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PULLBACK EXPONENTIAL ATTRACTORS FOR
DIFFERENTIAL EQUATIONS WITH VARIABLE DELAYS

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Abstract. We show how recent existence results for pullback exponential attractors can be applied to non-autonomous delay differential equations with time-varying delays. Moreover, we derive explicit estimates for the fractal dimension of the attractors.

As a special case, autonomous delay differential equations are also discussed, where our results improve previously obtained bounds for the fractal dimension of exponential attractors.

1. **Introduction.** We prove the existence of pullback exponential attractors for evolution processes generated by non-autonomous delay differential equations and derive explicit estimates for their fractal dimension. More precisely, we consider non-autonomous delay differential equations with a time-varying delay ρ of the form

$$\frac{d}{dt}x(t) = F(t, x(t - \rho(t))), \qquad t > s,$$
$$x_s = u \in \mathcal{C}, \qquad\qquad s \in \mathbb{R}, \tag{1}$$

where $\rho : \mathbb{R} \to [0, h]$ is continuous and $h > 0$. Here, $\mathcal{C} = C^0([-h, 0], \mathbb{R}^n)$ denotes the Banach space of continuous functions from $[-h, 0]$ into $\mathbb{R}^n$ endowed with the supremum norm

$$\|g\| = \sup_{t \in [-h, 0]} |g(t)|, \qquad g \in \mathcal{C}.$$

Moreover, if $x \in C^0([-h, T]; \mathbb{R}^n)$ for some $T > 0$, we denote by x_s, $s \in [0, T]$, the function in $\mathcal{C}$ defined by

$$x_s(\theta) = x(s + \theta), \qquad \theta \in [-h, 0].$$

Writing $f(t, x_t) = F(t, x(t - \rho(t)))$ problem (1) can be reformulated in an abstract framework as

$$\begin{aligned}
\frac{d}{dt}x(t) &= f(t, x_t), & t &> s, \\
x_s &= u \in \mathcal{C}, & s &\in \mathbb{R},
\end{aligned} \tag{2}$$

where we assume that the function $f : \mathbb{R} \times \mathcal{C} \longrightarrow \mathbb{R}^n$ is continuous and a bounded map, i.e., it maps bounded sets into bounded sets. The abstract formulation includes a larger set of examples such as, e.g. integro-differential equations (see [1], [2]), and our results can be extended to more general settings. However, in order to simplify the presentation we restrict our analysis to problem (1).

Exponential attractors have been introduced by Eden, Foias, Nicolaenco and Temam in the autonomous setting in [7]. An exponential attractor is a compact subset of finite fractal dimension, that is positively invariant and attracts all bounded subsets at an exponential rate. Due to the exponential rate of convergence, exponential attractors are more robust under perturbations than global attractors. They contain the global attractor, and hence, proving that an exponential attractor exists is one way of proving the existence and finite fractal dimension of the global attractor.

The first existence proof for exponential attractors in [7] was formulated for semigroups in Hilbert spaces, it is non-constructive and based on the squeezing property. An alternative method and explicit construction of exponential attractors for semigroups in Banach spaces was developed by Efendiev, Miranville and Zelik in [9] using the so-called smoothing property of the semigroup. This latter approach has more recently been extended for non-autonomous problems. In [9] non-autonomous exponential attractors were constructed for discrete time evolution processes based on the notion of uniform attractors. Subsequently, the method has been generalised for continuous time processes applying the weaker concept of pullback attractors (see [4], [5], [6], [12]).

While global attractors for delay differential equations have been extensively studied, in both, the autonomous and non-autonomous case, the existence of exponential attractors has only been shown for autonomous problems, namely, in [10] for finite delays and in [13] for unbounded delays. We aim to address this problem in the non-autonomous setting, which has not yet been considered so far.

The existence of global pullback attractors for non-autonomous delay differential equations of the form (1) has first been established in [2], for more general settings we refer to [1]. As in [2] we will distinguish two cases, namely, strongly dissipative and weakly dissipative nonlinearities. We extend the results in [2] by showing that not only the global pullback attractor, but also a pullback exponential attractor for problem (1) exists (which implies the existence and finite fractal dimension of the global pullback attractor). Moreover, we derive explicit estimates for the fractal dimension. Our proofs are based on the general existence theorems for pullback exponential attractors in [4], [5]. For applications with weakly dissipative nonlinearities, we need to slightly generalise the main result in [4].

In order to illustrate how the existence of exponential attractors can be shown based on the smoothing property we first analyse autonomous problems. We generalise the results in [10], where autonomous scalar delay differential equations with finite delays were considered, and improve previously obtained estimates for the fractal dimension of exponential attractors.

The outline of our paper is as follows: In Section 2 we recall basic notions and results from the theory of infinite dimensional dynamical systems and introduce the notion of exponential attractors, in the autonomous and non-autonomous setting. In Section 3, we prove the existence of exponential attractors for autonomous delay differential equations. Section 4 contains our main results. We first recall and slightly generalise previous existence results for pullback exponential attractors and subsequently, we apply them to non-autonomous delay differential equations with time varying delays. Hereby, the cases of weakly and strongly dissipative nonlinearities are distinguished.

2. **Preliminaries.** In this section, we collect several basic notions from the theory of infinite dimensional dynamical systems that we will need in the sequel and recall the notion of exponential attractors. First, we address the autonomous setting and subsequently, we formulate the corresponding concepts in the non-autonomous framework.

In this section, X will always denote a Banach space with norm $\|\cdot\|$.

2.1. **Semigroups and attractors in the autonomous setting.**

Definition 2.1. The family of operators $S(t) : X \to X$, $t \geq 0$, is called a **semigroup** if it satisfies the following properties:

$$S(t) \circ S(s) = S(s + t), \qquad \forall t, s \geq 0,$$
$$S(0) = \mathrm{Id},$$
$$(t, x) \mapsto S(t)x \qquad \text{is continuous,}$$

where Id denotes the identity in X and $\circ$ the composition of operators.

Definition 2.2. A subset $\mathcal{A} \subset X$ is the **global attractor** for the semigroup $S(t), t \geq 0$, if

- $\mathcal{A}$ is nonempty and compact,
- $\mathcal{A}$ is strictly invariant, i.e.,

$$S(t)\mathcal{A} = \mathcal{A}, \qquad \forall t \geq 0,$$

- and it attracts all bounded subsets of X, i.e., for every bounded $D \subset X$

$$\lim_{t \to \infty} \mathrm{dist}_H(S(t)D, \mathcal{A}) = 0,$$

where $\mathrm{dist}_H(\cdot, \cdot)$ denotes the Hausdorff semi-distance in X, i.e.

$$\mathrm{dist}_H(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|.$$

For the proof of the following existence result for global attractors we refer to [3], Corollary 2.13.

Theorem 2.3. *Let $S(t) : X \to X$, $t \geq 0$, be a semigroup in X. Then, the semigroup possesses a global attractor $\mathcal{A}$ if and only if there exists a compact subset K that attracts all bounded subsets of X. Moreover, in this case $\mathcal{A} = \omega(K)$, where*

$$\omega(K) = \bigcap_{s \geq 0} \overline{\bigcup_{r \geq s} S(r)K}$$

denotes the ω-limit set of K.

Exponential attractors are compact subsets of finite fractal dimension that contain the global attractor and attract all bounded subsets at an exponential rate.

