MAZUR'S INEQUALITY AND LAFFAILLE'S THEOREM

Christophe Cornut

MAZUR'S INEQUALITY AND LAFFAILLE'S THEOREM

CHRISTOPHE CORNUT

Abstract. We look at various questions related to filtrations in p-adic Hodge theory, using a blend of building and Tannakian tools. Specifically, Fontaine and Rapoport used a theorem of Laffaille on filtered isocrystals to establish a converse of Mazur's inequality for isocrystals. We generalize both results to the setting of (filtered) G-isocrystals and also establish an analog of Totaro's $\otimes$-product theorem for the Harder-Narasimhan filtration of Fargues.

1. Introduction

Many spaces that show up in p-adic Hodge theory are closely related to buildings, if only because they involve filtrations and lattices (or norms), which are respectively classified by Tits and Bruhat-Tits buildings. Quite naturally, building theoretical tools have thus occasionally been used to study these spaces, as in [22], [25] or [3]. In continuation with [7], this paper tries to promote the idea that a more systematic use of buildings sometimes leads to streamlined proofs and new results in p-adic Hodge theory, and certainly already provides an enlightening new point of view on various classical notions, simply by recasting them in a metric geometry framework.

For instance in [7], we showed that the Newton decomposition of an isocrystal (V, φ) over an algebraically closed residue field k of characteristic $p > 0$ can be identified with the translation vector of φ, viewing φ as a semi-simple isometry of the extended Bruhat-Tits building of $G = GL(V)$ – this still holds when k is merely perfect. Filtrations $\mathcal{F}$ on V come into the picture in the metric guise of non-expanding maps on the building, and the weak admissibility of a filtered isocrystal $(V, \varphi, \mathcal{F})$ is related to the joint dynamic of the $(\varphi, \mathcal{F})$-actions on the building: this was somehow first observed by Laffaille [19], who showed that $(V, \varphi, \mathcal{F})$ is weakly admissible if and only if V contains a strongly divisible lattice, giving rise to a special point of the building fixed by the composition of φ and $\mathcal{F}$. This criterion for weak admissibility implies that V contains a lattice L such that L and φL are in a given relative position μ if and only if there is a weakly admissible filtration $\mathcal{F}$ of type μ on V. Assuming that the residue field k is algebraically closed, Fontaine and Rapoport [12] gave a criterion for the existence of such an $\mathcal{F}$, thereby establishing a converse to the already known necessary condition for the existence of L – Mazur's inequality. Our first target is the generalization of these results to more general reductive groups G (Theorem 7 and 10). Note that the converse of Mazur's inequality for unramified groups was already established by [13], following an entirely different strategy suggested by [16]. Our method yields a somewhat weaker result: we say nothing about the existence of *hyperspecial* (strongly divisible) points.

2000 *Mathematics Subject Classification.* $14F30$, $20G25$.

Key words and phrases. Filtered G-isocrystals, Mazur's inequality, Harder-Narasimhan filtrations and tensor products.

Here is a rough dictionary that may help the reader digest the previous paragraph and warm-up for the sequel. Norms on V correspond to points on the (extended) Bruhat-Tits building X of G, and lattices correspond to a G-orbit X° of hyperspecial points in X. There is a canonical G-equivariant distance d on X with nice convexity properties: (X,d) is a complete CAT(0)-space, of which X° is a closed discrete subset. The $\mathbb{R}$-graduations $\mathcal{G}$ and (non-increasing) $\mathbb{R}$-filtrations $\mathcal{F}$ on V correspond to asymptotic classes of, respectively, (constant speed) geodesic lines $\ell : \mathbb{R} \to X$, and geodesic rays $c : \mathbb{R}_+ \to X$. In particular, $\mathbb{R}$-filtrations act on X by non-expanding maps as follows: $x + \mathcal{F} = c(1)$, where c is the unique geodesic ray in $\mathcal{F}$ emanating from $x \in X$. If $x \in X^\circ$ corresponds to a lattice L in V and $\mathcal{F}$ is a $\mathbb{Z}$-filtration, then $x + \mathcal{F}$ also belongs to X° and corresponds to the lattice

$$L + \mathcal{F} = \sum_{i\in\mathbb{Z}} p^{-i} L \cap \mathcal{F}^i.$$

The G-orbits of $\mathbb{R}$-filtrations on V are classified by their type $t(\mathcal{F}) \in \mathbb{R}^r_\geq$, where

$$\mathbb{R}^r_\geq = \{(x_1, \cdots, x_r) : x_1 \geq \cdots \geq x_r\}$$

is usually identified with the set of concave polygons over $[0,r]$ whose brake points have integral x-coordinate. We equip $\mathbb{R}^r_\geq$ with its usual partial order, given by

$$(x_1, \cdots, x_r) \leq (x'_1, \cdots, x'_r) \iff \begin{cases} \sum_{i=1}^j x_i \leq \sum_{i=1}^j x'_i & \forall j \in \{1, \cdots, r-1\} \\ \sum_{i=1}^r x_i = \sum_{i=1}^r x'_i. \end{cases}$$

We set $\mathbb{Q}^r_\geq = \mathbb{R}^r_\geq \cap \mathbb{Q}^r$ and $\mathbb{Z}^r_\geq = \mathbb{R}^r_\geq \cap \mathbb{Z}^r$. The formula $\mathbf{d}(x, x+\mathcal{F}) = t(\mathcal{F})$ defines a G-invariant vector-valued distance $\mathbf{d} : X \times X \to \mathbb{R}^r_\geq$, whose composition with the standard euclidean norm on $\mathbb{R}^r_\geq \subset \mathbb{R}^r$ is equal to the canonical distance d of X. The restriction of $\mathbf{d}$ to X° yields a bijection $G\backslash (X^\circ \times X^\circ) \simeq \mathbb{Z}^r_\geq$, which is the usual invariant describing the relative position of two lattices in V.

The Frobenius φ of V induces an isometry φ of (X,d) which preserves X°. It is a semi-simple isometry, which means that if $\min(\varphi) = \inf\{d(\varphi x, x) : x \in X\}$, then

$$\mathrm{Min}(\varphi) = \{x \in X : d(\varphi x, x) = \min(\varphi)\}$$

is *non-empty.* This is a closed, convex, φ-stable subset of X, equal to the disjoint union of the φ-stable geodesic lines of X. The common asymptotic class of these lines corresponds to the Newton $\mathbb{Q}$-graduation $\mathcal{G}_N$ of (V,φ). The induced pair of opposed Newton $\mathbb{Q}$-filtrations $(\mathcal{F}_N, \mathcal{F}_N^\iota)$ on V act on $x \in \mathrm{Min}(\varphi)$ as follows:

$$x + \mathcal{F}_N = \varphi^{-1}(x) \quad \text{and} \quad x + \mathcal{F}_N^\iota = \varphi(x).$$

In particular, the Newton type $t_N = t(\mathcal{F}_N) \in \mathbb{Q}^r_\geq$ is equal to $\mathbf{d}(\varphi(x), x)$ for every $x \in \mathrm{Min}(\varphi)$, and $\min(\varphi) = \|t_N\|$. Moreover, we have

$$\mathrm{Min}(\varphi) = X_\varphi(t_N) = X_\varphi(\mathcal{F}_N)$$

where for any $\mu \in \mathbb{R}^r_\geq$ and any $\mathbb{R}$-filtration $\mathcal{F}$ on V of type $t(\mathcal{F}) = \mu$,

$$X_\varphi(\mu) = \{x \in X : \mathbf{d}(\varphi x, x) = \mu\},$$
$$X_\varphi(\mathcal{F}) = \{x \in X : x = \varphi x + \mathcal{F}\}.$$

By definition, $X_\varphi(\mu)$ is a closed $\mathrm{Aut}(V,\varphi)$-stable subset of X, $X_\varphi(\mathcal{F})$ is a closed convex $\mathrm{Aut}(V,\varphi,\mathcal{F})$-stable subset of X, and we have a covering

$$X_\varphi(\mu) = \cup_{t(\mathcal{F})=\mu} X_\varphi(\mathcal{F}). \tag{1.1}$$

For $\mu \in \mathbb{Z}^r_{\geq}$ and any $\mathbb{Z}$-filtration $\mathcal{F}$ on V of type $t(\mathcal{F}) = \mu$, we set

$$X^\circ_\varphi(\mu) = X_\varphi(\mu) \cap X^\circ \quad \text{and} \quad X^\circ_\varphi(\mathcal{F}) = X_\varphi(\mathcal{F}) \cap X^\circ$$

so that $X^\circ_\varphi(\mu)$ is stable under $\mathrm{Aut}(V, \varphi)$, $X^\circ_\varphi(\mathcal{F})$ is stable under $\mathrm{Aut}(V, \varphi, \mathcal{F})$, and

$$X^\circ_\varphi(\mu) = \cup_{t(\mathcal{F})=\mu} X^\circ_\varphi(\mathcal{F}). \tag{1.2}$$

The subsets $X^\circ_\varphi(\mu)$ are usually called affine Deligne-Lusztig "varieties", and there is an extensive literature about them (and various generalizations): they show up in the description of special fibers of Shimura varieties, and indeed can be equipped with some sort of algebraic structure over the relevant residue field. The covering subsets $X^\circ_\varphi(\mathcal{F})$ are made of strongly divisible lattices in the sense of [18, 11], and they are related to stable lattices in crystalline Galois representations, but the covering (1.2) itself has not attracted much attention, even though it was crucially used in [12] to obtain a converse to Mazur's inequality using Laffaille's criterion for weak admissibility, as we shall now explain.

In this simple setting where $G = GL(V)$, Mazur's inequality and Laffaille's criterion can be stated as follows: for any $\mu \in \mathbb{Z}^r_{\geq}$ and any $\mathbb{Z}$-filtration $\mathcal{F}$ on V,

$$X^\circ_\varphi(\mu) \neq \emptyset \Longrightarrow \mu \geq t_N \text{ in } \mathbb{R}^r_{\geq}, \tag{1.3}$$

$$X^\circ_\varphi(\mathcal{F}) \neq \emptyset \iff (V, \varphi, \mathcal{F}) \text{ is (weakly) admissible.} \tag{1.4}$$

Let $\mathbf{Adm}_\varphi(\mu)$ be the set of all (weakly) admissible $\mathbb{Z}$-filtrations $\mathcal{F}$ of type $t(\mathcal{F}) = \mu$ on (V, φ). Using (1.2) and (1.4), the converse of Mazur's inequality (1.3) becomes

$$\mu \geq t_N \text{ in } \mathbb{R}^r_{\geq} \Longrightarrow \mathbf{Adm}_\varphi(\mu) \neq \emptyset. \tag{1.5}$$

This existence result is established by Fontaine and Rapoport in [12]: they show that any sufficiently generic $\mathbb{Z}$-filtration of type μ with $\mu \geq t_N$ is weakly admissible.

Forgetting hyperspecial points, let us now explain the results that we will eventually generalize in the first part of this paper – from $GL(V)$ to an arbitrary but unramified reductive group G. First, we have the following variants of Mazur's inequality and Laffaille's criterion: for any $\mu \in \mathbb{R}^r_{\geq}$ and any $\mathbb{R}$-filtration $\mathcal{F}$ on V,

$$X_\varphi(\mu) \neq \emptyset \Longrightarrow \mu \geq t_N \text{ in } \mathbb{R}^r_{\geq}, \tag{1.6}$$

$$X_\varphi(\mathcal{F}) \neq \emptyset \iff (V, \varphi, \mathcal{F}) \text{ is weakly admissible.} \tag{1.7}$$

Note that weak admissibility makes perfect sense for $\mathbb{R}$-filtrations. Let $\mathbf{Adm}_\varphi(\mu)$ be the set of all weakly admissible $\mathbb{R}$-filtrations $\mathcal{F}$ of type $t(\mathcal{F}) = \mu$ on (V, φ). Using (1.1) and (1.7) as above, the converse of Mazur's inequality (1.6) becomes

$$\mu \geq t_N \text{ in } \mathbb{R}^r_{\geq} \Longrightarrow \mathbf{Adm}_\varphi(\mu) \neq \emptyset. \tag{1.8}$$

As in [12], we establish the latter existence result by showing that any sufficiently generic $\mathbb{R}$-filtration of type μ with $\mu \geq t_N$ is weakly admissible, with a proof now based on a mixture of algebraic geometry and building theoretical tools.

Remark 1. The Fontaine-Rapoport method sketched above amounts to reduce a relatively hard existence problem in a CAT(0)-space (the converse of (1.3) or (1.6)) to an easier existence problem on the boundary of that space ((1.5) or (1.8)), using a suitable fixed point theorem ((1.4) or (1.7)). This line of thought has been used in other problems unrelated to p-adic Hodge theory, see [14] or the appendix of [17].

