



Inverse optical tomography through PDE constrained optimisation in L^∞

Article

Accepted Version

Katzourakis, N. (2019) Inverse optical tomography through PDE constrained optimisation in L^∞ SIAM Journal on Control and Optimization, 57 (6). pp. 4205-4233. ISSN 1095-7138 doi: <https://doi.org/10.1137/19M1239908> Available at <http://centaur.reading.ac.uk/87005/>

It is advisable to refer to the publisher's version if you intend to cite from the work. See [Guidance on citing](#).

To link to this article DOI: <http://dx.doi.org/10.1137/19M1239908>

Publisher: SIAM

All outputs in CentAUR are protected by Intellectual Property Rights law, including copyright law. Copyright and IPR is retained by the creators or other copyright holders. Terms and conditions for use of this material are defined in the [End User Agreement](#).

www.reading.ac.uk/centaur

CentAUR

Central Archive at the University of Reading

Reading's research outputs online

INVERSE OPTICAL TOMOGRAPHY THROUGH PDE CONSTRAINED OPTIMISATION IN L^∞

NIKOS KATZOURAKIS

ABSTRACT. Fluorescent Optical Tomography (FOT) is a new bio-medical imaging method with wider industrial applications. It is currently intensely researched since it is very precise and with no side effects for humans, as it uses non-ionising red and infrared light. Mathematically, FOT can be modelled as an inverse parameter identification problem, associated with a coupled elliptic system with Robin boundary conditions. Herein we utilise novel methods of Calculus of Variations in L^∞ to lay the mathematical foundations of FOT which we pose as a PDE-constrained minimisation problem in L^p and L^∞ .

CONTENTS

1. Introduction	1
2. Linear elliptic systems with Robin boundary conditions	5
3. Well-posedness of the direct Optical Tomography problem	8
4. The inverse problem through PDE-constrained minimisation	10
5. Kuhn-Tucker theory and Lagrange multipliers for the p -error	16
6. Kuhn-Tucker theory and Lagrange multipliers for the ∞ -error	23
Acknowledgments	27
References	27

1. INTRODUCTION

Fluorescent Optical Tomography (FOT) is a relatively new and still evolving bio-medical imaging method, which also has wider industrial applications. FOT is currently being very intensely studied, as it presents some advantages over more classical imaging methods which use X-rays, Gamma-rays, electromagnetic radiation and ultrasounds.

The goal of FOT is to reconstruct interior optical properties of an object (e.g. living tissue) by using light in the red (visible) and infrared (invisible) range. The principal current use of FOT is in medical applications (e.g. cancer tumours diagnosis and general prevention of various diseases), as well as in industrial applications (e.g. detecting structural flaws in superconductors), see e.g. [1, 3, 11, 12, 13, 25, 27, 32, 35, 36, 50, 52].

Key words and phrases. Nonlinear Inversion; Fluorescent Optical Tomography, Elliptic systems; Robin Boundary Conditions; Absolute minimisers; Calculus of Variations in L^∞ ; PDE-Constrained Optimisation; Kuhn-Tucker theory; Lagrange Multipliers.

The author has been partially financially supported by the EPSRC grant EP/N017412/1.

FOT improves on quite a few shortcomings of other popular imaging methods. The vast majority of currently available bio-imaging techniques image merely tissue structure variations created by tumours. However, since some features imaged are not specific to the presence of actual tumour cells, the unavoidable imaging of secondary effects might lead to false diagnoses. Additionally, imaging methods using X-rays and Gamma-rays actually use ionising radiation, which is harmful for humans and animals as it is potentially cancer-inducing itself. On the other hand, FOT is an imaging method which does not use harmful radiation and can be made specific to the presence of designated cell types. Therefore, FOT is more precise and with no side effects for humans.

Technically, the aim of FOT is to reconstruct the fluorophore distribution in a solid body from measurements of light intensity through detectors placed on the boundary. The highly diffusive nature of light propagation implies that in fact FOT forms a highly nonlinear and severely ill-posed inverse problem, hence mathematically it is a very challenging problem. FOT can be modelled by a coupled system of PDEs (partial differential equations) with \mathbb{C} -valued solutions and coefficients. The goal is to reconstruct a space-varying parameter in the system of PDEs in the interior of a body (e.g. living tissue).

Mathematically, FOT can be modelled as follows. Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set with C^1 boundary $\partial\Omega$ and $n \geq 3$. In medical applications $n = 3$, but from the mathematical viewpoint we may include greater dimensions without any ramifications. A fluorescent dye is injected into the body Ω and in order to determine the dye concentration $\xi = \xi(x)$, the body is illuminated by a red light source $s = s(x)$ placed on the boundary $\partial\Omega$. The wavelength of the light is adjusted to the excitation wavelength of the dye, in order to force it to fluoresce. The light diffuses inside the body, and wherever dye is present, fluorescent light in the infrared range is emitted that can then be detected again at the body surface using a camera and appropriate infrared filters. The goal is then to reconstruct the distribution $\xi = \xi(x)$ of the dye, from these obtained surface images.

Specifically, for time-periodic light sources modulated at a specific frequency, the following system of PDEs describes at any $x \in \Omega$ the \mathbb{C} -valued photon fluences $u = u(x)$ at the excitation wavelength and $v = v(x)$ at the fluorescent wavelength:

$$(1.1) \quad \begin{cases} -\operatorname{div}(A_\xi Du) + k_\xi u = S, & \text{in } \Omega, \\ -\operatorname{div}(A_\xi Dv) + k_\xi v = \xi hu, & \text{in } \Omega, \\ (A_\xi Du) \cdot \mathbf{n} + \gamma u = s, & \text{on } \partial\Omega, \\ (A_\xi Dv) \cdot \mathbf{n} + \gamma v = 0, & \text{on } \partial\Omega. \end{cases}$$

Here $u, v : \Omega \rightarrow \mathbb{C}$ are the solutions, Du, Dv are the gradients of the solutions (namely $D = (D_1, \dots, D_n)$ and $D_k \equiv \partial/\partial x_k$) and $\mathbf{n} : \partial\Omega \rightarrow \mathbb{R}^n$ is the outer unit normal vector field on the boundary, whilst the ξ -dependent coefficients A_ξ, k_ξ and the coefficients h, s, S, ξ, γ satisfy $\gamma > 0$ and

$$(1.2) \quad \begin{aligned} A_\xi &: \Omega \rightarrow \mathbb{R}_+^{n \times n}, \\ k_\xi, h, S &: \Omega \rightarrow \mathbb{C}, \\ s &: \partial\Omega \rightarrow \mathbb{C}, \\ \xi &: \Omega \rightarrow [0, \infty) \end{aligned}$$

with the real part of k_ξ being positive. We note that our general PDE notation will be either self-explanatory, or otherwise standard, as e.g. in the textbooks [24, 43]. For instance, $\mathbb{R}_+^{n \times n}$ symbolises the cone of real non-negative $n \times n$ matrices. In imaging applications where $n = 3$, the coefficients above take the following form:

$$(1.3) \quad \begin{cases} A_\xi(x) = \left(3(\mu_{ai}(x) + \mu'_s(x) + \xi(x))\right)^{-1} I_n, \\ k_\xi(x) = \mu_{ai}(x) + \xi(x) + i\omega c^{-1}, \\ h(x) = \phi(1 - i\omega\tau(x))^{-1}, \end{cases}$$

where I_n is the identity matrix in \mathbb{R}^n , the diffusion coefficient A_ξ describes the diffusion of photons, μ_{ai} is the absorption coefficient due to the endogenous chromophores, ξ is the absorption coefficient due to the exogenous fluorophore, μ'_s is the reduced scattering coefficient, ϕ is the quantum efficiency of the fluorophore, τ is the fluorophore lifetime and ω is the modulated light frequency and c the speed of light. Finally, S, s are the light sources. In applications, some authors model the problem with either boundary sources or interior light sources (see e.g. [27] versus [12, 13]). Mathematically, we may include both types of sources without difficulty.

To the best of our knowledge, although the FOT problem has been extensively studied computationally and numerically, it has not been considered from the purely analytical viewpoint. In this paper we utilise novel methods of Calculus of Variations in L^∞ in order to lay the rigorous mathematical foundations of the FOT problem. Motivated by developments underpinning the papers [38, 40, 41, 42], we pose FOT as a minimisation problem in L^∞ with PDE constraints as well as unilateral constraints, studying the direct as well as the inverse FOT problem, both in L^p for finite p and in L^∞ . Further, we derive the relevant variational inequalities in L^p for finite p and in L^∞ that the constrained minimisers satisfy, which involve (generalised) Lagrange multipliers. Additionally, we prove convergence of the corresponding L^p structures to the limiting L^∞ structures as $p \rightarrow \infty$, in a certain fashion that will become clear later.

Calculus of Variations in L^∞ is a modern area initiated by Aronsson in the 1960s (see [6]-[9]) who was the first to consider variational problems of functionals which are defined as a supremum. For a general pedagogical introduction we refer e.g. to [20, 37]. Except for the endogenous mathematical interest, L^∞ cost (or error) functionals are important for optimisation and control applications because by minimising the supremum of the cost rather than its average (as e.g. in the case of least square L^2 costs), we obtain better results as we achieve uniform smallness of the error and small area spike deviations are a priori excluded. Interesting theory and applications of L^∞ variational problems can be found e.g. in [10, 14, 15, 16, 17, 18, 19, 28, 45, 47, 48, 49].

For our purposes in this paper, it will be more convenient to rewrite (1.1) in vectorial rather than complex form because the real/vectorial form. Hence, for any complex valued function $f = f_R + if_I : \Omega \rightarrow \mathbb{C}$, we identify f with $(f_R, f_I)^\top : \Omega \rightarrow \mathbb{R}^2$ and we will consider the following generalisation of (1.1)-(1.3):

$$(1.4) \quad \begin{cases} (a) & -\operatorname{div}(A_\xi \bullet Du) + K_\xi u = S, & \text{in } \Omega, \\ (b) & -\operatorname{div}(A_\xi \bullet Dv) + K_\xi v = \xi Hu, & \text{in } \Omega, \\ (c) & (A_\xi \bullet Du)_n + \gamma u = s, & \text{on } \partial\Omega, \\ (d) & (A_\xi \bullet Dv)_n + \gamma v = 0, & \text{on } \partial\Omega. \end{cases}$$

Note that although the complex form is seemingly more neat than the tangled vectorial form, it is nonetheless more appropriate in order to apply standard PDE methods and results, a priori estimates, maximum principle results, lower semicontinuity arguments, etc, which are all stated for real-valued or vector-valued solutions and PDE coefficients. In (1.4) we have

$$(1.5) \quad A_\xi := A + r(\cdot, \xi)I_n, \quad r(x, t) := \frac{\lambda}{\kappa(x) + t}, \quad A_\xi \bullet Du := \begin{bmatrix} Du_R A_\xi \\ Du_I A_\xi \end{bmatrix},$$

with

$$(1.6) \quad K_\xi := K + \xi I_2, \quad K := \begin{bmatrix} k_R & -k_I \\ k_I & k_R \end{bmatrix}, \quad H := \begin{bmatrix} h_R & -h_I \\ h_I & h_R \end{bmatrix},$$

and

$$(1.7) \quad \begin{cases} u, v, S : \Omega \longrightarrow \mathbb{R}^2, & Du, Dv : \Omega \longrightarrow \mathbb{R}^{2 \times n}, \\ K, H : \Omega \longrightarrow \mathbb{R}^{2 \times 2}, & A : \Omega \longrightarrow \mathbb{R}_+^{n \times n}, \\ s : \partial\Omega \longrightarrow \mathbb{R}^2, & \xi : \Omega \longrightarrow [0, \infty), \\ \gamma, \lambda > 0, & \kappa : \Omega \longrightarrow (0, \infty). \end{cases}$$

We note that when dealing with derivatives of vector-valued functions, the conventions about column/row vectors are important. Therefore, we interpret u, v as column vectors and for each of the real scalar components u_R, u_I, v_R, v_I , we consider their gradients as row vectors. Similarly, for any matrix field $A = \sum_{k,l} A_{kl} e_k \otimes e_l$ (where $\{e_1, \dots, e_n\}$ is the standard Euclidean basis of column vectors and $a \otimes b := ab^\top$), it is customary to define

$$\operatorname{div}(A) := \sum_{k,l} (D_l A_{kl}) e_k,$$

namely the divergence of a matrix is the column vector of divergences of its rows. Therefore, for u we have

$$\operatorname{div}(A_\xi \bullet Du) = \begin{bmatrix} \operatorname{div}(Du_R A_\xi) \\ \operatorname{div}(Du_I A_\xi) \end{bmatrix} = \begin{bmatrix} \sum_l D_l \left(\sum_k (D_k u_R)(A_\xi)_{kl} \right) \\ \sum_l D_l \left(\sum_k (D_k u_I)(A_\xi)_{kl} \right) \end{bmatrix},$$

and similarly for v .

The table of contents gives an idea of the organisation of the material in this paper. After this Introduction, in Section 2 we delve into the study of the building stones of the FOT problem, namely of linear elliptic divergence PDE systems with Robin boundary conditions. To this end, we establish well-posedness of the direct problem for these in L^p for all $p \in [2, \infty)$ (Theorems 1-2). In Section 3 we establish the well-posedness of the direct FOT problem in the appropriate L^p spaces for all $p \in [2, \infty)$ (Theorem 3). In Section 4 we begin the study of the inverse FOT problem as a constrained minimisation problem with PDE and unilateral constraints, proving the existence of minimisers in L^p for all $p \in [2, \infty]$ and the convergence of the L^p minimisers to the L^∞ minimiser as $p \rightarrow \infty$ (Theorem 5). In Section 5 we establish the existence of generalised Lagrange multipliers to the L^p constrained minimisation problem and the relevant variational inequalities, by invoking the infinite-dimensional counterpart of the Kuhn-Tucker theory (Theorem 8). Finally, in Section 6 we establish the corresponding results for the extreme case of variational inequalities in L^∞ (Theorem 15). Let us conclude with the note that in the most recent paper [39] an alternative approach has been followed to optical

tomography which is mathematically distinct but of less relevance to applications, since the diffusion coefficients are assumed not to depend on the parameter ξ .

2. LINEAR ELLIPTIC SYSTEMS WITH ROBIN BOUNDARY CONDITIONS

In this section we begin with an auxiliary result of independent interest, namely the well-posedness of general linear divergence systems with Robin boundary conditions. Below we start with the case of the L^2 theory, which effectively is an application of the Lax Milgram theorem (see e.g. [24]).

