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Delayed boundary control of a heat equation under discrete-time point measurements

Anton Selivanov and Emilia Fridman

Abstract— **We consider a reaction-diffusion PDE under continuously applied boundary control that contains a constant delay. The point measurements are sampled in time and transmitted through a network with a time-varying delay. We construct an observer that predicts the value of the state allowing to compensate for the constant boundary delay. Using a time-varying injection gain, we ensure that the estimation error vanishes exponentially with a desired decay rate if the delays and sampling intervals are small enough while the number of sensors is large enough. The stability conditions, obtained via a Lyapunov-Krasovskii functional, are formulated in terms of linear matrix inequalities. By applying the backstepping transformation to the future state estimation, we derive a boundary controller that guarantees the exponential stability of the closed-loop system with an arbitrary decay rate smaller than that of the observer. The results are demonstrated by an example.**

I. Introduction

Networked control systems (NCSs) are systems with spatially distributed sensors, controllers, and actuators connected through a shared communication network. NCSs have become widespread due to great advantages they bring, such as long distance control, reduced system wiring, low cost, increased system agility, ease of reconfiguration, diagnosis, and maintenance [1], [2]. The main theoretical challenges caused by networked architecture are data sampling and transmission delays, which have been extensively studied for finite-dimensional systems. In particular, predictors, originally proposed for continuous-time measurements [3], [4], have been extended to discrete-time measurements for both static [5], [6], [7], [8] and dynamic feedback [9], [10], [11].

Another way to compensate for input delay is to use an observer that predicts the future value of the state. Such observer is obtained by shifting the plant in time and adding a correcting term, which is proportional to the difference between the last available measurement and correspondingly delayed observer output. The stability analysis consists in proving the observer's robustness with respect to measurement delays. This idea can be used to analyse chain observers [12], [13], [14], [15] and sequential predictors [16], [17], [18], [19], [20]. In [21] a time-varying injection gain was introduced in the observer to improve its exponential convergence under delayed measurements. This result was revisited in [22] to increase the period of a sampled-data system.

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A constant input delay can be compensated in a reaction-diffusion system by representing it as a PDE-PDE cascade [23], which is then analysed using the backstepping transformation [24], [25]. However, this method is hard to combine with data sampling. In [26], [27], [28], some qualitative stability results are provided for sampled-data infinite-dimensional systems of a general form. The same problem can be studied using Galerkin's method (see, e.g., [29], [30], [31] and references therein). The general idea is to approximate the PDE by a finite dimensional system that captures the dominant dynamics of the PDE. A drawback of such approach is the inherent loss of process information due to truncation before the controller design and stability analysis. Thus, it is difficult to guarantee the stability/performance for the original system.

Some qualitative stability results for state-feedback boundary control in the presence of data sampling have been recently obtained in [32]. The analysis is based on the Fourier method and Input-to-State Stability ideas of [33].

Quantitative stability results, formulated in terms of linear matrix inequalities, were obtained in [34], [35], [36], [37] using Lyapunov-Krasovskii functionals. These works concern parabolic systems with sampled measurements and controls applied through distributed shape functions and zero-order holds. So far, it is not clear whether such approach can be extended to state-feedback boundary control with sampled measurements or delayed input, since such control is represented by an unbounded operator.

In this paper, we design an observer-based boundary controller for a reaction-diffusion PDE with sampled measurements and continuous in time input that contains a constant delay. Inspired by the ideas of sequential predictors [16], [17], [18], [19], [20], we construct an observer that estimates the future value of the state using the sampled measurements. By introducing a time-varying injection gain [21] and performing the stability analysis in a manner similar to [34], [37], we show that the observation error exponentially vanishes with any desired decay rate if the delays and sampling are small enough while the number of sensors is large enough. Such observer allows to eliminate the constant input delay. Applying the backstepping transformation [24], [25] to the predictive state estimation, we obtain a target system that contains the exponentially vanishing estimation error in the differential equation. Proving its Input-to-State Stability with respect to this error, we guarantee the exponential stability of the closed-loop system with an arbitrary decay rate smaller than that of the observer.

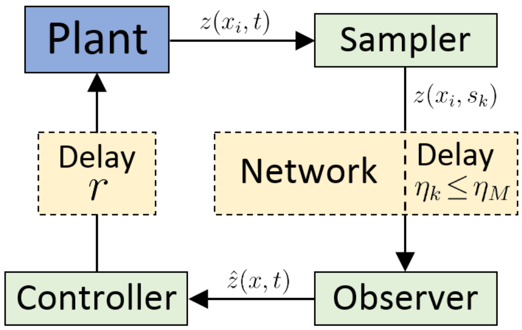

Fig. 1: System representation

Lemma 1 (Wirtinger inequality [38]): For $f \in \mathcal{H}^1(a,b)$,

$$\|f\|^2_{L^2} \le \frac{(b-a)^2}{\pi^2}\|f'\|^2_{L^2} \quad \text{if } f(a)=f(b)=0,$$
$$\|f\|^2_{L^2} \le \frac{4(b-a)^2}{\pi^2}\|f'\|^2_{L^2} \quad \text{if } f(a)=0 \text{ or } f(b)=0.$$