Definition 2.4. A subset $\mathcal{M} \subset X$ is called an **exponential attractor** for the semigroup $S(t), t \geq 0$, if

- $\mathcal{M}$ is nonempty and compact,
- $\dim_f(\mathcal{M}) < \infty$,
- $\mathcal{M}$ is positively invariant, i.e.,

$$S(t)\mathcal{M} \subset \mathcal{M}, \qquad t \geq 0,$$

- and there exists a constant $\omega > 0$ such that for every bounded set $D \subset X$

$$\lim_{t \to \infty} e^{\omega t} \mathrm{dist}_H(S(t)D, \mathcal{M}) = 0.$$

We recall that the fractal dimension of a precompact subset $A \subset X$ is defined as

$$\dim_f(A) = \lim_{\varepsilon \to 0} \frac{\ln(N_\varepsilon^X(A))}{\ln(\frac{1}{\varepsilon})},$$

where $N_\varepsilon^X(A)$ denotes the minimal number of ε-balls in X with centres in A needed to cover A.

Remark 1. It immediately follows from Theorem 2.3 that, if an exponential attractor $\mathcal{M}$ for the semigroup $S(t), t \geq 0$, exists then, there also exists the global attractor $\mathcal{A}$,

$$\mathcal{A} = \omega(\mathcal{M}) \subset \mathcal{M},$$

and the fractal dimension of $\mathcal{A}$ is finite.

2.2. **Evolution processes and pullback attractors.** The solutions of non-autonomous problems do not only depend on the elapsed time after starting, but also on the initial time. Hence, to describe the time evolution of non-autonomous dynamical systems a two-parameter family of operators is required.

Definition 2.5. Let $t, s, r \in \mathbb{R}$. The two-parameter family of operators $U(t, s) : X \to X, t \geq s$, is called an **evolution process** in X if it satisfies the following properties:

$$U(t, s) \circ U(s, r) = U(t, r), \qquad \qquad \forall t \geq s \geq r,$$
$$U(t, t) = \mathrm{Id}, \qquad \qquad \forall t \in \mathbb{R},$$
$$(t, s, x) \mapsto U(t, s)x \qquad \qquad \text{is continuous.}$$

Different concepts have been proposed to extend the notion of autonomous global attractors for evolution processes. Here, we consider the pullback approach.

Definition 2.6. Let $U(t, s), t \geq s$, be an evolution process in X. A family of non-autonomous sets $\mathcal{A} = \{\mathcal{A}(t) | t \in \mathbb{R}\}$ is said to be the **global pullback attractor** for U if it satisfies the following properties:

- $\mathcal{A}(t) \subset X$ is non-empty and compact for all $t \in \mathbb{R}$,
- $\mathcal{A}$ is strictly invariant, i.e.

$$U(t,s)\mathcal{A}(s) = \mathcal{A}(t) \qquad \forall t \geq s, \ s \in \mathbb{R},$$

- $\mathcal{A}$ pullback attracts all bounded sets, i.e. for every bounded subset $D \subset X$ and $t \in \mathbb{R}$,

$$\lim_{s \to \infty} \text{dist}_H(U(t, t-s)D, \mathcal{A}(t)) = 0,$$

- and $\mathcal{A}$ is minimal within the families of closed subsets that pullback attract all bounded subsets of X.

Next, we recall the generalisation of Theorem 2.3 for non-autonomous problems (see [3], Theorem 2.12).

Theorem 2.7. *Let $U(t,s), t \geq s$, be an evolution process in X. Then, the evolution process possesses a global pullback attractor if and only if there exists a family of compact subsets $\{K(t)|t \in \mathbb{R}\}$ that pullback attracts all bounded subsets of X. Moreover, in this case*

$$\mathcal{A}(t) = \overline{\bigcup_{D \subset X \, bounded} \omega(D,t)} \subset K(t), \qquad t \in \mathbb{R},$$

where

$$\omega(D,t) = \bigcap_{r \leq t} \overline{\bigcup_{s \leq r} U(t,s)D}$$

denotes the ω-limit set of D at time t.

Definition 2.8. Let $U(t,s), t \geq s$, be an evolution process in X. The family of non-empty compact subsets $\mathcal{M} = \{\mathcal{M}(t)|t \in \mathbb{R}\}$ is called a **pullback exponential attractor** for the evolution process U if

- $\mathcal{M}$ is positively invariant, i.e.

$$U(t,s)\mathcal{M}(s) \subset \mathcal{M}(t) \qquad \forall t \geq s,$$

- the fractal dimension of the sections $\mathcal{M}(t), t \in \mathbb{R}$, is uniformly bounded,

$$\sup_{t \in \mathbb{R}}\{\dim_f(\mathcal{M}(t))\} < \infty,$$

- and $\mathcal{M}$ exponentially pullback attracts all bounded sets, i.e. there exists a constant $\omega > 0$ such that for every bounded subset $D \subset X$ and every $t \in \mathbb{R}$

$$\lim_{s \to \infty} e^{\omega s}\text{dist}_H\big(U(t, t-s)D, \mathcal{M}(t)\big) = 0.$$

Remark 2. An immediate consequence of Theorem 2.7 is that, if a pullback exponential attractor $\mathcal{M}$ for an evolution process exists, then there also exists the global pullback attractor $\mathcal{A}$,

$$\mathcal{A}(t) \subset \mathcal{M}(t) \qquad \forall t \in \mathbb{R},$$

and the fractal dimension of the sections of $\mathcal{A}$ is uniformly bounded.

3. **Autonomous case.** In this section, we show the existence of exponential attractors for autonomous delay differential equations. The proof is based on the construction of exponential attractors using the smoothing property in [9] (see also [15]).

3.1. **A general existence result for exponential attractors.** The following theorem yields sufficient conditions for the existence of exponential attractors. More general results are available (e.g., see [9], [15]), but this version suffices when considering delay differential equations with a finite delay.

Theorem 3.1. *Let $S(t), t \geq 0$, be a semigroup in a Banach space W. We assume that the following properties hold:*

(H_0) *There exists another Banach space V such that the embedding $V \hookrightarrow \hookrightarrow W$ is dense and compact.*

(H_1) *There exists a bounded absorbing set $B \subset W$.*

(H_2) *The semigroup satisfies the smoothing property in B, i.e., there exist positive constants $\tilde{t}$ and κ such that*

$$\|S(\tilde{t})u - S(\tilde{t})v\|_V \leq \kappa \|u - v\|_W \qquad \forall u, v \in B.$$

(H_3) *The semigroup $S(t), t \geq 0$, is Hölder continuous in time within the interval $[\tilde{t}, 2\tilde{t}]$, i.e.*

$$\|S(t)u - S(s)u\|_W \leq c|t - s|^\alpha \qquad \forall u \in B, \ t, s \in [\tilde{t}, 2\tilde{t}],$$

for some constants $c > 0$ and $\alpha \in (0, 1]$.

Then, for every $\nu \in (0, \frac{1}{2})$ there exists an exponential attractor $\mathcal{M}^\nu$ in W, and its fractal dimension is bounded by

$$\dim_f(\mathcal{M}^\nu) \leq \log_{\frac{1}{2\nu}} \left(N_{\frac{\nu}{\kappa}}^W (B_1^V(0)) \right) + \frac{1}{\alpha}.$$

Proof. See [15], Theorem 3.6. □

We remark that, without loss of generality, one can assume that the absorbing set is positively invariant (see [15]).

