GEOMETRIC METHOD FOR GLOBAL STABILITY OF DISCRETE POPULATION MODELS

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ABSTRACT. A class of autonomous discrete dynamical systems as population models for competing species are considered when each nullcline surface is a hyperplane. Criteria are established for global attraction of an interior or a boundary fixed point by a geometric method utilising the relative position of these nullcline planes only, independent of the growth rate function. These criteria are universal for a broad class of systems, so they can be applied directly to some known models appearing in the literature including Ricker competition models, Leslie-Gower models, Atkinson-Allen models, and generalised Atkinson-Allen models. Then global asymptotic stability is obtained by finding the eigenvalues of the Jacobian matrix at the fixed point. An intriguing question is proposed: Can a globally attracting fixed point induce a homoclinic cycle?

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1. INTRODUCTION

We consider the discrete dynamical system

$$x(n) = T^n(x), \quad x(0) = x \in \mathbb{R}_+^N, \quad n = 1, 2, \dots, \tag{1}$$

where $\mathbb{R}_+^N$ is the the first orthant in $\mathbb{R}^N$ with $\mathbb{R}_+ = [0, +\infty)$ and the map $T : \mathbb{R}_+^N \to \mathbb{R}_+^N$ is defined by

$$T_i(x) = x_i G_i((Ax)_i), \quad i \in I_N = \{1, 2, \dots, N\}, \tag{2}$$

the entries of the $N \times N$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{pmatrix} \tag{3}$$

satisfy $a_{ii} > 0$ and $a_{ij} \geq 0$ for all $i, j \in I_N$, $(Ax)_i$ denotes the ith component of Ax. We assume that the functions $G_i \in C^1(\mathbb{R}_+, (0, +\infty))$ satisfy the following conditions:

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($a1$) G_i is decreasing on $\mathbb{R}_+$, $G_i(r_i) = 1$ for some $r_i > 0$ and $G_i'(r_i) < 0$.

($a2$) For $x \in \mathbb{R}_+^N$ and each $i \in I_N$, $\frac{\partial T_i(x)}{\partial x_i} > 0$ for $0 \le x_i < r_i$.

($a3$) For each nonempty $J \subset I_N$ and for any two points $x, y \in \mathbb{R}_+^N$ satisfying $y_i = x_i \in [0, r_i]$ for $i \in I_N \setminus J$, $0 = x_j < y_j \le r_j$ and $(Ay)_j \le r_j$ for $j \in J$ and $(Ay)_i = r_i$ for some $i \in J$, the functions $T_k(x + t(y - x)), k \in J$, are increasing for $t \in [0, 1]$.

(A simplified version of ($a3$) will be given in section 2 as $(a3)'$.)

The discrete dynamical system (1) defined above can be viewed as a population model of N competing species, where $x_i(n)$ denotes the population size of the ith species at time n (e.g. nth generation or end of nth time period). Indeed, if $a_{ij} > 0$, the existing population of the jth species reduces the growth rate G_i of the ith species; if $a_{ij} = 0$, although the current population of the jth species does not affect the next generation of the ith species $T_i(x)$ directly, it may reduce the growth rate $G_i((AT^n x)_i)(n \ge 1)$ of later generations. This reflects the nature of mutual competition between any two species.

System (1) with (2)–(3) and ($a1$)–($a3$) is competitive not only in ecological context, but the map T is also mathematically a competitive map under the additional condition that the spectral radius of the matrix $M(x) = \text{diag}(-\frac{x_i G_i'}{G_i})A$ satisfies

(4) $$\rho(M(x)) < 1, \forall x \in [0, r].$$

Recall that a general map $T : S \to T(S)$ for a set $S \subset \mathbb{R}_+^N$ is called *competitive* if $x < y$ whenever $T(x) < T(y)$ for $x, y \in S$, and *strongly competitive* (or *retrotone*) if $x \ll y$ whenever $T(x) < T(y)$ with $x, y \in S$ and $y \gg 0$ [35, 33, 15, 14] (see section 2 for detailed definition for "$<$" and "$\ll$"). By ($a1$) and ($a2$) we have $G_i' \le 0$ and $-\frac{x_i G_i'}{G_i} < 1$. Then (4) implies that $DT(x)^{-1} \ge 0$, so T is a competitive map. But since $DT(x)^{-1}$ may have zero entries, (4) does not imply strong competitiveness of T. However, we are not sure whether ($a1$)–($a3$) imply (4). Thus, without the condition (4) we are not sure whether T is a competitive map.

Since $a_{ii} > 0$ for all $i \in I_N$, by letting $y = Dx$ with $D = \text{diag}\{a_{11}, \dots, a_{NN}\}$ and $y(n) = Dx(n)$, we have

$$y(n+1) = DT(x(n)) = DT(D^{-1}y(n)) = \bar{T}(y(n)),$$

$$\bar{T}_i(y) = a_{ii}T_i(x) = y_i G_i((AD^{-1}y)_i) = y_i G_i((\bar{A}y)_i)$$

with $\bar{a}_{ii} = 1$ and $\bar{a}_{ij} \ge 0$. Without loss of generality, from now onward we assume that

(5) $$\forall i, j \in I_N, \quad a_{ii} = 1, \quad a_{ij} \ge 0.$$

System (1) with (2)–(5) and ($a1$)–($a3$) includes many known models as special instances. For example, if $G_i(u) = e^{r_i - u}, 0 < r_i \le 1$, then (2) becomes

(6) $$T_i(x) = x_i e^{r_i - (Ax)_i}, i \in I_N,$$

and systems with such a form for T are known as Ricker competition models. If $G_i(u) = \frac{1+r_i}{1+u}$ for $i \in I_N$, then

$$T_i(x) = \frac{b_i x_i}{1+(Ax)_i}, (b_i = 1 + r_i), i \in I_N, \tag{7}$$

and systems with such a form for T are known as Leslie-Gower competition models. If

$$G_i(u) = b + \frac{2(1-b)}{1+u}, 0 < b < 1, i \in I_N,$$

then

$$T_i(x) = bx_i + \frac{2(1-b)x_i}{1+(Ax)_i}, (r_i = 1), i \in I_N, \tag{8}$$

and systems with such a form for T are known as Atkinson-Allen models. If

$$G_i(u) = b_i + \frac{(1+r_i)(1-b_i)}{1+u}, 0 < b_i < 1, r_i > 0, i \in I_N,$$

then

$$T_i(x) = b_i x_i + \frac{(1+r_i)(1-b_i)x_i}{1+(Ax)_i}, i \in I_N. \tag{9}$$

Systems with such a form for T are known as generalised Atkinson-Allen Models. It can be checked that all these models defined by (6)–(9) and any system defined by a combination of (6)–(9) meet the requirement of $(a1)$–$(a3)$ (see Appendix 1). Thus, these models are typical examples of system (1) with (2)–(5) and $(a1)$–$(a3)$.

A more general discrete population model, known as Kolmogorov system, is (1) with T defined by

$$T_i(x) = x_i F_i(x), i \in I_N, \tag{10}$$

where $F : \mathbb{R}^N_+ \to \mathbb{R}^N_+$ is smooth enough with $F_i(x) > 0$ and $\frac{\partial F_i}{\partial x_j} \leq 0 (i \neq j)$. Hirsch [15], Herrera [13], Wang and Jiang [36] (see also [22] and the references therein) proved the existence and uniqueness of a carrying simplex $\Sigma \subset \mathbb{R}^N_+$ under certain conditions. Since Σ is a global attractor of (1) with (10) restricted to $\mathbb{R}^N_+ \setminus \{0\}$, the global dynamics of the system on $\mathbb{R}^N_+ \setminus \{0\}$ is essentially described by the dynamics on Σ if it exists. Then many researchers explored the behaviour of the dynamics of (1) with (10) on Σ. For a few examples, Jiang, Niu and Wang [25] investigated existence and local stability of heteroclinic cycles of competitive maps via carrying simplices; Herrera [14] investigated exclusion and dominance under conditions which guarantee the existence of Σ; Balreira, Elaydi and Luis [5] provided criteria for global stability of an interior fixed point under the existence of the carrying simplex.

For system (1) with T defined by (2) and (3) with $a_{ij} > 0$ for all $i, j \in I_N$ and each G_i decreasing, Franke and Yakubu [8, 9] described the competition model and studied the exclusion of some species and proved that k-weakly dominant G plus invariance of a set implies that the k-th species will survive and all other species will die out.

For Ricker models (1) with (6), Smith [34] analysed the dynamics in detail for two competing species ($N = 2$). Roeger [31] (see also [32]) studied the local dynamics near the interior fixed point and Neimark-Sacker bifurcations for the special $3D$ maps (6) with $r_1 = r_2 = r_3$. Hofbauer et al. [16] studied the long term survival of N species. Hirsch [15] showed that (1) with (6) possesses a carrying simplex under mild conditions. Gyllenberg et al. [12] classified all $3D$ Ricker maps (6) admitting a carrying simplex Σ and derived a total of 33 stable equivalence classes with a typical phase portrait on Σ given for each class. The authors of [5] applied their stability criteria to $3D$ Ricker models and derived a sufficient condition for the interior fixed point to be globally asymptotically stable when $r_1 = r_2 = r_3 < 1$ and $a_{ij} = a < 1 (i \neq j)$.

For Leslie-Gower models (1) with (7), Cushing et al. [7] thoroughly analysed the $2D$ model and showed convergence of every orbit to a fixed point. For $3D$ models, Jiang et al. [25] analysed the existence and stability of heteroclinic cycles. For N-dimensional models, Hirsch [15] and Herrera [14] verified the existence of the carrying simplex Σ under some conditions, but Jiang and Niu [24] showed the unconditional existence of Σ. Moreover, the authors of [24] classified all $3D$ Leslie-Gower models via the boundary dynamics on Σ and derived a total of 33 stable equivalent classes. For a special case of $3D$ models with $b_1 = b_2 = b_3$ and $a_{ij} = a(i \neq j)$, Balreira et al. [5] obtained a condition for global stability of the interior fixed point.

The Atkinson-Allen models (1) with (8) were first built as a plant competition model by Atkinson [3] and Allen et al. [2] (see also a discrete model in [30]). For $3D$ models, Jiang and Niu [23] proved the index formula for fixed points on the carrying simplex and, based on which, gave a complete classification of $3D$ models into 33 stable equivalent classes. The generalised Atkinson-Allen models (1) with (9) were proposed by Gyllenberg et al. [10] and a complete analysis for $3D$ models with (9) was given similar to [23].

In this paper, we are concerned with the global dynamics and, in particular, the global asymptotic stability of a boundary or interior fixed point, of the system (1) with (2)–(5) and $(a1)$–$(a3)$. There are some available methods and results for stability: the Liapunov function method initiated by LaSalle [27] for general discrete dynamical systems, the G-function method for asymptotic stability introduced by Bouyekhf and Gruyitch [6], the method of using convexity of the per-capita growth rate by Kon [26], the split Liapunov function method developed by Baigent and Hou [4] for Kolmogorov systems (1) with (10), the stability criteria for monotone Kolmogorov systems obtained by Balreira et al. [5] and a revised version by Gyllenberg et al. [11], and some stability results for specific systems (see the references in [6], [4], [5] and [11]). No doubt that all of these are precious contributions to the development of stability theory and methods for discrete dynamical systems. However, it is also obvious that the application of each of these methods or criteria has its limitation due to its conditions and requirements. For this reason, more alternative methods will be welcome and expected.

The aim of this paper is to provide a geometric method for global attraction and stability of a fixed point either in the interior or the boundary of $\mathbb{R}^N_+$ for (1) with (2)–(5) and

$(a1)$–$(a3)$. We do not rely on the construction of Liapunov functions, nor on the existence of the carrying simplex. The nature of global attraction will be simply derived by the relative position of the N nullcline hyperplanes defined by the component equations of $Ax = r$ within the N-dimensional cell $[0, r]$. Superficially, this method can be viewed as an extension to our discrete system (1) of the geometric method for Lotka-Volterra differential systems

$$\frac{dx_i}{dt} = x_i(r_i - (Ax)_i), i \in I_N, \tag{11}$$

and Kolmogorov differential systems

$$\frac{dx_i}{dt} = x_i f_i(x), i \in I_N,$$

buried in a large number of publications (e.g. [37, 1, 28, 29, 17, 18, 19, 20, 21]) since conditions derived for the discrete system here is similar to those for (11). But if we bear in mind the essential difference between an orbit $x(n), n \geq 1$, for (1) and an orbit $x(t), t \geq 0$, for (11) described below, we shall realise that this is not a simple extension as the similarity of the conditions for both discrete and continuous systems suggests: the former consists of isolated points and the latter is a smooth continuous curve; the former can go from one side to the other side of a plane without actual intersection with the plane, i.e. jumps over the plane, but the latter going from one side to the other of a plane must have a intersection point with the plane. Indeed, from later sections we shall appreciate the subtlety of the techniques employed to tackle some hard obstacles laying in the process of proofs of the main theorems.

The virtue of the method is that the derived criteria for global attraction only uses the matrix A and the point r, irrelevant to the functions G_k as long as $(a1)$–$(a3)$ are met, and the criteria can be applied to a broad class of systems (1) with (2)–(5) and $(a1)$–$(a3)$, universal to all the above models with T defined by (6)–(9). Then, by finding the eigenvalues of the Jacobian matrix $DT(x^*)$, we know that either the fixed point is globally asymptotically stable or a homoclinic cycle is induced.

The rest of the paper is organised as follows: 2. Notation and preliminaries. 3. Main results. 4. Some examples. 5. Proof of the main theorems. 6. Proof of Lemma 5.2. 7. Conclusion. Appendix 1. Proof of $(a1)$–$(a3)$ for models (6–(9). Appendix 2. Proof of Proposition 2.3. Appendix 3. Proof of Lemma 5.1.

2. Notation and preliminaries

Denote the interior of $\mathbb{R}^N_+$ by $\text{int}\mathbb{R}^N_+$ and the boundary of $\mathbb{R}^N_+$ by $\partial\mathbb{R}^N_+$, and define

$$\pi_i = \{x \in \mathbb{R}^N_+ : x_i = 0\}, i \in I_N. \tag{12}$$

Then π_i is the part of the ith coordinate plane restricted to $\mathbb{R}^N_+$ and is part of $\partial\mathbb{R}^N_+$. For any $u, v \in \mathbb{R}^N_+$, we write $u \ll v$ or $v \gg u$ if $v - u \in \text{int}\mathbb{R}^N_+$, $u \leq v$ or $v \geq u$ if $v - u \in \mathbb{R}^N_+$, and $u < v$ or $v > u$ if $u \le v$ but $u \ne v$. If $u \le v$, we define

$$[u, v] = \{x \in \mathbb{R}^N : u \le x \le v\}. \tag{13}$$

Then $[u, v]$ is a k-dimensional *cell* if $v - u$ has exactly k non-zero components. For any $u \in \mathbb{R}^N_+$ and $I \subset I_N$, we also introduce the following notation

$$\mathbb{R}^N_+(u) = \{x \in \mathbb{R}^N_+ : x \ge u\}, \tag{14}$$

$$\pi_i(u) = \{x \in \mathbb{R}^N_+(u) : x_i = u_i\}, \tag{15}$$

$$\Gamma_i = \{x \in \mathbb{R}^N_+ : (Ax)_i = r_i\}, i \in I_N. \tag{16}$$

Then Γ_i is the ith *nullcline* plane for T defined by (2)–(5) and $(a1)$–$(a3)$, i.e. $T_i(x) = x_i$ on Γ_i. We abuse the notation slightly by using 0 to denote scalar number zero, vector zero and the origin in $\mathbb{R}^N$. For any plane Γ in $\mathbb{R}^N_+$ with $0 \notin \Gamma$, $\mathbb{R}^N_+$ is divided into three mutually exclusive connected subsets Γ^-, Γ and Γ^+ with $0 \in \Gamma^-$ such that $\mathbb{R}^N_+ = \Gamma^- \cup \Gamma \cup \Gamma^+$. A point $x \in \mathbb{R}^N_+$ is said to be *below* (*on* or *above*) Γ if $x \in \Gamma^-$ ($x \in \Gamma$ or $x \in \Gamma^+$). A nonempty set $S \subset \mathbb{R}^N_+$ is said to be *on* Γ if $S \subset \Gamma$; S is said to be *below* (*above*) Γ if $S \subset (\Gamma^- \cup \Gamma)$ ($S \subset (\Gamma \cup \Gamma^+)$) but $S \not\subset \Gamma$; S is said to be *strictly below* (*strictly above*) Γ if $S \subset \Gamma^-$ ($S \subset \Gamma^+$).