II. Plant description and observer construction

We consider the system schematically presented in Fig. 1. The plant is governed by the reaction-diffusion PDE

$$\begin{aligned} &z_t(x,t) = z_{xx}(x,t) + az(x,t), \\ &d_L z(0,t) + (1-d_L)z_x(0,t) = 0, \\ &d_R z(1,t) + (1-d_R)z_x(1,t) = u(t-r), \end{aligned} \tag{1}$$

where $z\colon [0,1]\times[0,\infty) \to \mathbb{R}$ is a state and $u(t-r)$ with $r \ge 0$ is a delayed boundary control. Each constant $d_L, d_R \in \{0,1\}$ sets either the Dirichlet or Neumann boundary condition. If $u(t)=0$, the plant is unstable for a large enough reaction coefficient a.

We assume that N in-domain sensors measure the state at space points $0 \le x_1 < x_2 < \ldots < x_N \le 1$ at time instants $0 = s_0 < s_1 < s_2 < \ldots$ such that

$$s_{k+1} - s_k \le h, \qquad \lim s_k = \infty.$$

The measurements $z(x_i, s_k)$ are transmitted through a network with a time-varying delay $\eta_k \in [0, \eta_M]$ such that the observer/controller updating times $t_k = s_k + \eta_k$ form a non-decreasing sequence: $t_k \le t_{k+1}$.

We construct an observer that estimates the *future* value of the state: $\hat{z}(x,t) \approx z(x,t+r)$,

$$\begin{aligned} &\hat{z}_t(x,t) = \hat{z}_{xx}(x,t) + a\hat{z}(x,t) + Le^{-\alpha_o(t+r-s_k)}\times \\ &\quad \textstyle\sum_{i=1}^N b_i(x)[\hat{z}(x_i, s_k - r) - z(x_i,s_k)],\ t\in[t_k,t_{k+1}), \\ &d_L\hat{z}(0,t) + (1-d_L)\hat{z}_x(0,t) = 0, \\ &d_R\hat{z}(1,t) + (1-d_R)\hat{z}_x(1,t) = u(t), \\ &\hat{z}(\cdot,t) = 0, \quad t \le t_0. \end{aligned} \tag{2}$$

The observer (2) is obtained by shifting the plant (1) in time by r and introducing a correcting term. The time-varying injection gain $Le^{-\alpha_o(t+r-s_k)}$ will allow to guarantee that the observation error decays with the rate α_o [21]. The shape functions $b_i \in L^2(0,1)$ are given by

$$b_i(x) = \begin{cases} 0, & x \notin \Omega_i, \\ 1, & x \in \Omega_i, \end{cases} \tag{3}$$

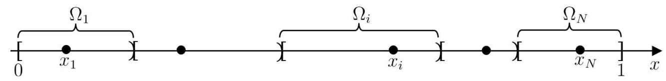

Fig. 2: Partition of $[0,1]$ for point measurements

where $\{\Omega_i\}$ is a partition of $[0,1]$ such that[1] $x_i \in \Omega_i$ (Fig. 2).

Due to (1), (2), the observation/prediction error $\bar{z}(x,t) = \hat{z}(x,t-r) - z(x,t)$ satisfies (if $u(t)=0$ for $t<t_0$)

$$\begin{aligned} &\bar{z}_t = \bar{z}_{xx} + a\bar{z}, \qquad\qquad t\in[0,t_0+r), \\ &\bar{z}_t = \bar{z}_{xx} + a\bar{z} + Le^{-\alpha_o(t-s_k)}\textstyle\sum_{i=1}^N b_i(x)\bar{z}(x_i,s_k) \\ &\qquad\qquad\qquad\qquad t\in[t_k+r, t_{k+1}+r), \\ &d_L\bar{z}(0,t) + (1-d_L)\bar{z}_x(0,t) = 0, \\ &d_R\bar{z}(1,t) + (1-d_R)\bar{z}_x(1,t) = 0, \\ &\bar{z}(\cdot,0) = -z(\cdot,0). \end{aligned} \tag{4}$$

In a manner similar to the proof of [39, Theorem 7.7], one can show that (4) has a unique strong solution on $[0,\infty)$ for the initial conditions $\bar{z}(\cdot,0) \in \mathcal{H}^1(0,1)$ subject to the boundary conditions.