Various criteria for weak admissibility of unramified filtered G-isocrystals are listed in section 2, including our generalization of Laffaille's criterion (the equivalence (1) $\iff$ (3) of Theorem 7). Mazur's inequality is addressed in section 3 (Theorem 10). In the ramified case, we establish another list of criteria for weak admissibility in section 4, using a Harder-Narasimhan filtration and its compatibility with tensor products. There are now many proofs of this compatibility, which essentially says that the tensor product of weakly admissible filtered isocrystals is weakly admissible. In the unramified case, this was an immediate consequence of Laffaille's criterion. Faltings [9] generalized Laffaille's proof to the unramified case, and Totaro [24] found yet another proof related to Geometric Invariant Theory. It turns out that Totaro's proof has a very clean building theoretical translation. Instead of just repeating his arguments in this smoother framework, we chose to treat the slightly more difficult case of semi-stable weakly admissible filtered isocrystals in section 5 (Theorem 15), where semi-stability here refers to the notion introduced by Fargues in connection with his own Harder-Narasimhan filtrations [10]. In particular, we prove that Fargues's filtration on weakly admissible filtered isocrystals is compatible with tensor products (Theorem 15), and moreover show that it is given by a convex projection (Lemma 13 and Proposition 16): the algebraically defined Fargues $\mathbb{Q}$-filtration $\mathcal{F}_F$ of a weakly admissible filtered isocrystal $(V,\varphi,\mathcal{F})$ is the best approximation (in a metric sense) of the Newton $\mathbb{Q}$-filtration $\mathcal{F}_N^\iota$ of (V,φ) by a filtration whose steps are weakly admissible subspaces of $(V,\varphi,\mathcal{F})$.

In sections 2 and 3, our filtered isocrystals are unramified (i.e. $K = K_0$ in standard notations). In the remaining sections, they are defined with respect to an extension L of the fraction field K of the Witt vectors $W(k)$, which we do not require to be finite or unramified. The residue field k is a perfect field of characteristic $p>0$, which is only required to be algebraically closed in Theorem 10. Throughout the paper, we work with Γ-filtrations (defined in 2.1), where Γ is a non-trivial subgroup of $\mathbb{R}$. For applications to p-adic Hodge theory, one should take $\Gamma=\mathbb{Z}$ for Hodge filtrations and $\Gamma=\mathbb{Q}$ for Newton, Harder-Narasimhan and Fargues filtrations.

The need to properly identify $\mathbb{R}$-filtrations with non-expanding operators on Bruhat-Tits buildings led us to write [5], which was initially meant to be the first part of this paper, but eventually grew way out of proportion. This will be our general background reference for everything pertaining to Γ-graduations, Γ-filtrations and their types. We similarly refer to [15, 20] for G-isocrystals, [21, 8] for Γ-filtered G-isocrystals, [1] for CAT(0)-spaces and [23, 5] for Bruhat-Tits buildings.

2. Unramified Filtered Isocrystals

In this section, we first review various basic notions related to graduations and filtrations (2.1), isocrystals (2.2), and unramified filtered isocrystals (2.3). We then add G-structures to them using the Tannakian framework (2.4), and gently shift from the latter to the building framework (2.5-2.9) using [5] for the translation. Our main result is Theorem 7, in particular the equivalence (1) $\iff$ (3) relating weak admissibility of an unramified filtered G-isocrystal $(\varphi,\mathcal{F})$ to the joint dynamic of the induced operators φ and $\mathcal{F}$ on the extended Bruhat-Tits building of G.

2.1. Let K be a field, $\Gamma\neq 0$ a subgroup of $\mathbb{R}$, V a finite dimensional K-vector space. A Γ-graduation on V is a collection of K-subspaces $\mathcal{G}=(\mathcal{G}_\gamma)_{\gamma\in\Gamma}$ such that $V=\oplus_{\gamma\in\Gamma}\mathcal{G}_\gamma$. A Γ-filtration on V is a collection of K-subspaces $\mathcal{F}=(\mathcal{F}^\gamma)_{\gamma\in\Gamma}$ for which there exists a Γ-graduation $\mathcal{G}$ such that $\mathcal{F}=\mathrm{Fil}(\mathcal{G})$, i.e. $\mathcal{F}^\gamma=\oplus_{\eta\geq\gamma}\mathcal{G}_\eta$ for every $\gamma \in \Gamma$. We call any such $\mathcal{G}$ a splitting of $\mathcal{F}$. If W is a K-subspace of V, then $\mathcal{F}|W = (\mathcal{F}^\gamma \cap W)_{\gamma \in \Gamma}$ is again a Γ-graduation on W. The degree of $\mathcal{F}$ is given by

$$\deg(\mathcal{F}) = \sum_\gamma \dim_K \left(\mathrm{Gr}^\gamma_{\mathcal{F}}\right) \cdot \gamma$$

where $\mathrm{Gr}^\gamma_{\mathcal{F}} = \mathcal{F}^\gamma / \mathcal{F}^\gamma_+$ with $\mathcal{F}^\gamma_+ = \cup_{\eta > \gamma} \mathcal{F}^\eta$ for every $\gamma \in \Gamma$. The scalar product of two Γ-filtrations $\mathcal{F}_1$ and $\mathcal{F}_2$ on V is defined by the analogous formula

$$\begin{aligned}\langle \mathcal{F}_1, \mathcal{F}_2 \rangle &= \sum_{\gamma_1, \gamma_2} \dim_K \left(\frac{\mathcal{F}_1^{\gamma_1} \cap \mathcal{F}_2^{\gamma_2}}{\mathcal{F}_{1,+}^{\gamma_1} \cap \mathcal{F}_2^{\gamma_2} + \mathcal{F}_1^{\gamma_1} \cap \mathcal{F}_{2,+}^{\gamma_2}} \right) \cdot \gamma_1 \gamma_2 \\ &= \sum_\gamma \gamma \cdot \deg \mathrm{Gr}^\gamma_{\mathcal{F}_1}(\mathcal{F}_2) = \sum_\gamma \gamma \cdot \deg \mathrm{Gr}^\gamma_{\mathcal{F}_2}(\mathcal{F}_1)\end{aligned}$$

where $\mathrm{Gr}^\gamma_{\mathcal{F}_i}(\mathcal{F}_j)$ is the Γ-filtration induced by $\mathcal{F}_j$ on $\mathrm{Gr}^\gamma_{\mathcal{F}_i}$. We denote by $\mathsf{Gr}^\Gamma(K)$ and $\mathsf{Fil}^\Gamma(K)$ the categories of Γ-graded and Γ-filtered finite dimensional K-vector spaces, and equip them with their usual structure of exact K-linear $\otimes$-categories. The Fil and Gr-constructions yield exact K-linear $\otimes$-functors

$$\mathrm{Fil} : \mathsf{Gr}^\Gamma(K) \leftrightarrow \mathsf{Fil}^\Gamma(K) : \mathrm{Gr}.$$

The formula $(\iota \mathcal{G})_\gamma = \mathcal{G}_{-\gamma}$ defines an involutive exact $\otimes$-endofunctor of $\mathsf{Gr}^\Gamma(K)$.

2.2. Let k be a perfect field of characteristic $p > 0$, $W(k)$ the ring of Witt vectors, $K = W(k)[\frac{1}{p}]$ its fraction field, σ the Frobenius of k, $W(k)$ or K. An isocrystal over k is a finite dimensional K-vector space V equipped with a σ-linear isomorphism φ. Since k is perfect, there is a canonical slope decomposition $(V, \varphi) = \oplus_{\lambda \in \mathbb{Q}} (V_\lambda, \varphi | V_\lambda)$. The φ-stable K-subspace V_λ is the union of all finitely generated $W(k)$-submodules L of V such that $\varphi^{(h)}(L) = p^d L$, where $\lambda = \frac{d}{h}$ with $(d, h) \in \mathbb{Z} \times \mathbb{N}$. We denote by $\mathsf{Iso}(k)$ the category of isocrystals over k. It is a $\mathbb{Q}_p$-linear (non-neutral) Tannakian category and the above slope decomposition yields an exact $\mathbb{Q}_p$-linear $\otimes$-functor

$$\nu : \mathsf{Iso}(k) \to \mathsf{Gr}^{\mathbb{Q}}(K).$$

2.3. A Γ-filtered isocrystal over k is an isocrystal (V, φ) together with a Γ-filtration $\mathcal{F}$ on V. The K-vector space V thus carries three filtrations: the *Hodge* Γ-filtration $\mathcal{F}_H = \mathcal{F}$ and the pair of opposed φ-stable *Newton* $\mathbb{Q}$-filtrations $\mathcal{F}_N = \mathsf{Fil}(\mathcal{G}_N)$ and $\mathcal{F}_N^\iota = \mathsf{Fil}(\iota \mathcal{G}_N)$ attached to the slope decomposition $\mathcal{G}_N = \nu(V, \varphi)$, given by

$$\mathcal{F}_N^\lambda = \oplus_{\eta \geq \lambda} V_\eta \quad \text{and} \quad \mathcal{F}_N^{\iota \lambda} = \oplus_{\eta \geq \lambda} V_{-\eta}.$$

For $\Gamma = \mathbb{Z}$, these objects are usually called unramified filtered isocrystals over k.

Lemma 2. *The following conditions are equivalent:*

1. $(V, \varphi, \mathcal{F})$ *is weakly admissible, i.e.:*
   (a) $\deg(\mathcal{F}_H) = \deg(\mathcal{F}_N)$ *and*
   (b) $\deg(\mathcal{F}_H | W) \leq \deg(\mathcal{F}_N | W)$ *for every φ-stable K-subspace W of V,*
2. *For every φ-stable Δ-filtration Ξ on V, $\langle \mathcal{F}_H, \Xi \rangle \leq \langle \mathcal{F}_N, \Xi \rangle$.*
3. *For every φ-stable Δ-filtration Ξ on V, $\langle \mathcal{F}_H, \Xi \rangle + \langle \mathcal{F}_N^\iota, \Xi \rangle \leq 0$.*

In (2) and (3), Δ is any non-trivial subgroup of $\mathbb{R}$.

Proof. For a φ-stable K-subspace W of V and a, b, δ in Δ with $a \leq b$, set

$$\Xi_{W,a,b}^{\delta} = \begin{cases} V & \text{if } \delta \leq a, \\ W & \text{if } a < \delta \leq b, \\ 0 & \text{if } b < \delta. \end{cases}$$

This defines a φ-stable Δ-filtration $\Xi_{W,a,b} = (\Xi_{W,a,b}^{\delta})_{\delta \in \Delta}$ on V with

$$\begin{aligned} \langle \mathcal{F}_H, \Xi_{W,a,b} \rangle &= a \cdot \deg(\mathcal{F}_H) + (b-a) \cdot \deg(\mathcal{F}_H | W) \\ \text{and} \quad \langle \mathcal{F}_N, \Xi_{W,a,b} \rangle &= a \cdot \deg(\mathcal{F}_N) + (b-a) \cdot \deg(\mathcal{F}_N | W) \end{aligned}$$

Now (2) implies that $\langle \mathcal{F}_H, \Xi_{W,a,b} \rangle \leq \langle \mathcal{F}_N, \Xi_{W,a,b} \rangle$ for every choice of W and $a \leq b$, from which (1) easily follows. Conversely, let Ξ be any φ-stable Δ-filtration on V. Write $\{\delta \in \Delta : \mathrm{Gr}_{\Xi}^{\delta} \neq 0\} = \{\delta_1 < \cdots < \delta_n\}$. Then Ξ^{δ_i} is a φ-stable K-subspace of V. Put $d_H(i) = \deg(\mathcal{F}_H | \Xi^{\delta_i})$ and $d_N(i) = \deg(\mathcal{F}_N | \Xi^{\delta_i})$. Then

$$\langle \mathcal{F}_H, \Xi \rangle = \sum_{i=1}^{n} \delta_i \cdot \Delta_H(i) \quad \text{and} \quad \langle \mathcal{F}_N, \Xi \rangle = \sum_{i=1}^{n} \delta_i \cdot \Delta_N(i)$$

where $\Delta_\star(i) = d_\star(i) - d_\star(i+1)$ for $1 \leq i < n$ and $\Delta_\star(n) = d_\star(n)$, so that also

$$\langle \mathcal{F}_H, \Xi \rangle = \sum_{i=1}^{n} \Delta_i \cdot d_H(i) \quad \text{and} \quad \langle \mathcal{F}_N, \Xi \rangle = \sum_{i=1}^{n} \Delta_i \cdot d_N(i)$$

where $\Delta_i = \delta_i - \delta_{i-1}$ for $1 < i \leq n$ and $\Delta_1 = \delta_1$. Now (1) implies that $d_H(i) \leq d_N(i)$ for all i and $d_H(1) = d_N(1)$ from which (2) follows since $\Delta_i > 0$ for $i > 1$. Note that with these notations, we also find that

$$\langle \mathcal{F}_N^{\iota}, \Xi \rangle = \sum_{i=1}^{n} \Delta_i \cdot d_N^{\iota}(i) \quad \text{with} \quad d_N^{\iota}(i) = \deg(\mathcal{F}_N^{\iota} | \Xi^{\delta_i}).$$

But $\deg(\mathcal{F}_N) + \deg(\mathcal{F}_N^{\iota}) = 0$ and more generally $\deg(\mathcal{F}_N|W) + \deg(\mathcal{F}_N^{\iota}|W) = 0$ for every φ-stable K-subspace W of V by functoriality of the slope $\mathbb{Q}$-graduation, thus $d_N(i) + d_N^{\iota}(i) = 0$ for all i. Therefore $\langle \mathcal{F}_N, \Xi \rangle + \langle \mathcal{F}_N^{\iota}, \Xi \rangle = 0$ for every φ-stable $\mathbb{R}$-graduation Ξ on V, which proves that (2) $\Leftrightarrow$ (3). $\square$

We denote by $\mathsf{Fil}^{\Gamma}\mathsf{Iso}(k)$ the category of Γ-filtered isocrystals over k and denote by $\mathsf{Fil}^{\Gamma}\mathsf{Iso}(k)^{wa}$ its full sub-category of weakly admissible objects, as defined in the previous lemma. These are both exact $\mathbb{Q}_p$-linear $\otimes$-categories, the smaller one is also abelian, and it is even a *neutral* Tannakian category when $\Gamma = \mathbb{Z}$ [4].