3.2. **Autonomous delay differential equations.** We show that Theorem 3.1 can easily be applied to autonomous delay differential equations with a constant, finite delay. Let $h > 0$ and $\mathcal{C} = C^0([-h, 0]; \mathbb{R}^n)$ be the Banach space of uniformly continuous functions equipped with the supremum norm

$$\|u\|_\mathcal{C} = \sup_{t \in [-h, 0]} \left\{ |u(t)| \right\}, \qquad u \in \mathcal{C}.$$

Moreover, if $x \in C^0([-h, T]; \mathbb{R}^n)$ for some $T > 0$, we denote by $x_s, \ s \in [0, T]$, the function in $\mathcal{C}$ defined by

$$x_s(\theta) = x(s + \theta), \qquad \theta \in [-h, 0].$$

We consider the delay differential equation

$$\begin{aligned} \frac{d}{dt}x(t) &= f(x(t-h)), & t &> 0, \\ x_0 &= u, & u &\in \mathcal{C}, \end{aligned} \tag{3}$$

where the nonlinearity $f : \mathbb{R}^n \to \mathbb{R}^n$ is globally Lipschitz continuous, i.e. there exists a constant $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y|, \qquad \forall x, y \in \mathbb{R}^n. \tag{4}$$

Here, we denote by $|\cdot|$ the norm in $\mathbb{R}^n$.

Under these assumptions for any $u \in \mathcal{C}$ there exists a unique solution $x(\cdot; u)$ of problem (3), which is defined for all times $t \geq 0$ and satisfies $x_t \in \mathcal{C}$ (e.g., see [14], Section 3.3). The associated semigroup $S(t) : \mathcal{C} \to \mathcal{C}, t \geq 0$, is defined by

$$S(t)u := x_t(\cdot; u) = x(\cdot + t; u), \qquad u \in \mathcal{C}.$$

By a different approach the existence of exponential attractors for autonomous delay differential equations has previously been proved in [10] in case of finite delays, and in [13] for equations with an infinite delay. For problems with a finite delay we show that such results can easily be deduced from Theorem 3.1.

Theorem 3.2. *We assume that there exists a bounded absorbing set B for the semigroup $S(t), t \geq 0$, in $\mathcal{C}$, and f satisfies* (4).

Then, for every $\nu \in (0, \frac{1}{2})$ there exists an exponential attractor $\mathcal{M}^\nu$, and its fractal dimension is bounded by

$$\dim_f(\mathcal{M}^\nu) \leq \log_{\frac{1}{2\nu}} \left(N^{\mathcal{C}}_{\frac{\nu}{\kappa}}(B_1^{\mathcal{C}^1}(0)) \right) + 1,$$

where $\kappa = 1 + L(1 + h)$ and $\mathcal{C}^1 = C^1([-h, 0])$ denotes the space of continuously differentiable functions equipped with the norm

$$\|u\|_{\mathcal{C}^1} = \sup_{t \in [-h,0]} \left\{ |u(t)| + |u'(t)| \right\} = \|u\|_{\mathcal{C}} + \|u'\|_{\mathcal{C}}, \qquad u \in \mathcal{C}^1.$$

Proof. **(i) Smoothing property** (H_2)**:** Let $u, v \in \mathcal{B}$ and $S(t)u = x_1(\cdot + t; u)$, $S(t)v = x_2(\cdot + t; v)$, $t \geq 0$, denote the corresponding solutions of (3). We will prove the smoothing property for $\tilde{t} = h$. To this end we observe that

$$\|S(h)u - S(h)v\|_{\mathcal{C}^1}$$
$$= \|x_1(\cdot + h; u) - x_2(\cdot + h; v)\|_{\mathcal{C}^1}$$
$$= \|x_1(\cdot + h; u) - x_2(\cdot + h; v)\|_{\mathcal{C}} + \|x_1'(\cdot + h; u) - x_2'(\cdot + h; v))\|_{\mathcal{C}}.$$

If $t \in [-h, 0]$, then integrating the equation (3) from 0 to $t + h$ yields

$$|x_1(t + h; u) - x_2(t + h; v)|$$
$$\leq |x_1(0; u) - x_2(0; v)| + \int_{-h}^{t} |f(x_1(s; u)) - f(x_2(s; v))| ds$$
$$\leq \|u - v\|_{\mathcal{C}} + L \int_{-h}^{t} |x_1(s; u) - x_2(s; v)| ds$$
$$\leq \|u - v\|_{\mathcal{C}} + Lh\|u - v\|_{\mathcal{C}} = (1 + Lh)\|u - v\|_{\mathcal{C}}.$$

Hence, taking the supremum over $t \in [-h, 0]$ it follows that

$$\|S(h)u - S(h)v\|_{\mathcal{C}} = \|x_1(\cdot + h; u) - x_2(\cdot + h; v)\|_{\mathcal{C}} \leq (1 + Lh)\|u - v\|_{\mathcal{C}}. \quad (5)$$

Moreover, if $t \in [-h, 0]$, then (3) implies that

$$|x_1'(t + h; u) - x_2'(t + h; v)| \leq |f(x_1(t; u)) - f(x_2(t; v))|$$
$$\leq L|x_1(t; u) - x_2(t; v)| \leq L\|u - v\|_{\mathcal{C}},$$

and, taking the supremum over $t \in [-h, 0]$ we obtain

$$\|x_1'(\cdot + h; u) - x_2'(\cdot + h; v)\|_{\mathcal{C}} \leq L\|u - v\|_{\mathcal{C}}. \quad (6)$$

Summing up the estimates (5) and (6) yields the smoothing property with $\tilde{t} = h$,

$$\|S(h)u - S(h)v\|_{\mathcal{C}^1} \leq (1 + L + Lh)\|u - v\|_{\mathcal{C}}.$$

(ii) Lipschitz continuity in time (H_3)**:** First, observe that w.l.o.g. we can assume that B is positively invariant. Indeed, if necessary we replace B by the set $\widetilde{B} = \bigcup_{t \geq T_B} S(t)B$, where T_B denotes the absorbing time corresponding to the set B.

We prove that the semigroup is Lipschitz continuous in time within the interval $[h, 2h]$. To this end let $s, t \in [0, h], t > s$, and $u \in B$. We denote the corresponding solution of (3) by $x_t(\cdot; u) = S(t)u, t \geq 0$, and observe that

$$\|S(h+t)u - S(h+s)u\|_{\mathcal{C}} = \|x(\cdot + h + t; u) - x(\cdot + h + s; u)\|_{\mathcal{C}}$$
$$= \sup_{\tau \in [-h,0]} \left\{ |x(\tau + h + t; u) - x(\tau + h + s; u)| \right\}$$
$$= \sup_{\tau \in [0,h]} \left\{ |x(\tau + t; u) - x(\tau + s; u)| \right\}.$$

If $\tau \in [0, h]$, then $\tau + t, \tau + s \in [0, 2h]$, and (3) implies that

$$|x(\tau + t; u) - x(\tau + s; u)| = \left| \int_{\tau+s}^{\tau+t} x'(r)dr \right| = \left| \int_{\tau+s}^{\tau+t} f(x(r-h))dr \right|$$
$$\leq \int_{\tau+s}^{\tau+t} |f(x(r-h))|dr \leq |t - s| \sup_{r \in [\tau+s, \tau+t]} \left\{ |f(x(r-h))| \right\}$$
$$\leq |t - s| \sup_{r \in [-h,h]} \left\{ |f(x(r))| \right\} \leq C|t - s|,$$

for some constant $C > 0$. Here, we used the positive invariance and boundedness of B and the continuity of f. Hence, taking the supremum over $\tau \in [0, h]$ in the above estimate it follows that

$$\|S(h+t)u - S(h+s)u\|_{\mathcal{C}} \leq C|t - s|,$$

for all $t, s \in [0, h]$, which shows (H_3) with $\alpha = 1$.