For $v \in \mathbb{R}^N_+$ and any $J \subset I_N$, we define $v^J \in \mathbb{R}^N_+$ by $v^J_j = v_j$ if $j \in J$ and $v^J_j = 0$ otherwise. Then $v^{I_N} = v$ and $v^{\emptyset} = 0$. Let $|J|$ denote the number of elements in J.

We view $\cap_{k \in \emptyset} \pi_k$ as $\mathbb{R}^N_+$ and denote the nonnegative half x_i-axis by X_i.

With the above notation, the condition $(a3)$ given in section 1 can be simplified as follows:

- $(a3)'$ For any nonempty $J \subset I_N$ and for any point $x \in [0, r]$ which is on Γ_i for some $i \in J$ but on or below Γ_j for all $j \in J$, the functions $T_j(x^{I_N \setminus J} + tx^J), j \in J$, are increasing for $t \in [0, 1]$.

Note from (5) and (16) that the plane Γ_i intersects the half axis X_i at a point Q_i, i.e. $\Gamma_i \cap X_i = \{Q_i\}$, with r_i as its ith component and 0 as other components. Thus, T has N axial fixed points $Q_i, i \in I_N$. Clearly, 0 is a repelling fixed point since the Jacobian matrix of T at the origin is $DT(0) = \text{diag}\{G_1(0), \ldots, G_N(0)\}$ with $G_i(0) > 1$ for all $i \in I_N$ by $(a1)$. We shall see that the point $r = (r_1, \ldots, r_N)^T \gg 0$ plays a very important role in this paper.

For any point $x \in \mathbb{R}^N_+$, the positive orbit $\gamma^+(x)$ is defined as

$$\gamma^+(x) = \{T^n(x) : n = 0, 1, 2, \cdots\}. \tag{17}$$

If T is invertible, then the negative orbit $\gamma^-(x)$ and the orbit $\gamma(x)$ can be defined as

$$\gamma^-(x) = \{(T^{-1})^n(x) : n = 0, 1, 2, \cdots\} \tag{18}$$

and $\gamma(x) = \gamma^+(x) \cup \gamma^-(x)$. The positive limit set $\omega(x)$ and the negative limit set $\alpha(x)$ are defined as usual: $\omega(x) = \cap_{n=0}^{\infty} \overline{\gamma^+(x(n))}$ and $\alpha(x) = \cap_{n=0}^{\infty} \overline{\gamma^-((T^{-1})^n(x))}$. A nonempty set $S \subset \mathbb{R}^N_+$ is said to be *invariant* if $T(S) = S$, *positive invariant* if $T(S) \subset S$, and *globally attracting* if $\omega(x) \subset S$ for all $x \in \mathbb{R}^N_+$.

Denote the open ball centred at $x \in \mathbb{R}^N$ with radius $\delta > 0$ by $\mathcal{B}(x, \delta)$. A fixed point $x^* \in \mathbb{R}^N_+$ is called *stable* if for any $\varepsilon > 0$ there is a $\delta > 0$ such that $x \in \mathcal{B}(x^*, \delta) \cap \mathbb{R}^N_+$ implies $x(n) = T^n(x) \in \mathcal{B}(x^*, \varepsilon) \cap \mathbb{R}^N_+$ for all integers $n > 0$. For a nonempty subset $J \subset I_N$ and a fixed point $x^* \in \mathbb{R}^N_+ \setminus \{0\}$ with $x^*_i > 0$ if and only if $i \in J$, x^* is said to be *globally attracting* if for any $x \in \mathbb{R}^N_+ \setminus (\cup_{i \in J} \pi_i)$ we have $\omega(x) = \{x^*\}$, and *globally asymptotically stable* if it is stable and globally attracting. Therefore, for a globally asymptotically stable fixed point $x^* > 0$, if $x^* \in \text{int}\mathbb{R}^N_+$ then it attracts all the interior points; if $x^* \in \partial\mathbb{R}^N_+$ it attracts not only the interior points but also the boundary points with $x_i > 0$ for all $i \in J$.

The principal idea of proving global attraction of a fixed point $x^* \in \mathbb{R}^N_+ \setminus \{0\}$ is as follows:

(i) If there is a point $u^{(0)} \geq 0$ such that $y \geq u^{(0)}$ for all $x \in \mathbb{R}^N_+ \setminus (\cup_{i \in J} \pi_i)$ and all $y \in \omega(x)$, then we can always find a point $v^{(0)} > u^{(0)}$ such that $\omega(x) \subset [u^{(0)}, v^{(0)}]$ for all $x \in \mathbb{R}^N_+ \setminus (\cup_{i \in J} \pi_i)$.

(ii) If $\omega(x) \subset [u^{(0)}, v^{(0)}]$ for all $x \in \mathbb{R}^N_+ \setminus (\cup_{i \in J} \pi_i)$ then we can always find a point $u^{(1)}$ with $u^{(0)} < u^{(1)} < v^{(0)}$ such that $\omega(x) \subset [u^{(1)}, v^{(0)}]$ for all $x \in \mathbb{R}^N_+ \setminus (\cup_{i \in J} \pi_i)$.

(iii) Then repetition of (i) and (ii) leads to $\omega(x) = \{x^*\}$ for all $x \in \mathbb{R}^N_+ \setminus (\cup_{i \in J} \pi_i)$ under the assumptions.

The proofs of these are technical, so we leave them to later sections except the proof for (i) (Proposition 2.1) with the following reason: At this stage we need to observe that the cell $[0, r]$ is positive invariant and globally attracting. Instead of giving a direct proof for this observation here, we prove Proposition 2.1 from which the observation immediately follows.

Let $J \subset I_N$ be a nonempty set and let $u \in \mathbb{R}^N_+$ such that $u_i = 0$ for each $i \in I_N \setminus J$ and $u \in \Gamma^-_j$ for all $j \in J$. Define $v \geq u$ by

$$\forall i \in I_N, \quad v_i = \begin{cases} u_i & \text{if} \quad u \in \Gamma_i \cup \Gamma^+_i, \\ r_i - (Au^{I_N \setminus \{i\}})_i & \text{if} \quad u \in \Gamma^-_i. \end{cases} \tag{19}$$

Then v is on or above Γ_i for all $i \in I_N$. Moreover, for any $w \geq u$, if $w_i > v_i$ for some $i \in I_N$ then w is above Γ_i since $(Aw)_i > v_i + (Au^{I_N \setminus \{i\}})_i = r_i$. From (19) we see that if $u = 0$ then $v = r$.

Proposition 2.1. *Assume the existence of a nonempty set $J \subset I_N$ and a point $u \in \mathbb{R}^N_+$ satisfying $u_i = 0$ for all $i \in I_N \setminus J$ and $u \in \Gamma^-_j$ for all $j \in J$ such that $\omega(x) \subset \mathbb{R}^N_+(u)$ for* *all* $x \in \mathbb{R}_+^N \setminus (\cup_{i\in J}\pi_i)$. *Then, for (1) with (2)–(5) and* $(a1)$–$(a3)$ *and* v *defined by (19),* $\omega(x) \subset [u,v]$ *for all* $x \in \mathbb{R}_+^N \setminus (\cup_{i\in J}\pi_i)$ *and* $T(y) \le v$ *for all* $y \in [u,v]$.

Proof. Take an arbitrary $i \in I_N$ and an arbitrary $x \in \mathbb{R}_+^N \setminus (\cup_{j\in J}\pi_j)$. Then $\omega(x) \subset \mathbb{R}_+^N(u)$ by assumption. Thus, for any small enough $\delta > 0$, there is N_1 such that $\gamma^+(T^{N_1}(x)) \subset \mathbb{R}_+^N(u')$, where $u' \in [0,u]$ with $u'_j = \max\{0, u_j - \delta\}$ for all $j \in I_N$. So $x(n) \ge u'$ for all $n \ge N_1$. Let $v' \ge u'$ be defined by (19) with the replacement of u, v by u' and v'. If $T_i^{N_1}(x) > v'_i$ then $T^{N_1}(x) \in \Gamma_i^+$ by (16) and (19), so $T_i^{N_1+1}(x) < T_i^{N_1}(x)$ and $x_i(n)$ is decreasing in n as long as $x(n) \in \Gamma_i^+$ for $n > N_1$. Thus, either there is an integer $K > N_1$ such that $x_i(K) \le v'_i$ or $x_i(n) > v'_i$ for all $n > N_1$. In the latter case, $x(n) \in \Gamma_i^+$ so the sequence $\{x_i(n)\}$ is decreasing and bounded below by v'_i. We claim that $\lim_{n\to+\infty} x_i(n) = v'_i$. Indeed, if $x_i(n) \downarrow \eta > v'_i$ as $n \to +\infty$, then $(Ax(n))_i \ge (Au'^{I_N\setminus\{i\}})_i + \eta > r_i$ so $G_i((Ax(n))_i) \le G_i((Au'^{I_N\setminus\{i\}})_i + \eta) = \xi < G_i(r_i) = 1$ for all $n > N_1$ and

$$x_i(N_1+k+1) = x_i(N_1+k)G_i((Ax(N_1+k))_i) \le \xi x_i(N_1+k) \le \xi^{k+1} x_i(N_1) \to 0$$

as $k \to +\infty$, a contradiction to $x_i(n) > v'_i$ for all $n > N_1$. This shows the above claim. In the former case, we show that $x_i(n) \in [u'_i, v'_i]$ for all $n > K$. If $x(K) \in \Gamma_i \cup \Gamma_i^+$, then $G_i((Ax(K))_i) \le 1$ so

$$x_i(K+1) = T_i(x(K)) = x_i(K)G_i((Ax(K))_i) \le x_i(K) \le v'_i.$$

If $x(K) \in \Gamma_i^-$, then $G_i((Ax(K))_i) > 1$ and there is $y \in \Gamma_i$ such that $(y - x(K))^{I_N\setminus\{i\}} = 0$,

$$x_i(K) < y_i = r_i - (Ax(K)^{I_N\setminus\{i\}})_i \le r_i - (Au'^{I_N\setminus\{i\}})_i = v'_i$$

and $y_i = T_i(y)$. By $(a2)$, $T_i(x(K)) < T_i(y) = y_i \le v'_i$. This shows that $T_i(x(K)) \le v'_i$ if $x_i(K) \le v'_i$. Repeating the above process, we obtain $x_i(n) \le v'_i$, so $x_i(n) \in [u'_i, v'_i]$, for all $n \ge K$. Then $\omega(x) \subset [u', v']$ follows from the arbitrariness of $i \in I_N$. By letting $\delta \to 0$ we obtain $\omega(x) \subset [u,v]$ for all $x \in \mathbb{R}_+^N \setminus (\cup_{j\in J}\pi_j)$.

For any $y \in [u,v]$ and $i \in I_N$, if $y \in (\Gamma_i \cup \Gamma_i^+)$ then $T_i(y) \le y_i \le v_i$. If $y \in \Gamma_i^-$ then there is a $\bar{y} \in \Gamma_i$ such that $(\bar{y} - y)^{I_N\setminus\{i\}} = 0$. By $(a2)$, $T_i(y) < T_i(\bar{y}) = \bar{y}_i \le v_i$. Hence, $T(y) \le v$. □

From Proposition 2.1 with $u = 0$ we see that $[0,r]$ is positive invariant and globally attracting on $\mathbb{R}_+^N$. Since $(a1)$ implies that 0 a repelling fixed point, we also observe that system (1) with (2)–(5) and $(a1)$–$(a3)$ on $\mathbb{R}_+^N \setminus \{0\}$ has a closed invariant set in $[0,r] \setminus \{0\}$ that is a global attractor.

Proposition 2.2. *There is a closed invariant set* $S_0 \subset [0,r] \setminus \{0\}$ *that is a global attractor of system (1) with (2)–(5) and* $(a1)$–$(a3)$ *on* $\mathbb{R}_+^N \setminus \{0\}$.

Proof. Let $S_1 = [0,r] \setminus \cap_{i\in I_N}\Gamma_i^-$. Since $0 < G_i((Ar)_i) \le 1$ and $G_i((Ar)_i) \le G_i((Ax)_i) \le G_i(0)$ for all $i \in I_N$ and $x \in [0,r]$, we have

$$\delta_0 = \min\{G_1((Ar)_1), \ldots, G_N((Ar)_N)\} \in (0,1].$$

Let

$$S_2 = \{x \in [0, r] : x = \delta y \text{ for some } y \in S_1, \delta \in [\delta_0, 1]\}.$$

Then S_2 is closed and $S_1 \subset S_2 \subset [0, r] \setminus \{0\}$. Since $T(x) \geq x, T(x) \neq x$ for all $x \in \cap_{i \in I_N} \Gamma_i^- \setminus \{0\}$ and $T(x) \geq \delta_0 x \in S_2$ for all $x \in S_1$, by the positive invariance of $[0, r]$ the set S_2 is positive invariant. As $[0, r]$ is globally attracting on $\mathbb{R}_+^N$ and 0 is repelling, we can easily check that S_2 is globally attracting on $\mathbb{R}_+^N \setminus \{0\}$. Then $S_0 = \cap_{n=0}^{\infty} T^n(S_2)$ meets the requirement. □

Note that under conditions which guarantee the existence of a carrying simplex Σ, we must have $S_0 = \Sigma$.

For a fixed $i \in I_N$, consider the following geometric condition

$$\forall k \in I_N \setminus \{i\}, \Gamma_k \cap [0, r] \cap \pi_i \text{ is strictly below } \Gamma_i \tag{20}$$

and the algebraic inequality

$$\max\{0, a_{ij}(r_j - (Ar^{I_N \setminus \{i,j\}})_j)\} < r_i - (Ar^{I_N \setminus \{i,j\}})_i. \tag{21}$$

Our next proposition establishes the relationship between (20) and (21).

Proposition 2.3. *Regarding (20) and (21) the following statements hold.*

(i) *Let $J \subset I_N$ with $|J| > 1$. If (20) holds for all $i \in J$, then (21) holds for all $i, j \in J$ with $i \neq j$.*

(ii) *Conversely, if (21) holds for a fixed $i \in I_N$ and all $j \in I_N \setminus \{i\}$, then (20) holds.*

(iii) *Hence, condition (20) holds for all $i \in I_N$ if and only if (21) holds for all $i, j \in I_N$ with $i \neq j$.*

Although the proof of this proposition is similar to that of Lemma 2.4 in [17], to ease the pain of juggling between different contexts under different notation, a self-contained proof of Proposition 2.3 is provided in Appendix 2 at the end of this paper.

3. Main results

We shall see that the global attraction of an interior or boundary fixed point x^*, even its existence, will be determined purely by the relative position of the N planes $\Gamma_1, \ldots, \Gamma_N$ on $[0, r] \cap \pi_i$ for all $i \in I_N$. If $x^* \in \text{int}\mathbb{R}_+^N$, the Jacobian matrix of T at x^* is

$$DT(x^*) = I + \text{diag}[x_1^* G_1'(r_1), \ldots, x_N^* G_N'(r_N)]A. \tag{22}$$

If $x^* \in \partial\mathbb{R}_+^N$ such that $x_j^* = 0$ if and only if $j \in J \subset I_N$, then $DT(x^*)$ is given by (22) with the replacement of 1 on the jth row of I by $G_j((Ax^*)_j)$ for all $j \in J$.

3.1. **Results for global attraction.** Our first expected result is for existence and global attraction of an interior fixed point.

Theorem 3.1. *Assume that* $\Gamma_j \cap [0, r] \cap \pi_i$ *is strictly below* Γ_i *for each* $i \in I_N$ *and every* $j \in I_N \setminus \{i\}$*. Then system (1) with (2)–(5) and* $(a1)$*–*$(a3)$ *has a unique interior fixed point* $x^* \in [0, r] \cap \mathrm{int}\mathbb{R}^N_+$ *and it is globally attracting in* $\mathrm{int}\mathbb{R}^N_+$*.*

Note that if the point $r^{I_N \setminus \{i\}}$ is below Γ_i, then the set $[0, r] \cap \pi_i = [0, r^{I_N \setminus \{i\}}]$ is strictly below Γ_i, so $\Gamma_j \cap [0, r] \cap \pi_i$ is strictly below Γ_i. This particular case of Theorem 3.1 is stated as a corollary below.