Proposition 1: For positive α_0, α_1 let there exist a scalar G and positive scalars S_i, R_i, p_i with $i=1,2$, such that[2]

$$\Phi < 0, \quad \alpha_1 p_2 \le 2p_1, \quad \begin{bmatrix} R_2 & G \\ G & R_2 \end{bmatrix} \ge 0,$$

with $\Phi = \{\Phi_{ij}\}$ being a symmetric matrix composed from

$$\begin{aligned} \Phi_{11} &= -R_1e^{-\alpha_1 r} + S_1 + 2p_1(a+\alpha_o) + \alpha_1 \\ &\qquad -\pi^2(2p_1 - \alpha_1 p_2)\tfrac{\max\{d_L,d_R\}}{4-3d_Ld_R}, \\ \Phi_{12} &= 1 - p_1 + p_2(a+\alpha_o), \\ \Phi_{13} &= R_1e^{-\alpha_1 r}, \\ \Phi_{14} &= \Phi_{16} = p_1L, \\ \Phi_{22} &= -2p_2 + r^2R_1 + (h+\eta_M)^2R_2, \\ \Phi_{24} &= \Phi_{26} = p_2L, \\ \Phi_{33} &= -(R_1+S_1-S_2)e^{-\alpha_1 r} - R_2e^{-\alpha_1\tau_M}, \\ \Phi_{34} &= \Phi_{45} = (R_2-G)e^{-\alpha_1\tau_M}, \\ \Phi_{35} &= Ge^{-\alpha_1\tau_M}, \\ \Phi_{44} &= -2(R_2-G)e^{-\alpha_1\tau_M} - \alpha_1, \\ \Phi_{55} &= -(R_2+S_2)e^{-\alpha_1\tau_M}, \\ \Phi_{66} &= -\tfrac{\alpha_1p_2\pi^2}{4\max_i|\Omega_i|^2}, \end{aligned}$$

where $\tau_M = h + \eta_M + r$. Then the system (4) is exponentially stable with the decay rate α_o, i.e.,

$$\|\bar{z}(\cdot,t)\|_{\mathcal{H}^1} \le \bar{C}e^{-\alpha_o t}\|\bar{z}(\cdot,0)\|_{\mathcal{H}^1}, \quad t\ge 0 \tag{5}$$

for some $\bar{C} > 0$. Moreover,

$$\|\sigma(\cdot,t)\|_{L^2} \le C_\sigma e^{-\alpha_o t}\|z(\cdot,0)\|_{\mathcal{H}^1}, \quad t \ge 0 \tag{6}$$

for some $C_\sigma > 0$, where

$$\sigma(x,t) = \textstyle\sum_{i=1}^N b_i(x)\bar{z}(x_i,t), \quad x\in[0,1],\ t\ge 0. \tag{7}$$

[1] It is reasonable to choose $\{\Omega_i\}$ that minimizes $\max_i |\Omega_i|$

[2] MATLAB codes for solving the LMIs are available at https://github.com/AntonSelivanov/CDC17

Proof: Let $\zeta(x,t)=e^{\alpha_o t}\bar{z}(x,t)$. For $t\geq t_0+r$, (4) implies

$$\begin{array}{l}\zeta_t=\zeta_{xx}+(a+\alpha_o)\zeta+L\sum_{i=1}^N b_i(x)\zeta(x_i,t-\tau(t)),\\ d_L\zeta(0,t)+(1-d_L)\zeta_x(0,t)=0,\\ d_R\zeta(1,t)+(1-d_R)\zeta_x(1,t)=0,\end{array}\tag{8}$$

where

$$\tau(t)=t-s_k,\quad t\in[t_k+r,t_{k+1}+r),\ k\in\mathbb{N}_0,\qquad r\leq\tau(t)\leq\tau_M=r+h+\eta_M.$$

Consider the Lyapunov-Krasovskii functional

$$V_\zeta=V_1+V_2+V_{S1}+V_{R1}+V_{S2}+V_{R2},\tag{9}$$

where

$$\begin{aligned}V_1&=\int_0^1\zeta^2(x,t)\,dx,\\ V_2&=p_2\int_0^1\zeta_x^2(x,t)\,dx,\\ V_{S1}&=S_1\int_0^1\int_{t-r}^t e^{-\alpha_1(t-s)}\zeta^2(x,s)\,ds\,dx,\\ V_{R1}&=rR_1\int_0^1\int_{-r}^0\int_{t+\theta}^t e^{-\alpha_1(t-s)}\zeta_s^2(x,s)\,ds\,d\theta\,dx,\\ V_{S2}&=S_2\int_0^1\int_{t-\tau_M}^{t-r} e^{-\alpha_1(t-s)}\zeta^2(x,s)\,ds\,dx,\\ V_{R2}&=(h+\eta_M)R_2\int_0^1\int_{-\tau_M}^{-r}\int_{t+\theta}^t e^{-\alpha_1(t-s)}\zeta_s^2(x,s)\,ds\,d\theta\,dx.\end{aligned}$$

Similarly to [40], we formally set $\zeta(\cdot,t)=\zeta(\cdot,0)$ for $t<0$ so that V_ζ is defined on[3] $[t_0+r-\tau_M,\infty)$. For $t\geq t_0+r$,

$$\begin{array}{l}\dot V_1+\alpha_1V_1=2\int_0^1\zeta\zeta_t+\alpha_1\int_0^1\zeta^2,\\ \dot V_2+\alpha_1V_2=2p_2\int_0^1\zeta_x\zeta_{xt}+\alpha_1p_2\int_0^1\zeta_x^2,\\ \dot V_{S1}+\alpha_1V_{S1}=S_1\int_0^1\zeta^2-S_1e^{-\alpha_1 r}\int_0^1\zeta^2(x,t-r)\,dx,\\ \dot V_{S2}+\alpha_1V_{S2}=S_2e^{-\alpha_1 r}\int_0^1\zeta^2(x,t-r)\,dx\\ \qquad\qquad -S_2e^{-\alpha_1\tau_M}\int_0^1\zeta^2(x,t-\tau_M)\,dx.\end{array}$$

