Aspects of graph colouring

Thesis

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ASPECTS OF GRAPH COLOURING

Jini Williams

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Dedicated

To Basil

Those that know it's history know
that combinatorics notoriously grow,
for every proof, as many conjectures;
For every known fact, far more are unknown
And each new idea requires a tome
And a series of lectures.

Publish or perish refers not to us:
The ideas that we cherish would turn to dust
If we never cared, nor dared, to share them:
If we never ignored the snide – 'who need them?'
Nor stopped ourselves thinking – 'Who'll ever read them?'

But history shows, that those who know, do try
To remember who did what, when, and why.
So, so long as it's printed and others can see it
What does it matter if nobody noticed

At the time?
That's life,
so be it.

ABSTRACT

The four-colour conjecture of 1852, and the total colouring conjecture of 1965, have sparked off many new concepts and conjectures. In this thesis we investigate many of the outstanding conjectures, establishing various related results, and present many conjectures of our own.

We give a brief historical introduction (Chapter 1) and establish some notation, terminology and techniques (Chapter 2). Next, in Chapter 3, we examine the use of latin squares to represent edge and total colourings. In Chapters 4 – 6 we deal with vertex, edge and total colourings respectively.

Various ways of measuring different aspects of graphs are presented, in particular, the 'colouring difference' between two edge-colourings of a graph (Chapter 5) and the 'beta parameter' (defined in Chapter 2 and used in Chapters 3 and 6); this is a measure of how far from a type 1 graph a type 2 graph can be. In Chapter 6 we derive an upper bound for the beta value of any near type 1 graph and give the exact results for all K_n. The number of ways of colouring K_n, and $K_{n,,n}$ are also quantified.

Chapter 6 also examines Hilton's concept of conformability. It is shown that every graph with at least Δ spines is conformable, and an extension to the concept, which we call G^*-conformability, is introduced. We then give new necessary conditions for a cubic graph to be type 1 in relation to G^*-conformability.

Various methods of manipulating graphs are considered and we present: a method to compatibly triangulate a graph $G - e$; a method of introducing a fourth colour thus

allowing a sequence of Kempe interchanges from any edge 3- colouring of a cubic graph to any other; and a method to re-colour a near type 1 graph within a certain bound on beta.

We end this thesis with a brief discussion on possible practical uses for colouring graphs.

A list of the main results and conjectures is given at the end of each chapter, but a short list of the principle theorems proven is given below.

Summary of principal results

Theorem 3.1.4 $\beta(K_n) = \begin{cases} 0, & (n \text{ odd}) \\ n/2, & (n \text{ even}). \end{cases}$ Main result (3.1) page 23

Theorem 3.4.7 *To isochromatism, the number of ways to colour $K_{n,n}$ is equal to*

$$3\omega + 2\theta + \rho$$

where

ω is the number of main classes of $n \times n$ latin squares in the set $\mathbf{C}_6$;

θ is the number of main classes of $n \times n$ latin squares in the set $\mathbf{C}_3$;

ρ is the number of main classes of $n \times n$ latin squares in the set $\mathbf{C}_1 \cup \mathbf{C}_2$.

Main result (3.3) Page 41

Corollary 3.4.8 *To isochromatism, there are only as many ways to colour K_n as there are main classes with symmetric representations.* Main result (3.4) page 41

Lemma 3.5.5 *A necessary condition for $\mu_S = \mu(K_{n,n} - E)$ to be a colouring of a critical set for $\mu(K_{n,n})$ is that the subgraph $\mu(E)$ has no potential $(x, y : a)$ swaps.*

Main result (3.5) page 44

Theorem 4.1.5 *Let F be a face of a simple plane graph G, and suppose μ: $V(G) \rightarrow \{c_1, c_2, c_3, c_4\}$ is a vertex colouring of G such that exactly three colours occur on the vertices of F. Then F can be compatibly triangulated.*

Main result (4.1) page 50

Theorem 4.2.2 *Let μ be a 4-colouring of $G - \{v_0\}$ where G is a critical triangulated graph and $d(v_0) = 5$. Let the colours of the neighbouring vertices $v_1, \ldots, v_5$ be as in the standard colouring, let $\phi_h = \phi({}_2[2, 4], {}_5[2, 3])$. Then if G has a Heawood colouring, $\phi_h < 2$.* Main result (4.2) page 56

Theorem 5.2.1 [Holroyd, Williams] *Every edge 3-colouring of a Class 1 (not necessarily planar) cubic graph can be obtained from every other edge 3- colouring of the same graph by a series of (edge) Kempe interchanges using at most four colours.* Main result (5.1) page 69

Theorem 5.3.13 *Two edge 3-colourings of a graph G with tri-star equi-covering have colour difference cd[$\lambda(\sigma)$] where $\lambda = |V(G)|/2$ and $\sigma = 0$.*

Main result (5.5) page 83

Theorem 5.3.14 *The maximal colour difference mcd(G)[$\lambda(\sigma)$] for any graph is such that $\lambda \leq |V(G)|/2$, equality being achieved only when G has a tri-star equi-covering.*

Main result (5.6) page 83

Lemma 6.1.8 *Every graph with at least Δ spines is conformable.*

Main result (6.3) Page 94

Theorem 6.1.12 *Any non-conformable irregular graph G is an induced subgraph of a type 2 conformable graph H of the same maximum degree, where $n(H) = n(G) + 1$.*

Main result (6.5) Page 97

Theorem 6.2.10 *Let G be regular cubic graph. Then a necessary condition for G to be type 1 is that G should have a G*-conformable vertex colouring $\mu(V^*)$ such that, for every pair of colour sets S_i and S_j, the subgraph $G - S_i - S_j$ has:*

(i) at least $(n_i + n_j)/2$ components;

(ii) at most $(n_i + n_j)/2$ components with less than four vertices.

(iii) an even number $2q$ of odd components, where $2q \leq \min\{n_i, n_j\}$.

Main result (6.9) Page 106

Theorem 6.2.13 *Let G be a semi-regular cubic graph. Then a necessary condition for G to be type 1 is that G should have a G*-conformable spine and vertex colouring*

$\mu(V^*)$ (as above) such that, for every pair of colour sets S_i and S_j, the subgraph $G - S_i - S_j$ has:

(i) at least $(n_i + n_j)/2 - \zeta(\{i, j\})$ components;

(ii) at most $(n_i + n_j)/2 - \zeta(\{i, j\})$ components with less than four vertices;

(iii) an odd number of odd components, bounded above by $\min\{n_i, n_j\}$, *except where $\{i, j\} = \{1, 2\}$, in which case an even number of odd components, bounded above by* $\min\{n_1, n_2\} - 1$.

Main result (6.12) Page 109

Theorem 6.3.3 *Let $\Delta(G) \geq 3$ and suppose there is a total $(\Delta + 1)$-colouring μ of $G - e$ such that $\mu(v_1)$, $\mu(s_1)$, $\mu(v_2)$ and $\mu(s_2)$ are not all distinct. Then $\beta \leq \Delta$.*

Main result (6.15) Page 115

Theorem 6.3.4 *Let G be a near Type 1 graph with $\Delta = 3$; then $\beta \leq 2$ unless all Type 1 total colourings μ of $G - e$ have $\mu(v_1) = \mu(v_2)$, $\mu(s_1) \neq \mu(s_2)$, when $\beta \leq 3$.*

Main result (6.16) Page 117

Theorem 6.3.10 *Let $q \geq 1$ and let G be a near Type 1 graph with $\beta(G) > 2(\Delta - 1) + (q - 1)(\Delta - 3) + (\Delta - 2^q + 1) = 2(\Delta+1) + q(\Delta - 3) - 2^q$; then*

$$\Delta(G) \geq 2^{q+2} - 1.$$

Main result (6.21) Page 137

Corollary 6.3.11 *Let G be a near Type 1 graph with $\Delta \geq 4$; then*

$$\beta(G) < \frac{3\Delta}{4} + (\Delta - 3)\log_2(\Delta + 1) + 5.$$

Main result (6.22) Page 138

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CONTENTS

CHAPTER 1

HISTORICAL INTRODUCTION

There is an old Welsh story about Hu Cadarn, the first ruler of Britain. In this story the boy Hu asked: 'Why do we only get barley grass when we throw barley seeds on the ground?' Nobody knew and, in trying to find out for himself, he invented the plough. But, even with the plough, all he can prove is that we only get barley grass when we throw barley seeds on the ground. He still doesn't have the answer to the question 'Why?' This answer is, however, known to Taliesyn in a different old Welsh story, but he never tells us what it is. We in the twenty-first century know that the answer is all to do with DNA, but even (or especially) genetic engineers are well aware that there is still a lot more to be found out about how and why.

There is similarly a not quite so old mathematical story about a young Francis Guthrie who, in 1852, asked his brother Frederick: 'Why can all maps be coloured with four colours?' [1] Nobody knew the answer and, in trying to find out, vertex colouring, edge colouring, total colouring, hyper graphs and other mathematics were all invented; but we are still a long way from discovering **why** 'four colours suffice'.

In 1879 Kempe provided a simple (apparent) proof that any critical map (that is, any map that 'only just' fails to be colourable with four colours) has no triangular, square or pentagonal regions. Since Euler's work led easily to the conclusion that all cubic maps have at least one region with fewer than six borders, no map was critical, hence

[1] Augustus De Morgan in a letter to Sir William Rowan Hamilton 23/10/1852: 'A student of mine asked me today to give him a reason for a fact which I did not know was a fact and do not yet. He says that if a figure be any how divided, and the compartments differently coloured, so that figures with any portion of common boundary [line] are differently coloured – four colours may be wanted but no more.'

all maps are four-colourable. Soon after, Tait gave an alternative proof using edge colouring. However, eleven years later (in 1890) Heawood realised that Kempe had made a mistake in the case when there is a pentagonal region, and the 'proof' was invalid. This made less of a stir than one would expect. Most mathematicians who were interested were prepared to accept the truth of the Four Colour assertion and thought that the answer to 'why?' would 'turn up'. The answer did not turn up and after a while interest revived, but it was not until 1976 that, after a lot of work and computer time, Appel and Haken [1.1] proved that we can indeed colour all maps with just four colours. But even now, at the start of the Third Millennium, we still haven't met our Taliesyn, for not even Appel and Haken have really explained 'why?'

We could, like most parents who have been asked a difficult question such as 'If it's time to go to sleep, then why isn't it dark?' pre-empt any further discussion by saying 'It doesn't matter; just do it!' Alternatively, being mathematicians rather than pragmatists, we can get out the globe and explain it to them, or at least promise to do so in the morning.

In the same way, when it comes to planar graphs and the Four Colour Theorem, we could simply ignore the fact that we don't know much more than Kempe and Heawood did over a hundred years ago. We could give Kempe's proof, up to where he went wrong, and ignore the rest. We couldn't in all honesty answer the question 'Why?' by quoting Appel and Haken, since to do so in detail would take more than a lifetime. Even Robertson, Sanders, Seymour and Thomas [1.2], in their not quite so long proof of the four colour theorem, state: 'We began by trying to read the A&H proof, but very soon gave this up'. Similarly there are many unanswered questions in graph theory and apparent results which have not yet been proven. One of these is

known as the 'Total Colouring Conjecture'. This proposes that for any graph G, $\chi''(G) \leq \Delta(G) + 2$, where $\Delta(G)$ is the maximum vertex degree and $\chi''(G)$ is the total chromatic number (see Chapter 2). This conjecture was proposed independently in 1965 by Mehdi Behzad [1.3], [1.4] and V.G. Vizing [1.5].

Other questions resulting from Guthrie's original question have been completely solved.

For vertex colouring we have a true upper bound due to Brooks [1.6], who proved that for any connected graph G the chromatic number $\chi(G)$ (see Chapter 2) is bounded by: $\chi(G) \leq \Delta(G) + 1$, the bound being attained if and only if G is a complete graph or an odd cycle, K_n or C_{2n+1}. However, determining $\chi(G)$ is in general an NP problem [1.7]; even determining whether $\chi(G) \leq 3$ is NP-complete [1.8].

For the chromatic index $\chi'(G)$ of a multigraph, Shannon [1.9] proved that

$$\chi'(G) \leq \frac{3\Delta(G)}{2}.$$

Later, Vizing proved that for a simple graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ see [1.5]. It is a corollary of the Four Colour Theorem that we need only $\Delta = 3$ colours to edge-colour a planar bridgeless cubic graph. Although it is known that most graphs, planar and non-planar, need only Δ colours, the only general way to find out whether just Δ are indeed necessary is to find such a colouring or show that it is one of a set of known graphs that cannot be so coloured. Again, the problem of determining the chromatic index of a graph or multigraph is NP-complete, even for regular graphs (see [1.7], [1.10], [1.11]).

This may be unsatisfactory, but it is more specific than in the problem of total colouring, where not only is the problem of determining the total colouring number NP-hard, [1.12], but we have no proof of a sensible or indeed probable upper bound. There are proofs for bounds such as $\frac{7}{5}\Delta + 3$ [1.13], $\Delta + c$ $(c \leq 10^{26})$ [1.14], but as no known graph requires even $\Delta + 3$ colours, the latter figure is clearly unrealistic.

It has been shown that the number of graphs requiring more than $\Delta + 1$ colours is very small. Also, should they exist at all, the number of graphs requiring $\Delta + 3$ colours is very much smaller [1.15]; never the less, except for a few categories of easily defined graphs, there is still no way of deciding beforehand how many colours will be needed. Thus there is no known formula for the classification of, nor reasonable upper bound for, the total chromatic number of graphs. However, this thesis attempts to shed light on why certain graphs have certain properties and introduces a few methods of constructing unusual graphs.

Basic definitions and results will be given in Chapter 2 to form a knowledge base for the remainder of the thesis. The structure of graphs and how these relate to other branches of mathematics such as Latin squares will be dealt with in Chapter 3.

Each of the main colourings for simple graphs (vertex, edge and total) will be examined in detail in Chapters 4 to 6, and other kinds of graphs such as hypergraphs will be mentioned where relevant. In these chapters we shall consider counter-examples to Kempe's original proof. Since the method of this erroneous 'proof' is effective for most graphs that have just one uncoloured pentagonal face, each counter-example must be quite exceptional and we can call such (almost) coloured graphs, *Heawood graph-colourings*. It seems then that an interesting area of study is to ask:- What makes a counter-example? Is it possible to create a graph-colouring that

generates new Heawood graph-colourings for a given set of Kempe interchanges? Is it possible to create a set of graph-colourings that are closed under a given set of Kempe interchanges? And so on.

Particular notice will be taken of Kempe chains, and although Kempe considered exchanging colours of only those Kempe chains which reached the pentagon boundary of an otherwise four-coloured graph, we take the same starting point but extend it to other Kempe chains and cycles.

In Chapter 6, we use the beta parameter introduced in Chapter 2. This is a new measure of classifying graphs with total colourings which measures how far from a type 1 graph a type 2 graph can be, though where this may lead is left to the next chapter.

Chapter 7 is concerned with models and applications. We find colourings for given graphs and create graph-growing mathematical models. Some suggestions will be offered as to how these models may be related to the real world and in the realms of recreational mathematics and artistic endeavours. Also the relevance of vertex and edge graph colouring to the real world is discussed, and some conjectures are proposed as to how the field of total colouring may also be related to applicable fields of study.

There are appendices at the end of the thesis which give further details of certain aspects of the thesis which, though relevant, tend to obscure the argument and need to be referred to only as confirmation that the argument is correct. These include a catalogue of 70 small type 2 graphs which are given 'best' $(\Delta + 1)$-colourings.

This thesis discusses the problem surrounding the major outstanding colouring conjectures. For example, Chapter 5 reconsiders the Four Colour Theorem , while the beta values of Chapter 6 shed light on the problem of classification. It is hoped that the various aspects of graph colouring that are here presented will contribute further knowledge in this field and help prepare the way for when these major issues can be dealt with.

CHAPTER 2

BASIC DEFINITIONS

2.1 Graphs and Colourings

2.1.1 Graphs

We will take basic definitions of graphs as in most standard text books (for example, *Total Colourings of Graphs* by H.P. Yap [2.1]), with the following additions and modifications.

- Unless otherwise stated, a graph is always assumed to be simple and connected.
- A plane triangulated graph (that is, a connected plane graph all of whose faces are triangles) is said to be a **plane triangulation**; if just one face is non-triangular, it is said to be a **near triangulation.**

Given any graph G, we denote:

- the number of vertices of G by $n(G)$;
- the number of edges of G by $m(G)$;
- the degree of any vertex v of G by $d(v)$;
- the maximum vertex degree by $\Delta(G)$;
- the difference $\Delta(G) - d(v)$ by $\mathrm{def}(v)$, the **deficiency** of v;
- the sum of vertex deficiencies, $\sum_{v \in V(G)} \mathrm{def}(v)$, by $\mathrm{def}(G)$, the **deficiency** of G.

Let t be any non-negative integer; then the ***t*-deficiency**, $\mathrm{def}_t(v)$, of a vertex v of a graph G is defined by

$$\mathrm{def}_t(v) = \mathrm{def}(v) + t.$$

Note that, if G has a total colouring using $\Delta(G) + 1 + t$ colours, then this quantity represents the number of colours from the colour set that are not present on v or any incident edge.

The ***t*-deficiency** of G is then given by

$$\mathrm{def}_t(G) = \sum_{v \in V(G)} \mathrm{def}_t(v) = \mathrm{def}(G) + tn(G).$$

From now on the argument (G) will be omitted where there is no risk of confusion.

The vertices of degree Δ of a graph are said to be its **major vertices**; the others are **minor vertices.**

Two graphs G and H are said to be **isomorphic** if there is a bijection $\phi : V(G) \to V(H)$ such that $\phi(v)$ is adjacent to $\phi(w)$ in H if and only if v is adjacent to w in G.

2.1.2 Colourings

We assume the usual definitions of vertex, edge and total colourings of a graph (see [2.1]). Given any vertex [resp edge, total] colouring of a graph G, a **colour class** is a subset of $V(G)$ [resp $E(G)$, $E(G) \cup V(G)$] that is the inverse image of a single colour.

We define a **semi-total colouring** of a graph G to be a function μ from $E(G) \cup V(G)$ to a colour set, such that any two adjacent edges have distinct colours and every

vertex has a colour distinct from those of its incident edges. This concept is similar to total colouring except that we do not require adjacent vertices to have distinct colours.

A (vertex, edge, total or semi-total) colouring of a graph using x colours is said to be an **x-colouring.**

We refer to [2.1] for the standard definitions of the **chromatic number, chromatic index** and **total chromatic number** of a graph G, denoted by $\chi(G)$, $\chi'(G)$ and $\chi''(G)$ respectively.

Let G and H be isomorphic graphs, with colourings μ, λ respectively (of the same type: vertex, edge, etc). Then (G, μ) and (H, λ) are said to be **isochromatic** if there is an isomorphism $\phi : G \rightarrow H$ and a bijection θ from the colour set of μ to that of λ such that $\lambda \circ \phi = \theta \circ \mu$.

Let G be any graph. By Vizing [1.5] we have $\Delta \leq \chi' \leq \Delta + 1$, and G is said to be **class 1** if $\chi' = \Delta$ and **class 2** if $\chi' = \Delta + 1$.

For semi-total colourings, the situation is straightforward.

Lemma 2.1.1 *Exactly* $(\Delta + 1)$ *colours are required for a semi-total colouring of a graph G.*

Proof. At least $\Delta + 1$ colours are required since Δ colours are required on the edges incident to a major vertex and one more is needed for the vertex itself. By Vizing's theorem [1.5], the edges of G can be coloured with $\Delta + 1$ or less colours. Each vertex v has at most Δ edges, hence at least one colour is available for the vertex. ■

The situation for the total chromatic number is not so simple. It is necessary to have a different colour on each edge and vertex hence, $\Delta+1 \leq \chi''$, and it is conjectured (but not proven) that $\Delta+1 \leq \chi'' \leq \Delta+2$ (see Behzad and Vizing, [1.4] and [1.5]). Thus, a graph G is said to be **type 1** if $\chi'' = \Delta+1$ and **type 2** otherwise.

In studying the total chromatic number of a graph G, it can be useful to consider a semi-total colouring μ of G with $\Delta + 1$ colours and to ask: 'how close is μ to being a total colouring?' More precisely, we define the parameters β_μ and β as follows:

Given any semi-total colouring μ of G using $(\Delta + 1)$ colours, β_μ is defined to be the number of edges $e = vw$ of G such that $\mu(v) = \mu(w)$. Over all semi-total colourings of G using $\Delta + 1$ colours, β is the minimum value of β_μ.

Consider the problem of constructing a $(\Delta + 1)$-semi-total colouring of G. Given any edge colouring of G (using either Δ or all $\Delta + 1$ colours), then if G is regular, there will be exactly one choice of colour for each vertex; otherwise, there will be vertices where there are several colour choices. It is useful, in the context of total and semi-total colourings, to 'regularise' G by adding def(v) **spines** at each vertex v of degree less than Δ. When we consider colourings of graphs using $(\Delta + 1 + t)$ colours, then we add (def(v) $+ t$) spines to each $v \in V(G)$ to regularise the graph. A spine can be visualised as half of a potential edge (they are sometimes called *semi-edges* or *dangling edges* in the literature). In the case of a total or semi-total colouring of G using $\Delta + 1$ colours, the spines at any vertex v will be assumed to be coloured with the 'spare' colours (those not used on v or its incident edges). The concept of a graph will be extended throughout this thesis, to assume the addition of spines to the 'deficient' vertices as above.

A **plane graph** (or **map**) is a drawing of a graph in the plane without crossings. A **face** of a connected plane graph is a connected component of the complement of the drawing. The number of faces of a plane graph G (including the infinite face) is denoted by $f(G)$. A face colouring of a map is **proper** if and only if no two faces with a common edge are assigned the same colour. The **geometric dual** of a plane map M is a plane graph $G = M^*$ as described above, with a vertex of M^* corresponding to each face of M and *vice versa*, two vertices of M^* being adjacent if and only if the corresponding faces of M share an edge. Thus, where M is cubic, every face of M^* will be a triangle and *vice versa*. The geometric dual of a proper face colouring of a map M is then a proper vertex colouring of M^*.

The term **planar graphs** will be used for graphs, as above, which could be represented as a drawing in the plane without crossings, regardless to whether this has actually been done. Similarly **non-planar graphs** cannot be drawn in the plane without crossings.

2.2 Representing Colourings

Apart from the usual graphic representation of a graph it is often useful to represent the graph as a partial symmetric latin square or a triangular array based on such a square. The main advantage of such a representation is clarity. Graphical colour labels in dense non-planar graphs are not always clear and Kempe chain operations in such graphs are often easiest to follow on a triangular array.

Lemma 2.2.1 *Any given edge, total or semi-total colouring of a graph can be represented as a (possibly partial) latin square.*

Proof. To obtain a (possibly partial) latin square from an edge or total or semi-total colouring of any graph, we need first to label the vertices, $v_1, \ldots, v_n$. Then we colour each of the elements a_{ij} and a_{ji} in the square with the colour of the edge from v_i to v_j and each element a_{ii} in the square with the colour of the vertex v_i. Where the given colouring is an edge colouring, we fill the element a_{ii} in the square with any colour which has not been used on the edges (thus, in effect, extending the edge colouring to a semi-total colouring). Where the graph is not a complete graph, all remaining elements have no entries. These can be left empty or filled with a symbol such as '#' or '*' to indicate that there is no edge between the two given vertices. ■

Corollary 2.2.2 *Any given edge, total or semi-total colouring of a graph can be represented as a (possibly partial) triangular array comprised of the leading diagonal and either the top right hand triangle or the bottom left hand triangle.* ■

Both the (possibly partial) latin square and the triangular array are unambiguous. The latter has the further advantage of having just one element for each edge and is thus easier to manipulate by Kempe chains.

The **cyclic Cayley table** corresponds to the cyclic latin square in which $a_{ij} = i + j - 1 \pmod{n}$.

Lemma 2.2.3 *Every K_n can be assigned a total or semi-total colouring using a cyclic Cayley table.*

Proof. We can have a colouring of K_n because the cyclic Cayley table is symmetric and the leading diagonal provides the vertex colours. When n is odd these are all different, $a_{ij} = 2i - 1 \pmod{n}$, and the colouring is a total colouring; when n is even the colouring is a semi-total colouring. ■

2.3 Kempe Chains

The oldest colouring tool is still the most effective, namely the concept of a **Kempe chain.** These can be defined for all of the colourings mentioned in Subsection 2.1.2.

Where G has a vertex colouring μ, a **vertex Kempe chain** with respect to μ is a maximal set of vertices coloured with just two colours and inducing a connected subgraph of G. More explicitly, let the colour set C be listed as $\{c_1, \ldots, c_x\}$ and vertex set $V(G)$ as $\{v_1, \ldots, v_n\}$, and suppose that $\mu(v_i) = c_p$. Then the **[*p, q*] vertex Kempe chain at v_i** is the maximal set of vertices of G, all of which have colour c_p or c_q, that includes v_i and induces a connected subgraph of G. In the context of vertex colourings, this Kempe chain will be denoted by ${}_i[p, q]$.

Let G have a total or semi-total colouring μ. A **Kempe chain** with respect to μ is a maximal connected set of elements (i.e. vertices, edges or spines) of G coloured with just two colours, say c_p and c_q. There are six possibilities here.

1. The chain could be the edge set of a circuit. If v_i is any vertex on the circuit, then we call the chain the **[*p, q*] Kempe chain, or circuit, through v_i** and denote it by ${}_i]p, q[$ or ${}_i]p, q[_i$. (Note that the colour of v_i cannot be either c_p or c_q in this case).

2. The chain could be $\{v_i\} \cup P \cup \{v_j\}$ where v_i and v_j are the vertices at the ends of a path whose edge set is P. If $\mu(v_i) = c_p$, then we call this chain the **[*p, q*] Kempe chain starting at v_i** and denote it by ${}_i[p, q]_j$. (Note that the colour of v_j may be c_p or c_q depending on the parity of the path length).

3. The chain could be $\{v_i\} \cup P \cup \{s_j\}$ where P is as above and $\mu(v_i) = c_p$, but $\mu(v_j)$ is neither c_p nor c_q; here, s_j is a spine at v_j. We denote this chain by ${}_i[p, q[_j$.

4. Similarly, the chain could be $\{s_i\} \cup P \cup \{v_j\}$, with $\mu(v_i)$ neither c_p nor c_q but $\mu(v_j)$ equal to one of the colours; this chain is denoted by ${}_i]p, q]_j$.

5. If neither end vertex has either of the colours c_p or c_q, the chain would be $\{s_i\} \cup P \cup \{s_j\}$; this chain is denoted by ${}_i]p, q[_j$.

6. Finally, the chain could involve no edges. Thus there is some vertex v_i such that the chain consists of just v_i and a spine at v_i (if $\mu(v_i) = c_p$ or c_q) or of two spines at v_i (otherwise). Only the first possibility is useful; in this case we denote it by ${}_i[p, q[_i$. The other is never referred to further in this thesis.

For any type of Kempe chain, the corresponding **Kempe interchange** or **swap** is an operation in which the two colours on the chain are exchanged, the colours on the remainder of the graph being held constant. The swap is denoted by the symbol "+" and changing the bracket style of the notation for the Kempe chain; for example, the swap corresponding to the chain ${}_i[p, q]_j$ is denoted by $+ {}_i(p, q)_j$. If μ_0 is the original colouring, then the resultant colouring after the colour swap in chain ${}_i[p, q]_j$ is denoted by $\mu_0 + {}_i(p, q)_j$. Similarly, for ${}_i]p, q[_j$, we use the same notation: the resultant colouring is denoted by $\mu_0 + {}_i(p, q)_j$.

Let G have an edge colouring μ. An **edge Kempe chain** with respect to μ is a maximal connected set of edges or spines of G coloured with just two colours, say c_p and c_q. These are defined as for semi-total colourings above except that there are only two possibilities, a circuit and a path.

1. The chain could be the edge set of a circuit. If v_i is any vertex on the circuit, then $[p, q]$ is called the **edge Kempe chain through** v_i and denoted by ${}_i]p, q[_i$ or ${}_i]p, q[$.

2. The chain could be $\{s_i\} \cup P \cup \{s_j\}$ where v_i and v_j are the vertices at the ends of a path whose edge set is P, with spines s_i and s_j each coloured either c_p or c_q. If $\mu(s_i) = c_p$, then we call this chain the **[*p*, *q*] edge Kempe chain starting at v_i** and denote it by ${}_i[p, q]_j$.

In Chapters 5 and 6, we shall make use of the idea of starting with an edge or semi-total $(\Delta + 1)$-colouring and altering it. As we shall see, a Δ-edge-colouring and a $(\Delta + 1)$-edge colouring can both be regarded as a semi-total $(\Delta + 1)$-colouring.

The following lemma follows from the definitions above.

Lemma 2.3.1 *Let μ be a semi-total colouring of G; then for each colour pair c_p, c_q, each vertex v_i of G lies on or is at the end of a Kempe chain ${}_j[p,q]$, where $i = j$ if and only if the colour c_p is on vertex v_i or on a spine at v_i.* ■

2.4 Webs of Kempe Chains

In the following chapters we shall often use Kempe interchanges with the intention of converting one colouring of a graph into another. In the context of achieving a total colouring from a semi-total colouring, it is important to note that two $[p, q]$ Kempe chains in a semi-total colouring, say ${}_i[p, q]_j$ and ${}_r[p, q]_s$, may be incompatible with a total colouring, in that a vertex at an end of one has the same colour (c_p or c_q) as a vertex at an end of the other, and these vertices are adjacent. If (say) v_j and v_s are adjacent with $\mu(v_j) = c_p$ and $\mu(v_s) = c_q$, then they may become incompatible if one of the chains is subjected to a Kempe interchange. Hence the following definitions. Let μ be an edge or semi-total colouring of a graph G using colours $c_1, \ldots, c_{\Delta+1}$; for any

pair of colours c_p, c_q, two $[p, q]$ Kempe chains ${}_i[p, q]_j$ and ${}_r[p, q]_s$ are said to be **web-adjacent** if one of v_i or v_j is adjacent to one of v_r or v_s, the adjacent vertices each independently having one of the colours c_p, c_q. The set of all $[p, q]$ Kempe chains is said to be the **$[p, q]$ Kempe web**, denoted by Σ_μ **$[p, q]$**, for the colouring μ. The **$[p, q]$ Kempe web** is partitioned into **web components**, Σ_i**$[p, q]$**, where v_i is a vertex in that component. Where two vertices v_i and v_j are coloured c_p, c_q; they are **part of the same web component** if and only if it is possible to get from one to another *via* adjacent $[p, q]$ Kempe chains. In total-, semi total- and edge-colourings every complete circuit is a separate web component.

Example 2.4.1. $G = C_7^2$.

The graph $G = C_7^2$ has $V(G) = \{v_1, \ldots, v_7\}$ and $e_{ij} \in E(G)$ if and only if $i - j = 1, 2, 5$ or 6 (where the subscripts are considered modulo 7). This can be given the semi-total colouring μ represented both by the diagram and triangular array in Figure 2.4.1 below.

$\circ$	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_1	1	2	3	*	*	4	5
v_2		5	4	3	*	*	1
v_3			5	1	2	*	*
v_4				4	5	2	*
v_5					3	1	4
v_6						5	3
v_7							2

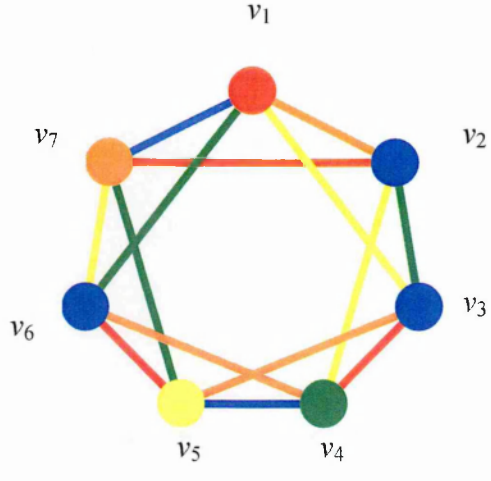


Figure 2.4.1

The web of [1,2] Kempe chains consists of all the 1, 2 entries see (Figure 2.4.2):

$\circ$	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_1	1	2	e_{13}	*	*	e_{16}	e_{17}
v_2		v_2	e_{23}	e_{24}	*	*	1
v_3			v_3	1	2	*	*
v_4				v_4	e_{45}	2	*
v_5					v_5	1	e_{57}
v_6						v_6	e_{67}
v_7							2

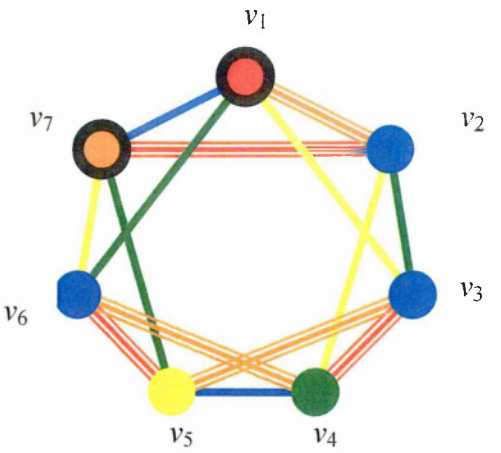


Figure 2.4.2

The web Σ_μ [1,2], coloured in the diagram as [red, orange] has two components, each component comprising one chain, one is a path the other is a circuit:

1) the chain ${}_1[1,2]_7$, consists of the edges e_{12} and e_{27} and the end vertices v_1 and v_7;
2) the circuit ${}_3[1,2]_3$, consists of the edges e_{34}, e_{46}, e_{65} and e_{53}.

There are no other webs in μ with a circuit component. In every other web, the separate paths have adjacent vertices; Σ_μ [1,2] is the only web with two components.

The [1,3], [1,4], [2,3], [2,4], and [3,4]-webs each consist of a single Kempe chain.

The [1,5], [2,5], and [3,5]-webs each have just one connected component comprising a pair of web-adjacent Kempe chains connected by two edges to form a circuit.

For example, $\Sigma_\mu[1, 5]$ comprises the Kempe chain ${}_3[1, 5]_6$ (having edges e_{34}, e_{45}, e_{56} and the end vertices v_3, v_6) and the Kempe chain ${}_2[1, 5]_1$ (having edges e_{27}, e_{71} and the end vertices v_2, v_1); these are web-adjacent by the edges e_{23} and e_{16}, so that the Kempe chains and the web-adjacency edges form a circuit. Finally, the $\Sigma_\mu[4,5]$-web, is comprised of a pair of web-adjacent Kempe chains which do not form a simple circuit. (There are three web-adjacency edges, e_{42}, e_{43} and e_{46}).

$\circ$	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_1	v_1	e_{12}	e_{13}	*	*	4	5
v_2		5	4	e_{24}	*	*	e_{27}
v_3			5	e_{34}	e_{35}	*	*
v_4				4	5	e_{46}	*
v_5					v_5	e_{56}	4
v_6						5	e_{67}
v_7							v_7

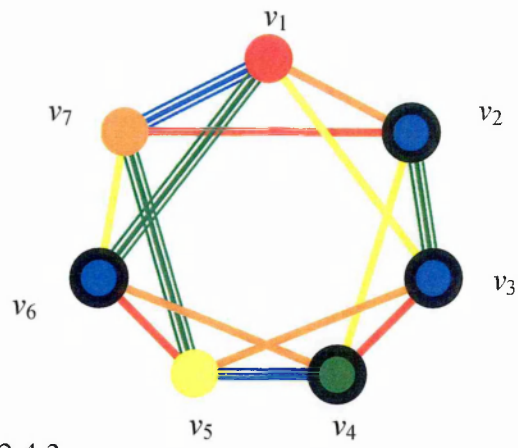


Figure 2.4.3

In general the next edge of a Kempe chain ${}_i[p, q]_j$ can be traversed on an array as follows. Suppose we are at an entry 'q' at a_{ij}, having just arrived from a vertex v_i. From position a_{ij} there will be two entries 'p': the one previously traversed is a_{ix} or a_{xi}; the other is either a_{jk} or a_{kj}. If a_{kj}, follow the column from a_{ij}, to a_{kj}. If a_{jk}, from a_{ij}, follow the column to a_{jj} and then follow the row to a_{jk}. Each successive edge is traversed in a similar way.

Other terms and notation will also be presented in context at appropriate points in the text.

Summary of Chapter 2:

We have presented the basic notation that will be used in the thesis and introduced the new concepts of semi-total colouring and web-adjacency. Since there are currently several different definitions and notations for many of the concepts defined in this thesis, we hope that any reader more familiar with these others, will bear in mind that 'decisions become harder to make as the number of alternatives increases' [2.2] and forgive us if our decisions have not coincided with their own preferences.

CHAPTER 3

GRAPH COLOURINGS AND LATIN SQUARES

3.1 Overview: Latin Squares and Edge Colourings

3.1.1 Introduction

Many combinatorial structures including colourings of graphs can be represented by latin squares. In particular (see Section II.1.2 on page 98 of [3.1]), a latin square of side n can represent the following different combinatorial objects:

- The multiplication table of a quasigroup on n elements;
- a 1-factorization of the complete bipartite graph $K_{n,n}$;
- an edge partition of the complete tripartite graph $K_{n,,n,,n}$ .into triangles;
- a set of n^2 mutually non-attacking rooks on a $n \times n \times n$ board;
- a single error detecting code of word length 3, with n^2 words from an n-symbol alphabet.

In this section, we will also add equivalences relating to K_n and total colourings. The fact that so many apparently different structures are equivalent means that insight into each structure leads to insight in every other. A theorem relating to latin squares such as: *we can always find a latin square of order $n \geq 3$ which has a transversal*, has an analogue for every other equivalent structure. In colouring, this is: for all $n \geq 3$, we can always find an n-edge colouring of $K_{n,n}$ in which a set of n edges, one of each colour, forms a 1-factor, see the remark following the proof of Lemma 8 of [3.2].

We shall explore the relationship between latin squares and colourings of $K_{n,n}$ and K_n. A theorem about the beta parameters of a semi-total colouring of K_n will be presented. We then comment on a conjecture by Mahmoodian.

3.1.2 Latin squares and edge-colourings of $K_{n,n}$

A single **latin square** is an $n \times n$ array of objects each with three distinguishing factors i, j and k (two of which determine its position in the array), where each factor has a range of n values that it can take, such that each value of each factor occurs just once with any other. A set of **mutually orthogonal latin squares, MOLs,** is an $n \times n$ array of objects each with up to $(n + 1)$ distinguishing factors $1, \ldots, n + 1$, such that each factor occurs just once with every other.

Usually, a single latin square is represented as an $n \times n$ array $\{a_{ij}\}$ of cells in which each cell a_{ij} contains just one entry from a set S of n symbols, say $\{1, \ldots, n\}$, such that each symbol occurs exactly once in each row and once in each column. Where the row set is denoted by V and the column set by B; let the rows be called v_i, $i = 1, 2, \ldots, n$, the columns b_j, $j = 1, 2, \ldots, n$ and the entry $a_{ij} = k$ where $k \in S$.

It is well established that an edge colouring of a complete $n \times n$ bipartite graph $K_{n,n}$ can be represented by a latin square and *vice versa* as follows.

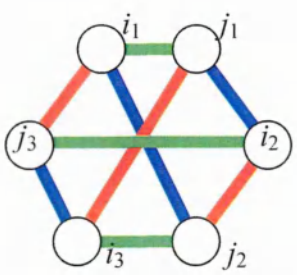
A complete bipartite graph $K_{n,n}$ has vertices in two distinct sets v_i, $i = 1, 2, \ldots, n$ and b_j, $j = 1, 2, \ldots, n$ such that each v_i is adjacent to each b_j but no two vertices v_i are adjacent (nor two b_j's). We can assign a proper edge-colouring to the edges of $K_{n,n}$ using just $\Delta = n$ colours. Each such colouring can be represented as an $n \times n$ square where each element a_{ij} represents the colour used on the edge (v_i, b_j). Every ordering of the vertices of a proper edge colouring of $K_{n,n}$ generates a latin square since each

vertex has one and just one edge of each colour. Every latin square generates an ordering of the vertices of a proper edge colouring of $K_{n,n}$ since each colour is on one and just one entry of each row and column.

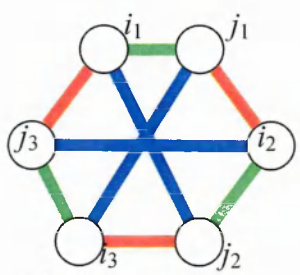
Any set $A = \{A_1, \ldots, A_k\}$ of k mutually orthogonal $n \times n$ latin squares can be represented as an $n \times n$ array $\{a_{ij}\}$, each a_{ij} now being a k-tuple $(a_{ij1}, a_{ij2}, \ldots, a_{ijk})$, the symbols a_{ijp} (for fixed p) corresponding to the latin square A_p. Alternatively, we may represent each cell by a $(k + 2)$-tuple, the first two positions of which record the row and column in the array. Thus, the system is represented by a set S of n^2 objects, each object a $(k + 2)$-tuple $(i, j, a_{ij1}, \ldots, a_{ijk})$. From the $(k + 2)$-tuple $(i, j, a_{ij1}, a_{ij2}, \ldots, a_{ijk})$, the index pair can now be chosen at random to generate further sets of MOLs. Note that, if we choose any two distinct coordinates (say l, m), then for each ordered pair (i, j): $(i, j = 1, \ldots, n)$, there is exactly one $s \in S$ with $s_l = i$ and $s_m = j$.

Example 3.1.1 ($n = 3$) $(i, j: k, l)$

(1, 1: 1, 1), (1, 2: 2, 2), (1, 3: 3, 3), (2, 1: 2, 3), (2, 2: 3, 1), (2, 3: 1, 2), (3, 1: 3, 2), (3, 2: 1, 3), (3, 3: 2, 1)

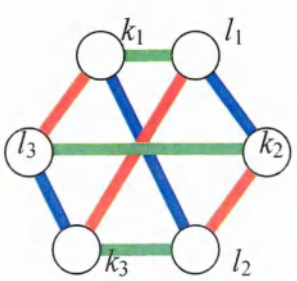


k	j_1	j_2	j_3
i_1	1	2	3
i_2	2	3	1
i_3	3	1	2

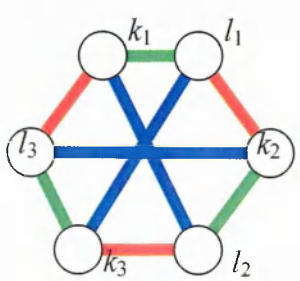


l	j_1	j_2	j_3
i_1	1	2	3
i_2	3	1	2
i_3	2	3	1

Figure 3.1.1(i, j)



j	l_1	l_2	l_3
k_1	1	3	2
k_2	3	2	1
k_3	2	1	3



i	l_1	l_2	l_3
k_1	1	2	3
k_2	3	1	2
k_3	2	3	1

Figure 3.1.1 (k,l)

A latin square A is said to be **in column standard form** if the elements in the first column are in natural order: $a_{i1} = i$ $(i = 1, \ldots, n)$. (*c.f.* row standard form).

A latin square A is said to be **in standard form** if the elements in the first row and column are in natural order: $a_{1i} = a_{i1} = i$ $(i = 1, \ldots, n)$.

Lemma 3.1.1 *A latin square can always be reduced to column standard form by applying a suitable permutation of S.*

Proof. Let A be an $n \times n$ latin square. Apply the permutation $\pi(a_{i1}) = i$ $(i = 1, \ldots, n)$ to all the entries of A; then the first column is now in natural order. ■

Lemma 3.1.2 *Let P and Q be orthogonal latin squares. Then their column standard forms are also orthogonal.*

Proof. If we apply any permutation π_1 to the entries of P and any permutation π_2 to the entries of Q, then by definition the resulting squares are orthogonal. ■

Lemma 3.1.3 *No more than $n - 1$ mutually orthogonal latin squares are possible.*

Proof. Let $P_1, \ldots, P_m$ be mutually orthogonal, $n \times n$ latin squares. By Lemma 3.1.2, we may assume that they are all in column standard form. Then $(P_i)_{11} = 1$ $(i = 1, \ldots, m)$. Now each P_i $(i = 1, \ldots, m)$ must have the symbol 1 on row 2, and these must occur in distinct cells $(2, l_i)$ where none of the l_i can equal 1. Thus, $m \leq n - 1$ as required. ■

A **complete set of mutually orthogonal latin squares** is a set of $(n - 1)$ mutually orthogonal $(n \times n)$ latin squares.

3.1.3 Beta parameter

An $n \times n$ symmetric latin square A may represent a semi-total colouring μ of K_n as follows: $a_{ii} = \mu(v_i)$ $(i = 1, \ldots, n)$;

$$a_{ij} = a_{ji} = \mu(e_{(i,j)}) \quad (i, j = 1, \ldots, n,\ i \neq j).$$

(Note that, for vertices of G labelled $v_1, \ldots, v_n$, we denote the edge v_iv_j by $e_{(i,j)}$)

Using the definition of the beta values given in Chapter 2, in the case of K_n we can express β_μ as follows: let $S_1, S_2, \ldots, S_k$ be the non-empty vertex colour classes; then

$\beta_\mu = \sum_{i=1}^{k} \binom{|S_i|}{2}$. This representation allows us to evaluate $\beta(K_n)$.

Main result (1)

Theorem 3.1.4 $\beta(K_n) = \begin{cases} 0, & (n \text{ odd}) \\ n/2, & (n \text{ even}). \end{cases}$

Proof. Note that $\Delta(K_n) = n - 1$.

Where colouring μ is the cyclic Cayley table in which $a_{ij} = i + j - 1 (\text{mod } n)$, from Lemma 2.2.3, we have:

Case 1: n is odd. Colouring μ is a total colouring of K_n and so $\beta(K_n) = 0$.

Case 2: n is even. Here, the cyclic Cayley table has the odd entries $1, 3, \ldots, n - 1$ each occurring twice on the diagonal, hence in μ, $\beta(K_n) = n/2$. To show that this is the best for all colourings of K_n, we establish that there can be no colouring with more than $n/2$ distinct entries on the diagonal. Since K_n can always be represented by a latin square, A, and n is even, every entry occurs an even number of times. Since A is

symmetric, every entry also occurs in an off-diagonal position an even number of times. Hence, every entry occurs on the diagonal an even number of times. Thus, as required, there cannot be more than $n/2$ distinct diagonal entries. ■

3.2 Conjugates

3.2.1 Colourings of conjugates

The row set, column set and entry set of a latin square may be permuted (set-wise) to produce other, **conjugate,** latin squares. Therefore we let the identity permutation of the initial latin square be called the **(i, j, k)-conjugate**. Let each entry of the square be $a_{ij} = k$, as above. Each element of the latin square can be regarded as an ordered triple (i, j, k) where in the (i, j, k)-conjugate, i and j give the positional information and k the information concerning the entry. There are six possible arrangements of symbols $\{i, j, k\}$; these are (i, j, k), (i, k, j), (j, i, k), (j, k, i), (k, i, j) and (k, j, i). For each of these arrangements we can choose to create a latin square by letting the rows be represented by the symbol in the first position, the columns by the symbol in the middle position and the entry in the latin square by the symbol in the last position. This will give us at most six different latin squares; for each arrangement (x, y, z) of $\{i, j, k\}$ the (y, x, z)-conjugate of L is the transpose of the (x, y, z)-conjugate.

Lemma 3.2.1 *Let x, y, z be any permutation of i, j, k. If the (x, y, z)-conjugate of a latin square is symmetric, then the (x, z, y)- and (y, z, x)-conjugates are identical.*

Proof. Let $L = L^T$ be the (x, y, z)- (and the (y, x, z)-) conjugate and let M and N be respectively the (x, z, y)- and (y, z, x)-conjugates. Then, for any $p, q = 1, 2, \ldots, n$, the (p, q) entry of M is the integer r such that the (p, r) entry of L is q. Thus the (r, p) entry of L is also q, and thus the (p, q) entry of N is r. Thus $M = N$. ■

Main result (2)

Corollary 3.2.2 *If any three conjugates of a latin square are symmetric, then all six conjugates are identical.* ■

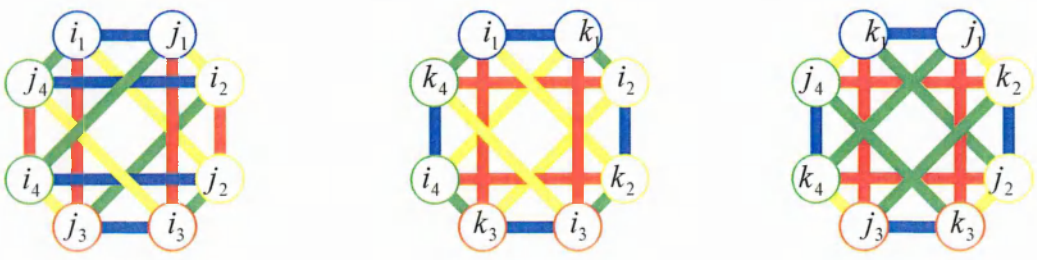
Where an $n \times n$ latin square corresponds to an edge colouring of $K_{n,n}$, we may refer to the entries as **colours,** and consider two latin squares to be **isochromatic** if the two colourings of $K_{n,n}$ are isochromatic. If λ_1 and λ_2 are colourings of $K_{n,n}$ generated by different conjugates of the same latin square, then we will say that λ_1 is **conjugate** to λ_2. The **conjugate colourings** of K_n or $K_{n,n}$ are the colourings generated by the conjugates of L where L is a latin square derived from the colouring of K_n or $K_{n,n}$.

Example 3.2.1

Consider the following set of triples (i, j, k):

(1, 1, 1), (1, 2, 2), (1, 3, 3), (1, 4, 4), (2, 1, 2), (2, 2, 3), (2, 3, 4), (2, 4, 1), (3, 1, 3), (3, 2, 4), (3, 3, 1), (3, 4, 2), (4, 1, 4), (4, 2, 1), (4, 3, 2), (4, 4, 3).

These can be represented as three colourings of $K_{4,4}$ and the six conjugate latin squares (i, j, k), (i, k, j), (j, i, k), (j, k, i), (k, i, j) and (k, j, i) in Figures 3.2.1, K, J, I:



K	j_1	j_2	j_3	j_4
i_1	1	2	3	4
i_2	2	3	4	1
i_3	3	4	1	2
i_4	4	1	2	3

J	k_1	k_2	k_3	k_4
i_1	1	2	3	4
i_2	4	1	2	3
i_3	3	4	1	2
i_4	2	3	4	1

I	j_1	j_2	j_3	j_4
k_1	1	4	3	2
k_2	2	1	4	3
k_3	3	2	1	4
k_4	4	3	2	1

Figure 3.2.1

Note that the other three conjugates give the same three colourings as above:

K	i_1	i_2	i_3	i_4
j_1	1	2	3	4
j_2	2	3	4	1
j_3	3	4	1	2
j_4	4	1	2	3

J	i_1	i_2	i_3	i_4
k_1	1	4	3	2
k_2	2	1	4	3
k_3	3	2	1	4
k_4	4	3	2	1

I	k_1	k_2	k_3	k_4
j_1	1	2	3	4
j_2	4	1	2	3
j_3	3	4	1	2
j_4	2	3	4	1

Figure 3.2.2

3.2.2 Colourings derived from MOLs

We now consider the set of n^2, $(k+2)$-tuples representing k mutually orthogonal squares as above, and we may assume that they are in column standard form. We note that MOLs have been studied in many combinatorial contexts such as in the study of mutually equiorthogonal frequency hypercubes (see [3.3]), but feel that the relationship with graph colourings is of sufficient merit to deserve further study in its own right. As noted in Section 3.1.2, the orthogonality property implies that we could have chosen any of the $(k+2)$ coordinates to be our row indicator and any other to be our column indicator. Hence we could take any of the $\binom{k+2}{2}$ conjugates and obtain another set of k MOLS (not necessarily in column standard form) from the same data: $(x_1, x_2, i, j, \ldots, x_i \ldots, x_k)$, since, by definition, there are no two elements with the same entries in two different orthogonal squares.

