $(p-2)$-form, respectively, so that $A = A^{(cl)} + A^{(q)}$. In order that A vanishes, both parts must vanish separately. We have:

$$A^{(cl)} = \frac{d(d^T\omega)}{p+1} - \frac{T \wedge \delta^T\omega}{n-p+1} - \frac{1}{2}dT \underset{1}{\wedge} \omega\,, \tag{4.8}$$

$$A^{(q)} = \frac{\delta(\delta^T\omega)}{n-p+1} - \frac{1}{6(p+1)}T \underset{3}{\wedge} d^T\omega + \frac{1}{12}dT \underset{3}{\wedge} \omega\,. \tag{4.9}$$

We shall sometimes refer to $A^{(cl)}$ as the 'classical' anomaly, as this anomaly shows up even in the semi-classical spinning particle limit.

For convenience, we state some alternative forms of the operator L_ω. Re-writing in terms of the Levi-Civita connection, $\nabla_{X_a} = \nabla^T_{X_a} + \frac{1}{4}X_a \lrcorner\, T$, we find

$$\begin{aligned} L_\omega &= (X^a \lrcorner\, \omega + e^a \wedge \omega)\nabla_{X_a} + \frac{p}{p+1}d\omega - \frac{n-p}{n-p+1}\delta\omega \\ &\quad -\frac{1}{4}T \wedge \omega + \frac{3-p}{4(p+1)}T \underset{1}{\wedge} \omega + \frac{n-p-3}{8(n-p+1)}T \underset{2}{\wedge} \omega + \frac{1}{24}T \underset{3}{\wedge} \omega\,. \end{aligned} \tag{4.10}$$

Using further the correspondence (2.4), one can rewrite this operator in 'gamma-matrix notations'. Defining $\tilde{L}_\omega = (p-1)!L_\omega$, we find

$$\begin{aligned} \tilde{L}_\omega &= \left[\omega^a{}_{b_1\ldots b_{p-1}}\gamma^{b_1\ldots b_{p-1}} + \frac{1}{p(p+1)}\omega_{b_1\ldots b_p}\gamma^{ab_1\ldots b_p}\right]\nabla_a + \frac{1}{(p+1)^2}(d\omega)_{b_1\ldots b_{p+1}}\gamma^{b_1\ldots b_{p+1}} \\ &\quad -\frac{n-p}{n-p+1}(\delta\omega)_{b_1\ldots b_{p-1}}\gamma^{b_1\ldots b_{p-1}} - \frac{1}{24}T_{b_1b_2b_3}\omega_{b_4\ldots b_{p+3}}\gamma^{b_1\ldots b_{p+3}} + \frac{3-p}{8(p+1)}T^a{}_{b_1b_2}\omega_{ab_3\ldots b_{p+1}}\gamma^{b_1\ldots b_{p+1}} \\ &\quad +\frac{(n-p-3)(p-1)}{8(n-p+1)}T^{ab}{}_{b_1}\omega_{abb_2\ldots b_{p-1}}\gamma^{b_1\ldots b_{p-1}} + \frac{(p-1)(p-2)}{24}T^{abc}\omega_{abcb_1\ldots b_{p-3}}\gamma^{b_1\ldots b_{p-3}}\,. \end{aligned} \tag{4.11}$$

4.2 (Anti)-Commuting operators for massive Dirac

We now deduce Corollaries 1.2 and 1.3. Suppose now that the anomaly A vanishes, then

$$\mathcal{D}L_\omega = \omega\mathcal{D}^2 + \frac{(-1)^p}{p+1}d^T\omega\mathcal{D} + \frac{(-1)^p}{n-p+1}\delta^T\omega\mathcal{D}\,. \tag{4.12}$$

Let us further define

$$K_\omega \equiv L_\omega - (-1)^p \omega \mathcal{D} = 2X^a \lrcorner\, \omega \nabla^T_{X_a} + \frac{p}{p+1} d^T\omega - \frac{n-p}{n-p+1}\delta^T\omega + T \underset{1}{\wedge} \omega - \frac{1}{6} T \underset{3}{\wedge} \omega\,, \quad (4.13)$$

$$M_\omega \equiv L_\omega + (-1)^p \omega \mathcal{D} = 2e^a \wedge \omega \nabla^T_{X_a} + \frac{p}{p+1} d^T\omega - \frac{n-p}{n-p+1}\delta^T\omega + T \wedge \omega - \frac{1}{2} T \underset{2}{\wedge} \omega\,. \quad (4.14)$$