Using Jensen's inequality [41, Proposition B.8],

$$\begin{aligned}&\dot V_{R1}+\alpha_1V_{R1}=\\ &r^2R_1\int_0^1\zeta_t^2(x,t)\,dx-rR_1\int_0^1\int_{t-r}^t e^{-\alpha_1(t-s)}\zeta_s^2(x,s)\,ds\,dx\\ &\leq r^2R_1\int_0^1\zeta_t^2(x,t)\,dx-R_1e^{-\alpha_1 r}\int_0^1\big(\zeta(x,t)-\zeta(x,t-r)\big)^2dx.\end{aligned}$$

Jensen's inequality and reciprocally convex approach [42, Theorem 1] allow to obtain[4]

$$\begin{aligned}&\dot V_{R2}+\alpha_1V_{R2}\leq(h+\eta_M)^2R_2\int_0^1\zeta_t^2(x,t)\,dx-e^{-\alpha_1\tau_M}\times\\ &\int_0^1\begin{bmatrix}\zeta(x,t-r)-\zeta(x,t-\tau(t))\\ \zeta(x,t-\tau(t))-\zeta(x,t-\tau_M)\end{bmatrix}^T\begin{bmatrix}R_2&G\\ G&R_2\end{bmatrix}\begin{bmatrix}\zeta(x,t-r)-\zeta(x,t-\tau(t))\\ \zeta(x,t-\tau(t))-\zeta(x,t-\tau_M)\end{bmatrix}dx.\end{aligned}$$

Instead of replacing ζ_t with the right-hand side of (8), we employ the descriptor method [43]. Namely, (8) implies

$$\begin{aligned}0=2\int_0^1&[p_1\zeta(x,t)+p_2\zeta_t(x,t)][-\zeta_t(x,t)+\zeta_{xx}(x,t)\\ &+(a+\alpha_o)\zeta(x,t)+L\textstyle\sum_{i=1}^N b_i(x)\zeta(x_i,t-\tau(t))]\,dx,\end{aligned}$$

which right-hand side will be added to $\dot V_\zeta$. Denote

$$\kappa(x,t)=\zeta(x_i,t)-\zeta(x,t),\quad x\in\Omega_i,\ i\in 1\!:\!N.\tag{10}$$

Then the latter can be rewritten as

$$\begin{aligned}0=2\textstyle\sum_{i=1}^N\int_{\Omega_i}&[p_1\zeta(x,t)+p_2\zeta_t(x,t)][-\zeta_t(x,t)+\zeta_{xx}(x,t)\\ &+(a+\alpha_o)\zeta(x,t)+L\kappa(x,t-\tau(t))\\ &+L\zeta(x,t-\tau(t))]\,dx.\end{aligned}\tag{11}$$

Integrating by parts, we obtain

$$\begin{array}{l}2p_1\sum_{i=1}^N\int_{\Omega_i}\zeta\zeta_{xx}=-2p_1\sum_{i=1}^N\int_{\Omega_i}\zeta_x^2,\\ 2p_2\sum_{i=1}^N\int_{\Omega_i}\zeta_t\zeta_{xx}=-2p_2\int_0^1\zeta_{xt}\zeta_x=-\dot V_2.\end{array}\tag{12}$$

Since $\alpha_1p_2\leq 2p_1$, Lemma 1 implies

$$\begin{array}{l}0\leq(2p_1-\alpha_1p_2)\max\{d_L,d_R\}\times\\ \qquad\left[\int_0^1\zeta_x^2(x,t)\,dx-\frac{\pi^2}{4-3d_Ld_R}\int_0^1\zeta^2(x,t)\,dx\right].\end{array}\tag{13}$$

Denote $[x_i^L,x_i^R)=\Omega_i$. Since $\kappa(x_i,t)=0$ and $\kappa_x=-\zeta_x$,

$$\begin{aligned}\int_{\Omega_i}\kappa^2=\int_{x_i^L}^{x_i}\kappa^2+\int_{x_i}^{x_i^R}\kappa^2&\overset{\text{Lem.1}}{\leq}\frac{4|\Omega_i|^2}{\pi^2}\left[\int_{x_i^L}^{x_i}\zeta_x^2+\int_{x_i}^{x_i^R}\zeta_x^2\right]\\ &\leq\frac{4\max_i|\Omega_i|^2}{\pi^2}\int_{\Omega_i}\zeta_x^2.\end{aligned}\tag{14}$$