Example 3.2.2 The mutually orthogonal set of 4×4 latin squares (i, j, x, y, z):

(1, 1, 1, 1, 1), (1, 2, 2, 2, 2), (1, 3, 3, 3, 3), (1, 4, 4, 4, 4), (2, 1, 2, 4, 3), (2, 2, 1, 3, 4), (2, 3, 4, 2, 1), (2, 4, 3, 1, 2), (3, 1, 3, 2, 4), (3, 2, 4, 1, 3), (3, 3, 1, 4, 2), (3, 4, 2, 3, 1), (4, 1, 4, 3, 2), (4, 2, 3, 4, 1), (4, 3, 2, 1, 4), (4, 4, 1, 2, 3)

can be represented by rows = j, columns = i, or by rows = x ,columns = y, to give the sets of mutually orthogonal latin squares in Figures 3.2.3 and 3.2.4 which are also represented by their graph colourings.

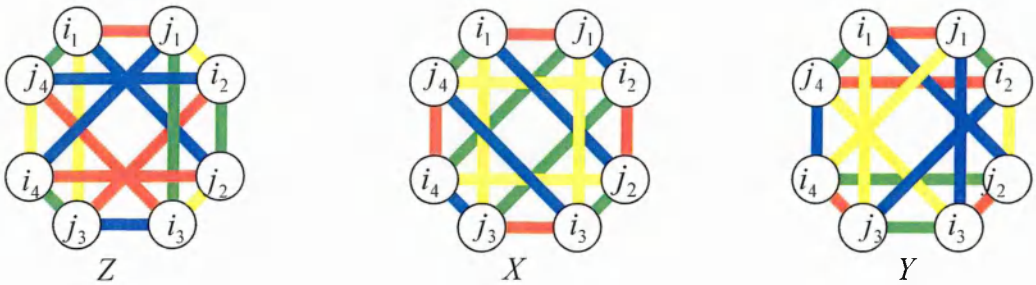


Figure 3.2.3

Z	i_1	i_2	i_3	i_4
j_1	1	3	4	2
j_2	2	4	3	1
j_3	3	1	2	4
j_4	4	2	1	3

X	i_1	i_2	i_3	i_4
j_1	1	2	3	4
j_2	2	1	4	3
j_3	3	4	1	2
j_4	4	3	2	1

Y	i_1	i_2	i_3	i_4
j_1	1	4	2	3
j_2	2	3	1	4
j_3	3	2	4	1
j_4	4	1	3	2

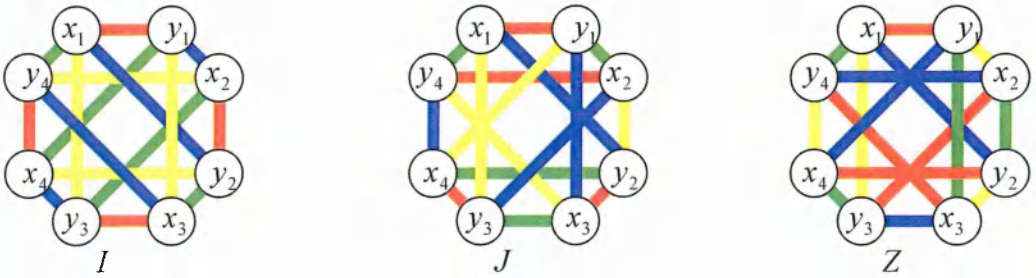


Figure 3.2.4

I	y_1	y_2	y_3	y_4
x_1	1	4	2	3
x_2	4	1	3	2
x_3	2	3	1	4
x_4	3	2	4	1

J	y_1	y_2	y_3	y_4
x_1	1	4	2	3
x_2	3	2	4	1
x_3	4	1	3	2
x_4	2	3	1	4

Z	y_1	y_2	y_3	y_4
x_1	1	3	4	2
x_2	4	2	1	3
x_3	2	4	3	1
x_4	3	1	2	4

■

We can see from the figures that the three colourings in $\{(X, Y, Z)/\ i, j\}$ are isochromatic to those in $\{(I, J, Z)/\ x,y\}$. Moreover, in each set of three, we find that by re-labelling they are mutually isochromatic. For example, I and J become the same when we re-label as follows: $I(x_2) = x_3$, $I(x_3) = x_4$, $I(x_4) = x_2$. Hence we could choose colours to make each set of colourings look the same.

3.3 Isochromatic Conjugates

3.3.1 Isotopy

Two latin squares L, M are **isotopic** if there are bijections from the row, column and entry sets of L to the respective sets of M, that map L to M.

Given two such latin squares, the bijections mentioned in the definition are equivalent to exchanging the orders of the vertices within the rows and columns, and re-labelling the entries of L, to give us M. That is, we perform a permutation on the vertex set (that gives an automorphism of the graph) and a permutation on the colour set. Thus, the corresponding coloured graphs are isochromatic.

Remark 3.3.1 It follows that two isochromatic edge colourings λ and μ of $K_{n,n}$ can be represented by two $n \times n$ latin squares L and M which are either isotopic, or such that L is isotopic to M^T. Conversely, if L and M are thus related, then they represent isochromatic edge colourings of $K_{n,n}$. ■

Since every conjugate of a latin square is also a latin square, we can see that every colouring of $K_{n,n}$ has conjugates which are also colourings of (various copies of) $K_{n,n}$. That is, the original $K_{n,n}$ has vertex set $V \cup B$ and colour set S, while the conjugates also give complete bipartite graphs with vertex sets $V \cup S$ and $B \cup S$.

The colourings of $K_{n,n}$ generated by the conjugates of any $n \times n$ latin square can be seen to have a very close relationship.

In the first 4×4 example, 3.2.1 above, there are three distinct conjugates, Figure 3.3.1 (a), (b), (c). One latin square is symmetric, the other two are transpositions of each other. To isochromatism, all three can be represented by the same colouring, Figure 3.3.1 (d); we must however re-order one of the vertex sets.

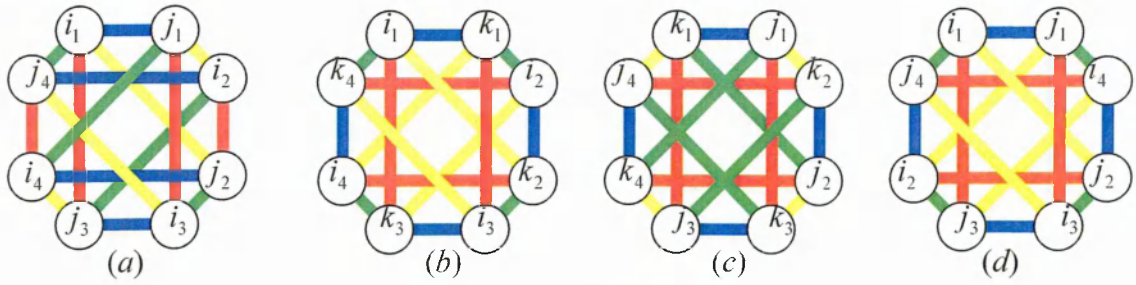


Figure 3.3.1

This follows because example 3.2.1 is in fact the cyclic Cayley table with $n = 4$, and so it is isotopic to all its conjugates by the following theorem of Denes and Keedwell.

Theorem 3.3.1 (Denes and Keedwell [3.4, Theorem 4.2.2]) *Let A be an $n \times n$ cyclic Cayley table or any group table. Then every conjugate of A is isotopic to A.* ■

Corollary 3.3.2 *Let A be an $n \times n$ cyclic Cayley table representing an edge colouring, λ, of $K_{n,n}$ then every conjugate of A represents λ to isochromatism.*

Proof. We have seen above that isotopic $n \times n$ latin squares represent the same colouring of $K_{n,n}$ to isochromatism. ■

We now consider other circumstances under which we find isochromatic conjugates.

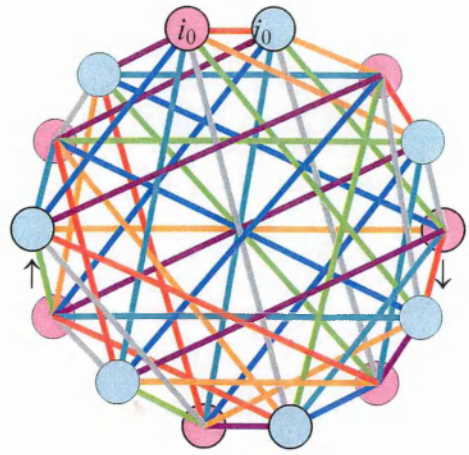
***Lemma* 3.3.3** *Two colourings of $K_{n,n}$ are isochromatic only if the webs have the same distribution of paths and circuits.* ■

***Corollary* 3.3.4** *There exist latin squares A such that not every conjugate of A represents the same colouring of $K_{n,n}$ to isochromatism.*

Proof. That in general the conjugates are not isochromatic can been seen by considering the bi-colour circuits. Consider the 7×7 latin square A (see [3.1], Table II.1.7, latin square 7.56), shown in Figure 3.3.2.

Figure 3.3.2

K	j_0	j_1	j_2	j_3	j_4	j_5	j_6
i_0	0	1	2	3	4	5	6
i_1	1	0	3	2	5	6	4
i_2	2	3	0	4	6	1	5
i_3	3	4	6	5	1	0	2
i_4	4	5	1	6	2	3	0
i_5	5	6	4	0	3	2	1
i_6	6	2	5	1	0	4	3



The corresponding edge-colouring of $K_{7,7}$ has the following set of circuits for each pair of colours:

$]0, 1[= C_4 \cup C_{10}$,	$]0, 2[= C_4 \cup C_{10}$,	$]0, 3[= C_4 \cup C_{10}$,
$]0, 4[= C_4 \cup C_{10}$,	$]0, 5[= C_6 \cup C_8$,	$]0, 6[= C_{14}$,
$]1, 2[= C_{14}$,	$]1, 3[= C_{14}$,	$]1, 4[= C_4 \cup C_4 \cup C_6$,
$]1, 5[= C_{14}$,	$]1, 6[= C_4 \cup C_{10}$,	$]2, 3[= C_4 \cup C_{10}$,
$]2, 4[= C_{14}$,	$]2, 5[= C_{14}$,	$]2, 6[= C_4 \cup C_{10}$,
$]3, 4[= C_{14}$,	$]3, 5[= C_{14}$,	$]3, 6[= C_4 \cup C_4 \cup C_6$,
$]4, 5[= C_{14}$,	$]4, 6[= C_4 \cup C_{10}$,	$]5, 6[= C_6 \cup C_8$.

Note that only two of the above webs, $]0, 5[$ and $]5, 6[$, are $C_6 \cup C_8$.

However the circuit colourings in the conjugates are different.

The (i, k, j)-conjugate of A, is given in Figure 3.3.3.

J	k_0	k_1	k_2	k_3	k_4	k_5	k_6
i_0	0	1	2	3	4	5	6
i_1	1	0	3	2	6	4	5
i_2	2	5	0	1	3	6	4
i_3	5	4	6	0	1	3	2
i_4	6	2	4	5	0	1	3
i_5	3	6	5	4	2	0	1
i_6	4	3	1	6	5	2	0

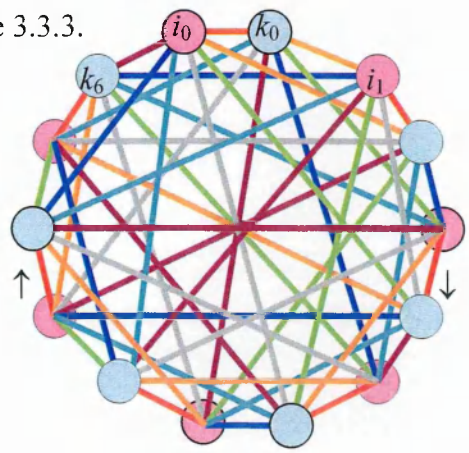


Figure 3.3.3

The edge-colouring circuits are as follows:

$]0, 1[= C_4 \cup C_{10}$,	$]0, 2[= C_4 \cup C_{10}$,	$]0, 3[= C_6 \cup C_8$,
$]0, 4[= C_6 \cup C_8$,	$]0, 5[= C_{14}$,	$]0, 6[= C_{14}$,
$]1, 2[= C_4 \cup C_{10}$,	$]1, 3[= C_{14}$,	$]1, 4[= C_4 \cup C_4 \cup C_6$,
$]1, 5[= C_6 \cup C_8$,	$]1, 6[= C_4 \cup C_{10}$,	$]2, 3[= C_4 \cup C_4 \cup C_6$,
$]2, 4[= C_{14}$,	$]2, 5[= C_6 \cup C_8$,	$]2, 6[= C_4 \cup C_{10}$,
$]3, 4[= C_{14}$,	$]3, 5[= C_4 \cup C_{10}$,	$]3, 6[= C_6 \cup C_8$,
$]4, 5[= C_4 \cup C_{10}$,	$]4, 6[= C_6 \cup C_8$,	$]5, 6[= C_{14}$.

Note that here six webs: $]0, 4[$, $]1, 5[$, $]2, 5[$, $]4, 6[$, $]0, 3[$ and $]3, 6[$ are $C_6 \cup C_8$.

The (k, j, i)-conjugate is given in Figure 3.3.4.

I	j_0	j_1	j_2	j_3	j_4	j_5	j_6
k_0	0	1	2	5	6	3	4
k_1	1	0	4	6	3	2	5
k_2	2	6	0	1	4	5	3
k_3	3	2	1	0	5	4	6
k_4	4	3	5	2	0	6	1
k_5	5	4	6	3	1	0	2
k_6	6	5	3	4	2	1	0

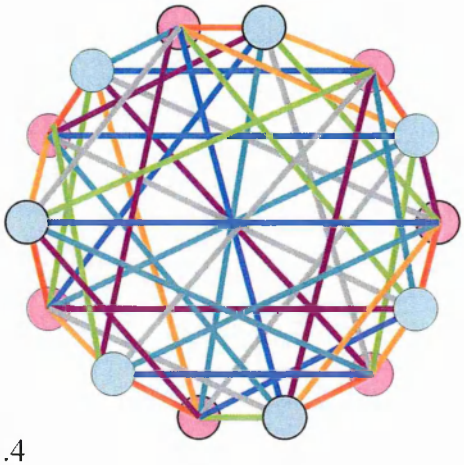


Figure 3.3.4

The edge-colouring circuits are as follows:

$]0, 1[= C_4 \cup C_{10}$,	$]0, 2[= C_4 \cup C_{10}$,	$]0, 3[= C_4 \cup C_4 \cup C_6$,
$]0, 4[= C_{14}$,	$]0, 5[= C_4 \cup C_{10}$,	$]0, 6[= C_{14}$,
$]1, 2[= C_4 \cup C_{10}$,	$]1, 3[= C_{14}$,	$]1, 4[= C_{14}$,
$]1, 5[= C_6 \cup C_8$,	$]1, 6[= C_{14}$,	$]2, 3[= C_6 \cup C_8$,
$]2, 4[= C_6 \cup C_8$,	$]2, 5[= C_4 \cup C_4 \cup C_6$,	$]2, 6[= C_4 \cup C_{10}$,
$]3, 4[= C_{14}$,	$]3, 5[= C_4 \cup C_{10}$,	$]3, 6[= C_{14}$,
$]4, 5[= C_4 \cup C_{10}$,	$]4, 6[= C_{14}$,	$]5, 6[= C_6 \cup C_8$.

Note that four webs: $]1, 5[$, $]2, 3[$, $]2, 4[$ and $]5, 6[$ have $C_6 \cup C_8$.

Since each conjugate has a different number of webs with circuits $C_6 \cup C_8$, there is no isochromatism between any of them. ■

3.3.2 Uniqueness of colourings of K_n and $K_{n,n}$

When the leading diagonal of a latin square is a transversal, all the vertex colours of K_n are different. When this is a symmetric latin square we have a proper total colouring of K_n. As only odd values of n give a proper total colouring of K_n, this can only occur when n is odd. It has been noted (in different words) by Yap [2.1 page 119] that up to isochromatism, K_3 and K_5 have just one (Δ+1)-total colouring whereas K_7 has two. The unique colourings of K_3 and K_5 are both cyclic Cayley tables and hence, from Corollary 3.2.5, all conjugates are isochromatic. In general this is not the case but we will show that for $K_{n,n}$ there are at most three different conjugate colourings and for K_n, at most two. We must note here that the definition of 'the same' in [2.1], Yap, and the definition of 'UTC, uniquely total colourable' in [3.4], Akbari et al., are different: UTC is only concerned with graphs that have a fixed vertex labelling. Both allow colourings to be the same if we change the labels of the colours, but only [2.1] allow two colourings to be the same if we change the labels of the vertices. Since the definition by Yap is contained in our definition of isochromatism, this is the one that we shall use in general. However, we have to use the UTC definition in Chapter 5 and also in Section 6.4 in order to make use of existing theorems. But which definition is in use will be made clear at the appropriate time.

By [3.4, Theorem 4.2.1], a latin square L and its conjugates form 1, 2, 3 or 6 distinct isotopy classes. We shall say that L belongs to the system ψ_1, ψ_2, ψ_3 or ψ_6 accordingly.

The proof also implies that when $L \in \psi_c$ then each class consists of $6/c$ of the conjugates. We now describe the isotopy classes for each ψ_c.

In ψ_1, all conjugates are isotopic.

In ψ_2, the (i, j, k), (j, k, i) and (k, i, j) conjugates are isotopic, as are the (j, i, k), (i, k, j) and (k, j, i) conjugates.

If a latin square L belongs to ψ_3, then there are three possibilities for the isotopy classes. The three pairs of isotopic conjugates may be:

(i) the (i, j, k) and (j, i, k) conjugates; the (i, k, j) and (j, k, i) conjugates; the (k, j, i) and (k, i, j) conjugates;

(ii) the (i, j, k) and (i, k, j) conjugates; the (j, i, k) and (k, i, j) conjugates; the (k, j, i) and (j, k, i) conjugates;

(iii) the (i, j, k) and (k, j, i) conjugates; the (j, i, k) and (j, k, i) conjugates; the (i, k, j) and (k, i, j) conjugates.

In ψ_6, each conjugate is a distinct isotopy class.

Lemma 3.3.5 *If a latin square L belongs to the system ψ_1 or ψ_2, then all conjugate colourings of the corresponding $K_{n,n}$ are isochromatic. If L belongs to the system ψ_3, then there are two non-isochromatic conjugate colourings of the corresponding $K_{n,n}$. Finally, if L belongs to the system ψ_6, then there are three non-isochromatic conjugate colourings of the corresponding $K_{n,n}$.*

Proof. By definition, each latin square isotopic to L generates an isochromatic colouring of $K_{n,n}$. Therefore, if a latin square is isotopic to all of its conjugates, then all conjugate colourings are isochromatic.

If L belongs to ψ_2, then the colourings corresponding to the (i, j, k), (j, k, i) and (k, i, j) conjugates are isochromatic, as are those corresponding to the (j, i, k), (i, k, j) and

(k, j, i) conjugates. However, the colourings corresponding to the (i, j, k) and (j, i, k) conjugates are also isochromatic, by Remark 3.3.1. Thus all conjugate colourings are isochromatic.

Now suppose L belongs to ψ_3. There are three cases to consider, corresponding to the above possibilities for the isotopy classes.

Case (i) Here, each pair of isotopic conjugates gives isochromatic colourings. Also, the colourings corresponding to the (i, k, j) and (k, i, j) conjugates are isochromatic, by Remark 3.3.1. Thus, all four of the (i, k, j), (j, k, i), (k, j, i) and (k, i, j) conjugates are isochromatic. However, none of the latter correspond to L or L^T, and so by Remark 3.3.1 they are not isochromatic to the (i, k, j) and (k, i, j) conjugates.

Cases (ii) and (iii) follow by analogy.

Finally, suppose L belongs to ψ_6. Then, by Remark 3.3.1, two conjugates give isochromatic colourings if and only if the corresponding latin squares are transposes of each other. Thus, there are three non-isochromatic colourings, corresponding to: the (i, j, k) and (j, i, k) conjugates; the (j, k, i) and (k, j, i) conjugates; and the (k, i, j) and (i, k, j) conjugates. ■

Corollary 3.3.6 *If any three conjugate colourings λ of $K_{n,n}$ have symmetric latin squares, then all conjugate colourings are identical.* ■

Note that Corollary 3.3.6 also follows from Corollary 3.2.2.

Corollary 3.3.7 *Every (semi-)total colouring λ of K_n has no other conjugate colouring to isochromatism.*

Proof. A latin square can only represent a (semi-)total colouring λ of K_n if it is symmetric. If the conjugate colouring μ is also symmetric, then it is identical to λ, if it is asymmetric then it is not a (semi-)total colouring. If μ is isotopic to a symmetric latin square θ, then θ has two identical conjugates one of which is isotopic to λ, hence all conjugates are isochromatic to λ. ■

Example 3.3.8

The colourings μ_A and μ_B with the latin squares A and B representing the two different colourings of K_7 mentioned in [2.1] are given in Figure 3.3.5, below:

A	j_1	j_2	j_3	j_4	j_5	j_6	j_7
i_1	1	2	3	4	5	6	7
i_2	2	3	4_A	5	6	7	1_A
i_3	3	4_A	5	6	7	1_A	2
i_4	4	5	6	7	1_A	2_A	3
i_5	5	6	7	1_A	2_A	3	4_A
i_6	6	7	1_A	2_A	3	4_A	5
i_7	7	1_A	2	3	4_A	5	6

B	j_1	j_2	j_3	j_4	j_5	j_6	j_7
i_1	1	2	3	4	5	6	7
i_2	2	3	1_B	5	6	7	4_B
i_3	3	1_B	5	6	7	4_B	2
i_4	4	5	6	7	2_B	1_B	3
i_5	5	6	7	2_B	4_B	3	1_B
i_6	6	7	4_B	1_B	3	2_B	5
i_7	7	4_B	2	3	1_B	5	6

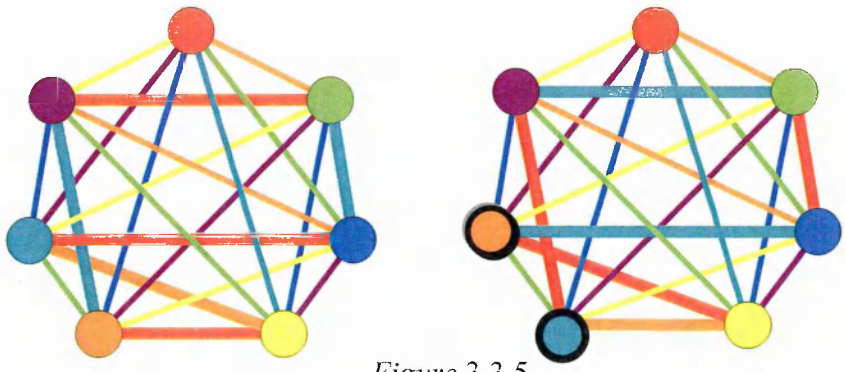


Figure 3.3.5

The symbols that differ in these latin squares are indicated by the suffix A or B. In the graph colouring the differences are shown as thicker lines.

From Corollary 3.3.2, since A is a cyclic Cayley table all the conjugates are isochromatic to μ_A. Call the other colouring the (i, j, k) –conjugate of B. The

symmetry in μ_B ensures that the (i, k, j)- and (j, k, i)- conjugates are identical to each other but they are not symmetric and hence are not isochromatic to μ_B.

Let the (i, k, j)-conjugate of B be C. The colourings μ_B and μ_C of $K_{7,7}$ from B and C respectively, are also shown in Figures 3.3.6 and 3.3.7. That they are not the same can be seen by the distribution of chains, since

$$\mu_B: \quad 18 \times (C_{14}) \cup 6 \times (C_4) \cup 3 \times (C_6) \quad \text{and}$$

$$\mu_C: \quad 9 \times (C_{14}) \cup 6 \times (C_4) \cup 6 \times (C_6) \cup 6 \times (C_8) \cup 6 \times (C_{10})$$

B	j_1	j_2	j_3	j_4	j_5	j_6	j_7
i_1	1	2	3	4	5	6	7
i_2	2	3	1	5	6	7	4
i_3	3	1	5	6	7	4	2
i_4	4	5	6	7	2	1	3
i_5	5	6	7	2	4	3	1
i_6	6	7	4	1	3	2	5
i_7	7	4	2	3	1	5	6

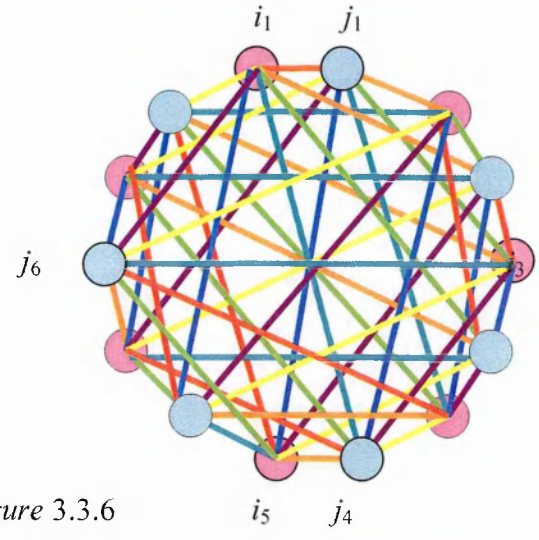


Figure 3.3.6

C, the (i, k, j)- conjugate of B

C	k_1	k_2	k_3	k_4	k_5	k_6	k_7
i_1	1	2	3	4	5	6	7
i_2	3	1	2	7	4	5	6
i_3	2	7	1	6	3	4	5
i_4	6	5	7	1	2	3	4
i_5	7	4	6	5	1	2	3
i_6	4	6	5	3	7	1	2
i_7	5	3	4	2	6	7	1

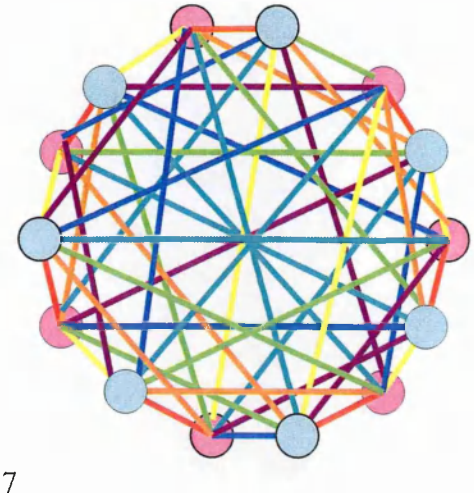


Figure 3.3.7

D, the (j, k, i)- conjugate of B is identical to C. ■

3.4 Main Classes of Colourings

3.4.1 Sub-squares

When a set of m elements appear on the same m rows and m columns, this is called an **$m\times m$ sub-square.**

Theorem 3.4.1 *The number of $m\times m$ sub-squares is the same in every conjugate.*

Proof. Taking any typical $m\times m$ sub-square in A, suppose its rows and columns are the intersections of rows $i_1, \ldots, i_m$ with columns $j_1, \ldots, j_m$ of A. Then there are exactly m distinct entries (colours), $k_1, \ldots, k_m$, in the positions occupied by the sub square. In (for example) the (i, k, j)-conjugate, B, there will be a corresponding $m\times m$ sub-square whose rows and columns are the intersections of rows $i_1, \ldots, i_m$ with columns $k_1, \ldots, k_m$ of B, and whose distinct entries are $j_1, \ldots, j_m$. Thus, there is a 1-to-1 correspondence between the $m\times m$ sub-squares of A and B; and a similar argument applies to the other conjugates. ■

Lemma 3.4.2 *A 3×3 sub-square has just three Kempe circuits.*

Proof. Let the colours in the sub-square be a, b and c. No circuit can have more than six elements. If the 3×3 had a 2×2 sub-square, then it would not be possible to have just three colours and so would not be a sub square, therefore every circuit has six elements. Every set of two colours must be in just one of these circuits, therefore we have $]a, b[$, $]a, c[$, and $]c, b[$. ■

Theorem 3.4.3 *The number of Kempe circuits in all 2×2 and 3×3 sub-squares is the same in every conjugate.*

Proof. From Theorem 3.4.1 above, every 2×2 sub-square has a one to one correspondence with a 2×2 sub-square in each conjugate; therefore the number of

such circuits is invariant. Since for each 3×3 sub-square there are just three circuits, the number of Kempe circuits remains the same. ■

However the same is not necessarily true of other circuits, as was seen in Example 3.3.8. The elements in a general m-circuit C_m in a colouring λ with latin square A, will appear on just two columns in the (i, k, j)-conjugate, colouring θ; this is not to say that there is no isochromatic circuit C_m in θ, but its existence is not automatic.

3.4.2 Main classes

Two latin squares L and M, are **main class isotopic** if L is isotopic to a conjugate of M. A **main class of latin squares** is an equivalence class under main class isotopy. The edge-colourings of $K_{n,n}$ corresponding to a main class of $n\times n$ latin squares is said to be a **main class of colourings**.

By [3.4, Theorem 4.2.1] the main classes of latin squares fall into four sets, depending on whether L belongs to the system ψ_1, ψ_2, ψ_3 or ψ_6 of Section 3.3.2.

We now mention an important theorem by Pittenger [3.5] which we rephrase in terms of colourings of $K_{n,n}$. We recall, Chapter 2, that where μ is a colouring of $K_{n,n}$, a Kempe chain swap $\mu_2 = \mu + {}_i(x, y)$ will swap the colours of the circuit ${}_i]x, y[$ which goes through v_i. Where c_{n+1} is a dummy colour not used on any edge of μ, or μ_3 where $\mu(e_{ij}) = c_a$, $\mu(e_{kl}) = c_a$, and where $c_x \neq c_y$ and both $\mu(e_{il}) = c_x$, $\mu(e_{kj}) = c_y$ are in the same circuit ${}_i]x, y[$, then a **$(x, y{:}a)$ swap,** $\mu_3 = \mu + {}_{ik}(x, y{:}a)$, is such that either

$\mu_{3.x} = \mu + {}_i(a, n{+}1)_j + {}_k(a, n{+}1)_l + {}_i(a, x)_l + {}_k(a, y)_j + {}_j(y, x)_l + {}_i(x, n{+}1)_j + {}_k(y, n{+}1)_l$ or

$\mu_{3.y} = \mu + {}_i(a, n{+}1)_j + {}_k(a, n{+}1)_l + {}_i(a, x)_l + {}_k(a, y)_j + {}_i(x, y)_k + {}_i(y, n{+}1)_j + {}_k(x, n{+}1)_l$.

See the example Figure 3.4.1 where $\mu_1 = \mu_2 + {}_{1,3}(4, 2{:}0)_x$.

Should we need to distinguish which particular swap that we are making we shall denote $\mu_{3.x}$ by $_{i,k}(x, y{:}a)$ because $\mu_{3.x}(e_{ij}) = c_x$, and $\mu_{3.y}$ by $_{i,k}(y, x{:}a)$ because $\mu_{3.y}(e_{ij}) = c_y$. These swaps are reversible as shall be shown in section 3.5.

K	j_1	j_2	j_3	j_4	j_5	j_6
i_1	0	1	2	3	4	5
i_2	1	2	3	4	5	0
i_3	2	3	4	5	0	1
i_4	3	4	5	0	1	2
i_5	4	5	0	1	2	3
i_6	5	0	1	2	3	4

μ_1

K	j_1	j_2	j_3	j_4	j_5	j_6
i_1	4	1	2	3	0	5
i_2	1	2	3	4	5	0
i_3	0	3	4	5	2	1
i_4	3	4	5	0	1	2
i_5	2	5	0	1	4	3
i_6	5	0	1	2	3	4

μ_2

figure 3.4.1

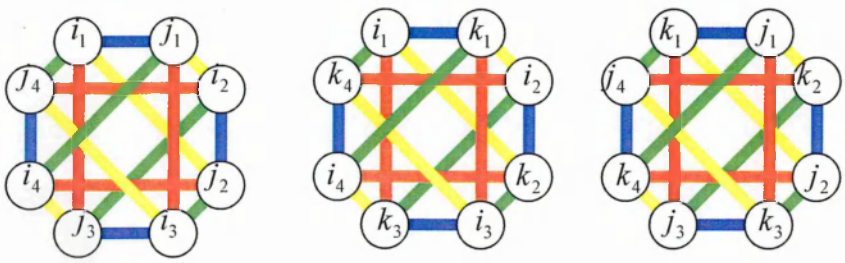
Theorem 3.4.4 [Pittenger – Re-phrased] *It is possible to swap from one main class colouring of $K_{n,n}$ to any other via the cyclic Cayley table by performing Kempe chain and (x, y:a) swaps.* ■

That the $(x, y{:}a)$ swap is necessary can be seen from the cyclic Cayley table μ_1 where n is a prime number. This has all Kempe circuits of length $2n$ and hence all $\mu_2 = \mu_1 + {}_i(x, y)$ are isochromatic to μ_1. Compare the latin squares μ_A and μ_B in figure 3.3.5. The [1, 3]-web show that these are not isochromatic since all circuits are C_{14} in μ_A but we have C_4, C_4 and C_6 in μ_B. In most cases we need only Kempe chains to swap between main classes. In the case of $K_{4,4}$ we can swap any 2×2 subsquare in A to get B; as shown below.

Theorem 3.4.5 (Denes and Keedwell [3.4, p 129]) *There are only two* 4×4 *main class latin squares: the cyclic group and the Klein group. Every* 4×4 *latin square is isotopic to all it's conjugates.* ■

Corollary 3.4.6 *To isochromatism, there are just two main class colourings of $K_{4,4}$.*

Proof. To isochromatism there is just one colouring of the main class $K_{4,4}$ given in Example 3.2.1. Call this main class A. A is the cyclic Cayley table which we know has the same colouring for every conjugate. Since we cannot have any 3×3 subsquares, the only other main class 4×4 latin square (known as a Klein group) is B where $B = A + i_2(1, 3)$. That A and B are not isochromatic can be seen from the fact that every web in B has two components, each a C_4, whereas there are only two such webs in A, the remaining webs having just one component, a C_8. Not only are the conjugates of B isochromatic, their latin squares are identical. See figures below.



(a) (b) (c)

Figure 3.4.1: *The Klein group conjugate colourings of* $K_{4,4}$.

K_B	j_1	j_2	j_3	j_4
i_1	1	2	3	4
i_2	2	1	4	3
i_3	3	4	1	2
i_4	4	3	2	1

J_B	k_1	k_2	k_3	k_4
i_1	1	2	3	4
i_2	2	1	4	3
i_3	3	4	1	2
i_4	4	3	2	1

I_B	j_1	j_2	j_3	j_4
k_1	1	2	3	4
k_2	2	1	4	3
k_3	3	4	1	2
k_4	4	3	2	1

Figure 3.4.2: *The Klein group main class* 4×4 *latin square and conjugates.* ■

Main result (3)

Theorem 3.4.7 *To isochromatism, the number of ways to colour $K_{n,n}$ is equal to*

$$3\omega + 2\theta + \rho$$

where

ω *is the number of main classes of* $n \times n$ *latin squares in the system* ψ_6;

θ *is the number of main classes of* $n \times n$ *latin squares in the system* ψ_3;

ρ *is the number of main classes of* $n \times n$ *latin squares in the union* $\psi_1 \cup \psi_2$.

Proof. The result follows directly from Lemma 3.3.5. ■

Main result (4)

Corollary 3.4.8 *To isochromatism, there are only as many ways to (semi)-total colour K_n as there are main classes of $n \times n$ latin squares containing a symmetric latin square.*

Proof. By definition, each latin square isotopic to L generates an isochromatic colouring of K_n. From Corollary 3.3.7, a (semi)-total colouring λ of K_n has no other conjugate colouring to isochromatism. L can represent λ only if it is symmetric. Hence there is only one isochromatic colouring for each main class containing a symmetric latin square. ■

3.5 Critical Sets

We can apply the parallel between latin squares and graph colourings to various other concepts involving latin squares. For example hypercubes [3.6] and secret sharing schemes [3.7]. The latter are related to the following problem: Given a particular edge-colouring μ of K_n, consider $\mu_{Sn} = \mu(K_n - E)$ where $E \subseteq E(K_n)$. What is the maximum value of $|E|$ such that μ is the unique edge-colouring of K_n to which μ_{Sn} can be extended? By Corollary 3.4.8, this is the same number for every latin square in each main class. Hence, although it is not yet known how many edges are neccessarily in μ_{Sn} for every n, we know that, to isochromatism, we can restrict our search to just one latin square for each symmetric main class.

In a similar way, for edge colourings of $K_{n,n}$, we can re-phrase a conjecture by Mahmoodian, who defines a **critical set**, μ_S, to be a minimal set of coloured edges $\mu(K_{n,n} - E)$ of $K_{n,n}$ such that μ_S has a unique extension to an edge colouring μ of $K_{n,n}$.

Conjecture 3.5.1 [Mahmoodian [3.8] – Re-phrased] *For any edge colouring μ of $K_{n,n}$ the number of edges in any critical set, $\mu_S = \mu(K_{n,n} - E)$, is greater than or equal to $\lfloor n^2/4 \rfloor$.* ■

We can now consider the answer to this conjecture in relation to the circuits in $K_{n,n}$. However, we must point out that in terms of the original conjecture, two latin squares are considered to be different if the vertex labels are different, hence two isochromatic colourings are not necessarily considered to be the same.

When considering $\mu_S = \mu(K_{n,n} - E)$ and $\mu(E)$ to be restrictions of a colouring of $K_{n,n}$ where the vertex labels are fixed, and $\mu_S(E)$ to be the uncoloured edges in $(K_{n,n} - \mu_S)$, we get the following lemmas:

Lemma 3.5.2 *A necessary condition for μ_S to be a colouring of a critical set for $\mu(K_{n,n})$ is that the subgraph $(K_{n,n} - E)$ has no more than one vertex with degree Δ.*

Proof. If there were two vertices, v_1 and v_2 in E with vertex degree Δ and the graph colouring μ_S could be extended to a colouring μ of $K_{n,n}$, then, since all vertices are isomorphic, μ_S could also be extended to a colouringμ_1 of $K_{n,n}$ where the labels of v_1 and v_2 are exchanged. Hence there are at least two reconstructions possible. ■

Lemma 3.5.3 *A necessary condition for $\mu_S = \mu(K_{n,n} - E)$ to be a colouring of a critical set for $\mu(K_{n,n})$ is that the subgraph $\mu(E)$ has no Kempe circuits.*

Proof. If any set $\mu(E)$ contained edges of colours $\{1, 2\}$, say, which are a Kempe circuit ${}_x[1, 2]$ in $\mu(K_{n,n})$, then $\mu(E)$ is also the set removed from $\mu_1(K_{n,n})$, where $\mu_1(K_{n,n}) = \mu(K_{n,n}) + {}_x(1, 2)$. Hence μ_E, can be extended to at least two possible colourings μ and μ_1 of $K_{n,n}$. ■

Corrollary 3.5.4 *A necessary condition for μ_S to be a colouring of a critical set is that μ_E has at least $(n-1)$ different colours.*

Proof. If there were two colours not in μ_E, then by Lemma 3.5.3 all circuits using these colours have been removed. Hence there are at least two reconstructions possible for each such circuit. ■

We recall Pittenger's Theorem, and consider $(x, y: a)$ swaps. A **potential $(x, y: a)$ swap** is a set of edges that in could be recoloured by an $(x, y: a)$ swap. Note that there

are two potential swaps ${}_{ik}(x, y: a)$ and ${}_{ki}(y, x: a)$ for each set of four edges which obey the necessary conditions for the existence of an $(x, y: a)$ swap.

Where $\mu_3 = \mu + {}_{ik}(x, y : a)$, we have $\mu_3(e_{ij}) = c_x$, $\mu_3(e_{kl}) = c_y$, $c_x \neq c_y$ and both edges $\mu_3(e_{ij}) = c_x$, $\mu_3(e_{kl}) = c_y$ are in the same circuit ${}_i]x, y[$, by construction since the two parts of the original circuit (one now swapped) are now connected by $\mu_3(e_{ij}) = c_x$ and $\mu_3(e_{kl}) = c_y$, instead of by $\mu(e_{il}) = c_x$, $\mu(e_{kj}) = c_y$.

Hence we can define $\mu_{3.x} + {}_{ik}(x, y :a)$ to be the inverse of swap $\mu + {}_{ik}(x, y :a)$ above, and $\mu_{3.y} + {}_{ki}(y, x :a)$ to be the inverse of swap $\mu + {}_{ki}(y, x :a)$, which is to say that if

$\mu_{3.x} = \mu + {}_i(a, n+1)_j + {}_k(a, n+1)_l + {}_i(a, x)_l + {}_k(a, y)_j + {}_j(y, x)_l + {}_i(x, n+1)_j + {}_k(y, n+1)_l$ then

$\mu = \mu_{3.x} + {}_i(a, n+1)_l + {}_k(a, n+1)_j + {}_i(a, x)_j + {}_k(a, y)_l + {}_j(x, y)_l + {}_i(x, n+1)_l + {}_k(y, n+1)_j$.

Similarly if

$\mu_{3.y} = \mu + {}_i(a, n+1)_j + {}_k(a, n+1)_l + {}_i(a, x)_l + {}_k(a, y)_j + {}_i(x, y)_k + {}_i(y, n+1)_j + {}_k(x, n+1)_l$ then

$\mu = \mu_{3.y} + {}_i(a, n+1)_l + {}_k(a, n+1)_j + {}_i(a, y)_j + {}_k(a, x)_l + {}_i(y, x)_k + {}_i(x, n+1)_l + {}_k(y, n+1)_j$.

Therefore we can see that

Main result (5)

Lemma 3.5.5 *A necessary condition for $\mu_S = \mu(K_{n,n} - E)$ to be a colouring of a critical set for $\mu(K_{n,n})$ is that the subgraph $\mu(E)$ has no potential $(x, y: a)$ swaps.*

Proof. If any potential $(x, y: a)$ swaps existed in $\mu(E)$ then the set $\mu_S = \mu(K_{n,n} - \mathrm{E})$ is also extendable to $\mu_2 = \mu + {}_{ik}(x, y: a)$. ■

Note that the smallest potential $\mu + {}_{ik}(x, y: a)$ swap has just six edges.

Figure 3.5.1

μ	j_1	j_2	j_3
i_1	a	x	...
i_2	y	a	...
i_3	x	y	...

$\mu + {}_{ik}(x, y : a) = \mu_2$

μ_2	j_1	j_2	j_3
i_1	x	a	...
i_2	a	y	...
i_3	y	x	...

The relevant part of the latin squares for these six edges are shown in figure 3.5.1. Any critical set would require at least one of these edges; where we are considering two colourings to be the same to isochromatism, somewhere in the critical set there would have to be another edge of the same colour.

Lemma 3.5.6 *The conditions in* 3.5.2 – 3.5.5 *above are not sufficient.*

Proof. In the following example $\mu_E(K_{n,n} - E)$ has at least one edge of each of the n colours, these have an element from each circuit, there are no potential $(x, y: a)$ swaps, nevertheless, there are the two given reconstructions possible.

μ_E	j_1	j_2	j_3	j_4
i_1	1			
i_2			4	
i_3				2
i_4		3		

μ_1	j_1	j_2	j_3	j_4
i_1	1	2	3	4
i_2	2	1	4	3
i_3	3	4	1	2
i_4	4	3	2	1

μ_2	j_1	j_2	j_3	j_4
i_1	1	4	2	3
i_2	3	2	4	1
i_3	4	1	3	2
i_4	2	3	1	4

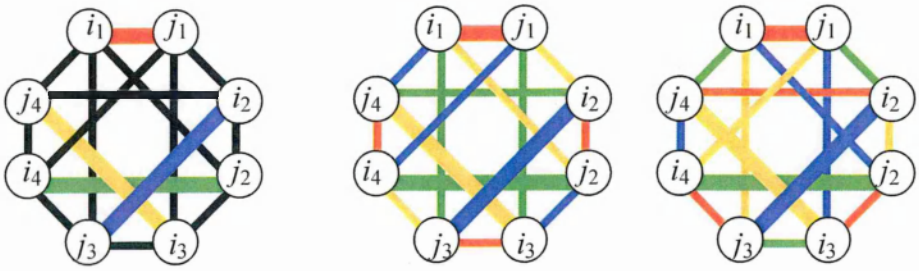


Figure 3.4.2

Note however that these colourings are the same to isochromatism. ■

It is possible that to isochromatism the problem of reconstruction is simplified and we present the following conjecture.

Conjecture 3.5.7 *To isochromatism, necessary and sufficient conditions for a colouring $\mu(K_{n,n} - E)$ to be a critical set for $\mu(K_{n,n})$, are that the colouring of $\mu(E)$ has no Kempe-circuits nor any potential $(x, y : a)$ swap.*

Justification. That these conditions are necessary has been shown in lemmas above. That they are also sufficient may follow from further consideration of Pittenger's theorem. Suppose that two colourings μ and μ_1 can be obtained from $\mu(K_{n,n} - E)$. They can be swapped from one to another by a sequence of Kempe chain and $(x, y: a)$ swaps. The colourings cannot be close enough to be swapped directly as there are no edges in $\mu(E)$ that would allow such a swap. If any circuit $[x, y]$ or $(x, y: a)$ swap μ_2 alters the colour of all the edges $\{x, y\}$ in μ_S then the colourings are isochromatic.

Therefore, the first swap must be a transitional colouring with a different set of colours in $\mu_1(K_{n,n} - E)$. At each stage some of the edges in $\mu_S = \mu(K_{n,n} - E)$ may be swapped but they would then need to be re-swapped back again. Since we have elements from every chain and $(x, y: a)$ swap of the original colouring, it may be possible to prove that to re-swap, the colourings μ and μ_1 need to be isochromatic. ■

We now consider how many elements are in μ_S. Given a colouring μ of $K_{n,n}$. There are n elements of each colour. Consider colours c_x, c_y and c_a. There are $n(n-1)/2$ pairs of elements c_a. Let one pair be $\mu(e_{ij}) = c_a$ and $\mu(e_{kl}) = c_a$ incident with edges e_{il} and e_{kj}. Let $\mu(e_{il}) = c_x$, then $\mu(e_{kj}) = c_y$ and both e_{il} and e_{kj} are in the same circuit $_i]x, y[$. This is the centre of a potential $(x, y: a)$-swap. Therefore, either we have an element from $_i]x, y[$ and one of the edges c_a in any μ_S, or we have at least two elements on the circuit in μ_S as we need at least one edge from each part of the circuit [x, y] otherwise it could be swapped. It would then appear that we need at least two elements in μ_S for each complete circuit C_{2n}. The number of shared edges is indicated by the following result by Cooper et al. Refering to cyclic Cayley tables, Cooper et al, [3.9] state that:

Theorem 3.5.8 (Cooper et al. - Re-phrased) *Let L be a cyclic Cayley table of order n. Then L contains a minimal critical set of size* $\lfloor n^2/4 \rfloor$. ■

Example 3.5.1

μ_E	j_1	j_2	j_3	j_4
i_1	1			4
i_2		3		
i_3				
i_4	4			

μ	j_1	j_2	j_3	j_4
i_1	1	2	3	4
i_2	2	3	4	1
i_3	3	4	1	2
i_4	4	1	2	3

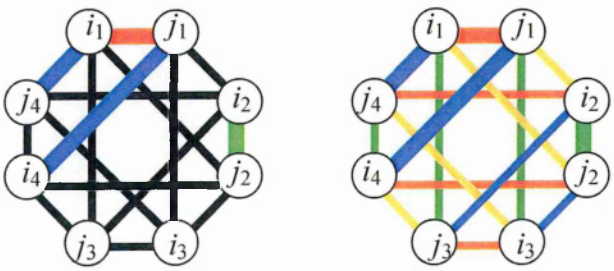


Figure 3.4.3 ■

Now we refer back to the conjecture by Mahmoodian, 3.4.9. Since a graph which has a representation composed entirely of 2×2 subsquares has $\lfloor n^2/4 \rfloor$ edge disjoint circuits, hence needs at least $\lfloor n^2/4 \rfloor$ elements in any critical set. Similarly, though the cyclic Cayley tables for prime n, have just $n(n-1)/2$ circuits C_{2n} yet by the theorem they also need $\lfloor n^2/4 \rfloor$ elements in any critical set. Since the result is true for the extremes, the Mahmoodian conjecture seems potentially provable, to isochromatism, by considering Kempe circuits and $(x, y : a)$ swaps.

Summary of Chapter 3:

In this chapter we have discussed the relationship between latin squares and graph colourings arriving at a number of useful results in particular the enumeration of the number of different edge colourings of $K_{n,n}$ and the number of (semi-) total-colourings of K_n. We discussed the Mahmoodian conjecture and related this to graph colourings.

Summary of main results by Jini Williams in Chapter 3

Theorem 3.1.4 $\beta(K_n) = \begin{cases} 0, & (n \text{ odd}) \\ n/2, & (n \text{ even}). \end{cases}$ Main result (1) page 23

Corollary 3.2.2 *If any three conjugates of a latin square are symmetric, then all six conjugates are identical.* Main result (2) page 25

Theorem 3.4.7 *To isochromatism, the number of ways to colour $K_{n,n}$ is equal to*

$$3\omega + 2\theta + \rho$$

where

ω is the number of main classes of $n \times n$ latin squares in the set $\mathbf{C}_6$;

θ is the number of main classes of $n \times n$ latin squares in the set $\mathbf{C}_3$;

ρ is the number of main classes of $n \times n$ latin squares in the set $\mathbf{C}_1 \cup \mathbf{C}_2$.

Main result (3) Page 41

Corollary 3.4.8 *To isochromatism, there are only as many ways to (semi)-total colour K_n as there are main classes of $n \times n$ latin squares containing a symmetric latin square.* Main result (4) page 41

Lemma 3.5.5 *A necessary condition for $\mu_S = \mu(K_{n,n} - E)$ to be a colouring of a critical set for $\mu(K_{n,n})$ is that the subgraph $\mu(E)$ has no potential $(x, y: a)$ swaps.*

Main result (5) page 44

CHAPTER 4

VERTEX COLOURINGS

4.1 Planar Graphs and Kempe's Argument

The problem of colouring planar maps is conveniently represented in terms of the vertex colouring of plane triangulations. The main interest in this context concerns 4-colourings; in addition there are two important theorems concerning the conditions under which a planar graph is 3-colourable.