Then, by using

$$\begin{aligned}\mathcal{D}\omega &= \Big(e^a \nabla^T_{X_a} + \frac{1}{2}T\Big)\omega = e^a\omega + d^T\omega - \delta^T\omega + \frac{1}{2}T\omega \\ &= (-1)^p\omega\mathcal{D} + 2X^a \lrcorner\, \omega\nabla^T_{X_a} + d^T\omega - \delta^T\omega + \frac{1}{2}[T,\omega]_{p+1} \\ &= -(-1)^p\omega\mathcal{D} + 2e^a \wedge \omega\nabla^T_{X_a} + d^T\omega - \delta^T\omega + \frac{1}{2}[T,\omega]_{p}\,,\end{aligned}$$

we find that these new operators satisfy

$$[\mathcal{D}, K_\omega]_p = \frac{2(-1)^p}{n-p+1}\delta^T\omega\mathcal{D}\,, \qquad [\mathcal{D}, M_\omega]_{p+1} = \frac{2(-1)^p}{p+1} d^T\omega\mathcal{D}\,. \quad (4.15)$$

We note that more generally, if the anomaly A does not vanish, then it will appear as the right-hand-side of both of these commutators. Let us now consider the cases when ω is a GKY and GCCKY separately.

4.2.1 GKY and corresponding operator

When ω is a GKY tensor ($\delta^T\omega = 0$), the conditions for vanishing of the anomalies $A^{(cl)}$ and $A^{(q)}$ reduce to

$$d(T \underset{1}{\wedge} \omega) + \frac{p+1}{2}\, dT \underset{1}{\wedge} \omega = 0\,, \qquad dT \underset{3}{\wedge} \omega - \frac{2}{p+1} T \underset{3}{\wedge} d^T\omega = 0\,, \quad (4.16)$$

and the operator K_ω, obeying $[\mathcal{D}, K_\omega]_p = 0$ becomes

$$K_\omega = 2X^a \lrcorner\, \omega\nabla^T_{X_a} + \frac{p}{p+1} d^T\omega + T \underset{1}{\wedge} \omega - \frac{1}{6} T \underset{3}{\wedge} \omega\,. \quad (4.17)$$

The first condition (4.16) is present already at the spinning particle level (see, e.g., [22] where it is derived for a GKY 2-form). It may be useful to rewrite K_ω in terms of the Levi-Civita connection. A calculation in the Clifford algebra gives

$$K_\omega = 2X^a \lrcorner\, \omega\nabla_{X_a} + \frac{p}{p+1} d\omega + \frac{1-p}{2(p+1)} T \underset{1}{\wedge} \omega - \frac{1}{2} T \underset{2}{\wedge} \omega + \frac{1}{12} T \underset{3}{\wedge} \omega\,. \quad (4.18)$$

Introducing $\tilde{K}_\omega \equiv K_\omega (p-1)!/2$, we find

$$\begin{aligned}\tilde{K}_\omega &= \omega^a{}_{b_1\dots b_{p-1}}\gamma^{b_1\dots b_{p-1}}\nabla_a + \frac{1}{2(p+1)^2}(d\omega)_{b_1\dots b_{p+1}}\gamma^{b_1\dots b_{p+1}} \\ &\quad + \frac{1-p}{8(p+1)} T^a{}_{b_1 b_2}\omega_{a b_3\dots b_{p+1}}\gamma^{b_1\dots b_{p+1}} - \frac{p-1}{4} T^{ab}{}_{b_1}\omega_{a b b_2\dots b_{p-1}}\gamma^{b_1\dots b_{p-1}} \\ &\quad + \frac{(p-1)(p-2)}{24} T^{abc}\omega_{abc b_1\dots b_{p-3}}\gamma^{b_1\dots b_{p-3}}\,. \end{aligned} \quad (4.19)$$

The first two terms correspond to symmetry operator discussed in [10, 11] in the absence of torsion. The third term is a 'leading' torsion correction, present already at the spinning particle level [4]. The last two terms are 'quantum corrections' due to the presence of torsion.