2.4. Fix a reductive group G over $\mathbb{Q}_p$, let $\mathsf{Rep}(G)$ be the Tannakian category of algebraic representations of G on finite dimensional $\mathbb{Q}_p$-vector spaces, and let

$$V : \mathsf{Rep}(G) \to \mathsf{Vect}(\mathbb{Q}_p)$$

be the natural fiber functor. A Γ-graduation on $V_K = V \otimes K$ (resp. Γ-filtration on V_K, G-isocrystal over k or Γ-filtered G-isocrystal over k) is a factorization of

$$V_K : \mathsf{Rep}(G) \to \mathsf{Vect}(K)$$

through the natural fiber functor of the relevant category $\mathsf{Gr}^{\Gamma}(K)$ (resp. $\mathsf{Fil}^{\Gamma}(K)$, $\mathsf{Iso}(k)$, $\mathsf{Fil}^{\Gamma}\mathsf{Iso}(k)$). We require these factorizations to be exact and compatible with the $\otimes$-products and their neutral objects. The set of all such factorizations is the set of K-valued points of a smooth and separated $\mathbb{Q}_p$-scheme $\mathbb{G}^{\Gamma}(G)$ (resp. $\mathbb{F}^{\Gamma}(G)$, G, $G \times \mathbb{F}^{\Gamma}(G)$). The G-isocrystal attached to $b \in G(K)$ maps $\rho \in \mathsf{Rep}(G)$ to

$(V_K(\rho), \rho(b) \circ \mathrm{Id} \otimes \sigma)$ (this would be $(V, \rho) \mapsto (V_K, \rho(b)\sigma)$ in standard notations). We identify b with the corresponding Frobenius element $\varphi = (b, \sigma)$ in $G(K) \rtimes \langle\sigma\rangle$. For $\rho \in \mathsf{Rep}(G)$, we denote by $\mathcal{G}(\rho)$, $\mathcal{F}(\rho)$ or $\varphi(\rho)$ the Γ-graduation, Γ-filtration or Frobenius on $V_K(\rho) = V(\rho) \otimes K$ attached to a Γ-graduation $\mathcal{G}$, Γ-filtration $\mathcal{F}$ or Frobenius φ on V_K. We say that a Γ-filtered G-isocrystal

$$(\varphi, \mathcal{F}) : \mathsf{Rep}(G) \to \mathsf{Fil}^{\Gamma}\mathsf{Iso}(k)$$

is weakly admissible if it factors through the subcategory $\mathsf{Fil}^{\Gamma}\mathsf{Iso}(k)^{wa}$ of $\mathsf{Fil}^{\Gamma}\mathsf{Iso}(k)$.

Remark 3. The G-isocrystals or Γ-filtered G-isocrystals considered in this paper are trivial in the sense that the underlying fiber functor is required to be the trivial fiber functor V_K. We caution our reader not to apply our results carelessly on more general G-isocrystals or Γ-filtered G-isocrystals, unless he or she has checked that the underlying fiber functors are at least isomorphic (over K) to the trivial one.

2.5. There is a G-equivariant sequence of morphisms of $\mathbb{Q}_p$-schemes

$$\mathbb{G}^{\Gamma}(G) \xrightarrow{\mathrm{Fil}} \mathbb{F}^{\Gamma}(G) \xrightarrow{t} \mathbb{C}^{\Gamma}(G)$$

where $\mathbb{C}^{\Gamma}(G) = G\backslash\mathbb{G}^{\Gamma}(G) = G\backslash\mathbb{F}^{\Gamma}(G)$ and the Fil-morphism is induced by the previous Fil-functors. We set $\mathbf{G}^{\Gamma}(G_K) = \mathbb{G}^{\Gamma}(G)(K)$, $\mathbf{F}^{\Gamma}(G_K) = \mathbb{F}^{\Gamma}(G)(K)$ and denote by $\mathbf{C}^{\Gamma}(G_K)$ the image of $\mathbf{F}^{\Gamma}(G_K)$ in $\mathbb{C}^{\Gamma}(G)(K)$ under the type morphism t, giving rise to a $G(K)$-equivariant sequence of surjective maps [5, 4.1]

$$\mathbf{G}^{\Gamma}(G_K) \xrightarrow{\mathrm{Fil}} \mathbf{F}^{\Gamma}(G_K) \xrightarrow{t} \mathbf{C}^{\Gamma}(G_K).$$

Then $\mathbf{C}^{\Gamma}(G_K) = G(K)\backslash\mathbf{G}^{\Gamma}(G_K) = G(K)\backslash\mathbf{F}^{\Gamma}(G_K)$, and it is a monoid. When $\Gamma = \mathbb{R}$, it is the usual (relative) Weyl cone attached to G over K, and it comes equipped with a partial order (the dominance order). If $S \subset G_K$ is a maximal K-split torus and $B \subset G_K$ is a minimal parabolic subgroup of G_K containing the centralizer of S, then $\mathbf{C}^{\Gamma}(G_K) \simeq \mathrm{Hom}^+(X^*(S), \Gamma)$, where $\mathrm{Hom}^+(X^*(S), \Gamma)$ is the monoid of all morphisms $f : X^*(S) \to \Gamma$ which are non-negative on the roots of S in the Lie algebra of B [5, 4.1.10]; when $\Gamma = \mathbb{R}$, $f_1 \leq f_2$ if and only if $f_2 - f_1$ is a non-negative linear combination of B-positive (relative) coroots [5, 2.4 and 5.1.2].

The Newton slope decomposition of section 2.2 yields a morphism

$$\nu : G(K) \to \mathbf{G}^{\mathbb{Q}}(G_K)$$

such that $\nu(g \diamond b) = g \cdot \nu(b)$ for $g, b \in G(K)$ with $g \diamond b = gb\sigma(g)^{-1}$, i.e. a $G(K)$-equivariant morphism $\varphi \mapsto \nu(\varphi)$ from the $G(K)$-coset $G(K) \cdot \sigma$ of Frobenius elements in $G(K) \rtimes \langle\sigma\rangle$ (with the action by conjugation) to $\mathbf{G}^{\mathbb{Q}}(G_K)$.

All of these constructions are covariantly functorial in G, Γ and k.

2.6. A Γ-filtered G-isocrystal $(\varphi, \mathcal{F}) \in G(K) \cdot \sigma \times \mathbf{F}^{\Gamma}(G_K)$ thus again defines three filtrations on V_K: the *Hodge* Γ-filtration $\mathcal{F}_H = \mathcal{F}$ in $\mathbf{F}^{\Gamma}(G_K)$ and the pair of opposed φ-stable *Newton* $\mathbb{Q}$-filtrations $\mathcal{F}_N = \mathrm{Fil}(\mathcal{G}_N)$ and $\mathcal{F}_N^{\iota} = \mathrm{Fil}(\iota\mathcal{G}_N)$ in $\mathbf{F}^{\mathbb{Q}}(G_K)$ attached to the slope $\mathbb{Q}$-graduation $\mathcal{G}_N = \nu(\varphi)$ in $\mathbf{G}^{\mathbb{Q}}(G_K)$.

2.7. On the other hand, a Γ-filtered G-isocrystal $(\varphi, \mathcal{F})$ also "acts" on the extended Bruhat-Tits building $\mathbf{B}^e(G_K)$ of G over K, through the classical action of the group $G(K) \rtimes \langle\sigma\rangle$ on $\mathbf{B}^e(G_K)$ and the canonical $G(K) \rtimes \langle\sigma\rangle$-equivariant $+$-operation

$$+ : \mathbf{B}^e(G_K) \times \mathbf{F}^{\mathbb{R}}(G_K) \to \mathbf{B}^e(G_K)$$

which is defined in [5, 6.2.5]. We recall that the extended Bruhat-Tits building is the direct product of the usual (or reduced) isogeny-invariant Bruhat-Tits building $\mathbf{B}(G_K)$ by $X_*(A)\otimes\mathbb{R}$, where A is the maximal K-split torus in the center of G_K and $X_*(A)$ is the group of cocaracters of A. The extended Bruhat-Tits building $\mathbf{B}^e(G_K)$ and vectorial Tits building $\mathbf{F}^{\mathbb{R}}(G_K)$ are covered by apartments, both indexed by the maximal K-split subtori of G_K; for any such torus, the corresponding apartment in the Bruhat-Tits building is canonically equipped with the structure of an affine space whose underlying vector space is the matching apartment in the vectorial Tits building, and the above $+$-map is obtained by gluing the resulting faithfully transitive actions on all apartments. To simplify our notations, we set

$$X = \mathbf{B}^e(G_K) = \mathbf{B}(G_K) \times X_*(A) \otimes \mathbb{R}.$$

For $\rho \in \mathsf{Rep}(G)$, $\varphi(\rho)$ and $\mathcal{F}(\rho)$ similarly act on

$$X(\rho) = \mathbf{B}^e(GL(V(\rho))_K).$$

Definition 4. For a Γ-filtered G-isocrystal $(\varphi, \mathcal{F})$, we set

$$X_\varphi(\mathcal{F}) = \{x \in X : \varphi(x) + \mathcal{F} = x\} = \{x \in X : \alpha(x) = x\}$$

where $\alpha(x) = \varphi(x) + \mathcal{F}$ for all x in X. For $\rho \in \mathsf{Rep}(G)$, we set

$$X_\varphi(\mathcal{F})(\rho) = \{x \in X(\rho) : \varphi(\rho)(x) + \mathcal{F}(\rho) = x\} = \{x \in X(\rho) : \alpha(\rho)(x) = x\}$$

where $\alpha(\rho)(x) = \varphi(\rho)(x) + \mathcal{F}(\rho)$ for all x in $X(\rho)$.

2.8. As explained in [5, §5], the choice of a faithful $\tau \in \mathsf{Rep}(G)$ induces a bunch of numerical functions on our buildings. The $G(K) \rtimes \langle\sigma\rangle$-invariant "scalar product"

$$\langle -, - \rangle_\tau : \mathbf{F}^{\mathbb{R}}(G_K) \times \mathbf{F}^{\mathbb{R}}(G_K) \to \mathbb{R}, \qquad \langle \mathcal{F}_1, \mathcal{F}_2 \rangle_\tau = \langle \mathcal{F}_1(\tau), \mathcal{F}_2(\tau) \rangle$$

yields a $\langle\sigma\rangle$-invariant length function

$$\|-\|_\tau : \mathbf{C}^{\mathbb{R}}(G_K) \to \mathbb{R}_+, \qquad \|t(\mathcal{F})\|_\tau = \|\mathcal{F}\|_\tau = \langle \mathcal{F}, \mathcal{F} \rangle_\tau^{\frac{1}{2}}$$

whose composition with the $G(K) \rtimes \langle\sigma\rangle$-equivariant "vector valued distance"

$$\mathbf{d} : \mathbf{B}^e(G_K) \times \mathbf{B}^e(G_K) \to \mathbf{C}^{\mathbb{R}}(G_K), \qquad \mathbf{d}(x, x + \mathcal{F}) = t(\mathcal{F})$$

yields a genuine $G(K) \rtimes \langle\sigma\rangle$-invariant distance

$$d_\tau : \mathbf{B}^e(G_K) \times \mathbf{B}^e(G_K) \to \mathbb{R}_+$$

which turns $\mathbf{B}^e(G_K)$ into a complete CAT(0)-space. The above $+$-map defines a $G(K) \rtimes \langle\sigma\rangle$-equivariant "action" of $\mathbf{F}^{\mathbb{R}}(G_K)$ on $X = \mathbf{B}^e(G_K)$ by non-expanding maps, which yields a $G(K) \rtimes \langle\sigma\rangle$-equivariant identification of $\mathbf{F}^{\mathbb{R}}(G_K)$ with the cone $\mathscr{C}(\partial X)$ on the visual boundary ∂X of X (acting on X as in [6]). Under this identification, $\partial X = \{\xi \in \mathbf{F}^{\mathbb{R}}(G_K) : \|\xi\|_\tau = 1\}$. On the other hand, the Frobenius φ is an isometry of (X, d_τ). Thus α is a non-expanding map of (X, d_τ). In particular,

$$X_\varphi(\mathcal{F}) \neq \emptyset \iff \alpha \text{ has bounded orbits.}$$

Lemma 5. *The isometry φ of (X, d_τ) is semi-simple, i.e.* $\mathrm{Min}(\varphi) \neq \emptyset$ *where*

$$\begin{array}{rrcl} & \mathrm{Min}(\varphi) & = & \{x \in X : d_\tau(x, \varphi(x)) = \min(\varphi)\} \\ \textit{with} & \min(\varphi) & = & \inf\{d_\tau(x, \varphi(x)) : x \in X\}. \end{array}$$

Moreover, $\varphi(x) = x + \mathcal{F}_N^\iota$ for every $x \in \mathrm{Min}(\varphi)$ and $\mathscr{C}(\partial \mathrm{Min}(\varphi))$ is the fixed point set of φ acting on $\mathscr{C}(\partial X) = \mathbf{F}^{\mathbb{R}}(G_K)$, i.e. the set of φ-stable $\mathbb{R}$-filtrations on V_K.