In this chapter, a graph G is said to be **(vertex) p-chromatic** if $\chi(G) \leq p$.

If an independent set C of vertices can be deleted from a graph G so that $G - C$ is $(p - 1)$-chromatic, then clearly G is p-chromatic. Thus, the following theorems of Heawood and Grötzsch have immediate corollaries concerning the 4-chromaticity of planar graphs.

Theorem 4.1.1 (Heawood [4.1]) *A plane triangulation is* 3*-chromatic if and only if all vertices have even degrees.* ■

Corollary 4.1.2 *Let G be a plane graph, and suppose that an independent set of vertices C can be deleted from G, and edges added to G – C, to form a triangulation all of whose vertices have even degree. Then G is 4-chromatic.*

Proof. Let H be the triangulation described. By the theorem, $\chi(H) = 3$ and so $\chi(G - C) \leq 3$; thus, since C is independent, $\chi(G) \leq 4$. ■

Theorem 4.1.3 (Grötzsch [4.2]) *Every planar graph G without triangles is 3-chromatic. Moreover, any vertex 3-colouring of a 4-circuit or a 5-circuit in G can be extended to a 3-colouring of all of $V(G)$.* ■

Corollary 4.1.4 *If an independent set C of vertices can be deleted from a plane graph G such that $G - C$ is triangle-free, then G is 4-chromatic.*

Proof. Since $G - C$ is 3-chromatic and C is independent, this is immediate. ■

Corollary 4.1.4 provides a sufficient, but not a necessary, condition for a planar graph to be 4-chromatic; there are many 4-chromatic graphs (for example K_4), from which it is not possible to delete an independent vertex set and leave a triangle-free graph.

Given a simple plane graph G with a vertex colouring μ and one or more non-triangular faces, the operation of **compatibly triangulating** G consists of adding further edges (but no further vertices) to G. This operation partitions the non-triangular faces into triangles in such a way that μ remains a vertex colouring.

Main result (1)

Theorem 4.1.5 *Let F be a face of a simple plane graph G, and suppose μ: $V(G) \to \{c_1, c_2, c_3, c_4\}$ is a vertex colouring of G such that exactly three colours occur on the vertices of F. Then F can be compatibly triangulated.*

Proof. We proceed by induction on the face degree m of F. Since G is simple, $m > 2$, and there is nothing to prove if $m = 3$. Let the vertices of F, in order, be $v_1, \ldots, v_m$. If $m = 4$, then just two of the colours on the vertices of F are the same, so that either $v_1 \neq v_3$ or $v_2 \neq v_4$, and we may triangulate F.

Now suppose the theorem is false, and let G, with face F, be a counterexample where m is a minimum. We may assume that the colours of $v_1, \ldots, v_m$ are c_1, c_2, c_3.

If c_1 (for example) occurs only once, then we may triangulate F by joining that vertex to every other vertex of F. Thus we may assume that each colour occurs at least twice, and that $\mu(v_1) = c_1$, $\mu(v_m) = c_3$. Let j be the least integer such that $\mu(v_j) = c_2$.

If $j = 2$, then add the edge $v_m v_2$, cutting off a triangle of F, to form a face F_0. Since c_1 occurs at least once more, the inductive assumption implies that F_0 can be triangulated.

If $j > 2$, then by adding the edge $v_m v_j$ we partition F into two faces, F_1 (containing v_1) and F_2. Clearly, three colours occur on the vertices of F_1. If the colour c_1 occurs on F_2, then this is also true of F_2 and, by induction, F can be triangulated. Otherwise, remove the edge $v_m v_j$ and add instead the edge $v_1 v_j$. Since $\mu(v_2) = 3$, this edge partitions F into two faces each containing three colours, and the inductive argument again applies. ■

Corollary 4.1.6 *Let G be a plane graph with vertex four-colouring μ, and let S be a colour class of μ. Then there is a subset R of S such that edges may be added to $G - R$ to form a triangulation all of whose vertices have even degree.*

Proof. With respect to μ, let the colour set be $\{c_1, c_2, c_3, c_4\}$ and let S be the set of vertices coloured c_4. Let $R = \{v \in S$: the neighbours of v belong to all three colour classes $c_1, c_2, c_3\}$. For each $v \in S - R$, we may re-colour v with the colour from $\{c_1, c_2, c_3\}$ not present on its neighbours. The resultant colouring, μ_1, is such that R is the set of vertices coloured c_4. With respect to μ_1, each face F of $G - R$,that is each F of $\mu_1(G - R)$ has vertices of exactly three colours and by Theorem 4.1.5, any non-triangular face of $\mu_1(G - R)$ may be compatibly triangulated; this triangulation is such that by Theorem 4.1.1 each vertex has even degree. ■

A fundamental property of plane graphs, vital for all approaches to colouring theory of such graphs, is Euler's formula. We recall from Chapter 2 that we use n, m and f respectively for the numbers of vertices, edges and faces of a plane graph.

Theorem 4.1.7 (Euler [4.3]) *Let g be a connected plane graph; then* $n - m + f = 2$. ■

Corollary 4.1.8 (Kempe [4.4]) *In any plane cubic graph, there must be a face with five or fewer boundary edges.* ■

The geometric dual of Corollary 4.1.8 is:

Corollary 4.1.9 *In any plane triangulated graph, there must be a vertex of degree at most* 5. ■

Kempe's attempted proof of the Four Colour Theorem [4.4] can be restated in modern terms (using vertex colouring) as follows:

'If there are planar graphs G of chromatic number 5, then there are **5-critical** graphs, that is, graphs G of chromatic number 5 such that deleting any vertex reduces the chromatic number. Let G be such a graph. Then there is a vertex 4-colouring μ of $G - \{v_0\}$ where by Corollary 4.1.9, $d(v_0) \leq 5$.

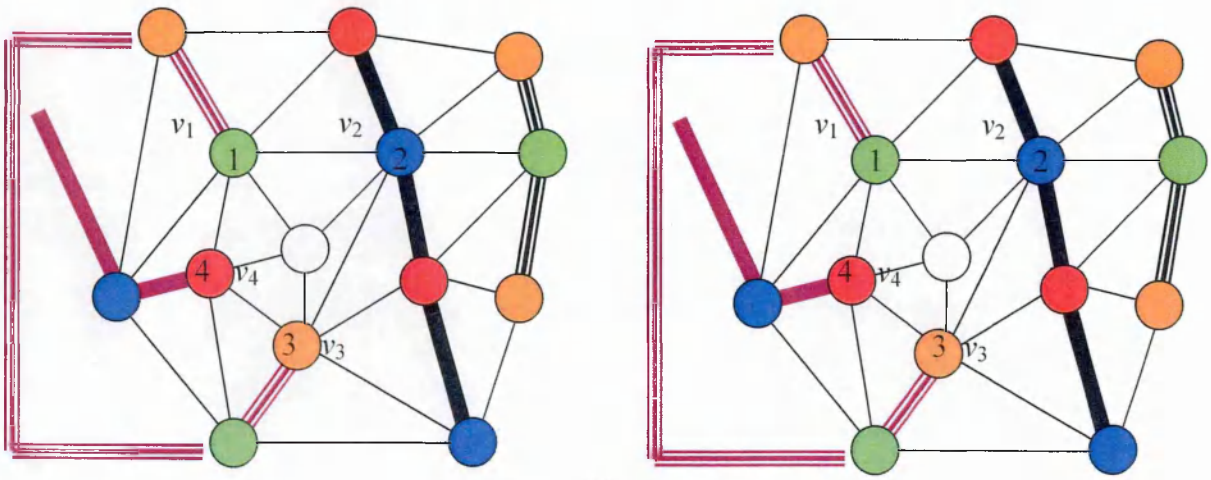


Figure 4.1.1

'If $d(v_0) < 4$, then clearly a 4-colouring of $G - \{v_0\}$ can be extended to a 4-colouring of G. If $d(v_0) = 4$, then the neighbours of v_0 must all receive different colours. Let us suppose that the neighbours in cyclic order are $v_1, \ldots, v_4$ with $\mu(v_i) = c_i$ $(i = 1, \ldots, 4)$. If there is a $[1, 3]$ Kempe chain that includes only one of v_1, v_3, then the corresponding Kempe interchange will allow G to be 4-coloured (see Figure 4.1.1). Thus there must be a Kempe chain ${}_1[1, 3]_3$; but together with the path $v_1v_0v_3$ this produces a closed Jordan curve with v_2 and v_4 on opposite sides. Thus there cannot be a Kempe chain ${}_2[2, 4]_4$, and so a Kempe interchange on the chain containing (say) v_2 will allow G to be 4-coloured, contradicting the assumption that G is 5-critical'.

Thus, any 5-critical planar graph has only vertices of degree five or more and at least one vertex of degree 5. The number of vertices of degree five in a critical graph was shown to be at least 13 out of a minimum number of 26 vertices by Errera in 1925 [4.5] and at least 15 out of a minimum number of 32 vertices by Franklin in 1938 [4.6]. (Note that these results were expressed in a face-colouring rather than a vertex-colouring context.) By 1968 it was proved by Ore and Stemple [4.7] that a 5-critical planar graph must have at least 40 vertices.

The case with $d(v_0) = 5$ is where Kempe made his mistake in 1879 which was noted by Heawood eleven years later.

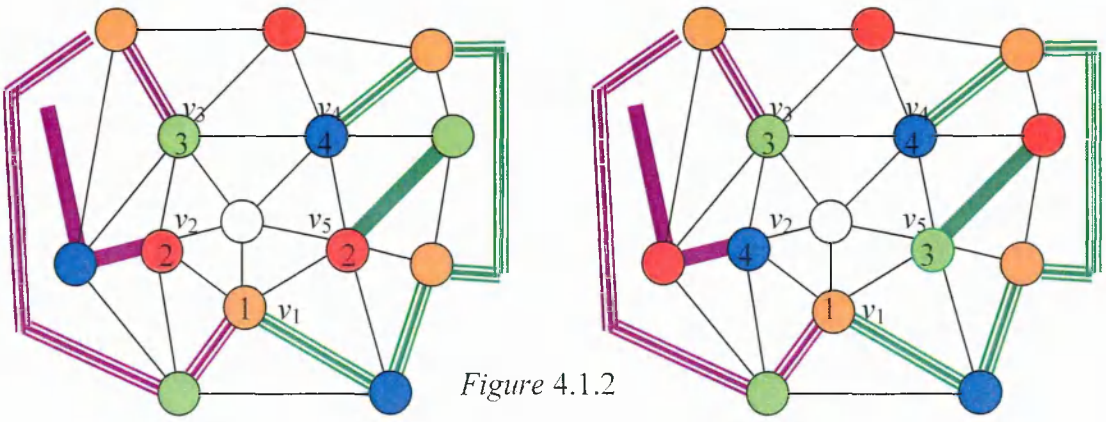


Figure 4.1.2

Translated into the vertex-colouring context, Kempe said in effect:

'Where $d(v_0) = 5$ and v_0 is the only vertex of the critical graph G not assigned a colour; when the neighbours of v_0 in cyclic order are $v_1, \ldots, v_5$, then one of these (say v_1) is adjacent to two of the same colour (v_2 and v_5); let $\mu(v_1) = c_1$, $\mu(v_2) = \mu(v_5) = c_2$; then since all four colours must appear on the neighbours of v_0, we may set $\mu(v_3) = c_3$, $\mu(v_4) = c_4$ (see Figure 4.1.2). In this context this will be called the **standard labelling.** If there is a Kempe chain ${}_1[1, 3]$ (resp ${}_1[1, 4]$) that does not include v_3 (resp v_4), then the corresponding Kempe interchange will allow a 4-colouring of G. Hence there are Kempe chains ${}_1[1, 3]_3$ and ${}_1[1, 4]_4$. But if ${}_1[1, 3]_3$ exists then ${}_2[2, 4]_4$ cannot exist, since the union of ${}_1[1, 3]_3$ with the path $v_1v_0v_3$ is a closed Jordan curve with v_2 and v_4 on opposite sides, and similarly if ${}_1[1, 4]_4$ exists then ${}_5[2, 3]_3$ cannot exist. Hence, both the interchanges ${}_2(2, 4)$ and ${}_5(2, 3)$ can be made and G can now be 4-coloured, giving a contradiction. Hence every planar graph can be 4-coloured.'

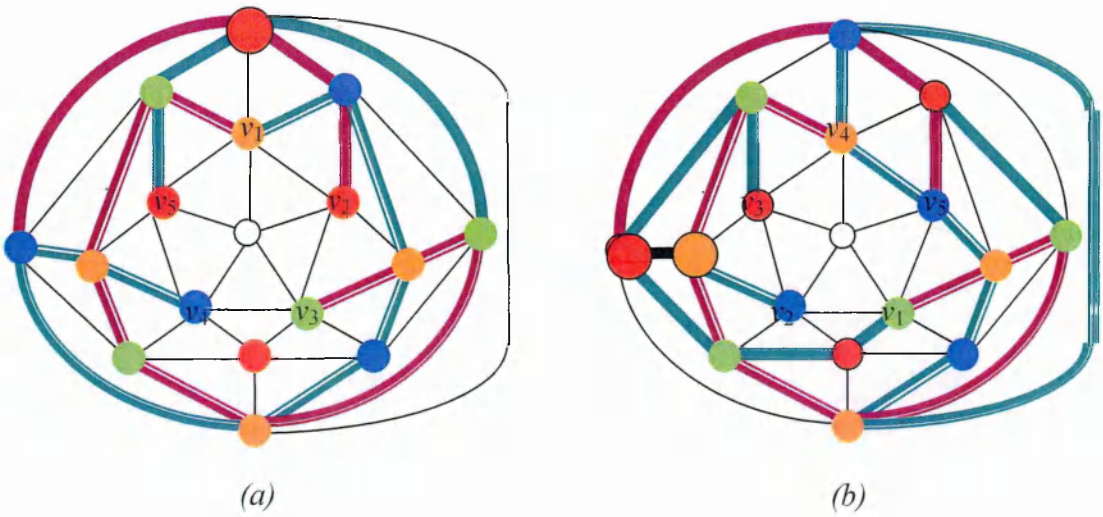


Figure 4.1.3

Heawood, however, noticed that although neither ${}_2[2, 4]_4$ nor ${}_5[2, 3]_3$ exist in the original colouring of $G - \{v_0\}$, performing either of the corresponding Kempe

interchanges could create the other Kempe chain; see Figure 4.1.3. This is because the chains ${}_2[2, 4]$ and ${}_5[2, 3]$ may possess shared or adjacent vertices, as in Figure 4.1.3 above. In (a), the chains possess the shared vertex shown enlarged, while in (b) a pair of adjacent vertices of the two chains are shown enlarged. The graph in Figure 4.1.3 has been lost and rediscovered many times and will be called the **Kittell graph** after the earliest known discoverer, see [4.8], and denoted by η_8 for brevity. Other work on Kittel's graphs can be found on [4.9]. The colouring of η_8 depicted in Figure 4.1.3(*a*) will be called the **Kittell colouring**, $\mu_0(\eta_8)$.

In $\mu_0(\eta_8)$ the edges of the vertex chains [1, 4] / [2, 3] and [1, 3] / [2, 4] have been highlighted. After the operation $\mu_1(\eta_8) = \mu_0(\eta_8) + {}_2(2, 4)$, the colouring $\mu_1(\eta_8)$ can be re-labelled with the standard labelling Figure 4.1.3(*b*).

4.2 Heawood Graphs

Given a supposed 5-critical triangulation G with a vertex v_0 of degree 5, we may (as in Section 4.1) label the 4-colouring μ_0 of $G - \{v_0\}$ with the standard colouring

$$\mu_0(v_1) = c_1,\ \mu_0(v_2) = c_2,\ \mu_0(v_3) = c_3,\ \mu_0(v_4) = c_4,\ \mu_0(v_5) = c_2.$$

Then we may perform a Kempe interchange and thus 4-colour G unless:

(1) ${}_1[1, 3] = {}_1[1, 3]_3$;

(2) ${}_1[1, 4] = {}_1[1, 4]_4$;

(3) ${}_5[2, 3]_3$ is a Kempe chain of $\mu_1 = \mu_0 + {}_2(2, 4)$;

(4) ${}_2[2, 4]_4$ is a Kempe chain of $\mu_2 = \mu_0 + {}_5(2, 3)$.

We call these four conditions the **Heawood conditions**; any colouring of $G - \{v_0\}$ obeying these conditions is a **Heawood colouring**. Note that, if a colouring μ_0 of $G - \{v_0\}$ does not obey these conditions, then G is 4-chromatic by Kempe's argument, and hence any non-Heawood colouring of $G - \{v_0\}$ may be called a **Kempe colouring**. The following lemma is an immediate consequence of the definitions.

Lemma 4.2.1 *Let μ be a Heawood colouring of $G - \{v_0\}$ as above, and suppose there is a Kempe interchange that transforms μ into a Kempe colouring. Then G is 4-chromatic.* ■

Let C_1, C_2 be two vertex Kempe chains with respect to a vertex 4-colouring of G. Then the **distance** $\phi(C_1, C_2)$ is the minimum number of edges in any path from a vertex in C_1 to a vertex in C_2. Let $\phi_h = \phi({}_2[2, 4], {}_5[2, 3])$. Kempe's argument is correct in all cases where $\phi_h \geq 2$. Heawood's counter example had $\phi_h = 1$. Various other counter-examples will be studied in detail in this section, together with specific examples where Kempe's algorithm also successfully colours G in cases where $\phi_h = 1$ and $\phi_h = 0$.

Main result (2)

Theorem 4.2.2 *Let μ be a vertex 4-colouring of $G - \{v_0\}$ where G is a critical graph and $d(v_0) = 5$. Let the colours of the neighbouring vertices $v_1, \ldots, v_5$ be as in the standard colouring, with ϕ_h as above. Then if G has a Heawood colouring, $\phi_h < 2$.*

Proof. If $\phi_h \geq 2$, then $\mu + {}_2(2, 4) = \mu_2$ where μ_2 is such that no vertices adjacent to ${}_5[2,3]$ have been affected. Hence $\mu_3 = \mu_2 + {}_5(2, 3)$ is a Kempe colouring of G allowing $\mu_4 = \mu_3 + {}_0(0, 2)$ to colour the previously uncoloured v_0 with colour c_2 such that $\mu_4(v_0) = c_2$. Therefore G has a vertex 4-colouring and was not a critical graph. ■

At any vertex of $G - \{v_0\}$ there are just three possible Kempe chains. A Heawood colouring μ of $G - \{v_0\}$ with the standard labelling, has just the following Kempe chains involving the neighbours of v_0:

$$_1[1, 3]_3,\ _1[1, 4]_4,\ _2[1, 2]_5,\ _2[2, 3]_3,\ _2[2, 4],\ _3[3, 4]_4,\ _4[4, 2]_5,\ _5[2, 3].$$

We will call the corresponding set of eight Kempe interchanges, S_0. If each interchange in S_0 produces a further Heawood colouring, then μ is said to be **completely Heawood** or **8H**.

Example 4.2.1: *colourings of* η_8

Figure 4.1.3 (a) showed the Kittell colouring $\mu_0(\eta_8)$. The edges of some of the chains were highlighted. Figure 4.1.3 (b) is labelled to show that $\mu_1(\eta_8) = \mu_0(\eta_8) + {}_2(2, 4)$ is still a Heawood colouring and the relevant edges are similarly highlighted. In the Kittell colouring, the chains $_1[1, 3]_3$, $_1[1, 4]_4$, $_2[1, 2]_5$ and $_3[3, 4]_4$ each use every occurrence of the given colours and hence any combination of the corresponding interchanges leaves the colouring isochromatic to μ_0. For example, see $\mu_0(\eta_8) + {}_1(1, 3)$ in Figure 4.2.1 (a) below. Similarly, either of the interchanges $_4(4, 2)_5$ or $_2(2, 3)_3$ result in colourings isochromatic to either of the interchanges $_2(2, 4)$ or $_5(2, 3)$ hence they all correspond to $\mu_0(\eta_8) + {}_2(2, 3)_3$, shown in Figure 4.2.1 (b).

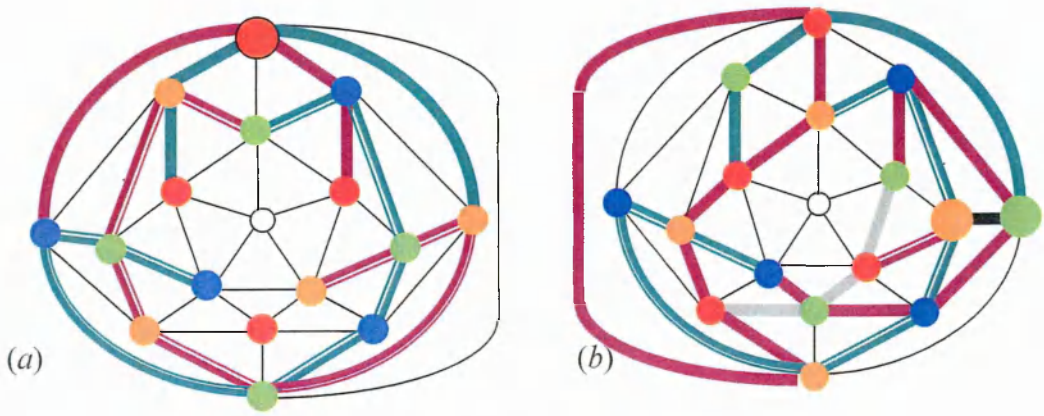


Figure 4.2.1

If n of the eight interchanges S_0 produce a further Heawood colouring, then μ is said to be ***n*-Heawood** or ***nH***.

Example 4.2.2: *the Errera graph*

The colourings of the near triangulation of a graph, which we will call the **Errerra graph,** was shown to have a sequence of $7H$ colourings by Errera [4.10]; these are shown in Figure 4.2.2 below. They have since been presented in papers by various authors for example, Holroyd and Miller [4.11] and Hutchinson and Wagon [4.12]. There is in each case one chain such that the corresponding Kempe interchange produces a Kempe colouring as shown (highlighted and exchanged) in Figures 4.2.3. Note that as these are near triangulations the uncoloured vertex, v_0, is not shown but assumed to be outside the graph with edges to the vertices $v_1, \ldots, v_5$ with the standard colouring

$$\mu_0(v_1) = c_1,\ \mu_0(v_2) = c_2,\ \mu_0(v_3) = c_3,\ \mu_0(v_4) = c_4,\ \mu_0(v_5) = c_2.$$

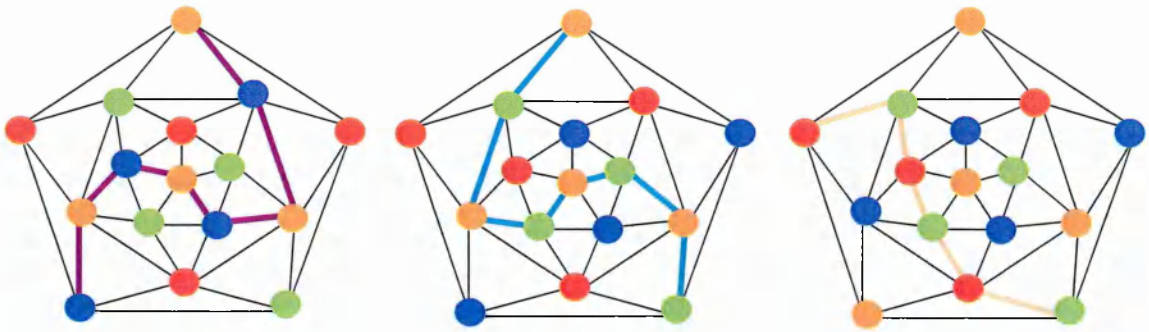


Figure 4.2.2 *The Errera graph with 7H colourings*

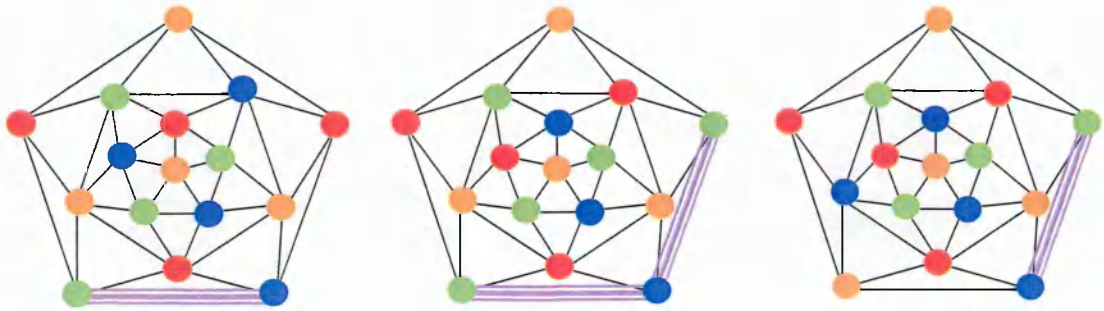


Figure 4.2.3 *Kempe colouring of the Errera graph*

Although in any Heawood colouring μ_0, there are eight Kempe chains involving the neighbours of v_0, in many cases $\mu_0 + {}_2(2, 4)$ and $\mu_0 + {}_5(2, 4)$ are isochromatic. This occurs whenever ${}_1[1, 3]_3$ is a tree containing all the vertices of these colours and is particularly likely to occur in small graphs since there are often insufficient vertices to have more than two [2, 4] chains. That is the case in both the above examples, where if we have two [2, 4] chains then $\phi_h \geq 2$ and the graph has a Kempe colouring.

Let μ be a colouring of $G - \{v_0\}$ where G is a plane triangulation and $d(v_0) = 5$. We say that μ is **Kempe-colourable** if a sequence of Kempe interchanges from μ will produce a colouring of $G - \{v_0\}$ that allows v_0 to be coloured. Although the four colour theorem has been proven, this still allows for the theoretical existence of an infinitely $8H$ colourable graph. However, we conjecture that no such colouring exists. Moreover, we make the more specific conjecture as follows.

Conjecture 4.2.3 *Let $G - \{v_0\}$ be as above. Then every 4-colouring is Kempe-colourable. There is no graph which can be given an 8H colouring which will continue to be 8H after an arbitrary sequence of interchanges in S_0. In particular, no colouring remains Heawood after, to isochromatism of the standard colouring, an infinite sequence of alternating ${}_2[2, 4]$, and ${}_3[3, 4]_4$ exchanges.* ■

Although unproven, an out line of our justification for conjecture 4.2.3 is as follows. It would appear that after a number of $\mu + {}_3(3, 4)_4 + {}_2(2, 4)$ exchange sequences, we arrive at a colouring μ_1 for which the ${}_3[3, 4]_4$ chain is not the complete set of all vertices colours c_3, c_4. When we then exchange the ${}_3[3, 4]_4$ chain, we either prevent the ${}_1[1, 3]_3$ from existing or separate this from ${}_1[1, 4]_4$, and so get a Kempe colouring. We believe that an argument should be possible, by using Theorem 4.2.2 to establish

a particular set of vertices and paths which would require the existence of $K_{3,3}$. This, with Kuratowski's Theorem [4.13] and the non existence of the paths of $K_{3,3}$, would prove the four colour theorem without the need for a computer.

Note however, that there are many heuristics for finding colourings, see [4.14]. A study of such heuristics may result in the discovery other infinite chain sequences that are incompatible Kuratowski's Theorem.

4.3 Rotatable Triangles

Suppose that in a colouring μ of G we have a triangle $t = \{v_i, v_j, v_k\}$ with $\mu(v_i) = c_p$, $\mu(v_j) = c_q$, $\mu(v_k) = c_r$, such that, if we rotate the colours on t (so that the colour of v_i becomes c_q, that of v_j becomes c_r and that of v_k becomes c_p), the new colours do not clash with any of the remaining colours. Then we shall call t a **rotatable triangle.** The operation of altering the colours in this way will be denoted by $_{i,j,k}(p,q,r)$, so that the new colouring is denoted by $\mu_x = \mu + {}_{i,j,k}(p,q,r)$.

Theorem 4.3.1 *Let G be a plane graph with vertex 4-colouring μ and a triangle $t = \{v_i, v_j, v_k\}$, where $\mu(v_i) = c_p$, $\mu(v_j) = c_q$ and $\mu(v_k) = c_r$. Let N_i, N_j, N_k denote the sets of vertices of $G - t$ adjacent to v_i, v_j, v_k respectively. If the operation $_{i,j,k}(p,q,r)$ is possible, then just two colours appear on each of the sets N_i, N_j, N_k.*

Proof. Vertex v_i can be recoloured with colour c_q only if the colour c_q is not present on N_i. Since the colour c_p is not present on N_i, the only colours on N_i are c_r and the fourth colour. Similar arguments hold for N_j and N_k. ■

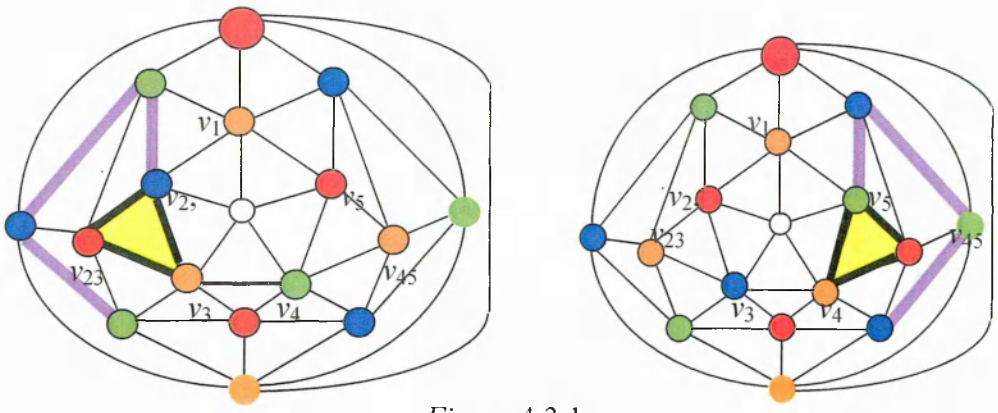


Figure 4.3.1

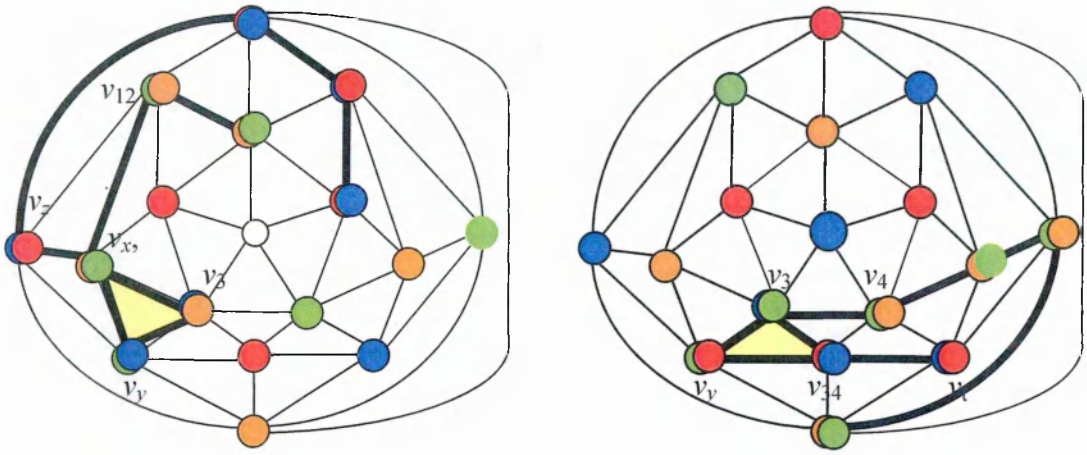
In the Kittell colouring, Figure 4.1.3 (a), the given colouring μ of G - v_0 has two rotatable triangles adjacent to the uncoloured vertex. Triangle $t = \{v_2, v_3, v_{23}\}$, where $\mu(v_2) = c_2$, $\mu(v_3) = c_3$ and $\mu(v_{23}) = c_1$, can be rotated by $\mu + {}_{2,3,23}(2,3,1) = \mu_x$ where $\mu_x(v_2) = c_3$, $\mu_x(v_3) = c_1$ and $\mu_x(v_{23}) = c_2$. See Figure 4.3.1. This is now colourable by $\mu_x + {}_2(3, 4)_x + {}_0(0, 3)$. Similarly we could colour G by

$$\mu + {}_{5,4,45}(2,4,1) + {}_5(3,4)_y + {}_0(0, 4).$$

However, a rotatable triangles are rare. But there are similar components which are much more common.

4.4 Rotatable Spiders

Suppose that in a colouring μ of G we have a triangle $t = \{v_i, v_j, v_k\}$ with $\mu(v_i) = c_p$, $\mu(v_j) = c_q$, $\mu(v_k) = c_r$, such that it is not rotatable. If we rotate the colours on t as in Section 4.3, we have a set R of vertices in $G - t$ which are now the same as the colours on adjacent vertices. If we can recolour these R vertices by exchanging the vertex colours on non intersecting, but possibly multi coloured, paths leading from them, then the operation of altering the colours in this way is denoted by $\mu_x = \mu + {}_{i,j,k}(p,q,r)^*$. We will call the triangle t and the set of non intersecting paths, a **rotatable triangle spider**.



(*a*) (*b*)

Figure 4.4.1

Two examples of rotatable triangle spiders in the Kittell colouring are shown above in Figure 4.4.1. In (*a*) $t = \{v_3, v_x, v_y\}$, where $\mu(v_3) = c_3$, $\mu(v_x) = c_1$ and $\mu(v_y) = c_4$.

Let $\mu_1 = \mu_0 + {}_{3,x,y}(3, 1, 4)$. Then $\mu_1(v_3) = c_1$, $\mu_1(v_x) = c_4$ and $\mu_1(v_y) = c_3$. Here the set $R(\mu_1)$ is comprised of $\mu_1(v_z) = c_3$ and $\mu_1(v_{12}) = c_4$. We can independently interchange paths $\mu_2 = \mu_1 + {}_z(3, 2) + {}_{12}(1, 4)$ to get a Kempe colouring which by $\mu_3 = \mu_2 + {}_2(2,3) + {}_0(0,3)$ becomes a proper colouring.

In (*b*) $t = \{v_3, v_y, v_{34}\}$, where $\mu(v_3) = c_3$, $\mu(v_y) = c_4$ and $\mu(v_{34}) = c_2$.

Let $\mu_1 = \mu_0 + {}_{3,y,34}(3, 4, 2)$. Then $\mu_1(v_3) = c_4$, $\mu_1(v_y) = c_2$ and $\mu_1(v_{34}) = c_3$. Here the set $R(\mu_1)$ is comprised of $\mu_1(v_4) = c_4$ and $\mu_1(v_t) = c_3$. We can independently interchange $\mu_2 = \mu_1 + {}_4(4, 1) + {}_t(2, 3) + {}_0(0,3)$ to get a proper colouring.

There are three kinds of rotatable triangle spiders: those that take the colouring to another Heawood colouring, those that take it to a Kempe colouring and those that take it to a proper colouring, i.e. allow the missing vertex to be coloured. All three are shown for the Errera colouring, (Figure 4.4.2).

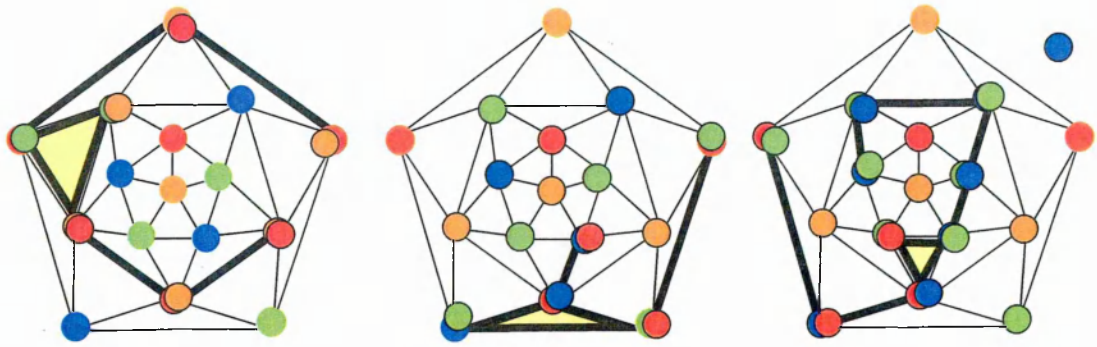


Figure 4.4.2

Note however that since any two Kempe chains from different points of the triangle could require the same alternating colour, their chains can cross. Hence a rotatable triangle spider is not available for all triangles. Nevertheless we feel that the following conjecture is justified.

Main conjecture (3)

Conjecture 4.4.1 *In any Heawood colouring μ, there is always at least one rotational triangle spider leading directly to either a Kempe or proper colouring.* ■

This conjecture is not a mere restatement of the four colour theorem. This is a way of recolouring. However this claim must be tempered by the knowledge that, as has been noted by other combinatorists: nearly every method of recolouring works.

Summary of chapter 4:

In this chapter we consider the various aspects of the (vertex) 4-Colour Theorem. We presented a way to compatibly triangulate a graph and several methods to swap from the standard uncompleateable colouring to a completable colouring under specific circumstances.

Summary of main results by Jini Williams in Chapter 4

Theorem 4.1.5 *Let F be a face of a simple plane graph G, and suppose μ: $V(G) \rightarrow \{c_1, c_2, c_3, c_4\}$ is a vertex colouring of G such that exactly three colours occur on the vertices of F. Then F can be compatibly triangulated.*

Main result (1) page 50

Theorem 4.2.2 *Let μ be a vertex 4-colouring of $G - \{v_0\}$ where G is a critical graph and $d(v_0) = 5$. Let the colours of the neighbouring vertices $v_1, \ldots, v_5$ be as in the standard colouring, with ϕ_h as above. Then if G has a Heawood colouring, $\phi_h < 2$.*

Main result (2) page 56

Conjecture 4.4.1 *In any Heawood colouring μ, there is always at least one rotational triangle spider leading directly to either a Kempe or proper colouring.*

Main conjecture (3) page 63

CHAPTER 5

EDGE COLOURINGS

5.1 Planar Graphs

A **cubic map** is a plane graph of regular vertex degree 3. It was recognised as long ago as 1880 by Tait [5.1] that colouring the edges of such a map M with three colours is equivalent to colouring the faces of M with four colours. If the face colour set is $\{1, 2, 3, 4\}$, then a pair of adjacent faces can have just one of six colour pairs. This is also equivalent to saying that in the dual of the four coloured map, an edge connects vertices in just one of six possible colour pairs. In the cubic graph we may associate a colour pair and it's complementary pair with a colour in the colour set $\{a, b, c\}$, and thus give a 3-edge colouring to G. For example, we may associate the pairs $\{1, 2\}$, $\{3, 4\}$ with a; $\{1, 3\}$, $\{2, 4\}$ with b; $\{1, 4\}$, $\{2, 3\}$ with c. Then, given a proper face 4-colouring of M, we may colour each edge using the colour a, b or c in direct correspondence with the colours of the faces and obtain a proper edge 3-colouring.

Importantly, Tait also proved the converse result: that every proper edge 3-colouring of M gives rise to a face 4-colouring that is unique to isochromatism. Hence, if a partial edge 3-colouring of M can be altered to allow the edge-colouring to be completed, then we may translate this into a face 4-colouring. Thus, since every face 4-colouring of a cubic map corresponds by duality to a vertex 4-colouring of the dual triangulation, the results of Chapter 4 on vertex 4-colourings of triangulations have analogies in terms of edge 3-colourings of cubic maps. See Figure 5.1.1.

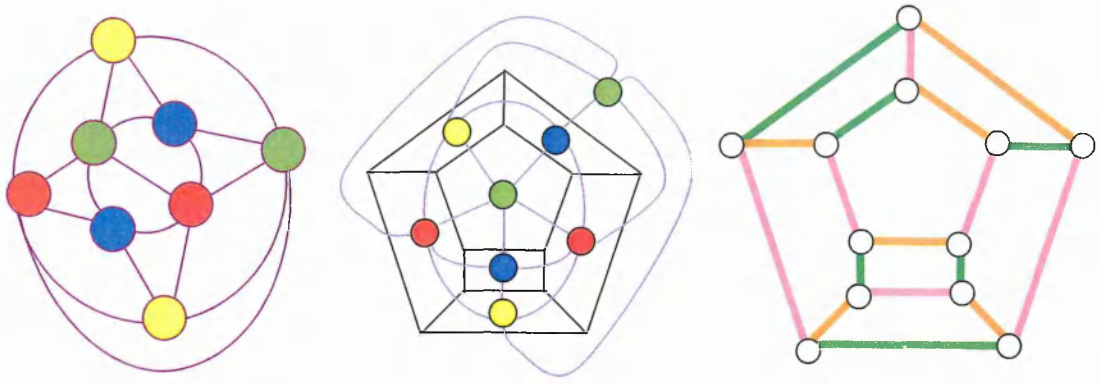


Figure 5.1.1

The Zhou Shuguo [5.2] *graph colouring as vertex, map and edge colourings.*

In this form, the Kittell graph colouring $\mu_0(\eta_8)$, of Chapter 4 becomes Figure 5.1.2.

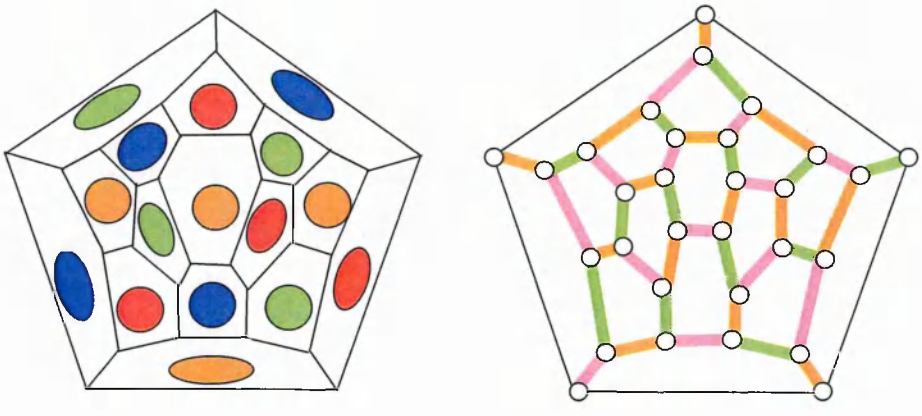
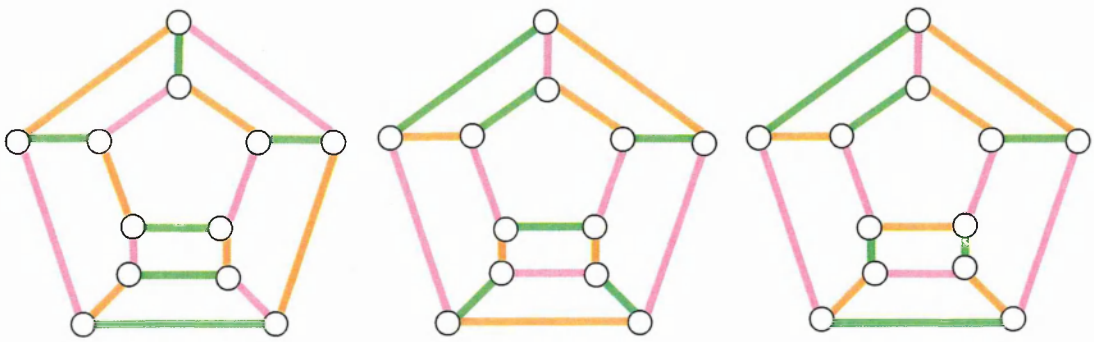


Figure 5.1.2

An exchange of one known vertex or edge colouring of a graph G for another known colouring of the same graph will be called a **transition.** A vertex transition is usually possible via a sequence of vertex Kempe chain or rotatable triangle exchanges; each of these has a corresponding set of edge-colour exchanges. For a vertex Kempe chain [1, 2], or part thereof, the corresponding set of edge Kempe chains [*b, c*] can be quite large as there are disconnected chains [*b, c*] corresponding to each circuit in [1, 2]. Similarly a single edge-colour exchange [*b, c*] may correspond to a set of vertex chain exchanges where several nested circuits [3, 4] and [1, 2] have their colours swapped.

Hence, although we can obtain the same result by either vertex- or edge-exchanges, there are certain transitions which are simplified when using edge, rather than vertex, exchanges and *vice versa*. In both, an infinitely 8-Heawood colouring, with the eight Kempe interchanges S_0 described in Chapter 4, could still exist.

There are, however, edge 3-colourings with corresponding vertex 4-colourings, between which a transition is not possible using just the given colour set. Consider the edge 3-colourings of the graph G given in Figure 5.1.3. We name this the **Zhou ShuGuo** graph [5.2] after its discoverer. These are, to isochromatism, the only edge 3-colourings of G; we denote these by μ_1, μ_2, μ_3.



μ_1 μ_2 μ_3

Figure 5.1.3

From the figure, it is clear that any Kempe exchange in μ_1 results in an isochromatic colouring so that there is no sequence of Kempe exchanges from μ_1 to μ_2 nor from μ_1 to μ_3, whereas we may move between μ_2 and μ_3 using Kempe exchanges.

In the 1990 Zhou ShuGuo attempted to prove the Four-Colour Theorem and used this graph as an example of the need to go beyond Kempe exchanges; although the attempt was unsuccessful, the example does suggest that reliance on Kempe exchanges is insufficient for a successful proof, as we also conjectured in Chapter 4.

Where n is a prime number and where μ_1 is a graph colouring of $K_{n,n}$ corresponding to the cyclic Cayley table; in Chapter 3, we showed that there was no transition to any other colouring of $K_{n,n}$ *via* Kempe chain interchanges alone, but that, nevertheless, this could be done by using Pittenger's interchanges. These interchanges, however, do not necessarily exist for non bipartite graphs. Not only do they require the existence of a four circuit, but consider ${}_{1,2}(x, y: a)$ swap: where the vertices of the four cycle are v_1, b_1, v_2 and b_2 in a bipartite graph, we know that we have a Kempe chain $v_1]x, y[v_2$, where v_1 is not adjacent to v_2 , hence we have a transition to a proper edge colouring, but in a general graph there is no guarantee that $v_1]x, y[$ will not return to an adjacent vertex. In Figure 5.1.3 colouring μ_1 has many four cycles but none have a valid Pittenger interchange. For example, consider the four cycle v_6, v_8, v_{12} and v_7, in this case $\mu_2 = \mu_1 + {}_{6,8}(a, b: c)$ would give us $\mu_1(e_{6,8}) = \mu_1(e_{7,12}) = c$ but ${}_6]a, b[_7$ is not a transition to a proper edge colouring.

We now show that for edge colourings, that there is an alternative transition *via* a sequence of Kempe exchanges from μ_1 to μ_3 when we introduce a fourth, dummy, colour which does not appear in either colouring. See Figure 5.1.4.

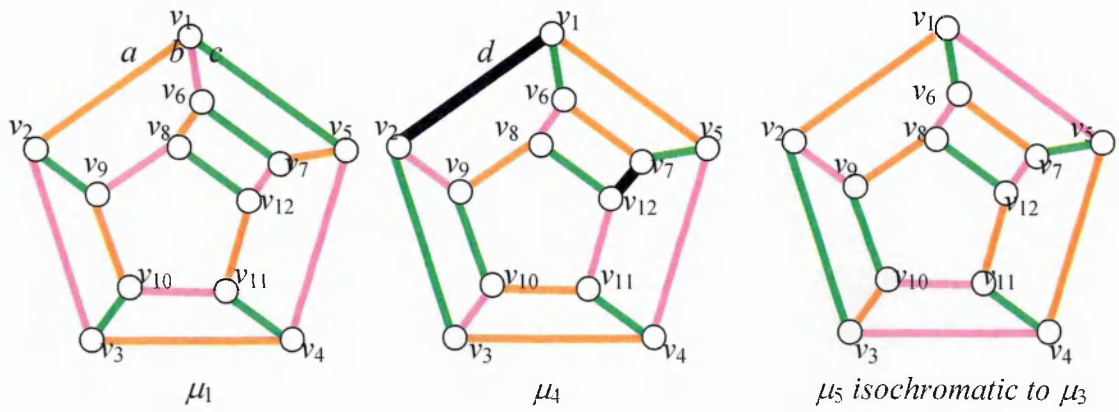


Figure 5.1.4

Here, μ_1 is as above; μ_4 is an intermediate edge 4-colouring; and μ_5 is an edge 3-colouring isochromatic with μ_3. The transition from μ_1 to μ_5 is obtained as follows:

$\mu_4 = \mu_1 + {}_1(a, d)_2 + {}_7(b, d)_{12} + {}_{12}(a, b)_1 + {}_2(b, c)_2 + {}_1(c, a)_1$;

$\mu_5 = \mu_4 + {}_{12}(b, a)_1 + {}_7(b,d)_{12} + {}_1(a,d)_2$, isochromatic to μ_3.

Note that this is not the only route from μ_1 to μ_5.

5.2 Edge 4-colour Transitions

It will now be shown to be possible to find a transition between any awkward set and another known edge 3-colouring. This is a method for all cubic graphs, planar or non-planar. However, it is only relevant to the four colour theorem if we accept that there is a proper 3-edge colouring of the graph. Furthermore, unlike in works such as by Yasuyuki Tsukui [5.3], the method below preserves the structure of G at all stages.

Main theorem (1)

Theorem 5.2.1 [Holroyd and Williams] *Every edge 3-colouring of a class 1 (not necessarily planar) cubic graph can be obtained from every other edge 3-colouring of the same graph by a series of (edge) Kempe interchanges using at most four colours.*

Proof. Let G be a regular class 1 cubic graph. Let μ_1 and μ_2 be any two edge 3-colourings of G, using the colours a, b, c. We shall transform μ_2 to μ_1 by a sequence of Kempe interchanges involving the temporary use of a fourth colour, d. The set F of edges e such that $\mu_1(e) = c$ is a 1-factor of G. By using $+ {}_x(d, \mu_2(e_{xy}))_y$ for every edge $e_{xy} \in F$ we transform μ_2 to a temporary colouring such that each edge in F is coloured d. Then $E(G) - F$ is a union of even circuits. In each circuit of $E(G) - F$,

we must convert each edge coloured c to a or b then, if necessary, exchange these colours throughout the circuit to match the colours in μ_1.