4.2.2 GCCKY and corresponding operator

When ω is a GCCKY tensor ($d^T\omega = 0$), the conditions for vanishing of the anomalies $A^{(cl)}$ and $A^{(q)}$ reduce to

$$A^{(cl)} = -\frac{T\wedge\delta^T\omega}{n-p+1} - \frac{1}{2}dT \underset{1}{\wedge}\omega = 0\,, \qquad A^{(q)} = \frac{-1}{2(n-p+1)}\delta(T\underset{2}{\wedge}\omega) + \frac{1}{12}dT\underset{3}{\wedge}\omega = 0\,, \tag{4.20}$$

and the operator M_ω, obeying $[\mathcal{D}, M_\omega]_{p+1} = 0$ becomes

$$M_\omega = 2e^a\wedge\omega\nabla^T_{X_a} - \frac{n-p}{n-p+1}\delta^T\omega + T\wedge\omega - \frac{1}{2}T\underset{2}{\wedge}\omega\,. \tag{4.21}$$

Again, we may express this operator in terms of the Levi-Civita connection as

$$M_\omega = 2e^a\wedge\omega\nabla_{X_a} - \frac{n-p}{n-p+1}\delta\omega - \frac{1}{2}T\wedge\omega + T\underset{1}{\wedge}\omega + \frac{n-p-1}{4(n-p+1)}T\underset{2}{\wedge}\omega\,, \tag{4.22}$$

and introducing $\tilde{M}_\omega \equiv M_\omega p!/2$, we find

$$\begin{aligned}\tilde{M}_\omega &= \omega_{b_1\dots b_p}\gamma^{ab_1\dots b_p}\nabla_a - \frac{p(n-p)}{2(n-p+1)}(\delta\omega)_{b_1\dots b_{p-1}}\gamma^{b_1\dots b_{p-1}}\\ &\quad -\frac{1}{24}T_{b_1b_2b_3}\omega_{b_4\dots b_{p+3}}\gamma^{b_1\dots b_{p+3}} + \frac{p}{4}T^a{}_{b_1b_2}\omega_{ab_3\dots b_{p+1}}\gamma^{b_1\dots b_{p+1}}\\ &\quad +\frac{p(p-1)(n-p-1)}{8(n-p+1)}T^{ab}{}_{b_1}\omega_{abb_2\dots b_{p-1}}\gamma^{b_1\dots b_{p-1}}\,.\end{aligned} \tag{4.23}$$

The first two terms correspond to symmetry operator discussed by [10] in the absence of torsion.

We noted above that GCCKY tensors form an algebra under the wedge product. Consider now $\omega = \omega_1\wedge\omega_2$, where ω_i are GCCKY tensors. From Lemma 3.1 we know that $\delta^T\circ(n-\pi+1)^{-1}$ is a graded derivation on the exterior algebra of GCCKY forms. Since T is a 3-form, we deduce that $T\wedge\delta^T\circ(n-\pi+1)^{-1}$ is a derivation. A short calculation shows that

$$dT\underset{1}{\wedge}(\omega_1\wedge\omega_2) = (dT\underset{1}{\wedge}\omega_1)\wedge\omega_2 + \omega_1\wedge(dT\underset{1}{\wedge}\omega_2) \tag{4.24}$$

so that

$$A^{(cl)}(\omega_1\wedge\omega_2) = A^{(cl)}(\omega_1)\wedge\omega_2 + \omega_1\wedge A^{(cl)}(\omega_2)\,. \tag{4.25}$$

Hence we have derived the following lemma:

Lemma 4.1. *Let ω_1, ω_2 are GCCKY tensors for which $A^{(cl)}(\omega_i) = 0$. Then the classical anomaly vanishes also for a GCCKY tensor $\omega = \omega_1\wedge\omega_2$, $A^{(cl)}(\omega_1\wedge\omega_2) = 0$.*

4.3 Examples

In [15] we show that the charged Kerr-NUT spacetimes of Chow [16] admit the towers of GCCKY and GKY forms for which the anomaly vanish. We additionally demonstrate the separability of the Dirac equation directly.