(iii) Existence of the exponential attractor: The hypotheses of Theorem 3.1 are satisfied. Consequently, for every $\nu \in (0, \frac{1}{2})$ there exists an exponential attractor $\mathcal{M}^\nu$ in $\mathcal{C}$ for the semigroup S, and its fractal dimension is bounded by

$$\dim_f(\mathcal{M}^\nu) \leq \log_{\frac{1}{2\nu}} \left(N_{\frac{\nu}{\kappa}}^{\mathcal{C}}(B_1^{\mathcal{C}^1}(0)) \right) + 1,$$

where $\kappa = 1 + L + Lh$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In the scalar case, using known estimates for the entropy numbers of the embedding $\mathcal{C}^1 \hookrightarrow \mathcal{C}$ we can derive an explicit estimate for the fractal dimension of the exponential attractor in Theorem 3.2. Moreover, we can determine the optimal value ν that minimises the bound for the fractal dimension. Theorem 3.3 improves the estimate for the fractal dimension of the exponential attractor obtained in [10] (Theorem 3.2).

Remark 3. We recall that for scalar equations the entropy numbers of the embedding $\mathcal{C}^1 \hookrightarrow \mathcal{C}$ are explicitly known, which yields an estimate for the growth of $N_\varepsilon^{\mathcal{C}}(B_1^{\mathcal{C}^1}(0))$. In fact, there exists a positive constant c such that

$$\log_2 \left(N_\varepsilon^{\mathcal{C}}(B_1^{\mathcal{C}^1}(0)) \right) \leq c\varepsilon^{-1} \qquad\qquad\qquad\qquad (7)$$

(see [8], Section 3.3).

Theorem 3.3. *Let the hypotheses of Theorem 3.2 be satisfied and let $n = 1$. Then, there exists an exponential attractor $\mathcal{M}$ such that its fractal dimension is bounded by*

$$dim_f(\mathcal{M}) \leq c\ln(4)e(1 + L + Lh) + 1,$$

where c is the constant in (7).

Proof. By Theorem 3.2 for every $\nu \in (0, \frac{1}{2})$ there exists an exponential attractor $\mathcal{M}^\nu$, and its fractal dimension is bounded by

$$\dim_f(\mathcal{M}^\nu) \leq \log_{\frac{1}{2\nu}}\left(N_{\frac{\nu}{\kappa}}^{\mathcal{C}}(B_1^{\mathcal{C}^1}(0))\right) + 1 = \frac{\log_2\left(N_{\frac{\nu}{\kappa}}^{\mathcal{C}}(B_1^{\mathcal{C}^1}(0))\right)}{\log_2(\frac{1}{2\nu})} + 1$$

$$\leq c\kappa\ln(2)\frac{\frac{1}{\nu}}{\ln(\frac{1}{2\nu})} + 1 = \tilde{c}d(\nu) + 1,$$

where we used (7), and $\kappa = 1 + L + Lh$. We observe that

$$\lim_{\nu\to 0} d(\nu) = \lim_{\nu\to\frac{1}{2}} d(\nu) = \infty,$$

and d attains its minimum in $(0, \frac{1}{2})$ for $\nu_0 = \frac{1}{2e}$. Inserting ν_0 in the estimate above yields the bound for the fractal dimension stated in the theorem. $\square$

4. **Non-autonomous case.** In this section, we prove the existence of pullback exponential attractors for non-autonomous delay differential equations with time varying delay of the form (1). As in [2], [3] we will distinguish two cases, namely, weakly and strongly dissipative nonlinearities.

4.1. **General existence theorems for pullback exponential attractors.** We recall existence results for pullback exponential attractors for non-autonomous evolutions processes. As in the autonomous setting, we present particular versions that suffice for applications to delay differential equations and refer for more general results to [4], [5], [15].

Different from Theorem 3.1, in the non-autonomous setting the Hölder continuity in time (H_3) can be omitted, and only the Lipschitz continuity w.r.t. the phase space is needed (cf. (A_3) below). The first existence result is formulated for evolution processes that are strongly bounded dissipative, i.e., it is assumed that a fixed bounded uniformly pullback absorbing set exists.

Theorem 4.1. *Let $U(t, s), t \geq s$, be an evolution process in a Banach space W. Moreover, we assume that for some $t_0 \in \mathbb{R}$ the following properties are satisfied:*

(A_0) *Let V be another Banach space such that the embedding $V \hookrightarrow\hookrightarrow W$ is dense and compact.*

(A_1) *There exists a bounded set $B \subset W$ that pullback absorbs all bounded sets at times $t \leq t_0$, i.e. for every bounded set $D \subset W$ there exists $T_D \geq 0$ such that*

$$\bigcup_{t\leq t_0} U(t, t - s)D \subset B \qquad \forall s \geq T_D.$$

(A_2) *The evolution process $U(t, s), t \geq s$, satisfies the smoothing property in B, i.e. there exist positive constants $\tilde{t} > 0$ and κ such that*

$$\|U(t, t - \tilde{t})u - U(t, t - \tilde{t})v\|_V \leq \kappa\|u - v\|_W \qquad \forall u, v \in B, t \leq t_0.$$

(A_3) *The evolution process $U(t,s), t \geq s$, is Lipschitz continuous in B, i.e. for all $t \in \mathbb{R}$, $t - \tilde{t} \leq s \leq t$, there exists $L_{t,s} \geq 0$ such that*

$$\|U(t,s)u - U(t,s)v\|_W \leq L_{t,s}\|u - v\|_W \qquad \forall u, v \in B.$$

Then, for every $\nu \in (0, \frac{1}{2})$ there exists a pullback exponential attractor $\mathcal{M}^\nu$ in W, and the fractal dimension of its section is bounded by

$$\dim_f(\mathcal{M}^\nu(t)) \leq \log_{\frac{1}{2\nu}}\left(N^W_{\frac{\nu}{\kappa}}(B_1^V(0))\right) \qquad \forall t \in \mathbb{R}.$$

Proof. See Theorem 3.4 and Remark 3 in [4]. Moreover, it can be observed from the proofs, that it suffices to assume properties (A_1) and (A_2) for all $t \leq t_0$ (see also [6]). In this case, the construction of the pullback exponential attractor is valid for all $t \leq t_0$. Then, using the Lipschitz continuity of the process (A_3), the sections $\mathcal{M}^\nu(t)$ for $t > t_0$ can be defined as $\mathcal{M}^\nu(t) = U(t,t_0)\mathcal{M}^\nu(t_0)$. $\qquad \square$

Corollary 1. *Let $U(t,s), t \geq s$, be an evolution process in a Banach space W. If the hypotheses $(A_0), (A_1)$ and (A_2) are satisfied, then the evolution process $U(t,s), t \geq s$, possesses a global pullback attractor, and the fractal dimension of its sections is bounded by*

$$\dim_f(\mathcal{A}(t)) \leq \inf_{\nu \in (0, \frac{1}{2})}\left\{\log_{\frac{1}{2\nu}}\left(N^W_{\frac{\nu}{\kappa}}(B_1^V(0))\right)\right\} \qquad \forall t \in \mathbb{R}.$$

Proof. See Theorem 3.4 in [4] and the proof of Theorem 3.1 in [5]. Moreover, it can be observed from the proofs, that it suffices to assume properties (A_1)–(A_2) for all $t \leq t_0$. $\qquad \square$

Next, we recall a more general existence theorem allowing that the pullback exponential attractor is unbounded in the past. More precisely, assumption (A_1) can be relaxed, and the fixed bounded pullback absorbing set B be replaced by a time-dependent family of absorbing sets $B = \{B(t)\}_{t \in \mathbb{R}}$ with certain properties.

The following theorem slightly generalises Theorem 3.4 in [4]. In particular, the hypotheses $(\widetilde{A}_1)$ and $(\widetilde{A}_2)$ are only required for $t \leq t_0$ (not for all $t \in \mathbb{R}$) and the positive invariance of the absorbing sets is only assumed for discrete time steps. Both generalisations are essential when we apply the result to prove the existence of pullback exponential attractors for non-autonomous delay differential equations with weakly dissipative nonlinearities.