Proof. The first two assertions are established in [7] when k is algebraically closed and the general case follows from [1, II.6.2 (4)]. The final claim holds true for any semi-simple isometry φ of a CAT(0)-space (X,d). Indeed, $\partial\mathrm{Min}(\varphi)$ is contained in $(\partial X)^{\varphi=\mathrm{Id}}$ by [1, II.6.8 (4)]. Conversely, suppose that $\xi\in\partial X$ is fixed by φ and choose some x in $\mathrm{Min}(\varphi)\neq\emptyset$. Then for every $t\geq 0$,

$$\varphi(x+t\xi)=\varphi(x)+t\varphi(\xi)=\varphi(x)+t\xi,$$

thus $x+t\xi$ also belongs to $\mathrm{Min}(\varphi)$ since

$$\min(\varphi)\leq d(x+t\xi,\varphi(x+t\xi))=d(x+t\xi,\varphi(x)+t\xi)\leq d(x,\varphi(x))=\min(\varphi)$$

by convexity of the CAT(0)-distance d, therefore $\xi\in\partial\mathrm{Min}(\varphi)$. □

Remark 6. We should rather write $\min_\tau(\varphi)$ to reflect the dependency of $\min(\varphi)$ on the choice of τ, but the subset $\mathrm{Min}(\varphi)$ of X really does not depend upon that choice: it is precisely the set of all $x\in X$ such that $\varphi(x)=x+\mathcal{F}_N^\iota$.

2.9. For any $\rho\in\mathsf{Rep}(G)$, applying these constructions to $GL(V(\rho))$ and its tautological faithful representation on $V(\rho)$, we obtain a canonical distance d_ρ on $X(\rho)$ for which $\varphi(\rho)$ is an isometry while $\alpha(\rho)$ is a non-expanding map. Moreover:

$$X_\varphi(\mathcal{F})(\rho)\neq\emptyset\iff\alpha(\rho)\text{ has bounded orbits.}$$

Here the cones $\mathscr{C}(\partial X(\rho))$ and $\mathscr{C}(\partial\mathrm{Min}(\varphi(\rho)))$ are respectively identified with the set of all $\mathbb{R}$-filtrations on $V_K(\rho)$ and its subset of $\varphi(\rho)$-stable $\mathbb{R}$-filtrations.

2.10. Fix a faithful $\tau\in\mathsf{Rep}(G)$ and some subgroup $\Delta\neq 0$ of $\mathbb{R}$. Then:

Theorem 7. *The following conditions are equivalent:*

(1) $(\varphi,\mathcal{F})$ *is weakly admissible.*
(1_ρ) *For every* $\rho\in\mathsf{Rep}(G)$, $(V_K(\rho),\varphi(\rho),\mathcal{F}(\rho))$ *is weakly admissible.*
(1_τ) $(V_K(\tau),\varphi(\tau),\mathcal{F}(\tau))$ *is weakly admissible.*
(2^Δ) *For every* φ*-stable* Δ*-filtration* Ξ *on* V_K,

$$\langle\mathcal{F}_H,\Xi\rangle_\tau\leq\langle\mathcal{F}_N,\Xi\rangle_\tau\,.$$

(2^Δ_ρ) *For every* $\rho\in\mathsf{Rep}(G)$ *and every* φ*-stable* Δ*-filtration* Ξ *on* $V_K(\rho)$,

$$\langle\mathcal{F}_H(\rho),\Xi\rangle\leq\langle\mathcal{F}_N(\rho),\Xi\rangle\,.$$

(2^Δ_τ) *For every* φ*-stable* Δ*-filtration* Ξ *on* $V_K(\tau)$,

$$\langle\mathcal{F}_H(\tau),\Xi\rangle\leq\langle\mathcal{F}_N(\tau),\Xi\rangle\,.$$

(3) $X_\varphi(\mathcal{F})\neq\emptyset$.
(3_ρ) *For every* $\rho\in\mathsf{Rep}(G)$, $X_\varphi(\mathcal{F})(\rho)\neq\emptyset$.
(3_τ) $X_\varphi(\mathcal{F})(\tau)\neq\emptyset$.
(4) α *has bounded orbits in* (X,d_τ).
(4_ρ) *For every* $\rho\in\mathsf{Rep}(G)$, $\alpha(\rho)$ *has bounded orbits in* $(X(\rho),d_\rho)$.
(4_τ) $\alpha(\tau)$ *has bounded orbits in* $(X(\tau),d_\tau)$.

Remark 8. It follows from the proof of Lemma 2 that here also

$$\langle\mathcal{F}_N,\Xi\rangle_\tau+\langle\mathcal{F}_N^\iota,\Xi\rangle_\tau=0$$

for every φ-stable $\mathbb{R}$-filtration Ξ on V_K.

Proof. Obviously $(1)\Leftrightarrow(1_\rho)$, $(2^\Delta_\tau)\Rightarrow(2^\Delta)$ and $(x_\rho)\Rightarrow(x_\tau)$ for $x\in\{1,2,3,4\}$. We have already seen that $(3_x)\Leftrightarrow(4_x)$ for all $x\in\{\emptyset,\rho,\tau\}$ and Lemma 2 shows that $(1_\rho)\Leftrightarrow(2^\Delta_\rho)$ and $(1_\tau)\Leftrightarrow(2^\Delta_\tau)$. Set $(2_x)=(2^{\mathbb{R}}_x)$ for $x\in\{\emptyset,\rho,\tau\}$ and let $(2')$, $(3')$ and $(4')$ be (2), (3) and (4) for the base change $(\varphi',\mathcal{F}')$ of $(\varphi,\mathcal{F})$ to an algebraic closure k' of k. Then obviously $(2_x)\Rightarrow(2^\Delta_x)$, $(2')\Rightarrow(2)$ and $(4)\Leftrightarrow(4')$ – thus also $(3')\Leftrightarrow(3)$. The main theorem of [6] asserts that $(2)\Leftrightarrow(3)$ provided that φ satisfies a certain decency condition, which holds true by the main result of [7] when k is algebraically closed. Therefore $(2')\Leftrightarrow(3')$. We will show below that $(2^\Delta)\Rightarrow(2)$ (2.11), $(2)\Rightarrow(2')$ (2.12) and $(4)\Rightarrow(4_\rho)$ (2.13). Since $(3_\rho)\Leftrightarrow(2_\rho)$ and $(3_\tau)\Leftrightarrow(2_\tau)$ are special cases of $(3)\Leftrightarrow(3')\Leftrightarrow(2')\Leftrightarrow(2)$ (applied respectively to $G=GL(V(\rho))$ and $G=GL(V(\tau))$), the theorem follows. □

Remark 9. Apart from the added generality of allowing $\Gamma\neq\mathbb{Z}$, the equivalence of most of these conditions is either well-known or trivial and can be found in various places. For instance $(1)\Leftrightarrow(1_\tau)$ follows from [21, 1.18] and $(1_\tau)\Leftrightarrow(3_\tau)$ is Laffaille's criterion for weak admissibility [18, 3.2]. Our only new contribution is $(2)\Leftrightarrow(3)$, which is essentially our metric reformulation of Laffaille's algebraic proof in [6].

2.11. $(2^\Delta)\Rightarrow(2)$**.** Suppose that (2) does not hold: there exists a φ-stable $\mathbb{R}$-filtration Ξ on V_K such that $\langle\mathcal{F}_H,\Xi\rangle_\tau>\langle\mathcal{F}_N,\Xi\rangle_\tau$. By [5, 4.1.13], there exists a maximal split torus S of G_K with character group $M=\mathrm{Hom}_K(S,\mathbb{G}_{m,K})$ and a morphism $f:M\to\mathbb{R}$ such that for every $\rho\in\mathsf{Rep}(G)$ and $\gamma\in\mathbb{R}$,

$$\Xi(\rho)^\gamma=\oplus_{m\in M,\,f(m)\geq\gamma}V(\rho)_m\quad\text{in}\quad V_K(\rho)=V(\rho_K)$$

where $V(\rho_K)=\oplus_{m\in M}V(\rho)_m$ is the weight decomposition of the restriction of ρ_K to S. The image of f is a finitely generated subgroup Q of $\mathbb{R}$. Since Δ is a non-trivial subgroup of $\mathbb{R}$, there exists a sequence of morphisms $q_n:Q\to\Delta$ such that $\frac{1}{n}q_n:Q\to\mathbb{R}$ converges simply to the given embedding $Q\hookrightarrow\mathbb{R}$. The formula

$$\Xi_n(\rho)^\gamma=\oplus_{m\in M,\,q_n\circ f(m)\geq\gamma}V(\rho)_m$$

then defines a Δ-filtration $\Xi_n\in\mathbf{F}^\Delta(G_K)$ with $\frac{1}{n}\Xi_n\to\Xi$ in $(\mathbf{F}^{\mathbb{R}}(G_K),d_\tau)$, thus also

$$\langle\mathcal{F}_H,\Xi_n\rangle_\tau=n\left\langle\mathcal{F}_H,\tfrac{1}{n}\Xi_n\right\rangle_\tau>n\left\langle\mathcal{F}_N,\tfrac{1}{n}\Xi_n\right\rangle_\tau=\langle\mathcal{F}_N,\Xi_n\rangle_\tau$$

for $n\gg0$. For any τ in $\mathsf{Rep}(G)$, (m_1,m_2) in $\{m\in M:V(\rho)_m\neq0\}$ and $n\gg0$,

$$f(m_1)-f(m_2)=f(m_1-m_2)\quad\text{and}\quad q_n\circ f(m_1)-q_n\circ f(m_2)=q_n\circ f(m_1-m_2)$$

have the same sign in $\{0,\pm1\}$, thus for every ρ in $\mathsf{Rep}(G)$ and $n\gg0$,

$$\{\Xi_n(\rho)^\gamma:\gamma\in\Delta\}=\{\Xi(\rho)^\gamma:\gamma\in\mathbb{R}\}.$$

In particular, $\Xi_n(\tau)$ is fixed by $\varphi(\tau)$ for $n\gg0$. But then Ξ_n is a φ-stable Δ-filtration on V_K by [5, 4.2.10 or 3.11.12], thus (2^Δ) also does not hold.

2.12. $(2)\Rightarrow(2')$**.** Suppose that $(2')$ does not hold. Then by [6, Theorem 1, (4)], there is a unique ξ in $\partial\mathrm{Min}(\varphi')$ such that $\langle\mathcal{F}_H,\xi\rangle_\tau+\langle\mathcal{F}_N^\iota,\xi\rangle_\tau>0$ is maximal. This uniqueness implies that $\xi\in\mathbf{F}^{\mathbb{R}}(G_{K'})$ is fixed by the group $\mathrm{Gal}(k'/k)$ of continuous automorphisms of K'/K. Therefore $\xi\in\mathbf{F}^{\mathbb{R}}(G_K)$ and (2) also does not hold.

2.13. (4) $\Rightarrow$ (4_ρ). Suppose that (4) holds and fix $\rho \in \mathsf{Rep}(G)$ corresponding to a morphism $f : G \to H = GL(V(\rho))$. By [5, 5.8.4], there is a finite Galois extension L of $\mathbb{Q}_p$ and for any factor $\tilde{K}$ of $L \otimes K$, a map $f : \mathbf{B}^e(G_{\tilde{K}}) \to \mathbf{B}^e(H_{\tilde{K}})$ such that

$$f(g \cdot x) = f(g) \cdot f(x), \quad f(x + \mathcal{G}) = f(x) + f(\mathcal{G}) \quad \text{and} \quad f(\theta \cdot x) = \theta \cdot f(x)$$

for every $x \in \mathbf{B}^e(G_{\tilde{K}})$, $g \in G(\tilde{K})$, $\mathcal{G} \in \mathbf{F}^{\mathbb{R}}(G_{\tilde{K}})$ and $\theta \in \mathrm{Aut}(\tilde{K}/\mathbb{Q}_p)$. Fix an extension $\tilde{\sigma} \in \mathrm{Aut}(\tilde{K}/\mathbb{Q}_p)$ of $\sigma \in \mathrm{Aut}(K/\mathbb{Q}_p)$, let $\tilde{\varphi} = (b, \tilde{\sigma})$ be the corresponding extension of $\varphi = (b, \sigma)$, write $\tilde{\alpha}$ and $\tilde{\alpha}(\rho)$ for the induced extensions of α and $\alpha(\rho)$ to $\mathbf{B}^e(G_{\tilde{K}}) \supset \mathbf{B}^e(G_K)$ and $\mathbf{B}^e(H_{\tilde{K}}) \supset \mathbf{B}^e(H_K)$. Then $f(\tilde{\alpha}(x)) = \tilde{\alpha}(\rho)(f(x))$ for every $x \in \mathbf{B}^e(G_{\tilde{K}})$. Since α has a fixed point x in $\mathbf{B}^e(G_K)$, $f(x)$ is a fixed point of $\tilde{\alpha}(\rho)$ in $\mathbf{B}^e(H_{\tilde{K}})$, thus $\tilde{\alpha}(\rho)$ has bounded orbits, and so does $\alpha(\rho) = \tilde{\alpha}(\rho)|\mathbf{B}^e(H_K)$.