Therefore, for any $\alpha_2>0$,

$$\begin{aligned}&-\alpha_2\sup_{\theta\in[t-\tau_M,t]}V_\zeta(\theta)\leq-\alpha_2V_\zeta(t-\tau(t))\\ &\leq-\alpha_2\sum_{i=1}^N\int_{\Omega_i}\zeta^2(x,t-\tau(t))dx-\alpha_2p_2\sum_{i=1}^N\int_{\Omega_i}\zeta_x^2(x,t-\tau(t))dx\\ &\leq-\alpha_2\sum_{i=1}^N\int_{\Omega_i}\zeta^2(x,t-\tau(t))dx\\ &\qquad-\frac{\alpha_2p_2\pi^2}{4\max_i|\Omega_i|^2}\sum_{i=1}^N\int_{\Omega_i}\kappa^2(x,t-\tau(t))dx.\end{aligned}$$

Consider the matrix Ψ that coincides with Φ except for

$$\begin{array}{l}\Psi_{44}=-2(R_2-G)e^{-\alpha_1\tau_M}-\alpha_2,\\ \Psi_{66}=-\frac{\alpha_2p_2\pi^2}{4\max_i|\Omega_i|^2}.\end{array}$$

Since $\Phi<0$ is a strict inequality, $\Psi<0$ for large enough $\alpha_2<\alpha_1$. By adding the right-hand sides of (11), (13) to $\dot V_\zeta$ and using (12), we obtain

$$\begin{array}{l}\dot V_\zeta+\alpha_1V_\zeta-\alpha_2\sup_{\theta\in[t-\tau_M,t]}V_\zeta(\theta)\\ \quad\leq\sum_{i=1}^N\int_{\Omega_i}\psi^T(x,t)\Psi\psi(x,t)\,dx\\ \quad\quad-(1-\max\{d_L,d_R\})(2p_1-\alpha_1p_2)\|\zeta_x(\cdot,t)\|_{L^2}^2\end{array}$$

with $\psi(x,t)=\mathrm{col}\{\zeta,\zeta_t,\zeta(x,t-r),\zeta(x,t-\tau(t)),\zeta(x,t-\tau_M),\kappa(x,t-\tau(t))\}$. Since $\Psi<0$ and $2p_1\geq\alpha_1p_2$,

$$\dot V_\zeta(t)\leq-\alpha_1V_\zeta(t)+\alpha_2\sup_{\theta\in[t-\tau_M,t]}V_\zeta(\theta),\quad t\geq t_0+r.$$

The Halanay inequality [44, Lemma 4.2] implies

$$V_\zeta(t)\leq e^{-\bar\alpha(t-t_0-r)}\sup_{\theta\in[t_0+r-\tau_M,t_0+r]}V_\zeta(\theta),\quad t\geq t_0+r,\tag{15}$$

where $\bar\alpha$ is a unique and positive solution of $\bar\alpha=\alpha_1-\alpha_2e^{\bar\alpha\tau_M}$.

[3] This is required for (15) to be meaningful

[4] Similar calculation is given in [37, (A.1)] in more details

For $t \in [0, t_0+r)$, (4) implies (8) with $L=0$. Then calculations similar to the above imply $\dot{V}_\zeta(t) \le \delta V_\zeta(t)$ for $t \in [0, t_0+r)$ with large enough δ. Therefore,

$$V_\zeta(t) \le e^{\delta t} V_\zeta(0) \le e^{\delta(t_0+r)} V_\zeta(0), \quad t \in [0, t_0+r].$$

Moreover, since we set $\zeta(\cdot,t)=\zeta(\cdot,0)$ for $t<0$,

$$V_\zeta(t) = V_\zeta(0), \quad t \in [t_0+r-\tau_M, 0].$$

Consequently,

$$\sup_{\theta \in [t_0+r-\tau_M, t_0+r]} V_\zeta(\theta) \le e^{\delta(t_0+r)} V_\zeta(0) \le C_V \|\zeta(\cdot,0)\|^2_{\mathcal{H}^1}$$

for some $C_V>0$. Recalling that $\zeta(x,t)=e^{\alpha_o t}\bar{z}(x,t)$, the latter and (15) yield

$$\begin{aligned}\|\bar{z}(\cdot,t)\|^2_{\mathcal{H}^1} &= e^{-2\alpha_o t}\|\zeta(\cdot,t)\|^2_{\mathcal{H}^1} \le \tfrac{e^{-2\alpha_o t}}{\min\{1,p_2\}} V_\zeta(t)\\ &\le \bar{C}^2 e^{-2\alpha_o t}\|\zeta(\cdot,0)\|^2_{\mathcal{H}^1} = \bar{C}^2 e^{-2\alpha_o t}\|\bar{z}(\cdot,0)\|^2_{\mathcal{H}^1}\end{aligned}$$

for $t \ge 0$ with some $\bar{C}>0$. This proves (5).