Another, more general, situation where the anomaly simplifies dramatically is the case of a Kähler with torsion (KT) or hyper-Kähler with torsion (HKT) manifold [7, 8]. Such manifolds have a range of applications in physics from supersymmetric sigma models with Wess-Zumino term [24] to the construction of solutions in five dimensional de Sitter supergravity [25].

A Hermitian manifold possesses an integrable complex structure J and a metric g compatible with J so that $g(JX, JY) = g(X, Y)$. The Kähler form F is defined to be $F(X, Y) = g(JX, Y)$. A result of Gauduchon [23] shows that there exists a unique connection ∇^T with skew-symmetric torsion such that

$$\nabla^T g = 0, \qquad \nabla^T J = 0. \tag{4.26}$$

A Hermitian manifold with this choice of connection is called a KT manifold. The holonomy of such a connection lies in $U(n)$. If the torsion vanishes, then the manifold is in fact Kähler.

The Kähler form is clearly preserved by the connection ∇^T and is therefore trivially a GCKY form. Since $d^T F = \delta^T F = 0$, the anomaly simplifies to give

$$A = \frac{1}{4}\left[dT, F\right]. \tag{4.27}$$

Which may or may not vanish. A KT-manifold is said to be *strong* if T is closed. In this case we find the anomaly vanishes and F generates a symmetry of the Dirac operator. In fact, since F is both GKY and GCCKY we have a symmetry of the massive Dirac equation.

A hyper-Hermitian manifold possesses three integrable complex structures I_1, I_2, I_3 satisfying

$$I_i I_j = -\delta_{ij} + \epsilon_{ijk} I_k \tag{4.28}$$

and a metric g compatible with the I_i so that

$$g(X, Y) = g(I_1 X, I_1 Y) = g(I_2 X, I_2 Y) = g(I_3 X, I_3 Y). \tag{4.29}$$

If there exists a connection with skew-symmetric torsion ∇^T such that

$$\nabla^T g = 0, \qquad \nabla^T I_1 = \nabla^T I_2 = \nabla^T I_3 = 0, \tag{4.30}$$

then ∇^T is a HKT-connection and M is a HKT manifold. Note that by Gauduchon's result, if ∇^T exists it is unique. The holonomy of the connection in this case lies in $Sp(n)$. We may define the triplet of Kähler forms by $F_i(X, Y) = g(I_i X, Y)$. Each Kähler form is again a GCKY, with anomaly

$$A_i = \frac{1}{4}\left[dT, F_i\right]. \tag{4.31}$$

As in the KT case, a *strong* HKT structure has a closed T and therefore the Kähler forms generate symmetries of the Dirac operator.

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A Appendix

A.1 Clifford Algebra Relations

In this appendix we gather various identities for the Clifford product used in the main text. We define the brackets as follows:

$$[\alpha,\beta]_p = \alpha\beta + (-1)^p \beta\alpha. \tag{A.1}$$

and will make use of the shorthand $[\alpha,\beta]_\pm = \alpha\beta \pm \beta\alpha$.

Let $\alpha \in \Omega^q(M)$ and $\beta \in \Omega^p(M)$, $q \le p$. Then, the following formulae can be derived from the defining Clifford algebra relations (2.2):[2]

$$\alpha\beta = \sum_{n=0}^{q} \frac{(-1)^{n(q-n)+[n/2]}}{n!} \alpha \underset{n}{\wedge} \beta\,, \quad \beta\alpha = (-1)^{pq} \sum_{n=0}^{q} \frac{(-1)^{n(q-n+1)+[n/2]}}{n!} \alpha \underset{n}{\wedge} \beta\,. \tag{A.2}$$

The corresponding brackets can be easily deduced. In particular, in the main text we shall use the following relations for $q \le 4$:

Let $\alpha \in \Omega^1(M), \beta \in \Omega^p(M)$, then

$$\alpha\beta = \alpha \underset{0}{\wedge} \beta + \alpha \underset{1}{\wedge} \beta, \tag{A.3}$$

$$\beta\alpha = (-1)^p(\alpha \underset{0}{\wedge} \beta - \alpha \underset{1}{\wedge} \beta), \tag{A.4}$$

$$[\alpha,\beta]_p = 2\alpha \underset{0}{\wedge} \beta, \tag{A.5}$$

$$[\alpha,\beta]_{p+1} = 2\alpha \underset{1}{\wedge} \beta. \tag{A.6}$$

[2] The derivation of these relations goes as follows. Let us calculate a coefficient standing by the term with n contraction, $(\alpha\beta)_n$. We have

$$\begin{aligned}(\alpha\beta)_n &= \alpha_{a_1\dots a_q}(e^{a_1\dots a_q}\beta)_n = (-1)^{n(q-n)}\alpha_{a_1\dots a_n a_{n+1}\dots a_q}(e^{a_{n+1}\dots a_q a_1\dots a_n}\beta)_n \\ &= \frac{(-1)^{n(q-n)}q!}{(q-n)!n!}\alpha_{a_1\dots a_n a_{n+1}\dots a_q} e^{a_{n+1}\dots a_q} \wedge X^{a_1} \lrcorner X^{a_2}\dots X^{a_n}\lrcorner\beta\,.\end{aligned}$$

Using now the fact that

$$\alpha_{a_1\dots a_n a_{n+1}\dots a_q} e^{a_{n+1}\dots a_q} = \frac{(q-n)!}{q!} X^{a_n}\lrcorner X^{a_{n-1}}\dots X^{a_1}\lrcorner\alpha = (-1)^{[n/2]}\frac{(q-n)!}{q!}X^{a_1}\lrcorner X^{a_2}\dots X^{a_n}\lrcorner\alpha\,,$$

we arrive at the first formula (A.2). The derivation of the second formula is analogous.

Let $\alpha \in \Omega^2(M), \beta \in \Omega^p(M)$, then

$$\begin{aligned}
\alpha\beta &= \alpha \underset{0}{\wedge} \beta - \alpha \underset{1}{\wedge} \beta - \tfrac{1}{2}\alpha \underset{2}{\wedge} \beta, && \text{(A.7)}\\
\beta\alpha &= \alpha \underset{0}{\wedge} \beta + \alpha \underset{1}{\wedge} \beta - \tfrac{1}{2}\alpha \underset{2}{\wedge} \beta, && \text{(A.8)}\\
[\alpha, \beta]_+ &= 2\alpha \underset{0}{\wedge} \beta - \alpha \underset{2}{\wedge} \beta, && \text{(A.9)}\\
[\alpha, \beta]_- &= -2\alpha \underset{1}{\wedge} \beta. && \text{(A.10)}
\end{aligned}$$

Let $\alpha \in \Omega^3(M), \beta \in \Omega^p(M)$, then

$$\begin{aligned}
\alpha\beta &= \alpha \underset{0}{\wedge} \beta + \alpha \underset{1}{\wedge} \beta - \tfrac{1}{2}\alpha \underset{2}{\wedge} \beta - \tfrac{1}{6}\alpha \underset{3}{\wedge} \beta, && \text{(A.11)}\\
\beta\alpha &= (-1)^p(\alpha \underset{0}{\wedge} \beta - \alpha \underset{1}{\wedge} \beta - \tfrac{1}{2}\alpha \underset{2}{\wedge} \beta + \tfrac{1}{6}\alpha \underset{3}{\wedge} \beta), && \text{(A.12)}\\
[\alpha, \beta]_p &= 2\alpha \underset{0}{\wedge} \beta - \alpha \underset{2}{\wedge} \beta, && \text{(A.13)}\\
[\alpha, \beta]_{p+1} &= 2\alpha \underset{1}{\wedge} \beta - \tfrac{1}{3}\alpha \underset{3}{\wedge} \beta. && \text{(A.14)}
\end{aligned}$$