Theorem 1 (Well-posedness in $W^{1,2}$). *Let $\Omega \Subset \mathbb{R}^n$ be a domain with C^1 boundary and let also $\mathbf{n} : \partial\Omega \rightarrow \mathbb{R}^n$ be the outer unit normal vector field. Consider the next boundary value problem with Robin boundary conditions*

$$(2.1) \quad \begin{cases} -\operatorname{div}(\mathbf{B} \cdot \mathbf{D}u) + Lu = f - \operatorname{div}F, & \text{in } \Omega, \\ (\mathbf{B} \cdot \mathbf{D}u - F)\mathbf{n} + \gamma u = g, & \text{on } \partial\Omega, \end{cases}$$

where the coefficients satisfy $\mathbf{B} : \Omega \rightarrow \mathbb{R}_+^{n \times n}$, $L : \Omega \rightarrow \mathbb{R}^{2 \times 2}$, $f : \Omega \rightarrow \mathbb{R}^2$, $F : \Omega \rightarrow \mathbb{R}^{2 \times n}$, $g : \partial\Omega \rightarrow \mathbb{R}^2$ and $\gamma > 0$. We suppose that there exists $\beta_0 > 0$ such that

$$(2.2) \quad \begin{cases} \mathbf{B} \in L^\infty(\Omega; \mathbb{R}_+^{n \times n}), & \mathbf{B} \cdot \mathbf{D}u := \begin{bmatrix} \mathbf{D}u_R \mathbf{B} \\ \mathbf{D}u_I \mathbf{B} \end{bmatrix}, & \sigma(\mathbf{B}) \subseteq \left[\beta_0, \frac{1}{\beta_0} \right], \\ L \in L^\infty(\Omega; \mathbb{R}^{2 \times 2}), & L := \begin{bmatrix} l_R & -l_I \\ l_I & l_R \end{bmatrix}, & l_R \geq \beta_0 \end{cases}$$

and also that

$$(2.3) \quad f \in L^2(\Omega; \mathbb{R}^2), \quad F \in L^2(\Omega; \mathbb{R}^{2 \times n}), \quad g \in L^2(\partial\Omega; \mathbb{R}^2).$$

Then, (2.1) has a unique weak solution in $W^{1,2}(\Omega; \mathbb{R}^2)$ satisfying

$$(2.4) \quad \begin{cases} \int_{\Omega} [\mathbf{B} : (\mathbf{D}u^\top \mathbf{D}\psi) + (Lu) \cdot \psi] d\mathcal{L}^n + \int_{\partial\Omega} [\gamma u \cdot \psi] d\mathcal{H}^{n-1} \\ = \int_{\Omega} [f \cdot \psi + F : \mathbf{D}\psi] d\mathcal{L}^n + \int_{\partial\Omega} [g \cdot \psi] d\mathcal{H}^{n-1}, \end{cases}$$

for all $\psi \in W^{1,2}(\Omega; \mathbb{R}^2)$. In addition, there exists $C > 0$ depending only on the coefficients and the domain such that

$$(2.5) \quad \|u\|_{W^{1,2}(\Omega)} \leq C \left(\|f\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \right).$$

In (2.4), the notation “:” symbolises the Euclidean (Frobenius) inner product in the matrix space $\mathbb{R}^{n \times n}$ and “.” the Euclidean inner product in \mathbb{R}^2 .

Proof. As we have already mentioned, the aim is to apply of the Lax Milgram theorem. (Note that the matrix L involved in the zeroth order term is not symmetric, therefore this is not a direct consequence of the Riesz representation theorem.) To this end, we define the bilinear form

$$\mathbb{B} : W^{1,2}(\Omega; \mathbb{R}^2) \times W^{1,2}(\Omega; \mathbb{R}^2) \longrightarrow \mathbb{R}$$

by setting

$$\mathbb{B}[u, \psi] := \int_{\Omega} [\mathbf{B} : (\mathbf{D}u^\top \mathbf{D}\psi) + (Lu) \cdot \psi] d\mathcal{L}^n + \int_{\partial\Omega} [\gamma u \cdot \psi] d\mathcal{H}^{n-1}.$$

Since B, L are L^∞ , we immediately have by Hölder inequality that

$$|\mathbb{B}[u, \psi]| \leq C \|u\|_{W^{1,2}(\Omega)} \|\psi\|_{W^{1,2}(\Omega)}$$

for some $C > 0$ and all $u, \psi \in W^{1,2}(\Omega; \mathbb{R}^2)$. Further, since

$$\begin{aligned} (Lu) \cdot u &= [u_R, u_I] \begin{bmatrix} l_R & -l_I \\ l_I & l_R \end{bmatrix} \begin{bmatrix} u_R \\ u_I \end{bmatrix} \\ &= l_R (u_R)^2 + l_R (u_I)^2 \\ &= l_R |u|^2 \\ &\geq \beta_0 |u|^2, \end{aligned}$$

we have that

$$\begin{aligned} \mathbb{B}[u, u] &\geq \beta_0 \left(\|Du\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 \right) + \gamma \|u\|_{L^2(\partial\Omega)}^2 \\ &\geq \beta_0 \|u\|_{W^{1,2}(\Omega)}^2, \end{aligned}$$

for any $u \in W^{1,2}(\Omega; \mathbb{R}^2)$. Hence, the bilinear form \mathbb{B} is bi-continuous and coercive, therefore the conditions of the Lax-Milgram theorem are satisfied. Thus, for any functional $\Phi \in (W^{1,2}(\Omega; \mathbb{R}^2))^*$, there exists a unique $u \in W^{1,2}(\Omega; \mathbb{R}^2)$ such that

$$\mathbb{B}[u, \psi] = \langle \Phi, \psi \rangle, \quad \forall \psi \in W^{1,2}(\Omega; \mathbb{R}^2).$$

To conclude, it suffices to show that

$$\langle \Phi, \psi \rangle := \int_{\Omega} [f \cdot \psi + F : D\psi] d\mathcal{L}^n + \int_{\partial\Omega} [g \cdot \psi] d\mathcal{H}^{n-1}$$

indeed defines an element of the dual space $(W^{1,2}(\Omega; \mathbb{R}^2))^*$, and also establish the a priori estimate. Indeed, by the trace theorem in $W^{1,2}(\Omega; \mathbb{R}^2)$, there is a $C > 0$ depending on Ω which allows to estimate

$$\begin{aligned} |\langle \Phi, \psi \rangle| &\leq \left(\|f\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)} \right) \|\psi\|_{W^{1,2}(\Omega)} + \|g\|_{L^2(\partial\Omega)} \|\psi\|_{L^2(\partial\Omega)} \\ &\leq C \left(\|f\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \right) \|\psi\|_{W^{1,2}(\Omega)}, \end{aligned}$$

which shows that Φ is indeed a bounded linear functional. The choice of $\psi := u$ together with Young inequality with $\varepsilon > 0$, gives

$$|\langle \Phi, u \rangle| \leq \frac{C^2}{4\varepsilon} \left(\|f\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \right)^2 + \varepsilon \|u\|_{W^{1,2}(\Omega)}^2.$$

When combining the above estimate with the lower bound for $\mathbb{B}[u, u]$, we conclude by choosing $\varepsilon < \beta_0/2$. \square

Note that in the above proof we have employed the common practice of denoting by C a generic constant whose value might change from step to step in an estimate. This practice will be utilised in the sequel freely. Now we show that the obtained unique weak solution to (2.1) is in fact more regular if the coefficients permit it.

Theorem 2 (Well-posedness in $W^{1,p}$). *In the setting of Theorem 1, consider again the boundary value problem (2.1) with Robin boundary conditions. In addition to the assumptions in Theorem 1, we also suppose that*

$$B \in VMO(\mathbb{R}^n; \mathbb{R}_+^{n \times n}), \quad f \in L^{\frac{np}{n+p}}(\Omega; \mathbb{R}^2), \quad F \in L^p(\Omega; \mathbb{R}^{2 \times n}), \quad g \in L^p(\partial\Omega; \mathbb{R}^2),$$

for some

$$p > \frac{2n}{n-2}.$$

Then, the unique weak solution of (2.1) lies in the space $W^{1,p}(\Omega; \mathbb{R}^2)$. In addition, there exists $C > 0$ depending only on the coefficients, the domain and p such that

$$(2.6) \quad \|u\|_{W^{1,p}(\Omega)} \leq C \left(\|f\|_{L^{\frac{np}{n+p}}(\Omega)} + \|F\|_{L^p(\Omega)} + \|g\|_{L^p(\partial\Omega)} \right).$$

Note that the space $VMO(\mathbb{R}^n)$ is the standard space of (scalar) functions of vanishing mean oscillation, defined as the closure of the space of continuous functions $C_0^0(\mathbb{R}^n)$ vanishing at infinity in the space $BMO(\mathbb{R}^n)$ of bounded mean oscillation (see e.g. [24]) and $VMO(\mathbb{R}^n; \mathbb{R}_+^{n \times n})$ is the convex set of those positive matrix-valued maps with VMO components. The stronger assumption of the leading matrix coefficient being VMO instead of merely L^∞ is required in order to obtain the higher integrability result stated above, which in turn relies on an estimate which is known to be true only under this stronger assumption.

Proof. The key ingredient is to apply the well-know estimate for the Robin boundary value problem for linear divergence elliptic equations which has the exact same form as (2.6), but applies to the scalar version of the problem (2.1) with $L \equiv 0$, see [5, 21, 23, 29, 33, 44, 46]. Hence, we need some arguments to show that it is still true in the more general case of (2.1). To this end, we rewrite (2.1) component-wise as

$$\begin{cases} -\operatorname{div}(Du_R B) = \left\{ f_R - (l_R u_R - l_I u_I) \right\} - \operatorname{div} F_R, & \text{in } \Omega, \\ (Du_R B - F_R) \cdot \mathbf{n} + \gamma u_R = g_R, & \text{on } \partial\Omega, \\ \\ -\operatorname{div}(Du_I B) = \left\{ f_I - (l_R u_I + l_I u_R) \right\} - \operatorname{div} F_I, & \text{in } \Omega, \\ (Du_I B - F_I) \cdot \mathbf{n} + \gamma u_I = g_I, & \text{on } \partial\Omega. \end{cases}$$

By applying the scalar estimate to the each of the boundary value problems separately, we have

$$(2.7) \quad \|u_R\|_{W^{1,p}(\Omega)} \leq C \left(\|f_R\|_{L^{\frac{np}{n+p}}(\Omega)} + \|F_R\|_{L^p(\Omega)} + \|g_R\|_{L^p(\partial\Omega)} + \|L\|_{L^\infty(\Omega)} \|u\|_{L^{\frac{np}{n+p}}(\Omega)} \right)$$

and also

$$(2.8) \quad \|u_I\|_{W^{1,p}(\Omega)} \leq C \left(\|f_I\|_{L^{\frac{np}{n+p}}(\Omega)} + \|F_I\|_{L^p(\Omega)} + \|g_I\|_{L^p(\partial\Omega)} + \|L\|_{L^\infty(\Omega)} \|u\|_{L^{\frac{np}{n+p}}(\Omega)} \right).$$

Note now that since by assumption $p > \frac{2n}{n-2}$, we have

$$2 < \frac{np}{n+p} < p.$$

Hence, by the L^p interpolation inequalities, we can estimate

$$\|u\|_{L^{\frac{np}{n+p}}(\Omega)} \leq \|u\|_{L^2(\Omega)}^\lambda \|u\|_{L^p(\Omega)}^{1-\lambda}, \quad \text{where } \lambda = \frac{2p}{n(p-2)}.$$

By the Young inequality (for $a, b \geq 0$, $\varepsilon > 0$, $r > 1$ and $r/(r-1) = r'$)

$$(2.9) \quad ab \leq \left\{ \frac{r-1}{r} (\varepsilon r)^{\frac{1}{1-r}} \right\} b^{\frac{r}{r-1}} + \varepsilon a^r,$$

for the choice $r := (1-\lambda)^{-1}$, we have

$$r = \frac{n(p-2)}{p(n-2)-2n}, \quad \frac{r}{r-1} = \frac{n(p-2)}{2p}, \quad 1-\lambda = \frac{n(p-2)}{p(n-2)-2n},$$

and hence we can further estimate

$$(2.10) \quad \begin{aligned} \|u\|_{L^{\frac{np}{n+p}}(\Omega)} &\leq \|u\|_{L^2(\Omega)}^\lambda \|u\|_{L^p(\Omega)}^{1-\lambda} \\ &= \left(\|u\|_{L^2(\Omega)} \right)^{\frac{2p}{n(p-2)}} \left(\|u\|_{L^p(\Omega)} \right)^{\frac{p(n-2)-2n}{n(p-2)}} \\ &\leq \left\{ \frac{r-1}{r} (\varepsilon r)^{\frac{1}{1-r}} \right\} \left(\|u\|_{L^2(\Omega)} \right)^{\frac{2p}{n(p-2)} \frac{r}{r-1}} + \left(\|u\|_{L^p(\Omega)} \right)^{\frac{p(n-2)-2n}{n(p-2)} r} \\ &= \left\{ \frac{2p}{n(p-2)} \left(\frac{\varepsilon n(p-2)}{p(n-2)-2n} \right)^{-\frac{p(n-2)-2n}{2p}} \right\} \|u\|_{L^2(\Omega)} + \varepsilon \|u\|_{L^p(\Omega)} \\ &=: C(\varepsilon, p, n) \|u\|_{L^2(\Omega)} + \varepsilon \|u\|_{L^p(\Omega)}. \end{aligned}$$

By (2.7), (2.8) and (2.10), by choosing $\varepsilon > 0$ small enough, we infer that

$$\|u\|_{W^{1,p}(\Omega)} \leq C \left(\|f\|_{L^{\frac{np}{n+p}}(\Omega)} + \|F\|_{L^p(\Omega)} + \|g\|_{L^p(\partial\Omega)} + \|u\|_{L^2(\Omega)} \right).$$

The desired estimate (2.6) ensues by combining the above estimate with our earlier $W^{1,2}$ estimate (2.5) from Theorem 1, together with Hölder inequality and the fact that $\min\{p, \frac{np}{n+p}\} > 2$. The theorem has been established. \square

3. WELL-POSEDNESS OF THE DIRECT OPTICAL TOMOGRAPHY PROBLEM

In this section we utilise the well-posedness results of Section 2 to show that the direct problem of Fluorescent Optical Tomography is well posed.