3. Mazur's Inequality

In this section, we explain our variant of Mazur's inequality (3.2) and prove its converse (Theorem 10, $\nu \leq \mu^\sharp \Longrightarrow X_\varphi(\mu) \neq \emptyset$), following the Fontaine-Rapoport strategy, i.e. using a covering of our affine Deligne-Lusztig sets (the $X_\varphi(\mu)$'s) by our subsets of strongly divisible points (the $X_\varphi(\mathcal{F})$'s, see 3.1 and 3.3), whose non-emptiness is characterized by our variant of Laffaille's theorem (Theorem 7).

3.1. Fix a Frobenius $\varphi = b \cdot \sigma \in G(K) \rtimes \langle \sigma \rangle$. For $\mu \in \mathbf{C}^{\mathbb{R}}(G_K)$, we define

$$X_\varphi(\mu) = \{x \in X : \mathbf{d}(\varphi(x), x) = \mu\}.$$

For any $x \in X_\varphi(\mu)$, there is an $\mathcal{F} \in \mathbf{F}^{\mathbb{R}}(G_K)$ of type μ such that $x = \varphi(x) + \mathcal{F}$, i.e. $x \in X_\varphi(\mathcal{F})$. Conversely, $X_\varphi(\mathcal{F}) \subset X_\varphi(\mu)$ for any $\mathcal{F}$ of type μ, thus

$$X_\varphi(\mu) = \cup_{\mathcal{F} \in \mathbf{F}^{\mathbb{R}}(G_K) : t(\mathcal{F}) = \mu} X_\varphi(\mathcal{F}) = \cup_{\mathcal{F} \in \mathbf{Adm}_\varphi(\mu)} X_\varphi(\mathcal{F})$$

and $X_\varphi(\mu) \neq \emptyset \Leftrightarrow \mathbf{Adm}_\varphi(\mu) \neq \emptyset$ by the previous theorem, where

$$\mathbf{Adm}_\varphi(\mu) = \left\{\mathcal{F} \in \mathbf{F}^{\mathbb{R}}(G_K) : (\varphi, \mathcal{F}) \text{ is weakly admissible and } t(\mathcal{F}) = \mu\right\}.$$

3.2. Write $\nu = t(\mathcal{F}_N)$. As $\mathcal{F}_N \in \mathbf{F}^{\mathbb{Q}}(G_K)$ is fixed by $\varphi = b \cdot \sigma$ and the type map $t : \mathbf{F}^{\mathbb{Q}}(G_K) \to \mathbf{C}^{\mathbb{Q}}(G_K)$ is $G(K)$-invariant, ν belongs to the fixed point set of σ in $\mathbf{C}^{\mathbb{Q}}(G_K)$. Since $\mathbf{C}^{\mathbb{Q}}(G_K)$ is a uniquely divisible commutative monoid on which $\langle \sigma \rangle$ acts with finite orbits, averaging over these orbits yields a retraction $\mu \mapsto \mu^\sharp$ onto the fixed point set of σ. When k is algebraically closed, [7, Théorème 6] shows that

$$X_\varphi(\mu) \neq \emptyset \Longrightarrow \nu \leq \mu^\sharp \quad \text{in } \left(\mathbf{C}^{\mathbb{Q}}(G_K), \leq\right)$$

where $\leq$ is the dominance partial order on $\mathbf{C}^{\mathbb{Q}}(G_K)$. This is still true over any perfect field k, because if k' is an algebraic closure of k and $K' = \mathrm{Frac}(W(k'))$, then $\mathbf{C}^{\mathbb{Q}}(G_K) \hookrightarrow \mathbf{C}^{\mathbb{Q}}(G_{K'})$ is a $\langle \sigma \rangle$-equivariant embedding of partially ordered commutative monoids. We will establish below that conversely, $\nu \leq \mu^\sharp$ implies $X_\varphi(\mu) \neq \emptyset$ when k is algebraically closed and G is unramified over $\mathbb{Q}_p$. The latter assumption seems superfluous, but some variant of the former is really needed: if $k = \mathbb{F}_p$ and $\varphi = \sigma$ (i.e. $b = 1$), then $\nu = 0$ but $X_\varphi(\mu) \neq \emptyset \iff \mu = 0$.

3.3. The subset $X_\varphi(\mu)$ of $X = \mathbf{B}^e(G_K)$ is closed for the canonical topology of X, but it is typically not convex, also $X_\varphi(\nu) = \mathrm{Min}(\varphi)$ is convex by [1, II.6.2]. On the other hand, $X_\varphi(\mathcal{F})$ is always closed and convex by [2, Theorem 1.3]. The group

$$J = \mathrm{Aut}^{\otimes}(V_K, \varphi) = \{g \in G(K) : \varphi g = g\varphi\} = \{g \in G(K) : g \diamond b = b\}$$

acts on $X_\varphi(\mu)$ and $\mathbf{Adm}_\varphi(\mu)$, and $j \in J$ maps $X_\varphi(\mathcal{F})$ to $X_\varphi(j \cdot \mathcal{F})$. The convex projection $p : X \twoheadrightarrow \mathrm{Min}(\varphi)$ is non-expanding, J-equivariant, and of formation compatible with algebraic base change on k. Indeed if k' is a Galois extension of k and $p' : X' \twoheadrightarrow \mathrm{Min}(\varphi')$ is the corresponding projection, then p' is also $\mathrm{Gal}(k'/k)$-equivariant. It thus maps $X = X'^{\mathrm{Gal}(k'/k)}$ [23, 2.6.1] to $\mathrm{Min}(\varphi') \cap X = \mathrm{Min}(\varphi)$ [1, II.6.2], from which easily follows that $p'|X$ equals p. The projection p restricts to a quasi-isometric J-equivariant embedding $p : X_\varphi(\mu) \to \mathrm{Min}(\varphi)$, which is even a J-equivariant quasi-isometry (as defined in [1, I.8.14]) when k is algebraically closed and $X_\varphi(\mu) \neq \emptyset$. Indeed, the restriction of p to $X_\varphi(\mu)$ has bounded fibers and J acts cocompactly on $\mathrm{Min}(\varphi)$ when k is algebraically closed by [7]. In this case, the boundary of $X_\varphi(\mu)$ equals that of $\mathrm{Min}(\varphi)$, i.e. it is the fixed point set of φ in the boundary of X. The boundary of $X_\varphi(\mathcal{F})$ is described in [6, Theorem 1].

3.4. We now establish our second main result:

Theorem 10. *If G is unramified over $\mathbb{Q}_p$ and k is algebraically closed, then*

$$X_\varphi(\mu) \neq \emptyset \iff \mathbf{Adm}_\varphi(\mu) \neq \emptyset \iff \nu \leq \mu^\sharp.$$

Proof. It only remains to show that $\nu \leq \mu^\sharp$ implies $\mathbf{Adm}_\varphi(\mu) \neq \emptyset$. By [5, 5.1.2],

$$\nu \leq \mu^\sharp \iff \forall z \in \mathbf{C}^{\mathbb{R}}(G_K) : \left\langle \mu^\sharp, z \right\rangle_\tau^{tr} \leq \langle \nu, z \rangle_\tau^{tr}$$

where $\langle -, - \rangle_\tau^{tr} : \mathbf{C}^{\mathbb{R}}(G_K) \times \mathbf{C}^{\mathbb{R}}(G_K) \to \mathbb{R}$ is defined in [5, 4.2.5] and given by

$$\langle \mu_1, \mu_2 \rangle_\tau^{tr} \quad = \quad \inf \left\{ \langle \mathcal{F}_1, \mathcal{F}_2 \rangle_\tau : (\mathcal{F}_1, \mathcal{F}_2) \in \mathbf{F}^{\mathbb{R}}(G_K)^2,\ t(\mathcal{F}_1, \mathcal{F}_2) = (\mu_1, \mu_2) \right\}.$$

This "scalar product" $\langle -, - \rangle_\tau^{tr}$ is σ-invariant and additive in both variables. It follows that $\left\langle \mu^\sharp, z \right\rangle_\tau^{tr} = \langle \mu, z \rangle_\tau^{tr}$ for every $z \in \mathbf{C}^{\mathbb{R}}(G_K)$ fixed by σ, so that

$$\nu \leq \mu^\sharp \Longrightarrow \forall z \in \mathbf{C}^{\mathbb{R}}(G_K)^{\sigma = \mathrm{Id}} : \langle \mu, z \rangle_\tau^{tr} \leq \langle \nu, z \rangle_\tau^{tr}.$$

Suppose therefore that $\nu \leq \mu^\sharp$. Then for any $\mathcal{F}_H \in \mathbf{F}^{\mathbb{R}}(G_K)$ of type $t(\mathcal{F}_H) = \mu$ and every φ-stable $\Xi \in \mathbf{F}^{\mathbb{R}}(G_K)$, since $t(\Xi) = z$ is fixed by σ and $t(\mathcal{F}_N) = \nu$,

$$\langle t(\mathcal{F}_H), t(\Xi) \rangle_\tau^{tr} \leq \langle t(\mathcal{F}_N), t(\Xi) \rangle_\tau^{tr} \leq \langle \mathcal{F}_N, \Xi \rangle_\tau.$$

Thus $\mathcal{F}_H$ will belong to $\mathbf{Adm}_\varphi(\mu)$ if it happens to be in generic (= transverse) relative position [5, 4.2.5] with respect to every φ-stable $\Xi \in \mathbf{F}^{\mathbb{R}}(G_K)$, for then

$$\langle \mathcal{F}_H, \Xi \rangle_\tau = \langle t(\mathcal{F}_H), t(\Xi) \rangle_\tau^{tr} \leq \langle \mathcal{F}_N, \Xi \rangle_\tau.$$

It is therefore sufficient to show that the set of φ-stable $\mathbb{R}$-filtrations on V_K is a thin subset of $\mathbf{F}^{\mathbb{R}}(G_K)$ in the sense of [5, 4.1.19]. Since k is algebraically closed, we may assume that φ satisfies a decency equation as in [7, 2.8] for some integer $s > 0$, in which case every φ-stable filtration is also fixed by σ^s, i.e. defined over the fixed field $\mathbb{Q}_{p^s}$ of σ^s in K, a finite (unramified) extension of $\mathbb{Q}_p$. Now since G is unramified over $\mathbb{Q}_p$, it admits a reductive model over $\mathbb{Z}_p$, which we also denote by G. We may now apply the thinness criterion of [5, 4.1.19], thus obtaining that the whole of $\mathbf{F}^{\mathbb{R}}(G_{\mathbb{Q}_{p^s}})$ is a thin subset of $\mathbf{F}^{\mathbb{R}}(G_K)$, which proves the theorem. □

4. Filtered Isocrystals

4.1. The hardest implication in Theorem 7 seems to be $(2^{\Delta}) \Rightarrow (1)$:

$$(2^{\Delta}) \Rightarrow (2) \Rightarrow (2') \Rightarrow (4') \Rightarrow (4'_{\tau}) \Rightarrow (3'_{\tau}) \Rightarrow (2'_{\tau}) \Rightarrow (2_{\tau}) \Rightarrow (1_{\tau}) \Rightarrow (1).$$

The key steps are: the fixed point theorem of [6], used in $(2') \Rightarrow (4')$, and some functoriality of Bruhat-Tits buildings, used in $(4') \Rightarrow (4'_{\tau})$. However, the theory of Harder-Narasimhan filtrations provides an easier proof of a more general result.

4.2. Fix an extension L of $K = W(k)[\frac{1}{p}]$ and denote by $\mathsf{Fil}^{\Gamma}_L\mathsf{Iso}(k)$ the category of triples $(V, \varphi, \mathcal{F})$ where (V, φ) is an isocrystal over k and $\mathcal{F} = \mathcal{F}_H$ is a Γ-filtration on $V_L = V \otimes_K L$. It is an exact $\otimes$-category equipped with Harder-Narasimhan filtrations attached to the slope function which is defined (for $V \neq 0$) by

$$\mu(V, \varphi, \mathcal{F}) = \frac{\deg(\mathcal{F}_H) - \deg(\mathcal{F}_N)}{\dim_K V} \in \mathbb{Q}[\Gamma] \subset \mathbb{R}.$$

Here $\mathbb{Q}[\Gamma]$ is the $\mathbb{Q}$-subspace of $\mathbb{R}$ spanned by Γ. We now have four filtrations at our disposal: the *Hodge* Γ-filtration $\mathcal{F} = \mathcal{F}_H$ on V_L, the φ-stable (opposed) *Newton* $\mathbb{Q}$-filtrations $\mathcal{F}_N$ and $\mathcal{F}_N^{\iota}$ on $V = V_K$, and the φ-stable *Harder-Narasimhan* $\mathbb{Q}[\Gamma]$-filtration $\mathcal{F}_{HN} = \mathcal{F}_{HN}(V, \varphi, \mathcal{F})$ on $V = V_K$.