Let $\alpha \in \Omega^4(M), \beta \in \Omega^p(M)$, then

$$\begin{aligned}
\alpha\beta &= \alpha \underset{0}{\wedge} \beta - \alpha \underset{1}{\wedge} \beta - \tfrac{1}{2}\alpha \underset{2}{\wedge} \beta + \tfrac{1}{6}\alpha \underset{3}{\wedge} \beta + \tfrac{1}{24}\alpha \underset{4}{\wedge} \beta, && \text{(A.15)}\\
\beta\alpha &= \alpha \underset{0}{\wedge} \beta + \alpha \underset{1}{\wedge} \beta - \tfrac{1}{2}\alpha \underset{2}{\wedge} \beta - \tfrac{1}{6}\alpha \underset{3}{\wedge} \beta + \tfrac{1}{24}\alpha \underset{4}{\wedge} \beta, && \text{(A.16)}\\
[\alpha, \beta]_+ &= 2\alpha \underset{0}{\wedge} \beta - \alpha \underset{2}{\wedge} \beta + \tfrac{1}{24}\alpha \underset{4}{\wedge} \beta, && \text{(A.17)}\\
[\alpha, \beta]_- &= 2\alpha \underset{1}{\wedge} \beta - \tfrac{1}{3}\alpha \underset{3}{\wedge} \beta. && \text{(A.18)}
\end{aligned}$$

A.2 Contracted Wedge Product Identities

In order to work with the contracted wedge product introduced in Section 2.1, it is convenient to derive some identities to help with calculations. We collect some results here. Firstly some algebraic results.

Lemma A.1. *Suppose $\alpha \in \Omega^p(M)$, then the interior derivative $X \lrcorner$ satisfies*

$$X \lrcorner (\alpha \underset{n}{\wedge} \beta) = (-1)^n (X \lrcorner \alpha) \underset{n}{\wedge} \beta + (-1)^p \alpha \underset{n}{\wedge} (X \lrcorner \beta)\,. \tag{A.19}$$

Proof. By induction. The $n = 0$ case is trivial. Suppose true for $n-1$, we have

$$\begin{aligned}
X \lrcorner (\alpha \underset{n}{\wedge} \beta) &= X \lrcorner (X_a \lrcorner \alpha \underset{n-1}{\wedge} X^a \lrcorner \beta)\\
&= (-1)^{n-1}(X \lrcorner X_a \lrcorner \alpha \underset{n-1}{\wedge} X^a \lrcorner \beta) + (-1)^{p-1}(X_a \lrcorner \alpha \underset{n-1}{\wedge} X \lrcorner X^a \lrcorner \beta)\\
&= (-1)^n (X \lrcorner \alpha) \underset{n}{\wedge} \beta + (-1)^p \alpha \underset{n}{\wedge} (X \lrcorner \beta)\,.
\end{aligned}$$

□

Lemma A.2. *Suppose* $\omega \in \Omega^1(M)$ *and* $\alpha \in \Omega^p(M)$, *then*

$$
\begin{aligned}
(\omega \wedge \alpha) \underset{n}{\wedge} \beta &= (-1)^n \omega \wedge (\alpha \underset{n}{\wedge} \beta) + n\, \alpha \underset{n-1}{\wedge} (\omega^\sharp \lrcorner\, \beta)\,, &\text{(A.20)}\\
\alpha \underset{n}{\wedge} (\omega \wedge \beta) &= (-1)^p \omega \wedge (\alpha \underset{n}{\wedge} \beta) + n\, (\omega^\sharp \lrcorner\, \alpha) \underset{n-1}{\wedge} \beta\,. &\text{(A.21)}
\end{aligned}
$$

Proof. Consider (A.20). By linearity it suffices to consider the case $\omega = e^a$. The $n = 0$ case is straightforward. Suppose true for $n-1$. Then

$$
\begin{aligned}
(e^a \wedge \alpha) \underset{n}{\wedge} \beta &= X_b \lrcorner\, (e^a \wedge \alpha) \underset{n-1}{\wedge} X^b \lrcorner\, \beta = \alpha \underset{n-1}{\wedge} X_a \lrcorner\, \beta - e^a \wedge (X_b \lrcorner\, \alpha) \underset{n-1}{\wedge} X^b \lrcorner\, \beta\\
&= \alpha \underset{n-1}{\wedge} X_a \lrcorner\, \beta - (-1)^{n-1} e^a \wedge (X_b \lrcorner\, \alpha \underset{n-1}{\wedge} X^b \lrcorner\, \beta) - (n-1) X_b \lrcorner\, \alpha \underset{n-1}{\wedge} X^a \lrcorner\, X^b \lrcorner\, \beta\\
&= (-1)^n e^a \wedge (\alpha \underset{n}{\wedge} \beta) + n\, \alpha \underset{n-1}{\wedge} (X^a \lrcorner\, \beta).
\end{aligned}
$$