Let C be a circuit of $E(G) - F$. We can immediately swap $_x]b, c[_y$ for every $e_{xy} \in C$ that has colour c and is between two edges of colour a, and also swap $_z]a, c[_w$ for every $e_{zw} \in C$ that has colour c and is between two edges of colour b. If no edges coloured c remain then, by a final swap $_p]a, b[_q$ if necessary, we have the edges of C coloured as in μ_1. If just one edge coloured c remains, then (as C is an even circuit) this edge lies between edges of the same colour, and can be coloured a or b as above. We may therefore assume that there are at least two distinct edges, e_{wx} and e_{yz}, coloured c and all the edges on the path from x to y are coloured alternately a and b. We now exchange colours a and b on this path. Now each of the edges e_{wx} and e_{yz} has the same colour on each side, and we may proceed as before. Eventually, therefore, we produce a sequence of Kempe interchanges that give the edges in C the same colours as in μ_1. We proceed in the same way with the other circuits of $E(G) - F$, then finally apply $+ (d, c)$ to each edge in F, to give us a colouring identical to μ_1. We have used only four colours and the theorem is proven. ■

5.3 Colour Difference

5.3.1 Equi-nets and trans-nets

The concept of the *transition net* between two (face or edge) colourings of a cubic graph has been previously explored by Zhou ShuGuo, F. Holroyd and F. Loupekine [5.4] in unpublished work. We here introduce a related concept.

Let μ_1 and μ_2 be two edge 3-colourings of a cubic graph G, using colour sets

$W_1 = \{A, B, C\}$ and $W_2 = \{a, b, c\}$ respectively. (It is possible that $A = a$, etc.) There are six bijections from W_1 to W_2; for convenience, we denote them by ϕ, θ, ρ, α, β, γ as follows.

$$\phi: A \mapsto a,\ B \mapsto b,\ C \mapsto c;\ \ \theta: A \mapsto b,\ B \mapsto c,\ C \mapsto a;\ \ \rho: A \mapsto c,\ B \mapsto a,\ C \mapsto b;$$

$$\alpha: A \mapsto a,\ B \mapsto c,\ C \mapsto b;\ \ \beta: A \mapsto c,\ B \mapsto b,\ C \mapsto a;\ \ \gamma: A \mapsto b,\ B \mapsto a,\ C \mapsto c.$$

For each of these bijections, ζ, we define a partition of $E(G)$ into an **equi-net** E_ζ and a **trans-net** T_ζ as follows:

$$E_\zeta = \{e \in E(G):\ \mu_2(e) = \zeta \circ \mu_1(e)\},\ \ T_\zeta = \{e \in E(G):\ \mu_2(e) \neq \zeta \circ \mu_1(e)\}.$$

Figure 5.3.1 shows the six such partitions into equi-nets (pink) and trans-nets (blue) of two colourings μ_1 and μ_2 of a cube. All colourings of the labelled cube and the relevant equi-nets will be shown, in a more concise manner, in Section 5.3.2, figures 5.3.8 and 5.3.9 on page 80.

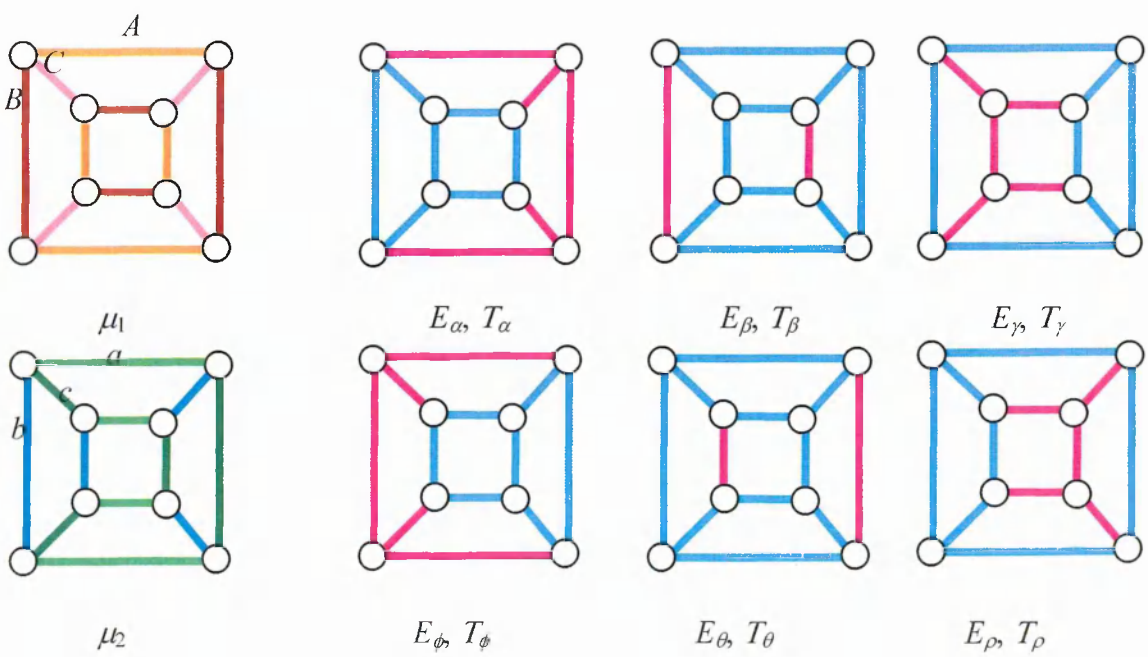


Figure 5.3.1

Each maximal connected set of edges in any equi-net will be called a **component.** For every two colourings μ_1 and μ_2 of G using colour sets W_1, W_2 as above, where the colour on an edge e_{ij} is c_A in μ_1 and c_a in μ_2, the edges e_{ix}, e_{iy} incident with e_{ij}, have colours c_B and c_C in μ_1 and colours c_b and c_c in μ_2 and edges e_{jw}, e_{jz} incident with e_{ij}, have colours c_B and c_C in μ_1 and colours c_b and c_c in μ_2, then we will say that the edge has **the same local colouring** w.r.t. μ_1 and μ_2. Where however, the edges e_{ij}, e_{ix}, e_{iy} are as above but edges e_{jw}, e_{jz}, have colours c_B and c_C in μ_1 and colours c_c and c_b in μ_2 then we will say that the edge has **a different local colouring** w.r.t. μ_1 and μ_2. Figure 5.3.2 shows (*a*) e_{ij}, an edge with the same local colouring and (b) e_{xy}, an edge with a different local colouring w.r.t. μ_1 and μ_2. Note that on the given example only $\mu_2(e_{xy})$ is equivalent to the *c-reduction* of [5.3] and that in all cases where both colourings have this same local colouring, we need only consider the smaller planar graph formed by removing the edge and joining the spines of the same colour. The same transformation for $\mu_1(e_{ij})$ and $\mu_2(e_{ij})$ could result in a non planar graph.

(*a*) *Same local colouring* (*b*) *Different local colouring*

Figure 5.3.2

A graph is **covered** by a set of components if every edge of the graph is in at least one component. A graph is **partitioned** by a set of components if every edge of the graph is in just one component. Coverings of graphs have been discussed by many authors, see [5.5] and [5.6], but we believe the following to be a new aspect.

Lemma 5.3.1 *For any two edge-colourings of a graph G, with the equi-nets defined as above, G is partitioned by the components of:*

(i) E_ϕ, E_θ, E_ρ; *and*

(ii) $E_\alpha, E_\beta, E_\gamma$.

Proof. (*i*) Any edge coloured A in μ_1 that is not in E_ϕ has colour b or c in μ_2. Hence it is in either E_θ or E_ρ but not both. A similar argument follows for the edges coloured B in μ_1, and for those coloured C.

(*ii*) Any edge coloured A in μ_1 that is not in E_α has colour b or c in μ_2. Hence it is in one of E_γ or E_β but not both. As in part (i), similar arguments follow for the colours B and C. Hence the graph is uniquely covered by the given components. ■

We will call both these partitions, **equi-coverings**. Where we wish to distinguish between them we will call them **equi-covering-(*i*)** and **equi-covering-(*ii*)** for E_ϕ, E_θ, E_ρ and $E_\alpha, E_\beta, E_\gamma$ respectively. In an equi-covering, we will call any vertex where components from at least two distinct equi-nets are incident a **junction vertex**. Any vertex that is not a junction vertex will be called an **inner vertex**.

Lemma 5.3.2 *Every junction vertex in an equi-covering has components from all three equinets.*

Proof. First note that the two equi-coverings correspond to the following 3×3 MOLs.

i	A	B	C
π	a	b	c
ρ	c	a	b
θ	b	c	a

ii	A	B	C
α	a	c	b
β	c	b	a
γ	b	a	c

Each element a, for example, represents the inclusion of an edge, coloured a in μ_2, in the given equi-net (row) matched with the given edge coloured the colour of the column heading in colouring μ_1.

Suppose that there was a junction joining just two components. One component has just one edge at the junction. Therefore the other two edges are in another component. From the latin squares we can see that if any two edges at a vertex are in an equi-net then the third edge is forced to be in the same equi-net; which is a contradiction. Hence all three equi-nets meet at a junction. ■

If μ_1 and μ_2 are edge-colourings of a graph G, with colour sets W_1, W_2, such that $\mu_2 = \zeta \circ \mu_1$ where ζ is a bijection from W_1 to W_2, then we say that μ_2 is **autochromatic** to μ_1, and the bijection is an **autochromatism**.

Lemma 5.3.3 *Where μ_2 is autochromatic to μ_1, the two equi-coverings corresponding to μ_1 and μ_2 are such that: one has just one component, namely $E(G)$, and the other is a partition into the colour classes of μ_1.*

Proof. Let the colours be A, B and C in μ_1 and a, b, c in μ_2. Again consider:

i	A	B	C
π	a	b	c
ρ	c	a	b
θ	b	c	a

ii	A	B	C
α	a	c	b
β	c	b	a
γ	b	a	c

The bijection ζ is one of ϕ, θ, ρ, α, β, γ, and hence the corresponding equi-net is $E(G)$ and the other two equi-nets of this equi-covering are empty. Equally, each equi-net of the other equi-covering corresponds to one colour class. For example, if $\zeta = \alpha$, then

$$E_\phi = \{e \in E(G)\colon\ \mu_1(e) = A\};\ E_\theta = \{e \in E(G)\colon\ \mu_1(e) = B\};$$

$$E_\rho = \{e \in E(G)\colon \mu_1(e) = C\}.$$ ■

A decomposition of the edges of a graph into three sets of independent three-armed stars S_3 such that at any vertex we have either three incident stars or just one star, as shown in figure 5.3.3, will be call a **tri-star structure**.

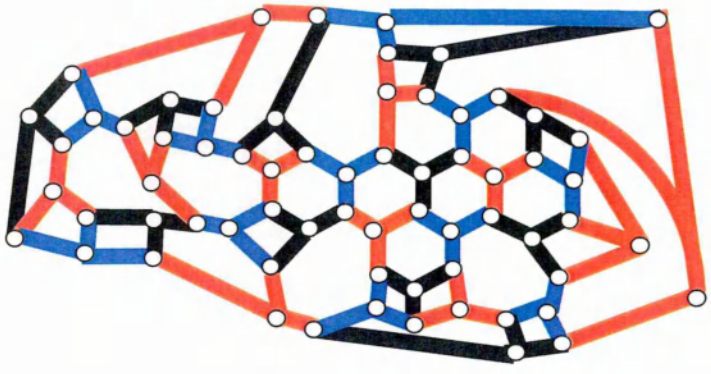


Figure 5.3.3

An equi-covering with the tri-star structure will be called a **tri-star equi-covering**. Not all cubic graphs can have this structure; in particular, $|E(G)|$ must be a multiple of 9 for a tri-star decomposition to exist at all.

Tri-star equi-coverings can be found from planar face colourings, see figures 5.3.10 and 5.3.11 (page 87) where four face-colourings of the hexagonal prism are shown, two of which give a pair of edge 3-colourings that produce a tri-star equi-covering. They can also be found from non planar edge colourings, see figures 5.3.12 (page 89).

Lemma. 5.3.4 *Where a vertex v is a junction vertex in equi-covering-(i), then it is an inner vertex in equi-covering-(ii) and vice versa..*

Proof. A junction vertex in any equi-covering has just one edge from each equi-net. These correspond to three bijections from W_1 to W_2. From the latin squares one can see that that these must all be in the same equi-net, but in the other equi-covering. ■

Lemma 5.3.5 *A graph with the tri-star structure has as many junction vertices as components.*

Proof. Each vertex is either an inner vertex, at a centre of a star, or a junction vertex incident with three stars. If for any colour c_x there are m inner vertices they are adjacent to $3m$ junction vertices. This must include all the junction vertices therefore there are exactly $3m$ junction vertices. Each of these has an edge from three different stars so there are at least m stars of both other colours. There cannot be more than m stars of any other colour as there are only $3m$ junction vertices, hence there are m stars of each colour, and the number of components and junctions is equal. ■

Main result (2)

Theorem 5.3.6 *If equi-covering-(i) of any two colourings of a graph G, has the tri-star structure, then so has equi-covering-(ii) and vice versa.*

Proof. In a tri-star-equi-covering there is just one inner vertex in each component. From Lemmas 5.3.4 and 5.3.5, in equi-covering-(*i*) we have as many components, and hence as many inner vertices, as we have components. Each inner vertex becomes a junction and each junction becomes a component in equi-covering-(*ii*). We have as many components as before, each with at least three edges, therefore they must have exactly three edges and other equi-covering-(i*i*) must also have tri-star structure. ■

Hence, we can now say that such pairs of colourings have **tri-star-equi-covering**.

Lemma 5.3.7 *In a cubic graph with tri-star structure, every circuit is even.*

Proof. In any circuit, the junction vertices and inner vertices must alternate. ■

Although Lemma 5.3.7 is true for both planar and non planar graphs, where the graph is planar these circuits may refer to faces, or groups of connected faces.

Corollary 5.3.8 *A cubic graph with a tri-star structure is triangle-free.* ■

Lemma 5.3.9 *A graph with a tri-star structure and no 4-circuits is non-planar.*

Proof. In a planar graph with a tri-star structure and no 4-circuits, every face must have at least six edges. From Corollary 4.1.8 we know that every planar graph has at least one face with at most five edges, therefore, the graph is non-planar. ■

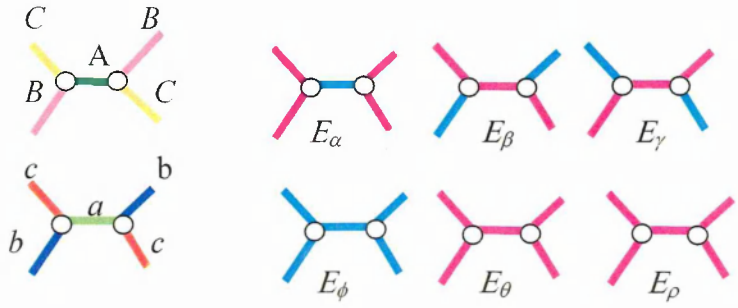
Main result (3)

Theorem 5.3.10 *Let two colourings of G be μ_1 and μ_2. At least one equi-net has a component consisting of a single edge unless they have tri-star equi-covering.*

Proof. Let the two colourings of G be μ_1 and μ_2 with equi-nets E_ϕ, E_α, E_β, E_γ E_θ, and E_ρ. We can call the colour on any edge e_{12} c_A in μ_1 and c_a in μ_2.

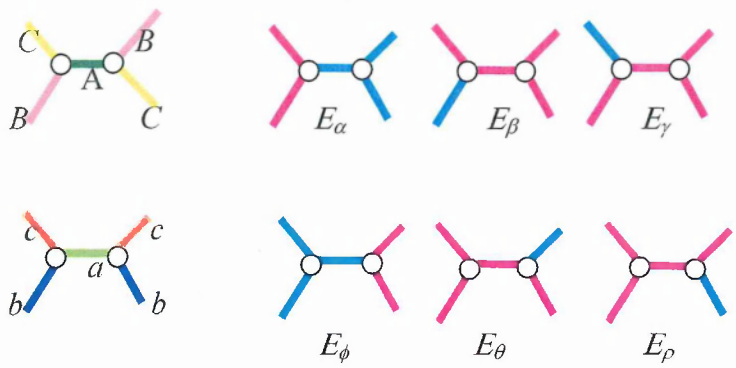
If the edges incident to edge e_{12} have the same local colouring w.r.t. μ_1 and μ_2, then we can call the colours on the respective edges c_B in μ_1 and c_b μ_2 and c_C in μ_1 and c_c μ_2 so that all five edges are in the same component of equinet E_ϕ and hence e_{12} is a single-edge component of E_α. See figure 5.3.4.

The edge was chosen at random and the colours are arbitrary. Hence, only if all the edges are coloured in such a way that they each have a different local colouring w.r.t. μ_1 and μ_2, can we have an equi-net with no component consisting of a single edge. Therefore, we can see that the colourings of the graph G with no single equi-net edge are such that, at any vertex, the edges are all as in Figure 5.3.5.



Same local colouring w.r.t. μ_1 and μ_2: equi net edges in blue.

Figure 5.3.4



Different local colouring w.r.t. μ_1 and μ_2; equi net edges in blue

Figure 5.3.5

By Lemma 5.3.2, no component can have just two edges in an equi-net; therefore, if any of the edges in equinets E_β, E_γ, E_θ or E_ρ are not in a larger component then they are themselves a component consisting of a single edge. If any edge is in a larger structure that is not a tri-star then it must have the same local colouring and hence it is a single edge component of a different equi-net. Therefore, for any colourings μ_1 and μ_2 with no such component, every equi-net must have the tri-star structure. ■

5.3.2 A measure for colour difference

We now consider two different edge-colourings and count how often the edge colours appear together, **match up,** when one colouring is laid on top of the other, **overlaid**. This has the same sense of maximal difference as we had in orthogonality when we considered latin squares in Chapter 3. We now show that equi-coverings measure this **maximal colour difference** and show the importance of the tri-star equi-covering. When G is planar, the difference in edge colourings is reflected by the difference in face colourings.

Lemma 5.3.11 *If G is a planar graph with two colourings μ_1 and μ_2 which have tri-star equi-covering, then, in the corresponding face-colouring, where μ_1 has an edge e_{12} whose local colouring gives figure* 5.3.6 (*a*), *then μ_2 gives Figure* 5.3.6 (*b*) *to automorphism of the colours.*

Proof. If the local colouring was as in figure 5.3.6 (*a*) in both, then there is an equi-net with figure 5.3.6 (*c*) and hence no tri-star equi-covering. Only with a different local colouring (*b*) can we get a tri-star equi-covering (*d*).

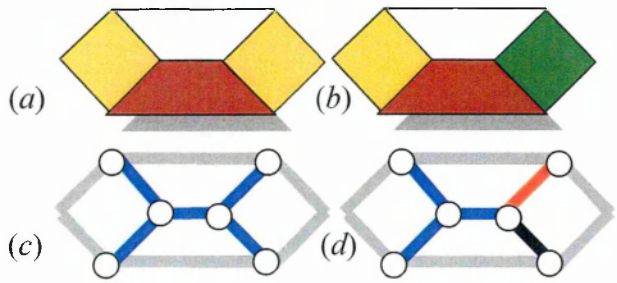
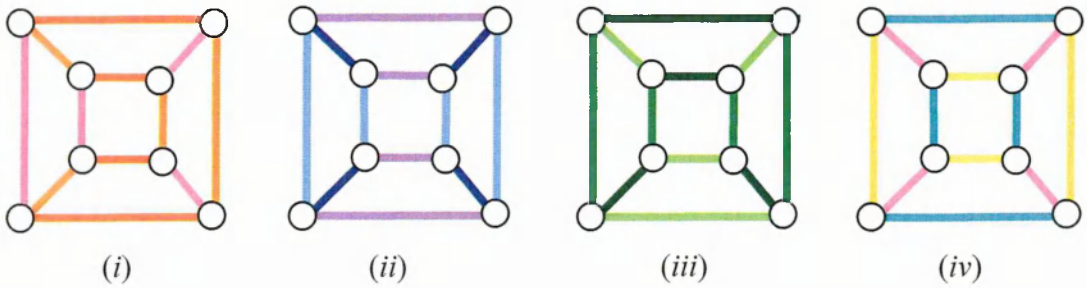


Figure 5.3.6 ■

With this result in mind, we consider how different two colourings can be in terms of their equi-coverings.

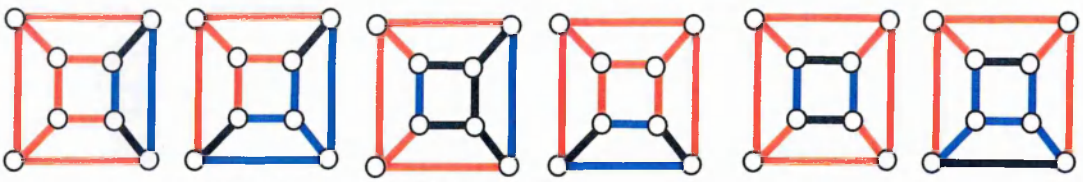


Colourings of the cube.

Figure 5.3.7

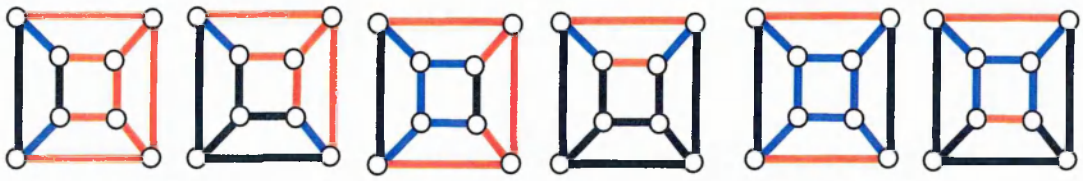
Figure 5.3.7 shows all the non-automorphic colourings of the cube when we consider a fixed vertex labelling. The full set of equi-coverings for Figure 5.3.7 is given in Figures 5.3.8 and 5.3.9 below.

The two equi-coverings for each pair of colourings are isochromatic but this is not true in general. Asymmetric graphs, for example, cannot have isochromatic equi-coverings since they would need to be automorphisms: from Lemma 5.3.4 we can see that this is not possible.



(*i*)/(*ii*) (*i*)/(*iii*) (*i*)/(*iv*) (*ii*)/(*iii*) (*ii*)/(*iv*) (*iii*)/(*iv*)

Figure 5.3.8 *Equi-covering (i)*



(*i*)/(*ii*) (*i*)/(*iii*) (*i*)/(*iv*) (*ii*)/(*iii*) (*ii*)/(*iv*) (*iii*)/(*iv*)

Figure 5.3.9 *Equi-covering (ii)*

We now consider the problem of quantifying the difference between any two colourings of the same graph. We believe that the following is a sensible measure of this difference.

Main concept (4)

The **colouring difference,** ***cd*[$\lambda(\sigma)$]** is the greatest number of components with more than one edge (**large components**), λ, that appear in both equi-coverings, qualified by the number of components that have just one edge (**solitary components**), σ.

For a pair of automorphic colourings, by Lemma 5.3.3, one equi-covering has just one large and no solitary components therefore the colouring difference is 1(0). In the

case of the cube above, since (i)/(ii), (ii)/(iii) and (ii)/(iv) have just one large and four solitary components in each equi-covering, they are said to have colouring difference 1(4). All others pairs have two large and two solitary components in each equi-covering and are said to have colouring difference 2(2). A colouring difference of 1(4) is deemed greater than 1(0) but less than 2(2). For any graph G, the colour difference between any two planar colourings with the greatest possible number of large components and of solitary components in both equi-coverings will be said to be the **maximal colouring difference** for G (***mcd*(*G*)**$[\lambda(\sigma)]$). Since colour difference is defined in relation to edge colourings, the equi-coverings can relate to finite non-planar cubic 3-edge colourings where again $cd[\lambda(\sigma)]$ and $mcd(G)[\lambda(\sigma)]$ give the maximum number of components that could be considered to be the same.

Theorem 5.3.12 *The maximal colouring difference for the cube is cd*[2(2)].

Proof. This follows from figures 5.3.8 and 5.3.9 above, where all possible colourings of the cube are compared. ■

The reason that we must consider the solitary components separately is that, as we found in Theorem 5.3.10, solitary components arise when ever the local colouring is the same, hence they indicate a measure of similarity as well as difference. This becomes clear when we look at the following examples.

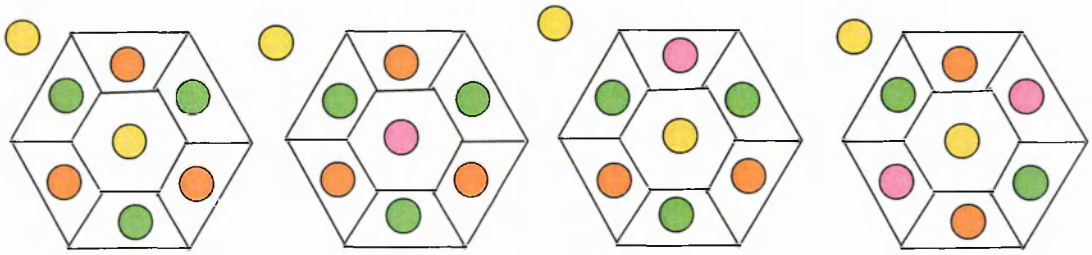


Figure 5.3.10

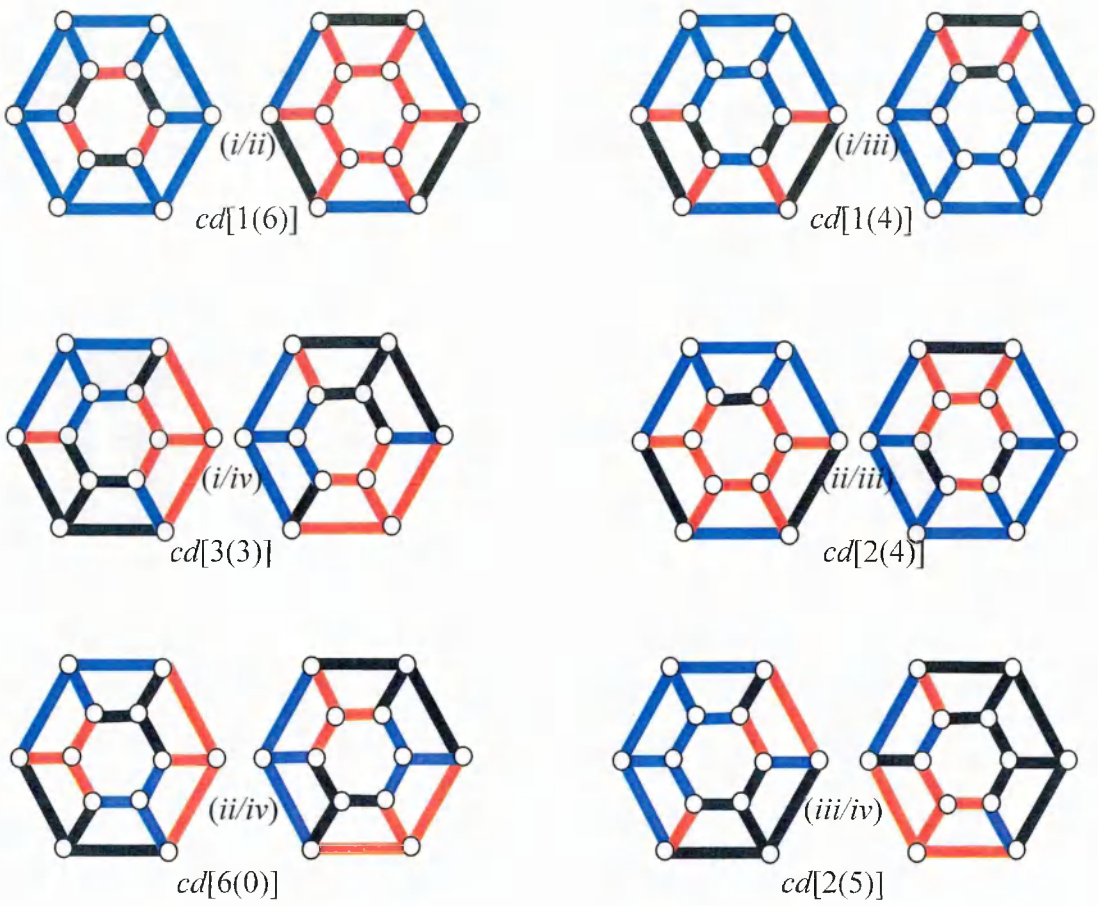


Figure 5.3.11

Figure 5.3.10 shows a selection of four out of the full set of twelve colourings of the hexagonal prism. We can see, in the equi-coverings, figure 5.3.11, that the colour differences range from *cd*[1(4)] for (i/iii), which have just one, four sided face, differently coloured, to *cd*[6(0)] for (ii/iv) where no more than three faces are the same to automorphism. Note that we have *cd*[1(6)] for (i/ii) since all faces are matched to (i) except for one hexagonal face. By this measure of difference, the difference caused by changing a hexagon is greater than that caused by changing a square. We feel that this is logical. Note that in all cases, the equi-coverings point out all possible groups of faces that could be considered the same.

Main result (5)

Theorem 5.3.13 *Two edge 3-colourings of a graph G with tri-star equi-covering have colour difference* $cd[\lambda(\sigma)]$ where $\lambda = |V(G)|/2$ *and* $\sigma = 0$.

Proof. In a tri-star structure there are no small components, and the large components are in one-to-one correspondence with the inner vertices. Thus the number of large components is $|V(G)|/2$. ■

Main result (6)

Theorem 5.3.14 *The maximal colour difference* $mcd(G)[\lambda(\sigma)]$ *for any graph is such that* $\lambda \leq |V(G)|/2$, *equality being achieved only when G has a tri-star equi-covering.*

Proof. In any large component the number of edges is at least 3. Hence, we can have at most one third as many large components as edges. Since G is cubic we have $3V(G)/2$ edges and therefore $mcd(G)[\lambda(\sigma)]$ has $\lambda \leq V(G)/2$, the inequality being strict if there are *either* any edges not in large components *or* any large components with more than three edges. ■

5.3.3 Colour difference in the infinite plane

We can also consider face colourings of the infinite plane and their related 3-edge colourings. In these cases we do not have a value for the colour difference but have a measure for how often particular colours will match up when overlaid.

From the proof of Theorem 5.3.10, we know that if an edge is a single-edge component of any equi-net then it has the same local colouring in both colourings. Hence every such local group of four colours will match up.

Consider the octagon-square tessellation with a tri-star structure in Figure 5.3.12. This can be represented by the planar colourings μ_1 and μ_2, Figures 5.3.14 and 5.3.15.

It is only possible to join up the edges of the given plane colouring, both vertically and horizontally, as a non planar graph. Since the edge 3-colourings of this non planar graph have equi-coverings with the tristar structure, they have maximal colouring difference $V(G)/2$. Where $n = V(G)$, in this example $n = 144$, although we have no value for n in the infinite plane we can use the value for the smallest repetitive section of the tri-star structure to give a measure of **finite colouring difference, $fcd[\lambda(\sigma)]$.** In this case there is a section of twenty-four vertices (Figure 5.3.13) which can be used to build up the whole of the block. In this case $\lambda = 24/2$. giving $fcd[12(0)]$. Note that both the octagon and the square colouring in μ_1 is based on the cyclic Cayley table order four and the octagon (only) colouring μ_2 is based on the cyclic Cayley table order three.

Figure 5.3.12

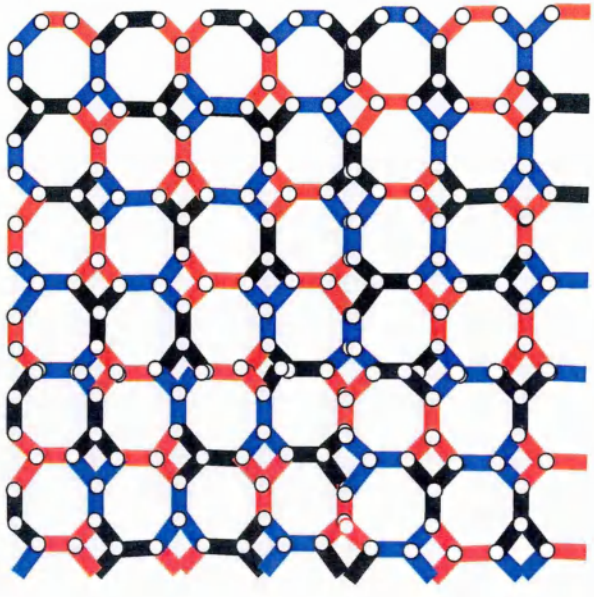


Figure 5.3.13

Figure 5.3.14
μ_1

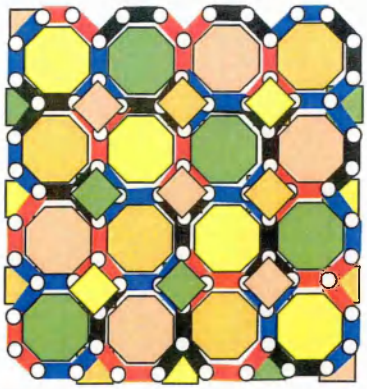
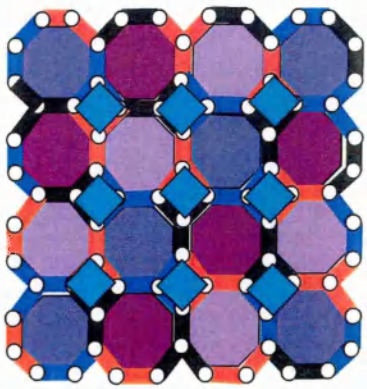


Figure 5.3.15
μ_2



Another graph with tristar structure in the infinite plane and in any non planar graph formed as before, is shown in Figure 5.3.18. Two planar colourings with this edge colouring as an equi-covering are shown in Figures 5.3.16 and 5.3.17. The smallest repetitive structure has six vertices. Hence these colourings can be said to have finite colouring difference *fcd*[3(0)].

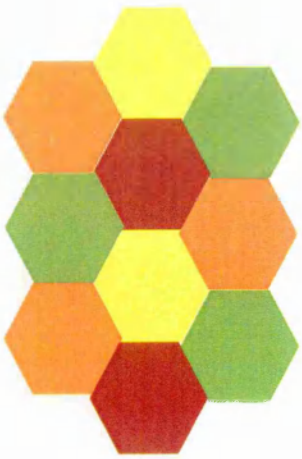


Figure 5.3.16

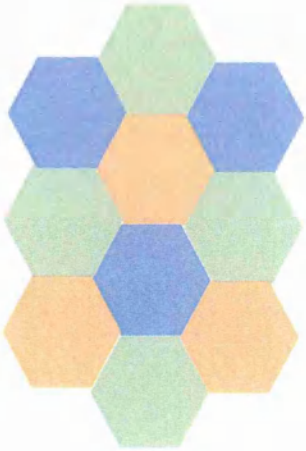


Figure 5.3.17

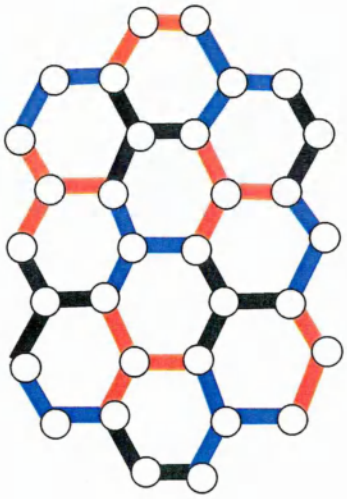


Figure 5.3.18

In all these colourings, the value of the colour difference is easy to compute without the necessity to decide in advance which colours are to be considered the same. Hence this is a device useful whenever a comparison between two apparently random colourings of the same graph is required.

Summary of chapter 5:

We discuss the existence of planar graphs with known edge 3-colourings such that have no transition sequence using Kempe (edge)-interchanges. We then show that there is a way of getting round the problem by making use of a temporary colour. We then study a measure of difference between two edge 3-colourings of the same graph.

Summary of main results by Jini Williams in Chapter 5

Theorem 5.2.1 [Holroyd and Williams] *Every edge 3-colouring of a class 1 (not necessarily planar) cubic graph can be obtained from every other edge 3-colouring of the same graph by a series of (edge) Kempe interchanges using at most four colours.*

Main result (1) page 69

Theorem 5.3.6 *If equi-covering-(i) of any two colourings of a graph G, has the tri-star structure, then so has equi-covering-(ii) and vice versa.* Main result (2) page 76

Theorem 5.3.10 *Let two colourings of G be μ_1 and μ_2. At least one equi-net has a component consisting of a single edge unless they have tri-star equi-covering.*

Main result (3) page77

The **colouring difference,** $cd[\lambda(\sigma)]$ is the greatest number of components with more than one edge (**large components**), λ, that appear in both equi-coverings, qualified by the number of components that have just one edge (**solitary components**), σ.

Main concept (4) page 80

Theorem 5.3.13 *Two edge 3-colourings of a graph G with tri-star equi-covering have colour difference* $cd[\lambda(\sigma)]$ *where* $\lambda = |V(G)|/2$ *and* $\sigma = 0$.

Main result (5) page 83

Theorem 5.3.14 *The maximal colour difference* $mcd(G)[\lambda(\sigma)]$ *for any graph is such that* $\lambda \leq |V(G)|/2$, *equality being achieved only when G has a tri-star equi-covering.*

Main result (6) page 83

CHAPTER 6

TOTAL COLOURING

6.1 Graphs and Total Colourings

6.1.1 Complete graphs

In the context of $(\Delta + 1)$-colourings we regularise G by adding spines to the vertices so that each vertex is incident with Δ edges and spines, see Chapter 2.

A **spine and vertex colouring** of G with respect to the colour set $C = \{c_1, c_2, \ldots, c_{\Delta+1}\}$ is a colouring of the spines and vertices of G such that:

(i) the spine colours at any vertex are distinct from each other and from the vertex colour;

(ii) the restriction to the vertices is a proper vertex colouring.

We use the notation $\mu(V)$ for such a colouring, to emphasise that it is defined only on vertices and spines (but may possibly be extendable to a total colouring that will then be denoted by μ).

For any vertex v, a colour is said to be **present at** v if it is the colour either of v or of a spine at v. For $i = 1, \ldots, \Delta + 1 + t$, we denote by S_i the set of vertices of G at which the colour c_i is present. (The colouring $\mu(V)$ will always be defined by the context.)

In a semi-total colouring, if two adjacent vertices have the same colour, then they and the edge between them, are said to be **opposed.** The number of edges which are opposed is the **β-number.** A total colouring is a semi-total colouring with $\beta = 0$, which is to say, that no two adjacent vertices have the same colour. The **β-number of**

G is the least number of opposed edges possible in any semi-total colouring of G. The β-number of a type 1 graph is $\beta = 0$. The β-number of a type 2 graph is a value such that $\beta > 0$. Given a total or semi-total colouring of G using $\Delta + t$ colours, for each colour c_i there is a set of m_i edges having colour c_i, these edges being collectively incident with $2m_i$ vertices. There remain $V - 2m_i$ vertices having c_i present either as a vertex colour or as a spine colour; we denote this set of vertices by S_i. If there are insufficient independent vertices for a colour to be on $V - 2m_i$ independent vertices, then, in a total colouring, S_i must include spines.

The total colourings of complete graphs have been completely classified. The results for odd n have been known since total colouring was discovered [6.1] and those for even n, theorem 6.1.2 below, were proven by A Hilton in 1998 [6.2]. We will not present a proof to this theorem but will present certain observations relating to it which lead to a conjecture which we explore further in Section 6.3.

Lemma 6.1.1 *The complete graphs of odd order are all type* 1.

Proof The cyclic Cayley tables of Chapter 2 (Lemma 2.2.3 page 13) provide a proper total colouring for each such graph. ■

In stating the following theorem, we use Hilton's notation: $e(\overline{G})$ and $\alpha'(\overline{G})$ are respectively the number of edges, and the maximum number of independent edges, in the complement $\overline{G}$ of G.

Theorem 6.1.2 (Hilton's Theorem [6.2]) *Suppose G is a graph of order* $2n$ *having* $\Delta(G) = 2n-1$. *Then G is type 1 if and only if*

$$e(\overline{G}) + \alpha'(\overline{G}) \geq n. \qquad \blacksquare$$

Observations on Hilton's Theorem. Consider the semi-total colouring $\mu_0(K_{2n})$ corresponding to the cyclic Cayley table for $2n$. By construction, every vertex has an odd colour. From Theorem 3.1.4, these are in pairs, giving $\beta_0 = n$.

Vertices v_1 and v_{n+1} have colour c_1 and are joined by an edge $e_{(1,n+1)}$ with $\mu(e_{(1,n+1)}) = c_{n+1}$. Similarly:

$$\mu(v_2) = \mu(v_{n+2}) = c_3,\ \mu(e_{(2,n+2)}) = c_{n+3};\ \ldots.$$

$$\mu(v_m) = \mu(v_{n+m}) = c_{2m-1},\ \mu(e_{(m,n+m)}) = c_{n+2m-1}(\text{mod } 2n);\ \ldots.$$

$$\mu(v_n) = \mu(v_{2n}) = c_{2n-1},\ \mu(e_{(n,2n)}) = c_{n-1}.$$

There are two cases: n is even and n is odd; but we will only consider odd n here.

Odd n: By construction all the opposed vertices are coloured with odd colours and the corresponding beta edges are coloured with even colours since $n + 2m - 1$ is even. Since no vertex is even, every even colour, c_{2k}, is in a chain ${}_x[c_{2x-1}, c_{2k}]_{n+x}$, from every vertex v_x to the other vertex of the same colour. Moreover, the construction of the colouring is such that every vertex colour is in a chain ${}_x[c_{2x-1}, c_{2x}]_{n+x}$, which uses every edge of both colours. Therefore, since all vertices are initially isomorphic, we can choose to colour K_{2n} in such a way that the set A of $\alpha'(\overline{G})$ independent edges correspond to the $a_{x,n+x}$ elements in the cyclic Cayley table which have the even colours c_{n+2x-1}. We can then let the set B consisting of the other $(e(\overline{G}) - \alpha'(\overline{G}))$ dependent edges have different even colours until either all even colours are used or we have a triangle in B which forces an edge to have an odd colour, such as in graph Number (69) in our catalogue, Appendix (6.1). Triangles are resolvable special cases but since doing so here, would throw no light on the point that we wish to make, we shall ignore them. When we remove just the set A from K_{2n}, we have a semi-total colouring μ_1 of $G_1 = K_{2n} - A$. If $n - \alpha'(\overline{G}) > 0$ then we have $\beta_1 = n - \alpha'(\overline{G})$ and the

colouring is not a proper total colouring. We still need to change the colour on another $n - \alpha'(\overline{G})$ vertices. We have a possible transition to a proper total colouring with the Kempe interchange sequence

$$\mu_2 = \mu_1 + \sum_{i=1}^{W} x_i [c_{n+2x_i-1}, c_{n+2x_i}] y_i, \text{ where } y_i = (x_i + (n+1)/2) \text{ and } W \leq \alpha'(\overline{G}).$$

In each case v_y is a vertex which was coloured with an odd colour c_{n+2x} in $\mu_1(K_{2n})$, but has the even colour c_{n+2x-1} in the semi-total colouring $\mu_2(G_1)$, with c_{n+2x} now on the spine. Since the colours c_{n+2x-1} are all different, there is no other vertex of that colour, and the vertices are no longer opposed. If $n - 2\alpha'(\overline{G}) > 0$ then we have $\beta_2 = n - 2\ \alpha'(\overline{G})$ and the colouring is not a proper total colouring. We must then consider the spines caused by the removal of the set B of $e(\overline{G}) - \alpha'(\overline{G})$ dependent edges. There are as many even colours as odd colours, therefore, there are still $(n - 2\alpha'(\overline{G}))$ even colours that can potentially be used in transitional chains ${}_x[c_{n+2x+2j-1}, c_{n+2x}]_{zj}$, where v_{zj} is a vertex still opposed in μ_2. It can be shown that these transitions exist, but for now we leave this discussion with a concluding observation: Deleting an edge in $\mu_i(K_{2n})$, allows a Kempe interchange to an initially opposed vertex allowing this vertex to be given a colour not used on any of its neighbours; while deleting an independent opposed edge in $\mu_i(K_{2n})$, allows the two vertices to remain the same colour and still allows a different initially opposed vertex to be given a colour not used on any of its neighbours. If, when all the spines have different colours in some colouring μ_3, we have $\beta_3 > 0$, then $e(\overline{G}) + \alpha'(\overline{G}) < n$, but, if $e(\overline{G}) + \alpha'(\overline{G}) \geq n$ then we can make the required swaps to eliminate all opposition. ■

We will refer again to Hilton's Theorem in Sections 6.1.2 and 6.4. However, our observation was included order to indicate that the following conjectures are known to

be true for complete graphs. We shall discuss beta values further in Section 6.3, and in Appendix (6.1) we have a catalogue of 70 small type 2 graphs where it can be seen that the following conjectures are true in those cases.

Lemma 6.1.3 *Every type 2 graph contains an edge-critical type 2 subgraph.*

Proof. Given a type 2 graph G, we may remove edges systematically until a type 1 graph is found we then replace that edge and continue. If no more edges can be removed then the subgraph is critical. ■

Similarly we also have **β–critical graphs** for semi-total colourings. In a β-critical graph removing any edge will reduce the value of β, though not necessarily by the same number. As we saw in discussing Hilton's Theorem, removing some edges reduces β by two, others by just one.

Lemma 6.1.4 *Any graph G contains a β-critical subgraph H with $\beta(H) = \beta(G)$.* ■

Main conjecture (1)

Conjecture 6.1.5 *Removing any set of p edges from a β-critical graph G will reduce β by between p and $2p$.*

Main conjecture (2)

Conjecture 6.1.6 *Adding an edge to a β-critical graph G will increase β by at most* 2.

We believe that the beta-classification of graphs is an important an area of study in it's own right: not just because it would help solve the total colouring conjecture. To be able to classify graphs by their β-number would give us deeper insight into the fundamental underlying structure of total colourings.

6.1.2 Conformable Graphs

We now consider Chetwynd and Hilton's Conformability Lemma [6.3]. We shall however present our own proof of this lemma as it leads on more readily.

Let μ be a vertex colouring of a graph G, using $\Delta + 1$ colours. We shall order the colours by parity, as follows: the first $q(\mu)$ colours, $c_1, \ldots, c_{q(\mu)}$, each occur on an even number of vertices while the remaining $r(\mu)$ colours each occur on an odd number of vertices, giving $q(\mu) + r(\mu) = \Delta + 1$.

We use the same notation in the case that G is a type 1 graph and μ is a total colouring of G, by restricting μ to the vertices.

Lemma 6.1.7 *(the Conformability Lemma)*

Let G be a type 1 graph and μ a proper total colouring of G using $\Delta + 1$ colours.

(1) *If n is even, then $def(G) \geq r(\mu)$;*

(2) *if n is odd, then $def(G) \geq q(\mu)$.*

Proof. If n is even, then for $i = 1, \ldots, \Delta + 1$ the number of vertices that are not incident with an edge coloured c_i is even. Hence S_i is even.

For each of the $r(\mu)$ colours c_i used on an odd number of vertices, there is also an odd number of vertices that are neither coloured c_i nor incident with an edge coloured c_i and hence there is an odd (and thus non-zero) number of spines for each of the $r(\mu)$ colours c_i. Thus $def(G) \geq r(\mu)$.

Similarly, if n is odd, then for any colour c_i the number of vertices that are not incident with an edge coloured c_i is also odd. For each of the $q(\mu)$ colours c_j used on an even number of vertices, there is an odd number of vertices that are not coloured c_j, and hence there is an odd (and therefore non-zero) number of spines with each of the $q(\mu)$ colours c_j. Thus $def(G) \geq q(\mu)$. ■

Chetwynd and Hilton define a graph G to be **conformable** if it has a proper vertex $(\Delta + 1)$-colouring $\mu = \mu(V)$ such that where $q(\mu) + r(\mu) = \Delta + 1$ as above,

$$def(G) \geq \begin{cases} r(\mu) & \text{(if } n \text{ is even);} \\ q(\mu) & \text{(if } n \text{ is odd).} \end{cases}$$

Such a colouring of G is said to be a **conformable colouring**. It follows from this lemma that every type 1 graph is conformable and every non-conformable graph is type 2. However, there are many conformable type 2 graphs; for example, $K_{3,3} - \{e\}$, Figure 6.1.1. Further conformable type 2 graphs will be found in Appendix (6.1).

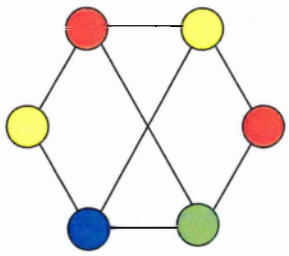


Figure 6.1.1

Main result (3)

Lemma 6.1.8 *Every graph with at least Δ spines is conformable.*

Proof. Let G be a graph with at least Δ spines. That is, $def(G) \geq \Delta$. By Brooks' Theorem [1.6], there is a proper vertex colouring μ of G using just Δ colours; that is, $c_{\Delta+1}$ is unused and hence $q(\mu) \geq 1$. But $q(\mu) + r(\mu) = \Delta + 1$. Thus $r(\mu) \leq \Delta$ and if n is

even the graph is conformable. Only if n is odd and $q(\mu) = \Delta+1$ would the graph be non conformable. However, no colouring can have every colour on an even number of vertices when n is odd. Therefore, there is at least one colour on an odd number of vertices and $r(\mu) \geq 1$ and $q(\mu) \leq \Delta$ giving.

$$def(G) \geq \Delta \geq \begin{cases} r(\mu) & \text{(if } n \text{ is even);} \\ q(\mu) & \text{(if } n \text{ is odd).} \end{cases}$$

and G is conformable. ■

When n is even, since every edge joins two vertices, G can have at most $\Delta n/2$ edges. However when n and Δ are odd, the number of edges is at most one less. A graph is called **overfull** if $|E(G)| > \Delta(G) \left\lfloor V(G)/2 \right\rfloor$.

This means that, for even n, a graph is never overfull and for odd n, a graph is overfull only if the number of spines is less than Δ. This gives the following corollary.

Corollary 6.1.9 *For odd n, if G is not overfull then G is conformable.*

Conjectures on overfull graphs and conformability were explored in [6.4] by Hilton, Holroyd and Zhao, where the author of this thesis, Jini Williams, was also mentioned.

Main result (4)

Lemma 6.1.10 *Let G be a non-connected graph, each of whose components is of maximum degree Δ. Then G is conformable if every component is conformable.*

Proof. Let m be the least integer such that there is a non-conformable graph G with m conformable components each of maximum degree Δ. Let G_1 be one component and let $G_2 = G - G_1$. Then, since G_2 has $m - 1$ components, it is conformable. Thus,

there is a vertex $(\Delta + 1)$-colouring μ of G whose restriction to each of G_1 and G_2 obeys the appropriate inequality. Let:

s be the number of colours on an even number of vertices of G_1 and an even number of vertices of G_2;

t be the number of colours on an odd number of vertices of G_1 and an odd number of vertices of G_2;

u be the number of colours on an even number of vertices of G_1 and an odd number of vertices of G_2;

v be the number of colours on an odd number of vertices of G_1 and an even number of vertices of G_2.

Case 1: $n(G_1)$ and $n(G_2)$ are odd.

Then $def(G_1) \geq s + u$, $def(G_2) \geq s + v$, so $def(G) \geq 2s + u + v \geq u + v$ and G is conformable.

Case 2: $n(G_1)$ and $n(G_2)$ are even.