Lemma 11. *The following conditions are equivalent:*

1. $(V, \varphi, \mathcal{F})$ *is weakly admissible, i.e.:*
   (a) $\deg(\mathcal{F}_H) = \deg(\mathcal{F}_N)$ *and*
   (b) $\deg(\mathcal{F}_H|W) \leq \deg(\mathcal{F}_N|W)$ *for every φ-stable K-subspace W of V,*
2. *For every φ-stable Δ-filtration Ξ on V,* $\langle \mathcal{F}_H, \Xi \rangle \leq \langle \mathcal{F}_N, \Xi \rangle$.
3. *For every φ-stable Δ-filtration Ξ on V,* $\langle \mathcal{F}_H, \Xi \rangle + \langle \mathcal{F}_N^{\iota}, \Xi \rangle \leq 0$.
4. $\langle \mathcal{F}_H, \mathcal{F}_{HN} \rangle \leq \langle \mathcal{F}_N, \mathcal{F}_{HN} \rangle$.
5. $\langle \mathcal{F}_H, \mathcal{F}_{HN} \rangle + \langle \mathcal{F}_N^{\iota}, \mathcal{F}_{HN} \rangle \leq 0$.
6. $\mathcal{F}_{HN} = 0$.

In (2) *and* (3), Δ *is any non-trivial subgroup of* $\mathbb{R}$.

Proof. One proves $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ and $(4) \Leftrightarrow (5)$ as in Lemma 2, moreover $(1) \Leftrightarrow (6)$ by definition of $\mathcal{F}_{HN}$ and obviously $(6) \Rightarrow (4), (5)$. Finally $(4) \Rightarrow (6)$ because:

$$\begin{aligned}
\langle \mathcal{F}_H, \mathcal{F}_{HN} \rangle - \langle \mathcal{F}_N, \mathcal{F}_{HN} \rangle &= \sum_{\gamma} \gamma \cdot \left(\deg \mathrm{Gr}^{\gamma}_{\mathcal{F}_{HN}}(\mathcal{F}_H) - \deg \mathrm{Gr}^{\gamma}_{\mathcal{F}_{HN}}(\mathcal{F}_N) \right) \\
&= \sum_{\gamma} \gamma \cdot \dim_K \left(\mathrm{Gr}^{\gamma}_{\mathcal{F}_{HN}} \right) \cdot \mu \left(\mathrm{Gr}^{\gamma}_{\mathcal{F}_{HN}} \right) \\
&= \sum_{\gamma} \gamma^2 \cdot \dim_K \left(\mathrm{Gr}^{\gamma}_{\mathcal{F}_{HN}} \right) \\
&= \langle \mathcal{F}_{HN}, \mathcal{F}_{HN} \rangle
\end{aligned}$$

using the definition of $\mathcal{F}_{HN}$ for the third equality. □

4.3. For a reductive group G over $\mathbb{Q}_p$, a Γ-filtered G-isocrystal over L is an exact $\otimes$-functor factorization $(\varphi, \mathcal{F})$ of the fiber functor $V_K : \mathsf{Rep}(G) \to \mathsf{Vect}(K)$ trough the natural fiber functor $\mathsf{Fil}^{\Gamma}_L\mathsf{Iso}(k) \to \mathsf{Vect}(K)$. It again induces four filtrations on the relevant fiber functors: the *Hodge* Γ-filtration $\mathcal{F}_H = \mathcal{F}$ on V_L, the φ-stable (opposed) *Newton* $\mathbb{Q}$-filtrations $\mathcal{F}_N$ and $\mathcal{F}_N^\iota$ on V_K, and the φ-stable *Harder-Narasimhan* $\mathbb{Q}[\Gamma]$-filtration $\mathcal{F}_{HN} = \mathcal{F}_{HN}(\varphi, \mathcal{F})$ on V_K defined by

$$\forall \tau \in \mathsf{Rep}(G): \qquad \mathcal{F}_{HN}(\tau) = \mathcal{F}_{HN}(V_K(\tau), \varphi(\tau), \mathcal{F}(\tau)).$$

It is not at all obvious that this formula indeed defines a filtration on V_K: it does yield a factorization $\mathcal{F}_{HN} : \mathsf{Rep}(G) \to \mathsf{Fil}^{\mathbb{Q}[\Gamma]}(K)$ of V_K, but we have to check that the latter is exact and compatible with the $\otimes$-products and their neutral objects. Since every exact sequence in $\mathsf{Rep}(G)$ is split (we are in characteristic 0), exactness here amounts to additivity, which is obvious. The issue is here the compatibility of Harder-Narasimhan filtrations with $\otimes$-products, where we need to use the main result of [24]. We say that $(\varphi, \mathcal{F}) : \mathsf{Rep}(G) \to \mathsf{Fil}_L^\Gamma \mathsf{Iso}(k)$ is weakly admissible if it factors through the full subcategory $\mathsf{Fil}_L^\Gamma \mathsf{Iso}(k)^{wa}$ of weakly admissible objects.

4.4. Fix a faithful $\tau \in \mathsf{Rep}(G)$ and some subgroup $\Delta \neq 0$ of $\mathbb{R}$. Then:

Theorem 12. *The following conditions on $(\varphi, \mathcal{F})$ are equivalent:*

(1) $(\varphi, \mathcal{F})$ *is weakly admissible.*
(1_ρ) *For every $\rho \in \mathsf{Rep}(G)$, $(V_K(\rho), \varphi(\rho), \mathcal{F}(\rho))$ is weakly admissible.*
(1_τ) $(V_K(\tau), \varphi(\tau), \mathcal{F}(\tau))$ *is weakly admissible.*
(2^Δ) *For every φ-stable Δ-filtration Ξ on V_K,*

$$\langle \mathcal{F}_H, \Xi \rangle_\tau \leq \langle \mathcal{F}_N, \Xi \rangle_\tau .$$

(2^Δ_ρ) *For every $\rho \in \mathsf{Rep}(G)$ and every φ-stable Δ-filtration Ξ on $V_K(\rho)$,*

$$\langle \mathcal{F}_H(\rho), \Xi \rangle \leq \langle \mathcal{F}_N(\rho), \Xi \rangle .$$

(2^Δ_τ) *For every φ-stable Δ-filtration Ξ on $V_K(\tau)$,*

$$\langle \mathcal{F}_H(\tau), \Xi \rangle \leq \langle \mathcal{F}_N(\tau), \Xi \rangle .$$

(5) $\mathcal{F}_{HN} = 0$.
(5_ρ) *For every $\rho \in \mathsf{Rep}(G)$, $\mathcal{F}_{HN}(\rho) = 0$.*
(5_τ) $\mathcal{F}_{HN}(\tau) = 0$.

Proof. The following implications are obvious, or given by the previous lemma:

$$\begin{array}{ccccccc}
(1) & \Longleftrightarrow & (1_\rho) & \Longrightarrow & (1_\tau) & & \\
 & & \Updownarrow & & \Updownarrow & & \\
 & & (2_\rho) & \Longrightarrow & (2_\tau) & \Longrightarrow & (2^\Delta) \\
 & & \Updownarrow & & \Updownarrow & & \\
(5) & \Longleftrightarrow & (5_\rho) & \Longrightarrow & (5_\tau) & &
\end{array}$$

We have $(2^\Delta) \Leftrightarrow (2^\mathbb{R}) \Leftrightarrow (2^{\mathbb{Q}[\Gamma]})$ as in 2.11, $(5_\tau) \Rightarrow (5)$ by [5, 4.2.10 or 3.11.12] and finally $(2^{\mathbb{Q}[\Gamma]}) \Rightarrow (5_\tau)$ since $(2^{\mathbb{Q}[\Gamma]})$ with $\Xi = \mathcal{F}_{HN}$ gives

$$\langle \mathcal{F}_H(\tau), \mathcal{F}_{HN}(\tau) \rangle \leq \langle \mathcal{F}_N(\tau), \mathcal{F}_{HN}(\tau) \rangle$$

which implies $\mathcal{F}_{HN}(\tau) = 0$ by the previous lemma. $\square$

4.5. The above proof of $(2^\Delta) \Rightarrow (1)$ looks much easier, but it still uses Totaro's theorem to the effect that the Harder-Narasimhan filtration is compatible with $\otimes$-products. Conversely, the above theorem implies that weakly admissible objects are stable under $\otimes$-product: apply $(1_\tau) \Rightarrow (1_\rho)$ with $G = GL(V_1) \times GL(V_2)$, $\tau = \rho_1 \boxplus \rho_2$ and $\rho = \rho_1 \boxtimes \rho_2$ with ρ_i the tautological representation of $GL(V_i)$. In particular, Theorem 7 yields another proof of Totaro's result when $L = K$, which was of course already known to Laffaille: if M_i is a strongly divisible lattice in V_i, then $M_1 \otimes M_2$ is a strongly divisible lattice in $V_1 \otimes V_2$.

5. Weakly admissible filtered isocrystals

In this section, we define the Fargues $\mathbb{Q}$-filtration on weakly admissible Γ-filtered isocrystals over L (5.1), show that it is compatible with tensor products (Theorem 15), and compute it as a convex projection (Lemma 13 and Proposition 16).

5.1. Weakly admissible Γ-filtered isocrystals over L have a Harder-Narasimhan filtration of their own, defined by Fargues in [10, §9] when $\Gamma = \mathbb{Z}$, for the slope function $\mu = \deg / \dim$, where the degree function is now given by

$$\deg(V, \varphi, \mathcal{F}) = -\deg(\mathcal{F}_H) = -\deg(\mathcal{F}_N) = \deg(\mathcal{F}_N^\iota) \in \mathbb{Q}.$$

An object $(V, \varphi, \mathcal{F})$ in $\mathsf{Fil}_L^\Gamma \mathsf{Iso}(k)^{wa}$ is thus again equipped with four filtrations: the *Hodge* Γ-filtration $\mathcal{F} = \mathcal{F}_H$ of V_L by L-subspaces, the pair of opposed *Newton* $\mathbb{Q}$-filtrations $(\mathcal{F}_N, \mathcal{F}_N^\iota)$ of V by φ-stable K-subspaces, and the *Fargues* $\mathbb{Q}$-filtration $\mathcal{F}_F$ of V by (φ-stable) weakly admissible K-subspaces. Note that the previous *Harder-Narasimhan* Γ-filtration $\mathcal{F}_{HN}$ on V by φ-stable K-subspaces is now trivial!

5.2. For a weakly admissible object $(V, \varphi, \mathcal{F})$ in $\mathsf{Fil}_L^\Gamma \mathsf{Iso}(k)$ and a subgroup $\Delta \neq 0$ of $\mathbb{R}$, we denote by $\mathbf{F}^\Delta(V)$ the set of all Δ-filtrations on V, by $\mathbf{F}^\Delta(V, \varphi) \subset \mathbf{F}^\Delta(V)$ its subset of Δ-filtrations by φ-stable K-subspaces, and by $\mathbf{F}^\Delta(V, \varphi, \mathcal{F}) \subset \mathbf{F}^\Delta(V, \varphi)$ its subset of Δ-filtrations by weakly admissible φ-stable K-subspaces. Moreover, we equip $\mathbf{F}^\mathbb{R}(V)$ with the φ-invariant CAT(0)-distance [5, 4.2.10] defined by

$$d(\mathcal{F}_1, \mathcal{F}_2) = \sqrt{\|\mathcal{F}_1\|^2 + \|\mathcal{F}_2\|^2 - 2\langle \mathcal{F}_1, \mathcal{F}_2 \rangle}.$$

Then $\mathbf{F}^\mathbb{R}(V, \varphi)$ and $\mathbf{F}^\mathbb{R}(V, \varphi, \mathcal{F})$ are closed and convex subsets of $\mathbf{F}^\mathbb{R}(V)$. In fact:

Lemma 13. *For $(V, \varphi, \mathcal{F}) \in \mathsf{Fil}_L^\Gamma \mathsf{Iso}(k)^{wa}$ and a subgroup $0 \neq \Delta \subset \mathbb{R}$ as above,*

$$\mathbf{F}^\Delta(V, \varphi, \mathcal{F}) = \left\{ \Xi \in \mathbf{F}^\Delta(V, \varphi) : \langle \mathcal{F}_H, \Xi \rangle = \langle \mathcal{F}_N, \Xi \rangle \right\}.$$

For $\Delta = \mathbb{R}$, this is a closed convex subset of $\mathbf{F}^\mathbb{R}(V)$. Moreover $\mathcal{F}_F \in \mathbf{F}^\mathbb{Q}(V, \varphi, \mathcal{F})$ is the image of $\mathcal{F}_N^\iota \in \mathbf{F}^\mathbb{Q}(V, \varphi)$ under the convex projection $p : \mathbf{F}^\mathbb{R}(V, \varphi) \to \mathbf{F}^\mathbb{R}(V, \varphi, \mathcal{F})$.