Theorem 3 (Well-posedness of the direct FOT problem). *In the setting of Section 1, consider the boundary value problem (1.4) and suppose that the coefficients $A, K, H, \xi, r, s, S, \kappa, \lambda, \gamma$ satisfy (1.5)-(1.7), where $\Omega \Subset \mathbb{R}^n$ is a domain with C^1 boundary and $n \geq 3$.*

In addition, we also suppose that there exists $a_0 > 0$ such that

$$(3.1) \quad \begin{cases} A \in UC(\mathbb{R}^n; \mathbb{R}_+^{n \times n}), & K, H \in L^\infty(\Omega; \mathbb{R}^{2 \times 2}), \\ \kappa \in C(\overline{\Omega}), & k_R, \kappa \geq a_0 \text{ on } \Omega, \quad \gamma, \lambda > 0, \end{cases}$$

where “ UC ” is the space of bounded uniformly continuous functions. We further assume that for some $m > n$ we have

$$(3.2) \quad S \in L^{\frac{nm}{2n+m}}(\Omega; \mathbb{R}^2), \quad s \in L^{\frac{m}{2}}(\partial\Omega; \mathbb{R}^2)$$

and also

$$p > \max \left\{ n, \frac{2n}{n-2} \right\}.$$

Then, for any $M > 0$ and

$$\xi \in C(\overline{\Omega}; [0, M]),$$

the boundary value problem (1.4) has a unique weak solution

$$(u, v) \in W^{1, \frac{m}{2}}(\Omega; \mathbb{R}^2) \times W^{1, p}(\Omega; \mathbb{R}^2)$$

which satisfies

(3.3)

$$\begin{cases} (a) & \int_{\Omega} [A_{\xi} : (Du^{\top} D\phi) + (K_{\xi}u - S) \cdot \phi] d\mathcal{L}^n + \int_{\partial\Omega} [(\gamma u - s) \cdot \phi] d\mathcal{H}^{n-1} = 0, \\ (b) & \int_{\Omega} [A_{\xi} : (Dv^{\top} D\psi) + (K_{\xi}v - \xi H u) \cdot \psi] d\mathcal{L}^n + \int_{\partial\Omega} [\gamma u \cdot \psi] d\mathcal{H}^{n-1} = 0, \end{cases}$$

for all test functions

$$(\phi, \psi) \in W^{1, \frac{m}{m-2}}(\Omega; \mathbb{R}^2) \times W^{1, \frac{p}{p-1}}(\Omega; \mathbb{R}^2).$$

In addition, there exists $C > 0$ depending only on M , the coefficients, p and the domain such that

$$(3.4) \quad \begin{cases} (a) & \|u\|_{W^{1, \frac{m}{2}}(\Omega)} + \|u\|_{L^m(\Omega)} \leq C \left(\|S\|_{L^{\frac{nm}{2n+m}}(\Omega)} + \|s\|_{L^{\frac{m}{2}}(\partial\Omega)} \right), \\ (b) & \|v\|_{W^{1, p}(\Omega)} + \|v\|_{C^{0, 1-\frac{p}{p}}(\bar{\Omega})} \leq C \|\xi\|_{L^\infty(\Omega)} \|u\|_{L^m(\Omega)}. \end{cases}$$

Proof. The goal is to apply Theorems 1-2. To this end, we begin by showing that under (1.5)-(1.7) and (3.1) the diffusion matrix A_{ξ} satisfies the assumptions of these results. Since

$$A_{\xi}(x) = A(x) + r(x, \xi)I_n, \quad r(x, t) = \frac{\lambda}{\kappa(x) + t}$$

and

$$\frac{\lambda}{\|\kappa\|_{L^\infty(\Omega)} + M} \leq r(x, \xi(x)) \leq \frac{\lambda}{a_0}$$

the positive bounded uniformly continuous function

$$r(\cdot, \xi(\cdot)) : \bar{\Omega} \ni x \mapsto r(x, \xi(x)) \in (0, \infty)$$

can be extended to a positive bounded uniformly continuous function $\tilde{r} : \mathbb{R}^n \rightarrow (0, \infty)$ with the same upper and lower bounds as those of $r(\cdot, \xi(\cdot))$. Then, since A is bounded and uniformly continuous on \mathbb{R}^n with values in $\mathbb{R}_+^{n \times n}$, the matrix valued mapping

$$\tilde{A} := A + \tilde{r}I_n : \mathbb{R}^n \rightarrow \mathbb{R}_+^{n \times n}$$

is bounded and uniformly continuous, valued in the positive matrices and its eigenvalues are uniformly bounded on $\bar{\Omega}$ away from zero. Additionally, it is evident that $K_{\xi} = K + \xi I_2 \in L^\infty(\Omega; \mathbb{R}^{2 \times 2})$ and that it satisfies the structural assumptions in (2.2). Hence, by Theorems 1-2 applied to the Robin boundary value problem (1.4)(a)-(1.4)(c) for $p = m/2$ and noting that

$$\frac{m}{2} > \frac{n}{2} > \frac{2n}{n-2},$$

for any $S \in L^{\frac{nm}{2n+m}}(\Omega; \mathbb{R}^2)$ and any $s \in L^{\frac{m}{2}}(\partial\Omega; \mathbb{R}^2)$ there exists a unique solution $u \in W^{1, \frac{m}{2}}(\Omega; \mathbb{R}^2)$ satisfying (3.3)(a) for all $\phi \in W^{1, \frac{m}{m-2}}(\Omega; \mathbb{R}^2)$, as well as the estimate (3.4)(a). The only thing which is not already stated in the estimate (2.6) is the estimate on $\|u\|_{L^m(\Omega)}$, which follows by the Sobolev inequalities.

Again by Theorems 1-2 applied to the Robin boundary value problem (1.4)(b)-(1.4)(d), there exists a unique solution $v \in W^{1,p}(\Omega; \mathbb{R}^2)$ satisfying (3.3)(b) for all $\psi \in W^{1, \frac{p}{p-1}}(\Omega; \mathbb{R}^2)$. Further, by applying (2.6), by Hölder inequality we estimate

$$\begin{aligned} \|v\|_{W^{1,p}(\Omega)} &\leq C \|\xi H u\|_{L^{\frac{np}{n+p}}(\Omega)} \\ &\leq C \|\xi\|_{L^\infty(\Omega)} \|u\|_{L^{\frac{np}{n+p}}(\Omega)} \\ &\leq C \|\xi\|_{L^\infty(\Omega)} \|u\|_{L^m(\Omega)}, \end{aligned}$$

since $m > n$. The estimate (3.4)(b) therefore follows by the above estimate together with the Sobolev inequalities. The proof is complete. \square

4. THE INVERSE PROBLEM THROUGH PDE-CONSTRAINED MINIMISATION

Now that the forward fluorescent optical tomography problem is understood, we proceed with the solvability of the inverse problem associated with (1.4). Throughout this and subsequent sections we assume that the hypotheses of Theorem 3 are satisfied for a domain $\Omega \Subset \mathbb{R}^n$ with $n \geq 3$ and which *from now is assumed to have $C^{1,1}$ regular boundary*.

Fix an integer $N \in \mathbb{N}$, $m > n$, $M, \delta, \alpha > 0$ and $p > \max\left\{n, \frac{2n}{n-2}\right\}$. Consider Borel sets

$$(4.1) \quad \{\Gamma_1, \dots, \Gamma_N\} \subseteq \partial\Omega$$

and light sources

$$(4.2) \quad \{S_1, \dots, S_N\} \subseteq L^{\frac{nm}{2n+m}}(\Omega; \mathbb{R}^2), \quad \{s_1, \dots, s_N\} \subseteq L^{\frac{m}{2}}(\partial\Omega; \mathbb{R}^2)$$

in the interior and on the boundary respectively. Let also

$$(4.3) \quad \{v_1^\delta, \dots, v_N^\delta\} \subseteq L^\infty(\partial\Omega; \mathbb{R}^2)$$

be predicted approximate values of the solution v of (1.4)(b)-(1.4)(d) on the boundary $\partial\Omega$, at noise (error) level δ . Suppose that for any $i \in \{1, \dots, N\}$, the pair (u_i, v_i) solves (1.4) with data (S_i, s_i, ξ) . For the N -tuple of solutions $(u_1, \dots, u_N; v_1, \dots, v_N)$, we will be using the notation

$$(\vec{u}, \vec{v}) \in W^{1, \frac{m}{2}}(\Omega; \mathbb{R}^{2 \times N}) \times W^{1,p}(\Omega; \mathbb{R}^{2 \times N})$$

and understand the vectors of solutions $(u_i)_{i=1 \dots N}$ and $(v_i)_{i=1 \dots N}$ as being $\mathbb{R}^{2 \times N}$ -valued. Similarly, we will understand the corresponding vectors of test functions as

$$(\vec{\phi}, \vec{\psi}) \in W^{1, \frac{m}{m-2}}(\Omega; \mathbb{R}^{2 \times N}) \times W^{1, \frac{p}{p-1}}(\Omega; \mathbb{R}^{2 \times N}).$$

The goal of the inverse problem associated with (1.4) is to *determine a non-negative $\xi \in L^p(\Omega, [0, \infty))$ such that the errors $|(v_i - v_i^\delta)|_{\Gamma_i}$ which describe the misfit between the predicted approximate solution and the actual solution become as small as possible*. We will minimise the error in L^∞ by means of approximations in L^p for large p and then take the limit $p \rightarrow \infty$. The benefit of minimisation in L^∞ is that one can achieve uniformly small error rather than on average. Since no reasonable error functional is coercive in the admissible class of N -tuples of PDE solutions without additional constraints, we add an extra Tykhonov-type regularisation term $\alpha \|\xi\|$ for a small parameter $\alpha > 0$ and some appropriate norm.

In view of the above observations, we define for $p > \max\left\{n, \frac{2n}{n-2}\right\}$ the functional

$$(4.4) \quad E_p(\vec{u}, \vec{v}, \xi) := \sum_{i=1}^N \|v_i - v_i^\delta\|_{\dot{L}^p(\Gamma_i)} + \alpha \|D^2 \xi\|_{\dot{L}^m(\Omega)}, \quad (\vec{u}, \vec{v}, \xi) \in \mathfrak{X}^p(\Omega)$$

and its limiting counterpart

$$(4.5) \quad E_\infty(\vec{u}, \vec{v}, \xi) := \sum_{i=1}^N \|v_i - v_i^\delta\|_{L^\infty(\Gamma_i)} + \alpha \|D^2 \xi\|_{L^m(\Omega)} \quad (\vec{u}, \vec{v}, \xi) \in \mathfrak{X}^\infty(\Omega),$$

where the dotted \dot{L}^p -functionals are the next regularisations of the respective norms

$$(4.6) \quad \|g\|_{\dot{L}^p(\Gamma_i)} := \left(\int_{\Gamma_i} (|g|_{(p)})^p d\mathcal{H}^{n-1} \right)^{1/p}, \quad \|f\|_{\dot{L}^m(\Omega)} := \left(\int_{\Omega} (|f|_{(m)})^m d\mathcal{L}^n \right)^{1/m}$$

and $|\cdot|_{(p)}$ is a regularisation of the Euclidean norm away from zero in the corresponding space, given by

$$(4.7) \quad |\cdot|_{(p)} := \sqrt{|\cdot|^2 + p^{-2}}.$$

The slashed integral symbolises the average with respect to either the Lebesgue measure \mathcal{L}^n or the Hausdorff measure \mathcal{H}^{n-1} . The respective admissible classes $\mathfrak{X}^p(\Omega)$ and $\mathfrak{X}^\infty(\Omega)$ are defined by

$$(4.8) \quad \mathfrak{X}^p(\Omega) := \left\{ \begin{array}{l} (\vec{u}, \vec{v}, \xi) \in \mathcal{W}^p(\Omega) : \text{for any } i \in \{1, \dots, N\}, (u_i, v_i, \xi) \text{ satisfies} \\ \quad \quad \quad 0 \leq \xi \leq M \text{ a.e. on } \Omega \\ \text{and} \\ \left\{ \begin{array}{ll} (a)_i & -\operatorname{div}(A_\xi \cdot Du_i) + K_\xi u_i = S_i, & \text{in } \Omega, \\ (b)_i & -\operatorname{div}(A_\xi \cdot Dv_i) + K_\xi v_i = \xi H u_i, & \text{in } \Omega, \\ (c)_i & (A_\xi \cdot Du_i) \mathbf{n} + \gamma u_i = s_i, & \text{on } \partial\Omega, \\ (d)_i & (A_\xi \cdot Dv_i) \mathbf{n} + \gamma v_i = 0, & \text{on } \partial\Omega, \end{array} \right. \\ \text{for } A, K, H, S_i, s_i, \kappa, \xi, \lambda, \gamma, p \text{ satisfying the hypotheses (1.5)} \\ \text{(1.6), (1.7), (3.1) and (4.1)-(4.3)} \end{array} \right\}$$

and

$$(4.9) \quad \mathfrak{X}^\infty(\Omega) := \bigcap_{n < p < \infty} \mathfrak{X}^p(\Omega),$$

whilst the Banach space $\mathcal{W}^p(\Omega)$ involved in the definition of the admissible class $\mathfrak{X}^p(\Omega)$ is

$$(4.10) \quad \mathcal{W}^p(\Omega) := W^{1, \frac{m}{2}}(\Omega; \mathbb{R}^{2 \times N}) \times W^{1,p}(\Omega; \mathbb{R}^{2 \times N}) \times W^{2,m}(\Omega).$$

Note that $\mathfrak{X}^\infty(\Omega)$ is a subset of a Frechét space, rather than of a Banach space, but this will not cause any added difficulties. In particular, note that each $\mathfrak{X}^p(\Omega)$ is subset of the Banach space $\mathcal{W}^p(\Omega)$, but $\mathfrak{X}^\infty(\Omega)$ is *not* " $\mathfrak{X}^p(\Omega)$ for $p = \infty$ " because our a priori estimates are not true in $W^{1,\infty}(\Omega)$, but only in $W^{1,p}(\Omega)$ for any $p \in [2, \infty)$ with a constant $C = C(p)$ depending on p . (This reminiscent to the classical strict inclusion $L^\infty(\Omega) \subsetneq \bigcap_{1 < p < \infty} L^p(\Omega)$ which is true when $\mathcal{L}^n(\Omega) < \infty$: any $f \in$

$\bigcap_{1 < p < \infty} L^p(\Omega)$ satisfies $\|f\|_{L^p(\Omega)} \leq C(p) < \infty$, but there exist f 's with $C(p) \rightarrow \infty$ as $p \rightarrow \infty$ for which $\|f\|_{L^\infty(\Omega)} = \infty$.)

Remark 4. It might be quite surprising that in the Tikhonov term we include the L^m norm of the Hessian of ξ , rather than as one would expect the L^m norm of either the gradient or ξ itself. It turns out that one cannot regularise enough by adding “ $+\alpha\|\xi\|_{\dot{L}^m(\Omega)}$ ” to obtain minimisers (this would be redundant anyway because of the unilateral constraint). On the other hand, by adding “ $+\alpha\|D\xi\|_{\dot{L}^m(\Omega)}$ ”, one can indeed recover all the results up to and including Section 5, but not the results of Section 6, as we cannot obtain the variational inequalities in L^∞ without higher regularity in the coefficients of the PDE systems in (4.8) due to the emergence of quadratic gradient terms.