Corollary 3.2. *Assume that* $r^{I_N \setminus \{i\}}$ *is below* Γ_i *for all* $i \in I_N$*. Then the conclusion of Theorem 3.1 holds.*

The next result is for existence and global attraction of a general boundary fixed point.

Theorem 3.3. *Let* $J \subset I_N$ *with* $1 \leq |J| \leq N - 1$*. Assume that the following conditions hold.*

(i) *For each* $i \in J$ *and every* $j \in I_N \setminus \{i\}$*,* $\Gamma_j \cap [0, r] \cap \pi_i$ *is strictly below* Γ_i*.*

(ii) *For each* $k \in I_N \setminus J$*, there is an* $i_k \in J$ *such that* $\Gamma_k \cap (\cap_{j \in I_N \setminus J} \pi_j)$ *is below* Γ_{i_k}*.*

Then system (1) with (2)–(5) and $(a1)$*–*$(a3)$ *has a fixed point* $x^* \in [0, r]$ *with* $x^*_i > 0$ *if and only if* $i \in J$ *and* x^* *is globally attracting.*

Remark 1. Note from Theorem 3.1 that condition (i) in Theorem 3.3 ensures that the $|J|$-dimensional subsystem has a globally attracting fixed point in $\mathrm{int}\mathbb{R}^{|J|}_+$, so system (1) has a fixed point $x^* \in [0, r]$ with $x^*_i > 0$ if and only if $i \in J$. From the proof given later we shall see that the global attraction of x^* in $\mathbb{R}^N_+ \setminus (\cup_{j \in J} \pi_j)$ requires that

$(ii)'$ x^* is on or above Γ_k for every $k \in I_N \setminus J$.

Since $x^* \in \cap_{j \in J} \Gamma_j$, it is clear that condition (ii) in Theorem 3.3 guarantees $(ii)'$ above. Thus, if we know x^* already, we may replace (ii) by the weaker condition $(ii)'$. However, for global asymptotic stability of x^*, we require that every eigenvalue λ of $DT(x^*)$ satisfies $|\lambda| < 1$ (see Theorem 3.9 in the next subsection). Since each $G_k((Ax^*)_k)$ for $k \in I_N \setminus J$ is an eigenvalue of $DT(x^*)$ and $G_k((Ax^*)_k) < 1$ if x^* is above Γ_k, for global asymptotic stability, x^* must be above Γ_k for all $k \in I_N \setminus J$.

Corollary 3.4. *Assume that* $r^{I_N \setminus \{i\}}$ *is below* Γ_i *for all* $i \in J$*. Then there is a fixed point* $x^* \in \mathbb{R}^N_+$ *such that* $x^*_i > 0$ *if and only if* $i \in J$*. If* x^* *is on or above* Γ_j *for all* $j \in I_N \setminus J$*, then* x^* *is globally attracting.*

So far the above results are stated under geometric conditions in terms of the relative position of the nullcline planes restricted to $[0, r] \cap \pi_i$ for $i \in I_N$. For convenience in checking these geometric conditions, we need to find equivalent algebraic conditions in terms of a_{ij} and r_i only. Actually, Proposition 2.3 serves this purpose. By Proposition 2.3 (iii), Theorem 3.1 can be restated as follows.

Theorem 3.5. *Assume that*

$$\forall i, j \in I_N (i \neq j),\ \max\{0, a_{ij}(r_j - (Ar^{I_N \setminus \{i,j\}})_j)\} < r_i - (Ar^{I_N \setminus \{i,j\}})_i. \tag{23}$$

Then system (1) with (2)–(5) and (a1)–(a3) has a unique fixed point $x^* \in [0, r] \cap \mathrm{int}\mathbb{R}^N_+$ and it is globally attracting.

Note that the point $r^{I_N \setminus \{i\}}$ is below Γ_i if and only if $(Ar^{I_N \setminus \{i\}})_i < r_i$, which is the same as $(Ar)_i < 2r_i$. Then Corollary 3.2 can be restated as follows.

Corollary 3.6. *Assume that $Ar \ll 2r$. Then the conclusion of Theorem 3.5 holds.*

The condition that $\Gamma_k \cap (\cap_{j \in I_N \setminus J} \pi_j)$ is below Γ_{i_k} means that $a_{i_k j} r_k \leq a_{kj} r_{i_k}$ for all $j \in J$. Then, by Proposition 2.3 (ii), we restate Theorem 3.3 and Corollary 3.4 below.

Theorem 3.7. *Let $J \subset I_N$ with $1 \leq |J| \leq N - 1$. Assume that the following conditions hold.*

(i) *For each $i \in J$ and every $j \in I_N \setminus \{i\}$,*

$$\max\{0, a_{ij}(r_j - (Ar^{I_N \setminus \{i,j\}})_j)\} < r_i - (Ar^{I_N \setminus \{i,j\}})_i.$$

(ii) *For each $k \in I_N \setminus J$, there is an $i_k \in J$ such that $a_{i_k j} r_k \leq a_{kj} r_{i_k}$ for all $j \in J$.*

Then system (1) with (2)–(5) and (a1)–(a3) has a fixed point $x^* \in [0, r]$ with $x_i^* > 0$ if and only if $i \in J$ and x^* is globally attracting.

Corollary 3.8. *Assume that $(Ar)_i < 2r_i$ for all $i \in J$. Then there is a fixed point $x^* \in \mathbb{R}^N_+$ such that $x_i^* > 0$ if and only if $i \in J$. If $(Ax^*)_j \geq r_j$ for al $j \in I_N \setminus J$, then x^* is globally attracting.*

3.2. **Results for global asymptotic stability.** In this subsection, we first state a general theorem for global asymptotic stability when global attraction is know. Then we give a particular case of Theorem 3.7 when an axial fixed point is globally asymptotically stable.

Theorem 3.9. *Assume that $x^* \in \mathbb{R}^N_+ \setminus \{0\}$ is a globally attracting fixed point. Then the following conclusions hold: (i) If each eigenvalue λ of $DT(x^*)$ satisfies $|\lambda| < 1$, then x^* is globally asymptotically stable. (ii) If $DT(x^*)$ is invertible and has an eigenvalue λ satisfying $|\lambda| > 1$, then there is a homoclinic cycle.*

Proof. (i) If each eigenvalue λ of $DT(x^*)$ satisfies $|\lambda| < 1$, then x^* is locally asymptotically stable. Then the global asymptotic stability of x^* follows from its local stability and global attraction. (ii) If $DT(x^*)$ is invertible and has an eigenvalue λ satisfying $|\lambda| > 1$, then there is a point $x \in \mathbb{R}^N_+ \setminus (\{x^*\} \cup_{i \in J} \pi_i)$ such that $\alpha(x) = \{x^*\}$. By the global attraction of x^*, we also have $\omega(x) = \{x^*\}$. Then $\gamma(x)$ with x^* forms a homoclinic cycle. □

Remark 2. We shall see examples in section 4 demonstrating the application of Theorem 3.9 (i). But we are not sure whether examples of the case for Theorem 3.9 (ii) exist. See the open problems in section 7.

The next theorem is for global stability of an axial fixed point Q_i.

Theorem 3.10. *Assume that* $\Gamma_j \cap [0, r] \cap \pi_i$ *is strictly below* Γ_i *and* Q_i *is above* Γ_j *for some* $i \in I_N$ *and all* $j \in I_N \setminus \{i\}$*. Then* Q_i *is globally asymptotically stable.*

The following is a particular case of Theorem 3.10 with a simple condition that $r^{I_N \setminus \{i\}}$ is below Γ_i.

Corollary 3.11. *Assume that* $r^{I_N \setminus \{i\}}$ *is below* Γ_i *and* Q_i *is above* Γ_j *for some* $i \in I_N$ *and all* $j \in I_N \setminus \{i\}$*. Then* Q_i *is globally asymptotically stable.*

The condition that the axial fixed point Q_i is above Γ_j holds if and only if $a_{ji} r_i > r_j$. Then, by Proposition 2.3 (ii), Theorem 3.10 and Corollary 3.11 can be restated as follows.

Theorem 3.12. *Assume that* $r_j < a_{ji} r_i$ *and*

$$\max\{0, a_{ij}(r_j - (Ar^{I_N \setminus \{i,j\}})_j)\} < r_i - (Ar^{I_N \setminus \{i,j\}})_i \tag{24}$$

for some $i \in I_N$ *and all* $j \in I_N \setminus \{i\}$*. Then the axial fixed point* Q_i *is globally asymptotically stable.*

Corollary 3.13. *Assume that* $(Ar)_i < 2r_i$ *and* $r_j < a_{ji} r_i$ *for some* $i \in I_N$ *and all* $j \in I_N \setminus \{i\}$*. Then* Q_i *is globally asymptotically stable.*

3.3. Combination of the results for global attraction and Theorem 3.9.

After stating the criteria for global attraction of an interior or boundary fixed point and the additional condition required for global asymptotic stability, we are now able to combine Theorems 3.1, 3.3 and 3.9 into a unified version of these results.

Theorem 3.14. *Let* $J \subset I_N$ *with* $J \neq \emptyset$*. Assume that for each* $i \in J$ *and every* $j \in I_N \setminus \{i\}$*,* $\Gamma_j \cap [0, r] \cap \pi_i$ *is strictly below* Γ_i*. Then system (1) with (2)–(5) and* (a1)*–*(a3) *has a fixed point* $x^* \in \mathbb{R}^N_+$ *such that* $x^*_i > 0$ *if and only if* $i \in J$*. If either* $J = I_N$ *or*

(i) *for each* $k \in I_N \setminus J$*, there is an* $i_k \in J$ *such that* $\Gamma_k \cap (\cap_{j \in I_N \setminus J} \pi_j)$ *is below* Γ_{i_k}*,*

then x^* *is globally attracting. In addition, if*

(ii) *every eigenvalue* λ *of* $DT(x^*)$ *satisfies* $|\lambda| < 1$*,*

then x^* *is globally asymptotically stable.*

Combination of Corollary 3.2, Corollary 3.4 and Theorem 3.9 gives the following corollary.

Corollary 3.15. *Let $J \subset I_N$ with $J \neq \emptyset$. Assume that $r^{I_N \setminus \{i\}}$ is below Γ_i for each $i \in J$. Then system (1) with (2)–(5) and (a1)–(a3) has a fixed point $x^* \in \mathbb{R}^N_+$ such that $x^*_i > 0$ if and only if $i \in J$. If either $J = I_N$ or*

(i) *x^* is on or above Γ_j for all $j \in I_N \setminus J$,*

then x^* is globally attracting. In addition, if

(ii) *every eigenvalue λ of $DT(x^*)$ satisfies $|\lambda| < 1$,*

then x^* is globally asymptotically stable.

Using Proposition 2.3 and combining Theorems 3.5, 3.7 and 3.9, we obtain the following theorem.

Theorem 3.16. *Let $J \subset I_N$ with $J \neq \emptyset$. Assume that*

$$\max\{0, a_{ij}(r_j - (Ar^{I_N \setminus \{i,j\}})_j)\} < r_i - (Ar^{I_N \setminus \{i,j\}})_i$$

for each $i \in J$ and every $j \in I_N \setminus \{i\}$. Then system (1) with (2)–(5) and (a1)–(a3) has a fixed point $x^* \in \mathbb{R}^N_+$ such that $x^*_i > 0$ if and only if $i \in J$. If either $J = I_N$ or

(i) *for each $k \in I_N \setminus J$, there is an $i_k \in J$ such that $a_{i_k j} r_k \leq a_{kj} r_{i_k}$ for all $j \in J$,*

then x^* is globally attracting. In addition, if

(ii) *every eigenvalue λ of $DT(x^*)$ satisfies $|\lambda| < 1$,*

then x^* is globally asymptotically stable.

Combination of Corollary 3.6, Corollary 3.8 and Theorem 3.9 gives the following corollary.

Corollary 3.17. *Let $J \subset I_N$ with $J \neq \emptyset$. Assume that $(Ar)_i < 2r_i$ for all $i \in J$. Then system (1) with (2)–(5) and (a1)–(a3) has a fixed point $x^* \in \mathbb{R}^N_+$ such that $x^*_i > 0$ if and only if $i \in J$. If either $J = I_N$ or*

(i) *$(Ax^*)_j \geq r_j$ for all $j \in I_N \setminus J$,*

then x^* is globally attracting. In addition, if

(ii) *every eigenvalue λ of $DT(x^*)$ satisfies $|\lambda| < 1$,*

then x^* is globally asymptotically stable.

4. Some examples

The results obtained in section 3 above are for system (1) with (2)–(5) and $(a1)$–$(a3)$. Since the Ricker models (6), the Leslie-Gower models (7), the Atkinson-Allen models (8), the extended Atkinson-Allen models (9) and models formed by any combination of (6)–(9) with (3) and (5) satisfy $(a1)$–$(a3)$, these results can be directly applied to such models.

We first apply our results to these four models with $N = 2$ and derive simple conditions for global asymptotic stability. Then we analyse the global stability for three dimensional systems of these four models with a circulant matrix A and $r = r_0(1,1,1)^T$ for $r_0 > 0$. An example of 4-dimensional system is given to demonstrate the global asymptotic stability of an axial fixed point. A final example is for a system of combination of the four models having a general boundary fixed point.

Example 4.1. Consider the two-dimensional system (1) with (2)–(5) and $(a1)$–$(a3)$. Then, with $\alpha \geq 0$ and $\beta \geq 0$, the matrix A and the component equations of $Ax = r$ for Γ_1 and Γ_2 can be written as

$$A = \begin{pmatrix} 1 & \alpha \\ \beta & 1 \end{pmatrix}, \quad \Gamma_1 = \{x \in \mathbb{R}^2_+ : x_1 + \alpha x_2 = r_1\}, \quad \Gamma_2 = \{x \in \mathbb{R}^2_+ : \beta x_1 + x_2 = r_2\}.$$

The system has axial fixed points Q_1 and Q_2 with coordinates $(r_1, 0)$ and $(0, r_2)$ respectively. Then Q_1 is below (above) Γ_2 if and only if $\beta r_1 < r_2$ ($\beta r_1 > r_2$), and Q_2 is below (above) Γ_1 if and only if $\alpha r_2 < r_1$ ($\alpha r_2 > r_1$). If there is an interior fixed point x^*, then $x_1^* = \frac{r_1 - \alpha r_2}{1-\alpha\beta}$, $x_2^* = \frac{r_2 - \beta r_1}{1-\alpha\beta}$ and

$$DT(x^*) = \begin{pmatrix} 1 + x_1^* G_1'(r_1) & \alpha x_1^* G_1'(r_1) \\ \beta x_2^* G_2'(r_2) & 1 + x_2^* G_2'(r_2) \end{pmatrix}.$$

The two eigenvalues $\lambda_{1,2}$ of $DT(x^*)$ are

$$\lambda_{1,2} = 1 + \frac{1}{2}[(x_1^* G_1'(r_1) + x_2^* G_2'(r_2)) \pm Z],$$

where

$$Z = \sqrt{(x_1^* G_1'(r_1) + x_2^* G_2'(r_2))^2 - 4x_1^* x_2^* G_1'(r_1) G_2'(r_2)(1 - \alpha\beta)}.$$

By $(a1)$, $G_1'(r_1) < 0$ and $G_2'(r_2) < 0$. If $1 - \alpha\beta > 0$ then $Z > 0$ and $\lambda_2 < \lambda_1 < 1$. To require $|\lambda_{1,2}| < 1$, we only need $\lambda_2 > -1$ or $\lambda_2 - 1 > -2$. This is equivalent to

$$4 + (x_1^* G_1'(r_1) + x_2^* G_2'(r_2)) > Z.$$

If $4 + (x_1^* G_1'(r_1) + x_2^* G_2'(r_2)) \geq 0$, the above inequality can be simplified as

$$4 + 2(x_1^* G_1'(r_1) + x_2^* G_2'(r_2)) + x_1^* x_2^* G_1'(r_1) G_2'(r_2)(1 - \alpha\beta) > 0.$$

As $x_1^* x_2^* G_1'(r_1) G_2'(r_2)(1 - \alpha\beta) > 0$, if

$$2 + (x_1^* G_1'(r_1) + x_2^* G_2'(r_2)) \geq 0, \tag{25}$$

then $|\lambda_{1,2}| < 1$. By Corollaries 3.11 and 3.2 and Theorem 3.9, we obtain the following.