Then $def(G_1) \geq t + v$, $def(G_2) \geq t + u$, so $def(G) \geq 2t + u + v \geq u + v$ and G is conformable.

Case 3: $n(G_1)$ is odd and $n(G_2)$ is even.

Then $def(G_1) \geq s + u$, $def(G_2) \geq t + u$, so $def(G) \geq 2u + s + t \geq s + t$ and G is conformable.

Case 4: $n(G_1)$ is even and $n(G_2)$ is odd.

Then $def(G_1) \geq t + v$, $def(G_2) \geq s + v$, so $def(G) \geq 2v + s + t \geq s + t$ and G is conformable. ■

Lemma 6.1.11 *The only connected irregular graphs with* $\Delta = 2$ *are paths, all of which are conformable.*

Proof. Only cycle graphs are regular with $\Delta = 2$. Path graphs are type 1 and can be coloured with the colours as in Figure 6.1.2. They are therefore conformable.

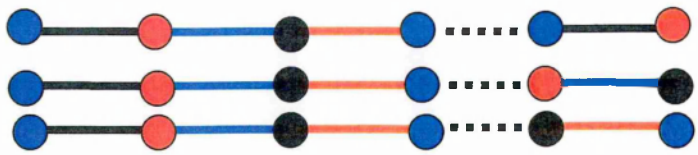


Figure 6.1.2

Main result (5)

Theorem 6.1.12 *Any non-conformable irregular graph G is an induced subgraph of a type* 2 *conformable graph H of the same maximum degree, where* $n(H) = n(G) + 1$.

Proof. Let G be a non-conformable (hence type 2) irregular graph. There is at least one vertex, v, with at least one spine. Attach a vertex w to this spine, adjacent only to v. Then w has $\Delta - 1$ spines, and thus $def(H) \geq \Delta - 1$. By Lemma 6.1.8, H is conformable if any vertex other than w also has a spine. When there is no such spine $def(H) = \Delta - 1$. Where G has vertex colouring μ, which is the closest possible to conformable, let H have the same colouring on all vertices except for vertex w. Call this colouring θ.

(*i*) G has an even number of vertices.

We cannot have just one colour in μ on an odd number of vertices hence $r(\mu) \geq 2$.

We can choose w to have a colour in $q(\mu)$. Giving $q(\theta) \geq 3$ and $r(\theta) \leq \Delta - 2$.

Hence, $def(H) = \Delta - 1 > r(\theta)$ and H is conformable.

(*ii*) G has an odd number of vertices.

As G has at least one spine, $q(\mu) \geq 2$. We can choose w to have a colour in $r(\mu)$.

Giving $r(\theta) \geq 3$, $q(\theta) \leq \Delta - 2$, hence, $def(H) = \Delta - 1 > q(\theta)$ and H is conformable. ■

It is conjectured by Chetwynd and Hilton [6.3] and qualified by Hamilton [6.5] that:

Conjecture 6.1.13 (The Conformability Conjecture) *Let G be a graph satisfying* $\Delta(G) \geq \frac{1}{2}(|V(G)|+1)$. *Then G is type* 2 *if and only if G contains a subgraph H with* $\Delta(G) = \Delta(H)$ *which is either non-conformable, or, when* $\Delta(G)$ *is even, consists of* $K_{\Delta(G)+1}$ *with one edge subdivided.* ■

This conjecture if proven, would yield the following set of important corollaries.

Corollary 6.1.14, Corollary (1) to conjecture 6.1.13. *Let G be a type* 2 *graph that does not consist of an odd complete graph with one edge subdivide. If every type* 2 *graph G satisfying* $\Delta(G) \geq \frac{1}{2}(|V(G)|+1)$, *contains a subgraph H with* $\Delta(G) = \Delta(H)$ *which is non-conformable then every critical subgraph H of G is non-conformable.*

Proof. Suppose that the smallest non-conformable subgraph H of G, $\Delta(G) = \Delta(H)$, was not critical. $\Delta(G) = \Delta(H) \geq \frac{1}{2}(|V(G)|+1) \geq \frac{1}{2}(|V(H)|+1)$. Since H is not critical we could remove one edge an have a smaller type 2 graph G_i satisfying $\Delta(G_i) = \Delta(H) \geq \frac{1}{2}(|V(H)|+1) = \frac{1}{2}(|V(G_i)|+1)$. Hence there would be another smaller type 2 subgraph H_i of G_i, $\Delta(G) = \Delta(G_i) = \Delta(H_i)$ which were not a critical graph and H would not be the smallest. This is a contradiction, therefore the smallest non-conformable subgraph H of G would be critical. But if G were a critical graph then it would be type 2; hence, as it could not have a smaller non conformable subgraph that is critical, the graph itself would need to be non conformable. Hence every critical subgraph H of G would be non-conformable. ■

Corollary 6.1.15, Corollary (2) to conjecture 6.1.13. *Let G be a type* 2 *graph that does not consist of an odd complete graph with one edge subdivided. If every type* 2 *graph G satisfying* $\Delta(G) \geq \frac{1}{2}(|V(G)|+1)$, *contains a subgraph H with* $\Delta(G) = \Delta(H)$

which is non-conformable then every graph with at most two maximum vertices of degree $\Delta(G) \geq \frac{1}{2}(|V(G)|+1)$, *is type* 1.

Proof. Every graph G with just one vertex of maximum degree $\Delta(G)$ has $def(G) \geq V(G)|-1$ and $\Delta(G) \leq V(G)|-1$. Hence $def(G) \geq \Delta(G)$ and by Lemma 6.1.3 the graph is conformable. Any subgraph H with $\Delta(G) = \Delta(H)$ would also have just one vertex of maximum degree and therefore could not have $def(H) < \Delta(H)$. Since G has no non conformable subgraph, if the conjecture is true then G is type 1.

Every graph G with just two vertices of maximum degree $\Delta(G)$ has $def(G) \geq V(G)|-2$. Hence $def(G) \geq \Delta(G)$ when $V(G)|-2 \geq \Delta(G)$. By Hilton's Theorem [6.2], the only type 2 graphs with $V(G)|-2 < \Delta(G)$ are of the form K_n-E, all which have more than two vertices of maximum degree. Hence all graphs with just two vertices of maximum degree satisfying $\Delta(G) \geq \frac{1}{2}(|V(G)|+1)$, have $def(G) \geq \Delta(G)$ and from Lemma 6.1.3 are conformable. If G has a non-conformable subgraph H with the same degree then it is not critical. The critical subgraph H must contain either or both the major vertices, but this must also have $def(H) \geq \Delta(H)$ for the same reason and hence is conformable, which is a contradiction. Hence, either the conjecture is invalid or all graphs with just two major vertices satisfying $\Delta(G) \geq \frac{1}{2}(|V(G)|+1)$ are type 1. ■

Corollary 6.1.16, Corollary (3) to conjecture 6.1.13. *Let G be a type 2 graph that does not consist of an odd complete graph with one edge subdivided. If every type 2 graph G satisfying* $\Delta(G) \geq \frac{1}{2}(|V(G)|+1)$, *contains a subgraph H with* $\Delta(G) = \Delta(H)$ *which is non-conformable then the total colour conjecture holds for all graphs with at most two maximum vertices degree* $\Delta(G) \geq \frac{1}{2}|V(G)|$.

Proof. If from Corollary 6.1.15 all graphs with at most two major vertices satisfying $\Delta(H) \geq \frac{1}{2}(|V(H)|+1)$ were type 1, then any type 2 graph H has at least three major vertices with $\Delta(H) \geq \frac{1}{2}(|V(H)|+1)$. Where H has degree $\Delta(H) = (|V(H)|-1)$ we know that the graph satisfies the total colouring conjecture. Where H has degree $\Delta(H) < (|V(H)|-1)$, we could create a graph G by introducing a new edge $e_{(1,2)}$ between any major vertex v_1 in H and any vertex v_2 to which v_1 is not adjacent. This gives us a graph with at most two major vertices. If the conjecture were valid, G would be type 1 and hence colourable with $\Delta(G) + 1 = \Delta(H) + 2$ colours. Let μ be such a colouring. When we remove $e_{(1,2)}$ from G we have H coloured with $\Delta(H) + 2$ colours and the total colouring conjecture would be proven. ■

The corollary above also follows more fully from the following conjecture:

Conjecture 6.1.17 *Every critical type* 2 *graph has more than two major vertices.* ■

Zhang Zhongfu, a major contributor to this study [6.6], also posed a question in [2.1] which rephrased becomes:

Conjecture 6.1.18 *Every critical type 2 graph has more than one major vertex.* ■

The proof of either of these conjectures would hold for all graphs. Moreover, we feel that it would probably be easier to prove the total colouring conjecture *via* proving 6.1.17 than *via* the conformability conjecture 6.1.13. Studies of the conformability conjecture can be found in works such as [6.6] and [6.7].

We now, for completeness, propose our own conjecture.

Main conjecture (6)

Conjecture 6.1.19 [Holroyd and Williams] *Every type* 2 *critical graph G satisfying* $\Delta(G) < \frac{1}{2}(|V(G)|-1)$ *is conformable.* ■

Since $\Delta(G) < \frac{1}{2}(|V(G)|+1)$ for $n = 2$ is $\Delta(G) < 1\frac{1}{2}$ it is clear that K_2 is not conformable, hence the need to amend the equation.

We also need to point out that in an analysis of the first 50 critical graphs as catalogued by Hamilton Hilton and Hind [6.8], only 19 (38%) are non-conformable see Appendix (6.1). Similarly in the 100 smallest type 2 graphs as catalogued by Marek Kubal , [6.9], only 20 (20%) are non-conformable. It must be noted that each of the graphs in this second list either appears in Appendix (6.1) or consists of a type 1 appendage attached to one of the said critical graphs by a set of spines. From the large number on conformable type 2 graphs, it is clear that conformability on its own is insufficient and that we must refine the concept.

6.2 G*-conformability

6.2.1 G*-conformable graphs

Since it is clear that conformability is insufficient to ensure that a graph is type 1, we must find a more restrictive variant of the concept. We now make the following definition.

Recall that S_i is the set of vertices of G at which the colour c_i is present on a spine or vertex.

A **spine and vertex colouring of G with respect to the colour set $C = \{c_1, c_2, \ldots, c_{\Delta+1+t}\}$** is as in section 6.1 except that we now have $(\Delta + 1 + t)$ colours and hence at any vertex, at least one, and up to $(\Delta + t)$, spines.

Consider the problem of finding a total $(\Delta + 1 + t)$-colouring for a graph G (where $t \geq 0$). We could define a vertex colouring μ of G to be **$(\Delta + 1 + t)$-conformable** if (in analogy with the definition in Section 6.1):

$$\mathrm{def}_t(G) \geq \begin{cases} r(\mu) & \text{(if } n \text{ is even);} \\ q(\mu) & \text{(if } n \text{ is odd)} \end{cases}$$

where we recall from Chapter 2 that $def_t(G) = \sum_{v \in V(G)} def_t(v) = def(G) + t|V(G)|$.

However, since $|V(G)| \geq \Delta + 1$, any vertex $(\Delta + 1 + t)$-colouring has at least Δ spines. Hence, from Lemma 6.1.3, the vertex colouring will be $(\Delta + 1 + t)$-conformable for any graph, and the concept is not useful.

Therefore we say that G is **G^*-conformable** for $(\Delta + 1 + t)$ colours, if it has a spine and vertex $(\Delta + 1 + t)$-colouring such that $G - S_i$ has a 1-factor for each $i = 1, \ldots, \Delta + 1 + t$. A G^*-conformable colouring will be denoted by $\mu(V^*)$.

Theorem 6.2.1 *If G has a total $(\Delta + 1 + t)$-colouring, then G is G^*-conformable for $(\Delta + 1 + t)$ colours.*

Proof. Let μ be such a total colouring. Then for each $i = 1, \ldots, \Delta + 1 + t$, the set $\{e \in E(G)\colon\ \mu(e) = c_i\}$ is a 1-factor of $G - S_i$, and so the restriction $\mu(V)$ is G*-conformable. ■

For the remainder of this section, we assume $t = 0$; that is, we study $(\Delta + 1)$-colourings.

Main result (7)

Corollary 6.2.2 *A graph which has no G^*-conformable colouring is type 2.* ■

We let $V(G - W_i)$ denote the set of vertices left in G when the vertices W_i are removed and $|V(G - W_i)|$ mean the number vertices that are in $V(G - W_i)$. We let $|S_j|$ denote the number of vertices in S_j. Similarly $(G - (S_i \cup S_j))$ will denote the number of vertices left after all vertices in S_i and S_j and both are removed. We let $|S_j \cup S_i|$ denote the number of vertices in S_i and S_j or both, hence $|G - (S_i \cup S_j)| = |(G - S_i) - S_j|$. For a connected graph G, the subgraphs $G - S_i$ may be disconnected.

For any graph X, we denote by $o(X)$ the number of **odd components**, that is, the number of connected components that have an odd number of vertices.

Lemma 6.2.3 *A necessary condition for G to be G*-conformable is that it should have a spine and vertex colouring such that no subgraph $G - S_i$ has an odd component.*

Proof. Clearly, if $G - S_i$ has an odd component, then it cannot have a 1-factor. ■

We owe the following definition to theorem to Tutte [6.10].

Definition. **(Tutte) 1-barrier.** A graph G has a **1-barrier** if there is a set of vertices W in G such that $o(G - W) > |W|$.

Theorem 6.2.4 (Tutte) *G has a 1-factor if and only if it does not have a 1-barrier.* ■

Lemma 6.2.5 *$\mu(V)$ is not a G*-conformable colouring of G if there is a colour c_i and a set of vertices $W \subset V(G - S_i)$ such that $o(G - (S_i \cup W)) > |W|$.*

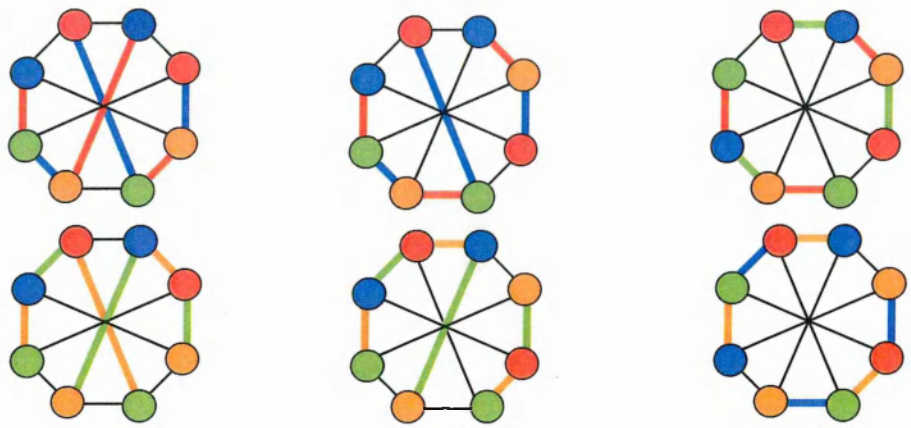
Proof. Assume that $\mu(V)$ has such a colour. By Tutte's Theorem 6.3.5, the subgraph $G - S_i$ does not have a 1-factor and hence $\mu(V)$ is not G*-conformable. ■

Corollary 6.2.6 *A graph is type 2 if it has no spine and vertex colouring such that $o(G - (S_i \cup S_j)) \leq |S_j \cup S_i|$ for each pair of colours c_i, c_j.*

Proof. Take $W = S_j \cup S_i$. ■

Theorem 6.2.7 *It is possible for a graph to be G*-conformable and also type 2.*

Proof. Figure 6.2.1 shows six copies of graph Number14 in Appendix (6.1).



μ_x μ_y μ_z,

Figure 6.2.1

Two copies of three G^*-conformable colourings: μ_x, μ_y and μ_z, are given in order to show the corresponding 1-factors for all four colours. It can easily be verified that these are, to isochromatism, the only G^*-conformable vertex colourings possible. For each of μ_x, μ_y, μ_z and for every colour, $G - S_i$ the 1-factor shown are the only 1-factors possible. However, at least one edge is forced to be in $G - S_i$ and $G - S_j$ for some colours $c_i \neq c_j$, hence the graph is type 2. ■

Main result (8)

Corollary 6.2.8 *A G^*-conformable graph which is also type 2 cannot have disjoint 1-factors for all $G - S_i$.*

Proof. Where G is G^*-conformable there is no 1-barrier in $G - S_i$, and thus there is always a 1-factor. Where G is type 2 and yet G*-conformable, the said 1-factors cannot be disjoint, else the graph would be type 1. ■

Observation 6.2.9 *Of the first 50 type 2 graphs as catalogued by Hamilton, Hilton and Hind* – [6.9] *only the one shown in figure 6.2.1 theorem 6.29. has G^*-conformability.*

See appendix (6.1). ■

6.2.2 Cubic graphs

We now study the case of cubic graphs; since regular cubic graphs have no spines in the current context, spine and vertex colourings become just vertex colourings. In this and the next subsection, we denote $|S_i|$ by n_i.

Main result (9)

Theorem 6.2.10 *Let G be regular cubic graph. Then a necessary condition for G to be type 1 is that G should have a G*-conformable vertex colouring $\mu(V^*)$ such that, for every pair of colour sets S_i and S_j, the subgraph $G - S_i - S_j$ has:*

(i) at least $(n_i + n_j)/2$ components;

(ii) at most $(n_i + n_j)/2$ components with less than four vertices;

(iii) an even number $2q$ of odd components, where $2q \leq \min\{n_i, n_j\}$.

Proof. Note that G has an even number of vertices. Suppose that G is type 1 and let μ be a total 4-colouring of G; then $\mu(V^*)$ is a G^*-conformable vertex colouring. For each vertex v with $\mu(v) = c_i$, and for each other colour c_j, we have a chain $\mu([i, j])$ in $\mu(G)$ which ends at a vertex with colour either c_i or c_j. As G is a cubic graph and the vertices have been coloured with only four colours, each chain passes through alternate vertices of the other two colours. No two of these chains can be incident to the same edge except at the end vertices c_i or c_j. For example consider $\mu([1, 2])$. The edges in the chain are alternately c_1 and c_2 and the incident vertices are alternately c_3 and c_4. An edge not on the chain but incident with a vertex coloured c_3 has colour c_4 and *vice versa*; for any such edge the other incident vertex must have colour c_1 or c_2. Therefore each chain creates a distinct connected component in $G - S_i - S_j$.

(*i*), (*ii*): Hence we have at least as many components as chains. We have exactly $(n_i + n_j)/2$ chains, and hence exactly $(n_i + n_j)/2$ of the components of $G - S_i - S_j$ are chains. We may possibly also have circuits $\mu([i, j])$ each giving distinct even components in $G - S_i - S_j$. Since each circuit has at least four vertices, parts (*i*) and (*ii*) follow.

(*iii*): Where $\mu([i, j])$ has both end vertices the same colour, c_i or c_j, it has an even number of incident vertices and hence creates an even component of $G - S_i - S_j$. Thus each odd component has exactly one vertex of each colour c_i, c_j; so there must be at most $\min\{n_i, n_j\}$ of these. By Theorem 6.2.1 (since $|V(G)|$ is even), n_i and n_j must be even, and hence the number of odd components of $G - S_i - S_j$ must be even. ■

We can now see why number 14 in our catalogue (Appendix 6.1) is type 2. To isochromatism there are just three distinct conformable vertex colourings, all three are G^*-conformable. These were shown in figure 6.2.1 where they were called μ_x, μ_y and μ_z. Although in every case, for all choices of c_i and c_j we have (*iii*) an even number of odd components, for some choices of c_i and c_j we also have $G - S_i - S_j$ with either (*i*) less than $(n_i + n_j)/2$ components, or (*ii*) more than $(n_i + n_j)/2$ components with less than four vertices.

We now conjecture that:

Main conjecture (10)

Conjecture 6.2.11 *Let G be a regular cubic graph. Then the necessary condition for G to be type* 1, *stated in theorem* 6.2.10, *is also sufficient.*

Justification This could only be untrue if there existed components of $G - S_i - S_j$ from $\mu(V^*)$, with edges that could not be allocated a chain $\mu([i, j])$. We believe that it should be possible to show that no such components can exist under the given constraints. ■

We can also consider '*acyclic proper colourings of graphs*'. A proper k-colouring of a graph G is **acyclic** if for each pair of colours, the subgraph induced by that pair has no cycles [6.11]. See also Boiron, Sopena and Vignal [6.12].

With this definition, we feel that the following conjecture would be easier to prove.

Main Conjecture (11)

Conjecture 6.2.12 *Let G be regular cubic graph. If G has an acyclic G*-conformable vertex colouring $\mu(V^*)$ with colour sets S_i and S_j, such that every $G - S_i - S_j$ has*

(i) exactly $(n_i + n_j)/2$ components;

(ii) an even number $2q$, of odd components, where $2q \leq \min\{n_i, n_j\}$;

then G is a type 1 graph.

Justification From the previous lemmas and theorems, if the vertex colouring could not be extended to the edges of the graph, then G is type 2. However, now that we have ruled out the existence of circuits and have a path for every pair of vertices, we feel that a proof is possible. ■

6.2.3 Semi-regular cubic graphs

A **semi-regular graph** was defined in [6.13] to be a graph G with odd degree, $\Delta(G)$, and $\text{def}(G) = 1$ (so that G has exactly one spine and an odd number of vertices). In the following lemmas and theorems we consider G to be a semi-regular cubic graph with the spine on v_1, with a G*-conformable spine and vertex colouring $\mu(V^*)$ such that $\mu(v_1) = c_1$ and $\mu(s_1) = c_2$.

It is convenient in this subsection to define the following function ζ on the pairs $\{i, j\}$ of distinct elements of $\{1,2,3,4\}$. (Note that ζ depends on $\mu(V^*)$).

$\zeta(\{1, 2\}) = 1$;

$\zeta(\{2, 3\}) = 1$ if one vertex adjacent to v_1 has colour c_2 and the other has colour c_2 or c_3;

$\zeta(\{2, 4\}) = 1$ if one vertex adjacent to v_1 has colour c_2 and the other has colour c_2 or c_4;

$\zeta(\{i, j\}) = 0$ otherwise.

Main result (12)

Theorem 6.2.13 *Let G be a semi-regular cubic graph. Then a necessary condition for G to be type 1 is that G should have a G*-conformable spine and vertex colouring $\mu(V^*)$ (as above) such that, for every pair of colour sets S_i and S_j, the subgraph $G - S_i - S_j$ has:*

(i) at least $(n_i + n_j)/2 - \zeta(\{i, j\})$ components;

(ii) at most $(n_i + n_j)/2 - \zeta(\{i, j\})$ components with less than four vertices;

(iii) an odd number of odd components, bounded above by $\min\{n_i, n_j\}$, *except where* $\{i, j\} = \{1, 2\}$, *in which case it has an even number of odd components, bounded above by* $\min\{n_1, n_2\} - 1$.

Proof. Suppose that G is type 1 and let μ be a total 4-colouring of G; then $\mu(V^*)$ is a G^*-conformable vertex colouring. Parts (*i*) and (*ii*) follow as in the proof of theorem 6.2.10, except that, if $\zeta(\{i, j\}) = 1$, then there is a $\mu([i, j])$ chain starting at v_1 that again does not create a connected component in $G - S_i - S_j$.

(*iii*): If $2 \notin \{i, j\}$, then (arguing as in the proof of theorem 6.2.12) each odd component of $G - S_i - S_j$ corresponds to a chain ending in one vertex of each colour c_i, c_j. Since n_i and n_j are odd (by Theorem 6.2.1), the number of these must be odd. The argument also holds for $G - S_2 - S_3$ and $G - S_2 - S_4$, since the exceptional cases when a vertex adjacent

to v_1 has colour c_2 do not affect the argument. In the case of $G - S_1 - S_2$, we must discount the trivial [1, 2] chain at v_1, yielding an even number of odd components. ■

We now make the equivalent conjectures as in the case of regular graphs.

Main conjecture (13)

Conjecture 6.2.14 *Let G be a semi-regular cubic graph. Then the necessary condition for G to be type 1, stated in Theorem* 6.2.13, *is also sufficient.* ■

Main conjecture (14)

Conjecture 6.2.15 *Let G be a semi-regular cubic graph. If G has an acyclic G^*-conformable vertex colouring $\mu(V)$ with $\mu(v_1) = c_1$, $\mu(s_1) = c_2$ and if, for each pair S_i, S_j of colour sets, the subgraph $G - S_i - S_j$:*

(i) has exactly $(n_i + n_j)/2 - \zeta(\{i, j\})$ components;

(ii) has an odd number of odd components, bounded above by $\min\{n_i, n_j\}$, *except where* $\{i, j\} = \{1, 2\}$, *in which case an even number of odd components, bounded above by* $\min\{n_1, n_2\} - 1$, *then G is a type* 1 *graph.* ■

We feel that it should be possible to prove these conjecture by considering the edges that link $S_i \cap S_j$ and $G - S_i - S_j$.

Since the conditions required by type 1 graphs, as given above, could be easily tested by a computer, such tests could cut down the number of calculations required in any attempt to classify a graph. This is especially true in the case of (semi)-regular cubic graphs, where we have even more counter conditions. The problem of classification is still NP-hard in general [1.12] but is only occasionally difficult for individual graphs; the cubic graphs with the above conditions which do not quickly yield a solution for either type

will be very rare. In the appendix to (6. 2), we present outlines of algorithms that would quickly classify relatively small graphs, though defining the limits and degree of difficulty is beyond the scope of this thesis. By considering 1-factors individually, rather than in reference to other edges already coloured, the computation is simplified; though the number of steps is clearly dependent on the size of n.

The starting point for these algorithms is Brooks' theorem and any vertex colouring would depend on using a sensible method for finding new vertex colourings. However, since we have no desire to find a minimal value for c, the problem is easier than that usually studied. It seems reasonable to predict that a maximal use of colours, an 'equalised colouring' similar to that studied in relation to edge colouring [6.13] should give a quicker result.

Although this new concept of conformability works well for most graphs, there are still a few graphs which are both G*-conformable and type 2. G*-conformability begins with a vertex spine colouring, the alternative approach, beginning with an edge colouring is considered in the next section.

6.3 Semi-total Colourings and the β parameter

6.3.1 Introduction

Let G be any graph and let μ be a semi-total colouring of G using $\Delta + 1$ colours. A **beta edge** of G (with respect to μ) is an edge $e_{1,2}$ such that $\mu(v_1) = \mu(v_2)$. We recall (Chapter 2) that β_μ is the number of such edges, and that

$$\beta = \min\{\beta_\mu\colon\ \mu \text{ a semi-total colouring of } G \text{ using } \Delta + 1 \text{ colours}\}.$$

A **non-triangular edge** of a graph is an edge that does not lie on any triangle. A **critical edge** of a type 2 graph, is an edge whose deletion results in a type 1 graph. If G is a type 1 graph and e is an edge of $\overline{G}$, then we say that e is **critical for G** if $G \cup \{e\}$ is type 2. A **near type 1 graph** is a connected type 2 graph with a non-triangular critical edge. In particular, any critical graph with a non-triangular edge is near type 1. **Surcritical** graphs are obtained from critical graphs by adding edges of $\overline{G}$ without increasing the maximum degree. The seventy smallest critical and surcritical type 2 graphs are given β-colourings in Appendix (6.1). As we have already conjectured, we believe that adding any critical edge to a type 1 graph can result in at most two beta edges. However, we will now concentrate on that which we can prove and the main result of this section is that for $\Delta \geq 4$:

$$\beta(G) < \frac{3\Delta}{4} + (\Delta - \lambda)\log_2(\Delta + 1) + 5,$$

where λ is a parameter that we show to be at least 3 (though this value can almost certainly be improved).

6.3.1 Graphs with small maximum degree

We begin with the straightforward case $\Delta \leq 2$.

Theorem 6.3.1 *Let G be a near type 1 graph with $\Delta \leq 2$. Then $\beta \leq \Delta$.*

Proof. *Case* 1: $\Delta = 1$. The only graph to consider is K_2, for which $\beta = 1$.

Case 2: $\Delta = 2$. The cycles C_n ($n \equiv 1$ or $2 \pmod 3$), are the only connected type 2 graphs with $\Delta = 2$ and they are all near type 1. Denote their vertices, in cyclic order, by $v_1, \ldots, v_n$. Consider the semi-total colouring μ of C_n (where $n = 3i + 1$ or $3i + 2$) as follows. List the vertices and edges in cyclic order: $v_1, e_{1,2}, v_2, e_{2,3}, \ldots, v_n, e_{(n,1)}$, and allocate the colours cyclically from v_1 to $e_{(3i-1,3i)}$:

$\mu(v_1) = c_1, \mu(e_{1,2}) = c_2, \mu(v_2) = c_3, \ldots, \mu(v_{3i-1}) = c_3, \mu(e_{3i-1,3i}) = c_1$.

Then:

where $n = 3i + 1$, define $\mu(v_{3i}) = c_3, \mu(e_{(3i,3i+1)}) = c_2, \mu(v_{3i+1}) = c_1, \mu(e_{(3i+1,1)}) = c_3$;

where $n = 3i + 2$, define $\mu(v_{3i}) = c_2, \mu(e_{(3i,3i+1)}) = c_3, \mu(v_{3i+1}) = c_2, \mu(e_{(3i+1,3i+2)}) = c_1$, $\mu(v_{3i+2}) = c_2, \mu(e_{(3i+2,1)}) = c_3$.

In each case, we have $\beta_\mu = 2$. Thus, $\beta \leq 2$ (and in fact $\beta = 2$). ■

Throughout the remainder of the section, we assume that G is near type 1, with a non-triangular critical edge $e_{1,2}$. The vertices v_1, v_2 are said to be the **central vertices.** We then cut $e_{1,2}$, to form spines s_1 and s_2 at v_1 and v_2. The other edges incident with the central vertices are the **central edges**, and the vertices adjacent to the central vertices are the **satellite vertices.** A semi-total colouring of $G - e_{1,2}$ in which $\mu(s_1) = \mu(s_2)$ is said to be a ***G*-colouring** (because the spines may be rejoined to form a semi-total colouring of G). This context should be assumed unless otherwise stated. We now define the following notation.

In a (total or semi-total) colouring μ of $G - e_{1,2}$, the **number of neighbours** of a vertex v_z that have colour c_i is denoted by $N(\mu, v_z, c_i)$ **or** $N(v_z, c_i)$ for brevity, when μ is unambiguous. Frequently, an (i, j) Kempe chain will be assumed to end at v_z but it will not be known whether $\mu(v_z)$ is c_i or c_j; we then write

$$N(v_z, c_i : c_j) = \max\{N(v_z, c_i), N(v_z, c_j)\}.$$

Lemma 6.3.2 *Let G be as assumed above, with maximum degree Δ.*

(i) *Suppose μ and θ are G-colourings such that:*

(a) $\mu(v_1) = c_i$, $\mu(v_2) = c_j$, $\theta(v_1) = c_j$, $\theta(v_2) = c_i$, *the satellite vertices having the same colours in μ and θ;*

(b) *where φ is the number of beta edges not incident with either v_1 nor v_2,* $\beta_\mu(G) = \varphi + N(v_1, c_i) + N(v_2, c_j)$ *and* $\beta_\theta(G) = \varphi + N(v_1, c_j) + N(v_2, c_i)$;

(c) *in $G - e_{1,2}$, v_1 and v_2 each have at least λ adjacent vertices whose colour is neither c_i nor c_j.*

Then $\beta(G) \leq \varphi + (\Delta - \lambda - 1)$.

(ii) *Suppose μ and θ are G-colourings with the same vertex colours except only that $\mu(v_w) = c_i$, $\theta(v_w) = c_j$, $\mu(v_z) = c_z$, $\theta(v_z) = c_k$, $\mu(v_y) = c_x$, $\theta(v_y) = c_y$, where $N(\mu, v_z, c_z) = 0$, $N(\theta, v_y, c_y) = 0$ and v_w has at least ϕ adjacent vertices whose colour is neither c_i nor c_j. Then, where φ is the number of beta edges not incident with v_w, v_y nor v_z,*

$$\beta(G) \leq \varphi + (\Delta - 1) + \tfrac{1}{2}(\Delta - \phi).$$

Proof. (*i*) Since the central vertices have degree $\Delta - 1$ in $G - e_{1,2}$:

$$N(v_1, c_i) + N(v_1, c_j) \leq (\Delta - \lambda - 1), \quad N(v_2, c_i) + N(v_2, c_j) \leq (\Delta - \lambda - 1).$$

Therefore $\beta_\mu + \beta_\theta = 2\varphi + (N(v_1, c_i) + N(v_2, c_j) + N(v_1, c_j) + N(v_2, c_i))$

$$\leq 2\varphi + 2(\Delta - \lambda - 1).$$

In each of μ and θ, the number of beta edges incident with neither v_1 nor v_2 is the same, φ. We need only consider the beta edges at vertices v_1 and v_2. If β_μ has $N(v_1, c_i) + N(v_2, c_j) > (\Delta - \lambda - 1)$, then β_θ has $N(v_1, c_j) + N(v_2, c_i)) < (\Delta - \lambda - 1)$ and *vice versa*. Therefore at least one of β_μ and β_θ is bounded above by $\varphi + (\Delta - \lambda - 1)$, as required. Hence,

$$\beta \leq \varphi + (\Delta - \lambda - 1).$$

Part (*ii*) follows similarly. ■

Main result (15)

Theorem 6.3.3 *Let* $\Delta(G) \geq 3$ *and suppose there is a total* $(\Delta + 1)$*-colouring* μ *of* $G - e_{1,2}$ *such that* $\mu(v_1)$, $\mu(s_1)$, $\mu(v_2)$ *and* $\mu(s_2)$ *are not all distinct. Then* $\beta \leq \Delta$.

Proof. The case $\mu(v_1) \neq \mu(v_2)$, $\mu(s_1) = \mu(s_2)$ does not arise since this would imply that G is type 1. Up to isochromatism there are four other cases, as follows. See Figure 6.3.1.

Case 1: $\mu(v_1) = \mu(v_2) = c_1$, $\mu(s_1) = \mu(s_2) = c_2$.

Then μ may be interpreted as a semi-total colouring of G with $\beta_\mu = 1$, simply by re-joining the spines. Thus, $\beta = 1$.

Case 2: $\mu(v_1) = c_1$, $\mu(s_1) = c_2$, $\mu(v_2) = c_2$, $\mu(s_2) = c_1$.

Let $\mu_1 = \mu + {}_1(1, 2)_1$; thus, the colours at v_1 and s_1 are simply exchanged. The spines may be re-joined, and we have a semi-total colouring of G whose only beta edges can be incident with v_1. Thus, $\beta_1 \leq \Delta$.

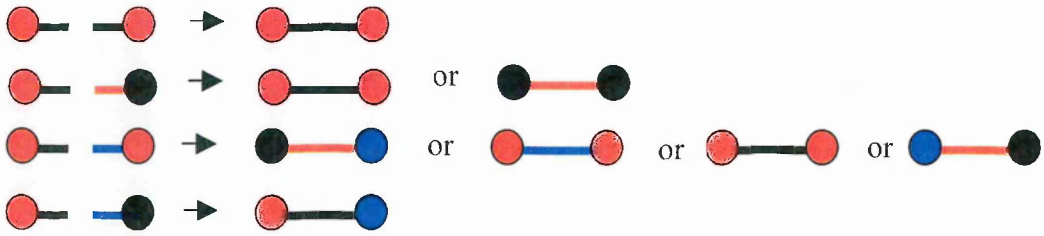


Figure 6.3.1

Case 3: $\mu(v_1) = \mu(v_2) = c_1$, $\mu(s_1) = c_2$, $\mu(s_2) = c_3$.

Let $\mu_2 = \mu + {}_1(1, 2)_1 + {}_2(1, 3)_2$; thus, spine and vertex colours are interchanged at both the vertices v_1, v_2. Again the spines may now be re-joined to form a semi-total colouring of G. We have $N(\mu_2, c_2, v_1) \leq \Delta - 1$, $N(\mu_2, c_3, v_2) \leq \Delta - 1$. There are no other beta edges; thus $\beta \leq 2(\Delta - 1)$. This bound may be improved as follows.

Consider the ${}_1]2, 3[_y$ Kempe chain; if $y \neq 2$, then the colouring $\mu_3 = \mu + {}_1(2, 3)_y$ has at most $(\Delta - 1)$ beta edges at v_y, and the beta edge $e_{1,2}$. Thus, $\beta \leq \Delta$. Alternatively, if the above chain terminates at v_2, then let $\mu_4 = \mu_2 + {}_1(2, 3)_2$. Then the pair of colourings μ_2, μ_4 obey the conditions of Lemma 6.3.2, with $\lambda = 1$ (because of the Kempe chain) and $x = 0$, to give the result.

$$\beta \leq \Delta - 2.$$

Case 4: $\mu(v_1) = c_1$, $\mu(s_1) = c_2$, $\mu(v_2) = c_2$, $\mu(s_2) = c_3$.

Let $\mu_5 = \mu + {}_2(2, 3)_2$; thus, the colours at v_2 and s_2 are simply exchanged. The only possible beta edges are incident with v_2, and $e_{1,2}$ is not a beta edge. Thus, $\beta_5 \leq \Delta - 1$.

The conclusion is that $\beta \leq \Delta$. ■

The result when $\Delta = 3$ is a special case.

Main result (16)

Theorem 6.3.4 *Let G be a near type 1 graph with $\Delta = 3$; then $\beta \leq 2$ unless all type 1 total colourings μ of $G - e_{1,2}$ have $\mu(v_1) = \mu(v_2)$, $\mu(s_1) \neq \mu(s_2)$ when $\beta \leq 3$.*

Proof. Note that the condition $\mu(v_1) = \mu(v_2)$, $\mu(s_1) \neq \mu(s_2)$ corresponds to Case 3 in Theorem 6.3.3 and $\beta \leq 3$. Thus we must verify that we have $\beta \leq 2$ in Cases 1, 2 and 4 above, and also (Case 5) when $\mu(v_1)$, $\mu(s_1)$, $\mu(v_2)$ and $\mu(s_2)$ are distinct.

Cases 1 and 4 follow directly from the proof of Theorem 6.3.3, see Figure 6.3.1.

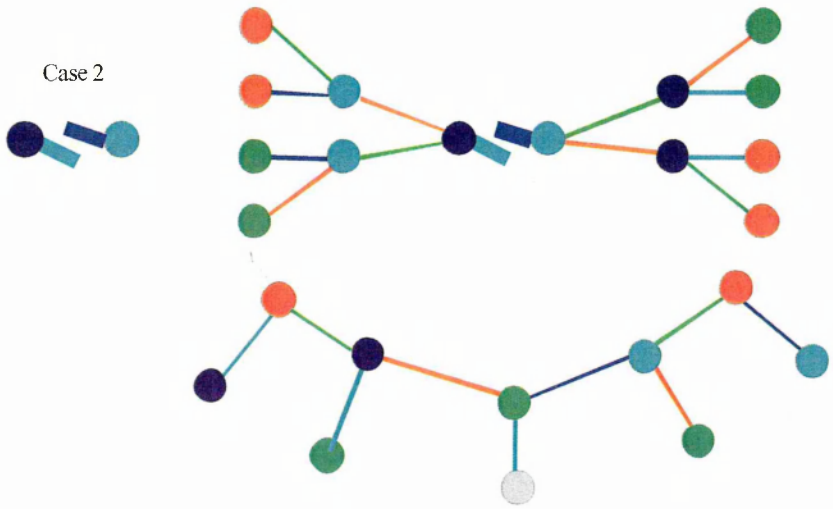


Figure 6.3.2

Case 2: $\mu(v_1) = c_1$, $\mu(s_1) = c_2$, $\mu(v_2) = c_2$, $\mu(s_2) = c_1$.

By Theorem 6.3.3, $\beta \leq 3$; let us assume that $\beta = 3$. See Figure 6.3.2.

There is a minimum-length (and hence chordless) path P in $G - e_{1,2}$ from v_1 to v_2; let us re-number the vertices of G such that the vertices on this path are (in order) $v_1, \ldots, v_q$ (so that v_1 retains its label and the vertex that was v_2 becomes v_q). For $i = 1, \ldots, q$ let the vertex adjacent to v_i and not on P (where it exists) be labelled w_i. If $q = 3$ then the edges $e_{1,2}$ and $e_{2,3}$, and the vertex v_2, must have colours distinct from each other and from c_1, c_2, giving a contradiction; thus $q > 3$.

If $\mu(v_2) \neq c_2$ or $\mu(w_1) \neq c_2$, then $\mu_1 = \mu + {}_1(1, 2)_1$ has at most two beta edges at v_1, giving a contradiction. Thus, $\mu(v_2) = \mu(w_1) = c_2$. Now let $\mu_2 = \mu_1 + {}_1(2, x_1)_2$ where $\mu(e_{1,2}) = c_{x_1}$. The assumption $\beta = 3$ implies $\mu(v_3) = \mu(w_2) = c_{x_1}$.

Now for $i = 2, \ldots, q-1$ let $\mu(e_{i,i+1}) = c_{x_i}$.

Suppose that there is some first y (not equal to $q-2$) such that $\mu(v_{y+2}) \neq c_{x_y}$ and $\mu(w_{y+1}) \neq c_{x_y}$. Then we can do a sequence of Kempe interchanges, each involving just one edge and its incident vertices, to obtain a G-colouring $\mu_y = \mu_1 + {}_1(2, x_1)_2 + \ldots + {}_y(x_{y-1}, x_y)_{y+1}$ with at most two beta edges.

Suppose otherwise that $\mu(v_{y+2}) = \mu(w_{y+1}) = c_{x_y}$ for $y = 1, \ldots, q-2$. Then, since $\mu(v_q) = c_2$, we must have $\mu(e_{q-2,\, q-1}) = c_2 = \mu(w_{q-1})$. Starting an analogous argument from v_q rather than v_1, we must have $\mu(v_{q-1}) = c_1$, $\mu(e_{q-1,\, q}) = c_3$ or c_4; assume c_3. But now $\mu + {}_q(1, 2)_q + {}_q(1, 3)_{q-1}$ has at most two beta edges, contradicting the assumption that $\beta = 3$.

Case 5: Consider now the possibility that $\mu(v_1)$, $\mu(s_1)$, $\mu(v_2)$ and $\mu(s_2)$ are distinct, see Figure 6.3.3. We may assume $\mu(v_1) = c_1$, $\mu(s_1) = c_2$, $\mu(s_2) = c_3$, $\mu(v_2) = c_4$. There is a ${}_1]2, 3]_x$ Kempe chain to some vertex v_x where v_x is not v_1 but may be v_2.

Case 5(*a*): *The chain is* ${}_1]2, 3]_x$, $x > 2$.

Let $\mu_0 = \mu + {}_1(2, 3)_x$. This is a G-colouring, the only possible beta edges being at those at v_x that are not on the chain ${}_1[2, 3]_x$. Thus, $\beta_0 \leq 2$.

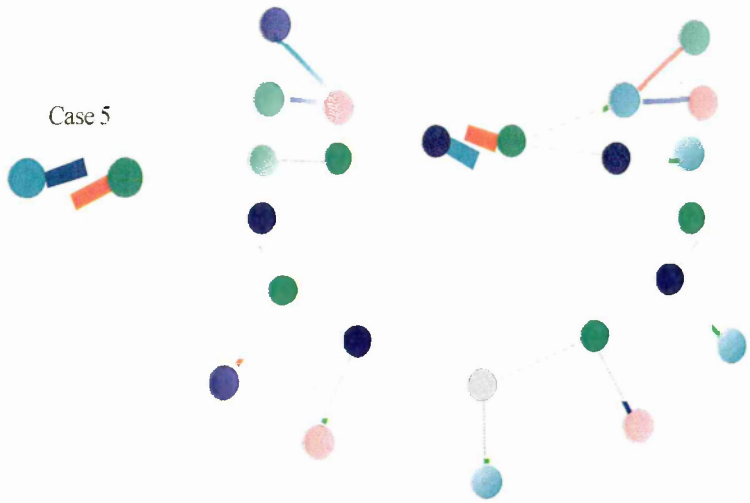


Figure 6.3.3

Case 5 (*b*): *The chain is* $_1]2, 3[_2$.

Since μ is a total colouring of $G - e_{1,2}$, and there are only four colours available, the vertices on the chain must alternate in colour between c_1 and c_4. Since $\mu(v_1) = c_1$ and $\mu(v_2) = c_4$, there are an even number of vertices on the chain, contradicting the fact that $\mu(s_1) = c_2$ and $\mu(s_2) = c_3$. Thus, this case cannot occur.

Thus, the only arrangement in which it is possible that $\beta > 2$ is the Case 3 arrangement with $\mu(v_1) = \mu(v_2)$, $\mu(s_1) \neq \mu(s_2)$. ■

Main conjecture (17)

Conjecture 6.3.5 *Let G be any critical cubic graph; then* $\beta(G) \leq 2$.

Justification No known critical cubic graph requires $\mu(v_1) = \mu(v_2)$, $\mu(s_1) \neq \mu(s_2)$ on all colourings of $G - e_{1,2}$ for any edge $e_{1,2}$, yet this would be need to be the case for every edge in any counter example. Therefore we conjecture that it is always possible to find a way to alter any colouring with this colour arrangement into one of the other cases, in which case $\beta \leq 2$. ■

6.3.3 $\Delta > 3$ and four distinct colours at v_1, v_2, s_1, s_2

The case $\Delta > 3$ where there are fewer than four colours at v_1, v_2, s_1, s_2 has been covered by Theorem 6.3.3. We now assume that these colours are distinct.

For the remainder of this section, we assume that μ is a total $(\Delta + 1)$-colouring of $G - e_{1,2}$, the colour labels being such that

$$\mu(v_1) = c_1,\ \mu(s_1) = c_2,\ \mu(s_2) = c_3,\ \mu(v_2) = c_4.$$

We refer to this as the **stage 0 colour arrangement** at v_1 and v_2, see Figure 6.3.4. The **stage 0 chains** (when they exist) are Kempe chains ${}_1]2, 3[_2$, ${}_1]2, 4]_2$, ${}_1[1, 3[_2$, ${}_1[1, 4]_2$.

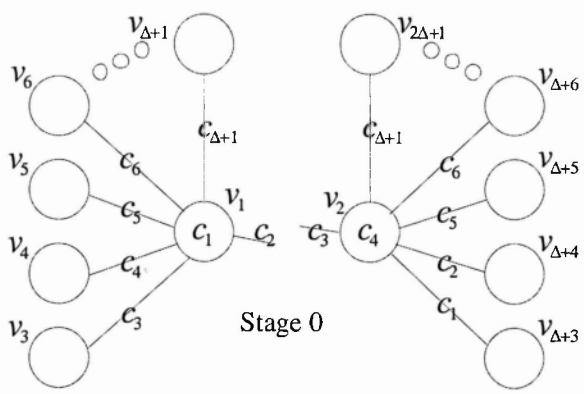


Figure 6.3.4

We let N_1 be the set of neighbours of v_1 and N_2 the set of neighbours of v_2, in $G - e_{1,2}$. (Thus, $N_1 \cup N_2$ is the set of satellite vertices). We denote the vertices in N_1 by $v_3, \ldots, v_{\Delta+1}$ and those in N_2 by $v_{\Delta+3}, \ldots, v_{2\Delta+1}$, the labels being chosen so that:

(i) $\mu(e_{(1,\,i)}) = c_i \quad (i = 3, \ldots, \Delta + 1)$;

(ii) $\mu(e_{(2,\,\Delta+3)}) = c_1$, $\mu(e_{(2,\,\Delta+4)}) = c_2$;

(iii) $\mu(e_{(2,\,\Delta+i)}) = c_i \quad (i = 5, \ldots, \Delta + 1)$.

Note that, as $e_{1,2}$ is a non-triangular edge, all the above vertices are distinct; note also that the label $v_{\Delta+2}$ is not used for any satellite vertex.

We say that μ obeys the **stage 0 inequalities** if

$$N(\mu, v_3, c_3) \leq \Delta - 3, N(\mu, v_4, c_4) \leq \Delta - 3,$$

$$N(\mu, v_{\Delta+3}, c_1) \leq \Delta - 3, N(\mu, v_{\Delta+4}, c_2) \leq \Delta - 3.$$

We now proceed to define stage t colour arrangements and inequalities for $t > 0$.

We say that μ has the **stage 1 colour arrangement** (Figure 6.3.5) if (possibly after re-numbering the colours):

(i) μ has the stage 0 colour arrangement;

(ii) $\mu(v_3) = c_5$, $\mu(v_{\Delta+3}) = c_6$;

(iii) $\mu(v_4) = c_7$, $\mu(v_{\Delta+4}) = c_8$.

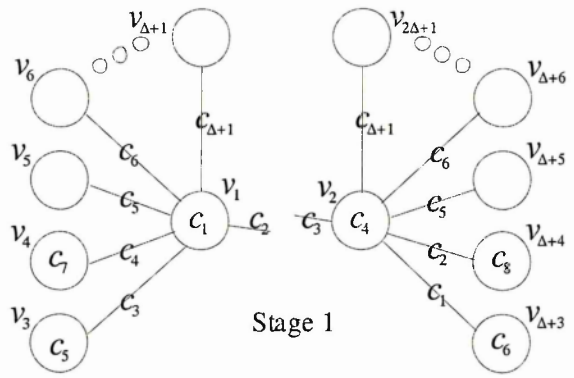


Figure 6.3.5

For $t > 1$, we say that μ has the **stage t colour arrangement** if (possibly after re-numbering the colours):

(i) μ has the stage 1 colour arrangement;

(ii) $\mu(v_i) = c_{2i-1}$ $(i = 5, \ldots, 2^{t+1})$;

(iv) $\mu(v_{\Delta+i}) = c_{2i}$ $(i = 5, \ldots, 2^{t+1})$.

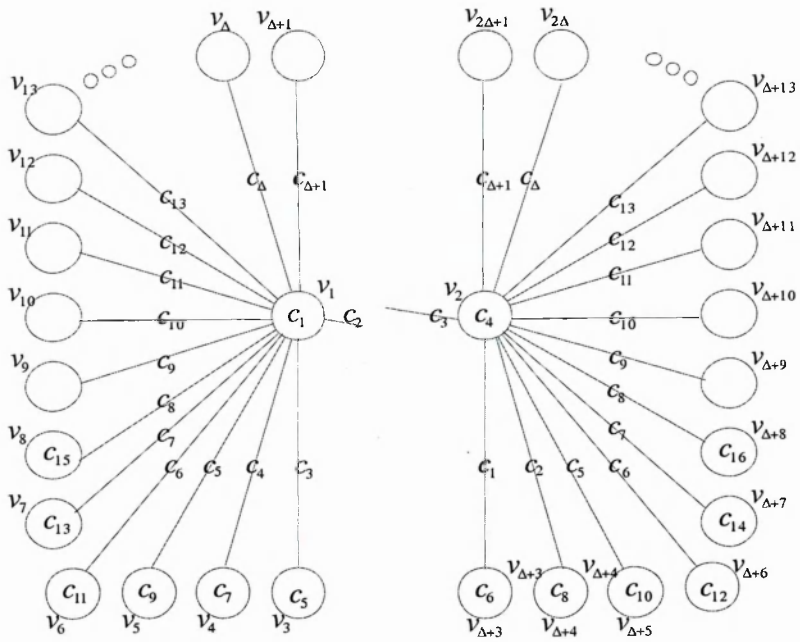


Figure 6.3.6

Note that, in this case, there are at least 2^{t+2} colours, and therefore edges $e_{i,1}$ and $e_{\Delta+i,2}$ and hence vertices v_i and $v_{\Delta+i}$ $(i = 3, \ldots, 2^{t+2})$. We say that μ obeys the **stage *t* inequalities if** μ obeys the stage 0 inequalities and

$$N(\mu, v_i, c_i) \leq \Delta - 3,\ N(\mu, v_{\Delta+i}, c_i) \leq \Delta - 3 \quad (i = 5, \ldots, 2^{t+2}).$$

Figure 6.3.6 shows the stage 2 arrangement.