Using the notation (10), $b_i(x)\zeta(x_i,t) = b_i(x)(\zeta(x,t)+\kappa(x,t))$ for any $x \in [0,1]$. Therefore,

$$\begin{aligned}\int_0^1 \sigma^2 &= \int_0^1 \left(\sum_{i=1}^N b_i(x)\zeta(x_i,t)\right)^2 dx\\ &= \int_0^1 (\zeta(x,t)+\kappa(x,t))^2 \left(\sum_{i=1}^N b_i(x)\right)^2 dx\\ &\le 2\int_0^1 \kappa^2 + 2\int_0^1 \zeta^2 \overset{(14)}{\le} 2\max\left\{1, \tfrac{4\max_i |\Omega_i|^2}{p_2\pi^2}\right\} V_\zeta(t)\\ &\le C_\sigma^2 e^{-2\alpha_o t}\|\bar{z}(\cdot,0)\|^2_{\mathcal{H}^1} = C_\sigma^2 e^{-2\alpha_o t}\|z(\cdot,0)\|^2_{\mathcal{H}^1}\end{aligned}$$

for $t \ge 0$ with some $C_\sigma>0$. This proves (6). ■

Remark 1: Using the standard arguments for time-delay systems [44], one can show that the LMIs of Proposition 1 are feasible for any given α_o if the delays r, η_M and sampling h are small enough while the maximum subdomain size $\max_i |\Omega_i|$ is small enough.

III. Boundary controller synthesis

A boundary controller for (1) is constructed based on the estimation $\hat{z}$ using the backstepping transformation [24], [25]

$$w(x,t) = \hat{z}(x,t) - \int_0^x k(x,y)\hat{z}(y,t)\,dy, \tag{16}$$

where $k(x,y)$ is the solution of

$$\begin{aligned}&k_{xx}(x,y) - k_{yy}(x,y) = \lambda k(x,y),\\ &k(x,x) = -\tfrac{\lambda}{2}x,\\ &d_L k(x,0) + (1-d_L)k_y(x,0) = 0\end{aligned} \tag{17}$$

with some $\lambda \in \mathbb{R}$. Such kernel $k(x,y)$ exists for any λ and is bounded (see, e.g., [25, Theorem 2.1]). Let

$$\begin{aligned}&u(t) = \int_0^1 k(1,y)\hat{z}(y,t)\,dy && \text{if } d_R = 1,\\ &u(t) = k(1,1)\hat{z}(1,t) + \int_0^1 k_x(1,y)\hat{z}(y,t)\,dy && \text{if } d_R = 0\end{aligned} \tag{18}$$

for $t \ge t_0$ and $u(t)=0$ for $t<t_0$. Then, performing calculations similar to those in [25, Chapter 2.2], we have

$$\begin{aligned}&w_t(x,t) = w_{xx}(x,t) - (\lambda-a)w(x,t) + v(x,t),\\ &d_L w(0,t) + (1-d_L)w_x(0,t) = 0,\\ &d_R w(1,t) + (1-d_R)w_x(1,t) = 0,\\ &w(\cdot,t_0) = 0\end{aligned} \tag{19}$$

for $t \ge t_0$, where

$$\begin{aligned}v(x,t) = {}& L e^{-\alpha_o(t+r-s_k)} \times\\ &\left[\sigma(x,s_k) - \int_0^x k(x,y)\sigma(y,s_k)\,dy\right],\ t \in [t_k, t_{k+1})\end{aligned}$$

with $\sigma(x,t)$ defined in (7).

Proposition 2: Under the assumptions of Proposition 1, if

$$\lambda > \alpha_c + a - \frac{\max\{d_L,d_R\}\pi^2}{4-3d_Ld_R+\pi^2}, \tag{20}$$

where $\alpha_c>0$, then the solutions of the system (19) satisfy

$$\|w(\cdot,t)\|_{\mathcal{H}^1} \le C_w e^{-\min\{\alpha_o,\alpha_c\}t}\|z(\cdot,0)\|_{\mathcal{H}^1}, \quad t \ge t_0 \tag{21}$$

with some $C_w>0$.

Proof: Consider $V_w = V_{w1} + V_{w2}$ with

$$V_{w1} = \int_0^1 w^2(x,t)\,dx, \quad V_{w2} = \int_0^1 w_x^2(x,t)\,dx.$$

We have

$$\dot{V}_{w1} = 2\int_0^1 ww_{xx} - 2(\lambda-a)\int_0^1 w^2 + 2\int_0^1 wv.$$

Since

$$\begin{aligned}&2\int_0^1 ww_{xx} = -2\int_0^1 w_x^2 && \text{(integration by parts)}\\ &2\int_0^1 wv \le 2\mu\int_0^1 w^2 + \tfrac{1}{2\mu}\int_0^1 v^2 && \text{(Young's inequality)}\end{aligned}$$

with an arbitrary $\mu>0$, we obtain

$$\dot{V}_{w1} \le -2\int_0^1 w_x^2 - 2(\lambda-a-\mu)\int_0^1 w^2 + \tfrac{1}{2\mu}\int_0^1 v^2.$$