Theorem 4.2. *The following statements hold for (1) with (2)–(5) and* $(a1)$–$(a3)$ *when* $N = 2$ *and* A *is given above.*

(i) *If* $\beta r_1 < r_2$ *and* $\alpha r_2 > r_1$, *then* Q_2 *is globally asymptotically stable.*

(ii) *If* $\beta r_1 > r_2$ *and* $\alpha r_2 < r_1$, *then* Q_1 *is globally asymptotically stable.*

(iii) *If* $\beta r_1 < r_2$ *and* $\alpha r_2 < r_1$, *then there is a globally attracting interior fixed point* x^*. *In addition, if (25) holds then* x^* *is globally asymptotically stable.*

Next, we check that the inequality (25) holds for each of the four models (6)–(9) under the condition $\beta r_1 < r_2$ and $\alpha r_2 < r_1$. Clearly, $\alpha\beta < 1$. For Ricker models, $G_i(u) = e^{r_i - u}$, so $G_i'(r_i) = -1$ and

$$\begin{aligned} 2 + (x_1^* G_1'(r_1) + x_2^* G_2'(r_2)) &= 2 - \frac{r_1 - \alpha r_2}{1 - \alpha\beta} - \frac{r_2 - \beta r_1}{1 - \alpha\beta} \\ &\geq 2 - \frac{r_1 - \alpha\beta r_1}{1 - \alpha\beta} - \frac{r_2 - \beta\alpha r_2}{1 - \alpha\beta} = 2 - r_1 - r_2. \end{aligned}$$

Then (25) follows from $0 < r_i \leq 1$. For Leslie-Gower models, $G_i(u) = \frac{1+r_i}{1+u}$, so $G_i'(r_i) = -\frac{1}{1+r_i}$ and

$$\begin{aligned} 2 + (x_1^* G_1'(r_1) + x_2^* G_2'(r_2)) &= 2 - \frac{r_1 - \alpha r_2}{(1+r_1)(1-\alpha\beta)} - \frac{r_2 - \beta r_1}{(1+r_2)(1-\alpha\beta)} \\ &\geq 2 - \frac{r_1}{1+r_1} - \frac{r_2}{1+r_2} > 0. \end{aligned}$$

For Atkinson-Allen models, $G_i(u) = b + \frac{2(1-b)}{1+u}$ with $0 < b < 1$ and $r_i = 1$, so $G_i'(r_i) = -\frac{1-b}{2}$ and

$$\begin{aligned} 2 + (x_1^* G_1'(r_1) + x_2^* G_2'(r_2)) &= 2 - \frac{(1-\alpha)(1-b)}{2(1-\alpha\beta)} - \frac{(1-\beta)(1-b)}{2(1-\alpha\beta)} \\ &\geq 2 - \frac{1-b}{2} - \frac{1-b}{2} = 1 + b > 0. \end{aligned}$$

For generalised Atkinson-Allen models, $G_i(u) = b_i + \frac{(1+r_i)(1-b_i)}{1+u}$, so $G_i'(r_i) = -\frac{1-b_i}{1+r_i}$ and

$$\begin{aligned} 2 + (x_1^* G_1'(r_1) + x_2^* G_2'(r_2)) &= 2 - \frac{(1-b_1)(r_1 - \alpha r_2)}{(1+r_1)(1-\alpha\beta)} - \frac{(1-b_2)(r_2 - \beta r_1)}{(1+r_2)(1-\alpha\beta)} \\ &\geq 2 - \frac{r_1}{1+r_1} - \frac{r_2}{1+r_2} > 0. \end{aligned}$$

Therefore, from Theorem 4.2 (iii) we see that if $\beta r_1 < r_2$ and $\alpha r_2 < r_1$, then there is a globally asymptotically stable interior fixed point x^* for the four models (6)–(9). Note that, under the conditions $\alpha > 0$ and $\beta > 0$, the results obtained here for Ricker and Leslie-Gower models is consistent with those given in [5].

Example 4.3. Consider the four models given by (6)–(9) with $N = 3$, $r_i = r_0 > 0$ for $i \in I_3$ and, with $\alpha \geq 0$ and $\beta \geq 0$,

$$A = \begin{pmatrix} 1 & \alpha & \beta \\ \beta & 1 & \alpha \\ \alpha & \beta & 1 \end{pmatrix}. \tag{26}$$

Then $Ax = r$ has a solution $x^* \in \mathrm{int}\mathbb{R}_+^3$ with $x_i^* = \frac{r_0}{1+\alpha+\beta}$ for $i \in I_3$. We now derive a condition for global attraction of x^* by using Theorem 3.5. For $i = 3$ and $j = 1$, (21) becomes

$$\max\{0, a_{31}(r_1 - (Ar^{\{2\}})_1)\} < r_3 - (Ar^{\{2\}})_3,$$

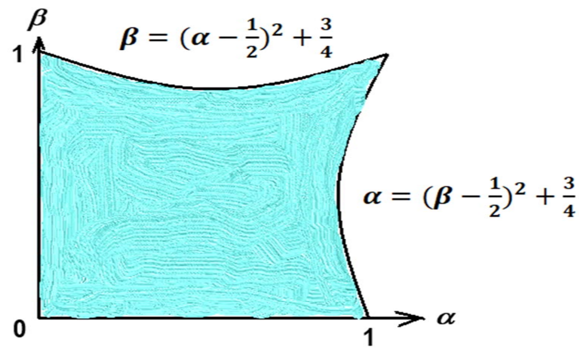

FIGURE 1. A stability region for (α, β) obtained by Theorems 3.5 and 3.9.

i.e.

$$\max\{0, \alpha(r_0 - \alpha r_0)\} < r_0 - \beta r_0. \tag{27}$$

As $r_0 > 0$, (27) implies that $\beta < 1$. For $i = 3$ and $j = 2$, (21) becomes

$$\max\{0, a_{32}(r_2 - (Ar^{\{1\}})_2)\} < r_3 - (Ar^{\{1\}})_3,$$

i.e.

$$\max\{0, \beta(r_0 - \beta r_0)\} < r_0 - \alpha r_0, \tag{28}$$

from which follows $\alpha < 1$. Then (27) and (28) can be simplified as

$$0 \leq \alpha < 1, 0 \leq \beta < 1, \alpha + \beta(1 - \beta) < 1, \beta + \alpha(1 - \alpha) < 1. \tag{29}$$

Then, by Theorem 3.5, x^* is globally attracting if α and β satisfy (29). The (α, β) region given by (29) is shown in Figure 1. Note that this region is independent of the value r_0 and it is a region for global attraction of x^*. (It may not be the largest region for global attraction. See comparisons given after the proof for global asymptotic stability.)

Next we show the global asymptotic stability of x^*, when (α, β) is in this region, for (6) with $0 < r_0 \leq 1$, for (7) with $0 < r_0 < +\infty$, for (8) with $r_0 = 1$, and for (9) with $b_i = b \in (0, 1)$ and $0 < r_0 < +\infty$.

For (6) with $r_i = r_0 \in (0, 1]$, we have $G_i(u) = e^{r_0 - u}$ and $G_i'(r_i) = -1$ so $DT(x^*) = I - \frac{r_0}{1+\alpha+\beta}A$. As A is a circulant matrix, and so is $DT(x^*)$, we can easily check that $DT(x^*)$ has eigenvalues $\lambda_1 = 1 - r_0 \in [0, 1)$ and

$$\lambda_{2,3} = \frac{1}{2(1 + \alpha + \beta)}\{[2(1 + \alpha + \beta) - r_0(2 - \alpha - \beta)] \pm i\sqrt{3}r_0|\alpha - \beta|\}.$$

Then $|\lambda_{2,3}| < 1$ if and only if

$$[2(1+\alpha+\beta) - r_0(2-\alpha-\beta)]^2 + 3r_0^2(\alpha-\beta)^2 < 4(1+\alpha+\beta)^2,$$

which is simplified to

$$r_0[(\alpha+\beta-2)^2 + 3(\alpha-\beta)^2] < 4(1+\alpha+\beta)(2-\alpha-\beta).$$

Since $0 < r_0 \leq 1$, the above inequality holds if

$$(\alpha+\beta-2)^2 + 3(\alpha-\beta)^2 < 4(1+\alpha+\beta)(2-\alpha-\beta),$$

which can be simplified to

$$2(\alpha^2+\beta^2) + \alpha\beta < 1 + 2(\alpha+\beta). \tag{30}$$

By $\alpha < 1$ and $\beta < 1$, we have $\frac{1}{2}(\alpha+\beta) < 1$ so

$$\begin{aligned} 2(\alpha^2+\beta^2) + \alpha\beta &\leq 2(\alpha^2+\beta^2) + \frac{1}{2}(\alpha^2+\beta^2) \\ &\leq 2(\alpha+\beta) + \frac{1}{2}(\alpha+\beta) < 1 + 2(\alpha+\beta). \end{aligned}$$

Thus, (30) holds and $|\lambda_{2,3}| < 1$. By Theorem 3.9, x^* is globally asymptotically stable.

For (7) with $r_i = r_0 \in (0, +\infty)$, we have $G_i(u) = \frac{1+r_0}{1+u}$ and $G_i'(r_i) = -\frac{1}{1+r_0}$ so $DT(x^*) = I - \frac{r_0}{1+r_0}\frac{1}{1+\alpha+\beta}A$ with $0 < \frac{r_0}{1+r_0} < 1$. For (8) with $b \in (0,1)$ and $r_i = 1$, $G_i(u) = b + \frac{2(1-b)}{1+u}$ and $G_i'(r_i) = -\frac{1-b}{2}$ so $DT(x^*) = I - \frac{1-b}{2}\frac{1}{1+\alpha+\beta}A$ with $\frac{1-b}{2} \in (0,1)$. For (9) with $b_i = b \in (0,1)$ and $r_i = r_0$, $G_i(u) = b + \frac{(1+r_0)(1-b)}{1+u}$ and $G_i'(r_i) = -\frac{1-b}{1+r_0}$, so $DT(x^*) = I - \frac{r_0(1-b)}{1+r_0}\frac{1}{1+\alpha+\beta}A$ with $\frac{r_0(1-b)}{1+r_0} \in (0,1)$. Then, by the same analysis as that for (6), the eigenvalues of $DT(x^*)$ satisfy $|\lambda| < 1$, so x^* is globally asymptotically stable.

We have mentioned in section 1 that the stability problem for three-dimensional Ricker models and Leslie-Gower models were dealt with in [4], [5] and [11]. (There might be other references, but these three are the latest.) Now it is time to compare our stability region in Figure 1 and those given in the references.

We first state that the stability region obtained by using the criteria for global asymptotic stability given in [5] and [11] is the open rectangle

$$\{(\alpha,\beta) : 0 < \alpha < 1, 0 < \beta < 1\} \tag{31}$$

for each of the four models under the assumption that the carrying simplex exists. For the four models with A given by (26), from Example 4.1 or by the computation given in [5] for general planar systems, we know that (31) is the condition for each axial fixed point to be a repeller on the carrying simplex and for the existence and global asymptotic stability of a fixed point $P_i \in \pi_i$ in the interior of each boundary plane π_i. We now show that each P_i is a saddle point. Note that P_3 has coordinates $(\frac{r_0(1-\alpha)}{1-\alpha\beta}, \frac{r_0(1-\beta)}{1-\alpha\beta}, 0)$. For Ricker models,

$DT(P_3)$ has an eigenvalue $e^{r_0-(AP_3)_3} = e^{r_0(1-\frac{\alpha(1-\alpha)+\beta(1-\beta)}{1-\alpha\beta})}$. Under the condition (31), we have

$$\begin{aligned}\alpha\beta+\alpha(1-\alpha)+\beta(1-\beta) &\leq \frac{1}{2}(\alpha^2+\beta^2)+\alpha-\alpha^2+\beta-\beta^2\\ &= -\frac{1}{2}[(1-\alpha)^2+(1-\beta)^2-2]<1,\end{aligned}$$

so $e^{r_0-(AP_3)_3}>1$ and P_3 is a saddle point. Similarly, P_1 and P_2 are also saddle points. By Theorem 2.1 in [11] we know that the interior fixed point x^* of Ricker models is globally asymptotically stable. The above analysis for each P_i to be a saddle point for Ricker models is also applicable to the other three models. By Remark 2.1 in [11], the interior fixed point x^* is globally asymptotically stable for all of the four models. Comparing our region in Figure 1 with (31) we see that neither of them is contained in the other. Obviously, our theorems permit $\alpha=0$ or $\beta=0$ or both, but the existence of carrying simplex requires $\alpha>0$ and $\beta>0$. If $\alpha>0$, $\beta>0$ and the carrying simplex exists, then the region given by (31) is larger than our region in Figure 1. In particular, the carrying simplex always exists for Leslie-Gower models [24]. However, for Ricker models with (26) and $r_i=r_0>0$, a sufficient condition for existence of a carrying simplex (e.g. Lemma 2.1 in [11]) requires that r_0 satisfies $0<r_0<\frac{1}{1+\alpha+\beta}$. Does a carrying simplex exist for Ricker models when $\frac{1}{1+\alpha+\beta}\leq r_0\leq 1$? In case a carrying simplex does not exist, the stability region (31) is invalid and our region in Figure 1 is safe.

Next, we compare our stability region in Figure 1 with the conditions obtained in [4] by using Liapunov function method. For Ricker models to have an interior fixed point x^* that is globally asymptotically stable, a sufficient condition given by Theorem 1.4 in [4] is that $0<\alpha$, $0<\beta$, $0<r_0<1$, $\alpha+\beta<2$, and either

$$3r_0(1+\alpha^2+\beta^2-\alpha-\beta-\alpha\beta)<(2-\alpha-\beta)(1+\alpha+\beta) \tag{32}$$

or

$$r_0(5-\alpha^2-\beta^2-5\alpha-5\beta+7\alpha\beta)\geq(2-\alpha-\beta)(1+\alpha+\beta). \tag{33}$$

(Note that the Ricker models given in [4] has the form $T_i(x)=x_i\exp(r_0(1-(Ax)_i))$, which is different from our (6). But a simple scaling $y=x/r_0$ will transform (6) to the form given in [4].) Clearly, this is not applicable when $\alpha=0$ or $\beta=0$ or both, so our stability result has an advantage in this case. When $\alpha>0$ and $\beta>0$, since the region in Figure 1 is independent of r_0 whereas both (32) and (33) depend on r_0, it is hard to see which one is better. For $\alpha=0.99,\beta=0.01,r_0=0.9$, the inequalities in (29) hold but calculation of the two sides of (32) gives $2.61981>2$ and calculation of the two sides of (33) gives $-0.81981<2$, so neither (32) nor (33) is met. For $\alpha=0.5,\beta=1.2,r_0=0.5$, (α,β) is neither in the region defined by (29) nor in the region defined by (31). But calculation of (32) gives $0.585<0.81$ so (32) is satisfied. These particular instances demonstrate that the stability condition for Ricker models obtained here and those in [5], [11] and [4] are mutual supplements. For Leslie-Gower models, a stability condition given by Theorem 1.3

in [4] is that $0 < \alpha$, $0 < \beta$, $\alpha + \beta < 2$, and either

$$(1 + r_0)(4\alpha^2 + 4\beta^2 - 4\alpha - 4\beta - \alpha\beta + 1) < 3(\alpha^2 + \beta^2 - \alpha - \beta - \alpha\beta + 1) \tag{34}$$

or

$$3(1 + r_0)(1 - 2\alpha - 2\beta + 3\alpha\beta) \geq 5(1 - \alpha - \beta) + 7\alpha\beta - \alpha^2 - \beta^2. \tag{35}$$

For $\alpha = 0.01, \beta = 0.99, r_0 = 3$, the inequalities in (29) are met. But calculation of the two sides of (34) gives $3.6436 > 2.9109$ and calculation of the two sides of (35) gives $-11.6436 < -0.9109$, so neither (34) nor (35) is satisfied. For $\alpha = 0.5, \beta = 1.2, r_0 = 1$, neither (29) nor (31) is not met, but (34) holds. Therefore, the stability condition for Leslie-Gower models obtained here and those in [5], [11] and [4] are mutual supplements.

Example 4.4. Consider the four models (6)–(9) with $N = 4$, $r_i = r_0 > 0$ for $i \in I_4$ and

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} & 2 \\ a_{21} & 1 & a_{23} & 3 \\ a_{31} & a_{32} & 1 & 4 \\ \frac{1}{4} & 0 & \frac{1}{2} & 1 \end{pmatrix}.$$

Then $(Ar^{I_4\setminus\{4\}})_4 = \frac{1}{4}r_0 + \frac{1}{2}r_0 < r_0$ so $[0, r^{I_4\setminus\{4\}}]$ is below Γ_4. The axial fixed point $Q_4 = (0, 0, 0, r_0)^T$ satisfies $(AQ_4)_1 = 2r_0 > r_0$, $(AQ_4)_2 = 3r_0 > r_0$, $(AQ_4)_3 = 4r_0 > r_0$. So Q_4 is above Γ_1, Γ_2 and Γ_3. Then, by Corollary 3.11, Q_4 is globally asymptotically stable.