Theorem 4.2. *Let $U(t,s), t \geq s$, be an evolution process in a Banach space W and (A_0) be satisfied. Moreover, we assume that for some $t_0 \in \mathbb{R}$ the following properties hold:*

$(\widetilde{A}_1)$ *There exists a family of bounded pullback absorbing sets $B = \{B(t)\}_{t \in \mathbb{R}}$ in W, i.e. for every bounded set $D \subset W$ and $t \leq t_0$, there exists $T_{D,t} \geq 0$ such that*

$$U(s, s - r)D \subset B(s) \qquad \forall r \geq T_{D,t}, \, s \leq t.$$

Moreover, there exists $\tilde{t} > 0$ such that

$$U(t, t - \tilde{t})B(t - \tilde{t}) \subset B(t) \qquad \forall t \leq t_0,$$

and the diameter of the family of absorbing sets $B = \{B(t)\}_{t \in \mathbb{R}}$ grows at most sub-exponentially in the past, i.e.

$$\lim_{t \to -\infty} \mathrm{diam}(B(t))e^{\gamma t} = 0 \qquad \forall \gamma > 0.$$

($\widetilde{A}_2$) *The evolution process $U(t,s), t \geq s$, satisfies the smoothing property in B, i.e. there exist a positive constant κ such that*

$$\|U(t, t-\tilde{t})u - U(t, t-\tilde{t})v\|_V \leq \kappa\|u-v\|_W \qquad \forall u, v \in B(t-\tilde{t}),\ t \leq t_0.$$

($\widetilde{A}_3$) *The evolution process $U(t,s), t \geq s$, is Lipschitz continuous in B, i.e. for all $t \in \mathbb{R}$, $t \leq s \leq t + \tilde{t}$, there exists $L_{s,t} \geq 0$ such that*

$$\|U(s,t)u - U(s,t)v\|_W \leq L_{s,t}\|u-v\|_W \qquad \forall u, v \in B(t).$$

Then, for every $\nu \in (0, \frac{1}{2})$ there exists a pullback exponential attractor $\mathcal{M}^\nu$ in W, and the fractal dimension of its sections is bounded by

$$\dim_f(\mathcal{M}^\nu(t)) \leq \log_{\frac{1}{2\nu}}\left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0))\right) \qquad \forall t \in \mathbb{R}.$$

Proof. See Theorem 3.4 in [4]. It can be observed from the proof, that it suffices to assume properties (A_1) and (A_2) for all $t \leq t_0$ (see also [6]). Moreover, the positive invariance of the family of pullback absorbing sets is only used to construct the pullback exponential attractor for discrete time evolution processes. Hence, it is sufficient to impose the positive invariance for discrete time steps. Under these assumptions the construction of the pullback exponential attractor in [4] is valid for all $t \leq t_0$. Then, using the Lipschitz continuity of the evolution process ($\widetilde{A}_3$), the sections $\mathcal{M}^\nu(t)$ for $t > t_0$ can be defined as $\mathcal{M}^\nu(t) = U(t, t_0)\mathcal{M}^\nu(t_0)$. $\qquad\square$

4.2. **Non-autonomous delay differential equations with time varying delay.** Finally, we consider the non-autonomous delay differential equation with a time varying delay (1),

$$\frac{d}{dt}x(t) = F(t, x(t-\rho(t))), \qquad t > s,$$

$$x_s = u \in \mathcal{C}, \qquad\qquad s \in \mathbb{R},$$

where $\rho : \mathbb{R} \to [0, h]$ is continuous. We recall that writing $f(t, x_t) = F(t, x(t-\rho(t)))$ problem (1) can be reformulated in an abstract framework as (2),

$$\frac{d}{dt}x(t) = f(t, x_t), \qquad\qquad t > s,$$

$$x_s = u \in \mathcal{C}, \qquad\qquad s \in \mathbb{R}.$$

We suppose that $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ is continuous and a bounded map, i.e. it maps bounded sets into bounded sets. Under these assumptions, for every $(s, u) \in \mathbb{R} \times \mathcal{C}$ there exists a unique solution $x(\cdot; s, u)$ of (2) defined on the maximal interval of existence $[s, \alpha_{s,u})$, where $\alpha_{s,u} > s + h$ (see [11], [1]). Moreover, either $\alpha_{s,u} = \infty$, or the solution blows up in finite time (see [14], Section 3.3). In the sequel, we assume that $\alpha_{s,u} = \infty$, and denote by $U(t, s) : \mathcal{C} \to \mathcal{C}, t \geq s$, the generated evolution process, i.e.

$$U(t, s)u := x_t(\cdot; s, u), \qquad (s, u) \in \mathbb{R} \times \mathcal{C}.$$

In the sequel, as in [2], [3] we distinguish the cases of weakly and strongly dissipative nonlinearities. If f is strongly dissipative, the existence of exponential attractors can be deduced from Theorem 4.1, while for weakly dissipative nonlinearities we need the generalized existence result stated in Theorem 4.2.

4.3. **The case of strong dissipativity.** The nonlinearity f and delay function ρ are assumed to satisfy the following hypotheses:

- We suppose that $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ in (1) is uniformly Lipschitz continuous w.r.t. the second variable, i.e., there exists a positive constant L such that

$$|F(t,x) - F(t,y)| \leq L|x - y| \qquad \forall t \in \mathbb{R}, x, y \in \mathbb{R}^n. \tag{8}$$

- The function $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ in (2) is strongly dissipative, i.e., there exists positive constants α and β such that

$$\langle f(t,\phi), \phi(0) \rangle \leq -\alpha |\phi(0)|^2 + \beta \qquad \forall \phi \in \Phi(h)\mathcal{C}, \tag{9}$$

  where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$ and

$$\Phi(h)\mathcal{C} = \{\phi \in \mathcal{C} | \phi = U(s + h, s)\psi \text{ for some } s \in \mathbb{R}, \psi \in \mathcal{C}\}$$

  is the set of functions in $\mathcal{C}$ that are realisable as solutions after time h.
- The delay function ρ is continuously differentiable and satisfies

$$\rho'(t) \leq \rho^* < 1 \qquad \forall t \in \mathbb{R}, \tag{10}$$

  for some constant $\rho^* < 1$.

Under the assumptions (8) and (9) it was proven in [2] that there exists a fixed, bounded uniformly pullback absorbing set for the evolution process $U(t,s), t \geq s$, and hence, the existence of the global attractor follows (see also [3], Chapter 10). Here, we show the existence of an exponential attractor which, in particular, implies the (existence and) finite dimensionality of the global attractor. To this end we additionally need the hypothesis (10).

Theorem 4.3. *Let $U(t,s), t \geq s$, be the evolution process in $\mathcal{C}$ generated by problem (2), and assume that (8) and (9) are satisfied.*

Then, for every $\nu \in (0, \frac{1}{2})$ there exist a pullback exponential attractor for $U(t,s)$, $t \geq s$, in $\mathcal{C}$, and its fractal dimension is bounded by

$$\dim_f(\mathcal{M}^\nu) \leq \log_{\frac{1}{2\nu}} \left(N_{\frac{\nu}{\kappa}}^{\mathcal{C}}(B_1^{\mathcal{C}^1}(0)) \right) + 1,$$

where $\kappa = (1 + L)\left(1 + \frac{Lh}{1 - \rho^*}\right) e^{\frac{Lh}{1 - \rho^*}}$. Moreover, the global pullback attractor exists, it is contained in the pullback exponential attractor $\mathcal{M}^\nu$, and its fractal dimension is finite.