Remark 6.3.1 For $(0 \leq j < t)$. If μ has the stage t colour arrangement, then it also has the stage j colour arrangement; the same is true of the stage t and stage j inequalities.

Remark 6.3.2 The stage t colour arrangement uses 2^{t+2} colours. Thus a necessary condition for G to have a colouring with the stage t colour arrangement is $\Delta \geq 2^{t+2} - 1$; in particular, the central vertices must each have degree at least $2^{t+2} - 1$.

Our proof of the general result hinges on an inductive argument that, if $\beta(G) > \Delta - 1 + (t + 1)(\Delta - \lambda) + (\Delta - 2^{t+1} + 1)$, then any total colouring of $G - e_{1,2}$ has

stage $(t + 1)$ colour arrangement and inequalities. Here, the parameter λ is a number such that we know that any vertex v_i or $v_{\Delta+i}$ has at least λ adjacent vertices with colour not equal to c_i (except that for $v_{\Delta+3}$, $v_{\Delta+4}$ the colour in question is c_1, c_2 respectively).

Remark 6.3.3 In this section we establish that, provided $\Delta \geq 4$, we may take $\lambda = 3$. It should be noted, however, that almost certainly this result can be improved, in that it should be possible to show that λ may be taken to increase with Δ.

Main result (18)

Theorem 6.3.6 *Let G have $\Delta \geq 4$, $\beta > 3\Delta - 7$, and let μ be a total $(\Delta + 1)$-colouring of $G - e_{1,2}$; then:*

(i) *the stage 0 chains exist and the stage 0 inequalities hold;*

(ii) *any total colouring μ of $G - e$ has stage 1 colour arrangement, and the following chains (the* **stage 1 chains***) exist:*

$${}_1[1, 5]_3,\ {}_1]2, 5]_3,\ {}_2[4, 5]_3;\quad {}_1]2, 6]_{\Delta+3},\ {}_2]3, 6]_{\Delta+3},\ {}_2[4, 6]_{\Delta+3};$$

$${}_1[1, 7]_4,\ {}_1]2, 7]_4,\ {}_2]3, 7]_4;\quad {}_1[1, 8]_{\Delta+4},\ {}_2]3, 8]_{\Delta+4},\ {}_2[4, 8]_{\Delta+4}.$$

Proof. By Theorem 6.3.3, the colours at v_1, s_1, v_2, s_2 are distinct and we may number them so that there is the stage 0 colour arrangement at v_1, v_2.

(*i*) Consider the Kempe chain ${}_1]2, 3]_x$. If $x \neq 2$, then the G-colouring $\mu + {}_1(2, 3)_x$ has at most $(\Delta - 1)$ beta edges, all at v_x, contradicting our assumption that $\beta > 3\Delta - 7$. Thus the chain is ${}_1]2, 3[_2$.

Next, consider ${}_1[1, 3]_x$. If $x \neq 2$, then $\mu + {}_1(1, 3)_x + {}_1(2, 3)_1$ has at most $(2\Delta - 3)$ beta edges: $(\Delta - 1)$ at v_x and at most $(\Delta - 2)$ at v_1 ($\mu(v_3) \neq c_2$ owing to the chain ${}_1]2, 3[_2$). This again contradicts our assumption (since $\Delta \geq 4$), so there is a chain ${}_1[1, 3[_2$.

Similarly, there is a chain ${}_1]2, 4]_2$.

Now consider the chain ${}_1[1, 4]_x$. If $x \neq 2$, then there is a G-colouring

$$\mu_1 = \mu + {}_1(1, 4)_x + {}_1(2, 4)_1 + {}_2(3, 4)_2 \text{ with}$$

$$\beta_1 = N(v_x, c_1 : c_4) + N(v_1, c_2) + N(v_2, c_3)$$

(since all the neighbours of v_x, v_1 and v_2 are the same in both in μ and μ_1).

Similarly there is a G-colouring

$$\mu_2 = \mu + {}_1(1, 4)_x + {}_1(2, 3)_2 + {}_1(3, 4)_1 + {}_2(2, 4)_2 \text{ with}$$

$$\beta_2 = N(v_x, c_1 : c_4) + N(v_1, c_3) + N(v_2, c_2).$$

Now $N(v_x, c_1 : c_4) \leq \Delta - 1$, and v_1 and v_2 each have at least one adjacent vertex coloured neither c_2 nor c_3. Thus by Lemma 6.3.2, $\beta \leq 2\Delta - 3$. Our assumption is again contradicted, and we conclude that there is a chain ${}_1[1, 4]_2$.

There are thus two stage 0 chains through each of the vertices v_3, v_4, $v_{\Delta+3}$, $v_{\Delta+4}$, which end at the central vertices. Therefore the stage 0 inequalities hold.

(*ii*) Let $\mu(v_3) = c_\alpha$; then $\alpha \neq 1, 3$, and we have seen that $\alpha \neq 2$. If $\alpha = 4$, then the G-colouring $\mu + {}_1(2, 4)_2 + {}_1(4, 3)_3$ has beta edges only at v_2 and v_3. Thus a stage 0 inequality implies $\beta \leq 2\Delta - 4$, contrary to assumption. Therefore $\alpha > 4$. Now consider each of the chains ${}_1[1, \alpha]_w$, ${}_1]2, \alpha]_x$, ${}_2[4, \alpha]_y$.

If $w \neq 3$, then the G-colouring $\mu_1 = \mu + {}_1(1, \alpha)_w + {}_1(\alpha, 3)_3 + {}_1(2, 3)_1$ has at most $(3\Delta - 7)$ beta edges: $(\Delta - 1)$ beta edges at v_w, $(\Delta - 3)$ at v_3 (as above) and $(\Delta - 3)$ at v_1 (because $\mu_1(v_3) \neq c_2$ and the ${}_1]2, 4]_2$ chain implies $\mu(v_4) \neq c_2$). This contradicts our assumption, and so the chain ${}_1[1, \alpha]_3$ exists.

If $x \neq 3$, then the G-colouring $\mu + {}_1(2, \alpha)_x + {}_1(\alpha, 3)_3$ has at most $(2\Delta - 4)$ beta edges: $(\Delta - 1)$ at v_x, $(\Delta - 3)$ at v_3. Again this contradicts our assumption, and ${}_1]2, \alpha]_3$ exists.

Remark 6.3.4. Before going on to consider the chain ${}_2[4, \alpha]_y$, we note that the existence of the chains established above implies that $N(\mu, v_1, c_2) \leq (\Delta - 4)$ (since v_3, v_4 and v_α cannot have colour c_2), and similarly that $N(\mu, v_1, c_3) \leq (\Delta - 4)$. By symmetry we also conclude that, where c_ϖ is the colour on $v_{\Delta+4}$, then $v_{\Delta+3}$, $v_{\Delta+4}$ and $v_{\Delta+\varpi}$ cannot have c_2 nor c_3 hence $N(\mu, v_2, c_2) \leq (\Delta - 4)$ and $N(\mu, v_2, c_3) \leq (\Delta - 4)$.

Now let us consider ${}_2[4, \alpha]_y$. If $y \neq 3$, then there also exists ${}_3[\alpha, 4]_z$ where $z \neq 2$. Consider the colouring

$$\mu_3 = \mu + {}_3(4, \alpha)_z + {}_1(2, 4)_q.$$

Since ${}_3(4, \alpha)_z$ may disrupt μ (${}_1]2, 4]_2$), we must consider the cases $q = 2$, $q \neq 2$.

Case 1: $q = 2$

Then $\mu_4 = \mu + {}_3(4, \alpha)_z + {}_1(2, 4)_2 + {}_1(4, 3)_3$ is a G-colouring with

$$\begin{aligned}\beta_4 &= N(v_z, c_4 : c_\alpha) + N(v_2, c_2) + N(v_3, c_3)\\ &\leq (\Delta - 1) + (\Delta - 4) + (\Delta - 3) = 3\Delta - 8,\end{aligned}$$

which contradicts our assumption.

Case 2: $q \neq 2$

Let $\mu_5 = \mu_3 + {}_1(4, 3)_3. = \mu + {}_3(\alpha, 4)_z + {}_1(2, 4)_q + {}_1(4, 3)_3.$

This a G-colouring with

$$\beta_5 = N(v_z, c_4 : c_\alpha) + N(v_q, c_2 : c_4) + N(v_3, c_3) \leq 2(\Delta - 1) + N(v_3, c_3).$$

We must also consider the G-colouring

$$\mu_6 = \mu + {}_1(2, 3)_2 + {}_3(4, \alpha)_z + {}_1(3, 4)_r.$$

If $r = 2$, then (arguing as for μ_4) $\mu_7 = \mu + {}_1(2, 3)_2 + {}_3(4, \alpha)_z + {}_1(3, 4)_2 + {}_1(4, 2)_3$ is a G-colouring with $\beta_7 \leq 3\Delta - 8$, contradicting our assumption.

If $r \neq 2$, then let $\mu_8 = \mu_6 + {}_1(4, 2)_3$. We have

$$\beta_8 = N(v_z, c_4 : c_\alpha) + N(v_r, c_3 : c_4) + N(v_3, c_2) \leq 2(\Delta - 1) + N(v_3, c_2).$$

Comparing β_5 and β_8 we can chose either μ_5 or μ_8 to get the lowest value for β as in part (*ii*) of Lemma 6.3.2 giving

$$\beta < 2(\Delta - 1) + \tfrac{1}{2}(\Delta - 3)$$

$$\leq 3\Delta - 7 \text{ provided that } \Delta \geq 5.$$

Thus, having established the case for $\Delta = 3$ in Theorem 6.3.4, we have contradicted our assumption except possibly when $\Delta = 4$. We have already established that $\mu(v_3)$ cannot be c_1, c_2, c_3 or c_4, and a similar argument applies to v_4, so that if $\Delta = 4$, both of these must have colour c_5. But then the ${}_1]1, 5]$ chain must end at both v_3 and v_4, which is a contradiction. This completes our analysis of chains at v_3.

Next, let $\mu(v_4) = c_\gamma$. We must consider each of the chains ${}_1[1, \gamma]_w$, ${}_1]2, \gamma]_x$, ${}_2]3, \gamma]_y$.

For ${}_1]2, \gamma]_x$ and ${}_2]3, \gamma]_y$, the arguments are similar to the above with the G-colourings with the highest beta values being:

$\theta_1 = \mu + {}_1(\gamma, 2)_x + {}_3(4, \gamma)_1 + {}_2(3, 4)_2$ where $x \neq 4$, and

$\theta_2 = \mu + {}_2(\gamma, 3)_y + {}_2(2, \gamma)_1 + {}_4(\gamma, 4)_1 + {}_2(2, 4)_2$, where $y \neq 4$, these both have

$\beta \leq (\Delta - 1) + (\Delta - 3) + (\Delta - 4) = 3\Delta - 8.$

The argument for ${}_1[1, \gamma]_w$ is as follows.

If $w \neq 4$, we must consider the G-colouring

$$\omega = \mu + {}_1(1, \gamma)_w + {}_1(\gamma, 4)_4 + {}_1(2, 4)_1 + {}_2(3, 4)_2;$$

this has at most $(\Delta - 1)$ beta edges at v_w and at most $(\Delta - 3)$ at each of v_4, v_1 and v_2.

However, if we let $\mu_0 = \mu + {}_1(2, 3)_2$, then this total colouring has the same properties as μ and we may draw the same conclusions regarding Kempe chains, giving:

$$\omega_0 = \mu_0 + {}_1(1, \gamma)_w + {}_1(\gamma, 4)_4 + {}_1(3, 4)_1 + {}_2(2, 4)_2.$$

The vertex colours of ω and ω_0 differ only at v_1 and v_2, and by Lemma 6.3.2 (using Remark 6.3.4),

$$\beta \leq (\Delta - 1) + (\Delta - 3) + (\Delta - 3) = 3\Delta - 7.$$

This contradicts our assumption, and we conclude the existence of ${}_1[1, \gamma]_4$.

The cases of the chains at $v_{\Delta+3}$ and $v_{\Delta+4}$ follow those for the chains at v_4 and v_3 respectively. Thus, letting $\mu(v_{\Delta+3}) = c_\delta$, $\mu(v_{\Delta+4}) = c_\varepsilon$, then none of γ, δ, ε can be 1, 2, 3 or 4, and there are chains

${}_1[1, \gamma]_4$, ${}_1]2, \gamma]_4$, ${}_2]3, \gamma]_4$,

${}_1]2, \delta]_{\Delta+3}$, ${}_2]3, \delta]_{\Delta+3}$, ${}_2[4, \delta]_{\Delta+3}$,

${}_1[1, \varepsilon]_{\Delta+4}$, ${}_2]3, \varepsilon]_{\Delta+4}$, ${}_2[4, \varepsilon]_{\Delta+4}$.

Thus α, γ, δ, ε must be distinct, since (for example) if $\gamma = \delta$, then the $(2, \gamma)$ chain starting at v_1 cannot end both at v_4 and at $v_{\Delta+3}$.

It follows that we may assume $\alpha = 5$, $\gamma = 7$, $\delta = 6$, $\varepsilon = 8$, so μ has a stage 1 colouring arrangement as required. Moreover, with these colour choices, the argument has also established the existence of the stage 1 chains. ■

In order to make the inductive step, we need additional terminology and notation.

Let a total colouring μ of $G - e_{1,2}$ have the stage t colour arrangement. Let ε_1 be a central edge, where $\mu(\varepsilon_1) = c_a$, $5 \leq a \leq 2^{t+2}$ (that is, ε_1 is not $e_{(1,3)}$, $e_{(1,4)}$, $e_{(2,\Delta+3)}$, or $e_{(2,\Delta+4)}$. There is a unique satellite vertex v_{w_1} in the stage t arrangement with

$\mu(v_{w_1}) = c_a$; let ε_2 denote the edge that joins v_{w_1} to v_1 or v_2. Let $\mu(\varepsilon_2) = c_b$. It may be that $b \leq 4$, in which case we stop. Otherwise, $b = \left\lceil \frac{a}{2} \right\rceil$ and there is a unique vertex v_{w_2} adjacent to v_1 or v_2 with $\mu(v_{w_2}) = c_b$. Let ε_3 denote the edge that joins v_{w_2} to v_1 or v_2. Continuing in this way, we eventually reach an edge ε_q with $\mu(\varepsilon_q) = c_1, c_2, c_3$ or c_4 (that is, one of the edges $e_{(1,3)}$, $e_{(1,4)}$, $e_{(2,\Delta+3)}$, or $e_{(2,\Delta+4)}$). We say that the sequence $X(\varepsilon_1) = \varepsilon_1, \ldots, \varepsilon_q$ is the **cascade sequence** of ε_1. The edge ε_1 is in the **cascade set** S_i ($i = 1, \ldots, 4$) according as $\mu(\varepsilon_q) = c_i$; its **branch number**, $\mathrm{br}(\varepsilon_1) \leq t + 1$, is the number of edges in its cascade sequence. We define the edges $e_{(2,\Delta+3)}$, $e_{(2,\Delta+4)}$, $e_{(1,3)}$, $e_{(1,4)}$ to belong to $S_1, \ldots, S_4$ respectively and to have branch number 1. If c_α is the colour of an edge belonging to S_i, we say that c_α is **associated** with S_i and we write $A(c_\alpha) = i$. The cascade sets partition the central edges with colours c_α ($1 \leq \alpha \leq 2^{t+2}$), and (for $z > 4$) the branch number of an edge ε with $\mu(\varepsilon) = c_z$ where $2^q < z \leq 2^{q+1}$ is $\mathrm{br}(\varepsilon) = q = \lceil \log_2 z \rceil - 1$.

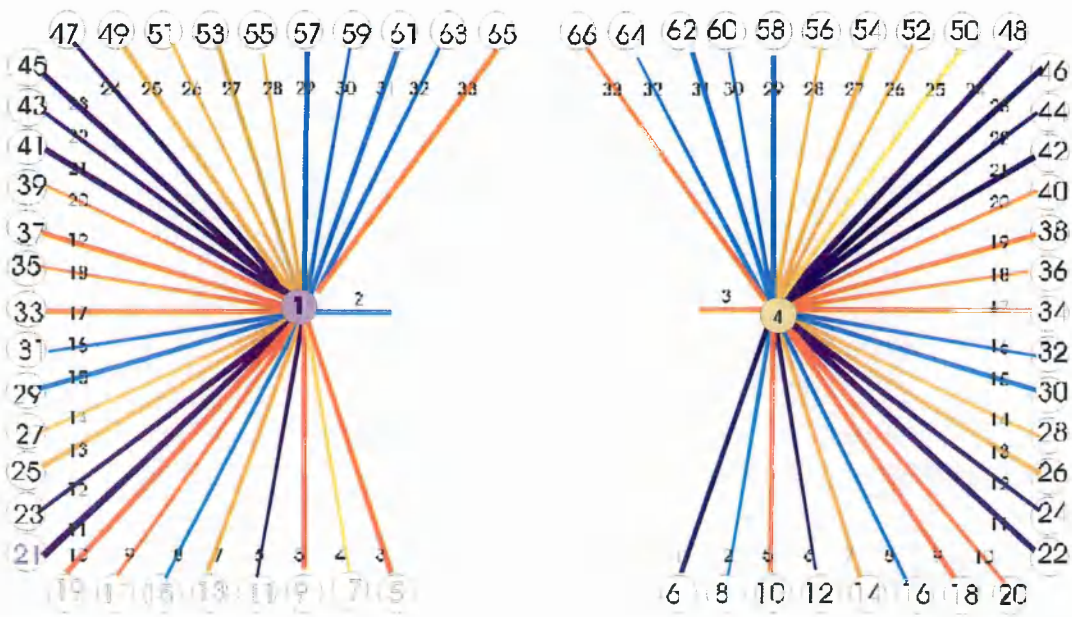


Figure 6.3.7

The stage 4 colour arrangement is shown in Figure 6.3.7. Note that the numbers refer to a total colouring. In the figure the cascade set S_1 is violet, S_2 is blue, S_3 is orange and S_4 is yellow.

We may now define the **stage t chains** as follows (though we do *not* at this stage imply their existence). For each z, $5 \leq z \leq 2^{t+2}$, there are three possible stage t chains ending at the vertex with colour c_z (in the stage t arrangement); they are the $[i, z]$ chains ($i = 1, \ldots, 4$) starting at v_1 or v_2, *except* that where c_z is associated with S_j the (j, z) chain is *not* a stage t chain. (Thus the chains at each stage also belong to the subsequent stages).

Let us assume that μ has the stage t colour arrangement and chains.

Briefly, our inductive step consists in taking a central edge ε, with $\mu(\varepsilon) = c_z$ where $2^{t+1} < z \leq 2^{t+2}$ and letting c_α denote the (currently unknown) colour of the vertex v_z or $v_{\Delta+z}$ incident with ε. Since both vertex v_1 and v_2 have the same properties by symmetry, we can assume that this vertex is v_z (i.e. that e is incident with v_1). Let us suppose that we can make an initial Kempe interchange that *either*:

(*i*) brings the colour c_α to v_1 or s_1; *or*:

(*ii*) brings the colour c_1 or c_2 to v_z,

and that does not involve any edge in $X(\varepsilon)$ or the satellite vertex incident with such an edge. Then the edge colour may next be exchanged for the above colour, to start a **Vizing fan exchange** that sequentially exchanges edge colours with colours at s_1 or v_1. (The name is in recognition of the *Vizing fan argument*, [6.14]). Then (as will be shown) we eventually produce a G-colouring ω with $\beta(\omega)$ below the required bound.

To be precise, an **initial exchange** for a central edge e_z (or $e_{\Delta+z}$ as the case may be) in S_i (where the colour of the incident satellite vertex is c_α) is a Kempe interchange ${}_j(k, \alpha)_q$ where:

(i) $k \in \{1,2,3,4\} \setminus \{i\}$;

(ii) either $j = z$ (or $\Delta + z$ as the case may be) or $j = \lceil \frac{k}{2} \rceil$;

(iii) the interchange does not use a central edge in S_i or end at a satellite vertex incident with such an edge.

We shall derive a bound for $\beta(G)$ on the assumption that an initial exchange exists.

First, we must consider in more detail the structure of a cascade sequence $X(\varepsilon_1)$. It is possible that all the edges in $X(\varepsilon_1)$ are incident with the same central vertex, but in general this will not be so, and then we partition the sequence into a sequence $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_f$, where each $\mathbf{F}_i$ is a sequence of edges incident with the same central vertex, the central vertices alternating between the $\mathbf{F}_i$. Each such subsequence is said to be a **Vizing fan**. The **branch number** of $\mathbf{F}_i$ is the number of edges in $\mathbf{F}_i$ and is denoted by br($\mathbf{F}_i$).

Note that with these definitions, in a stage t colouring the cascade sequences for the 'new' edges e_z, $e_{\Delta+z}$ ($2^{t+1} < z \leq 2^{t+2}$) each have branch number $t+1$.

Main result (19)

Lemma 6.3.7 *Let μ be a total $(\Delta + 1)$-colouring of $G - e_{1,2}$ with stage t colour arrangement, chains and inequalities; let $2^{t+1} < z \leq 2^{t+2}$; let $\varepsilon_1 = e_{(1,z)}$ or $e_{(2,\Delta+z)}$. If there is an initial exchange at ε_1, then $\beta(G) \leq 2(\Delta - 1) + t(\Delta - \lambda) + (\Delta - 2^{t+1} + 1)$ where $\lambda \geq 3$. Moreover $\beta(G) \leq (\Delta - 1) + (t + 1)(\Delta - \lambda) + (\Delta - 2^{t+1} + 1)$ where $\mathbf{F}_1$ is the complete cascade.*

Proof. We assume $\beta > (\Delta - 1) + (t + 1)(\Delta - \lambda) + (\Delta - 2^{t+1} + 1)$ and derive a contradiction. Various cases can occur, depending on the cascade set of ε_1 and the nature of the initial exchange. The arguments are very similar and give slightly different bounds on β depending on how many of the central vertices change colour. In this proof, we deal in detail with a case that gives the largest bound for β.

Thus, let us assume that $\varepsilon_1 \in S_4$ where $\varepsilon_1 = e_{(1,z)}$, and that $\mu(v_z) = c_\alpha$. (The argument is analogous for $\varepsilon_1 \in S_1$ in the case $\varepsilon_1 = e_{(2,\Delta+z)}$). Assume that there is an initial exchange E involving colours c_1, c_α and ending at a vertex v_x. Note that if ${}_1[\alpha, 1]_z$ does not exist then at most one of ${}_{x1}[\alpha, 1]_z$ or ${}_1[\alpha, 1]_{x2}$ can go through v_2. Hence we can choose which ever initial exchange E does not go through v_2. Let $\mu_E = \mu + E$; then $\mu_E(v_1) = \mu_E(v_z) = c_1$ or c_α. Let $\mu_A = \mu_E + {}_1(x, 2)_1$ (where $x = 1$ or α as the case may be); then

$$\mu_A(v_1) = c_2 \text{ and } \mu_A(s_1) = \mu_A(v_z).$$

Partition $X(\varepsilon_1)$ into Vizing fans $\mathbf{F}_1, \ldots, \mathbf{F}_f$, with $\mathrm{br}(\mathbf{F}_i) = l_i$ $(i = 1, \ldots, f)$. Note that since $2^{t+1} < z \leq 2^{t+2}$, we have $\sum_{i=1}^{f} \mathrm{br}(\mathbf{F}_i) = t + 1$. Let $\mathbf{F}_1 = \varepsilon_{1.1}, \varepsilon_{1.2}, \ldots, \varepsilon_{1.l_1}$, the colours on these edges being $c_{1.1}, \ldots, c_{1.l_1}$ respectively and the incident satellite vertices being $v_{1.1}, \ldots, v_{1.l_1}$. (Thus $c_{1.1} = c_z$ and $v_{1.1} = v_z$). Then we may sequentially exchange the edge colours on the $\varepsilon_{1.i}$ with the spine colour at one end and the (same) vertex colour on the other, to obtain the semi-total colouring of $G - e_{1,2}$:

$$\mu_1 = \mu_A + F_1 = \mu_A + {}_1(\alpha, c_{1.1})_{1.1} + {}_1(c_{1.1}, c_{1.2})_{1.2} + \ldots + {}_1(c_{1.l_1-1}, c_{1.l_1})_{1.l_1}.$$

Then, $\mu_1(s_1) = c_{1.l_1}$.

If $\mathbf{F}_1$ is the complete cascade, then $c_{1,l_1} = c_4$, and $\omega = \mu_1 + {}_2(4, 3)_2$ is a G-colouring:

$$\beta_\omega = N(\omega, v_x, 1 : \alpha) + \sum_{i=1}^{l_1} N(\omega, v_{1,i}, c_{1,i}) + N(\omega, v_2, c_3) + N(\omega, v_1, c_2)$$

$$\leq (\Delta - 1) + \sum_{i=1}^{l_1} N(\omega, v_{1,i}, c_{1,i}) + N(\omega, v_2, c_3) + N(\omega, v_1, c_2).$$

Now our assumption concerning E implies that we may replace ω by μ in the $N(,,)$ terms *except* if $c_\alpha = c_{1,i}$ for some i and v_x is adjacent to $v_{1,i}$. However, in this case the beta edge from v_x to $v_{1,i}$ is one of at most $(\Delta - 1)$ beta edges at v_x, and so we may avoid double-counting that edge and still conclude that

$$\beta_\omega \leq N(v_x, 1 : \alpha) + \sum_{i=1}^{l_1} N(v_{1,i}, c_{1,i}) + N(v_2, c_3) + N(v_1, c_2)\,.$$

But we also have ${}_1[2, 3[_2$, which has not been affected by the cascade, hence the G-colouring $\omega_0 = \mu_1 + {}_1(2, 3)_2 + {}_2(4, 2)_2$ has

$$\beta_{\omega_0} = N(\omega_0, v_x, 1 : \alpha) + \sum_{i=1}^{l_1} N(\omega_0, v_{1,i}, c_{1,i}) + N(\omega_0, v_2, c_2) + N(\omega_0, v_1, c_3)$$

$$\leq (\Delta - 1) + \sum_{i=1}^{l_1} N(v_{1,i}, c_{1,i}) + N(v_2, c_2) + N(v_1, c_3).$$

Moreover, by the stage t inequalities,

$$N(v_{1,i}, c_{1,i}) \leq \Delta - 3 \quad (i > 1).$$

Finally, none of the $(2^{t+1} - 2)$ vertices v_i $(3 \leq i \leq 2^{t+1})$ has colour c_2 or c_3. Hence, by Lemma 6.3.2, we have

$\beta \leq (\Delta - 1) + (t + 1)(\Delta - \lambda) + (\Delta - 2^{t+1} + 1)$, which contradicts our assumption.

Thus $\mathbf{F}_1$ is not the complete cascade.

Since, μ_1 has the colour $c_{1.l_1}$ on the spine s_1 and also on the non-central vertex incident with the first edge in $\mathbf{F}_2$ (which, we recall, is incident with v_2), then with respect to μ_1, there is a Kempe chain ${}_1]c_{1.l_1}, 3[_y$.

If $y \neq 2$, consider the G-colouring $\theta = \mu_1 + {}_1(c_{1.l_1}, 3)_y$. We have

$$\beta_\theta = N(\theta, v_x, 1 : \alpha) + \sum_{i=1}^{l_1} N(\theta, v_{1,i}, c_{1,i}) + N(\theta, v_y, 3 : c_{1.l_1}) + N(\theta, v_1, c_2).$$

Since none of the vertices adjacent to v_1 have had their colour changed to c_2, we have (arguing as before):

$$\beta_\theta \leq (\Delta - 1) + \sum_{i=1}^{l_1} N(\mu, v_{1,i}, c_{1,i}) + N(\mu, v_y, 3 : c_{1l_1}) + N(\mu, v_1, c_2)$$

$$\leq 2(\Delta - 1) + l_1(\Delta - \lambda) + (\Delta - 2^{t+1} + 1)$$

$$= 2(\Delta - 1) + l_1(\Delta - \lambda) + (\Delta - 2^{t+1} + 1).$$

But $l_1 < \sum_{i=1}^{f} \mathrm{br}(\mathbf{F}_i) = t + 1$. This contradicts our assumption concerning $\beta(G)$, and we conclude that $v_y = v_2$, and that we have the chain ${}_1]c_{1.l_1}, 3[_2$.

Denote by Q_1 the corresponding Kempe exchange ${}_1(c_{1.l_1}, 3)_2$, the first **side exchange.** Then, since $\mu_1 + Q_1$ has colour c_{1l_1} on s_2, we may now implement the Kempe exchanges F_2 corresponding to the Vizing fan $\mathbf{F}_2$.

Continuing in this fashion, we arrive at a semi-total colouring

$$\mu_f = \mu_A + F_1 + Q_1 + F_2 + \ldots + Q_{f-1} + F_f.$$

In checking that the side exchanges Q_i exist, the estimates of β increase for successive exchanges, since the number of Vizing fan exchanges increases, but are always below the bound $2(\Delta - 1) + t(\Delta - \lambda) + (\Delta - 2^{t+1} + 1)$ given in the statement of the lemma. The last edge in the sequence $\mathbf{F}_f$ must be $e_{1,4}$ (as we have assumed $\varepsilon_1 \in S_4$), and thus $\mu_f(s_1) = c_{f l_f} = c_4$, $\mu_f(s_2) = c_3$, $\mu_f(v_2) = c_4$. Thus, we have a G-colouring

$$\mu_{\text{final}} = \mu_f + {}_2(4, 3)_2,$$

with

$$\beta_{\text{final}} = N(\beta_{\text{final}}, v_x, 1 : \alpha) + \sum_{i=1}^{f}\sum_{j=1}^{l_j} N(\beta_{\text{final}}, v_{i,j}, c_{i,j})$$

$$+ N(\beta_{\text{final}}, v_1, c_2) + N(\beta_{\text{final}}, v_2, c_3)$$

$$\leq (\Delta - 1) + (t + 1)(\Delta - \lambda) + N(v_1, c_2) + N(v_2, c_3)$$

(since $\sum_{i=1}^{f} l_i = t + 1$).

We now make final use of Lemma 6.3.2. In the original colouring μ, there is a Kempe chain ${}_1]2, 3[_2$; let $\mu_0 = \mu + {}_1(2, 3)_2$. This colouring has the same properties as μ, and an analogous argument to the above leads to a final G-colouring with

$$\hat{\beta}_{\text{final}} \leq (\Delta - 1) + (t + 1)(\Delta - \lambda) + N(v_1, c_3) + N(v_2, c_2)$$

and whose vertex colours differ from those of μ_{final} only at v_1 and v_2.

Thus, by Lemma 6.3.2,

$$\beta(G) \leq (\Delta - 1) + (t + 1)(\Delta - \lambda) + (\Delta - 2^{t+1} + 1)$$

$$\leq 2(\Delta - 1) + t(\Delta - \lambda) + (\Delta - 2^{t+1} + 1).$$

and the result is proven for $\varepsilon_1 \in S_4$ such that

(i) ε_1 is adjacent to v_1 (rather than v_2);

(ii) the initial exchange involves the vertex colour rather than the spine colour at the central vertex.

By symmetry we can see that if ε_1 is adjacent to v_2, this will not affect the conclusion as we are in effect beginning from the second fan. Similarly if ε_1 belongs to S_1, symmetry with S_4 shows that this give us the same result.

Should ε_1 belong to cascade set S_2 or S_3, the initial exchange will involve only the spine colour. Since the sequence no longer affects the colours of v_1 or v_2 in the final semi-total colouring of G resulting in a lower bound for β than given by the above argument. ■

Main result (20)

Lemma 6.3.8 *Let G be a near type 1 graph with maximum degree $\Delta \geq 4$ and with a total $(\Delta + 1)$-colouring μ of $G - e_{1,2}$ having stage t colour arrangement, chains and inequalities. If $t \geq 0$, $\beta(G) > 2(\Delta - 1) + t(\Delta - \lambda) + (\Delta - 2^{t+1} + 1)$, then we may choose the colours c_z and $c_{\Delta+z}$ $(2^{t+1} < z \leq 2^{t+2})$ so that μ has the stage $(t + 1)$ colour arrangement, chains and inequalities.*

Proof. Let $2^{t+1} < z \leq 2^{t+2}$.

By Lemma 6.3.7, if $\beta(G) > 2(\Delta - 1) + t(\Delta - \lambda) + (\Delta - 2^{t+1} + 1)$ there is no initial exchange at either of the vertices v_z, $v_{\Delta+z}$. Suppose $\mu(v_z) = c_\alpha$ where $\alpha \leq 2^{t+2}$. Let v_w be the satellite vertex with $\mu(v_w) = c_\alpha$ (note that $w \leq 2^{t+1}$). Let $\varepsilon_z \in S_i$ and $\varepsilon_w \in S_j$.

We have an initial exchange unless each of the exchanges ${}_1(1, \alpha)_z$, ${}_1(2, \alpha)_z$, ${}_2(3, \alpha)_z$, ${}_2(4, \alpha)_z$ (except possibly ${}_m(i, \alpha)_z$ with $m = 1$ or 2 as appropriate) exist. However, the corresponding stage t chains ${}_m(k, \alpha)_w$ exist for $k \in \{1, 2, 3, 4\} \setminus \{i, j\}$. Thus at least one such chain must end both at v_z and at v_w, which is a contradiction. Therefore there is an initial exchange, contradicting Lemma 6.3.7. Thus, $\alpha > 2^{t+2}$.

A similar argument holds for the vertices $v_{\Delta+z}$. Thus, all of the vertices v_z, $v_{\Delta+z}$ ($2^{t+1} < z \leq 2^{t+2}$) have colours distinct from the colours c_α ($1 \leq \alpha \leq 2^{t+2}$). Choose v_y, v_z (for example), with $\mu(v_y) = c_\gamma$, $\mu(v_z) = c_\delta$. Let l be such that neither ε_y nor ε_z belong to S_l. Suppose that $l = 1$. Then, because there are no initial exchanges, there are ${}_1[1, \gamma]_y$ and ${}_1[1, \delta]_z$ chains, and so $c_\gamma \neq c_\delta$. The same conclusion follows (for example) for v_y, $v_{\Delta+z}$, so we conclude that the vertex colours at these vertices are all distinct. Therefore they can be chosen so that there is a stage $(t + 1)$ colour arrangement, giving also the stage $(t + 1)$ chains.

Finally we deal with the stage $(t + 1)$ inequalities. Consider a vertex v_z ($2^{t+1} < z \leq 2^{t+2}$); the argument for a vertex $v_{\Delta+z}$ is analogous and will not be given.

We can see that $\mu(v_1) \neq c_z$. For $i = 1, 2, 3$ let v_{w_i} be the vertex adjacent to v_z such $\mu(e_{z,wi}) = c_i$. If $\varepsilon_z \in S_3$ or S_4, then the existence of the stage $(t + 1)$ chains ${}_1[1, z]_y$, ${}_1]2, z]_y$ (where v_y is the satellite vertex coloured c_z) shows that neither v_{w_1} nor v_{w_2} can have colour c_z.

Suppose now $\varepsilon_z \in S_1$. We still have $\mu(v_{w_2}) \neq c_z$. Assume $\mu(v_{w_1}) = c_z$; then there is a Kempe chain ${}_1[1, z]_{w_1}$ and a (stage $(t + 1)$) chain ${}_2]3, z]_y$ that cannot end at v_{w_1}. This is

either disjoint from ${}_1[1, z]_{w_1}$ or goes through edge $\mu(e_{z,1}) = c_z$. In the latter case there is a vertex v_{w3} which is not c_z.

If the chains are disjoint then the G-colouring $\omega = \mu + {}_1(1, z)_{w_1} + {}_2(3, z)_y + {}_1(z, 2)_1$ has

$$\beta_\omega \leq N(w_1, 1) + N(y, 3) + N(1, z) \leq (\Delta - 1) + (\Delta - 3) + (\Delta - 2^{t+1} + 2),$$

contradicting our assumption that $\beta(G) > 2(\Delta - 1) + t(\Delta - \lambda) + (\Delta - 2^{t+1} + 1)$.

Thus $\mu(v_{w1}) \neq c_z$, and the stage $(t + 1)$ inequality for v_z holds.

If $\varepsilon_z \in S_2$, then the conclusion holds with only two Kempe interchanges rather than three. ■

Corollary 6.3.9 *Let G be a near type 1 graph with maximum degree $\Delta \geq 4$ and with a total $(\Delta + 1)$-colouring μ of $G - e_{1,2}$. If $q \geq 1$,*
$\beta(G) > 2(\Delta - 1) + (q\text{-}1)(\Delta - 3) + (\Delta - 2^q + 1)$, then we may choose the colours so that μ has the stage q colour arrangement, chains and inequalities.

Proof. The statement is true if $t = 1$ by Theorem 6.3.6, and thus (by induction, using Lemma 6.3.8) for $1 \leq q \leq t$.
(Note that the stage q inequalities allow the choice 3 for λ). ■

Main result (21)

Theorem 6.3.10 *Let $q \geq 1$ and let G be a near type 1 graph with*
$\beta(G) > 2(\Delta - 1) + (q\text{-}1)(\Delta - 3) + (\Delta - 2^q + 1) = 2(\Delta+1) + q(\Delta - 3) - 2^q$; then

$$\Delta(G) \geq 2^{q+2} - 1.$$

Proof. Since G is near type 1, there is a total $(\Delta + 1)$-colouring μ of $G - e_{1,2}$, whose colours may be ordered so that it has stage q colour arrangement. The number of vertex colours involved in such an arrangement is 2^{q+2}, hence the result. ■

Main result (22)

Corollary 6.3.11 *Let G be a near type 1 graph with* $\Delta \geq 4$; *then*

$$\beta(G) < \frac{3\Delta}{4} + (\Delta - 3)\log_2(\Delta + 1) + 5.$$

Proof. From Lemma 6.3.9, when $\beta(G) > 2(\Delta - 1) + (q - 1)(\Delta - 3) + (\Delta - 2^q + 1)$ we may choose the colours so that μ has the stage q colour arrangement, chains and inequalities.

From Theorem 6.3.10, in such cases $\Delta(G) \geq 2^{q+2} - 1$; $\Delta+1 \geq 2^{q+2}$; $\lfloor \log_2(\Delta+1) \rfloor \geq q + 2$. On the other hand, when $q > \lfloor \log_2(\Delta+1) \rfloor - 2$, then there is no stage q colour arrangement, hence

$$\beta(G) \leq 2(\Delta - 1) + (q - 1)(\Delta - 3) + (\Delta - 2^q + 1).$$

Let $q = \lfloor \log_2(\Delta+1) \rfloor - 1$. Thus $2^{q+1} - 1 \leq \Delta(G) < 2^{q+2} - 1$. Therefore, by the result of the theorem,

$$\begin{aligned}
\beta(G) &\leq 2(\Delta+1) + q(\Delta - 3) - 2^q; \\
&\leq 2(\Delta+1) + (\Delta - 3)(\log_2(\Delta + 1) - 1) - \frac{\Delta+1}{4} \\
&= \Delta + (\Delta - 3)\log_2(\Delta + 1) + 5 - \frac{\Delta+1}{4} \\
&< \frac{3\Delta}{4} + (\Delta - 3)\log_2(\Delta + 1) + 5, \text{ as required.}
\end{aligned}$$

■

6.4 Total Chromatic Numbers of Certain Graph Classes

The results in this subsection (with the exception of theorem 6.4.1) are joint with M.A. Seoud, A. E. Abd El Maqsoud and R.J. Wilson; see [6.15] and [6.16]. Note that the latter was published under Jini Williams' maiden name, J. Senior.

The **Cartesian product** $G \times H$ of two graphs G and H with vertex sets $V(G)$ and $V(H)$ is the graph with vertex $V(G) \times V(H)$ where the vertex $v_{(g_i,h_i)}$ is adjacent to the vertex $v_{(g_j,h_j)}$ whenever $g_i = g_j$ and h_i is adjacent to h_j, or $h_i = h_j$ and g_i is adjacent to g_j. See Figure 6.4.1. Note that there is copy of G for every vertex in H and *vice versa.*

In the first subsection, we present a general result concerning Cartesian products, which generalizes most of the work of [6.15] on Cartesian products involving the cycles C_n, the paths P_n and the stars S_n (where C_n, P_n and S_n are respectively the cycles, paths and stars with n vertices).

6.4.1 Cartesian products [6.15]

Main result (23)

Theorem 6.4.1 *Let G be a type* 1 *graph of maximum degree* Δ_1, *H a class* 1 *graph of maximum degree* Δ_2 *such that* $\chi(H) \leq \Delta_1 + 1$. *Then $G \times H$ is type* 1.

Proof. Let μ_1 be a total $(\Delta_1 + 1)$-colouring of G using the colour set $X = \{0, 1, \ldots, \Delta_1\}$ (considered as elements of the cyclic group Z_{Δ_1+1}); let $\mu_2(V_H)$ be a vertex colouring of H using X; and let $\mu_3(E_H)$ be an edge colouring of H using the colour set $Y = \{\Delta_1 + 1, \Delta_1 + 2, \ldots, \Delta_1 + \Delta_2\}$. For typographical convenience, let us denote the edges of $G \times H$ as follows. If g_1 is adjacent to g_2 in G by $e_{(g_1,g_2)}$ and h_1 adjacent to h_2 in H by $e_{(h_1,h_2)}$, and g_i, h_i are arbitrary vertices of G and H respectively,

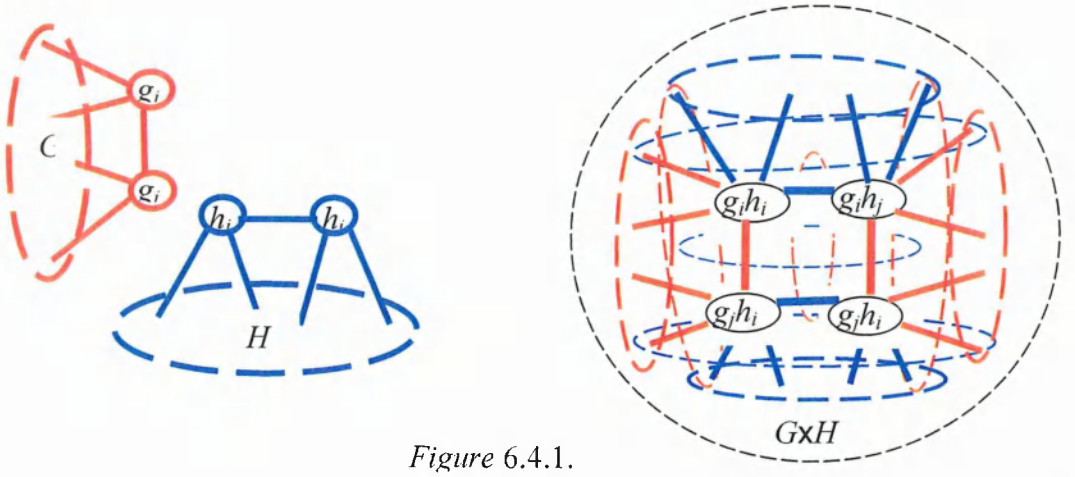


Figure 6.4.1.

the

n edge joining $v_{(g_1,h_i)}$ to $v_{(g_2,h_i)}$ is denoted by $e_{(1,2)g,h_i}$ and the edge joining $v_{(g_i,h_1)}$ to $v_{(g_i,h_2)}$ is denoted by $e_{(1,2)h,g_i}$. Now let μ be the following colouring of the vertices and edges of $G \times H$:

$$\mu(v_{(g_i,h_j)}) = \mu_1(v_{g_i}) + \mu_2(v_{h_j});$$

$$\mu(e_{(1,2)g,h_i}) = \mu_1(e_{1,2}) + \mu_2(v_{h_j});$$

$$\mu(e_{(1,2)h,g_i}) = \mu_3(E_H).$$

Hence we let the edge colouring $\mu_3(E_H)$ of H remain the same in each copy of H. We give every copy of G an isochromatic total colouring where each copy differs from μ_1 by the value $\mu_2(v_{h_j})$. Therefore two adjacent vertices of $G \times H$ of the form $v_{(g_i,h_1)}$ and $v_{(g_i,h_2)}$ have distinct colours, as do two adjacent vertices of the form $v_{(g_1,h_i)}$ and $v_{(g_2,h_i)}$. Given any vertex $v = v_{(g_i,h_j)}$ the colours of the edges joining v to a vertex of form $v_{(g_1,h_i)}$ all belong to X and are distinct from $\mu(e_{(g_i,h_j)})$; while the colours of the other adjacent edges are distinct and belong to Y. ■

Main result (24)

Corollary 6.4.2 (c.f. [6.15], Theorem 1) *Let $F = P_m \times S_n$; then G is type 1 if $m \geq 3$ or $n \geq 3$.*

Proof. Note first that $P_2 \times S_2 \cong C_4$, while $P_1 \times S_2 \cong P_2 \times S_1 \cong K_2$, so that these products are type 2.

Let $F = P_m \times S_n$ where $m \geq 3$ or $n \geq 3$. If $m \geq 3$, then take $G = P_m$, $H = S_n$ and apply the theorem; if $n \geq 3$, then take $G = S_n$, $H = P_m$. ■

Main result (25)

Corollary 6.4.3 (c.f. [6.15], Theorem 2) *Let $F = C_m \times S_n$ where $m \geq 3$, $n \geq 2$; then F is type 1 except when $F = C_5 \times S_2$, which is type 2.*

Proof. When $n = 2$, the result (including the anomaly at $m = 5$) follows from Chetwynd and Hilton [6.3]. Assuming $n \geq 3$, we have that S_n is type 1; moreover, $\chi(C_m) \leq \Delta_1 + 1 = 3$, and so we may apply the theorem, taking $G = S_n$, $H = C_m$. ■

Main result (26)

Corollary 6.4.4 (c.f. [6.15], Theorem 3) *Let $F = C_m \times P_n$ where $m \geq 3$, $n \geq 2$; then F is type 1 except when $m = 5$ and $n = 2$.*

Proof. The case $n = 2$ follows from corollary 6.4.3 since $P_2 = S_2$; thus let us assume $n \geq 3$. Take $G = P_n$, $H = C_m$ and proceed as in corollary 6.4.3. ■

Main result (27)

Corollary 6.4.5 (c.f. [6.15], Theorem 4) *Let $F = C_m \times C_n$ where $m \geq 3$, $n \geq 3$ and one of m, n is a multiple of 3; then F is type 1.*

Proof. We may assume that m is a multiple of 3. Then C_m is type 1, and we may take $G = C_m$, $H = C_n$ and proceed as in corollary 6.4.3. ■

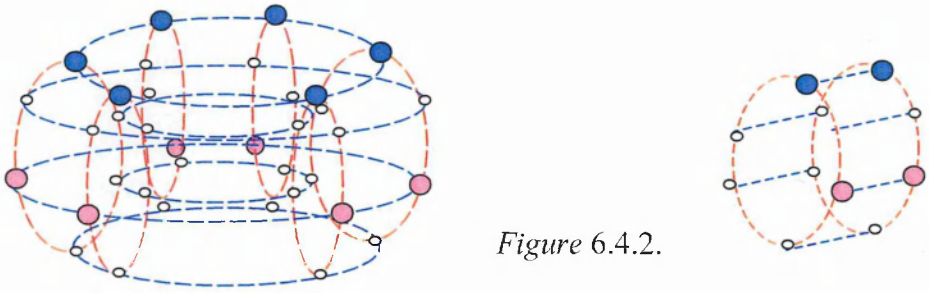


Figure 6.4.2.

The following cases, Theorem 4 of [6.15], are not covered by corollary 6.4.5.

Main result (28)

Theorem 6.4.6 *For* $F = C_m \times C_n$ *where* m *is even,* $n \neq 5$ *and* $m \geq 3$, $n \geq 3$; F *is type* 1.

Proof. The graph $C_m \times C_n$ can be seen as the net of a Taurus, see $C_6 \times C_6$ Figure 6.4.2. Let $G = C_m$, $H = C_n$, and denote the vertices of G and H (in natural order) by $g_1, \ldots, g_m$ and $h_1, \ldots, h_m$ respectively. We now use the notation for edges developed in the proof of theorem 6.4.1.

Let $m = 2p$. Temporarily delete all edges of the form $e_{(2i-1,\,2i)g,h}$ for $i = 1, \ldots, p$ and all $h \in V(H)$. The resulting graph E is a union of p copies of $P_2 \times C_n$. (see Figure 6.4.2) By Corollary 6.4.4, $P_2 \times C_n$ is type 1 and has a total colouring μ using the elements of Z_4. Now colour each component E_p using the same colouring μ; thus, since in every component we have $\mu(v_{(g_{p1},h_j)}) \neq \mu(v_{(g_{p2},h_j)})$ $(j = 1, \ldots, n)$ then in E we have

$$\mu(v_{(g_{2i},h_j)}) \neq \mu(v_{(g_{2i+1},h_j)}) \ (i = 1, \ldots, m, j = 1, \ldots, n) \text{ and also}$$

$$\mu(v_{(g_{2i},h_j)}) = \mu(v_{(g_{2i+2},h_j)}) \ (i = 1, \ldots, m, j = 1, \ldots, n).$$

It follows that, when we replace the edges $e_{(2i-1,\,2i)g,h}$ we have $\mu(v_{(g_{2i-1},h_j)}) \neq \mu(v_{(g_{2i},h_j)})$ $(i = 1, \ldots, m, j = 1, \ldots, n)$ and can colour these edges with a fifth colour, we now have a total 5-colouring (i.e. a type 1 colouring) of F. ■

6.4.2 Powers of Paths [6.13]

Let G be any graph. The kth **power** G (denoted by G^k) is the graph with vertex set $V(G)$, where v and w are adjacent in G^k if the distance from v to w in G is at most k. Thus, if the vertices of P_n are denoted by $v_1, \ldots, v_n$ in natural order, then v_i and v_j are adjacent in P_n^k if and only if $|i-j| \leq k$.

In the statement and proof of the following theorem, we use Hilton's notation (see page 89).

Main result (29)

Theorem 6.4.7 *If* $1 \leq k \leq n-1$, *then* $G = P_n^k$ *is type 1 except when n is even and* $(\alpha'(\overline{G})^2 + 3\alpha'(\overline{G})) < n$.