Example 4.5. Consider the three-dimensional system (1) with (2), where $r_1 = r_2 = r_3 = 1$,

$$G_1(u) = e^{1-u}, \quad G_2(u) = \frac{2}{1 + u}, \quad G_3(u) = b + \frac{2(1 - b)}{1 + u}$$

with $b \in (0, 1)$ and, with $\alpha \in [0, 1)$,

$$A = \begin{pmatrix} 1 & \alpha & \frac{1}{2} \\ \alpha & 1 & \frac{1}{2} \\ 1 & 1 & 1 \end{pmatrix}.$$

Note that $\pi_1 \cap \Gamma_1$ is given by $\alpha x_2 + \frac{1}{2}x_3 = 1$, $\pi_1 \cap \Gamma_2$ by $x_2 + \frac{1}{2}x_3 = 1$, and $\pi_1 \cap \Gamma_3$ by $x_2 + x_3 = 1$. From the three equations we see that $\pi_1 \cap \Gamma_3$ is below $\pi_1 \cap \Gamma_2$, which is below $\pi_1 \cap \Gamma_1$. By drawing the three lines we see that $[0, r] \cap \pi_1 \cap (\Gamma_2 \cup \Gamma_3)$ is strictly below Γ_1. Similarly, $[0, r] \cap \pi_2 \cap (\Gamma_1 \cup \Gamma_3)$ is strictly below Γ_2. We can check that $x^* = (\frac{1}{1+\alpha}, \frac{1}{1+\alpha}, 0)^T$ is a fixed point satisfying $(Ax^*)_3 = \frac{2}{1+\alpha} > 1$, so x^* is above Γ_3. Then, by Theorem 3.3, x^* is globally attracting. We show that the eigenvalues of $DT(x^*)$ satisfy $|\lambda| < 1$. From (22) and the lines below it we know that

$$G_3((Ax^*)_3) = b + \frac{2(1 - b)}{1 + \frac{2}{1+\alpha}} = b + \frac{2 + 2\alpha}{3 + \alpha}(1 - b)$$

is an eigenvalue. As $0 < \frac{2+2\alpha}{3+\alpha} < 1$, we see that $G_3((Ax^*)_3) \in (0,1)$. The other eigenvalues of $DT(x^*)$ are those of

$$\begin{pmatrix} 1+x_1^*G_1'(r_1) & \alpha x_1^*G_1'(r_1) \\ \alpha x_2^*G_2'(r_2) & 1+x_2^*G_2'(r_2) \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{1+\alpha} & -\frac{\alpha}{1+\alpha} \\ -\frac{\alpha}{2(1+\alpha)} & \frac{1+2\alpha}{2(1+\alpha)} \end{pmatrix}.$$

Its two eigenvalues are

$$\lambda_{1,2} = \frac{1}{4(1+\alpha)}(1+4\alpha \pm \sqrt{1+8\alpha^2}).$$

As $\alpha \in [0,1)$ implies $\sqrt{1+8\alpha^2} < 3$, we see that $\lambda_{1,2} \in (0,1)$. Thus, the eigenvalues of $DT(x^*)$ satisfy $|\lambda| < 1$ so that x^* is globally asymptotically stable by Theorem 3.9.

5. Proof of the main theorems

Note that Theorems 3.5–3.7 and Corollaries 3.6–3.8 are restatements of Theorems 3.1–3.3 and Corollaries 3.2–3.4. Note also that Corollaries 3.2–3.4 are particular cases of Theorems 3.1–3.3. Hence, we need only prove Theorems 3.1–3.3. We shall adopt the following strategy: prove the local stability of Q_i under the conditions of Theorem 3.10 first, then provide a unified proof for existence of a fixed point x^* with $x_j^* > 0$ if and only if $j \in J \subset I_N$, $1 \le |J| \le N$, and its global attraction under the conditions of Theorem 3.3. Then this proof is for Theorem 3.1 when $|J| = N$, i.e. $J = I_N$, so that $I_N \setminus J = \emptyset$ and condition (ii) in Theorem 3.3 is redundant. When $|J| = 1$, i.e. $J = \{i\}$ for some fixed $i \in I_N$, the proof is for global attraction of Q_i in Theorem 3.10.

Proof of Theorem 3.10. From (22) and the lines below it we know that

$$DT(Q_i) = \mathrm{diag}[G_1((AQ_i)_1), \ldots, G_N((AQ_i)_N)] + \tilde{A},$$

where the ith row of $\tilde{A}$ is $(r_iG_i'(r_i)a_{i1}, \ldots, r_iG_i'(r_i)a_{iN})$ and each of the other entries of $\tilde{A}$ is 0. Under the conditions of Theorem 3.10, Q_i is above Γ_j, so $(AQ_i)_j > r_j$ and the jth eigenvalue of $DT(Q_i)$ satisfies $G_j((AQ_i)_j) \in (0,1)$, for all $j \in I_N \setminus \{i\}$. The ith eigenvalue of $DT(Q_i)$ is $G_i((AQ_i)_i) + r_iG_i'(r_i)a_{ii} = 1 + r_iG_i'(r_i)$. By $(a2)$, $uG_i(u)$ is increasing so $\frac{d}{du}[uG_i(u)] = G_i(u) + uG_i'(u) \ge 0$ for $u \in [0, r_i]$. Thus, $1 + r_iG_i'(r_i) = G_i(r_i) + r_iG_i'(r_i) \ge 0$. By $(a1)$, $G_i'(r_i) < 0$ so $1 + r_iG_i'(r_i) < 1$. Since Q_i is globally attracting (to be proved later), by Theorem 3.9, Q_i is globally asymptotically stable. □

The lemma below is for existence of a fixed point x^* in Theorem 3.1.

Lemma 5.1. *Assume that*

$$\forall i, j \in I_N (i \ne j),\ \Gamma_j \cap [0, r] \cap \pi_i \text{ is strictly below } \Gamma_i. \tag{36}$$

Then there is a point $0 < x^* \le r$ *such that* $\cap_{i \in I_N} \Gamma_i = \{x^*\}$.

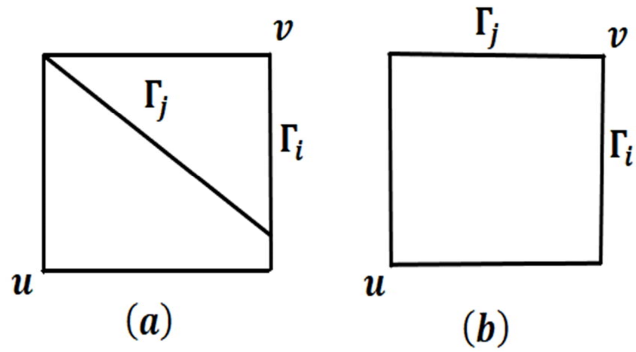

FIGURE 2. Configuration of $h([u,v] \cap (\cup_{k\in\{i,j\}}(\Gamma_k \cup \Gamma_k^-)))$ for $a_{ij} = 0$, (a) $a_{ji} > 0$, (b) $a_{ji} = 0$.

The proof of Lemma 5.1 is similar to that of Lemma 3.1 in [17] so we omit it here. Considering that the proof in [17] is not easy to follow under different context, for clarity we provide the proof of Lemma 5.1 in Appendix 3.

Let $J \subset I_N$ be any nonempty subset such that conditions (i) and (ii) of Theorem 3.3 hold. For this J, let $[u,v] \subset \mathbb{R}_+^N$ be a cell as described in Proposition 2.1, i.e. $\omega(x) \subset [u,v]$ for all $x \in \mathbb{R}_+^N \setminus (\cup_{j\in J}\pi_j)$ and $T(y) \leq v$ for all $y \in [u,v]$. Let $J_1 \subset I_N$ such that $v_i > u_i$ if and only if $i \in J_1$. Then $J \subset J_1$, $[u,v] \subset \pi_j$ and $[u,v]$ is above Γ_j for all $j \in I_N \setminus J_1$. Since the dynamics on π_j is not affected by the position of Γ_j on π_j, we need only consider the set $[u,v] \cap (\cup_{j\in J_1}(\Gamma_j \cup \Gamma_j^-))$. As $J \neq \emptyset$, if $J = J_1 = \{j\}$, then $v \in \Gamma_j$, u is below Γ_j but is on or above Γ_i for all $i \in I_N \setminus \{j\}$, $[u,v]$ is a line segment on the x_j-axis, so $[u,v] \cap (\cup_{i\in J_1}(\Gamma_i \cup \Gamma_i^-)) = [u,v] \cap (\Gamma_j \cup \Gamma_j^-) = [u,v]$ is a convex set. However, if $|J_1| = k \geq 2$, $[u,v] \cap (\cup_{i\in J_1}(\Gamma_i \cup \Gamma_i^-))$ may not be convex. In general, let $h([u,v] \cap (\cup_{i\in J_1}(\Gamma_i \cup \Gamma_i^-)))$ be a convex set containing $[u,v] \cap (\cup_{i\in J_1}(\Gamma_i \cup \Gamma_i^-))$ and bounded by the surface planes of $[u,v]$ and possibly a plane Γ such that $[u,v] \cap (\cup_{i\in J_1}(\Gamma_i \cup \Gamma_i^-))$ is below Γ and has as many touching points with Γ as possible. For example, when $|J_1| = 1$ with $J_1 = \{j\}$,

$$h([u,v] \cap (\cup_{i\in J_1}(\Gamma_i \cup \Gamma_i^-))) = h([u,v] \cap (\Gamma_j \cup \Gamma_j^-)) = h([u,v]) = [u,v].$$

When $|J_1| = 2$ with $J_1 = \{i,j\}$, there are following three cases for the configuration of $h([u,v] \cap (\cup_{k\in\{i,j\}}(\Gamma_k \cup \Gamma_k^-)))$.

Case 1: $a_{ij} = 0$. No matter whether $a_{ji} = 0$ or $a_{ji} > 0$, we always have

$$[u,v] \cap (\cup_{k\in\{i,j\}}(\Gamma_k \cup \Gamma_k^-)) = [u,v] \cap (\Gamma_i \cup \Gamma_i^-) = [u,v],$$

so $h([u,v] \cap (\cup_{k\in\{i,j\}}(\Gamma_k \cup \Gamma_k^-))) = [u,v]$ as shown in Figure 2.

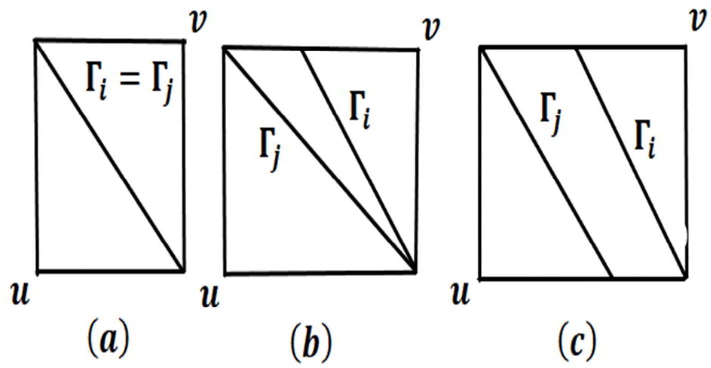

FIGURE 3. Configuration of $h([u,v] \cap (\cup_{k\in\{i,j\}}(\Gamma_k \cup \Gamma_k^-)))$ when $a_{ij} > 0, a_{ji} > 0$ and $[u,v] \cap \Gamma_j$ is on or below Γ_i. (a) $\Gamma_i = \Gamma_j$, (b) $u_j \geq 0$, (c) $u_j = 0$.

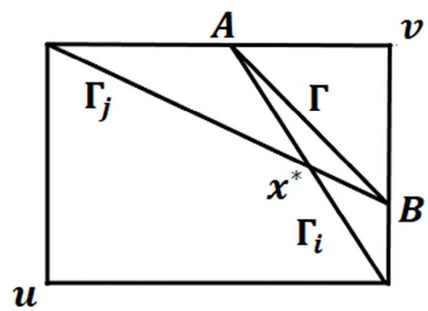

FIGURE 4. Configuration of $h([u,v] \cap (\cup_{k\in\{i,j\}}(\Gamma_k \cup \Gamma_k^-)))$ when there is a fixed point x^* in the interior of $[u,v]$.

Case 2: $a_{ij} > 0, a_{ji} > 0$, $[u,v] \cap \Gamma_j$ is on or below Γ_i. In this case, $\Gamma = \Gamma_i$ and

$$h([u,v] \cap (\cup_{k\in\{i,j\}}(\Gamma_k \cup \Gamma_k^-))) = [u,v] \cap (\cup_{k\in\{i,j\}}(\Gamma_k \cup \Gamma_k^-) = [u,v] \cap (\Gamma_i \cup \Gamma_i^-)$$

as shown in Figure 3.

Case 3: $[u,v] \cap \Gamma_i \cap \Gamma_j = \{x^*\}, u_k < x_k^* < v_k$ for $k \in \{i,j\}$. In this case, $[u,v] \cap (\cup_{k\in\{i,j\}}(\Gamma_k \cup \Gamma_k^-))$ is a proper subset of $h([u,v] \cap (\cup_{k\in\{i,j\}}(\Gamma_k \cup \Gamma_k^-)))$, which is bounded by the boundary lines of $[u,v]$ and the line Γ determined by A and B shown in Figure 4.

Let $P^{(i)} : \mathbb{R}^N \to \mathbb{R}^{N-1}$ be the projection such that $P^{(i)}$ simply omit the ith component of each $x \in \mathbb{R}^N$, i.e. $P^{(i)}_j(x) = x_j$ for all $j \in I_N \setminus \{i\}$ but $P^{(i)}(x)$ has no ith component.

Lemma 5.2. *Assume that* $[u, v]$ *as described in Proposition 2.1 contains* $\omega(x)$ *for all* $x \in \mathbb{R}^N_+ \setminus (\cup_{i\in J}\pi_i)$ *and* $J \subset J_1 \subset I_N$ *such that* $v_j > u_j$ *if and only if* $j \in J_1$*. Then* $\omega(x) \subset h([u, v] \cap (\cup_{k\in J_1}(\Gamma_k \cup \Gamma^-_k)))$ *for all* $x \in \mathbb{R}^N_+ \setminus (\cup_{i\in J}\pi_i)$*. Moreover, if* $|J_1| \geq 2$*, then for each* $k \in J$*,* $\omega(x) \subset [u_k, v_k] \times P^{(k)}(S_k(u))$ *for all* $x \in \mathbb{R}^N_+ \setminus (\cup_{i\in J}\pi_i)$*, where*

$$S_k(u) = h(\pi_k(u) \cap [u, v] \cap (\cup_{\ell\in J_1\setminus\{k\}}(\Gamma_\ell \cup \Gamma^-_\ell))).$$

As the proof of Lemma 5.2 is lengthy but technical, we shall dedicate the next section to it.

We now prove the existence of a fixed point x^* with $x^*_i \in (0, r_i]$ if and only if $i \in J$ and the global attraction of x^*.

Proof of Theorems 3.1–3.3 for global attraction. If $J = I_N$, the existence of a fixed point x^* with $0 \ll x^* \leq r$ follows from condition (i) of Theorem 3.3 and Lemma 5.1. If $J = \{i\}$ for some $i \in I_N$, then $x^* = Q_i$ is the required fixed point. In general, if $J \neq I_N$, we can view the $|J|$-dimensional subsystem on $\cap_{j\in I_N\setminus J}\pi_j$ as a system on $\mathbb{R}^{|J|}_+$. Then, by condition (i) and applying Lemma 5.1 to this $|J|$-dimensional system, we obtain a fixed point $x^* \in \mathbb{R}^N_+$ with $x^*_i \in (0, r_i]$ if and only if $i \in J$.

To show the global attraction of x^*, we let $u = u(t) = tx^*$, then $v = v(t)$ is defined by (19), for $t \in [0, 1)$. We show that for each $t \in [0, 1)$, $\omega(x) \subset [u(t), v(t)]$ for all $x \in \mathbb{R}^N_+ \setminus (\cup_{i\in J}\pi_i)$. Then the global attraction of x^* follows from $[u(1), v(1)] = \{x^*\}$ by letting $t \to 1$.