Proof. We verify the hypotheses of Theorem 4.1 for the spaces $W = \mathcal{C}$ and $V = \mathcal{C}^1$. Certainly, (A_0) is satisfied due to the compact embedding $\mathcal{C}^1 \hookrightarrow\hookrightarrow \mathcal{C}$,
(i) Uniformly pullback absorbing set (A_1): In [2] it was shown that at every time $t \in \mathbb{R}$ the set

$$B_0 = \left\{ u \in \mathcal{C} : \|u\| \leq 1 + \frac{\beta}{\alpha} \right\}$$

is uniformly pullback absorbing every bounded subset $D \subset \mathcal{C}$. Moreover, if D is a bounded subset and $d > 0$ such that $\|\phi\| \leq d$ for all $\phi \in D$, then, the absorbing time corresponding to D is given by $T_D = \frac{1}{2\beta} \log(de^{2\beta h})$, which is independent of t. This proves hypothesis (A_1).
(ii) Smoothing property (A_2): We show the smoothing property with respect to the spaces $\mathcal{C}$ and $\mathcal{C}^1$ for $\tilde{t} = h$.

Let $s \in \mathbb{R}$, $u, v \in B_0$ and $x_1(t) = x_1(t; u, s) = U(t, s)u$ and $x_2(t) = x_2(t; v, s) = U(t, s)v$, $t \geq s$, denote the corresponding solutions of (2). By assumption (8) it follows that

$$
|x_1(t) - x_2(t)|
$$

$$
= |u(s) - v(s)| + \int_s^t \big| F\big(\tau, x_1(\tau - \rho(\tau))\big) - F\big(\tau, x_2(\tau - \rho(\tau))\big) \big| d\tau
$$

$$
\leq \|u - v\|_\mathcal{C} + L \int_s^t |x_1(\tau - \rho(\tau)) - x_2(\tau - \rho(\tau))| d\tau
$$

$$
\leq \|u - v\|_\mathcal{C} + \frac{L}{1 - \rho^*} \int_{s - \rho(s)}^{t - \rho(t)} |x_1(\sigma) - x_2(\sigma)| d\sigma
$$

$$
\leq \|u - v\|_\mathcal{C} + \frac{L}{1 - \rho^*} \left(\int_{s-h}^s |x_1(\sigma) - x_2(\sigma)| d\sigma + \int_s^t |x_1(\sigma) - x_2(\sigma)| d\sigma \right)
$$

$$
\leq \|u - v\|_\mathcal{C} \left(1 + \frac{Lh}{1 - \rho^*} \right) + \int_s^t |x_1(\sigma) - x_2(\sigma)| d\sigma,
$$

where we applied the change of variables $\sigma = \tau - \rho(\tau)$ and used the assumption (10). Hence, Gronwall's lemma implies that

$$
|x_1(t) - x_2(t)| \leq \left(1 + \frac{Lh}{1 - \rho^*} \right) e^{\frac{L}{1 - \rho^*}(t - s)} \|u - v\|_\mathcal{C}.
$$

Let now $\theta \in [-h, 0]$ and $t = s + h + \theta$. Then, using this estimate and taking the supremum over $\theta \in [-h, 0]$ we obtain

$$
\|x_1(s + h + \cdot) - x_2(s + h + \cdot)\|_\mathcal{C} \leq \left(1 + \frac{Lh}{1 - \rho^*} \right) e^{\frac{Lh}{1 - \rho^*}} \|u - v\|_\mathcal{C}. \tag{11}
$$

On the other hand, we observe that

$$
|x_1'(s + h + \theta) - x_2'(s + h + \theta)|
$$

$$
\leq \big| F\big(s + h + \theta, x_1(s + h + \theta - \rho(s + h + \theta))\big)
$$

$$
- F\big(s + h + \theta, x_2(s + h + \theta - \rho(s + h + \theta))\big) \big|
$$

$$
\leq L|x_1(s + h + \theta - \rho(s + h + \theta)) - x_2(s + h + \theta - \rho(s + h + \theta))|
$$

$$
\leq L \sup_{\tau \in [-h, h]} \{|x_1(s + \tau) - x_2(s + \tau)|\}
$$

$$
\leq L\big(\|u - v\|_\mathcal{C} + \|x_1(s + h + \cdot) - x_2(s + h + \cdot)\|_\mathcal{C} \big)
$$

$$
\leq L \left(1 + \frac{Lh}{1 - \rho^*} \right) e^{\frac{Lh}{1 - \rho^*}} \|u - v\|_\mathcal{C}.
$$

Taking the supremum over $\theta \in [-h, 0]$ it follows that

$$
\|x_1'(s + h + \cdot) - x_2'(s + h + \cdot)\|_\mathcal{C} \leq L \left(1 + \frac{Lh}{1 - \rho^*} \right) e^{\frac{Lh}{1 - \rho^*}} \|u - v\|_\mathcal{C}. \tag{12}
$$

Finally, summing up the inequalities (11) and (12) we obtain

$$
\|U(s + h, s)u - U(s + h, s)v\|_{\mathcal{C}^1} = \|x_1(s + h + \cdot) - x_2(s + h + \cdot)\|_{\mathcal{C}^1}
$$

$$
\leq (1 + L) \left(1 + \frac{Lh}{1 - \rho^*} \right) e^{\frac{Lh}{1 - \rho^*}} \|u - v\|_\mathcal{C} = \kappa \|u - v\|_\mathcal{C},
$$

which proves the smoothing property with $\tilde{t} = h$ and

$$\kappa = (1 + L)\left(1 + \frac{Lh}{1 - \rho^*}\right)e^{\frac{Lh}{1-\rho^*}}.$$

(iii) Lipschitz continuity (A_3) : The Lipschitz continuity follows from the first estimate in (ii). In fact, we obtain

$$\|U(t,s)u - U(t,s)v\|_{\mathcal{C}} = \|x_1(t + \cdot) - x_2(t + \cdot)\|_{\mathcal{C}}$$
$$\leq \left(2 + \frac{Lh}{1 - \rho^*}\right)e^{\frac{L}{1-\rho^*}(t-s)}\|u - v\|_{\mathcal{C}}.$$

(iv) Existence of the pullback exponential attractor: The existence of a pullback exponential attractor and the estimate for its fractal dimension is an immediate consequence of Theorem 4.1. □

As in the autonomous case, for scalar equations we can use known estimates for the entropy numbers of the embedding $\mathcal{C}^1 \hookrightarrow \mathcal{C}$ to derive an explicit estimate for the fractal dimension and determine the optimal value of ν in Theorem 4.3.

Theorem 4.4. *Let the hypotheses of Theorem 4.3 be satisfied and let $n = 1$. Then, there exists a pullback exponential attractor $\mathcal{M}$ such that its fractal dimension is bounded by*

$$dim_f(\mathcal{M}) \leq c\ln(4)e(1 + L)\left(1 + \frac{Lh}{1 - \rho^*}\right)e^{\frac{Lh}{1-\rho^*}},$$

where c is the constant in (7).

Proof. Following the arguments in the proof of Theorem 3.3, the optimal bound for the fractal dimension immediately follows from Theorem 4.3. □

4.4. **The case of weak dissipativity.** Finally, we prove the existence of pullback exponential attractors in the case of weakly dissipative nonlinearities, in particular, neither the disspativity condition nor the Lipschitz continuity are assumed to be uniform in time.