Proof. For $\Xi \in \mathbf{F}^\Delta(V, \varphi)$, set $\{\delta \in \Delta : \mathrm{Gr}_\Xi^\delta \neq 0\} = \{\delta_1 < \cdots < \delta_n\}$. Then Ξ^{δ_i} is a φ-stable K-subspace of V. Put $d_H(i) = \deg(\mathcal{F}_H | \Xi_K^{\delta_i})$, $d_N(i) = \deg(\mathcal{F}_N | \Xi^{\delta_i})$. Then

$$\langle \mathcal{F}_H, \Xi \rangle - \langle \mathcal{F}_N, \Xi \rangle = \sum_{i=1}^n \delta_i \cdot (\Delta_H(i) - \Delta_N(i)) = \sum_{i=1}^n \Delta_i \cdot (d_H(i) - d_N(i))$$

where $\Delta_\star(i) = d_\star(i) - d_\star(i+1)$ for $1 \leq i < n$, $\Delta_\star(n) = d_\star(n)$, $\Delta_i = \delta_i - \delta_{i-1}$ for $1 < i \leq n$ and $\Delta_1 = \delta_1$, as in the proof of lemma 2. We now know that $d_H(i) \leq d_N(i)$ for all i with equality for $i = 1$. Since $\Delta_i > 0$ for $i > 1$, we obtain

$$\langle \mathcal{F}_H, \Xi \rangle = \langle \mathcal{F}_N, \Xi \rangle \iff \forall 1 \leq i \leq n : \quad d_H(i) = d_N(i).$$

The displayed equality immediately follows, which may also be written as

$$\begin{aligned}\mathbf{F}^{\Delta}(V,\varphi,\mathcal{F}) &= \left\{\Xi\in\mathbf{F}^{\Delta}(V,\varphi):\langle\mathcal{F}_H,\Xi\rangle+\langle\mathcal{F}_N^{\iota},\Xi\rangle=0\right\},\\ &= \left\{\Xi\in\mathbf{F}^{\Delta}(V,\varphi):\langle\mathcal{F}_H,\Xi\rangle+\langle\mathcal{F}_N^{\iota},\Xi\rangle\geq 0\right\}.\end{aligned}$$

Since the scalar product $\langle -,-\rangle$ is continuous and concave (see [5, 4.2.10]), this is (for $\Delta=\mathbb{R}$) a closed and convex subset of the closed and convex subset $\mathbf{F}^{\mathbb{R}}(V,\varphi)$ of the CAT(0)-space $\mathbf{F}^{\mathbb{R}}(V)$. Take now $\Xi=p(\mathcal{F}_N^{\iota})$, the convex projection of $\mathcal{F}_N^{\iota}$ to $\mathbf{F}^{\mathbb{R}}(V,\varphi,\mathcal{F})$. For $1\leq i\leq n$, a weakly admissible subspace W of $\mathrm{Gr}_{\Xi}^{\delta_i}$ and a small $\epsilon>0$, let $\Xi'=\Xi(i,W,\epsilon)$ be the φ-stable $\mathbb{R}$-filtration on V with $\mathrm{Gr}_{\Xi'}^{\delta}=\mathrm{Gr}_{\Xi}^{\delta}$ unless $\delta=\delta_i$ or $\delta_i+\epsilon$, where $\mathrm{Gr}_{\Xi'}^{\delta_i}=\mathrm{Gr}_{\Xi}^{\delta_i}/W$ and $\mathrm{Gr}_{\Xi'}^{\delta_i+\epsilon}=W$. By definition of Ξ,

$$\langle\Xi,\Xi\rangle-2\langle\mathcal{F}_N^{\iota},\Xi\rangle\leq\langle\Xi',\Xi'\rangle-2\langle\mathcal{F}_N^{\iota},\Xi'\rangle,$$

which easily unfolds to the following inequality, valid for small $\epsilon>0$:

$$\epsilon^2\dim_K W+2\epsilon\left(\delta_i\dim_K W-\deg(\mathcal{F}_N^{\iota}|W)\right)\geq 0.$$

It follows that $\delta_i\geq\mu(W)$. For $W=\mathrm{Gr}_{\Xi}^{\delta_i}$, we may also allow negative ϵ's with small absolute values in the above argument, this time obtaining $\delta_i=\mu(\mathrm{Gr}_{\Xi}^{\delta_i})$. Therefore $\mathrm{Gr}_{\Xi}^{\delta_i}$ is μ-semi-stable of slope δ_i for all $1\leq i\leq n$, and Ξ thus equals $\mathcal{F}_F$. □

5.3. Let now G be a reductive group over $\mathbb{Q}_p$ and let

$$(\varphi,\mathcal{F}):\mathsf{Rep}(G)\to\mathsf{Fil}_L^{\Gamma}\mathsf{Iso}(k)^{wa}$$

be a weakly admissible Γ-filtered G-isocrystal over L. We denote by

$$\mathbf{F}^{\Delta}(G_K,\varphi,\mathcal{F})\subset\mathbf{F}^{\Delta}(G_K,\varphi)\subset\mathbf{F}^{\Delta}(G_K)$$

the set of Δ-filtrations on the fiber functor $V_K:\mathsf{Rep}(G)\to\mathsf{Vect}(K)$ by respectively φ-stable and φ-stable weakly admissible K-subspaces. We also fix a faithful representation τ of G and equip $\mathbf{F}^{\mathbb{R}}(G_K)$ with the CAT(0)-metric defined by

$$d_\tau(\mathcal{F}_1,\mathcal{F}_2)=\sqrt{\|\mathcal{F}_1\|_\tau^2+\|\mathcal{F}_2\|_\tau^2-2\langle\mathcal{F}_1,\mathcal{F}_2\rangle_\tau}.$$

Lemma 14. *With notations as above,*

$$\mathbf{F}^{\Delta}(G_K,\varphi,\mathcal{F})=\left\{\Xi\in\mathbf{F}^{\Delta}(G_K,\varphi):\langle\mathcal{F}_H,\Xi\rangle_\tau=\langle\mathcal{F}_N,\Xi\rangle_\tau\right\}.$$

For $\Delta=\mathbb{R}$, this is a closed convex subset of $(\mathbf{F}^{\mathbb{R}}(G_K),d_\tau)$.

Proof. If $\Xi\in\mathbf{F}^{\Delta}(G_K,\varphi,\mathcal{F})$, then $\Xi(\tau)\in\mathbf{F}^{\Delta}(V_K(\tau),\varphi(\tau),\mathcal{F}(\tau))$, thus

$$\langle\mathcal{F}_H,\Xi\rangle_\tau=\langle\mathcal{F}_H(\tau),\Xi(\tau)\rangle=\langle\mathcal{F}_N(\tau),\Xi(\tau)\rangle=\langle\mathcal{F}_N,\Xi\rangle_\tau$$

by the previous lemma. Suppose conversely that $\Xi\in\mathbf{F}^{\Delta}(G_K,\varphi)$ satisfies this equation, choose a splitting of Ξ (which exists by [5, 3.11.3]) and use it to transport $(\mathrm{Gr}_\Xi(\varphi),\mathrm{Gr}_\Xi(\mathcal{F}))$ back from the fiber functor $\mathrm{Gr}_\Xi(V_K)$ to $(\varphi',\mathcal{F}')$ on V_K. Since $\mathsf{Fil}_L^{\Gamma}\mathsf{Iso}(k)^{wa}$ is a strictly full subcategory of $\mathsf{Fil}_L^{\Gamma}\mathsf{Iso}(k)$ which is stable under extensions, it is sufficient to establish that $(\varphi',\mathcal{F}')$ is weakly admissible, and this now follows from $(1_\tau)\Rightarrow(1)$ of theorem 12 since $(V_K(\tau),\varphi'(\tau),\mathcal{F}'(\tau))$ is weakly admissible by the previous lemma. For $\Delta=\mathbb{R}$, one shows as above that

$$\begin{aligned}\mathbf{F}^{\mathbb{R}}(G_K,\varphi,\mathcal{F}) &= \left\{\Xi\in\mathbf{F}^{\mathbb{R}}(G_K,\varphi):\langle\mathcal{F}_H,\Xi\rangle_\tau+\langle\mathcal{F}_N^{\iota},\Xi\rangle_\tau=0\right\}\\ &= \left\{\Xi\in\mathbf{F}^{\mathbb{R}}(G_K,\varphi):\langle\mathcal{F}_H,\Xi\rangle_\tau+\langle\mathcal{F}_N^{\iota},\Xi\rangle_\tau\geq 0\right\}\end{aligned}$$

is a closed convex subset of $\mathbf{F}^{\mathbb{R}}(G_K,\varphi)$. □

5.4. We may now establish the analog of Totaro's tensor product theorem for the Fargues filtration $\mathcal{F}_F$. In the classical setting of p-adic Hodge theory where $\Gamma = \mathbb{Z}$ and L is a totally ramified finite algebraic extension of K, an entirely different proof is given in [10, §9.2], using Fontaine's functor to work on the Galois side.

Theorem 15. *The Fargues filtration defines a* $\otimes$*-functor*

$$\mathcal{F}_F : \mathsf{Fil}_L^\Gamma \mathsf{Iso}(k)^{wa} \to \mathsf{Fil}_K^{\mathbb{Q}}.$$

Proof. The general formalism of Harder-Narasimhan filtrations yields the functoriality of $\mathcal{F}_F$ and reduces its compatibility with tensor products to the following statement: if $(V_i, \varphi_i, \mathcal{F}_i) \in \mathsf{Fil}_L^\Gamma \mathsf{Iso}(k)^{wa}$ is semi-stable of slope $\mu_i \in \mathbb{Q}$ for $i \in \{1,2\}$,

$$(V, \varphi, \mathcal{F}) = (V_1 \otimes V_2, \varphi_1 \otimes \varphi_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \in \mathsf{Fil}_L^\Gamma \mathsf{Iso}(k)^{wa}$$

is semi-stable of slope $\mu = \mu_1 + \mu_2$, which means that for every weakly admissible subobject X of V, $\deg(\mathcal{F}_N^\iota | X) \leq \deg(V(\mu)|X)$ with equality for $X = V$. Here $V(\nu)$ denotes the filtration on V such that $\mathrm{Gr}^\nu_{V(\nu)} = V$. Exactly as in Lemma 2, this condition is itself equivalent to the following one: for every $\Xi \in \mathbf{F}^{\mathbb{R}}(V, \varphi, \mathcal{F})$,

$$\langle \mathcal{F}_N^\iota, \Xi \rangle \leq \langle V(\mu), \Xi \rangle \quad (\text{or} : \langle \mathcal{F}_N^\iota, \Xi \rangle + \langle V(-\mu), \Xi \rangle \leq 0).$$

We adapt Totaro's proof to this setting. Applying the functor $\mathbf{F}^{\mathbb{R}}(-)$ to

$$GL(V_1) \times GL(V_2) \twoheadrightarrow G = GL(V_1) \times GL(V_2)/\Delta(\mathbb{G}_m) \hookrightarrow GL(V_1 \otimes V_2)$$

where Δ is the diagonal embedding yields a factorization

$$\mathbf{F}^{\mathbb{R}}(V_1) \times \mathbf{F}^{\mathbb{R}}(V_2) \twoheadrightarrow \mathbf{F}^{\mathbb{R}}(G) \hookrightarrow \mathbf{F}^{\mathbb{R}}(V_1 \otimes V_2)$$

of the map $(\mathcal{G}_1, \mathcal{G}_2) \mapsto \mathcal{G}_1 \otimes \mathcal{G}_2$, which identifies $\mathbf{F}^{\mathbb{R}}(G)$ with a closed and convex subset of $\mathbf{F}^{\mathbb{R}}(V_1 \otimes V_2)$. Let $p : \mathbf{F}^{\mathbb{R}}(V_1 \otimes V_2) \twoheadrightarrow \mathbf{F}^{\mathbb{R}}(G)$ be the convex projection and pick $(\Xi_1, \Xi_2) \in \mathbf{F}^{\mathbb{R}}(V_1) \times \mathbf{F}^{\mathbb{R}}(V_2)$ mapping to $p(\Xi) \in \mathbf{F}^{\mathbb{R}}(G)$ (this is Kempf's filtration of [24, §2]). Since p is φ-equivariant, $p(\Xi)$ belongs to $\mathbf{F}^{\mathbb{R}}(G, \varphi)$ and then also $\Xi_i \in \mathbf{F}^{\mathbb{R}}(V_i, \varphi)$ for $i \in \{1,2\}$. Moreover, for any $(\mathcal{G}_1, \mathcal{G}_2) \in \mathbf{F}^{\mathbb{R}}(V_{1,L}) \times \mathbf{F}^{\mathbb{R}}(V_{2,L})$,

$$\langle \mathcal{G}_1 \otimes \mathcal{G}_2, \Xi \rangle \leq \langle \mathcal{G}_1 \otimes \mathcal{G}_2, p(\Xi) \rangle \tag{5.1}$$

by [5, 5.5.12] since p is also the restriction to $\mathbf{F}^{\mathbb{R}}(V_1 \otimes V_2)$ of the convex projection from $\mathbf{F}^{\mathbb{R}}(V_{1,L} \otimes V_{2,L})$ onto $\mathbf{F}^{\mathbb{R}}(G_L)$. On the other hand, one checks easily that

$$\begin{aligned} \langle \mathcal{G}_1 \otimes \mathcal{G}_2, \mathcal{H}_1 \otimes \mathcal{H}_2 \rangle &= \langle \mathcal{G}_1, \mathcal{H}_1 \rangle \cdot \dim V_2 + \langle \mathcal{G}_2, \mathcal{H}_2 \rangle \cdot \dim V_1 \\ &\quad + \deg \mathcal{G}_1 \cdot \deg \mathcal{H}_2 + \deg \mathcal{H}_1 \cdot \deg \mathcal{G}_2 \end{aligned} \tag{5.2}$$

for every $(\mathcal{G}_i, \mathcal{H}_i) \in \mathbf{F}^{\mathbb{R}}(V_{i,L})$. We will now apply these two formulas to

$$\begin{aligned} \mathcal{F}_N^\iota(\text{on } V) &= \mathcal{F}_N^\iota(\text{on } V_1) \otimes \mathcal{F}_N^\iota(\text{on } V_2), \\ \mathcal{F}_H(\text{on } V_L) &= \mathcal{F}_H(\text{on } V_1) \otimes \mathcal{F}_H(\text{on } V_2), \\ V(-\mu) &= V_1(-\mu_1) \otimes V_2(-\mu_2), \\ p(\Xi) &= \Xi_1 \otimes \Xi_2. \end{aligned}$$