The main result in this section concerns the existence of E_p -minimisers in the admissible class $\mathfrak{X}^p(\Omega)$, the existence of E_∞ -minimisers in the admissible class $\mathfrak{X}^\infty(\Omega)$ and the approximability of the latter by the former as $p \rightarrow \infty$.

Theorem 5 (E_∞ -error minimisers, E_p -error minimisers and convergence as $p \rightarrow \infty$).
(A) *The functional E_p given by (4.4) has a unique constrained minimiser $(\vec{u}_p, \vec{v}_p, \xi_p)$ in the admissible class $\mathfrak{X}^p(\Omega)$:*

$$(4.11) \quad E_p(\vec{u}_p, \vec{v}_p, \xi_p) = \inf \left\{ E_p(\vec{u}, \vec{v}, \xi) : (\vec{u}, \vec{v}, \xi) \in \mathfrak{X}^p(\Omega) \right\}.$$

(B) *The functional E_∞ given by (4.5) has a constrained minimiser $(\vec{u}_\infty, \vec{v}_\infty, \xi_\infty)$ in the admissible class $\mathfrak{X}^\infty(\Omega)$:*

$$(4.12) \quad E_\infty(\vec{u}_\infty, \vec{v}_\infty, \xi_\infty) = \inf \left\{ E_\infty(\vec{u}, \vec{v}, \xi) : (\vec{u}, \vec{v}, \xi) \in \mathfrak{X}^\infty(\Omega) \right\}.$$

Additionally, there exists a subsequence of indices $(p_j)_1^\infty$ such that the sequence of respective E_{p_j} -minimisers $(\vec{u}_{p_j}, \vec{v}_{p_j}, \xi_{p_j})$ satisfies

$$(4.13) \quad \begin{cases} \vec{u}_p \rightharpoonup \vec{u}_\infty, & \text{in } W^{1, \frac{m}{2}}(\Omega; \mathbb{R}^{2 \times N}), \\ \vec{u}_p \rightarrow \vec{u}_\infty, & \text{in } L^{\frac{m}{2}}(\Omega; \mathbb{R}^{2 \times N}), \\ \vec{v}_p \rightharpoonup \vec{v}_\infty, & \text{in } W^{1, q}(\Omega; \mathbb{R}^{2 \times N}), \text{ for all } q \in (1, \infty), \\ \vec{v}_p \rightarrow \vec{v}_\infty, & \text{in } C(\bar{\Omega}; \mathbb{R}^{2 \times N}), \\ \xi_p \rightharpoonup \xi_\infty, & \text{in } W^{2, m}(\Omega), \\ \xi_p \rightarrow \xi_\infty, & \text{in } C^1(\bar{\Omega}), \end{cases}$$

as $p_j \rightarrow \infty$. Further

$$(4.14) \quad E_\infty(\vec{u}_\infty, \vec{v}_\infty, \xi_\infty) = \lim_{p_j \rightarrow \infty} E_{p_j}(\vec{u}_{p_j}, \vec{v}_{p_j}, \xi_{p_j}).$$

The proof of Theorem 5 is a consequence of the next two propositions, utilising the direct method of Calculus of Variations ([21]).

Proposition 6 (E_p -minimisers). *In the setting of Theorem 5, the functional E_p has unique constrained minimiser $(\vec{u}_p, \vec{v}_p, \xi_p)$ in the admissible class $\mathfrak{X}^p(\Omega)$, as claimed in (4.11).*

Proof. Let us begin by noting that $\mathfrak{X}^p(\Omega) \neq \emptyset$, and, as we will show right next, in fact it is a weakly closed subset of the reflexive Banach space $\mathcal{W}^p(\Omega)$ with cardinality greater or equal to that of $L^p(\Omega)$. Next note that there is an a priori energy bound

for the infimum of E_p , in fact uniform in p . Indeed, for each $i \in \{1, \dots, N\}$ let (u_{0i}, v_{0i}) be the solution to (1.4) with $\xi \equiv 0$ and sources (S_i, s_i) as in (4.2). Then, by Theorem 3, we have $v_{0i} \equiv 0$. Therefore, by (4.5)-(4.6) we infer that

$$(\vec{u}_0, \vec{0}, 0) \in \mathfrak{X}^p(\Omega)$$

for all $p \in [n, \infty]$, and also, by Hölder inequality and (4.3)-(4.7), we obtain

$$\begin{aligned} E_p(\vec{u}_0, \vec{0}, 0) &\leq \frac{N+1}{p} + E_\infty(\vec{u}_0, \vec{0}, 0) \\ &\leq \frac{N+1}{n} + \sum_{i=1}^N \|v_i^\delta\|_{L^\infty(\Gamma_i)} \\ &< \infty. \end{aligned}$$

Consider now (for fixed p) a minimising sequence $(\vec{u}^j, \vec{v}^j, \xi^j)_{j=1}^\infty$ of E_p in $\mathfrak{X}^p(\Omega)$. Then, for all large enough $j \in \mathbb{N}$ we have

$$0 \leq E_p(\vec{u}^j, \vec{v}^j, \xi^j) \leq 1 + \frac{N+1}{n} + \sum_{i=1}^N \|v_i^\delta\|_{L^\infty(\Gamma_i)}.$$

By Theorem 3, we have the estimates

$$(4.15) \quad \begin{cases} \|\vec{u}^j\|_{W^{1, \frac{m}{2}}(\Omega)} \leq C \sum_{i=1}^N \left(\|S_i\|_{L^{\frac{nm}{2n+m}}(\Omega)} + \|s_i\|_{L^{\frac{m}{2}}(\partial\Omega)} \right), \\ \|\vec{v}^j\|_{W^{1,p}(\Omega)} \leq C M \|\vec{u}^j\|_{W^{1, \frac{m}{2}}(\Omega)}. \end{cases}$$

By the above and (4.4), we have the estimate

$$\|D^2 \xi^j\|_{L^m(\Omega)} \leq \|D^2 \xi^j\|_{\dot{L}^m(\Omega)} \leq \frac{1}{\alpha} \left(1 + \frac{N+1}{n} + \sum_{i=1}^N \|v_i^\delta\|_{L^\infty(\Gamma_i)} \right).$$

Further, in view of the unilateral constraint, we readily have

$$\|\xi^j\|_{L^\infty(\Omega)} \leq M.$$

By the $C^{1,1}$ regularity of $\partial\Omega$, the interpolation inequalities in the Sobolev space $W^{2,m}(\Omega)$ (see e.g. [34, Theorem 7.28, p.173]) imply the existence of $C > 0$ independent of j such that

$$\|D\xi^j\|_{L^m(\Omega)} \leq C \|D^2 \xi^j\|_{L^m(\Omega)} + \|\xi^j\|_{L^m(\Omega)}.$$

Thus, the above estimates yield the uniform bound

$$(4.16) \quad \sup_{j \in \mathbb{N}} \|\xi^j\|_{W^{2,m}(\Omega)} < \infty.$$

By the estimates (4.15)-(4.18) and standard weak and strong compactness arguments, there exists a weak limit

$$(\vec{u}_p, \vec{v}_p, \xi_p) \in \mathcal{W}^p(\Omega)$$

and a subsequence $(j_k)_1^\infty$ such that along which we have

$$\left\{ \begin{array}{ll} \vec{u}^j \rightharpoonup \vec{u}_p, & \text{in } W^{1, \frac{m}{2}}(\Omega; \mathbb{R}^{2 \times N}), \\ \vec{u}^j \rightarrow \vec{u}_p, & \text{in } L^{\frac{m}{2}}(\Omega; \mathbb{R}^{2 \times N}), \\ \vec{v}^j \rightharpoonup \vec{v}_p, & \text{in } W^{1, p}(\Omega; \mathbb{R}^{2 \times N}), \\ \vec{v}^j \rightarrow \vec{v}_p, & \text{in } C(\bar{\Omega}; \mathbb{R}^{2 \times N}), \\ \xi^j \rightharpoonup \xi_p, & \text{in } W^{2, m}(\Omega), \\ \xi^j \rightarrow \xi_p, & \text{in } C^1(\bar{\Omega}), \end{array} \right.$$

as $j_k \rightarrow \infty$. We note that in this paper we will utilise the common practice of passing to subsequences as needed without perhaps explicit relabelling of the new subsequences. To show that in fact the limit $(\vec{u}_p, \vec{v}_p, \xi_p)$ lies in the admissible constrained class $\mathfrak{X}^p(\Omega)$, we argue as follows. Note now that the pointwise constraint

$$0 \leq \xi^j \leq M \quad \text{a.e. on } \Omega,$$

is weakly closed in $W^{2, m}(\Omega)$, namely the set

$$(4.17) \quad W^{2, m}(\Omega; [0, M]) = \left\{ \eta \in L^p(\Omega) : 0 \leq \eta \leq M \quad \text{a.e. on } \Omega \right\}$$

is weakly closed. This is an immediate consequence of the strong compactness of the embedding of $W^{2, m}(\Omega)$ into $C^1(\bar{\Omega})$. We thus infer that

$$(\vec{u}_p, \vec{v}_p, \xi_p) \in \mathfrak{X}^p(\Omega)$$

by passing to the weak limit in the equations $(a)_i - (d)_i$ defining (4.8), which is possible due to the modes of convergence the minimising sequence satisfies.

We finally show that the weak limit $(\vec{u}_p, \vec{v}_p, \xi_p) \in \mathfrak{X}^p(\Omega)$ is indeed the minimiser of E_p over the same space. To this end, note that for any $\alpha > 0$ the nonlinear functional $\alpha \|D^2(\cdot)\|_{L^m(\Omega)}$ is convex and strongly continuous on the reflexive space $W^{2, m}(\Omega)$, by (4.6)-(4.7). Therefore, it is weakly lower semi-continuous on the same space. Similarly, for each $i \in \{1, \dots, N\}$ the functional $\|\cdot - v_i^\delta\|_{L^p(\Gamma_i)}$ is strongly continuous on $L^p(\Gamma_i)$. Hence, we conclude that

$$\begin{aligned} E_p(\vec{u}_p, \vec{v}_p, \xi_p) &= \sum_{i=1}^N \|v_{pi} - v_i^\delta\|_{L^p(\Gamma_i)} + \alpha \|D^2 \xi_p\|_{L^m(\Omega)} \\ &\leq \liminf_{j_k \rightarrow \infty} \left\{ \sum_{i=1}^N \|v_i^j - v_i^\delta\|_{L^p(\Gamma_i)} + \alpha \|D^2 \xi^j\|_{L^m(\Omega)} \right\} \\ &= \liminf_{j_k \rightarrow \infty} E_p(\vec{u}^j, \vec{v}^j, \xi^j) \\ &= \inf \{ E_p : \mathfrak{X}^p(\Omega) \}. \end{aligned}$$

Finally, note that for any fixed p , the p -minimiser $(\vec{u}_p, \vec{v}_p, \xi_p)$ is unique. On the one hand, the functional E_p is strictly convex in ξ (and also in \vec{v} , but constant in \vec{u}). On the other hand, for any ξ , there exists a unique pair (\vec{u}, \vec{v}) satisfying the PDE constraints, as a consequence of Theorem 3. The proposition ensues. \square

Our next result below concerns the existence of minimisers for the L^∞ -error functional and approximation of those by minimisers of L^p functionals, completing the proof of Theorem 5.

Proposition 7 (E_∞ -minimisers). *In the above setting, the functional E_∞ given by (4.5) has a constrained minimiser $(\vec{u}_\infty, \vec{v}_\infty, \xi_\infty)$ in the admissible class $\mathfrak{X}^\infty(\Omega)$, as claimed in (4.12).*

Additionally, there exists a subsequence of indices $(p_j)_1^\infty$ such that the sequence of respective E_{p_j} -minimisers $(\vec{u}_{p_j}, \vec{v}_{p_j}, \xi_{p_j})$ constructed in Proposition 6 satisfy (4.13) as $p_j \rightarrow \infty$. Further the energies converge as claimed in (4.14).

Proof. We essentially continue from the proof of Proposition 6. The energy bound $(\vec{u}_0, \vec{0}, 0)$ constructed therein is uniform in p and also, in view of (4.8)-(4.9) we have $(\vec{u}_0, \vec{0}, 0) \in \mathfrak{X}^\infty(\Omega)$. Fix now $q > n$ and consider large enough $p \geq q$. Then, by Hölder inequality and minimality, we have the bound

$$\begin{aligned} E_q(\vec{u}_p, \vec{v}_p, \xi_p) &\leq E_p(\vec{u}_p, \vec{v}_p, \xi_p) \\ &\leq E_p(\vec{u}_0, \vec{0}, 0) \\ &\leq \frac{N+1}{n} + E_\infty(\vec{u}_0, \vec{0}, 0) \\ &\leq \frac{N+1}{n} + \sum_{i=1}^N \|v_i^\delta\|_{L^\infty(\Gamma_i)}, \end{aligned}$$

which is uniform in p . By the above estimate, we have

$$\|D^2 \xi_p\|_{L^m(\Omega)} \leq \|D^2 \xi_p\|_{L^m(\Omega)} \leq \frac{1}{\alpha} \left(\frac{N+1}{n} + \sum_{i=1}^N \|v_i^\delta\|_{L^\infty(\Gamma_i)} \right).$$

On the other hand, by the unilateral pointwise constraint, we immediately have

$$0 \leq \xi_p \leq M \quad \text{on } \bar{\Omega}.$$

Hence, by the interpolation inequalities in $W^{2,m}(\Omega)$, we deduce the uniform bound

$$(4.18) \quad \sup_{p \geq n} \|\xi_p\|_{W^{2,m}(\Omega)} < \infty.$$

By the above estimates, by Theorem 3 (see (4.15)) and by standard weak and strong compactness arguments together with a diagonal argument, there exists a limit

$$(\vec{u}_\infty, \vec{v}_\infty, \xi_\infty) \in \bigcap_{n < q < \infty} \mathcal{W}^q(\Omega)$$

and a subsequence $(p_j)_1^\infty$ such that the modes of convergence in (4.13) hold true as $p_j \rightarrow \infty$. Further, by passing to the limit as $p_j \rightarrow \infty$ in the equations (a) _{i} – (d) _{i} forming the admissible class (4.8) and the closed unilateral pointwise constraint $0 \leq \xi_p \leq M$, we see that in fact the limit $(\vec{u}_\infty, \vec{v}_\infty, \xi_\infty)$ lies in the admissible class $\mathfrak{X}^\infty(\Omega)$. It remains to show that $(\vec{u}_\infty, \vec{v}_\infty, \xi_\infty)$ is indeed a minimiser of E_∞ and that the energies converge as claimed. To this end, fix an arbitrary $(\vec{u}, \vec{v}, \xi) \in \mathfrak{X}^\infty(\Omega)$.