- We suppose that $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ in (1) is Lipschitz continuous locally in time, i.e., there exist a continuous function $m_1 : \mathbb{R} \to (0, \infty)$ such that

$$|F(t,x) - F(t,y)| \leq m_1(t)|x - y| \qquad \forall t \in \mathbb{R}, x, y \in \mathbb{R}^n. \tag{13}$$

- The function $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ in (2) is weakly dissipative, i.e., there exists a positive constant α and a continuous function $m_2 : \mathbb{R} \to (0, \infty)$ such that

$$\langle f(t, \phi), \phi(0)\rangle \leq (-\alpha + m_1(t))|\phi(0)|^2 + m_2(t) \qquad \forall \phi \in \Phi(h)\mathcal{C}, t \in \mathbb{R}. \tag{14}$$

- The function m_2 is non-decreasing and m_1, m_2 satisfy the integrability conditions

$$\int_{-\infty}^{t} m_1(s)ds < \infty, \qquad \int_{-\infty}^{t} e^{\varepsilon s}m_2(s)ds < \infty \qquad \forall \varepsilon > 0, \ t \in \mathbb{R}. \tag{15}$$

Under these hypotheses the existence of the global pullback attractor has been shown in [2], where it was not assumed that the function m_2 is non-decreasing. As the assumption on the delay function (10), this is an additional property that we need to prove the existence of a pullback exponential attractor, in particular, for showing the positive invariance of the family of pullback absorbing sets.

Theorem 4.5. *Let $U(t,s), t \geq s$, be the evolution process generated by problem (2). We assume that the hypotheses (13), (14) and (15) are satisfied and the delay function fulfills condition (10).*

Then, there exist a pullback exponential attractor for $U(t,s), t \geq s$, in $\mathcal{C}$. Moreover, the global pullback attractor exists, it is contained in the pullback exponential attractor, and its fractal dimension is finite. The bound on the fractal dimension is determined by m_1, ρ^* and the entropy properties of the embedding $\mathcal{C}^1 \hookrightarrow \mathcal{C}$.

Proof. We verify the hypotheses of Theorem 4.2 for the spaces $W = \mathcal{C}$ and $V = \mathcal{C}^1$. Certainly, (A_0) is satisfied due to the compact embedding $\mathcal{C}^1 \hookrightarrow\hookrightarrow \mathcal{C}$.

(i) Existence of a family of pullback absorbing sets $(\widetilde{A}_1)$: Let $\psi \in \mathcal{C}$ be given initial data, $\theta \in [-h, 0]$, $\tau \geq h$ and $x(\cdot; \psi, t - \tau)$ denote the solution of (2) with $x_{t-\tau} = \psi$. Taking the inner product of (2) with x and using the dissipativity assumption (14) we observe that

$$\frac{1}{2}\frac{d}{dt}|x(t)|^2 \leq (-\alpha + m_1(t))|x(t)|^2 + m_2(t),$$

and hence, Gronwall's lemma implies that

$$|x(t+\theta)|^2$$
$$\leq \|\psi\|_{\mathcal{C}}^2 e^{\int_{t-\tau}^{t+\theta} -2\alpha + 2m_1(r)dr} + 2\int_{t-\tau}^{t+\theta} m_2(r)e^{\int_r^{t+\theta} -2\alpha+2m_1(\xi)d\xi}dr$$
$$\leq \|\psi\|_{\mathcal{C}}^2 e^{-2\alpha\tau}e^{2\alpha h + 2\int_{-\infty}^t m_1(r)dr} + 2\int_{t-\tau}^{t+\theta} m_2(r)e^{-2\alpha(t+\theta-r)+2\int_{-\infty}^t m_1(\xi)d\xi}dr$$
$$\leq \|\psi\|_{\mathcal{C}}^2 e^{-2\alpha\tau}e^{2\alpha h + 2M_1(t)} + 2e^{2M_1(t)+2\alpha h}\int_{-\infty}^t e^{-2\alpha(t-r)}m_2(r)dr,$$

where $M_1(t) = \int_{-\infty}^t m_1(r)dr$. Consequently, if we set

$$r^2(t) = e^{2M_1(t)} + 2e^{2M_1(t)+2\alpha h}\int_{-\infty}^t e^{-2\alpha(t-r)}m_2(r)dr$$

it is clear that the family $\overline{B^{\mathcal{C}}}(0, r(t)), t \in \mathbb{R}$, is pullback absorbing all bounded subsets of $\mathcal{C}$, and the growth of $r(t)$ as $t \to -\infty$ is subexponential. Moreover, if $D \subset \mathcal{C}$ is a bounded set and $d > 0$ such that $\|\psi\| \leq d$ for all $\psi \in D$, then the corresponding absorbing time is

$$T_D = \frac{1}{2\alpha}\log(d^2 e^{2\alpha h}),$$

which is independent of $t \in \mathbb{R}$. Next, we show the positive invariance for discrete time steps. Let $s \in \mathbb{R}$, $\psi \in \overline{B^{\mathcal{C}}}(0, r(s))$ and x be the solution of (2) with $x_s = \psi$. Moreover, let $\theta \in [-h, 0]$ and $t \geq s + h$. Using the above estimate we observe that

$$|x(t+\theta)|^2$$
$$\leq \|\psi\|^2 e^{-2\alpha(t+\theta-s)+\int_s^{t+\theta} 2m_1(r)dr} + 2\int_s^{t+\theta} m_2(r)e^{-2\alpha(t+\theta-r)+\int_r^{t+\theta} 2m_1(\xi)d\xi}dr$$
$$\leq \left(e^{2M_1(s)} + 2e^{2\alpha h + 2M_1(s)}\int_{-\infty}^s e^{-2\alpha(s-r)}m_2(r)dr\right)e^{-2\alpha(t+\theta-s)+\int_s^{t+\theta} 2m_1(r)dr}$$

$$+ 2 \int_s^{t+\theta} m_2(r) e^{-2\alpha(t+\theta-r) + \int_r^{t+\theta} 2m_1(\xi)d\xi} dr$$

$$= e^{2M_1(s) + \int_s^{t+\theta} 2m_1(r)dr - 2\alpha(t+\theta-s)}$$

$$+ 2 e^{2\alpha h + 2M_1(s) + \int_s^{t+\theta} 2m_1(r)dr} \int_{-\infty}^s e^{-2\alpha(s-r+t+\theta-s)} m_2(r)dr$$

$$+ 2 \int_s^{t+\theta} m_2(r) e^{-2\alpha(t+\theta-r) + \int_r^{t+\theta} 2m_1(\xi)d\xi} dr$$

$$\leq e^{2M_1(t) - 2\alpha(t+\theta-s)} + 2 e^{2\alpha h + 2M_1(t)} \int_{-\infty}^s e^{-2\alpha(t+\theta-r)} m_2(r)dr$$

$$+ 2 \int_s^{t+\theta} m_2(r) e^{-2\alpha(t+\theta-r) + \int_r^{t+\theta} 2m_1(\xi)d\xi} dr$$

$$\leq e^{2M_1(t)} + 2 e^{2\alpha h + 2M_1(t)} \int_{-\infty}^s e^{-2\alpha(t+\theta-r)} m_2(r)dr$$

$$+ 2 e^{2M_1(t)} \int_s^{t+\theta} m_2(r) e^{-2\alpha(t+\theta-r)} dr$$

$$\leq e^{2M_1(t)} + 2 e^{2M_1(t) + 2\alpha h} \int_{-\infty}^{t+\theta} e^{-2\alpha(t+\theta-r)} m_2(r)dr.$$

Finally, using the change of variables $\tilde{r} = r - \theta$ leads to

$$|x(t+\theta)|^2 \leq e^{2M_1(t)} + 2 e^{2M_1(t) + 2\alpha h} \int_{-\infty}^t e^{-2\alpha(t-\tilde{r})} m_2(\tilde{r} + \theta) d\tilde{r}$$

$$\leq e^{2M_1(t)} + 2 e^{2M_1(t) + 2\alpha h} \int_{-\infty}^t e^{-2\alpha(t-r)} m_2(r)dr,$$

where, in the last step, we used that m_2 is non-decreasing. Taking the supremum over $\theta \in [-h, 0]$ and setting $t = s + h$ it follows that

$$\|x(s+h+\cdot)\|^2 \leq r^2(s+h),$$

which shows that the family of absorbing sets is positively invariant for discrete time steps $\tilde{t} = h$.