From Proposition 2.1 with $u = 0$ we know that $[u(0), v(0)] = [0, r]$ is positive invariant and $\omega(x) \subset [0, r]$ for all $x \in \mathbb{R}^N_+$. As $v = r \gg 0 = u$, we have $J_1 = I_N$. By Lemma 5.2 with $u = 0$, for each $i \in J$ with

$$S_i(0) = h(\pi_i \cap [0, r] \cap (\cup_{j\in I_N\setminus\{i\}}(\Gamma_j \cup \Gamma^-_j))),$$

$\omega(x) \subset [0, r_i] \times P^{(i)}(S_i(0))$ for all $x \in \mathbb{R}^N_+ \setminus (\cup_{j\in J}\pi_j)$. By condition (i), the set

$$\pi_i \cap [0, r] \cap (\cup_{j\in I_N\setminus\{i\}}(\Gamma_j \cup \Gamma^-_j))$$

is strictly below Γ_i. By the convexity and definition of the set $h(S)$ for a set S, $S_i(0)$ is strictly below Γ_i so $(Ay)_i < r_i$ for all $y \in S_i(0)$. Since $S_i(0)$ is compact and $S_i(0) \subset \pi_i$, there is a $\delta_i \in (0, \frac{1}{2}x^*_i)$ such that the set

$$[0, 2\delta_i] \times P^{(i)}(\overline{\mathcal{B}(S_i(0), \delta_i)}) = \{x \in \mathbb{R}^N_+ : x_i \in [0, 2\delta_i], x^{I_N\setminus\{i\}} \in \overline{\mathcal{B}(S_i(0), \delta_i)}\}$$

is strictly below Γ_i, so $2\delta_i < x^*_i \leq r_i$, where $\overline{\mathcal{B}(S_i(0), \delta_i)}$ is the closure of

$$\mathcal{B}(S_i(0), \delta_i) = \{y \in \pi_i : \exists x \in S_i(0) \text{ such that } |x - y| < \delta_i\}.$$

We show that

$$\forall x \in \mathbb{R}^N_+ \setminus (\cup_{j\in J}\pi_j), \;\; \omega(x) \cap ([0, \delta_i] \times P^{(i)}(S_i(0))) = \emptyset. \tag{37}$$

Suppose (37) is not true. Then, for some $x_0 \in \mathbb{R}^N_+ \setminus (\cup_{j\in J}\pi_j)$, there is $y^0 \in \omega(x_0) \cap ([0,\delta_i] \times P^{(i)}(S_i(0)))$ and an increasing sequence $\{n_k\}$ such that $T^{n_k}(x_0) \to y^0$ as $k \to \infty$. As $\omega(x_0) \subset [0, r_i] \times P^{(i)}(S_i(0))$, $y^0_i \leq \delta_i < 2\delta_i < r_i$, and $x_i \leq r_i$ implies $T_i(x) \leq r_i$ for any $x \in \mathbb{R}^N_+$, there is an integer $N_0 > 0$ such that

$$\forall n \geq N_0, T^n(x_0) \in [0, r_i] \times P^{(i)}(\mathcal{B}(S_i(0), \delta_i)).$$

For any point $x \in [0, r_i] \times P^{(i)}(\mathcal{B}(S_i(0), \delta_i))$ with $x_i > 2\delta_i$, there is a point $y \in \Gamma_i$ such that $(y - x)^{I_N \setminus \{i\}} = 0$. If $x \in \Gamma^-_i$, by $(a1)$ and $(a2)$ we have $x_i < T_i(x) < T_i(y) = y_i \leq r_i$. If $x \in \Gamma^+_i$, by $(a1)$ and $(a2)$ again, $2\delta_i < y_i = T_i(y) < T_i(x) < x_i \leq r_i$. Thus, if $T^n_i(x_0) > 2\delta_i$ for some $n \geq N_0$ then $T^{n+k}_i(x_0) > 2\delta_i$ for all $k \geq 0$. As $\lim_{k\to\infty} T^{n_k}_i(x_0) = y^0_i \leq \delta_i$, we must have

$$\forall n \geq N_0, T^n(x_0) \in [0, 2\delta_i] \times P^{(i)}(\mathcal{B}(S_i(0), \delta_i)).$$

Since the compact set $[0, 2\delta_i] \times P^{(i)}(\overline{\mathcal{B}(S_i(0), \delta_i)})$ is strictly below Γ_i, there is an $\bar{r}_i < r_i$ such that $(Ay)_i \leq \bar{r}_i$, so that $T_i(y) = y_iG_i((Ay)_i) \geq y_iG_i(\bar{r}_i) > y_i$, for all $y \in [0, 2\delta_i] \times P^{(i)}(\overline{\mathcal{B}(S_i(0), \delta_i)})$. Thus,

$$T^{N_0+k}_i(x_0) \geq T^{N_0}_i(x_0)(G_i(\bar{r}_i))^k \to +\infty \quad (k \to \infty),$$

a contradiction to the boundedness of $\{T^n(x_0)\}$. This shows (37).

Since (37) holds for all $i \in J$, there is a $\delta \in (0, 1]$ such that

$$\forall x \in \mathbb{R}^N_+ \setminus (\cup_{j\in J}\pi_j), \;\; \omega(x) \subset \mathbb{R}^N_+(u(\delta)),$$

where $u(\delta) = \delta x^*$. By Proposition 2.1,

$$\forall x \in \mathbb{R}^N_+ \setminus (\cup_{j\in J}\pi_j), \;\; \omega(x) \subset [u(\delta), v(\delta)], \tag{38}$$

and $T(y) \leq v(\delta)$ for all $y \in [u(\delta), v(\delta)]$. We claim that the supremum δ_0 of δ satisfying (38) is 1. If not, then $v(\delta_0) \neq u(\delta_0)$, $\omega(x) \subset [u(\delta_0), v(\delta_0)]$ for all $x \in \mathbb{R}^N_+ \setminus (\cup_{j\in J}\pi_j)$, and $T(y) \leq v(\delta_0)$ for all $y \in [u(\delta_0), v(\delta_0)]$. Now with new $J_1 \subset I_N$ such that $v_i(\delta_0) > u_i(\delta_0)$ if and only if $i \in J_1$, we see that $v_i(\delta_0) = u_i(\delta_0) = 0$ for $i \in I_N \setminus J_1$ and for any $i \in J_1$, by (19),

$$v_i(\delta_0) = r_i - (Au(\delta_0)^{I_N\setminus\{i\}})_i = r_i - \delta_0(Ax^*)_i + \delta_0x^*_i.$$

If $J_1 = J = \{i\}$ for some $i \in I_N$, then $[u(\delta_0), v(\delta_0)] = [\delta_0Q_i, Q_i]$ is on the x_i-axis and δ_0Q_i is on or above Γ_j for all $j \in I_N \setminus \{i\}$ and, by (38), $\omega(x) \subset [\delta_0Q_i, Q_i]$ for all $x \in \mathbb{R}^N_+ \setminus \pi_i$. By $(a2)$, $T_i(x)$ is increasing for $x_i \in [\delta_0r_i, r_i]$. So, for any $\delta_1 \in (\delta_0, 1]$, we have $\omega(x) \subset [\delta_1Q_i, Q_i] = [u(\delta_1), v(\delta_1)]$ for all $x \in \mathbb{R}^N_+ \setminus \pi_i$. This contradicts the definition of δ_0.

Now suppose $|J_1| \geq 2$ and we also derive a contradiction. By Lemma 5.2, we have

$$\forall i \in J, \forall x \in \mathbb{R}^N_+ \setminus (\cup_{j\in J}\pi_j), \omega(x) \subset [u_i(\delta_0), v_i(\delta_0)] \times P^{(i)}(S_i(u(\delta_0))).$$

We define an affine map $f : \mathbb{R}^N_+ \to \mathbb{R}^N_+(u(\delta_0))$ by

$$\forall x \in \mathbb{R}^N_+, f(x) = \delta_0x^* + (1 - \delta_0)x.$$

Then $f(x^*) = x^*$ and $f(x) - x^* = (1-\delta_0)(x - x^*)$, so f maps the line segment x^*x to $x^*f(x) \subset x^*x$, $[0, r]$ to $f([0, r]) = [u(\delta_0), f(r)]$, and $[0, r^{I_N\setminus\{i\}}]$ to $f([0, r^{I_N\setminus\{i\}}]) = [u(\delta_0), f(r^{I_N\setminus\{i\}})]$ for all $i \in I_N$.

For a fixed $i \in J$, $f(r^{I_N\setminus\{i\}}) = \delta_0 x^* + (1-\delta_0)r^{I_N\setminus\{i\}}$. For any $j \in I_N \setminus \{i\}$, if $j \in I_N \setminus J_1$ then $v_j(\delta_0) = 0 < (1-\delta_0)r_j = f_j(r^{I_N\setminus\{i\}})$; if $j \in J$ then

$$v_j(\delta_0) = r_j - \delta_0 r_j + \delta_0 x_j^* = f_j(r^{I_N\setminus\{i\}});$$

if $j \in J_1 \setminus J$ then $(Ax^*)_j \geq r_j$ so

$$v_j(\delta_0) \leq r_j - \delta_0 r_j = f_j(r^{I_N\setminus\{i\}}).$$

Thus, $u(\delta_0)^{\{i\}} + v(\delta_0)^{I_N\setminus\{i\}} \leq f(r^{I_N\setminus\{i\}})$ so

$$\pi_i(u(\delta_0)) \cap [u(\delta_0), v(\delta_0)] = [u(\delta_0), u(\delta_0)^{\{i\}} + v(\delta_0)^{I_N\setminus\{i\}}] \subset f([0, r^{I_N\setminus\{i\}}]).$$

For $j \in J_1 \setminus \{i\}$ and any $x \in \pi_i(u(\delta_0)) \cap [u(\delta_0), v(\delta_0)] \cap \Gamma_j$, we show that $(Ax)_i < r_i$ so that x is below Γ_i. Since $x \in f([0, r^{I_N\setminus\{i\}}])$ and f is invertible, then

$$y = f^{-1}(x) = \frac{1}{1-\delta_0}(x - \delta_0 x^*) \in [0, r^{I_N\setminus\{i\}}].$$

As $x \in \Gamma_j$ and x^* is on or above Γ_j by condition (ii) of Theorem 3.3, we have $(Ax^*)_j \geq r_j$ and $(Ax)_j = r_j$, so

$$(Ay)_j = \frac{1}{1-\delta_0}((Ax)_j - \delta_0(Ax^*)_j) \leq \frac{1}{1-\delta_0}(r_j - \delta_0 r_j) = r_j.$$

This shows that $y \in \pi_i \cap [0, r] \cap (\Gamma_j \cup \Gamma_j^-)$. By condition (i) of Theorem 3.3, y is below Γ_i so $(Ay)_i < r_i$. Then, as $(Ax^*)_i = r_i$, we have

$$(Ax)_i = \delta_0(Ax^*)_i + (1-\delta_0)(Ay)_i < \delta_0 r_i + (1-\delta_0)r_i = r_i$$

so x is below Γ_i. This shows that $\pi_i(u(\delta_0)) \cap [u(\delta_0), v(\delta_0)] \cap (\Gamma_j \cup \Gamma_j^-)$ is strictly below Γ_i for all $j \in J_1 \setminus \{i\}$. Then $S_i(u(\delta_0))$ is strictly below Γ_i. By Lemma 5.2, $[u_i(\delta_0), v_i(\delta_0)] \times P^{(i)}(S_i(u(\delta_0)))$ contains $\omega(x)$ for all $x \in \mathbb{R}^N_+ \setminus (\cup_{j\in J}\pi_j)$. Then the same argument as before leads to the existence of a number $\delta_i > \delta_0$ such that $[\delta_0 x_i^*, \delta_i x_i^*] \times P^{(i)}(S_i(u(\delta_0)))$ does not contain any point of $\omega(x)$ for any $x \in \mathbb{R}^N_+ \setminus (\cup_{j\in J}\pi_j)$. Then, for $\bar{\delta} = \min\{\delta_i : i \in J\} > \delta_0$, we have $\omega(x) \subset \mathbb{R}^N_+(u(\bar{\delta}))$ for all $x \in \mathbb{R}^N_+ \setminus (\cup_{j\in J}\pi_j)$. By Proposition 2.1 again, we have $\omega(x) \subset [u(\bar{\delta}), v(\bar{\delta})]$ for all $x \in \mathbb{R}^N_+ \setminus (\cup_{j\in J}\pi_j)$ and $y \in [u(\bar{\delta}), v(\bar{\delta})]$ implies $T(y) \leq v(\bar{\delta})$. But by the definition of δ_0, we should have $\bar{\delta} \leq \delta_0$, a contradiction. This shows that $\delta_0 = 1$, so x^* is globally attracting. □

6. Proof of Lemma 5.2

In order to prove Lemma 5.2, we first prove the following two lemmas. Note that the u in the next two lemmas are different from, but less restrictive than, the u in Proposition 2.1 as we do not require $u_j = 0$ for $j \in I_N \setminus J_1$ here.

Lemma 6.1. *For any $u \in [0, r]$, let $v \in [u, r]$ be defined by (19). Then,*

$$\forall w \in [v, r], \forall x \in [u, w], T(x) \le w. \tag{39}$$

Proof. Take $w \in [v, r], x \in [u, w], j \in I_N$ and fix them. If $x \in \Gamma_j \cup \Gamma_j^+$ then $T_j(x) \le x_j \le w_j$. If $x \in \Gamma_j^-$ then there is a $y \in \Gamma_j \cap [u, v] \subset \Gamma_j \cap [u, w]$ such that $y \ge x$ and $(y - x)^{I_N \setminus \{j\}} = 0$. By $(a2)$, $T_j(x) < T_j(y) = y_j \le w_j$. Thus, for all $x \in [u, w]$, $T(x) \le w$ and (39) holds. □

Lemma 6.2. *For u, v, w as in Lemma 6.1, let $J_1 \subset I_N$ such that $u \in \Gamma_i^-$ if and only if $i \in J_1$. For any $J_2 \subset I_N$ with $J_1 \subset J_2$, let $S = h([u, w] \cap (\cup_{j \in J_2}(\Gamma_j \cup \Gamma_j^-)))$. If $S \ne \emptyset$, then*

$$\forall x \in S, \exists y \in S \text{ such that } T(x) \le y. \tag{40}$$

Proof. We prove (40) by induction on $|J_1|$. When $|J_1| = 0$, i.e. $J_1 = \emptyset$, $[u, w]$ is above Γ_j for all $j \in I_N$. Thus, for all $x \in S$, $T(x) \le x$ so (40) holds. When $|J_1| = 1$, $J_1 = \{i\}$ for some $i \in I_N$. As u is below Γ_i but on or above Γ_j for all $j \in I_N \setminus \{i\}$, $[u, v]$ and $[u, w]$ are above Γ_j for all $j \in I_N \setminus \{i\}$. For any $x \in S$, if $x \in \Gamma_i \cup \Gamma_i^+$ then x is on or above Γ_k for all $k \in I_N$ so (40) holds with $y = x$. If $x \in \Gamma_i^-$, then there is a $y \in \Gamma_i \cap S$ such that $x \le y$ and $(y - x)^{I_N \setminus \{i\}} = 0$. By $(a2)$, $T_i(x) < T_i(y) = y_i$. For each $j \in I_N \setminus \{i\}$, as x is on or above Γ_j, we have $T_j(x) \le x_j = y_j$. Hence, $T(x) \le y$ and (40) holds when $|J_1| = 1$.

(IH) Suppose (40) holds when $0 \le |J_1| \le k$ for some positive integer $k < N$.