Since $p_j \geq q$ for any $q > 1$ and large enough $j \in \mathbb{N}$, by minimality we have

$$\begin{aligned}
E_\infty(\vec{u}_\infty, \vec{v}_\infty, \xi_\infty) &= \lim_{q \rightarrow \infty} E_q(\vec{u}_\infty, \vec{v}_\infty, \xi_\infty) \\
&\leq \liminf_{q \rightarrow \infty} \left(\liminf_{p_j \rightarrow \infty} E_q(\vec{u}_p, \vec{v}_p, \xi_p) \right) \\
&\leq \liminf_{p_j \rightarrow \infty} E_p(\vec{u}_p, \vec{v}_p, \xi_p) \\
&\leq \limsup_{p_j \rightarrow \infty} E_p(\vec{u}_p, \vec{v}_p, \xi_p) \\
&\leq \lim_{p_j \rightarrow \infty} E_p(\vec{u}, \vec{v}, \xi) \\
&= E_\infty(\vec{u}, \vec{v}, \xi),
\end{aligned}$$

for any $(\vec{u}, \vec{v}, \xi) \in \mathfrak{X}^\infty(\Omega)$. Hence $(\vec{u}_\infty, \vec{v}_\infty, \xi_\infty)$ is a minimiser of E_∞ over $\mathfrak{X}^\infty(\Omega)$. The particular choice of $(\vec{u}, \vec{v}, \xi) := (\vec{u}_\infty, \vec{v}_\infty, \xi_\infty)$ in the above inequality yields

$$\lim_{p_j \rightarrow \infty} E_p(\vec{u}_p, \vec{v}_p, \xi_p) = E_\infty(\vec{u}_\infty, \vec{v}_\infty, \xi_\infty).$$

The proof of Proposition 7 is now complete. \square

5. KUHN-TUCKER THEORY AND LAGRANGE MULTIPLIERS FOR THE p -ERROR

In this section we return the L^p -minimisation problem (4.11) solved in Theorem 5 for finite $p < \infty$ (Section 4). Given the presence of both PDE and unilateral constraints, in general one cannot have an Euler-Lagrange equation, but an one-sided variational inequality with Lagrange multipliers. The goal here is to derive the relevant variational inequality associated with (4.11). The main result is therefore the following.

Theorem 8 (The variational inequalities in L^p). *In the setting of Section 4 and under the same assumptions, for any $p > \max\{n, 2n/(n-2)\}$, there exist Lagrange multipliers*

$$(\vec{\phi}_p, \vec{\psi}_p) \in W^{1, \frac{m}{m-2}}(\Omega; \mathbb{R}^{2 \times N}) \times W^{1, \frac{p}{p-1}}(\Omega; \mathbb{R}^{2 \times N})$$

associated with the constrained minimisation problem (4.11) for E_p in the admissible class $\mathfrak{X}^p(\Omega)$, such that the constrained minimiser $(\vec{u}_p, \vec{v}_p, \xi_p) \in \mathfrak{X}^p(\Omega)$ satisfies the next three relations:

$$\begin{aligned}
(5.1) \quad & \frac{\alpha m}{p} \int_{\Omega} (D^2 \eta - D^2 \xi_p) : \mu(D^2 \xi_p) \, d\mathcal{L}^n \geq \\
& \sum_{i=1}^N \int_{\Omega} (\eta - \xi_p) \left\{ -(\mathbf{H}u_{pi}) \cdot \psi_{pi} + \dot{r}(\cdot, \xi_p) [Du_{pi} : D\phi_{pi} \right. \\
& \quad \left. + Dv_{pi} : D\psi_{pi}] + u_{pi} \cdot \phi_{pi} + v_{pi} \cdot \psi_{pi} \right\} \, d\mathcal{L}^n
\end{aligned}$$

for any $\eta \in W^{2,m}(\Omega, [0, M])$; further,

$$\begin{aligned}
(5.2) \quad & \int_{\partial\Omega} \vec{w} : d[\vec{v}_p(\vec{v}_p)] = \sum_{i=1}^N \left\{ \int_{\Omega} [A_{\xi_p} : (Dw_i^\top D\psi_{pi}) + (K_{\xi_p} w_i) \cdot \psi_{pi}] \, d\mathcal{L}^n \right. \\
& \quad \left. + \int_{\partial\Omega} (\gamma w_i) \cdot \psi_{pi} \, d\mathcal{H}^{n-1} \right\},
\end{aligned}$$

for any $\vec{w} \in W^{1,p}(\Omega; \mathbb{R}^{2 \times N})$, and finally

$$(5.3) \quad \begin{aligned} & \sum_{i=1}^N \left\{ \int_{\Omega} \left[A_{\xi_p} : (Dz_i^\top D\phi_{pi}) + (K_{\xi_p} z_i) \cdot \phi_{pi} \right] d\mathcal{L}^n + \int_{\partial\Omega} (\gamma z_i) \cdot \phi_{pi} d\mathcal{H}^{n-1} \right\} \\ &= \sum_{i=1}^N \int_{\Omega} \xi_p(\mathbf{H}z_i) \cdot \psi_{pi} d\mathcal{L}^n, \end{aligned}$$

for any $\vec{z} \in W^{1, \frac{m}{2}}(\Omega; \mathbb{R}^{2 \times N})$.

In the relations (5.1)-(5.2), $\mu(V)$ is defined for any $V \in L^m(\Omega; \mathbb{R}^{n \times n})$ as

$$(5.4) \quad \mu(V) := \frac{(|V|_{(m)})^{m-2} V}{\mathcal{L}^n(\Omega) (\|V\|_{L^m(\Omega)})^{m-1}},$$

$\dot{r}(x, t)$ symbolises the partial derivative of r with respect to t (recall (1.5)) and $\vec{\nu}_p(\vec{v})$ is a \vec{v} -dependent $\mathbb{R}^{2 \times N}$ -valued matrix Radon measure in $\mathcal{M}(\partial\Omega; \mathbb{R}^{2 \times N})$ given by

$$(5.5) \quad \vec{\nu}_p(\vec{v}) := \sum_{i=1}^N \left(\frac{(|v_i - v_i^\delta|_{(p)})^{p-2} (v_i - v_i^\delta)}{\mathcal{H}^{n-1}(\Gamma_i) (\|v_i - v_i^\delta\|_{L^p(\Gamma_i)})^{p-1}} \otimes e_i \right) \mathcal{H}^{n-1} \llcorner_{\Gamma_i}.$$

Note that $\vec{\nu}_p(\vec{v})$ is absolutely continuous with respect to the Hausdorff measure $\mathcal{H}^{n-1} \llcorner_{\partial\Omega}$. The notation $\{e_1, \dots, e_N\}$ symbolises the standard Euclidean basis of \mathbb{R}^N and “ \cdot ” symbolises the standard inner product in $\mathbb{R}^{2 \times N}$. Additionally, one may trivially compute that

$$\dot{r}(x, t) = -\frac{\lambda}{(\kappa(x) + t)^2}.$$

Remark 9. The reason that we obtain the three different relations (5.1)-(5.3) of which one is inequality and two are equations can be explained as follows. If one ignores the PDE constraints defining (4.8) (which give rise to the Lagrange multipliers), then the admissible class is in fact the Cartesian product of three sets, two of which are vector spaces (spaces for \vec{u} and \vec{v}), and one is a convex set (space of ξ), see (5.11) that follows. Hence, since the unilateral constraint is only for ξ , the variational inequality is only for this variable. The decoupling of the relations is merely a consequence of linear independence.

The proof of Theorem 8 consists of several particular sub-results. We begin with computing the Frechét derivative of the functional E_p .

Lemma 10. *The functional $E_p : \mathcal{W}^p(\Omega) \rightarrow \mathbb{R}$ given by (4.4)-(4.10) is Frechét differentiable and its derivative*

$$dE_p : \mathcal{W}^p(\Omega) \rightarrow (\mathcal{W}^p(\Omega))^*, \quad (\vec{u}, \vec{v}, \xi) \mapsto (dE_p)_{(\vec{u}, \vec{v}, \xi)},$$

is given by

$$(5.6) \quad (dE_p)_{(\vec{u}, \vec{v}, \xi)}(\vec{z}, \vec{w}, \eta) = p \int_{\partial\Omega} \vec{w} : d[\vec{\nu}_p(\vec{v})] + \alpha m \int_{\Omega} D^2 \eta : \mu(D^2 \xi) d\mathcal{L}^n$$

for any $(\vec{u}, \vec{v}, \xi), (\vec{z}, \vec{w}, \eta) \in \mathcal{W}^p(\Omega)$.

Proof. The Frechét differentiability of E_p follows from standard results on the geometry of Banach spaces and the p -regularisations of the norms, given by (4.6)-(4.7).

To compute the Frechét derivative, we use Gateaux differentiation. To this end, fix $(\vec{u}, \vec{v}, \xi), (\vec{z}, \vec{w}, \eta) \in \mathcal{W}^p(\Omega)$. Then, we have

$$\begin{aligned} (dE_p)_{(\vec{u}, \vec{v}, \xi)}(\vec{z}, \vec{w}, \eta) &= \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} E_p\left((\vec{u}, \vec{v}, \xi) + \varepsilon(\vec{z}, \vec{w}, \eta)\right) \\ &= \sum_{i=1}^N \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \left(\int_{\Gamma_i} (|v_i + \varepsilon w_i - v_i^\delta|_{(p)})^p d\mathcal{H}^{n-1} \right)^{\frac{1}{p}} \\ &\quad + \alpha \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \left(\int_{\Omega} (|D^2\xi + \varepsilon D^2\eta|_{(m)})^m d\mathcal{L}^n \right)^{\frac{1}{m}} \end{aligned}$$

which by the chain rule yields

$$\begin{aligned} (dE_p)_{(\vec{u}, \vec{v}, \xi)}(\vec{z}, \vec{w}, \eta) &= p \sum_{i=1}^N \left(\int_{\Gamma_i} (|v_i - v_i^\delta|_{(p)})^p d\mathcal{H}^{n-1} \right)^{\frac{1}{p}-1} \\ &\quad \cdot \int_{\Gamma_i} (|v_i - v_i^\delta|_{(p)})^{p-2} (v_i - v_i^\delta) \cdot w_i d\mathcal{H}^{n-1} \\ &\quad + \alpha m \left(\int_{\Omega} (|D^2\xi|_{(m)})^m d\mathcal{L}^n \right)^{\frac{1}{m}-1} \int_{\Omega} (|D^2\xi|_{(m)})^{m-2} D^2\xi : D^2\eta d\mathcal{L}^n. \end{aligned}$$

Hence, (5.6) follows in view of the definitions (5.4)-(5.5). The lemma ensues. \square

In order to derive the appropriate variational inequality that any minimiser as in (4.11) satisfies, we need to define a map which incorporates the PDE constraints of the admissible class $\mathfrak{X}^p(\Omega)$ in (4.8). We define

$$(5.7) \quad G : \mathcal{W}^p(\Omega) \longrightarrow \left[(W^{1, \frac{m}{m-2}}(\Omega; \mathbb{R}^2))^* \times (W^{1, \frac{p}{p-1}}(\Omega; \mathbb{R}^2))^* \right]^N$$

by setting

$$(5.8) \quad \left\langle G(\vec{u}, \vec{v}, \xi), (\vec{\phi}, \vec{\psi}) \right\rangle := \begin{bmatrix} \left\langle G_1^1(\vec{u}, \vec{v}, \xi), \phi_1 \right\rangle \\ \left\langle G_1^2(\vec{u}, \vec{v}, \xi), \psi_1 \right\rangle \\ \vdots \\ \left\langle G_N^1(\vec{u}, \vec{v}, \xi), \phi_N \right\rangle \\ \left\langle G_N^2(\vec{u}, \vec{v}, \xi), \psi_N \right\rangle \end{bmatrix} \in \mathbb{R}^{2N},$$

where, for each $i = 1, \dots, N$ and $j = 1, 2$, the mapping G_i^j is defined as

$$(5.9) \quad \begin{cases} \left\langle G_i^1(\vec{u}, \vec{v}, \xi), \phi_i \right\rangle := \\ \int_{\Omega} \left[A_\xi : (Du_i^\top D\phi_i) + (K_\xi u_i - S_i) \cdot \phi_i \right] d\mathcal{L}^n + \int_{\partial\Omega} [(\gamma u_i - s_i) \cdot \phi_i] d\mathcal{H}^{n-1} \end{cases}$$

and

$$(5.10) \quad \begin{cases} \left\langle G_i^2(\vec{u}, \vec{v}, \xi), \psi_i \right\rangle := \\ \int_{\Omega} \left[A_\xi : (Dv_i^\top D\psi_i) + (K_\xi v_i - \xi H u_i) \cdot \psi_i \right] d\mathcal{L}^n + \int_{\partial\Omega} [\gamma v_i \cdot \psi_i] d\mathcal{H}^{n-1}, \end{cases}$$

for any test functions

$$(\phi_i, \psi_i) \in W^{1, \frac{m}{m-2}}(\Omega; \mathbb{R}^2) \times W^{1, \frac{p}{p-1}}(\Omega; \mathbb{R}^2).$$

Let us also define for a fixed $M > 0$ the next convex weakly closed subset of the Banach space $\mathcal{W}^p(\Omega)$:

$$(5.11) \quad \mathcal{W}_M^p(\Omega) := W^{1, \frac{m}{2}}(\Omega; \mathbb{R}^{2 \times N}) \times W^{1, p}(\Omega; \mathbb{R}^{2 \times N}) \times W^{2, m}(\Omega; [0, M]).$$

Then, in view of (5.7)-(5.11), we may reformulate the admissible class $\mathfrak{X}^p(\Omega)$ of the minimisation problem (4.11) as

$$(5.12) \quad \mathfrak{X}^p(\Omega) = \left\{ (\vec{u}, \vec{v}, \xi) \in \mathcal{W}_M^p(\Omega) : G(\vec{u}, \vec{v}, \xi) = 0 \right\}.$$

With the aim of deriving the variational inequality which is the necessary condition of the minimisation problem (4.11), we compute the Frechét derivative of the mapping G above and prove that it is a submersion.