Proof. Note first that P_1 is type 1 and that, if $G = P_2^1$, then $\alpha'(\overline{G}) = 0$, hence the statement is true when $n = 1$ or 2.

Let $n > 2$; we consider four cases.

Case 1: $1 \leq k \leq (n-1)/2$.

Define a total colouring μ as follows:

$\mu(v_i) = c_\lambda$ where $1 \leq \lambda \leq 2k+1$ and $\lambda \equiv 2i-1 \pmod{(2k+1)}$;

$\mu(e_{(i,j)}) = c_\lambda$ where $1 \leq \lambda \leq 2k+1$ and $\lambda \equiv i+j-1 \pmod{(2k+1)}$.

This is a proper $(\Delta+1)$-total-colouring based on part of the cyclic Cayley table introduced in Chapter 3.

Case 2 $k = n-1$.

In this case, P_n^k is the complete graph K_n, which is type 1 if n is odd and type 2 if n is even.

Case 3 $n/2 \leq k \leq n-2$ *and n is odd.*

Here, we have $\Delta(P_n^k) = n-1 = \Delta(K_n)$, and P_n^k is a subgraph of K_n. Thus, since n is odd, then restricting a total n-colouring of K_n to P_n^k gives a $(\Delta+1)$-colouring.

Case 4 $n/2 \leq k \leq n-2$ *and n is even.*

Hilton [6.2] has shown that a graph G with n even and $\Delta = n-1$ is type 2 if

$$e(\overline{G}) + \alpha'(\overline{G}) < n/2.$$

When $G = P_n^k$, we have $e(\overline{G}) = (n-k)(n-k-1)/2$, and the following $i = n-k-1$ edges form an independent edge set in $\overline{G}$: $\{e_{(1,k+2)}, e_{(2,k+3)}, e_{(3,k+4)}, \ldots, e_{(i-1,\, n-1)}, e_{(i,\, n)}\}$. Since these edges use every available vertex in $\overline{G}$, this is a maximal set and

$$\alpha'(\overline{G}) = i = n-k-1.$$

It follows from Hilton's result that G is type 2 if

$$(n-k)(n-k-1)/2 + (n-k-1) < n/2$$

$$(n-k-1)^2 + 3(n-k-1) < n$$

and hence, as required, $\quad (\alpha'(\overline{G})^2 + 3\alpha'(\overline{G})) < n.$ ■

Summary of Chapter 6

In Section 6.1 we looked at the conformability conjecture and came to the conclusion that since every graph with at least Δ spines is conformable, the concept needs refining. We presented various conjectures with relation to the beta parameter.

In Section 6.2 we presented a necessary condition for a graph to be type 1, which we named G^*-conformability. This categorises the majority of known graphs, though there are still a few graphs that are G^*-conformable and type 2. We then imposed further conditions on cubic graphs, which we believe categorise them all.

In Section 6.3 we considered near type 1 graphs: graphs that become type 1 when an edge not belonging to a triangle is removed. We discussed the beta parameter and classification by β-number of such graphs. We discovered that where G is a nearly type 1 graph with $\Delta \geq 4$, then

$$\beta(G) < \frac{3\Delta}{4} + (\Delta - 3)\log_2(\Delta + 1) + 5.$$

In Section 6.4, we discussed the classification of certain Cartesian products and all powers of paths. We found several classes of cross product that are always type 1 and that for most m and n, that is: where m is even, $n \neq 5$ and $m \geq 3$, $n \geq 3$, then $C_m \times C_n$ is type 1 and that P_n^k is type 1 except when n is even and $(\alpha'(\overline{G})^2 + 3\alpha'(\overline{G})) < n$.

In Appendix (6.1), we present a catalogue of surcritical graphs with no more than ten vertices. In Appendix (6.2), we present an algorithm for checking for G^*-conformability which will either produce a colouring for G or prove that no such colouring exists. Clearly due to the effort involved, this would need to be done by computer.

Summary of main results by Jini Williams in Chapter 6

Conjecture 6.1.5 *Removing any set of p edges from a β-critical graph will reduce β by between p and 2p.* Main conjecture (1) page 92

Conjecture 6.1.6 *Adding an edge to a β-critical graph will increase β by at most 2.*

Main conjecture (2) page 92

Lemma 6.1.8 *Every graph with at least Δ spines is conformable.*
Main result (3) page 94

Lemma 6.1.10 *A non-connected graph, each whose components is of maximum degree Δ, is conformable if every component is conformable.*

Main result (4) page 95

Theorem 6.1.12 *Any non-conformable irregular graph G is an induced subgraph of a type 2 conformable graph H of the same maximum degree, where $n(H) = n(G) + 1$.*

Main result (5) page 97

Conjecture 6.1.19 [Holroyd and Williams] *Every type 2 critical graph G satisfying $\Delta(G) < \frac{1}{2}(|V(G)|-1)$ is conformable.* Main conjecture (6) page 101

Corollary 6.2.2 *A graph which has no G^*-conformable colouring is type 2.*

Main result (7) page 103

Corollary 6.2.8 *A G^*-conformable graph which is also type 2 cannot have disjoint 1-factors for all $G - S_i$.* Main result (8) page 105

Theorem 6.2.10 *Let G be regular cubic graph. Then a necessary condition for G to be type 1 is that G should have a G^*-conformable vertex colouring $\mu(V^*)$ such that, for every pair of colour sets S_i and S_j, the subgraph $G - S_i - S_j$ has:*

(i) at least $(n_i + n_j)/2$ components;

(ii) at most $(n_i + n_j)/2$ components with less than four vertices.

(iii) an even number $2q$ of odd components, where $2q \leq \min\{n_i, n_j\}$.

Main result (9) page 106

Conjecture 6.2.11 *Let G be a regular cubic graph. Then the necessary condition for G to be type 1, stated in Theorem 6.2.10, is also sufficient.*

Main conjecture (10) page 107

Conjecture 6.2.12 *Let G be regular cubic graph. If G has an acyclic G^*-conformable vertex colouring $\mu(V^*)$ with colour sets S_i and S_j, such that every $G - S_i - S_j$ has*

(i) exactly $(n_i + n_j)/2$ components;

(ii) an even number $2q$, of odd components, where $2q \leq \min\{n_i, n_j\}$;

then G is a type 1 graph. Main conjecture (11) page 108

Theorem 6.2.13 *Let G be a semi-regular cubic graph. Then a necessary condition for G to be type 1 is that G should have a G^*-conformable spine and vertex colouring*

$\mu(V^*)$ (as above) such that, for every pair of colour sets S_i and S_j, the subgraph $G - S_i - S_j$ has:

(i) at least $(n_i + n_j)/2 - \zeta(\{i, j\})$ components;

(ii) at most $(n_i + n_j)/2 - \zeta(\{i, j\})$ components with less than four vertices;

(iii) an odd number of odd components, bounded above by $\min\{n_i, n_j\}$, *except where* $\{i, j\} = \{1, 2\}$, *in which case an even number of odd components, bounded above by* $\min\{n_1, n_2\} - 1$.

Main result (12) page 109

Conjecture 6.2.14 *Let G be a semi-regular cubic graph. Then the necessary condition for G to be type 1, stated in Theorem* 6.2.13, *is also sufficient.*

Main conjecture (13) page 110

Conjecture 6.2.15 *Let G be a semi-regular cubic graph. If G has an acyclic G^*-conformable vertex colouring $\mu(V)$ with $\mu(v_1) = c_1$, $\mu(s_1) = c_2$ and if, for each pair S_i, S_j of colour sets, the subgraph $G - S_i - S_j$:*

(i) has exactly $(n_i + n_j)/2 - \zeta(\{i, j\})$ components;

(ii) has an odd number of odd components, bounded above by $\min\{n_i, n_j\}$, *except where* $\{i, j\} = \{1, 2\}$, *in which case an even number of odd components, bounded above by* $\min\{n_1, n_2\} - 1$, *then G is a type* 1 *graph.*

Main conjecture (14) page 110

Theorem 6.3.3 *Let $\Delta(G) \geq 3$ and suppose there is a total $(\Delta + 1)$-colouring μ of $G - e$ such that $\mu(v_1)$, $\mu(s_1)$, $\mu(v_2)$ and $\mu(s_2)$ are not all distinct. Then $\beta \leq \Delta$.*

Main result (15) Page 115

Theorem 6.3.4 *Let G be a near type 1 graph with $\Delta = 3$; then $\beta \leq 2$ unless all type 1 total colourings μ of $G - e$ have $\mu(v_1) = \mu(v_2)$, $\mu(s_1) \neq \mu(s_2)$, when $\beta \leq 3$.*

Main result (16) Page 117

Conjecture 6.3.5 *Let G be any critical cubic graph; then $\beta(G) \leq 2$.*

Main conjecture (17) Page 119

Theorem 6.3.6 *Let G have $\Delta \geq 4$, $\beta > 3\Delta - 7$, and let μ be a total $(\Delta + 1)$-colouring of $G - e$; then:*

(i) *the Stage 0 chains exist and the Stage 0 inequalities hold;*

(ii) *any total colouring μ of $G - e$ has Stage 1 colour arrangement, and the following chains (the* **Stage 1 chains***) exist:*

$${}_1[1, 5]_3,\ {}_1]2, 5]_3,\ {}_2[4, 5]_3;\quad {}_1]2, 6]_{\Delta+3},\ {}_2]3, 6]_{\Delta+3},\ {}_2[4, 6]_{\Delta+3};$$

$${}_1[1, 7]_4,\ {}_1]2, 7]_4,\ {}_2]3, 7]_4;\quad {}_1[1, 8]_{\Delta+4},\ {}_2]3, 8]_{\Delta+4},\ {}_2[4, 8]_{\Delta+4}.$$

Main result (18) Page 123

Lemma 6.3.7 *Let μ be a total $(\Delta + 1)$-colouring of $G - e_{1,2}$ with stage t colour arrangement, chains and inequalities; let $2^{t+1} < z \leq 2^{t+2}$; let $\varepsilon_1 = e_{(1,z)}$ or $e_{(2,\Delta+z)}$. If there is an initial exchange at ε_1, then $\beta(G) \leq 2(\Delta - 1) + t(\Delta - \lambda) + (\Delta - 2^{t+1} + 1)$ where $\lambda \geq 3$. Moreover $\beta(G) \leq (\Delta - 1) + (t + 1)(\Delta - \lambda) + (\Delta - 2^{t+1} + 1)$ where $\mathbf{F}_1$ is the complete cascade.*

Main result (19) Page 130

Lemma 6.3.8 *Let G be a near type 1 graph with maximum degree* $\Delta \geq 4$ *and with a total* $(\Delta + 1)$*-colouring* μ *of* $G - e_{1,2}$ *having stage t colour arrangement, chains and inequalities. If* $t \geq 0$, $\beta(G) > 2(\Delta - 1) + t(\Delta - \lambda) + (\Delta - 2^{t+1} + 1)$, *then we may choose the colours* c_z and $c_{\Delta+z}$ $(2^{t+1} < z \leq 2^{t+2})$ *so that* μ *has the stage* $(t + 1)$ *colour arrangement, chains and inequalities.* Main result (20) Page 135

Theorem 6.3.10 *Let* $q \geq 1$ *and let G be a near type 1 graph with*

$\beta(G) > 2(\Delta - 1) + (q\text{-}1)(\Delta - 3) + (\Delta - 2^q + 1) = 2(\Delta+1) + q(\Delta - 3) - 2^q$; *then*

$\Delta(G) \geq 2^{q+2} - 1.$ Main result (21) Page 137

Corollary 6.3.11 *Let G be a near type 1 graph with* $\Delta \geq 4$; *then*

$$\beta(G) < \frac{3\Delta}{4} + (\Delta - 3)\log_2(\Delta + 1) + 5.$$ Main result (22) Page 138

Theorem 6.4.1 *Let G be a type* 1 *graph of maximum degree* Δ_1, *H a class* 1 *graph of maximum degree* Δ_2 *such that* $\chi(H) \leq \Delta_1 + 1$. *Then* $G \times H$ *is type* 1.

Main result (23) Page 139

Corollary 6.4.2 (c.f. [6.15], Theorem 1) *Let* $F = P_m \times S_n$; *then G is type* 1 *if* $m \geq 3$ *or* $n \geq 3$. Main result (24) Page 140

Corollary 6.4.3 (c.f. [6.15], Theorem 2) *Let* $F = C_m \times S_n$ *where* $m \geq 3$, $n \geq 2$; *then F is type* 1 *except when* $F = C_5 \times S_2$, *which is type* 2. Main result (25) Page 141

Corollary 6.4.4 (c.f. [6.15], Theorem 3) *Let* $F = C_m \times P_n$ *where* $m \geq 3, n \geq 2$; *then* F *is type* 1 *except when* $m = 5$ *and* $n = 2$. Main result (26) Page 141

Corollary 6.4.5 (c.f. [6.15], Theorem 4) *Let* $F = C_m \times C_n$ *where* $m \geq 3, n \geq 3$ *and one of* m, n *is a multiple of* 3; *then* F *is type* 1. Main result (27) Page 141

Theorem 6.4.6 *Let* $F = C_m \times C_n$ *where* m *is even,* $n \neq 5$ *and* $m \geq 3, n \geq 3$; *then* F *is type* 1. Main result (28) Page 142

Theorem 6.4.7 *If* $1 \leq k \leq n-1$, *then* $G = P_n^k$ *is type 1 except when* n *is even and* $(\alpha'(\overline{G})^2 + 3\alpha'(\overline{G})) < n$. Main result (29) Page 143

CHAPTER 7

APPLICATIONS

We now consider how graph colourings can be applied in the non academic world. There are many applications already in use for vertex and edge colouring but there are as yet no known practical uses for total colouring. All possible attempts to apply total colouring are therefore conjectural, but some are more plausible than others. Therefore following suggestions are simply that: suggestions. Clearly the type of research required to justify or dismiss any of the following conjectural application of total colour is beyond the scope of this thesis. The appendices include algorithms for finding colourings which may be used in an application should one be found.

The fields of electric and electronic theory hold many applications for edge colouring. Therefore, the close link between total and edge colouring, leads us to feel that there are certain applications of edge colouring which would be improved if total colouring was taken into account.

The entertainment industry is always on the look out for new games: especially computer games. Although it is also true, that the number of games invented, exceeds the number that become commercially successful: as can be verified by visiting charity shops after Christmas. Nevertheless the number of different games that could be produced by using graph colouring is endless: the details of each are so variable that it would be impossible to mention them all. Therefore we present just one example from each of the main aspects of graph colouring that have been studied in this thesis.

(1) Patience squares (Chapter 3)

This is a straightforward card game of patience. The standard 52 card pack is well shuffled. The aim is to deal the cards into three separate 4x4 latin squares. Since we have thirteen cards in a pack there are four cards left over, these are placed as they arise in a separate 'exchange' pile. Each card is turned up when dealt and placed in up to three rows of four base cards. This can be done until a cards is dealt that has the same rank as one already on the table. As each repeat card is dealt, it is placed (partially) on top of one of the existing base cards. This must be done in such a way that no two cards have the same rank in any row or column. If there is no place for the card to go at the time it is dealt it goes to the discard pile. The aim is to have as few cards as possible in this pile at the end of the game. If at a further point in play, the top card of the discard pile can be used, then the player is allowed to use it. This frees the card underneath for use. The base cards use cards of different rank and the thirteenth value goes to the 'cheat' pile. The player is allowed to reallocate one top card to a vacant place by using a 'cheat' card just once, after which the 'cheat' card is turned face down to prove that it is has been used.

A beginner's level can be created for this game by including Jokers in the 'cheat pile'. A master's level on the other hand can insist that the squares produced are MOLs.

(2) Map colouring (Chapter 4)

The manipulation of graphs and colourings is a source of intellectual interest as well as amusement. A program could be produced where a map is generated either real or at random and the user has to find a four colouring in as short a time as possible. The graphics are, by virtue of the task involved, predisposed to being colourful and attractive. Alternatively, two colourings A and B of the same map could be presented where the task is to transform A into B by means of Kempe interchanges.

(3) Computer Jigsaws (Chapter 5)

This game is based on the task of finding two graph colourings of the plane where a tri-star equi-net has been given, as in figure 7.1. Pieces could me moved to show the colour allocated or the faces could be filled using the fill option with a limited palette.

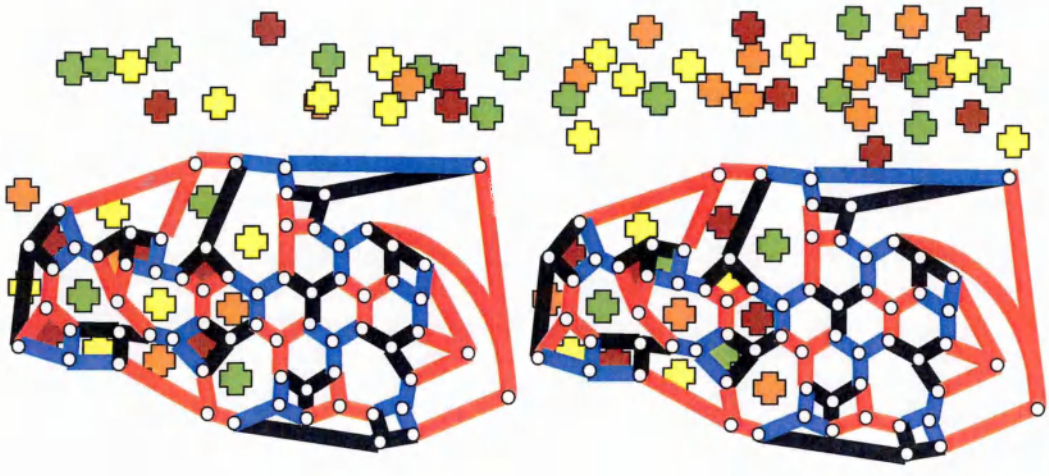


Figure 7.1

(4) Network colourings. (Chapter 6)

Here we have a graph with edges and vertices and, as with map colouring, each level could have successively more complicated graphs which the player would be required to total colour in a limited amount of time. The colours could be pre allocated with the aim of finding a transition from the given colour to another via Kempe and Pittenger interchanges.

But entertainment, though perhaps the most lucrative application, may not be the only way that total colouring could be put to use.

Dr David Caroliaro has already investigated the link between hypergraphs and fire prevention,[talk PGCC 2004 after presentation of doctoral thesis] and there could well be links to total colouring in similar circumstances.

Secret sharing

Consider also a secret sharing scheme, where we have a number of people with passwords in various locations. We need to change these passwords at regular intervals but do not want two people in neighbouring locations to have the same password. (For example, because we may want to know who is accessing the data). If we plot these people on to an adjacency graph we can assign the graph a total colouring and by means of exchanging the colour chains keep a record of which person has which password at any one time.

A similar colouring could be generated for the variation of the communications problem where we have fixed frequencies (which could be seen as coloured vertices) and acceptable interactions (which could be seen as defined edges) but could have unacceptable interactions such as interference whenever two people with similar frequencies meet within a certain distance (which could be adjacent vertices with the same colour).

Electrical impulse systems

It is well known that nerves are a bundle of nerve fibres. We will now consider a possible model for tracking the course of the electrical impulses that are transmitted by such fibres. Unless otherwise referenced, in this discussion of the model, all the quotes in italics below are from the same source, the Oxford World Encyclopaedia [7.1] as this gives a concise outline of the known facts. There are many other sources that would give a more thorough picture, but we only require an outline here.
We know that every nerve fibre '*carries undirectional signals to and from the brain independently of neighbouring fibres in the nerve*'. We can model these fibres as undirected edges of a graph.

Each fibre is insulated from the others '*to prevent electrical interference between adjacent fibres in the nerve*'. But although in physical terms these fibres initially run in parallel, they have different end points. Therefore let us suppose that in terms of our graph these bundles of fibres/edges are incident at one vertex only.

'*Signals carried in the fibre are called impulses because they travel as discrete bursts of electrical activity*'. Let us consider that each edge has a colour and that there is no colour in common in any bundle of fibres. We can consider each impulse to be an edge Kempe chain, affecting each edge in the chain from the moment that the colour is changed.

'*Each burst is followed by a short period when no further impulses can travel along the nerve. These spaces enable the information to be coded into bursts*'. It is important to finish with one interchange before starting the next, hence the time delay. Once the previous interchange has been completed there is no hindrance to the next interchange taking place.

'*Information at the receptor end of the nerve fibre is thus coded into burst patterns eventually to be translated into appropriate action by the brain.*' The different sequences of colours affect the eventual outcome in different ways. Not all chains will have the same end vertex as the chain could only end at a vertex or spine of an appropriate colour.

'*The junction between two nerve cells is called a synapse*' In our model this junction is a vertex.

'*The transfer of impulses is mediated by the release of chemicals called neurotransmitters across the gap of the synapse.*' The interchanges in our model will

be determined by the colours of the chains themselves. The neurotransmitters can therefore be modelled as colour assignments.

'*Amplification or moderation of the impulse can occur at this point.*' In our model the impulse could be stable if the chain ends at a vertex where there are no neighbours of that new colour; it would continue into a further impulse if there is only one neighbouring vertex with the new colour; it could be suppressed if it ends at a spine; it could initiate a different impulse, (be amplified), if there are two or more neighbouring vertices with the new colour.

'*Most fibres split into several smaller fibres at their ends enabling each neurone to communicate with many other fibres.*' That is to say that, in out model, most vertices are adjacent to more than two other vertices.

It was noted by Geary in [7.2] that '*memories are not fixed and immovable facts; they are emerging from an ever changing maze of neural firing formations and synaptic connections*'. In the same article Professor Steven Rose of The Open University was quoted as saying that '*memory is a dynamic property simultaneously residing everywhere and nowhere in the brain*'. Our model, would have exactly that structure: the end vertices of each impulse could be at any distance from the initial vertex and in any part of the network. We also note that the breakdown of the cell adhesion molecules in Alzheimer's disease in our model is mirrored by the breaking of edges.

The dual to this model has the edges as synapses and vertices as nerve fibres.
A total colouring model could, therefore, well prove useful in the field of neurology.

For an overview of current accepted knowledge in this field see Golden [7.3]

Art

The symbiotic relationship between Mathematics and Art is a study in itself and has generated much interest leading to numerous books, conferences and artworks. However, since graph colouring has not yet reached the popular realm we feel that it is important to note a few of the possible applications.

Take the colouring of the infinite sub triangulation of the plane in Appendix 7.3 as an example. An installation could be formed with hemispheres and spheres of decreasing diameter as follows:- Seven differently coloured large hemispheres are placed as vertices to form the seven colour tiling flower $\mu(\varpi_0)$. Medium sized spheres are chosen with diameters such that they are half the size of large hemispheres, these are joined on a frame overlaying the large hemispheres, the colours are chosen so that they form $\mu(\varpi_1)$. Small spheres are now chosen with diameters half the size of medium sized spheres. They are fixed on a frame overlaying the medium spheres. The colours are chosen so that they form $\mu(\varpi_2)$. Tiny spheres are now chosen with diameters half of that of the small hemispheres, the colours are chosen so that they form $\mu(\varpi_3)$. Depending on the size of the original hemispheres this pattern can be continued until the size is too small to be distinguished. The result is a fractal total colouring and an art work.

Now consider the graphs themselves. Molecular models are used in chemistry for practical purposes but are also interesting as works of art. Since total colouring has no known practical applications the three dimensional models of total coloured graphs are even more interesting from an artistic point of view as they are an abstract concept with a physical form.

Latin squares and arrays for edge and total colourings make excellent two dimensional paintings. MOL's and Mutually Orthogonal Arrays make excellent three dimensional and textile works.

Another work can be based on the observation that a single latin square is a set of $n \times n$ objects each with three distinguishing factors i, j and k, such that each factor occurs just once with any other. This is equivalent to $n \times n$ people being given differently coloured hats, coats and gloves in such a way that no two people are dressed the same to isochromatism. Applied to K_n, we have an $n \times n$ set of coloured edges and vertices. Each vertex and colour has been given an arbitrary , but consistent label. Therefore, people wearing matching hats and gloves could represent the vertices, in which case the edges are such that the hats do not match the gloves. However, since the edge from v_i to v_j is the same as from v_j to v_i is the person with green hat and red gloves is wearing the same colour coat as the person with red hat and green gloves, regardless of what that colour might be. Hence a truly modern installation would be a set of people dressed as above with no other remit than to socialise in the same room. Alternatively we could be more conventional and at set times, arrange our people on a grid such that they stand in rows of the same colour hat and in columns of the same colour gloves. There will be exactly one person per square. One conjugate is therefore, the same people arranged in rows of the same colour coat and in columns of the same colour hat, and another by gloves and coat.

The complete set of MOLs and conjugates could be displayed as a continuous display of slides when the original installation has to be disbanded.

There are innumerable ways of using total colour in the world of Art and an exhibition of such woks would be an exciting application for total colouring.

Summary of Chapter 7

In conclusion, just as Hu Cadarn was unable to say why barley grows from barley seed, we still do not know the answers to many of the questions that have arisen from the study of graph colourings. Unlike Hu, we have not yet invented the colouring plough, but we have found out many things in our search which may yet prove useful. Applications of total colour may not be in existence other than by chance, but there is much scope for future efforts. Each application of combinatorics which is presented, makes it possible for people working in other fields to harness the mathematics to their own endeavours, hence distributing awareness and increasing knowledge.

Appendix (6.1)

We present the seventy critical and surcritical type 2 graphs with no more than ten vertices.

Catalogue of Critical and Surcritical Graphs

This appendix revisits the Catalogue of Critical graphs Numbers 1-50 by Hamilton Hilton and Hind [6.8] Although many of the graphs also have other names, in this appendix each graph from [6.8] will be refered to as *HHHx*, where x is a number 1-50.

We present two copies of each graph. The first copy has coloured edges showing a best possible β-colouring. In order that the β-edges can be easily identified, they have been given a thicker line. The second copy has grey edges.

As well as the graphs identified in [6.8], where they exist, we show best β-colourings of further **surcritical** graphs where these are obtained from *HHHx* by adding new edges up to degree Δ, but no new vertices. These edges will be called **surcritical edges**. In some cases these surcritical graphs overlap, for instance the graphs *HHH*42 and *HHH*43 are both sub-graphs of $K_{5,5}$ and hence share this and $K_{5,5}-e$ as surcritical derivatives. The graphs will also be given a number which refers to its place in this catalogue, for instance $K_{5,5}$ is No. 57.

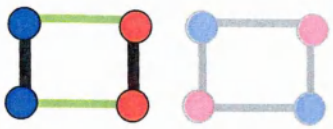
All possible further surcritical edges will be shown as thin dotted lines in the second (grey edged) copy. This copy, will be given a conformable vertex colouring where such a colouring exists, otherwise it will have all vertices coloured white.

We give a third copy of the only graph, *HHH*12, with a G^*-conformable colouring. Here multi-edges replace the relevant edges of the conformable vertex colouring.

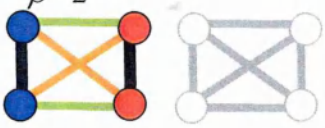
The Catalogue.