We show that (40) holds when $|J_1| = k + 1$. For any $x \in S$ such that $x \in \Gamma_i \cup \Gamma_i^+$ for some $i \in J_1$, let $J_3 \subset I_N$ such that x is below Γ_j if and only if $j \in J_3$. Since x is on or above Γ_k for all $k \in \{i\} \cup (I_N \setminus J_1)$, we have $J_3 \subset J_1 \setminus \{i\}$ so $0 \le |J_3| \le |J_1| - 1 = k$. As $u \le x \le w \le r$ and $u \le v \le w$, defining $x' \in [x, r]$ by (19), i.e.

$$x'_j = \begin{cases} x_j & \text{if} \quad j \in I_N \setminus J_3, \\ r_j - (Ax^{I_N \setminus \{j\}})_j & \text{if} \quad j \in J_3, \end{cases}$$

we have $x'_j = x_j \le w_j$ for $j \in I_N \setminus J_3$ and $x'_j \le r_j - (Au^{I_N \setminus \{j\}})_j = v_j \le w_j$ for $j \in J_3$. Thus, $x \le x' \le w \le r$. Then, by (IH),

$$\forall z \in h([x, w] \cap (\cup_{j \in J_2}(\Gamma_j \cup \Gamma_j^-))), \exists y \in h([x, w] \cap (\cup_{j \in J_2}(\Gamma_j \cup \Gamma_j^-)))$$

such that $T(z) \le y$. Note that

$$x \in h([x, w] \cap (\cup_{j \in J_2}(\Gamma_j \cup \Gamma_j^-))) \subset h([u, w] \cap (\cup_{j \in J_2}(\Gamma_j \cup \Gamma_j^-))) = S.$$

Thus, for this $x \in S$, there is a $y \in S$ such that $T(x) \le y$.

Note that $u \in \cap_{j \in J_1} \Gamma_j^-$ so $[u, w] \cap (\cap_{j \in J_1} \Gamma_j^-) \ne \emptyset$ and it is a subset of S. We next show that

$$\forall z \in [u, w] \cap (\cap_{j \in J_1} \Gamma_j^-), \exists y \in S \text{ such that } T(z) \le y.$$

From the previous paragraph we know that, for any $x \in S \cap \Gamma_i$ for some $i \in J_1$ but $x \in \Gamma_j \cup \Gamma_j^-$ for all $j \in J_1 \setminus \{i\}$, there is a $y \in h([x, w] \cap (\cup_{j \in J_2}(\Gamma_j \cup \Gamma_j^-))) \subset S$ such that $T(x) \le y$. If $x^{J_1} = 0$ then $p(t) = tx^{J_1} + x^{I_N \setminus J_1} = x$ for all $t \in [0, 1]$. Otherwise, these points $p(t)$ form a line segment $p(0)p(1) = x^{I_N \setminus J_1}x$. By $(a3)$, $T_j(p(t))$ for each $j \in J_1$ is increasing for $t \in [0,1]$. Thus, if $p(t) \in S$, then $p(t)$ is on or above Γ_k for all $k \in I_N \setminus J_1$, so

$$\forall j \in J_1, T_j(p(t)) \le T_j(x) \le y_j, \quad \forall k \in I_N \setminus J_1, T_k(p(t)) \le p_k(t) \le x_k \le y_k.$$

Hence, $T(p(t)) \le y$ if $p(t) \in S$. Since the set $S \cap (\cap_{j \in J_1}(\Gamma_j \cup \Gamma_j^-))$ consists of such points $p(t) \in S$, we have proved (40) when $|J_1| = k+1$. By induction, (40) holds for all J_1 with $0 \le |J_1| \le N$. □

Equipped with Lemma 6.2, we are now in a position to prove Lemma 5.2.

Proof of Lemma 5.2. By assumption, $[u, v]$ is as described in Proposition 2.1, so $u_j = 0$ for $j \in I_N \setminus J$ and $u \in \Gamma_i^-$ for all $i \in J$, and by (19), $v_j = u_j = 0$ for $j \in I_N \setminus J_1$ and $v_i = r_i - (Au^{I_N \setminus \{i\}})_i$ for $i \in J_1$. By Lemma 6.1,

$$\forall w \in [v, r], \forall x \in [u, w], T(x) \le w. \tag{41}$$

By Lemma 6.2 for $w = v$ and $J_2 = J_1$ with $S = h([u, v] \cap (\cup_{j \in J_1}(\Gamma_j \cap \Gamma_j^-)))$, we have

$$\forall x \in S, \exists y \in S \text{ such that } T(x) \le y. \tag{42}$$

We first show the conclusion

$$\forall x \in \mathbb{R}_+^N \setminus (\cup_{j \in J}\pi_j), \omega(x) \subset S. \tag{43}$$

By assumption, $\omega(x) \subset [u, v]$ for all $x \in \mathbb{R}_+^N \setminus (\cup_{j \in J}\pi_j)$. If $S = [u, v]$ then (43) is obviously true. If $S \ne [u, v]$ then (43) follows from

$$\forall x \in \mathbb{R}_+^N \setminus (\cup_{j \in J}\pi_j), \omega(x) \cap ([u, v] \setminus S) = \emptyset. \tag{44}$$

We now show the truth of (44) by contradiction. Suppose (44) is not true. Then there exist a point $x^0 \in \mathbb{R}_+^N \setminus (\cup_{j \in J}\pi_j)$ and a point $y^0 \in \omega(x^0) \cap ([u, v] \setminus S)$. Since $\omega(x^0)$ is invariant, there is a $y^1 \in \omega(x^0)$ such that $T(y^1) = y^0$. By the definition of the set S, $T(y^1) = y^0 \not\le x$ for any $x \in S$. Thus, by (42), $y^1 \notin S$. As $y^1 \in \omega(x^0) \subset [u, v]$, we must have $y^1 \in \omega(x^0) \cap ([u, v] \setminus S)$. Note that $[u, v] \setminus S$ is strictly above Γ_j for all $j \in J_1$. Thus, $y_j^0 = T_j(y^1) < y_j^1$ for all $j \in J_1$. As $y_i^0 = y_i^1 = 0$ for all $i \in I_N \setminus J_1$, we have $y^0 \le y^1 \le v$. For any $y \ge y^0$ and any $j \in J_1$, $(Ay)_j \ge (Ay^0)_j > r_j$ so $G_j((Ay)_j) \le G_j((Ay^0)_j) < G_j(r_j) = 1$. Thus,

$$\forall j \in J_1, \forall y \in [y^0, v], G_j((Ay)_j) \le \max_{i \in J_1} G_i((Ay^0)_i) = K < 1.$$

Then, by $y^1 \in [y^0, v]$, we have $y^0 \le Ky^1$. Repeating the above process, we obtain a sequence $\{y^n\} \subset \omega(x^0) \cap ([u, v] \setminus S)$ satisfying

$$\forall n \ge 0, y^0 \le y^n \le y^{n+1} \le v, \ y^n \le Ky^{n+1}.$$

It then follows that $y^0 \le K^n y^n \le K^n v$ for all $n \ge 0$. By letting $n \to \infty$, we obtain $y^0 = 0$, a contradiction to y^0 above Γ_j for all $j \in J_1$. This shows the truth of (44). Then (43) follows from (44).

Next, taking a fixed $k \in J$, we show that, if $|J_1| \geq 2$,

$$\forall x \in \mathbb{R}^N_+ \setminus (\cup_{j\in J}\pi_j), \omega(x) \subset [u_k, v_k] \times P^{(k)}(S_k(u)). \tag{45}$$

This is true if $S_k(u) = [u, u^k + v^{I_N\setminus\{k\}}]$ so $[u_k, v_k] \times P^{(k)}(S_k(u)) = [u, v]$ or when $S_k(u) \neq [u, u^k + v^{I_N\setminus\{k\}}]$,

$$\forall x \in \mathbb{R}^N_+ \setminus (\cup_{j\in J}\pi_j), \omega(x) \cap ([u, v] \setminus [u_k, v_k] \times P^{(k)}(S_k(u))) = \emptyset. \tag{46}$$

In order to show (46), we first show that

$$\forall x \in [u_k, v_k] \times P^{(k)}(S_k(u)), \exists y \in [u_k, v_k] \times P^{(k)}(S_k(u))) \text{ such that } T(x) \leq y. \tag{47}$$

For this purpose, we consider the $(N-1)$-dimensional system

$$y(n) = \tilde{T}^n(y), y = P^{(k)}(x), \tilde{T}(P^{(k)}(x)) = P^{(k)}(T(x))$$

for all $x \in \mathbb{R}^N_+$ with $x_k = u_k$. Then $\tilde{T}_j(y) = T_j(x)$ for all $j \in I_N \setminus \{k\}$. We can easily check that the assumptions $(a1)$–$(a3)$ for T also hold for $\tilde{T}$. By Lemma 6.1 with $w = u^{\{k\}} + v^{I_N\setminus\{k\}}$, we have $\tilde{T}(P^{(k)}(x)) \leq P^{(k)}(v)$ for all $x \in [u, u^{\{k\}} + v^{I_N\setminus\{k\}}]$. Then, by Lemma 6.2 with $w = u^{\{k\}} + v^{I_N\setminus\{k\}}$ and $J_2 = J_1 \setminus \{k\}$, we have

$$\forall x \in S_k(u), \exists z \in S_k(u) \text{ such that } \tilde{T}(P^{(k)}(x)) \leq P^{(k)}(z), \tag{48}$$

i.e. $T_j(x) \leq z_j$ for all $j \in I_N \setminus \{k\}$.

Now for each $x \in [u_k, v_k] \times P^{(k)}(S_k(u))$, there is $\bar{x} \in S_k(u)$ such that $x \geq \bar{x}$ and $P^{(k)}(\bar{x}) = P^{(k)}(x)$. By $(a1)$, $T_j(x) \leq T_j(\bar{x})$ for all $j \in I_N \setminus \{k\}$. For this $\bar{x}$, by (48) there is a $z \in S_k(u)$ such that $\tilde{T}(P^{(k)}(\bar{x}) \leq P^{(k)}(z)$. Since $x \in [u, v]$, by (41) with $w = v$ we have $T(x) \leq v$, so $T_k(x) \leq v_k$. Then, taking $y \in [u_k, v_k] \times P^{(k)}(S_k(u))$ with $P^{(k)}(y) = P^{(k)}(z)$ and $y_k = v_k$, we have $T(x) \leq y$. This shows (47).

We now show (46) by contradiction. Suppose (46) is not true. Then there exist a point $x^0 \in \mathbb{R}^N_+ \setminus (\cup_{j\in J}\pi_j)$, a point

$$y^0 \in \omega(x^0) \cap ([u, v] \setminus [u_k, v_k] \times P^{(k)}(S_k(u))),$$

and a point $\bar{y} \in [u, v] \setminus [u_k, v_k] \times P^{(k)}(S_k(u))$ such that $P^{(k)}(\bar{y}) = P^{(k)}(y^0)$ and $\bar{y}_k = u_k$. By the invariance of $\omega(x^0)$, there is a $y^1 \in \omega(x^0) \subset [u, v]$ such that $T(y^1) = y^0$. From the definition of $S_k(u)$ we know that $y^0 \not\leq x$ for any $x \in [u_k, v_k] \times P^{(k)}(S_k(u))$. Then, by (47), we must have $y^1 \notin [u_k, v_k] \times P^{(k)}(S_k(u))$ so

$$y^1 \in \omega(x^0) \cap ([u, v] \setminus [u_k, v_k] \times P^{(k)}(S_k(u))).$$

Note that the set $[u, v] \setminus [u_k, v_k] \times P^{(k)}(S_k(u))$ is strictly above Γ_j for every $j \in J_1 \setminus \{k\}$. Thus, y^1 is above Γ_j, so $y^0_j = T_j(y^1) < y^1_j$, for all $j \in J_1 \setminus \{k\}$. Let $K = \max_{j\in J_1\setminus\{k\}} G_j((Ay^0)_j)$. Then $0 < K < 1$ and

$$\forall j \in J_1 \setminus \{k\}, y^0_j = y^1_j G_j((Ay^1)_j) \leq y^1_j G_j((Ay^0)_j) \leq K y^1_j.$$

Repeating the above process, we obtain a sequence $\{y^n\} \subset \omega(x^0) \cap ([u,v] \setminus [u_k, v_k] \times P^{(k)}(S_k(u))$ such that

$$\forall j \in J_1 \setminus \{k\}, \forall n \geq 0, y_j^0 \leq y_j^n < y_j^{n+1} \leq v_j, y_j^n \leq K y_j^{n+1}.$$

It then follows from these that $y_j^0 \leq K^n y_j^n \leq v_j K^k \to 0$ as $n \to \infty$. Thus, $y_j^0 = 0$ for $j \in J_1 \setminus \{k\}$, so $\bar{y}^{I_N \setminus \{k\}} = (y^0)^{I_N \setminus \{k\}} = 0$ and $\bar{y} = u^{\{k\}}$ is above Γ_j for all $j \in J_1 \setminus \{k\}$. But u is below Γ_i for all $i \in J_1$ by assumption and $u^{\{k\}} \leq u$, so $u^{\{k\}}$ must be below Γ_j for $j \in J_1 \setminus \{k\} \neq \emptyset$, a contradiction. This shows the truth of (46). Then (45) follows from (46). □

7. Conclusion

For a class of discrete dynamical systems (1) with (2)–(5) and $(a1)$–$(a3)$, we have successfully applied the geometric method of using the relative position of the N nullcline planes Γ_i restricted to the cell $[0, r]$ to the study of global dynamics and established criteria for the system to have a globally attracting nontrivial fixed point x^*. Then global asymptotic stability of x^* or existence of a homoclinic cycle can be determined by the eigenvalues of the Jacobian $DT(x^*)$. We have demonstrated application of the criteria for global asymptotic stability of a fixed point by various concrete examples.

For global asymptotic stability of a fixed point of Kolmogorov type discrete dynamical systems, one typical theorem among the existing criteria in literature was given in [5] and a revised version was given in [11] by assuming the existence of a carrying simplex on a monotone region; another typical result was given in [4] by using split Lyapunov function method. Comparing our results with these two typical results, we reach the following conclusions:

(i) While both of the existing typical results deal with global asymptotic stability of an interior fixed point only, our results cover global asymptotic stability of both interior and boundary fixed points.

(ii) The requirement for existence of a carrying simplex in the existing results in [5] and [11] is very restrictive for systems (1) with (2)–(5) and $(a1)$–$(a3)$. Almost every available sufficient condition for existence of carrying simplex requires $a_{ij} > 0$ for all $i, j \in I_N$. But it is a great relief that we do not require the existence of a carrying simplex and only require $a_{ii} > 0, a_{ij} \geq 0$, and permit $a_{ij} = 0$, for distinct $i, j \in I_N$.

(iii) Although the theorems given in [4] are useful in general, and powerful in some particular cases, it is generally difficult in constructing suitable Liapunov functions. Fortunately, we are not bothered here with construction of Liapunov functions and we need only check geometric conditions relating to the relative position of the N nullcline planes restricted to $[0, r] \cap \partial \mathbb{R}_+^N$ or equivalent algebraic conditions using a_{ij} and r_k only.

(iv) From the comparisons given in section 4 at the end of Examples 4.1 and 4.3 we see that none of the three results by three different methods is better than others in general, but each has its advantages for some particular cases. Therefore, all of the results are supplements to each other and enrich the theory and methods of discrete dynamical systems.

Due to the restriction of time, space and the author's knowledge, there are a few problems that the author has not solved to make this paper more attractive and complete. Solutions to the problems below are expected for future investigation. (The author should be grateful to anyone who offers advice by private communications.)

1. For a discrete dynamical system in general, or for system (1) with (2)–(5) and $(a1)$–$(a3)$ in particular, is a globally attracting fixed point always stable so always globally asymptotically stable? A proof for YES or a counter-example for NO is expected.
2. CONJECTURE. Assume that x^* is a globally attracting fixed point and the Jacobian $DT(x^*)$ is invertible. Then either x^* is globally asymptotically stable or there is a homoclinic cycle. Is this conjecture correct?
3. If the answer to problem 1 above is YES, then our Theorem 3.9 (ii) is void. Otherwise, find an example that a fixed point x^* is globally attracting and $DT(x^*)$ has an eigenvalue with modulus greater than 1 so that x^* induces a homoclinic cycle.
4. For system (1) with (2)–(5), are the assumptions $(a1)$–$(a3)$ enough to guarantee the existence of a carrying simplex Σ so that the global attractor S_0 of the system on $\mathbb{R}^N_+ \setminus \{0\}$ obtained in Proposition 2.2 satisfies $S_0 = \Sigma$?

Appendix 1 Proofs of $(a1)$–$(a3)$ for models (6)–(9)

We check that models (6)–(9) and their combinations satisfy $(a1)$–$(a3)$.