Lemma 11. *The mapping G defined in (5.7)-(5.11) is a continuously Frechét differentiable submersion and its derivative*

$$(5.13) \quad dG : \mathcal{W}^p(\Omega) \longrightarrow \mathcal{L} \left(\mathcal{W}^p(\Omega), \left[(W^{1, \frac{m}{m-2}}(\Omega; \mathbb{R}^2))^* \times (W^{1, \frac{p}{p-1}}(\Omega; \mathbb{R}^2))^* \right]^N \right)$$

which maps

$$(\vec{u}, \vec{v}, \xi) \mapsto (dG)_{(\vec{u}, \vec{v}, \xi)}$$

is given by

$$(5.14) \quad \left\langle (dG)_{(\vec{u}, \vec{v}, \xi)}(\vec{z}, \vec{w}, \eta), (\vec{\phi}, \vec{\psi}) \right\rangle = \begin{bmatrix} \left\langle (dG_1^1)_{(\vec{u}, \vec{v}, \xi)}(\vec{z}, \vec{w}, \eta), \phi_1 \right\rangle \\ \left\langle (dG_1^2)_{(\vec{u}, \vec{v}, \xi)}(\vec{z}, \vec{w}, \eta), \psi_1 \right\rangle \\ \vdots \\ \left\langle (dG_N^1)_{(\vec{u}, \vec{v}, \xi)}(\vec{z}, \vec{w}, \eta), \phi_N \right\rangle \\ \left\langle (dG_N^2)_{(\vec{u}, \vec{v}, \xi)}(\vec{z}, \vec{w}, \eta), \psi_N \right\rangle \end{bmatrix}$$

where, for each $i \in \{1, \dots, N\}$ and $j \in \{1, 2\}$, we have

$$(5.15) \quad \left\{ \begin{array}{l} \left\langle (dG_i^1)_{(\vec{u}, \vec{v}, \xi)}(\vec{z}, \vec{w}, \eta), \phi_i \right\rangle = \\ \int_{\Omega} \left[A_\xi : (Dz_i^\top D\phi_i) + (K_\xi z_i) \cdot \phi_i \right] d\mathcal{L}^n \\ + \int_{\partial\Omega} (\gamma z_i) \cdot \phi_i d\mathcal{H}^{n-1} + \int_{\Omega} \left[\dot{r}(\cdot, \xi)(Du_i : D\phi_i) + u_i \cdot \phi_i \right] \eta d\mathcal{L}^n \end{array} \right.$$

and

$$(5.16) \quad \left\{ \begin{array}{l} \left\langle (dG_i^2)_{(\vec{u}, \vec{v}, \xi)}(\vec{z}, \vec{w}, \eta), \psi_i \right\rangle = \\ \int_{\Omega} \left[A_\xi : (Dw_i^\top D\psi_i) + (K_\xi w_i - H(\eta u_i + \xi z_i)) \cdot \psi_i \right] d\mathcal{L}^n \\ + \int_{\partial\Omega} (\gamma w_i) \cdot \psi_i d\mathcal{H}^{n-1} + \int_{\Omega} \left[\dot{r}(\cdot, \xi)(Dv_i : D\psi_i) + v_i \cdot \psi_i \right] \eta d\mathcal{L}^n, \end{array} \right.$$

for any test functions

$$(\vec{\phi}, \vec{\psi}) \in W^{1, \frac{m}{m-2}}(\Omega; \mathbb{R}^{2 \times N}) \times W^{1, \frac{p}{p-1}}(\Omega; \mathbb{R}^{2 \times N})$$

and any $(\vec{u}, \vec{v}, \xi), (\vec{z}, \vec{w}, \eta) \in \mathcal{W}^p(\Omega)$.

Proof. The mapping G is at most quadratic in all arguments and also continuously Frechét differentiable in the space $\mathcal{W}^p(\Omega)$. The form of the derivative can be computed by using directional Gateaux differentiation

$$(\mathrm{d}G)_{(\vec{u}, \vec{v}, \xi)}(\vec{z}, \vec{w}, \eta) = \left. \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \right|_{\varepsilon=0} G\left((\vec{u}, \vec{v}, \xi) + \varepsilon(\vec{z}, \vec{w}, \eta)\right).$$

The exact form of the Gateaux derivative of G is a simple consequence of the definitions of A_ξ, K_ξ and the next computations:

$$\begin{aligned} \left. \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \right|_{\varepsilon=0} H(\xi + \varepsilon\eta)(u_i + \varepsilon z_i) &= H(\eta u_i + \xi z_i), \\ \left. \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \right|_{\varepsilon=0} A_{\xi + \varepsilon\eta} : ((Du_i + \varepsilon Dz_i)^\top D\phi_i) &= A : \left. \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \right|_{\varepsilon=0} (Du_i + \varepsilon Dz_i)^\top D\phi_i \\ &\quad + \left. \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \right|_{\varepsilon=0} r(\cdot, \xi + \varepsilon\eta)(Du_i + \varepsilon Dz_i) : D\phi_i \\ &= A : (Dz_i^\top D\phi_i) + r(\cdot, \xi)(Dz_i : D\phi_i) \\ &\quad + \dot{r}(\cdot, \xi)(Du_i : D\phi_i) \eta \\ &= A_\xi : (Dz_i^\top D\phi_i) + \dot{r}(\cdot, \xi)(Du_i : D\phi_i) \eta, \end{aligned}$$

and

$$\begin{aligned} \left. \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \right|_{\varepsilon=0} K_{\xi + \varepsilon\eta}(u_i + \varepsilon z_i) \cdot \phi_i &= K \left. \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \right|_{\varepsilon=0} (u_i + \varepsilon z_i) \cdot \phi_i \\ &\quad + \left. \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \right|_{\varepsilon=0} (\xi + \varepsilon\eta)(u_i + \varepsilon z_i) \cdot \phi_i \\ &= (Kz_i) \cdot \phi_i + \xi(z_i \cdot \phi_i) + (u_i \cdot \phi_i) \eta \\ &= (K_\xi z_i) \cdot \phi_i + (u_i \cdot \phi_i) \eta. \end{aligned}$$

To conclude, we need to show that G is a submersion, namely that for any $(\vec{u}, \vec{v}, \xi) \in \mathcal{W}^p(\Omega)$, the differential at this point, which is

$$(\mathrm{d}G)_{(\vec{u}, \vec{v}, \xi)} : \mathcal{W}^p(\Omega) \longrightarrow \left[(W^{1, \frac{m}{m-2}}(\Omega; \mathbb{R}^2))^* \times (W^{1, \frac{p}{p-1}}(\Omega; \mathbb{R}^2))^* \right]^N$$

is surjective. To this end, for each $i \in \{1, \dots, N\}$ fix functionals

$$(\Phi_i, \Psi_i) \in (W^{1, \frac{m}{m-2}}(\Omega; \mathbb{R}^2))^* \times (W^{1, \frac{p}{p-1}}(\Omega; \mathbb{R}^2))^*.$$

This means that there exist

$$\begin{cases} (f_i, F_i) \in L^{\frac{m}{2}}(\Omega; \mathbb{R}^2) \times L^{\frac{m}{2}}(\Omega; \mathbb{R}^{2 \times n}), \\ (g_i, G_i) \in L^p(\Omega; \mathbb{R}^2) \times L^p(\Omega; \mathbb{R}^{2 \times n}), \end{cases}$$

such that the next representation formulas hold true (see e.g. [2])

$$(5.17) \quad \begin{cases} \langle \Phi_i, \phi_i \rangle = \int_{\Omega} (f_i \cdot \phi_i + F_i : D\phi_i) \, \mathrm{d}\mathcal{L}^n, \\ \langle \Psi_i, \psi_i \rangle = \int_{\Omega} (g_i \cdot \psi_i + G_i : D\psi_i) \, \mathrm{d}\mathcal{L}^n, \end{cases}$$

for any $\phi_i \in W^{1, \frac{m}{m-2}}(\Omega; \mathbb{R}^2)$ and $\psi_i \in W^{1, \frac{p}{p-1}}(\Omega; \mathbb{R}^2)$. Then, by (5.13)-(5.17), the surjectivity of the differential $G'(\vec{u}, \vec{v}, \xi)$ follows from the solvability in (z_i, w_i) (for the choice $\eta \equiv 0$) in the weak sense of the PDE systems

$$\begin{cases} -\operatorname{div}(A_\xi \bullet D z_i) + K_\xi z_i = f_i - \operatorname{div} F_i, & \text{in } \Omega, \\ (A \bullet D z_i - F_i) \mathbf{n} + \gamma z_i = 0, & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\operatorname{div}(A_\xi \bullet D w_i) + K_\xi w_i = (g_i + \xi H z_i) - \operatorname{div} G_i, & \text{in } \Omega, \\ (A \bullet D w_i - G_i) \mathbf{n} + \gamma w_i = 0, & \text{on } \partial\Omega, \end{cases}$$

for all $i \in \{1, \dots, N\}$ and with $A, K, H, u_i, \xi, \gamma, f_i, F_i, g_i, G_i$ being fixed coefficients and parameters. The solvability of the above systems follows from Theorems 1-3. The result is therefore complete. \square

Now we derive the variational inequality through the generalised Kuhn-Tucker theory of Lagrange multipliers.

Proposition 12 (The variational inequality). *For any $p > 2n/(n-2)$, there exist Lagrange multipliers*

$$(\vec{\phi}_p, \vec{\psi}_p) \in W^{1, \frac{m}{m-2}}(\Omega; \mathbb{R}^{2 \times N}) \times W^{1, \frac{p}{p-1}}(\Omega; \mathbb{R}^{2 \times N})$$

associated with the constrained minimisation problem (4.11) for E_p in the admissible class (5.12), such that the constrained minimiser $(\vec{u}_p, \vec{v}_p, \xi_p) \in \mathfrak{X}^p(\Omega)$ satisfies the inequality

$$(5.18) \quad \frac{1}{p} (dE_p)_{(\vec{u}_p, \vec{v}_p, \xi_p)}(\vec{z}, \vec{w}, \eta - \xi_p) \geq \sum_{i=1}^N \left\langle (dG_i^1)_{(\vec{u}_p, \vec{v}_p, \xi_p)}(\vec{z}, \vec{w}, \eta - \xi_p), \phi_{pi} \right\rangle + \sum_{i=1}^N \left\langle (dG_i^2)_{(\vec{u}_p, \vec{v}_p, \xi_p)}(\vec{z}, \vec{w}, \eta - \xi_p), \psi_{pi} \right\rangle,$$

for any (\vec{z}, \vec{w}, η) in the convex set $\mathcal{W}_M^p(\Omega)$ (see (5.11)).

Proof. In view of Lemmas 10-11, E_p is Fréchet differentiable and G is a continuously Fréchet differentiable submersion everywhere on $\mathcal{W}^p(\Omega)$, whilst the set $\mathcal{W}_M^p(\Omega)$ is convex and with non-empty interior (with respect to the norm topology). Hence, the hypotheses of the generalised Kuhn-Tucker theory are satisfied (see e.g. [51, p. 417-418, Theorem 48B & Corollary 48.10]). Therefore, there exists a Lagrange multiplier

$$\Lambda_p \in \left((W^{1, \frac{m}{m-2}}(\Omega; \mathbb{R}^{2 \times N}))^* \times (W^{1, \frac{p}{p-1}}(\Omega; \mathbb{R}^{2 \times N}))^* \right)^*$$

which by standard duality arguments regarding product Banach spaces and their dual spaces that it can be identified with a pair of functions

$$(\vec{\phi}_p, \vec{\psi}_p) \in W^{1, \frac{m}{m-2}}(\Omega; \mathbb{R}^{2 \times N}) \times W^{1, \frac{p}{p-1}}(\Omega; \mathbb{R}^{2 \times N})$$

such that, the constrained minimiser $(\vec{u}_p, \vec{v}_p, \xi_p)$ satisfies

$$(5.19) \quad \begin{aligned} & \frac{1}{p} (\mathrm{dE}_p)_{(\vec{u}_p, \vec{v}_p, \xi_p)} (\vec{z} - \vec{u}_p, \vec{w} - \vec{v}_p, \eta - \xi_p) \\ & - \sum_{i=1}^N \left\langle (\mathrm{dG}_i^1)_{(\vec{u}_p, \vec{v}_p, \xi_p)} (\vec{z} - \vec{u}_p, \vec{w} - \vec{v}_p, \eta - \xi_p), \phi_{pi} \right\rangle \\ & - \sum_{i=1}^N \left\langle (\mathrm{dG}_i^2)_{(\vec{u}_p, \vec{v}_p, \xi_p)} (\vec{z} - \vec{u}_p, \vec{w} - \vec{v}_p, \eta - \xi_p), \psi_{pi} \right\rangle \geq 0, \end{aligned}$$

for any (\vec{z}, \vec{w}, η) in the convex set $\mathcal{W}_M^p(\Omega)$. Recall now that (5.11) implies that the convex subset $\mathcal{W}_M^p(\Omega)$ of the Banach space $\mathcal{W}^p(\Omega)$ can be written as the Cartesian product of the vector spaces

$$W^{1, \frac{m}{2}}(\Omega; \mathbb{R}^{2 \times N}) \times W^{1, p}(\Omega; \mathbb{R}^{2 \times N})$$

with the convex set $W^{2, m}(\Omega, [0, M])$, we may replace \vec{z} by $\vec{z} + \vec{u}_p$ and we may also replace \vec{w} by $\vec{w} + \vec{v}_p$ in (5.19) to arrive at (5.18). The proof of Proposition 12 is now complete. \square

We now use Proposition 12 to deduce that the variational inequality takes the form (5.20) below, as a direct consequence of Lemmas 10-11, (5.6), (5.4), (5.5), (5.13)-(5.16).

Corollary 13. *In the setting of Proposition 12, in view of the form of the Frechét derivatives of E_p and G , the variational inequality (5.18) takes the form*

$$(5.20) \quad \begin{aligned} & \int_{\partial\Omega} \vec{w} : \mathrm{d}[\vec{v}_p(\vec{v}_p)] + \frac{\alpha m}{p} \int_{\Omega} (\mathrm{D}^2 \eta - \mathrm{D}^2 \xi_p) : \mu(\mathrm{D}^2 \xi_p) \, \mathrm{d}\mathcal{L}^n \\ & \geq \sum_{i=1}^N \left\{ \int_{\Omega} \left[\mathbf{A} : (\mathrm{D}z_i^\top \mathrm{D}\phi_{pi}) + (\mathbf{K}z_i) \cdot \phi_{pi} \right] \, \mathrm{d}\mathcal{L}^n + \int_{\partial\Omega} (\gamma z_i) \cdot \phi_{pi} \, \mathrm{d}\mathcal{H}^{n-1} \right\} \\ & + \sum_{i=1}^N \left\{ \int_{\Omega} \left[\mathbf{B} : (\mathrm{D}w_i^\top \mathrm{D}\psi_{pi}) + (\mathbf{L}w_i - \mathbf{H}((\eta - \xi_p)u_{pi} + \xi_p z_i)) \cdot \psi_{pi} \right] \, \mathrm{d}\mathcal{L}^n \right. \\ & + \int_{\partial\Omega} (\gamma w_i) \cdot \psi_{pi} \, \mathrm{d}\mathcal{H}^{n-1} \left. \right\} + \sum_{i=1}^N \int_{\Omega} (\eta - \xi_p) \left(\dot{r}(\cdot, \xi_p) \left[\mathrm{D}u_{pi} : \mathrm{D}\phi_{pi} \right. \right. \\ & \left. \left. + \mathrm{D}v_{pi} : \mathrm{D}\psi_{pi} \right] + u_{pi} \cdot \phi_{pi} + v_{pi} \cdot \psi_{pi} \right) \, \mathrm{d}\mathcal{L}^n \end{aligned}$$

for any $(\vec{z}, \vec{w}, \eta) \in \mathcal{W}_M^p(\Omega)$.