An entirely analogous calculation establishes (A.21). □

We now consider the action of ∇^T_X, δ^T and d^T on the contracted wedge product

Lemma A.3. ∇^T_X *is a derivation over the contracted wedge product:*

$$
\nabla^T_X (\alpha \underset{n}{\wedge} \beta) = \nabla^T_X \alpha \underset{n}{\wedge} \beta + \alpha \underset{n}{\wedge} \nabla^T_X \beta\,. \qquad \text{(A.22)}
$$

Moreover, when $\alpha \in \Omega^p(M)$, *then*

$$
\begin{aligned}
\delta^T (\alpha \underset{n}{\wedge} \beta) &= (-1)^n \delta^T \alpha \underset{n}{\wedge} \beta - (-1)^p \nabla^T_{X_a} \alpha \underset{n}{\wedge} X^a \lrcorner\, \beta\\
&\qquad + (-1)^p \alpha \underset{n}{\wedge} \delta^T \beta - (-1)^n X^a \lrcorner\, \alpha \underset{n}{\wedge} \nabla^T_{X_a} \beta\,. &\text{(A.23)}\\
d^T (\alpha \underset{n}{\wedge} \beta) &= (-1)^n (d^T \alpha \underset{n}{\wedge} \beta - n \nabla^T_{X_a} \alpha \underset{n-1}{\wedge} X^a \lrcorner\, \beta)\\
&\qquad + (-1)^p (\alpha \underset{n}{\wedge} d^T \beta - n X^a \lrcorner\, \alpha \underset{n}{\wedge} \nabla^T_{X_a} \beta)\,. &\text{(A.24)}
\end{aligned}
$$

Proof. Let us first prove (A.22). This is clearly true for $n = 0$. Suppose true for $n-1$, then

$$
\begin{aligned}
\nabla^T_X (\alpha \underset{n}{\wedge} \beta) &= \nabla^T_X (X_a \lrcorner\, \alpha \underset{n-1}{\wedge} X^a \lrcorner\, \beta)\\
&= \nabla^T_X (X_a \lrcorner\, \alpha) \underset{n-1}{\wedge} X^a \lrcorner\, \beta + X_a \lrcorner\, \alpha \underset{n-1}{\wedge} \nabla^T_X (X^a \lrcorner\, \beta)\\
&= \frac{1}{2} T(X_b, X^a, X_c) X^c \lrcorner\, \alpha \underset{n-1}{\wedge} X_a \lrcorner\, \beta + X_a \lrcorner\, \nabla^T_X \alpha \underset{n-1}{\wedge} X^a \lrcorner\, \beta\\
&\qquad + X_a \lrcorner\, \alpha \underset{n-1}{\wedge} \frac{1}{2} T(X_b, X_a, X^c) X_c \lrcorner\, \beta + X_a \lrcorner\, \alpha \underset{n-1}{\wedge} X^a \lrcorner\, \nabla^T_X \beta\\
&= \nabla^T_X \alpha \underset{n}{\wedge} \beta + \alpha \underset{n}{\wedge} \nabla^T_X \beta\,,
\end{aligned}
$$

using the anti-symmetry of the torsion. The properties (A.23) and (A.24) follow from (A.22), Lemmas A.1, A.2 and the fact that $\delta^T = -X^a \lrcorner\, \nabla^T_{X_a}$, $\delta^T = e^a \wedge \nabla^T_{X_a}$. □

A.3 Bianchi Identities

We derive some Bianchi identities by repeated application of the T-external derivative (2.8) to a general form ω. Working in a basis satisfying (2.9)

$$\begin{aligned} d^T d^T \omega &= e^a \wedge \nabla^T_{X_a} \left(e^b \wedge \nabla^T_{X_b} \right) \\ &= e^a \wedge \frac{1}{2} T(X_a X^b, X_c) e^c \wedge \nabla^T_{X_b} \omega + \frac{1}{2} e^a \wedge e^b \wedge \left[\nabla^T_{X_a}, \nabla^T_{X_b} \right] \omega \\ &= -X^a \lrcorner\, T \wedge \nabla^T_{X_a} \omega + \frac{1}{2} e^a \wedge e^b \wedge R(X_a, X_b) \omega \,. \end{aligned} \tag{A.25}$$