Ricker models (6): $G_i(u) = e^{r_i - u}$, $r_i \in (0, 1]$,

$$T_i(x) = x_i G_i((Ax)_i) = x_i e^{r_i - (Ax)_i}, i \in I_N.$$

Clearly, $G_i(r_i) = 1$, $G_i'(u) = -e^{r_i - u} < 0$, and for $x_i \in [0, r_i)$,

$$\frac{\partial T_i(x)}{\partial x_i} = (1 - x_i)e^{r_i - (Ax)_i} > (1 - r_i)e^{r_i - (Ax)_i} \geq 0.$$

So $(a1)$ and $(a2)$ are satisfied. For each nonempty $J \subset I_N$ and every $y \in [0, r] \cap \Gamma_i \cap_{j \in J \setminus \{i\}} (\Gamma_j \cup \Gamma_j^-)$ with $y_j > 0$ for all $j \in J$, we have $0 < (Ay^J)_j \leq (Ay)_j \leq r_j \leq 1$. Then

$$\begin{aligned} \frac{d}{dt} T_j(y^{I_N \setminus J} + ty^J) &= \frac{d}{dt}[ty_j e^{r_j - (Ay^{I_N \setminus J} + tAy^J)_j}] \\ &= y_j(1 - t(Ay^J)_j)e^{r_j - (Ay^{I_N \setminus J} + tAy^J)_j} \\ &\geq y_j(1 - t)e^{r_j - (Ay^{I_N \setminus J} + tAy^J)_j} > 0 \end{aligned}$$

for $t \in [0,1)$ and all $j \in J$. Thus, $(a3)$ holds.

Leslie-Gower models (7): $G_i(u) = \frac{1+r_i}{1+u}$, $r_i \in (0,+\infty)$,

$$T_i(x) = x_i G_i((Ax)_i) = \frac{(1+r_i)x_i}{1+(Ax)_i}, i \in I_N.$$

Obviously, $G_i(r_i) = 1$, $G_i'(u) = -\frac{1+r_i}{(1+u)^2} < 0$, and for $x_i \geq 0$,

$$\frac{\partial T_i(x)}{\partial x_i} = \frac{(1+r_i)(1+(Ax^{I_N\setminus\{i\}})_i)}{(1+(Ax)_i)^2} > 0.$$

So $(a1)$ and $(a2)$ are satisfied. For each nonempty $J \subset I_N$ and every $y \in [0,r] \cap \Gamma_i \cap_{j\in J\setminus\{i\}} (\Gamma_j \cup \Gamma_j^-)$ with $y_j > 0$ for all $j \in J$, we have

$$\begin{aligned}\frac{d}{dt}T_j(y^{I_N\setminus J} + ty^J) &= \frac{d}{dt}\frac{ty_j(1+r_j)}{1+(Ay^{I_N\setminus J})_j + t(Ay^J)_j} \\ &= \frac{y_j(1+r_j)(1+(Ay^{I_N\setminus J})_j)}{[1+(Ay^{I_N\setminus J})_j + t(Ay^J)_j]^2} > 0\end{aligned}$$

for $t \in [0,1)$ and all $j \in J$. Thus, $(a3)$ holds.

Atkinson-Allen models (8): $G_i(u) = b + \frac{2(1-b)}{1+u}$, $r_i = 1$, $0 < b < 1$,

$$T_i(x) = x_i G_i((Ax)_i) = bx_i + \frac{2(1-b)x_i}{1+(Ax)_i}, i \in I_N.$$

Since $G_i(1) = 1$, $G_i'(u) = -\frac{2(1-b)}{(1+u)^2} < 0$, and for $x_i \geq 0$,

$$\frac{\partial T_i(x)}{\partial x_i} = b + \frac{2(1-b)(1+(Ax^{I_N\setminus\{i\}})_i)}{(1+(Ax)_i)^2} > 0,$$

$(a1)$ and $(a2)$ are met. For each nonempty $J \subset I_N$ and every y as above, we have

$$\begin{aligned}\frac{d}{dt}T_j(y^{I_N\setminus J} + ty^J) &= \frac{d}{dt}[bty_j + \frac{2ty_j(1-b)}{1+(Ay^{I_N\setminus J})_j + t(Ay^J)_j}] \\ &= by_j + \frac{2y_j(1-b)(1+(Ay^{I_N\setminus J})_j)}{[1+(Ay^{I_N\setminus J})_j + t(Ay^J)_j]^2} > 0\end{aligned}$$

for $t \in [0,1)$ and all $j \in J$. So $(a3)$ is satisfied.

Generalised Atkinson-Allen models (9): $G_i(u) = b_i + \frac{(1+r_i)(1-b_i)}{1+u}$, $r_i \in (0,+\infty)$, $0 < b_i < 1$,

$$T_i(x) = x_i G_i((Ax)_i) = b_i x_i + \frac{(1+r_i)(1-b_i)x_i}{1+(Ax)_i}, i \in I_N.$$

Clearly, $G_i(r_i) = 1$, $G_i'(u) = -\frac{(1+r_i)(1-b_i)}{(1+u)^2} < 0$, and for $x_i \geq 0$,

$$\frac{\partial T_i(x)}{\partial x_i} = b_i + \frac{(1+r_i)(1-b_i)(1+(Ax^{I_N\setminus\{i\}})_i)}{(1+(Ax)_i)^2} > 0.$$

So $(a1)$ and $(a2)$ are fulfilled. For each nonempty $J \subset I_N$ and every y as above, we have

$$\begin{aligned}\frac{d}{dt}T_j(y^{I_N\setminus J}+ty^J) &= \frac{d}{dt}[b_j t y_j + \frac{(1+r_j)ty_j(1-b_j)}{1+(Ay^{I_N\setminus J})_j + t(Ay^J)_j}] \\ &= b_j y_j + \frac{(1+r_j)y_j(1-b_j)(1+(Ay^{I_N\setminus J})_j)}{[1+(Ay^{I_N\setminus J})_j+t(Ay^J)_j]^2} > 0\end{aligned}$$

for $t \in [0,1)$ and all $j \in J$. So $(a3)$ is met.

From the above detailed check we see that for system (1) with (2)–(5), for each $i \in I_N$, if $G_i(u)$ is taken to be any one of e^{r_i-u}, $\frac{1+r_i}{1+u}, b+\frac{2(1-b)}{1+u}$ or $b_i+\frac{(1+r_i)(1-b_i)}{1+u}$, then the system as a combination of (6)–(9) still satisfies $(a1)$–$(a3)$.

Appendix 2. Proof of Proposition 2.3

Proof of Proposition 2.3. (i) Under (20) for all $i \in J$, we suppose $(Ar^{I_N\setminus\{i,j\}})_i \geq r_i$ for some $i, j \in J$ with $i \neq j$. Then the point $r^{I_N\setminus\{i,j\}}$ is on or above Γ_i. As 0 is below Γ_i, we have

$$\emptyset \neq \left([0, r^{I_N\setminus\{i,j\}}] \cap \Gamma_i\right) \subset \left([0, r^{I_N\setminus\{i\}}] \cap \Gamma_i\right).$$

By (20), $[0, r^{I_N\setminus\{i\}}] \cap \Gamma_j = \Gamma_j \cap [0,r] \cap \pi_i$ is strictly below Γ_i, so $[0, r^{I_N\setminus\{i\}}] \cap \Gamma_i$ is strictly above Γ_j. Thus, $[0, r^{I_N\setminus\{i,j\}}] \cap \Gamma_i$ as a nonempty subset of $[0, r^{I_N\setminus\{i\}}] \cap \Gamma_i$ is strictly above Γ_j. On the other hand, as (20) also holds with the replacement of i by j, $[0, r^{I_N\setminus\{j\}}] \cap \Gamma_i = \Gamma_i \cap [0,r] \cap \pi_j$ is strictly below Γ_j, so $[0, r^{I_N\setminus\{i,j\}}] \cap \Gamma_i$ as a nonempty subset of $[0, r^{I_N\setminus\{j\}}] \cap \Gamma_i$ is strictly be low Γ_j, a contradiction to $[0, r^{I_N\setminus\{i,j\}}] \cap \Gamma_i$ strictly above Γ_j. This contradiction shows that $(Ar^{I_N\setminus\{i,j\}})_i < r_i$, i.e. $r^{I_N\setminus\{i,j\}}$ is below Γ_i and Γ_j for all $i, j \in J$ with $i \neq j$. Since $(Ar^{I_N\setminus\{i\}})_j \geq a_{jj}r_j = r_j$, $r^{I_N\setminus\{i\}}$ is on or above Γ_j. Hence, $[r^{I_N\setminus\{i,j\}}, r^{I_N\setminus\{i\}}] \cap \Gamma_j = \{z\}$, where $(z-r)^{I_N\setminus\{i,j\}} = 0$, $z_i = 0$ and $z_j = r_j - (Ar^{I_N\setminus\{i,j\}})_j > 0$. As $z \in [0, r^{I_N\setminus\{i\}}] \cap \Gamma_j$, by (20) z is below Γ_i so

$$(Az)_i = a_{ij}(r_j - (Ar^{I_N\setminus\{i,j\}})_j) + (Ar^{I_N\setminus\{i,j\}})_i < r_i.$$

Then (21) follows for all $i, j \in J$ with $i \neq j$.

(ii) If $r^{I_N\setminus\{i\}}$ is below Γ_i then $[0, r^{I_N\setminus\{i\}}] = [0,r] \cap \pi_i$ is strictly below Γ_i so (20) holds. If $r^{I_N\setminus\{i\}}$ is on or above Γ_i, by (21) we have $(Ar^{I_N\setminus\{i,j\}})_i < r_i$ so $r^{I_N\setminus\{i,j\}}$ is below Γ_i for all $j \in I_N \setminus \{i\}$. Thus, $[r^{I_N\setminus\{i,j\}}, r^{I_N\setminus\{i\}}] \cap \Gamma_i = \{P_j\}$. It can be checked that

$$[0, r^{I_N\setminus\{i\}}] \cap \Gamma_i = \left\{ \sum_{j\in I_N\setminus\{i\}} c_j P_j : c_j \geq 0, \sum_{j\in I_N\setminus\{i\}} c_j = 1 \right\}. \tag{49}$$

For each fixed $k \in I_N \setminus \{i\}$, we show that P_j is above Γ_k for all $j \in I_N \setminus \{i\}$. In fact, if $j \neq k$ then $(Ar^{I_N\setminus\{i,j\}})_k \geq a_{kk}r_k = r_k$, so $r^{I_N\setminus\{i,j\}}$ is on or above Γ_k. For $j = k$, if $r^{I_N\setminus\{i,k\}}$ is above Γ_k then P_k is above Γ_k as $P_k \geq r^{I_N\setminus\{i,k\}}$; if $r^{I_N\setminus\{i,k\}}$ is on or below Γ_k, as $[r^{I_N\setminus\{i,k\}}, r^{I_N\setminus\{i\}}] \cap \Gamma_k = \{z\}$ and z is below Γ_i, $[r^{I_N\setminus\{i,k\}}, r^{I_N\setminus\{i\}}] \cap \Gamma_k$ is strictly below Γ_i, so $[r^{I_N\setminus\{i,k\}}, r^{I_N\setminus\{i\}}] \cap \Gamma_i$ is strictly above Γ_k. Hence, P_k is above Γ_k. As $r^{I_N\setminus\{i\}} \geq P_k$, $r^{I_N\setminus\{i\}}$ must be above Γ_k. Since $r^{I_N\setminus\{i,j\}}$ is on or above Γ_k for all $j \in I_N \setminus \{i,k\}$, the line segment $[r^{I_N\setminus\{i,j\}}, r^{I_N\setminus\{i\}}] \setminus \{r^{I_N\setminus\{i,j\}}\}$, which contains P_j, is strictly above Γ_k. Therefore, P_j is above Γ_k for all $j \in I_N \setminus \{i\}$. By (49), $[0, r^{I_N\setminus\{i\}}] \cap \Gamma_i$ is strictly above Γ_k. Hence, $[0, r^{I_N\setminus\{i\}}] \cap \Gamma_k$ is strictly below Γ_i for all $k \in I_N \setminus \{i\}$, i.e. (20) holds.

(iii) This is a combination of (i) and (ii). □

Appendix 3. Proof of Lemma 5.1

Proof of Lemma 5.1. We prove the statement by induction on N. When $N = 2$, Γ_1 and Γ_2 are straight line segments in $\mathbb{R}^2_+$, (36) means that the fixed point Q_1 on Γ_1 is below Γ_2 and the fixed point Q_2 on Γ_2 is below Γ_1. This implies that Γ_1 has a point above Γ_2 and a point Q_1 below Γ_2, so Γ_1 and Γ_2 has a unique intersection point $x^* \in \mathrm{int}\mathbb{R}^2_+$, i.e. $Ax = r$ has a unique solution $x^* \gg 0$. That $x^* \leq r$ follows from $a_{ii} = 1$ and $a_{ij} \geq 0$.

Suppose the statement is true on $\mathbb{R}^{N-1}_+$. We show the truth of the statement on $\mathbb{R}^N_+$. Viewing π_N as $\mathbb{R}^{N-1}_+$ and $\Gamma_1 \cap \pi_N, \ldots, \Gamma_{N-1} \cap \pi_N$ as planes in $\mathbb{R}^{N-1}_+$, by (36) we have

$$\forall i \in I_{N-1}, \forall j \in I_{N-1} \setminus \{i\}, (\Gamma_j \cap [0,r] \cap \pi_i) \cap \pi_N \text{ is strictly below } \Gamma_i.$$

By the inductive hypothesis, $(\cap_{i\in I_{N-1}} \Gamma_i) \cap \pi_N = \{z^*\}$ with $z^*_N = 0$ and $0 < z^*_i \leq r_i$ for all $i \in I_{N-1}$. This implies that $A_0 z = \tilde{r}$ has unique solution $\tilde{z}^* \in \mathrm{int}\mathbb{R}^{N-1}_+$ with $0 \ll \tilde{z}^* \leq \tilde{r}$, where A_0 is the submatrix of A obtained by deleting the Nth row and column of A and $\tilde{r} \in \mathrm{int}\mathbb{R}^{N-1}_+$ is obtained from r by deleting the Nth component. Thus, A_0^{-1} exists and $\tilde{z}^* = A_0^{-1}\tilde{r}$. From this we deduce that the set

$$\cap_{i\in I_{N-1}} \Gamma_i = \{z \in \mathbb{R}^N_+ : \forall i \in I_{N-1}, (Az)_i = r_i\}$$

is a line segment in $\mathbb{R}^N_+$ and it can be written as

$$\cap_{i\in I_{N-1}} \Gamma_i = \{z(\delta) = (\tilde{z}(\delta)^T, \delta)^T : 0 \leq \delta \leq \delta_0 \leq r_N\},$$

where, with $\tilde{C}_N = (a_{1N}, \ldots, a_{(N-1)N})^T$,

$$\tilde{z}(\delta) = A_0^{-1}\{\tilde{r} - \delta\tilde{C}_N\} = \tilde{z}^* - \delta A_0^{-1}\tilde{C}_N.$$

As $\tilde{z}(0) = \tilde{z}^* \gg 0$ and $z(\delta)$ is continuous in δ, we have $\tilde{z}(\delta) \gg 0$ and $z(\delta) \gg 0$ for $\delta > 0$ small enough. Since z^* is below Γ_N by (36), $z(\delta)$ is also below Γ_N for $\delta > 0$ small enough. Let

$$\delta^* = \sup\{\delta \in (0, \delta_0] : z(\delta) \gg 0, z(\delta) \text{ is below } \Gamma_N\}.$$

Then $0 \leq z(\delta^*) \leq r$ and $z(\delta^*)$ is on or below Γ_N. We show that $z(\delta^*) \gg 0$ and $z(\delta^*) = \cap_{i\in I_N} \Gamma_i$. Suppose $z(\delta^*) \not\gg 0$, then $z_j(\delta^*) = 0$ for some $j \in I_{N-1}$ so $z(\delta^*) \in \Gamma_j \cap [0,r] \cap \pi_j$. By (36), for this j and all $k \in I_N \setminus \{j\}$, $\Gamma_k \cap [0,r] \cap \pi_j$ is strictly below Γ_j so $\Gamma_N \cap [0,r] \cap \pi_j$ is strictly below Γ_j. Thus, $\Gamma_j \cap [0,r] \cap \pi_j$ is strictly above Γ_N. As $z(\delta^*) \in \Gamma_j \cap [0,r] \cap \pi_j$, $z(\delta^*)$

is above Γ_N, a contradiction to $z(\delta^*)$ on or below Γ_N. Therefore, we must have $z(\delta^*) \gg 0$. Then, by the definition of δ^*, $z(\delta^*)$ cannot be below Γ_N so $z(\delta^*)$ must be on Γ_N. □

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