(ii) **Smoothing property** $(\widetilde{A}_2)$: Let $s \in \mathbb{R}$, $u, v \in B(s)$ and $x_1(t; u, s)$ and $x_2(t; u, s)$, $t \geq s$, denote the corresponding solutions of (2). In order to shorten notations, in the sequel we write $x_1(t) = x_1(t; u, s)$ and $x_2(t) = x_2(t; v, s), t \geq s$. For the difference of x_1 and x_2 we obtain

$$\|U(s+h, s)u - U(s+h, s)v\|_{\mathcal{C}^1} = \|x_1(s+h+\cdot) - x_2(s+h+\cdot)\|_{\mathcal{C}^1}$$

$$= \|x_1(s+h+\cdot) - x_2(s+h+\cdot)\|_{\mathcal{C}} + \|x_1'(s+h+\cdot) - x_2'(s+h+\cdot)\|_{\mathcal{C}}.$$

To estimate the first term, we use hypothesis (H_1) and (10). Let $\mu(\tau) := \tau - \rho(\tau)$. Then, $\mu' > 1$ and hence, μ as well as μ^{-1} are strictly increasing. Integrating the equation (2) from s to t we obtain

$$|x_1(t) - x_2(t)|$$

$$\leq \|u - v\|_{\mathcal{C}} + \int_s^t \left| F\big(\tau, x_1(\tau - \rho(\tau))\big) - F\big(\tau, x_2(\tau - \rho(\tau))\big) \right| d\tau$$

$$\leq \|u - v\|_{\mathcal{C}} + \int_s^t m_1(\tau)|x_1(\tau - \rho(\tau)) - x_2(\tau - \rho(\tau))|d\tau$$

$$\leq \|u - v\|_{\mathcal{C}} + \frac{1}{1 - \rho^*} \int_{s-\rho(s)}^{t-\rho(t)} m_1(\mu^{-1}(\sigma))|x_1(\sigma) - x_2(\sigma)|d\sigma$$

$$\leq \|u - v\|_{\mathcal{C}} + \frac{1}{1 - \rho^*} \Big(\int_{s-h}^s m_1(\mu^{-1}(\sigma))|x_1(\sigma) - x_2(\sigma)|d\sigma$$

$$+ \int_s^t m_1(\mu^{-1}(\sigma))|x_1(\sigma) - x_2(\sigma)|d\sigma \Big)$$

$$\leq \|u - v\|_{\mathcal{C}} \left(1 + \frac{1}{1 - \rho^*} \int_{s-h}^s m_1(\mu^{-1}(\sigma))d\sigma \right)$$

$$+ \frac{1}{1 - \rho^*} \int_s^t m_1(\mu^{-1}(\sigma))|x_1(\sigma) - x_2(\sigma)|d\sigma,$$

where we applied the change of variables $\sigma = \tau - \rho(\tau) = \mu(\tau)$. Gronwall's lemma implies that

$$|x_1(t) - x_2(t)|$$

$$\leq \|u - v\|_{\mathcal{C}} \left(1 + \frac{1}{1 - \rho^*} \int_{s-h}^s m_1(\mu^{-1}(\sigma))d\sigma \right) e^{\frac{1}{1-\rho^*} \int_s^t m_1(\mu^{-1}(\sigma))d\sigma}.$$

Let now $\theta \in [-h, 0]$. Replacing t by $s + h + \theta$ in the previous estimate and taking the supremum over all θ we obtain

$$\|x_1(s + h + \cdot) - x_2(s + h + \cdot)\|_{\mathcal{C}} \leq \|u - v\|_{\mathcal{C}} \left(1 + \frac{1}{1 - \rho^*} \widetilde{M}_1(s) \right) e^{\frac{1}{1-\rho^*} \widetilde{M}_1(s)}, \quad (16)$$

where $\widetilde{M}_1(s) = \int_{-\infty}^{s+h} m_1(\mu^{-1}(\sigma))d\sigma$.

On the other hand, we observe that

$$|x_1'(s + h + \theta) - x_2'(s + h + \theta)|$$

$$\leq \big| F\big(s + h + \theta, x_1(s + h + \theta - \rho(s + h + \theta))\big)$$

$$- F\big(s + h + \theta, x_2(s + h + \theta - \rho(s + h + \theta))\big) \big|$$

$$\leq m_1(s + h + \theta)|x_1(s + h + \theta - \rho(s + h + \theta)) - x_2(s + h + \theta - \rho(s + h + \theta))|$$

$$\leq \widehat{M}_1(s) \sup_{\tau \in [-h,h]} \big\{ |x_1(s + \tau) - x_2(s + \tau)| \big\}$$

$$\leq \widehat{M}_1(s) \Big(\|u - v\|_{\mathcal{C}} + \|x_1(s + h + \cdot) - x_2(s + h + \cdot)\|_{\mathcal{C}} \Big)$$

$$\leq \widehat{M}_1(s) \Big(1 + \Big(1 + \frac{1}{1 - \rho^*} \widetilde{M}_1(s) \Big) e^{\frac{1}{1-\rho^*} \widetilde{M}_1(s)} \Big) \|u - v\|_{\mathcal{C}}$$

where we used (16) in the last estimate and

$$\widehat{M}_1(s) = \sup \big\{ m_1(\tau) : \tau \in (-\infty, s + h] \big\}.$$

Taking the supremum over $\theta \in [-h, 0]$ it follows that

$$\|x_1'(s + h + \cdot) - x_2'(s + h + \cdot)\|_{\mathcal{C}} \leq \widehat{M}_1(s) \left(2 + \frac{\widetilde{M}_1(s)}{1 - \rho^*} \right) e^{\frac{\widetilde{M}_1(s)}{1-\rho^*}} \|u - v\|_{\mathcal{C}}. \quad (17)$$

Finally, summing up the inequalities (16) and (17) we obtain

$$\|U(s+h,s)u - U(s+h,s)v\|_{\mathcal{C}^1} = \|x_1(s+h+\cdot) - x_2(s+h+\cdot)\|_{\mathcal{C}^1}$$

$$\leq \left(1 + \widehat{M_1}(s)\right)\left(2 + \frac{\widetilde{M_1}(s)}{1-\rho^*}\right)e^{\frac{\widetilde{M_1}(s)}{1-\rho^*}}\|u-v\|_{\mathcal{C}} = \kappa(s)\|u-v\|_{\mathcal{C}},$$

which proves the smoothing property for $\tilde{t} = h$.

(iii) Lipschitz continuity $(\widetilde{A}_3)$: The Lipschitz continuity follows from the first step (ii).

(iv) Existence of the pullback exponential attractor: Let $t_0 \in \mathbb{R}$ be arbitrary. We observe that the constant in the smoothing property in part (ii) satisfies $\kappa(s) \leq \kappa(t_0)$ for all $s \leq t_0$. Hence, hypothesis $(\widetilde{A}_2)$ is satisfied with $\tilde{t} = h$ and $\kappa = \kappa(t_0)$. The Lipschitz continuity $(\widetilde{A}_3)$ was shown in (ii), and the existence of a family of pullback absorbing sets satisfying $(\widetilde{A}_1)$ in (i). Consequently, all assumptions of Theorem 4.2 are fulfilled, which implies the existence of a pullback exponential attractor. $\square$

Remark 4. In order to illustrate the ideas and simplify the presentation we assumed that the nonlinearity in (2) does not explicitly depend on the current state $x(t)$, but only on the delayed term $x(t - \rho(t))$. As in [1], [2] the results can be generalised to larger classes of nonlinearities and also to problems with unbounded delays. This will be subject of future work.

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