Using integration by parts, we have

$$\begin{aligned}\dot{V}_{w2} &= 2\int_0^1 w_x w_{xt} = -2\int_0^1 w_{xx}w_t\\ &= -2\int_0^1 w_{xx}^2 + 2(\lambda-a)\int_0^1 w_{xx}w - 2\int_0^1 w_{xx}v.\end{aligned}$$

Since

$$\begin{aligned}&2(\lambda-a)\int_0^1 w_{xx}w = -2(\lambda-a)\int_0^1 w_x^2 && \text{(int. by parts)},\\ &-2\int_0^1 w_{xx}v \le 2\int_0^1 w_{xx}^2 + \tfrac{1}{2}\int_0^1 v^2 && \text{(Young's inequality)},\end{aligned}$$

we obtain

$$\dot{V}_{w2} \le -2(\lambda-a)\int_0^1 w_x^2 + \tfrac{1}{2}\int_0^1 v^2.$$

Summing up, for any $\mu>0$

$$\begin{aligned}\dot{V}_w + 2\alpha_c V_w \le {}& -2(1+\lambda-a-\alpha_c)\|w_x\|^2_{L_2}\\ &- 2(\lambda-a-\alpha_c-\mu)\|w\|^2_{L_2} + \left(\tfrac{1}{2\mu}+\tfrac{1}{2}\right)\int_0^1 v^2.\end{aligned}$$

The condition (20) yields $1+\lambda-a-\alpha_c>0$. Then, using

$$-\|w_x\|^2_{L_2} \overset{\text{Lem.1}}{\le} -\tfrac{\max\{d_L,d_R\}\pi^2}{4-3d_Ld_R}\|w\|^2_{L_2},$$

and (20), for small enough $\mu>0$, we obtain

$$\dot{V}_w \le -2\alpha_c V_w + \left(\tfrac{1}{2\mu}+\tfrac{1}{2}\right)\int_0^1 v^2.$$

Since $k(x,y)$ is bounded, there exists $C_v>0$ such that

$$\begin{aligned}\int_0^1 v^2(x,t)\,dx &\le C_v e^{-2\alpha_o(t-s_k)}\|\sigma(\cdot,s_k)\|^2_{L^2}\\ &\overset{(6)}{\le} C_v C_\sigma^2 e^{-2\alpha_o t}\|z(\cdot,0)\|^2_{\mathcal{H}^1}.\end{aligned}$$

Summing up,

$$\dot{V}_w(t) \le -2\alpha_c V_w(t) + \left(\frac{1}{2\mu}+\frac{1}{2}\right) C_v C_\sigma^2 e^{-2\alpha_o t}\|z(\cdot,0)\|^2_{\mathcal{H}^1}.$$

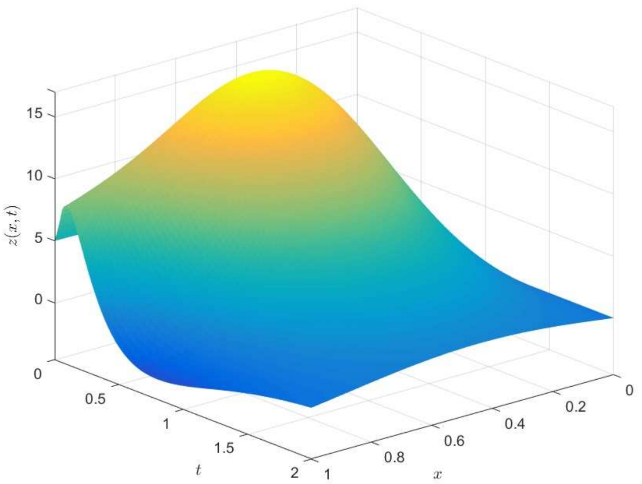

Fig. 3: The state $z(x,t)$

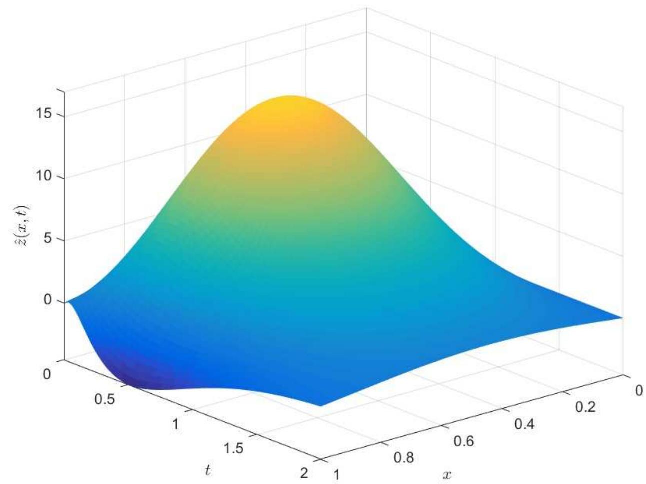

Fig. 4: The estimation/prediction $\hat{z}(x,t)$

If $\alpha_c \neq \alpha_o$, the comparison principle implies (21) (note that $V_w(t_0) = 0$). If (20) holds for $\alpha_c = \alpha_o$, it remains true for slightly larger $\alpha'_c > \alpha_c$. Then (21) holds for $\alpha'_c \neq \alpha_o$, what implies (21) for α_c. ■

Corollary 1: If the assumptions of Proposition 1 are satisfied, the observer-based boundary controller (2), (17), (18) with λ satisfying (20) makes the system (1) exponentially stable with the decay rate $\min\{\alpha_o, \alpha_c\}$, i.e.,

$$\|z(\cdot,t)\|_{\mathcal{H}^1} \leq C_z e^{-\min\{\alpha_o,\alpha_c\}t}\|z(\cdot,0)\|_{\mathcal{H}^1}, \quad t \geq 0. \quad (22)$$

with some $C_z > 0$.