We conclude this section by obtaining the further desired information on the variational inequality (5.20).

Lemma 14. *In the setting of Corollary 13, the variational inequality (5.20) for the constrained minimiser $(\vec{u}_p, \vec{v}_p, \xi_p)$ is equivalent to the triplet of relations (5.1)-(5.3).*

Proof. The inequality (5.1) follows by setting $\vec{z} = \vec{w} = 0$ in (5.20). The identity (5.2) follows by setting $\eta = \xi_p$ and $\vec{z} = 0$ in (5.20) and by recalling that $W^{1, p}(\Omega; \mathbb{R}^{2 \times N})$ is a vector space, so the inequality we obtain in fact holds for both $\pm w$. Finally, the identity (5.3) follows by setting $\eta = \xi_p$ and $\vec{w} = 0$ in (5.20) and

by recalling again that $W^{1, \frac{m}{2}}(\Omega; \mathbb{R}^{2 \times N})$ is a vector space, so the inequality holds for both $\pm z$. \square

6. KUHN-TUCKER THEORY AND LAGRANGE MULTIPLIERS FOR THE ∞ -ERROR

In this section we consider the L^∞ -minimisation problem (4.12) solved in part (B) of Theorem 5 (Section 4). The goal is to derive the relevant variational inequalities associated with the constrained minimisation of the functional E_∞ (see (4.5)) in the admissible class (4.9), by analogy to the results in Theorem 8 of Section 5. To this aim, let us set

$$C_\infty := \limsup_{p_j \rightarrow \infty} C_p,$$

where

$$C_p := \|\vec{\phi}_p\|_{W^{1, \frac{m}{m-2}}(\Omega)} + \|\vec{\psi}_p\|_{W^{1,1}(\Omega)}$$

and $(\vec{\phi}_p, \vec{\psi}_p)$ are the Lagrange multipliers associated with the constrained minimisation problem (4.12) (Theorem 8). The main result here is therefore the following.

Theorem 15 (The variational inequalities in L^∞). *In the setting of Section 5 and under the same assumptions, suppose additionally that $m > 2n$ and also*

$$(6.1) \quad A \in C^1(\Omega; \mathbb{R}_+^{n \times n}), \quad K, H \in W^{1, \infty}(\Omega; \mathbb{R}^{2 \times 2}), \quad \kappa \in C^1(\Omega), \quad \vec{S} \in W^{1, \frac{m}{2}}(\Omega; \mathbb{R}^{2 \times N}).$$

Then, there exists a subsequence $(p_j)_1^\infty$ and a limiting measure

$$\vec{v}_\infty \in \mathcal{M}(\partial\Omega; \mathbb{R}^{2 \times N})$$

such that

$$(6.2) \quad \vec{v}_p(\vec{v}_p) \xrightarrow{*} \vec{v}_\infty \text{ in } \mathcal{M}(\partial\Omega; \mathbb{R}^{2 \times N}),$$

as $p_j \rightarrow \infty$, where $\vec{v}_p(\vec{v}_p)$ is given by (5.5), and:

(I) If $C_\infty = 0$, then $\vec{v}_\infty = \vec{0}$.

(II) If $C_\infty > 0$, then there exist (rescaled) limiting Lagrange multipliers

$$(\vec{\phi}_\infty, \vec{\psi}_\infty) \in W^{1, \frac{m}{m-2}}(\Omega; \mathbb{R}^{2 \times N}) \times BV(\Omega; \mathbb{R}^{2 \times N}),$$

such that

$$(6.3) \quad \left(\frac{\vec{\phi}_p}{C_p}, \frac{\vec{\psi}_p}{C_p} \right) \xrightarrow{*} (\vec{\phi}_\infty, \vec{\psi}_\infty) \text{ in } W^{1, \frac{m}{m-2}}(\Omega; \mathbb{R}^{2 \times N}) \times BV(\Omega; \mathbb{R}^{2 \times N})$$

as $p_j \rightarrow \infty$. Then, for the above Lagrange multipliers, the constrained minimiser $(\vec{u}_\infty, \vec{v}_\infty, \xi_\infty) \in \mathfrak{X}^\infty(\Omega)$ satisfies the next three relations:

$$(6.4) \quad \begin{aligned} & \sum_{i=1}^N \int_{\Omega} (\eta - \xi_\infty) \left[(Hu_{\infty i}) \cdot \psi_{\infty i} - (u_{\infty i} \cdot \phi_{\infty i} + v_{\infty i} \cdot \psi_{\infty i}) \right] d\mathcal{L}^n \\ & \geq \sum_{i=1}^N \int_{\Omega} (\eta - \xi_\infty) \dot{r}(\cdot, \xi_\infty) (Du_{\infty i} : D\phi_{\infty i}) d\mathcal{L}^n \\ & \quad + \sum_{i=1}^N \int_{\Omega} (\eta - \xi_\infty) \dot{r}(\cdot, \xi_\infty) Dv_{\infty i} : d[D\psi_{\infty i}] \end{aligned}$$

for any $\eta \in C_{\xi_\infty}(\Omega; [0, M])$ (namely $\eta \in C(\Omega; [0, M])$ with $\eta = \xi_\infty$ on $\partial\Omega$); further,

$$(6.5) \quad \frac{1}{C_\infty} \int_{\partial\Omega} \vec{w} : d\vec{\nu}_\infty = \sum_{i=1}^N \left\{ \int_{\Omega} A_{\xi_\infty} : (Dw_i)^\top d[D\psi_{\infty i}] \right. \\ \left. + \int_{\partial\Omega} (K_{\xi_\infty} w_i) \cdot \psi_{\infty i} d\mathcal{L}^n + \int_{\partial\Omega} (\gamma w_i) \cdot \psi_{\infty i} d\mathcal{H}^{n-1} \right\},$$

for any $\vec{w} \in C_0^1(\bar{\Omega}; \mathbb{R}^{2 \times N})$, and finally

$$(6.6) \quad \sum_{i=1}^N \int_{\Omega} \left[A_{\xi_\infty} : (Dz_i)^\top D\phi_{\infty i} + (K_{\xi_\infty} z_i) \cdot \phi_{\infty i} - \xi_\infty(\mathbf{H}z_i) \cdot \psi_{\infty i} \right] d\mathcal{L}^n \\ = - \sum_{i=1}^N \int_{\partial\Omega} (\gamma z_i) \cdot \phi_{\infty i} d\mathcal{H}^{n-1}$$

for any $\vec{z} \in C^1(\bar{\Omega}; \mathbb{R}^{2 \times N})$.

Note the interesting fact that the limiting variational inequality (6.4) has no dependence on the regularisation parameter α , as the corresponding term in (5.1) is annihilated.

Proof. We begin by showing that for any $p > n$ and any $\vec{v} \in W^{1,p}(\Omega; \mathbb{R}^{2 \times N})$, we have the next total variation bound for the measure (5.5):

$$(6.7) \quad \|\vec{\nu}_p(\vec{v})\|(\partial\Omega) \leq N.$$

Indeed, by Hölder inequality we have

$$\|\nu_{p_i}(\vec{v})\|(\partial\Omega) \leq \frac{\int_{\Gamma_i} (|v_i - v_i^\delta|_{(p)})^{p-2} |v_i - v_i^\delta| d\mathcal{H}^{n-1}}{\left(\int_{\Gamma_i} (|v_i - v_i^\delta|_{(p)})^p d\mathcal{H}^{n-1} \right)^{\frac{p-1}{p}}} \\ \leq \frac{\int_{\Gamma_i} (|v_i - v_i^\delta|_{(p)})^{p-1} d\mathcal{H}^{n-1}}{\left(\int_{\Gamma_i} (|v_i - v_i^\delta|_{(p)})^p d\mathcal{H}^{n-1} \right)^{\frac{p-1}{p}}} \\ \leq 1,$$

for any $i \in \{1, \dots, N\}$. By the sequential weak* compactness of the spaces of Radon measures, the estimate (6.7) implies the existence of a subsequence $(p_j)_1^\infty$ and of the claimed limit measure $\vec{\nu}_\infty$ in (6.2).

Now we proceed with establishing (I) and (II) of the theorem.

(I) Suppose that $C_\infty = 0$. Then, it follows that

$$(6.8) \quad (\vec{\phi}_p, \vec{\psi}_p) \longrightarrow (\vec{0}, \vec{0}) \quad \text{in } W^{1, \frac{m}{m-2}}(\Omega; \mathbb{R}^{2 \times N}) \times BV(\Omega; \mathbb{R}^{2 \times N})$$

as $p_j \rightarrow \infty$, where $(\vec{\phi}_p, \vec{\psi}_p)$ are the Lagrange multipliers associated with the constrained minimisation problem (4.12). By (6.2) and (6.10), by passing to the limit

as $p_j \rightarrow \infty$ in (5.2), we obtain

$$\int_{\partial\Omega} \vec{w} : d\vec{\nu}_\infty = 0 = \lim_{p_j \rightarrow \infty} \sum_{i=1}^N \left\{ \int_{\Omega} \left[A_{\xi_p} : (Dw_i^\top D\psi_{pi}) + (K_{\xi_p} w_i) \cdot \psi_{pi} \right] d\mathcal{L}^n + \int_{\partial\Omega} (\gamma w_i) \cdot \psi_{pi} d\mathcal{H}^{n-1} \right\},$$

for any $\vec{w} \in C^1(\bar{\Omega}; \mathbb{R}^{2 \times N})$. Therefore, $\vec{\nu}_\infty = \vec{0}$, as claimed.

(II) Suppose now $C_\infty > 0$. Then, the desired relations (6.4)-(6.6) follow directly from (5.2)-(5.3) by rescaling the Lagrange multipliers $(\vec{\phi}_p, \vec{\psi}_p)$ and passing to the limit as $p_j \rightarrow \infty$, since the rescaled Lagrange multipliers $(\vec{\phi}_p/C_p, \vec{\psi}_p/C_p)$ are bounded in the product space

$$W^{1, \frac{m}{m-2}}(\Omega; \mathbb{R}^{2 \times N}) \times BV(\Omega; \mathbb{R}^{2 \times N})$$

and therefore the sequence is weakly* precompact. (Recall also that on a reflexive space the weak and the weak* topology coincide.) Note first that we have

$$\begin{aligned} \left| \int_{\Omega} D^2(\eta - \xi_p) : \mu(D^2\xi_p) d\mathcal{L}^n \right| &= \left| \int_{\Omega} D^2(\eta - \xi_p) : \frac{(|D^2\xi_p|^{(m)})^{m-2} D^2\xi_p}{(\|D^2\xi_p\|_{L^m(\Omega)})^{m-1}} d\mathcal{L}^n \right| \\ &\leq \int_{\Omega} |D^2(\eta - \xi_p)| \frac{(|D^2\xi_p|^{(m)})^{m-1}}{(\|D^2\xi_p\|_{L^m(\Omega)})^{m-1}} d\mathcal{L}^n \\ &\leq C \|D^2(\eta - \xi_p)\|_{L^m(\Omega)}, \end{aligned}$$

by the definition of μ and by Hölder inequality. In order to conclude, we need to justify the weak* convergence as $p_j \rightarrow \infty$ of the quadratic terms

$$Du_{pi} : \frac{D\phi_{pi}}{C_p}, \quad Dv_{pi} : \frac{D\psi_{pi}}{C_p}.$$

To this end, we will show that under the higher regularity assumptions on the coefficients, we in fact have the next strong modes of convergence for the p -minimisers:

$$(6.9) \quad Du_{pi} \longrightarrow Du_{\infty i} \quad \text{in } L_{\text{loc}}^{\frac{m}{2}}(\Omega; \mathbb{R}^2),$$

$$(6.10) \quad Dv_{pi} \longrightarrow Dv_{\infty i} \quad \text{in } C(\Omega; \mathbb{R}^2),$$

as $p_j \rightarrow \infty$, for all $i \in \{1, \dots, N\}$. Before proving (6.9)-(6.10), we demonstrate how to conclude by assuming them. Since we have

$$(6.11) \quad \frac{D\phi_{pi}}{C_p} \longrightarrow D\phi_{\infty i} \quad \text{in } L^{\frac{m}{m-2}}(\Omega; \mathbb{R}^2),$$

$$(6.12) \quad \frac{D\psi_{pi}}{C_p} \mathcal{L}^n \xrightarrow{*} D\psi_{\infty i} \quad \text{in } \mathcal{M}(\Omega; \mathbb{R}^2)$$

and also $\xi_p \longrightarrow \xi_\infty$ in $C^1(\bar{\Omega})$ as $p_j \rightarrow \infty$, by choosing any $\mathcal{O} \Subset \Omega$ with Lipschitz boundary (for instance the union of finitely many balls), $(\eta_{p_j})_1^\infty \subseteq W^{2,m}(\Omega; [0, M])$ and $\eta \in W^{2,m}(\Omega; [0, M])$ with

$$\eta_p \equiv \xi_p \quad \text{on } \Omega \setminus \mathcal{O}, \quad \eta_p \longrightarrow \eta \quad \text{in } W^{2,m}(\mathcal{O}) \subseteq C^1(\bar{\mathcal{O}}),$$

we have $\eta - \xi_p \in W_0^{2,m}(\mathcal{O})$ and

$$\eta_p - \xi_p \longrightarrow \eta - \xi_\infty \quad \text{in } W_0^{2,m}(\mathcal{O})$$

as $p_j \rightarrow \infty$. Hence, (6.4)-(6.6) follow by (6.9)-(6.12), together with the weak-strong continuity of the duality pairing between $L^{\frac{m}{2}}(\mathcal{O}; \mathbb{R}^2)$ and $L^{\frac{m}{m-2}}(\mathcal{O}; \mathbb{R}^2)$ and the weak*-strong continuity of the duality pairing between $C_0(\mathcal{O}; \mathbb{R}^2)$ and $\mathcal{M}(\mathcal{O}; \mathbb{R}^2)$, at least for test functions $\eta \in W_{\xi_\infty}^{2,m}(\mathcal{O}; [0, M])$. The general case for test functions $\eta \in C_{\xi_\infty}(\Omega; [0, M])$ follows by a standard approximation argument.