(No. 1) *HHH*1
$\beta=1$

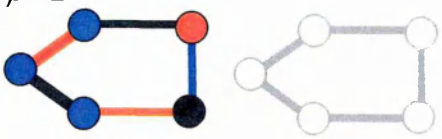
(No. 2) *HHH*2
$\beta=2$



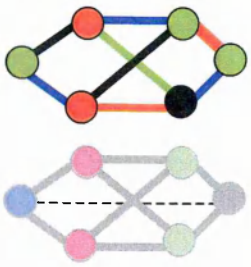
(No. 3) *HHH*3
$\beta=2$



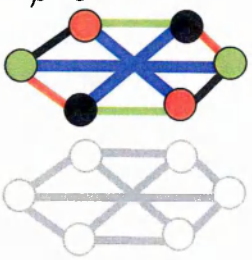
(No. 4) *HHH*4
$\beta=2$



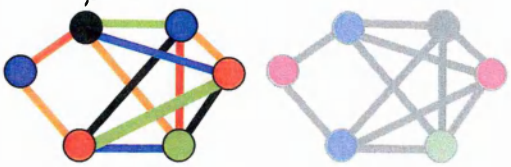
(No. 5) *HHH*5
$\beta=1$



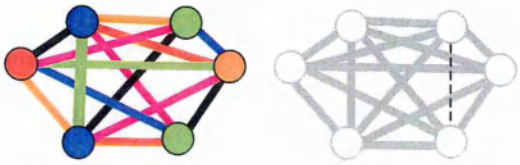
(No. 6) *HHH*5 +*e*
$\beta=3$



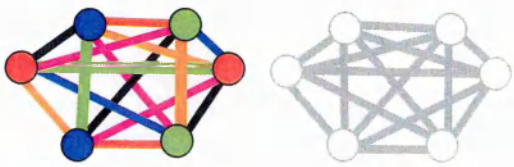
(No. 7) *HHH*6
$\beta=1$



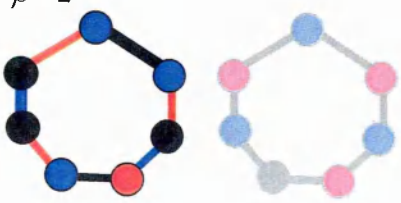
(No. 8) *HHH*7
$\beta=1$



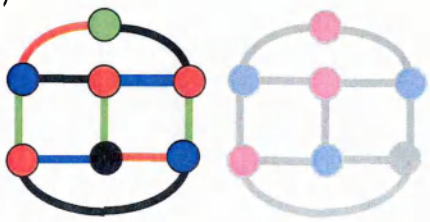
(No. 9) *HHH*7+*e*
$\beta=3$



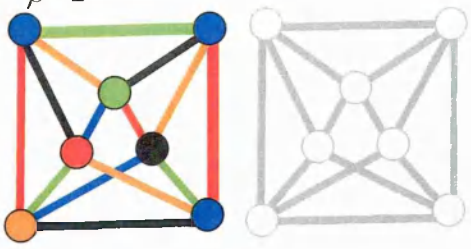
(No. 10) *HHH*8
$\beta=2$



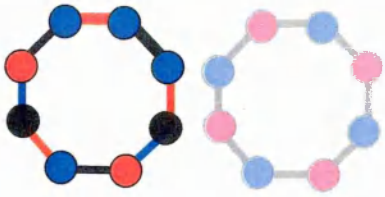
(No. 11) *HHH*9
$\beta=1$



(No. 12) *HHH*10
$\beta=2$

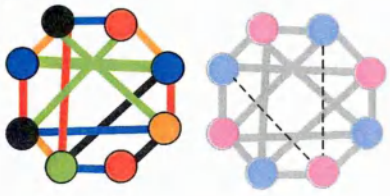


(No. 13) *HHH*11
$\beta=2$

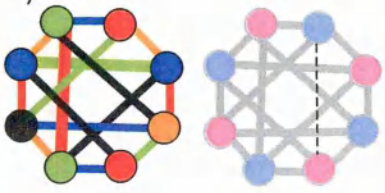


(No. 14) *HHH*12
$\beta=1$

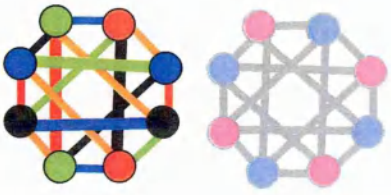
(No. 15) *HHH*13
$\beta=1$



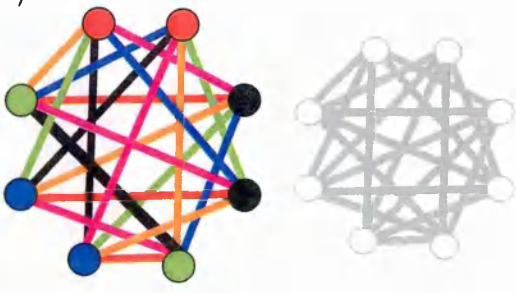
(No. 16) *HHH*13+*e*
$\beta=2$



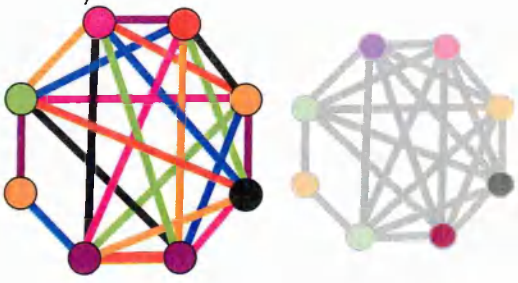
(No. 17) *HHH*13+2*e*
$\beta=4$



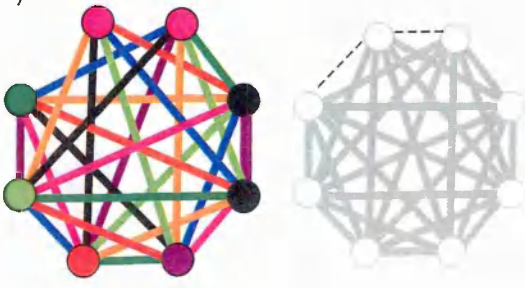
(No. 18) *HHH*14
$\beta=1$



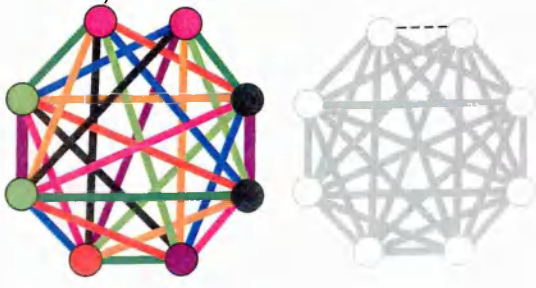
(No. 19) *HHH*15
$\beta=1$



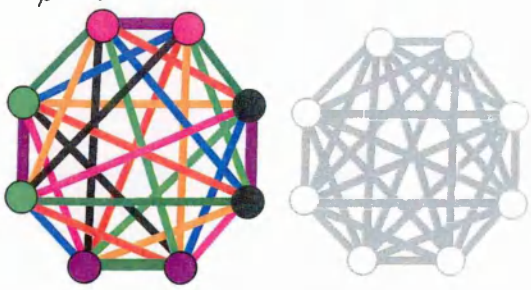
(No. 20) *HHH*16
$\beta=1$



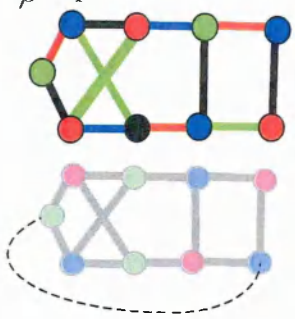
(No. 21) *HHH*16 + *e*
$\beta=2$



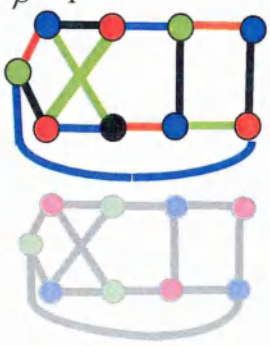
(No. 22) *HHH*16+2*e*
$\beta = 4$



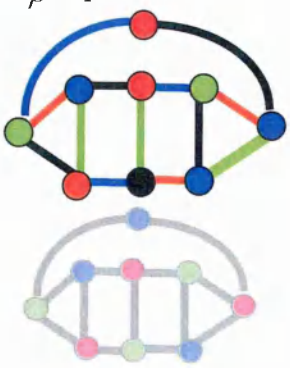
(No. 23) *HHH*17
$\beta = 1$



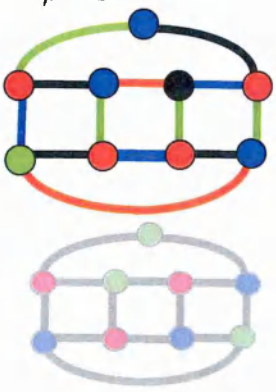
(No. 24) *HHH*17 + *e*
$\beta = 1$



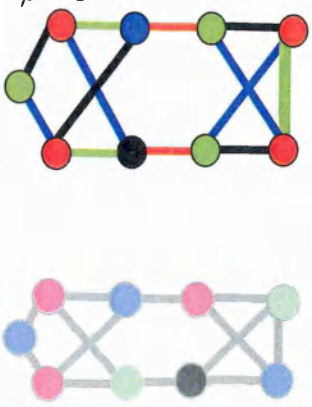
(No. 25) *HHH*18
$\beta = 1$



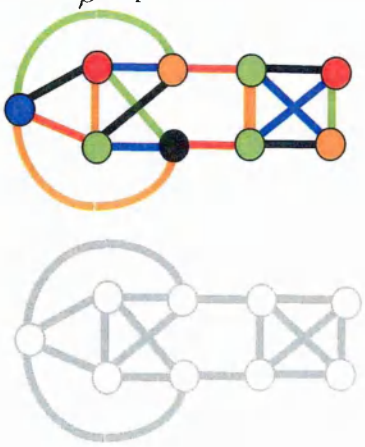
(No. 26) *HHH*19
$\beta = 1$



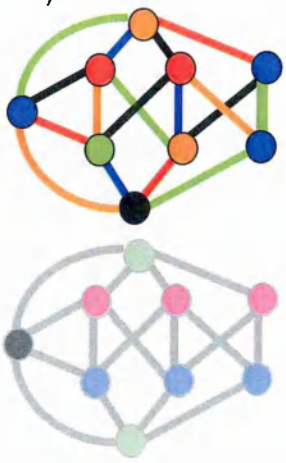
(No. 27) *HHH*20
$\beta = 1$



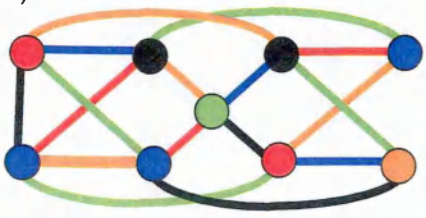
(No. 28) *HHH*21
$\beta = 1$

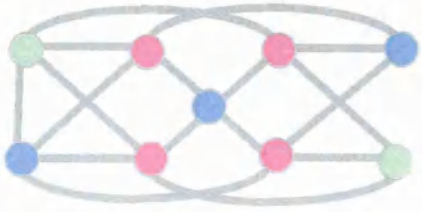


(No. 29) *HHH*22
$\beta = 1$



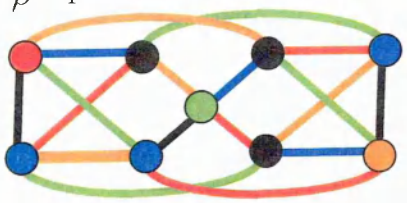
(No. 30) *HHH*23
$\beta = 1$

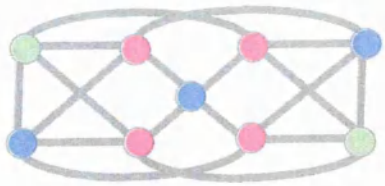




(No. 31) *HHH*23+*e*

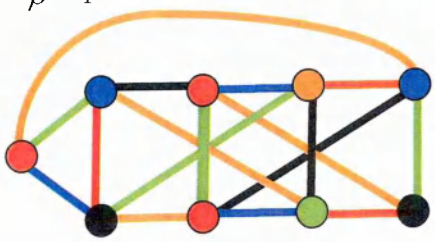
$\beta = 1$

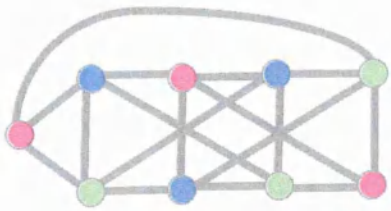




(No. 32) *HHH24*

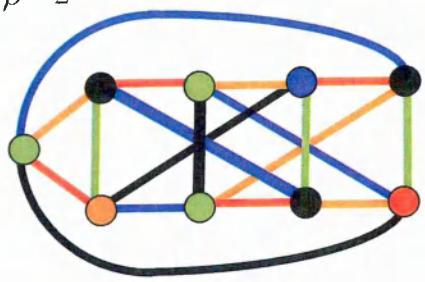
$\beta = 1$

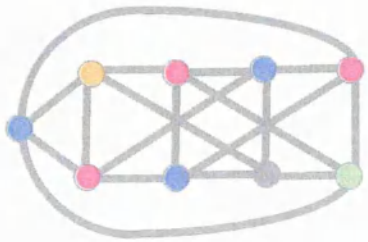




(No. 33) *HHH*24+*e*

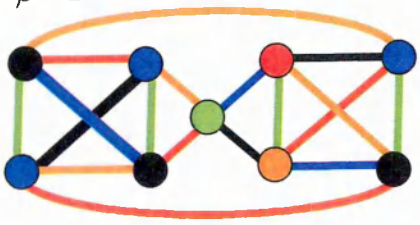
$\beta = 2$

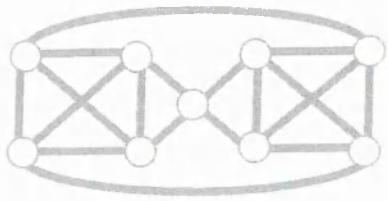




(No. 34) *HHH*25

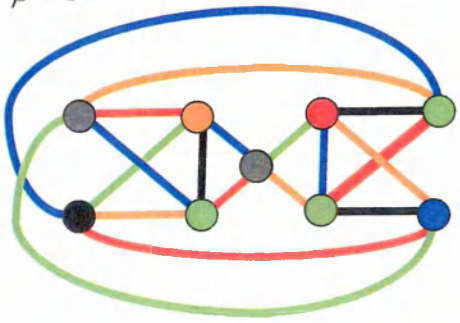
$\beta = 2$

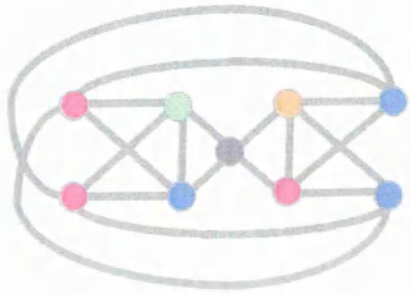




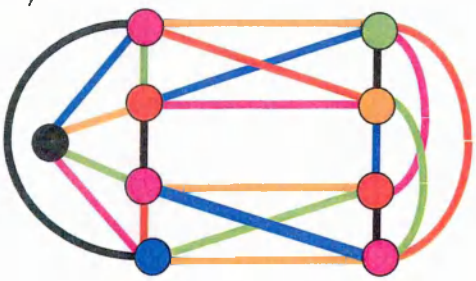
(No. 35) *HHH*26

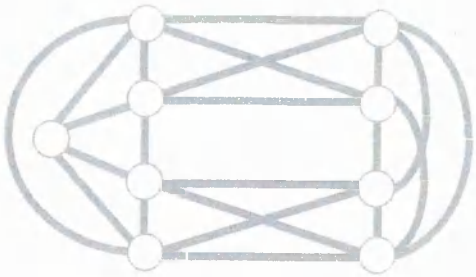
$\beta = 1$



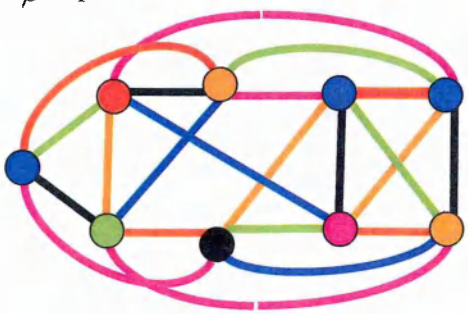


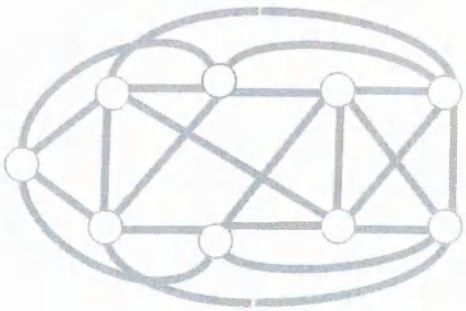
(No. 36) *HHH*27
$\beta = 1$



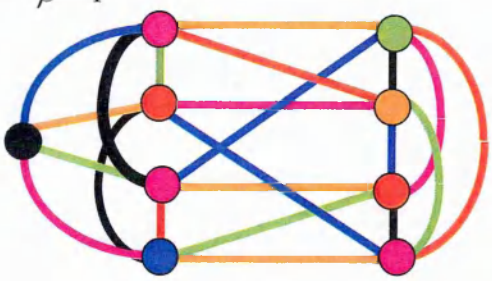


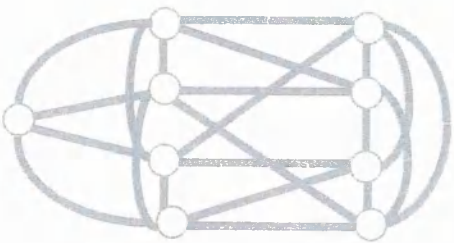
(No. 37) *HHH*28
$\beta = 1$



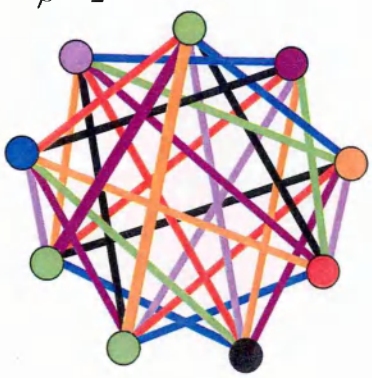


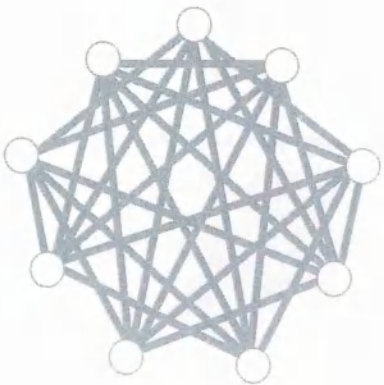
(No. 38) *HHH*29
$\beta = 1$





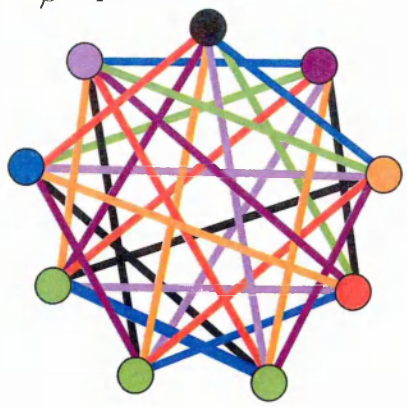
(No. 39) *HHH*30
$\beta = 2$

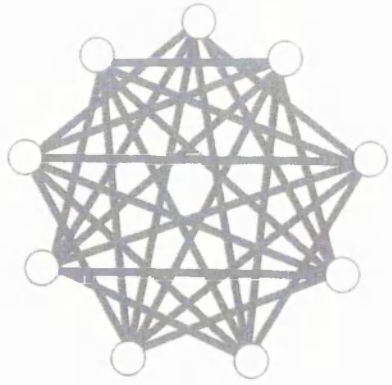




(No. 40) *HHH*31

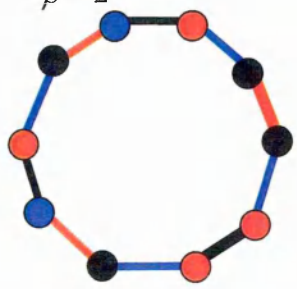
$\beta = 1$

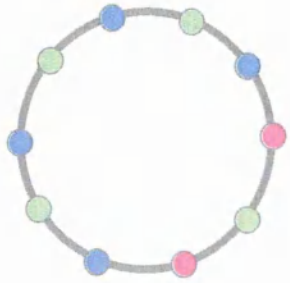




(No. 41) *HHH*32

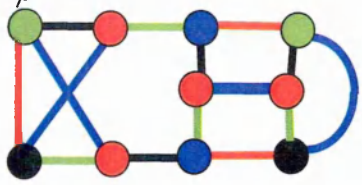
$\beta = 2$

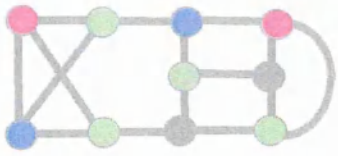




(No. 42) *HHH*33

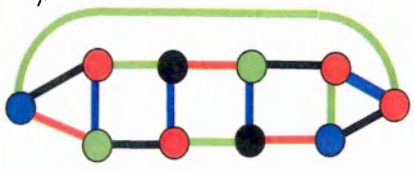
$\beta = 1$

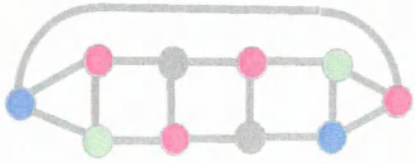




(No. 43) *HHH*34

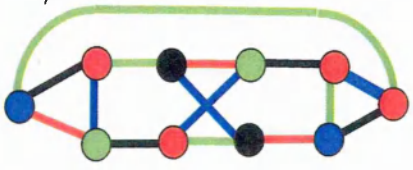
$\beta = 1$

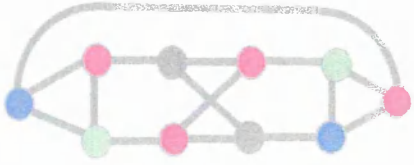




(No. 44) *HHH*35

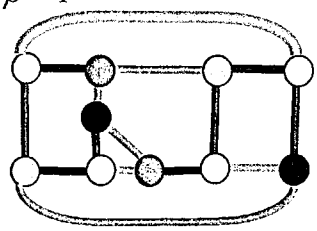
$\beta = 2$

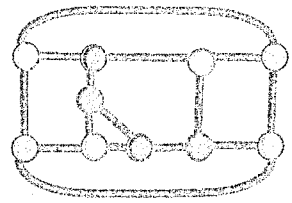




(No. 45) *HHH*36

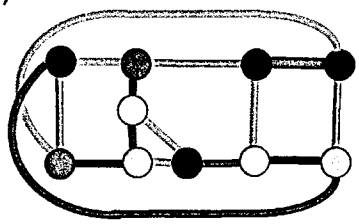
$\beta = 1$

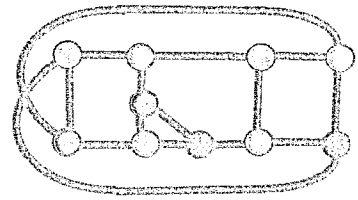




(No. 46) *HHH*37

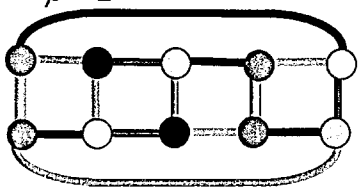
$\beta = 1$

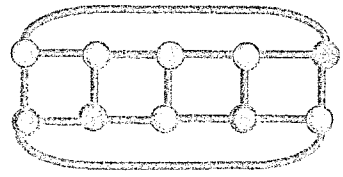




(No. 47) *HHH*38

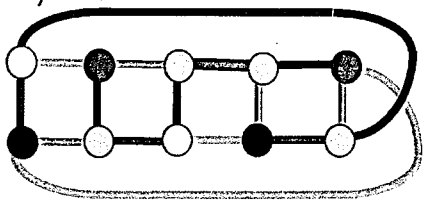
$\beta = 2$

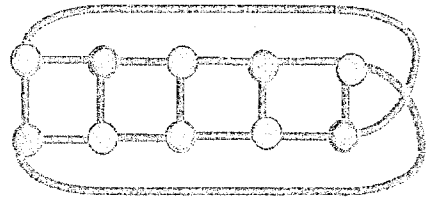




(No. 48) *HHH*39

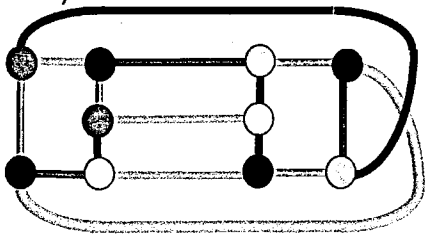
$\beta = 1$

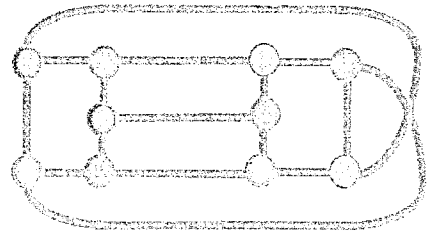




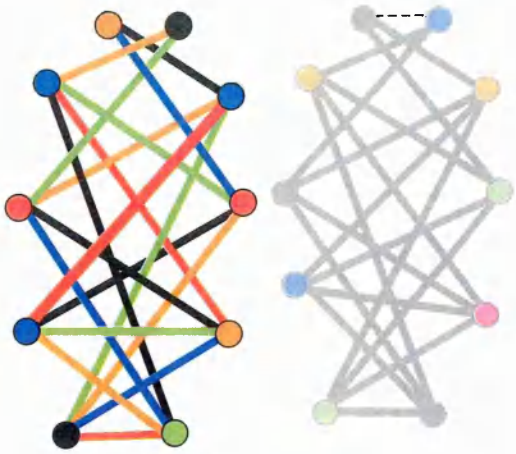
(No. 49) *HHH*40

$\beta = 1$

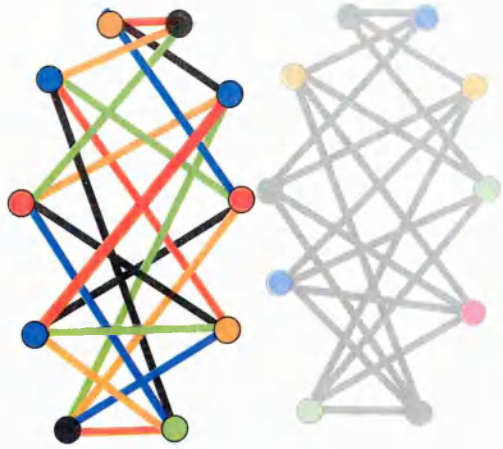




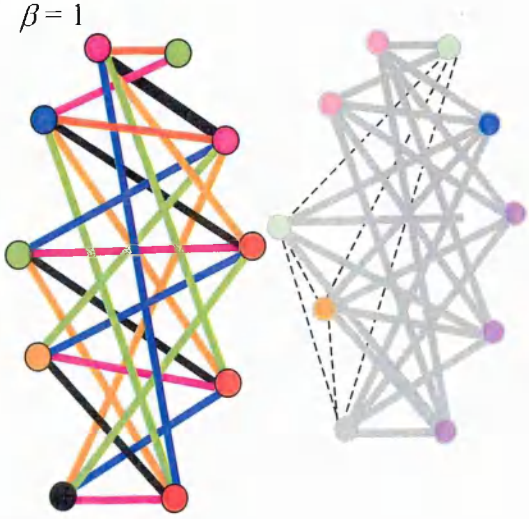
(No. 50) *HHH*41
$\beta = 1$



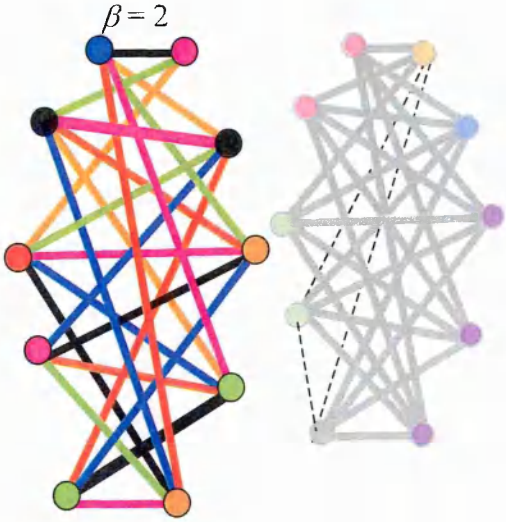
(No. 51) *HHH*41 + *e*
$\beta = 1$



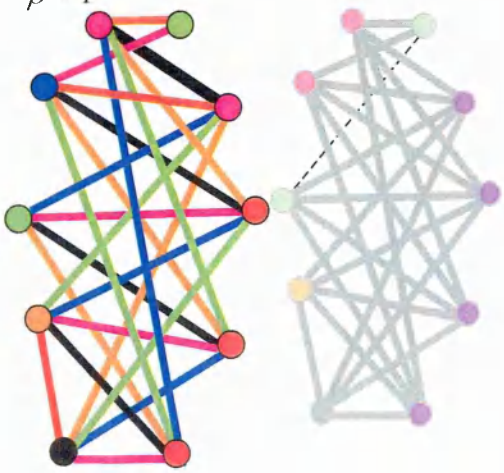
(No. 52) *HHH*42
$\beta = 1$



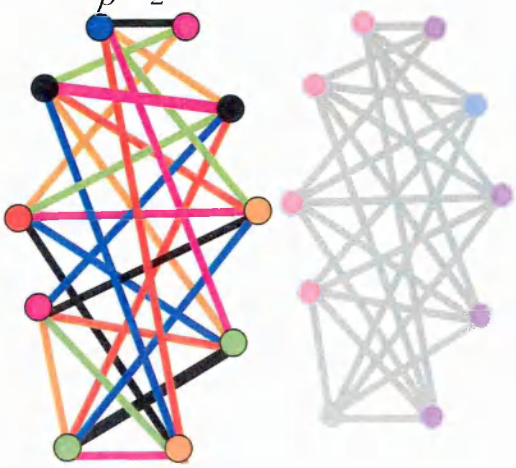
(No. 53) *HHH*42 + *e(a)*
$\beta = 2$



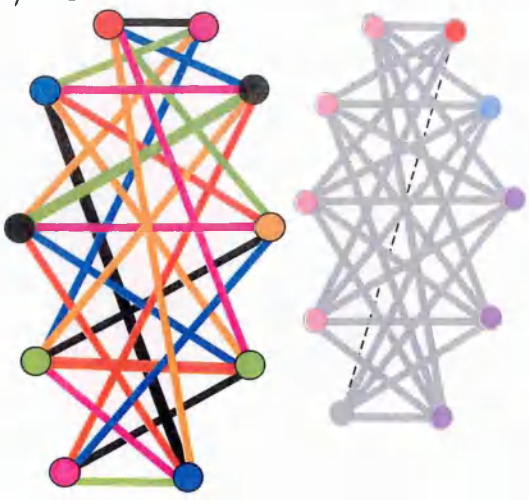
(No. 54) *HHH*42 + *e (b)*
$\beta = 1$



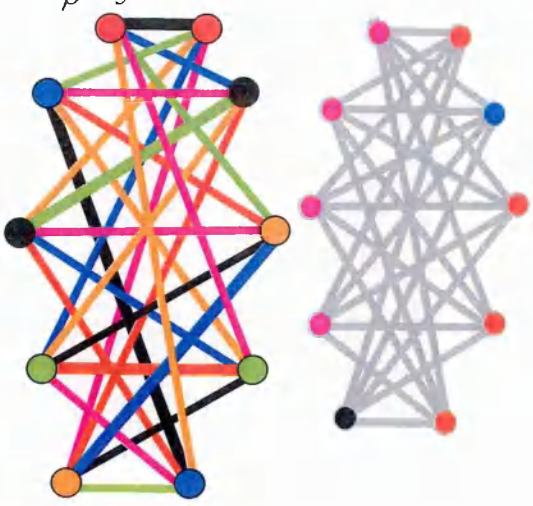
(No. 55) *HHH*42 + 2*e(a* +*b)*
$\beta = 2$



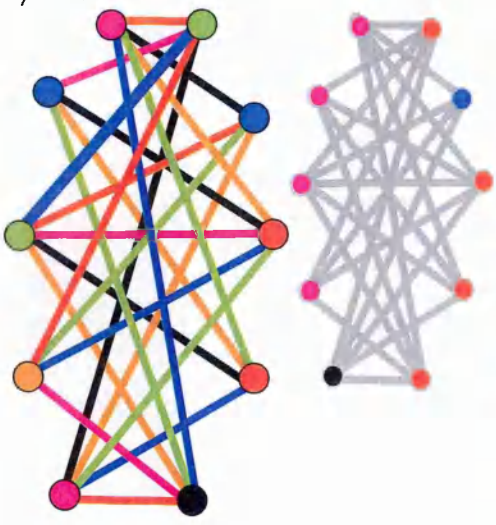
(No. 56) $HHH42 + 2e = HHH43 + e$
$\beta = 3$



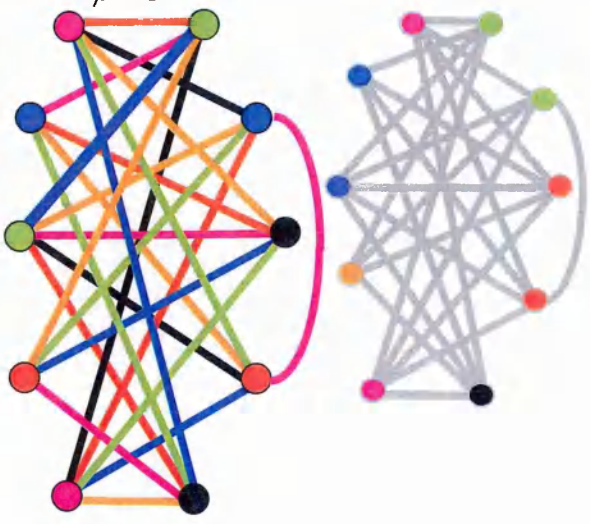
(No. 57) $HHH42 + 3e = HHH43 + 2e$
$\beta = 5$



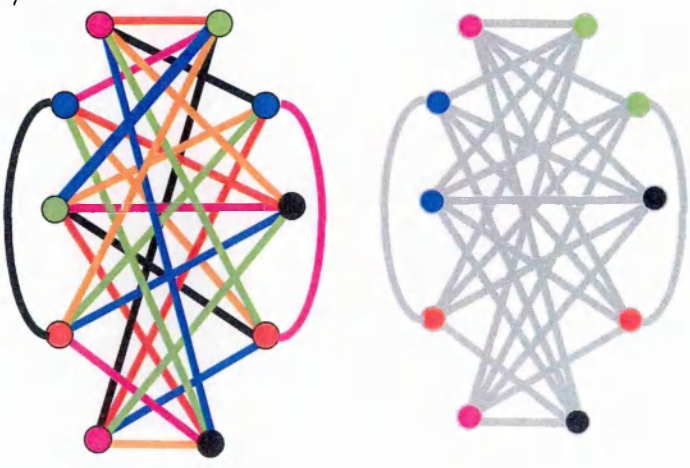
(No. 58) $HHH43$
$\beta = 1$



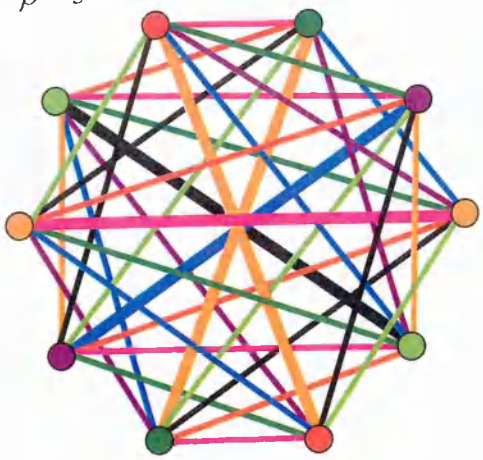
(No. 59) $HHH43+ e$
$\beta = 1$

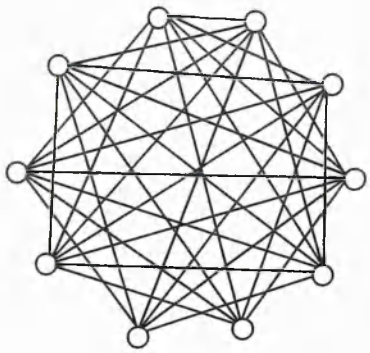


(No. 60) $HHH43+2e$
$\beta = 1$

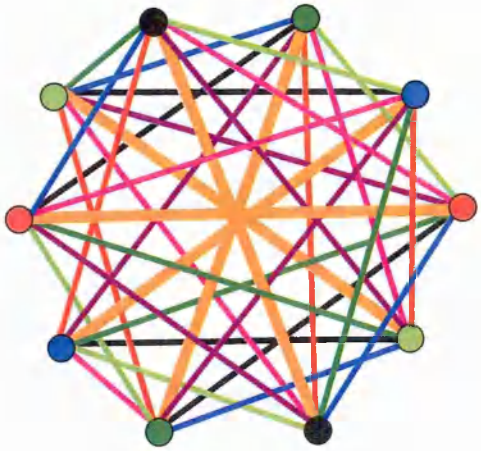


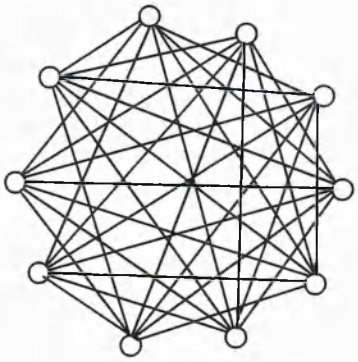
(No. 61) *HHH*44
$\beta = 5$



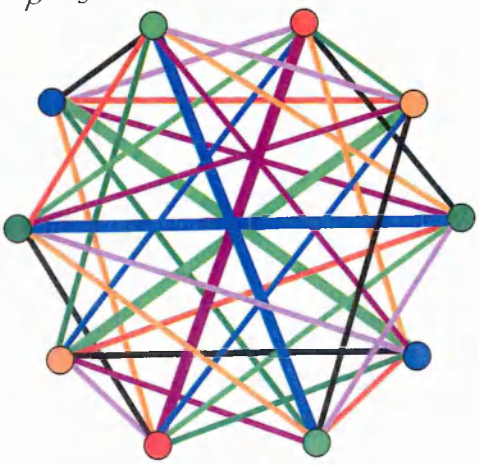


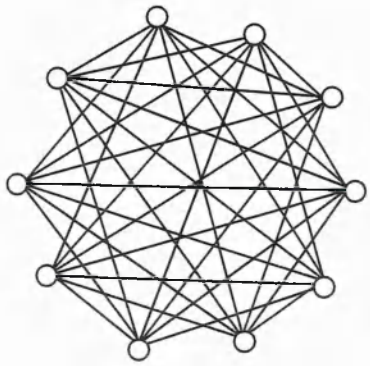
(No. 62) *HHH*45
$\beta = 5$





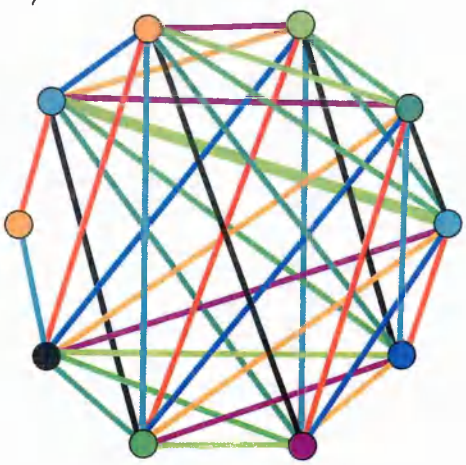
(No. 63) *HHH*46
$\beta = 5$

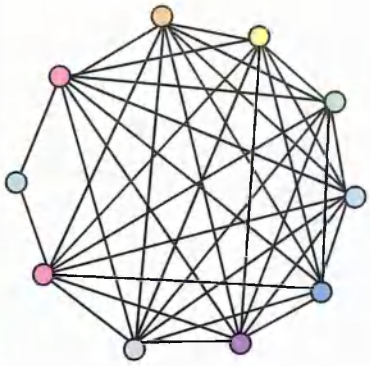




(No. 64) $HHH47$

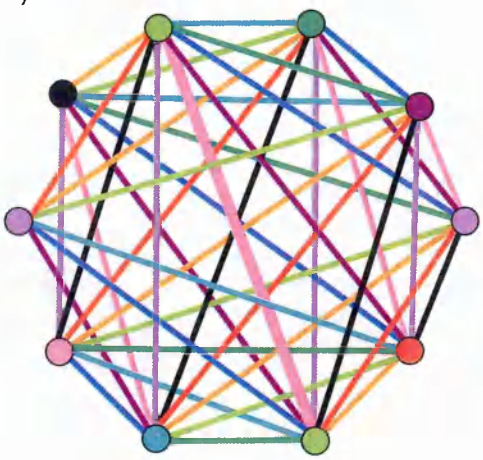
$\beta = 1$

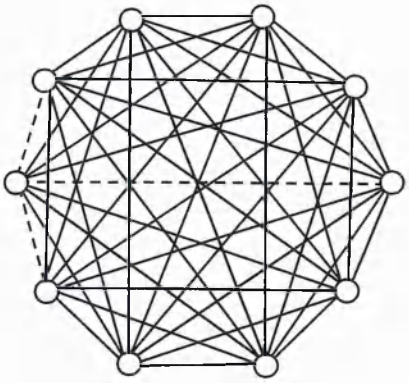




(No. 65) $HHH48$

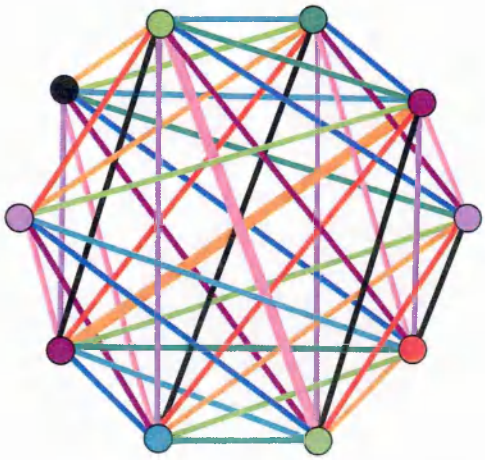
$\beta = 1$

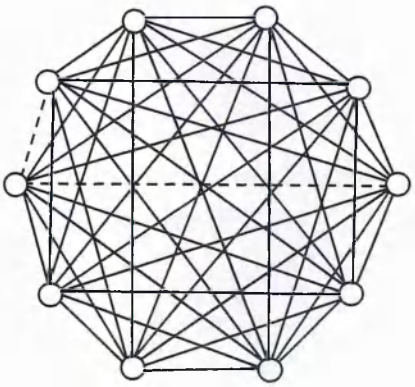




(No. 66) $HHH48 + e \;=\; HHH49 + e$

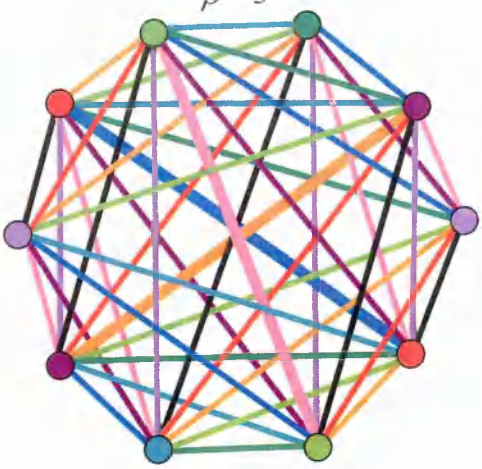
$\beta = 2$

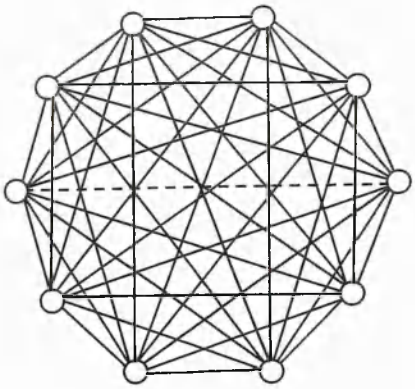




(No. 67) $HHH48{+}2e \;=\; HHH49 + 2e \;=\; HHH50 + e$

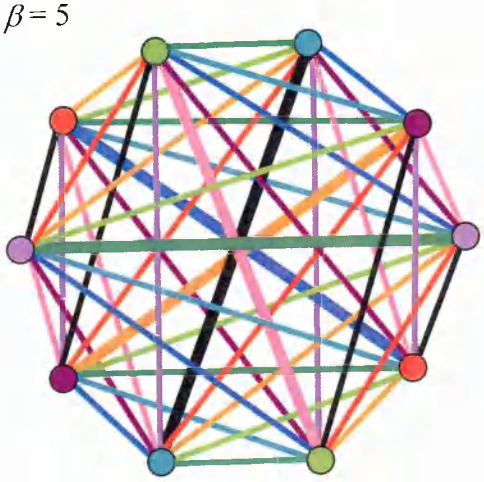
$\beta = 3$

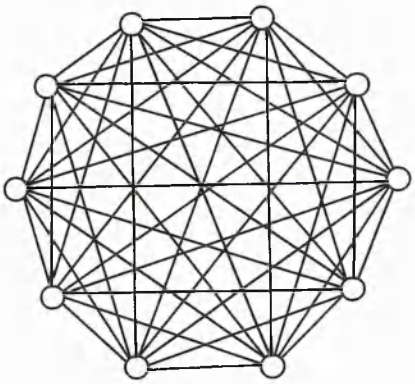




(No. 68) $HHH48+3e = HHH49 + 3e$

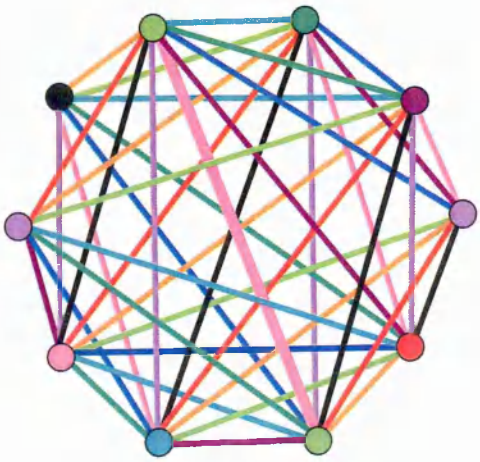
$\beta = 5$

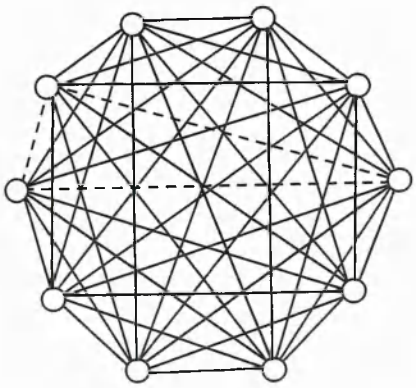




(No. 69) $HHH49$

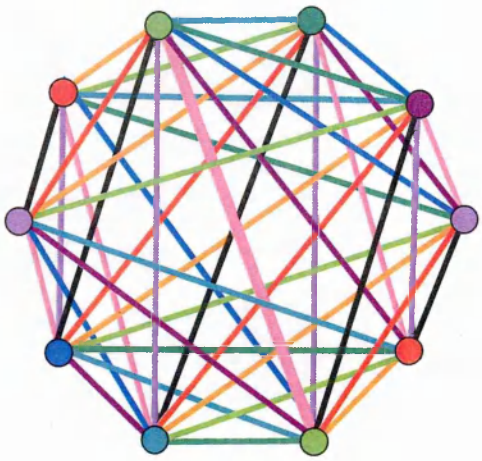
$\beta = 1$

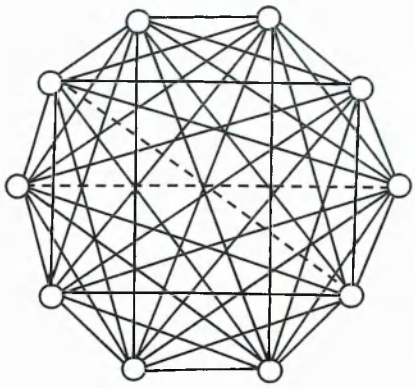




(No. 70) $HHH50$

$\beta = 1$





End of catalogue.

Appendix (6.2)

We present brief descriptions of algorithms for classifying graphs. For any graph G, we denote by $p(G)$ the number of **even components**, that is, the number of connected components with an even number of vertices. We denote by $q_{three}(G - (S_i \cup S_j))$ the components, odd and/or even, in $(G - (S_i \cup S_j))$ which have less than four vertices.

Algorithm for Regular Cubics

(1) Find a vertex colouring $\mu(V(G))$ that is non-isochromatic to any colouring previously tested. If all have been tested, the graph is type 2, therefore, stop.

(2) Check for (ordinary) conformability. If not conformable *go back to* (1).

(3) Count n_i for all i.

(4) Compare $o(G - (S_i \cup S_j))$ and $|S_j \cup S_i|$. If $o(G - (S_i \cup S_j)) > |S_j \cup S_i|$ then $\mu(V)$ is not G^*-conformable, *go back to* (1).

(5) Apply Tutte's condition to find a 1-factor in $G - S_i$ for each colour c_i. If for some colour c_i, no 1-factor can be found *go back to* (1)

(6) If $|o(G - (S_i \cup S_j)) + p(G - (S_i \cup S_j))| < (n_i + n_j)/2$, *go back to* (1).

(7) If $|q_{three}(G - (S_i \cup S_j))| < (n_i + n_j)/2$, *go back to* (1).

(8) If $|o(G - (S_i \cup S_j))| > 2q$, *or odd,* where $2q \leq \min\{n_i, n_j\}$ *go back to* (1).

(9) Find a new (untested) **set** of 1-factors, such that we have one 1-factor for each colour. If no new set can be found *go back to* (1). (This includes the case where no 1-factor can be found for some colour c_j)

(10) If in the set of 1-factors the same edge is in two 1-factors, *go back to* (9).

(12) If a set F_y is found which gives a proper total colouring: stop.

This method should result in a proper total colouring. If however an exhaustive search fails then since all possibilities have been tried then the graph is type 2. The

method for semi regular graphs is so similar we will not present it. However, since in general, graphs with degree greater than three are more likely to have just one connected components for all or most $(G - S_i - S_j)$ the method has fewer known conditions and hence though in the following algorithm the number of steps are fewer the number of operations within each step will be much longer as the general case is an NP-problem. Nevertheless, in practical terms, solutions for relatively small graphs such as those in appendix (6.1) have be achieved by this method.

Algorithm for general non cubic graphs

(1) Find a vertex colouring $\mu(V(G))$ that is non-isochromatic to any previous colouring. If all other vertex colourings have already been tested then the graph is type 2, therefore, stop.

(2) Check for (ordinary) conformability. If not conformable *go back to* (1).

(3) Find a spine-vertex colouring $\mu(V)$ with $\mu(V(G))$ that is non-isochromatic to any spine-vertex colouring that has previously been tested. If no new spine –vertex colourings exists *go back to* (1).

(4) Count n_i for all i.

(5) If $|o(G - (S_i \cup S_j))| > |(S_i \cup S_j)|$, *go back to* (3)

(6) Apply Tuttes condition to find a new (untested) set of 1-factors F_y in $G - S_i$ for each colour c_i. If no new set of 1-factors can be found, *go back to* (3).

(7) If any edge is in two 1-factors of F_y, *go back to* (6).

(8) This is a proper total colouring: stop.

Again this algorithm should result in a proper total colouring. If however an exhaustive search fails then since all possibilities have been tries then the graph is type 2. The method for semi regular graphs is so similar we will not present it.

Appendix (7.1)

We find an algorithm for finding a semi-total colouring with low beta number.

Colouring a Known Graph

The following method of colouring a known graph is based on the concept of a 'greedy algorithm'. Though it has not been proven to be the best possible method there are none, to our knowledge, that have been proven to work any better.

This method builds up a colouring using the webs of Kempe chains defined in Chapter 2. First choose a maximum value for t, *i.e.* we let $t \leq T$. Then we decide how hard we need to try to get a proper $(\Delta + T)$-total colouring, which is to say that we decide in advance how many iterations of each stage we are prepared to make before going on to the next stage, we call the number of iterations s_t for the stage t total colouring process and b_t for stage t β-colouring process. We iterate as follows from *.

*For each vertex in turn we attempt to find a (locally) total colouring with $\Delta+t$ colours with up to s_t iterations. If this fails, we find the best beta colouring that we can with $\Delta+t$ colours up to b_t iterations. When all vertices have been considered and $t = T$, we stop: but if $t < T$, we introduce the next colour $t + 1$ and begin again at *.

In more detail but nevertheless in outline we proceed as follows with the notation as above and below.

An **acceptable** edge colour for $e_{x,y}$ is one which is not on neither vertex v_x nor v_y.

An **acceptable** vertex colour for v_x is one which is not on any adjacent vertex.

An **acceptable** Kempe interchange from v_x is one which does not end at any vertex adjacent to another vertex of the other (new) colour.

An **acceptable** web from v_x is one which does not use any colour of any vertex adjacent to v_x nor does it include any vertex adjacent to v_x.

Step 0. Let t = 0. Chose a maximal vertex call it v_1 call the neighbours of v_1 : v_i where $i = 1, 2, ..., \Delta+1$. Consider each labelled vertex v_j, in numerical order, label any unlabelled neighbours v_j with the next (unused) label v_i. Eventually all vertices will be labelled.

Step 1. We now find an $(\Delta + 1)$-edge colouring and attempt to find a $(\Delta + 1)$-total colouring. Colour the edge $e_{1,i}$ with c_i as in Chapter 6. Where this is acceptable, colour vertex $v_{\Delta+1}$ with colour c_2 and colour each of the vertices v_i with c_{i+1} for all $i < \Delta+1$. (In the case of K_2 there is no colour c_{i+1} and we must go directly to step 2 . We continue with successive vertices v_i. At each vertex we first assigning acceptable colours to the edges incident with v_i then, if there are uncoloured edges, we use Kempe interchanges until a proper (local) colouring has been achieved at v_i. This can always be done *via* Vizing's theorem. We then assign acceptable vertex colours to the neighbours of v_i and then, if there are uncoloured vertices adjacent to v_i we look at a further $s_i \leq s_t$ Kempe chains until we find one which would give us a proper (local) colouring and when the first one is found we apply the said Kempe interchange and go to the next vertex v_{i+1}. If when $s_i = s_t$ we have not found a chain that would give us a proper local colouring then we apply the Kempe interchange that gave us the best beta value and go to the next vertex v_{i+1}. When all edges and vertices have been

assigned colours, if the colouring is a proper total colouring we stop. If the colouring is a semi-total colouring we go on to the next stage, first setting b=0.

Step 2.

We wish to find a proper $(\Delta + t + 1)$ total colouring. We have a subset of vertices (V_β) such that each vertex is in only if it is opposed. We arrange the vertices v_i of (V_β) in numerical order and consider the vertices with smallest i first. At each v_i we look at every available Kempe chain leading from a spine. If any acceptable interchange is found we apply it and go on to the next vertex in (V_β). If there are no acceptable Kempe interchanges, we consider every available web. We interchange the first one that is acceptable. If this has resulted in a type 1 graph we stop otherwise we go on to the nest vertex in (V_β). If however, no web is acceptable we apply any Kempe or web interchange that reduces beta providing all the new opposed vertices are in of (V_β) which we have not yet considered otherwise we do not alter anything but repeat the search w.r.t. the next vertex. This avoids including new vertices in (V_β). When we have looked at all vertices in (V_β) we let $b = b + 1$. If now, we have $b = b_t$ then we go on to the next step. If $b < b_t$ then we repeat this step for all opposed vertices.

Step 3. If the number of colours that we have been using in the last stage of step 2 is $(\Delta + t) = (\Delta + T)$ then we stop. This is the best beta colouring that we could achieve within our limitations. If however, $t < T$, then we introduce the next colour $c_{\Delta+t+1}$ as a spine at every vertex in G. Now we go back and repeat step 2 with this new set of spines. Clearly the first vertex in the new (V_β) will be assigned the colour $c_{\Delta+t+1}$.

Appendix (7.2)

We remark on a particular infinite set of nested total colourings.

Fractal Colouring

Consider the standard tessellation of seven hexagons each with a different colour. The central hexagon is never adjacent to any other hexagon of the same colour. When the piece is kept in the same orientation we can cover the infinite plane in such a way that any colour can be chosen as the central hexagon with sides adjacent to all six other colours. The dual of this graph is a regular triangulation of the plane, $\Delta = 6$. This dual is a proper vertex colouring each with just one neighbour of every other colour. This vertex colouring can be extended to several complete but different total colourings. A sample of the hexagonal tessellation and dual are shown in figure 7.2 1.

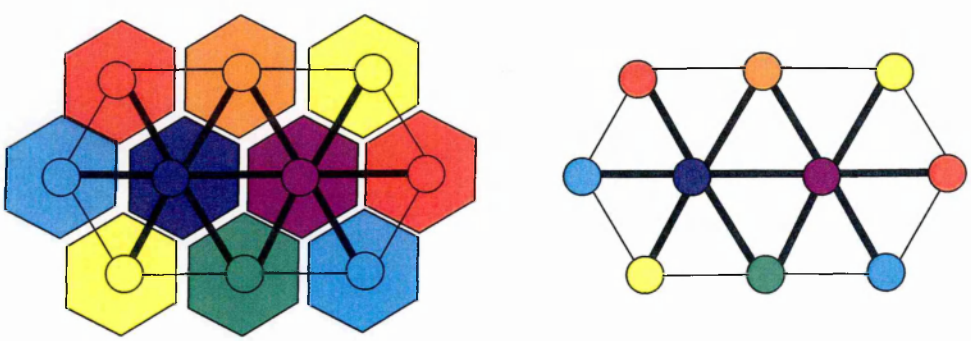
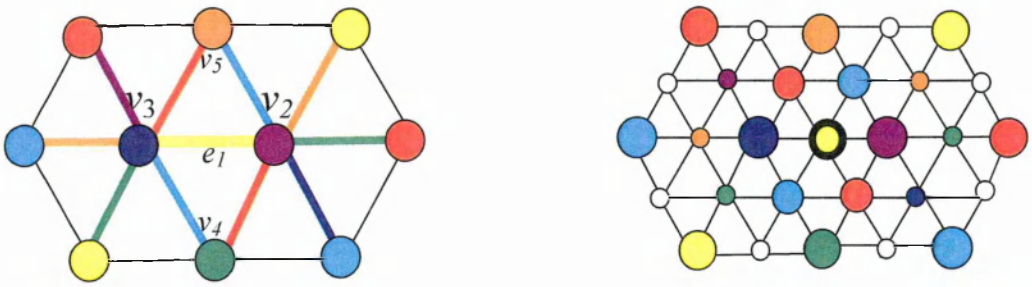


Figure 7.2.1

A **sub-triangulation** of the plane is derived from a triangulation of the plane where each of the original triangles is subdivided into four smaller triangles by joining the mid points of each edge across the plane of each triangle.

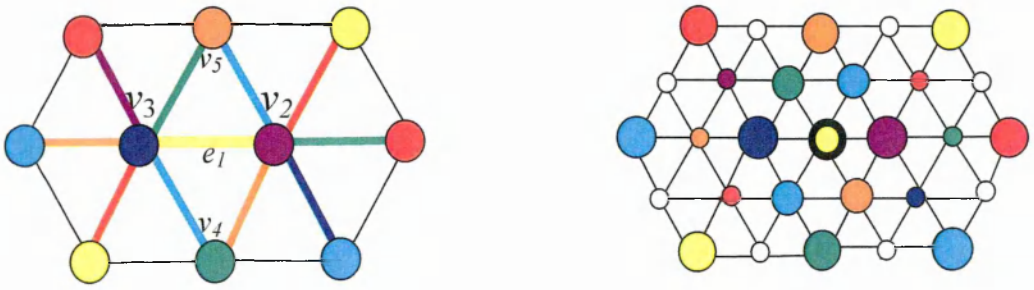
Lemma A.7.2.1 *Any total colouring of the triangulated plane, G_1, provides a proper vertex colouring of the sub-triangulation, G_2.*

Proof. Consider any total colouring of a triangulation of the plane. The vertices of the total colouring are already a proper vertex colouring of the plane. Consider such a colouring where the labels of the colours have not yet been fixed. Take any edge e_1 in G_1, call the colour c_1. Let the vertices incident with e_1 be v_2 and v_3 with colours c_2 and c_3 respectively. There are two incident triangles in the plane, call them $\{v_2, v_3$ and $v_4\}$ and $\{v_3\ v_2$ and $v_5\}$, see figures A.7.2.2 - A7.2.4. To create G_2 we replace edge e_1 with a new vertex, call it v_1 and two new edges $e_{1,2}$ and $e_{1,3}$. We do a similar operation to every other edge. Now in G_2, v_1 has neighbours $\{v_2, v_3, v_{24}, v_{25}, v_{34}, v_{35}\}$. Since the original graph had a total colouring, v_1 is not the same colour as either of the old vertices, v_2, v_3, to which it is adjacent nor is it the colour of an edges from v_2, v_3 to v_4 or v_5 in G_1, hence v_1 is not the same colour as the vertices to which it is now adjacent. Hence the colouring G_2 is a proper vertex colouring. ■



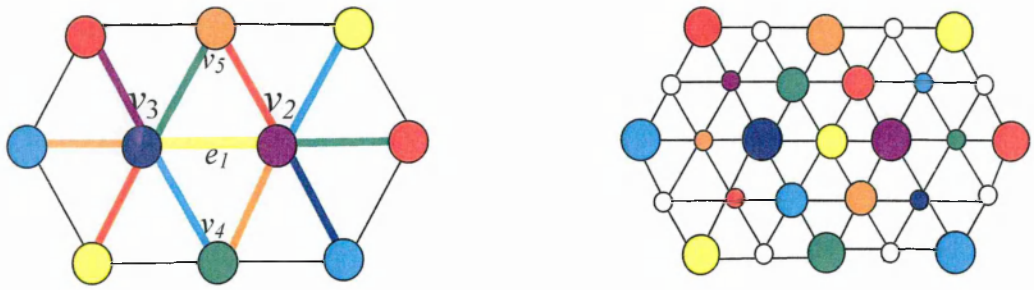
(*a*) (*b*)

Figure A.7.2.2



(*a*) (*b*)

Figure A.7.2.3



(*a*) $\theta(G_1)$ (*b*) $\mu(G_2)$.

Figure A.7.2.4

A **delete-triangulation** of the plane is the reverse of a sub-triangulation. G_{-1} is derived from a triangulation of the plane G_1, by removing every other vertex (call these, V_t) and all edges incident with them and then creating a single edge between the vertices in $(V - V_t)$ such that where v_1 is a vertex in V_t then we get a triangulation of the plane with an edge $e_{2,3}$ in G_{-1} where we had edges $e_{1,2}$ and $e_{1,3}$ in G_1.

Note that the reverse of lemma A.7.2.1 is not necessarily true. In the case of Figure A.7.2.2 the vertex colouring (*b*) only provides a total colouring for a delete-triangulation of the plane if we choose the particular vertices given in (a), the other set of vertices would give an improper colouring to the edges. For example, if we chose v_1 to be a vertex in G_{-1} then we would have two red edges and two blue edges incident with *e*.

Theorem A.7.2.2 *There is a sequence of total colourings which lead to isochromatic vertex colourings for an infinite number of sub-triangulations of the plane.*

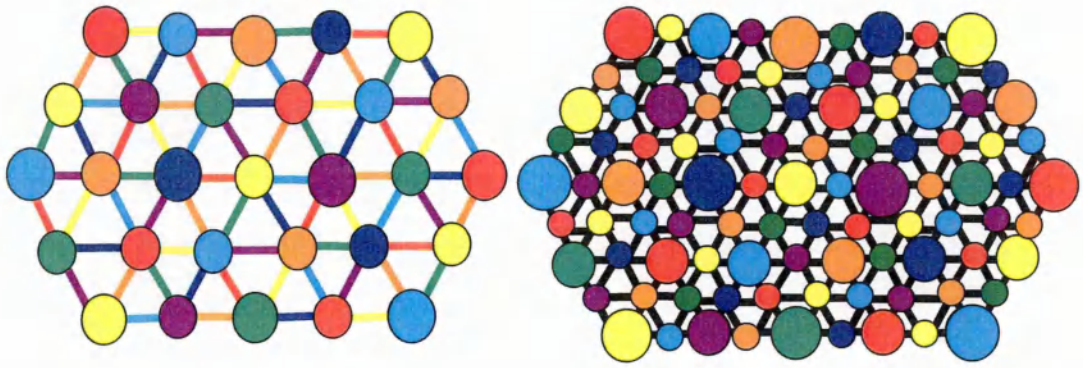
Proof. Again consider figure A.7.2.1. This is a standard seven colour vertex colouring of the triangulation of the plane G_1, which we will call $\mu(G_1)$. Note that we

can initially call any colour on a vertex, colour c_1. We can extend this fragment to the entire plane by repeating the sequence so that every horizontal has the colours in the same order $\{1, 2, 3, 4, 5, 6, 7\}$; this causes every forward slash diagonal to be $\{1, 4, 7, 3, 6, 2, 5\}$ and every backward slash diagonal to be $\{1, 3, 5, 7, 2, 4, 6\}$.

We can then total colour the edges to get the (specific) total colour in figure 7.2.4 (*a*). Call this colouring $\theta(G_1)$. From $\theta(G_1)$ we can get the vertex colouring of the sub-triangulation G_2 in figure 7.3.4 (*b*) Call it $\mu(G_2)$.
Since every vertex in $\mu(G_2)$ has neighbours of every other colour in the same order, we could re-label the colours to get our original colouring, hence $\mu(G_2)$ is isochromatic to $\mu(G_1)$. However, note that other total colourings such as those in figures 7.2.2 (*a*) and 7.2.3 (*a*) lead to vertex colourings 7.2.2 (*b*) and 7.2.3 (*b*) which are non isochromatic to $\mu(G_1)$ since there are now vertices which do not have every neighbour of a different colour.

In $\mu(G_2)$ we can call any colour c_1 on any vertex. From this vertex every horizontal has the colours in the sequence $\{1, 5, 2, 6, 3, 7, 4\}$ (which is $\{1, 4, 7, 3, 6, 2, 5\}$ reversed); every forward slash diagonal has the sequence $\{1, 6, 4, 2, 7, 5, 3\}$ (which is $\{1, 3, 5, 7, 2, 4, 6\}$ reversed); every backward slash diagonal has the sequence $\{1, 2, 3, 4, 5, 6, 7\}$.

We now total colour the edges of to $\mu(G_2)$ with the (specific) colouring in figure A.7.2.5 (*a*). Call this total colouring $\theta(G_2)$.



(*a*) $\theta(G_2)$ (*b*) $\mu(G_3)$.

Figure A.7.2.5

From this total colouring we can get the vertex colouring of the sub-triangulation in figure A.7.2.5 (*b*) Call it $\mu(G_3)$. Again we can call any colour c_1 on any vertex. Since we could re-label the colours to get our original colouring, $\mu(G_3)$ is also isomorphic to $\mu(G_1)$. From this vertex every horizontal has the colours in the sequence {1, 3, 5, 7, 2, 4, 6}; every forward slash diagonal has the sequence {1, 7, 6, 5, 4, 3, 2} (which is {1, 2, 3, 4, 5, 6, 7} reversed); every backward slash diagonal has the sequence {1, 5, 2, 6, 3, 7, 4}.

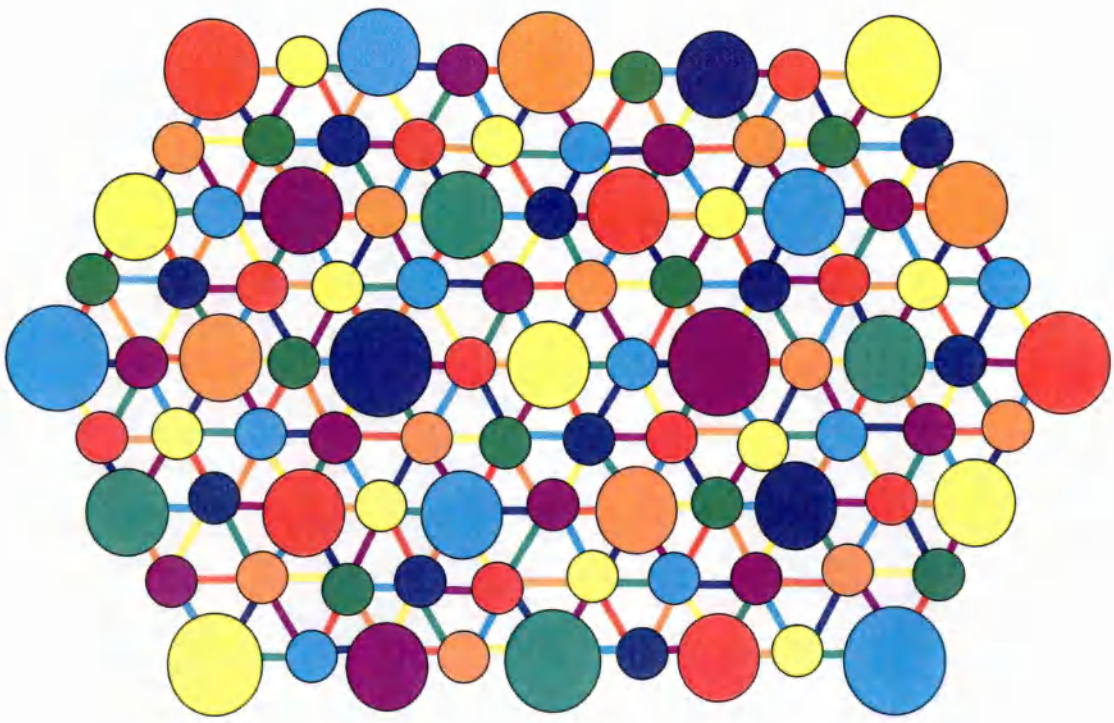


Figure A.7.2.6 $\theta(G_3)$

We now total colour the edges of $\mu(G_3)$ to get figure A.7.2.6. Call this $\theta(G_3)$. From this total colouring we can get the vertex colouring of the sub-triangulation in figure A.7.2.7 Call it $\mu(G_4)$.

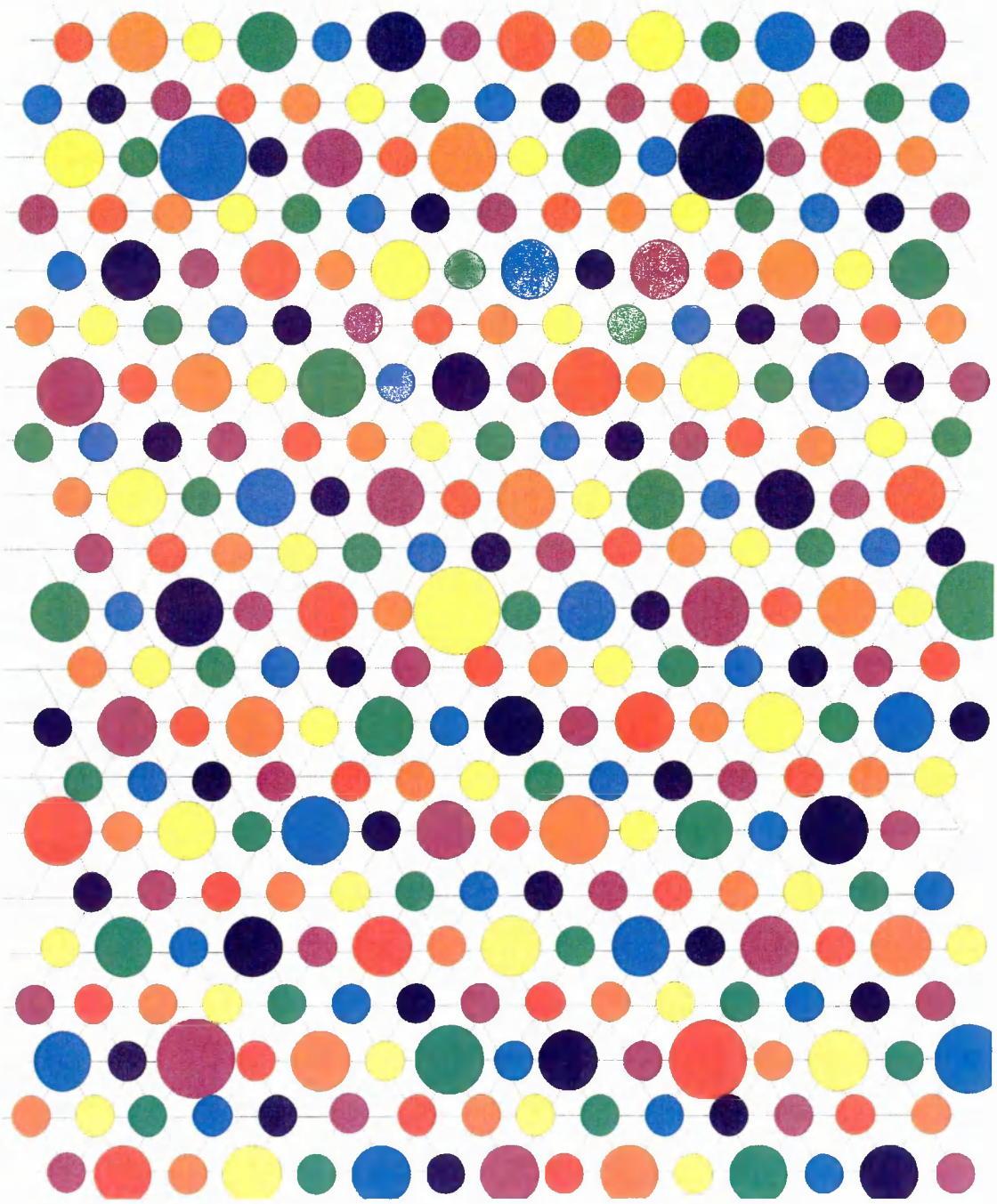


Figure A.7.2.7 $\mu(G_4)$

Again we can call any colour c_1 on any vertex. From this vertex every horizontal has the colours in the sequence $\{1, 2, 3, 4, 5, 6, 7\}$; every forward slash diagonal has the sequence $\{1, 4, 7, 3, 6, 2, 5\}$; every backward slash diagonal has the sequence$\{1, 3, 5, 7, 2, 4, 6\}$ this is identical to $\mu(G_1)$.

Since $\mu(G_4)= \mu(G_1)$ we know that we can continue to find total colourings and isomorphic vertex colourings for an infinite number of further sub-triangulations. ■

Since all the above colourings are standard vertex colourings based on a standard tiling pattern, we do not claim to have invented them. However, we believe that we are the first to have noticed that what we are looking at is an infinite set of nested total colourings.

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