Alternatively, we may express the T-exterior derivative as

$$\begin{aligned} d^T d^T \omega &= (d - T \underset{1}{\wedge})(d\omega - T \underset{1}{\wedge} \omega) \\ &= -T \underset{1}{\wedge} d^T \omega - d^T (T \underset{1}{\wedge} \omega) - T \underset{1}{\wedge} (T \underset{1}{\wedge} \omega) \,. \end{aligned} \tag{A.26}$$

Now, making use of Lemma A.3, we find

$$\frac{1}{2} e^a \wedge e^b \wedge R(X_a, X_b) \omega = d^T T \underset{1}{\wedge} \omega - \nabla^T_{X_b} T \wedge X^b \lrcorner\, \omega - T \underset{1}{\wedge} (T \underset{1}{\wedge} \omega). \tag{A.27}$$

By an analogous calculation we deduce that

$$\delta^T \delta^T \omega = \frac{1}{2} X^a \lrcorner\, T \underset{2}{\wedge} \nabla^T_{X_a} \omega + \frac{1}{2} X^a \lrcorner\, X^b \lrcorner\, R(X_a, X_b) \omega \tag{A.28}$$

and

$$X^a \lrcorner\, X^b \lrcorner\, R(X_a, X_b) \omega = -\delta^T T \underset{2}{\wedge} \omega - \nabla^T_{X_b} T \underset{2}{\wedge} X^b \lrcorner\, \omega - \frac{1}{2} T \underset{2}{\wedge} (T \underset{2}{\wedge} \omega). \tag{A.29}$$

We now deduce a Bianchi identity for the curvature forms R_{ab} defined by

$$R_{ab}(X, Y) = X_a \lrcorner\, R(X, Y) e_b \,.$$

These are anti-symmetric on their indices as may be shown by considering the curvature operator acting on $g^{-1}(e_a, e_b)$. We calculate

$$R_{ab} \wedge e^b = -R_{ba} \wedge e^b = -\frac{1}{2} (X_b \lrcorner\, R(X_c, X_d) e_a) e^b \wedge e^c \wedge e^d = -\frac{1}{2} e^c \wedge e^d \wedge R(X_c, X_d) e_a. \tag{A.30}$$

Making use of (A.28) we find

$$R_{ab} \wedge e^b = \nabla^T_{X_a} T - X_a \lrcorner\, d^T T + (X_a \lrcorner\, T) \underset{1}{\wedge} T \tag{A.31}$$

and readily deduce

$$\begin{aligned} R_{ab} \wedge e^a \wedge e^b &= -3 d^T T - 2 T \underset{1}{\wedge} T \,, &\text{(A.32)} \\ X^a \lrcorner\, (R_{ab} \wedge e^b) &= -\delta^T T. &\text{(A.33)} \end{aligned}$$

Now, recall the definition of $\mathcal{R}_{X,Y}$

$$\mathcal{R}_{X,Y} = -\frac{1}{4} X \lrcorner Y \lrcorner R_{ab} e^{ab}. \tag{A.34}$$

Making use of the rules for Clifford products we find

$$\begin{aligned} e^{ab}\mathcal{R}_{X_a,X_b} &= -\frac{1}{4} e^{ab} X_a \lrcorner Y_b \lrcorner R_{cd} e^{cd} = \frac{1}{2} R_{ab} e^{ab} \\ &= \frac{1}{2} R_{ab} \wedge e^a \wedge e^b - X_a \lrcorner (e^b \wedge R_{ab}) + \frac{1}{2} X^a \lrcorner X^b \lrcorner R_{ab} \\ &= -\frac{3}{2} d^T T - T \underset{1}{\wedge} T + \delta^T T - \frac{1}{2} s. \end{aligned} \tag{A.35}$$

Here s is the scalar curvature of the connection with torsion, which we take to be defined by

$$s = -X^a \lrcorner X^b \lrcorner R_{ab} = -X^a \lrcorner R(X_a, X_b) e^b. \tag{A.36}$$

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