Now we establish (6.9)-(6.10). Fix $i \in \{1, \dots, N\}$, $e \in \mathbb{R}^n$ with $|e| = 1$, $h \neq 0$ and $(\vec{u}, \vec{v}, \xi) \in \mathfrak{X}^p(\Omega)$ for some p large. Fix also $\zeta \in C_c^1(\Omega)$ and let $D_e^{1,h}$ symbolise the difference quotient with step size h along the direction of e . By testing in the weak form of the equations $(a)_i - (b)_i$ appearing in the constrained class (4.8) against test functions of the form

$$\phi_i = \psi_i := D_e^{1,-h}\zeta,$$

standard regularity arguments imply that the directional derivatives $D_e u_i$ and $D_e v_i$ solve weakly the divergence systems

$$(6.13) \quad -\operatorname{div}(A_\xi \bullet D(D_e u_i)) = \left\{ -K_\xi(D_e u_i) + D_e S_i - (D_e K_\xi)u_i \right\} + \operatorname{div}((D_e A_\xi) \bullet D u_i), \quad \text{in } \Omega,$$

and

$$(6.14) \quad -\operatorname{div}(A_\xi \bullet D(D_e v_i)) = \left\{ -K_\xi(D_e v_i) + \xi H(D_e u_i) + D_e(\xi H)u_i - (D_e K_\xi)v_i \right\} + \operatorname{div}((D_e A_\xi) \bullet D v_i), \quad \text{in } \Omega.$$

In view of (1.5)-(1.6) we have

$$D_e A_\xi = D_e A + D_e(r(\cdot, \xi))I_n, \quad D_e K_\xi = D_e K + (D_e \xi)I_2,$$

and

$$D_e(r(\cdot, \xi)) = -\frac{\lambda(D_e \kappa + D_e \xi)}{(\kappa + \xi)^2}.$$

Due to our assumption (6.1) and the embedding $W^{2,m}(\Omega) \subseteq C^1(\bar{\Omega})$, we have that

$$D_e A_\xi \in C(\Omega; \mathbb{R}_+^{n \times n}), \quad D_e K_\xi \in L^\infty(\Omega; \mathbb{R}^{2 \times 2})$$

as

$$D_e(r(\cdot, \xi)) \in C(\Omega), \quad K \in W^{1,\infty}(\Omega; \mathbb{R}^{2 \times 2}), \quad \xi, \kappa \in C^1(\Omega).$$

Further, in view of our assumptions and Hölder inequality we have that

$$\left. \begin{array}{l} -K_\xi(D_e u_i) + D_e S_i - (D_e K_\xi)u_i \\ (D_e A_\xi) \bullet D u_i \\ -K_\xi(D_e v_i) + \xi H(D_e u_i) + D_e(\xi H)u_i - (D_e K_\xi)v_i \\ (D_e A_\xi) \bullet D v_i \end{array} \right\} \in L^{\frac{m}{2}}(\Omega)$$

because

$$\begin{aligned} \xi, |H|, |D_e H|, |D_e(\xi H)|, |D_e K_\xi|, |K_\xi|, |D_e A_\xi| &\in L^\infty(\Omega), \\ |u_i|, |D_e u_i|, |v_i|, |D_e v_i|, |D_e S_i|, |D u_i|, |D v_i| &\in L^{\frac{m}{2}}(\Omega), \end{aligned}$$

for $p > 2m$. By the interior L^2 and $L^{\frac{m}{2}}$ regularity estimates for the systems (6.13)-(6.14) (see e.g. [31, Sections 4.3.1 & 7.1.2]), for any $\mathcal{O} \Subset \Omega$ there exists $C > 0$ independent of p such that

$$\|D^2 \vec{u}_p\|_{L^{\frac{m}{2}}(\mathcal{O})} + \|D^2 \vec{v}_p\|_{L^{\frac{m}{2}}(\mathcal{O})} \leq C.$$

Since by assumption $m > 2n$, by the Morrey estimate we have

$$\|D\vec{u}_p\|_{C^{0,1-\frac{2n}{m}}(\mathcal{O})} + \|D\vec{v}_p\|_{C^{0,1-\frac{2n}{m}}(\mathcal{O})} \leq C.$$

By standard compact embedding arguments in Hölder spaces, (6.9)-(6.10) ensue as a consequence of the above estimates. The proof of the theorem is complete. \square

Acknowledgments. The author would like to thank the anonymous referee for their constructive critique on an earlier version of this paper, which improved both the content and the presentation.

REFERENCES

1. G.S. Abdoulaev, K. Ren, A.H. Hielscher, *Optical tomography as a PDE-constrained optimization problem*, Inverse Problems 21, 1507 - 1530 (2005).
2. R.A. Adams, *Sobolev spaces*, second edition, Academic Press, 2012.
3. D. Alvarez, P. Medina, and M. Moscoso, *Fluorescence lifetime imaging from time resolved measurements using a shape-based approach*, Optics Express 17, 8843 - 8855 (2009).
4. L. Ambrosio, N. Fusco, D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford University Press, 2000.
5. C. Amrouche, C. Conca, A. Ghosh, T. Ghosh, *Uniform $W^{1,p}$ estimate for elliptic operator with Robin boundary condition in C^1 domain*, <https://arxiv.org/abs/1805.09519v3>.
6. G. Aronsson, *Minimization problems for the functional $\sup_x \mathcal{F}(x, f(x), f'(x))$* , Arkiv für Mat. 6 (1965), 33 - 53.
7. G. Aronsson, *Minimization problems for the functional $\sup_x \mathcal{F}(x, f(x), f'(x))$ II*, Arkiv für Mat. 6 (1966), 409 - 431.
8. G. Aronsson, *Extension of functions satisfying Lipschitz conditions*, Arkiv für Mat. 6 (1967), 551 - 561.
9. G. Aronsson, *On Certain Minimax Problems and Pontryagin's Maximum Principle*, Calculus of Variations and PDE 37, 99 - 109 (2010).
10. G. Aronsson, E.N. Barron, *L^∞ Variational Problems with Running Costs and Constraints*, Appl Math Optim 65, 53 - 90 (2012).
11. S. R. Arridge, *Optical tomography in medical imaging*, Inverse Problems 15, R41 - R93 (1999).
12. W. Bangerth, A. Joshi, *Nonlinear inversion for optical tomography*, Proceedings of the CT2008 - Tomography Confluence: An International Conference on the Applications of Computerized Tomography, Kanpur, India, February 2008. P. Munshi (ed.), American Institute of Physics, 2008.
13. W. Bangerth, A. Joshi, *Adaptive finite element methods for nonlinear inverse problems*, Proceedings of the 24rd ACM Symposium on Applied Computing, March 8-12, 2009, Honolulu, Hawaii. D. Shin (ed.), 1002-1006.
14. E.N. Barron, M. Bocea, R. Jensen, *Viscosity solutions of stationary Hamilton-Jacobi equations and minimizers of L^∞ functionals*, Proc. Amer. Math. 145(12), 5257 - 5265 (2017).
15. E.N. Barron, R. Jensen, *Minimizing the L^∞ norm of the gradient with an energy constraint*, Comm. Partial Differential Equations 30, 10-12, 1741 - 1772 (2005).
16. E. N. Barron, R. Jensen, C. Wang, *The Euler equation and absolute minimizers of L^∞ functionals*, Arch. Rational Mech. Analysis 157 (2001), 255 - 283.
17. M. Bocea, V. Nesi, *Γ -convergence of power-law functionals, variational principles in L^∞ , and applications*, SIAM J. Math. Anal., 39 (2008), 1550 - 1576.
18. M. Bocea, C. Popovici, *Variational principles in L^∞ with applications to antiplane shear and plane stress plasticity*, Journal of Convex Analysis Vol. 18 No. 2, (2011) 403 - 416.
19. T. Champion, L. De Pascale, F. Prinari, *Γ -convergence and absolute minimizers for supremal functionals*, COCV ESAIM: Control, Optimisation and Calculus of Variations (2004), Vol. 10, 14 - 27.
20. M. G. Crandall, *A visit with the ∞ -Laplacian*, in *Calculus of Variations and Non-Linear Partial Differential Equations*, Springer Lecture notes in Mathematics 1927, CIME, Cetraro Italy 2005.
21. B. Dacorogna, *Direct Methods in the Calculus of Variations*, 2nd Edition, Volume 78, Applied Mathematical Sciences, Springer, 2008.

22. D. Daners, *Robin boundary value problems on arbitrary domains*, Trans. AMS 352 (2), 4207-4236 (2000).
23. H. Dong, D. Kim, *Elliptic equations in divergence form with partially BMO coefficients*, Arch. Rat. Mech. Anal. 196(1):25 - 70, 2010.
24. L.C. Evans, *Partial differential equations*, Graduate Studies in Mathematics 19.1, AMS, 2010.
25. T. J. Farrell and M. S. Patterson, *Diffusion modeling of fluorescence in tissue*, in Handbook of Biomedical Fluorescence, M.-A. Mycek and B. W. Pogue, Eds., New York, Basel: Marcel Dekker Inc., 2003, ch. 2.
26. I. Fonseca, G. Leoni, *Modern methods in the Calculus of Variations: L^p spaces*, Springer Monographs in Mathematics, 2007.
27. M. Freiberger, H. Egger, H. Scharfetter, *Nonlinear Inversion in Fluorescent Optical Tomography*, IEEE Transactions on Biomedical Engineering 57:11, 2723 - 2729 (2010).
28. A. Garroni, V. Nesi, M. Ponsiglione, *Dielectric breakdown: optimal bounds*, Proceedings of the Royal Society A 457, issue 2014 (2011).
29. J. Geng, *$W^{1,p}$ estimates for elliptic problems with Neumann boundary conditions in Lipschitz domains*, Adv. Math., 229(4): 2427 - 2448 (2012).
30. M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, volume 105 of Annals of Mathematics Studies, Princeton University Press, Princeton, 1983.
31. M. Giaquinta, L. Martinazzi, *An Introduction to the Regularity Theory for Elliptic Systems, Harmonic Maps and Minimal Graphs*, Publications of the Scuola Normale Superiore 11, Springer, 2012.
32. A. P. Gibson, J. C. Hebden, and S. R. Arridge, *Recent advances in diffuse optical imaging*, Physics in Medicine and Biology 50(4), R1 - R43 (2005).
33. T. Ghosh, personal communication.
34. D. Gilbarg, N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, reprint of the 1998 edition, Springer.
35. A. Godavarty, M. J. Eppstein, C. Zhang et al., *Fluorescence-enhanced optical imaging in large tissue volumes using a gain-modulated ICCD camera*, Physics in Medicine and Biology 48 (12), 1701 - 1720 (2003).
36. A. Joshi, W. Bangerth, and W. M. Seivick-Muraca, *Adaptive finite element based tomography for fluorescence optical imaging in tissue*, Opt. Express 12, 5402 - 5417 (2004).
37. N. Katzourakis, *An Introduction to Viscosity Solutions for Fully Nonlinear PDE with Applications to Calculus of Variations in L^∞* , Springer Briefs in Mathematics, 2015, DOI 10.1007/978-3-319-12829-0.
38. N. Katzourakis, *An L^∞ regularisation strategy to the inverse source identification problem for elliptic equations*, SIAM Journal on Mathematical Analysis, to appear.
39. N. Katzourakis, *A minimisation problem in L with PDE and unilateral constraints*, ESAIM: Control, Optimisation and Calculus of Variations, to appear.
40. N. Katzourakis, R. Moser, *Existence, Uniqueness and Structure of Second Order Absolute Minimisers*, Archives for Rational Mechanics and Analysis, published online 06/09/2018, DOI: 10.1007/s00205-018-1305-6.
41. N. Katzourakis, T. Pryer, *2nd order L^∞ variational problems and the ∞ -Polylaplacian*, Advances in Calculus of Variations, Published Online: 27-01-2018, DOI: <https://doi.org/10.1515/acv-2016-0052> (in press).
42. N. Katzourakis, E. Parini, *The Eigenvalue Problem for the ∞ -Bilaplacian*, Nonlinear Differential Equations and Applications NoDEA 24:68, (2017).
43. N. Katzourakis, E. Varvaruca, *An Illustrative Introduction to Modern Analysis*, CRC Press / Taylor & Francis, Dec 2017.
44. C.E. Kenig, F. Lin, and Z. Shen, *Homogenization of elliptic systems with Neumann boundary conditions*, J. Amer. Math. Soc. 26(4), 901 - 937 (2013).
45. Q. Miao, C. Wang, Y. Zhou, *Uniqueness of Absolute Minimizers for L^∞ -Functionals Involving Hamiltonians $H(x, p)$* , Archive for Rational Mechanics and Analysis 223 (1), 141-198 (2017).
46. R. Nittka, *Regularity of solutions of linear second order elliptic and parabolic boundary value problems on Lipschitz domains*, J. Differential Equations 251, 860 - 880 (2011).
47. G. Papamikos, T. Pryer, *A Lie symmetry analysis and explicit solutions of the two-dimensional ∞ -Polylaplacian*, Studies in Applied Mathematics, Online 17 September 2018.
48. F. Prinari, *On the lower semicontinuity and approximation of L^∞ -functionals*, NoDEA 22, 1591 - 1605 (2015).

49. A.N. Ribeiro, E. Zappale, *Existence of minimisers for nonlevel convex functionals*, SIAM J. Control Opt., Vol. 52, No. 5, (2014) 3341 - 3370.
50. G. Zacharakis, J. Ripoll, R. Weissleder, and V. Ntziachristos, *Fluorescent protein tomography scanner for small animal imaging*, IEEE Trans. on Medical Imaging 24, 878 - 885 (2005).
51. E. Zeidler, *Nonlinear Functional Analysis and its Application III: Variational Methods and Optimization*, Springer-Verlag, 1985.
52. B. Zhu, A. Godavarty, *Near-Infrared Fluorescence-Enhanced Optical Tomography*, Hindawi Publishing Corporation, BioMed Research Inter. 2016, Article ID 5040814, 10 pages (2016).

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF READING, WHITEKNIGHTS,
PO BOX 220, READING RG6 6AX, UNITED KINGDOM

E-mail address: `n.katzourakis@reading.ac.uk`