Proof: The transformation (16) has an inverse, which is bounded in $\mathcal{H}^1$ norm (see, e.g., [25]). Therefore, there exists a constant $\tilde{C}$ such that

$$\|\hat{z}(\cdot,t)\|_{\mathcal{H}^1} \leq \tilde{C}\|w(\cdot,t)\|_{\mathcal{H}^1} \overset{(21)}{\leq} \tilde{C}C_w e^{-\min\{\alpha_o,\alpha_c\}t}\|z(\cdot,0)\|_{\mathcal{H}^1}$$

for $t \geq t_0$. Since $z(x,t) = \hat{z}(x,t-r) - \bar{z}(x,t)$, the latter together with (5) imply (22). ■

Remark 2: One can achieve an arbitrary decay rate in (22) if the delays and sampling are small enough while the number of sensors is large enough. This follows from Remark 1 and solvability of (17) for any λ satisfying (20).

IV. EXAMPLE

Consider the plant (1) with $a = 10$, $r = 0.05$, $d_L = 1$, $d_R = 0$, which is unstable with $u(t-r) = 0$. Assume there are $N = 10$ in-domain sensors transmitting point measurements with the sampling period $h = 0.01$ and time-varying network delay $\eta_k \leq \eta_M = 0.01$. The conditions of Proposition 1 are satisfied with $L = -10$, $\alpha_o = 0.48$, $\alpha_1 = 1$. Therefore, the observer (2) provides a prediction of the state that converges with the rate α_o. Taking $\alpha_c = 0.48$, we derive the boundary controller (18) with

$$k(1,1) = -\tfrac{\lambda}{2}, \quad k_x(1,y) = -\lambda y \frac{I_2\left(\sqrt{\lambda(1-y^2)}\right)}{1-y^2},$$

where $\lambda = a + \alpha_o - \pi^2/(4+\pi^2) + 10^{-5}$ and I_2 is the Modified Bessel Function. Corollary 1 guarantees exponential stability of the plant with the decay rate $\min\{\alpha_o, \alpha_c\} = 0.48$.

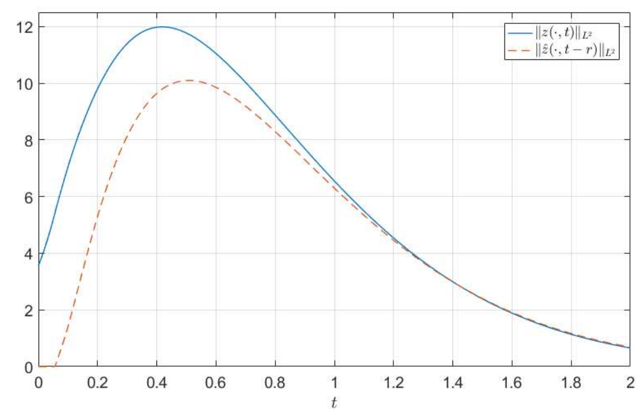

Fig. 5: $\|z(\cdot,t)\|_{L^2}$ (blue solid line) and $\|\hat{z}(\cdot,t-r)\|_{L^2}$ (red dashed line)

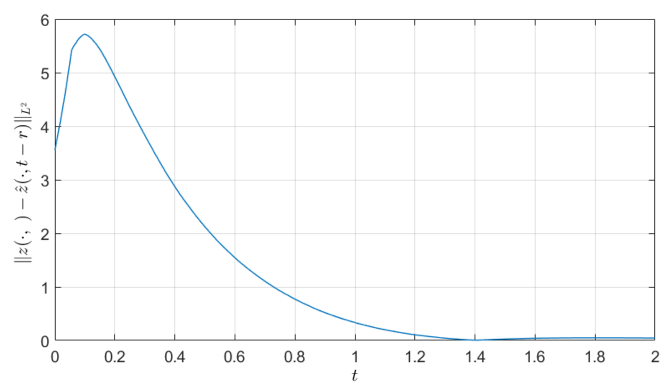

Fig. 6: The error $\|z(\cdot,t) - \hat{z}(\cdot,t-r)\|_{L^2}$

The numerical simulations were performed with

$$z(x,0) = 5\sin\left(\frac{\pi x}{2}\right)$$

and randomly chosen $\eta_k \in [0, 0.01]$ such that $t_k \leq t_{k+1}$. The results are presented in Figs. 3–6.

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