First recall that since $(V, \varphi, \mathcal{F})$ is weakly admissible (by Totaro's theorem!),

$$\langle \mathcal{F}_H, p(\Xi) \rangle + \langle \mathcal{F}_N^\iota, p(\Xi) \rangle \leq 0.$$

Since $\langle \mathcal{F}_H, \Xi \rangle + \langle \mathcal{F}_N^\iota, \Xi \rangle = 0$ by assumption on Ξ and Lemma 13, actually

$$\langle \mathcal{F}_H, p(\Xi) \rangle + \langle \mathcal{F}_N^\iota, p(\Xi) \rangle = 0$$

by (5.1). Applying formula (5.2) twice and grouping terms, we obtain

$$\begin{array}{rcl} 0 &=& (\langle \mathcal{F}_H, \Xi_1\rangle + \langle \mathcal{F}_N^\iota, \Xi_1\rangle) \cdot \dim V_2 \\ &+& (\langle \mathcal{F}_H, \Xi_2\rangle + \langle \mathcal{F}_N^\iota, \Xi_2\rangle) \cdot \dim V_1 \\ &+& (\deg(\mathcal{F}_H \text{ on } V_1) + \deg(\mathcal{F}_N^\iota \text{ on } V_1)) \cdot \deg \Xi_2 \\ &+& (\deg(\mathcal{F}_H \text{ on } V_2) + \deg(\mathcal{F}_N^\iota \text{ on } V_2)) \cdot \deg \Xi_1. \end{array}$$

Since V_1 and V_2 are weakly admissible, the first two terms are ≤ 0 and the last two trivial. Thus $\langle \mathcal{F}_H, \Xi_i\rangle + \langle \mathcal{F}_N^\iota, \Xi_i\rangle = 0$ for $i \in \{1,2\}$, i.e. $\Xi_i \in \mathbf{F}^{\mathbb{R}}(V_i, \varphi_i, \mathcal{F}_i)$. Now applying our two formulas (5.1) and (5.2) twice again also gives that

$$\begin{array}{rcl} \langle \mathcal{F}_N^\iota, \Xi\rangle + \langle V(-\mu), \Xi\rangle &\leq& \langle \mathcal{F}_N^\iota, p(\Xi)\rangle + \langle V(-\mu), p(\Xi)\rangle \\ &\leq& (\langle \mathcal{F}_N^\iota, \Xi_1\rangle + \langle V_1(-\mu_1), \Xi_1\rangle) \cdot \dim V_2 \\ &+& (\langle \mathcal{F}_N^\iota, \Xi_2\rangle + \langle V_2(-\mu_1), \Xi_2\rangle) \cdot \dim V_1 \\ &+& (\deg(\mathcal{F}_N^\iota \text{ on } V_1) + \deg(V_1(-\mu_1))) \cdot \deg \Xi_2 \\ &+& (\deg(\mathcal{F}_N^\iota \text{ on } V_2) + \deg(V_2(-\mu_2))) \cdot \deg \Xi_1. \end{array}$$

Since V_i is semi-stable of slope μ_i for $i \in \{1,2\}$, the first two terms are ≤ 0 and the last two trivial. Thus $\langle \mathcal{F}_N^\iota, \Xi\rangle + \langle V(-\mu), \Xi\rangle \leq 0$ for all $\Xi \in \mathbf{F}^{\mathbb{R}}(V, \varphi, \mathcal{F})$ and $(V, \varphi, \mathcal{F})$ is indeed semi-stable of slope μ. □

5.5. A weakly admissible Γ-filtered G-isocrystal $(\varphi, \mathcal{F})$ over L thus again yields four filtrations: the *Hodge* Γ-filtration $\mathcal{F}_H = \mathcal{F} \in \mathbf{F}^\Gamma(G_L)$, the opposed *Newton* $\mathbb{Q}$-filtrations $\mathcal{F}_N, \mathcal{F}_N^\iota \in \mathbf{F}^{\mathbb{Q}}(G_K, \varphi)$ and the Fargues $\mathbb{Q}$-filtration $\mathcal{F}_F \in \mathbf{F}^{\mathbb{Q}}(G_K, \varphi, \mathcal{F})$, which is obtained by composing the exact $\otimes$-functor $(\varphi, \mathcal{F})$ with the $\otimes$-functor of the previous proposition (the resulting $\otimes$-functor is exact because it is plainly additive and every exact sequence in $\mathsf{Rep}(G)$ is split). The former Harder-Narasimhan filtration $\mathcal{F}_{HN}$ of section 4 is trivial. We claim that, just as in the $GL(V)$-case:

Proposition 16. *The Fargues $\mathbb{Q}$-filtration $\mathcal{F}_F$ is the convex projection of $\mathcal{F}_N^\iota$.*

Proof. The map $\mathcal{G} \mapsto \mathcal{G}(\tau)$ identifies $\mathbf{F}^{\mathbb{R}}(G_K)$ with a φ-stable closed convex subset of $\mathbf{F}^{\mathbb{R}}(V_K(\tau))$. Let p be the convex projection from $\mathbf{F}^{\mathbb{R}}(V_K(\tau))$ to $\mathbf{F}^{\mathbb{R}}(G_K)$. It is φ-equivariant and thus maps $\mathbf{F}^{\mathbb{R}}(V_K(\tau), \varphi(\tau))$ to $\mathbf{F}^{\mathbb{R}}(G_K, \varphi)$. Moreover,

$$\langle \mathcal{H}(\tau), \mathcal{G}\rangle \leq \langle \mathcal{H}(\tau), p(\mathcal{G})(\tau)\rangle = \langle \mathcal{H}, p(\mathcal{G})\rangle_\tau$$

for every $\mathcal{H} \in \mathbf{F}^{\mathbb{R}}(G_K)$ (or $\mathbf{F}^{\mathbb{R}}(G_L)$) and $\mathcal{G} \in \mathbf{F}^{\mathbb{R}}(V_K(\tau))$, exactly as in the proof of the previous proposition. Suppose now that $\mathcal{G}$ belongs to $\mathbf{F}^{\mathbb{R}}(V_K(\tau), \varphi(\tau), \mathcal{F}(\tau))$. Then $p(\mathcal{G})$ also belongs to $\mathbf{F}^{\mathbb{R}}(G_K, \varphi, \mathcal{F})$. Indeed, since $(\varphi, \mathcal{F})$ is weakly admissible,

$$\langle \mathcal{F}_H, p(\mathcal{G})\rangle_\tau + \langle \mathcal{F}_N^\iota, p(\mathcal{G})\rangle_\tau \leq 0$$

but also $\langle \mathcal{F}_H(\tau), \mathcal{G}\rangle + \langle \mathcal{F}_N^\iota(\tau), \mathcal{G}\rangle = 0$ thus actually

$$\langle \mathcal{F}_H, p(\mathcal{G})\rangle_\tau + \langle \mathcal{F}_N^\iota, p(\mathcal{G})\rangle_\tau = 0$$

i.e. $p(\mathcal{G})$ belongs to $\mathbf{F}^{\mathbb{R}}(G_K, \varphi, \mathcal{F})$. Since p is non-expanding and $\mathcal{F}_N^\iota \in \mathbf{F}^{\mathbb{R}}(G_K)$,

$$d\left(\mathcal{F}_N^\iota(\tau), \mathcal{G}\right) \geq d\left(\mathcal{F}_N^\iota(\tau), p(\mathcal{G})(\tau)\right) = d_\tau\left(\mathcal{F}_N^\iota, p(\mathcal{G})\right) \geq d_\tau(\mathcal{F}_N^\iota, \mathcal{F}_F') = d(\mathcal{F}_N^\iota(\tau), \mathcal{F}_F'(\tau))$$

where $\mathcal{F}_F'$ is the convex projection of $\mathcal{F}_N^\iota \in \mathbf{F}^{\mathbb{R}}(G_K, \varphi)$ to $\mathbf{F}^{\mathbb{R}}(G_K, \varphi, \mathcal{F})$. Since this holds true for any $\mathcal{G} \in \mathbf{F}^{\mathbb{R}}(V_K(\tau), \varphi(\tau), \mathcal{F}(\tau))$, it follows that $\mathcal{F}_F'(\tau) = \mathcal{F}_F(\tau)$ by Lemma 13. But then $\mathcal{F}_F' = \mathcal{F}_F$ since $\mathbf{F}^{\mathbb{R}}(G_K) \hookrightarrow \mathbf{F}^{\mathbb{R}}(V_K(\tau))$ is injective. □

References

[1] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[2] P. Chaoha and A. Phon-on. A note on fixed point sets in CAT(0) spaces. *J. Math. Anal. Appl.*, 320(2):983–987, 2006.

[3] Miaofen Chen and Eva Viehmann. Affine Deligne-Lusztig varieties and the action of J. *J. Algebraic Geom.*, 27(2):273–304, 2018.

[4] Pierre Colmez and Jean-Marc Fontaine. Construction des représentations p-adiques semi-stables. *Invent. Math.*, 140(1):1–43, 2000.

[5] Christophe Cornut. Filtrations and Buildings. To appear in Memoirs of the AMS.

[6] Christophe Cornut. A fixed point theorem in Euclidean buildings. *Adv. Geom.*, 16(4):487–496, 2016.

[7] Christophe Cornut and Marc-Hubert Nicole. Cristaux et immeubles. *Bull. Soc. Math. France*, 144(1):125–143, 2016.

[8] J.-F. Dat, S. Orlik, and M. Rapoport. *Period domains over finite and p-adic fields*, volume 183 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2010.

[9] Gerd Faltings. Mumford-Stabilität in der algebraischen Geometrie. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 648–655. Birkhäuser, Basel, 1995.

[10] L. Fargues. Théorie de la réduction pour les groupes p-divisibles, Preprint.

[11] Jean-Marc Fontaine and Guy Laffaille. Construction de représentations p-adiques. *Ann. Sci. École Norm. Sup. (4)*, 15(4):547–608 (1983), 1982.

[12] Jean-Marc Fontaine and Michael Rapoport. Existence de filtrations admissibles sur des isocristaux. *Bull. Soc. Math. France*, 133(1):73–86, 2005.

[13] Qëndrim R. Gashi. On a conjecture of Kottwitz and Rapoport. *Ann. Sci. Éc. Norm. Supér. (4)*, 43(6):1017–1038, 2010.

[14] Michael Kapovich. Generalized triangle inequalities and their applications. In *International Congress of Mathematicians. Vol. II*, pages 719–741. Eur. Math. Soc., Zürich, 2006.

[15] R. E. Kottwitz. Isocrystals with additional structure. *Compositio Math.*, 56(2):201–220, 1985.

[16] Robert E. Kottwitz. On the Hodge-Newton decomposition for split groups. *Int. Math. Res. Not.*, (26):1433–1447, 2003.

[17] Shrawan Kumar. A survey of the additive eigenvalue problem. *Transform. Groups*, 19(4):1051–1148, 2014. With an appendix by M. Kapovich.

[18] Guy Laffaille. Groupes p-divisibles et modules filtrés: le cas peu ramifié. *Bull. Soc. Math. France*, 108(2):187–206, 1980.

[19] E. Landvogt. Some functorial properties of the Bruhat-Tits building. *J. Reine Angew. Math.*, 518:213–241, 2000.

[20] M. Rapoport and M. Richartz. On the classification and specialization of F-isocrystals with additional structure. *Compositio Math.*, 103(2):153–181, 1996.

[21] M. Rapoport and Th. Zink. *Period spaces for p-divisible groups*, volume 141 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1996.

[22] M. Rapoport and Th. Zink. A finiteness theorem in the Bruhat-Tits building: an application of Landvogt's embedding theorem. *Indag. Math. (N.S.)*, 10(3):449–458, 1999.

[23] Jacques Tits. Reductive groups over local fields. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 29–69. Amer. Math. Soc., Providence, R.I., 1979.

[24] Burt Totaro. Tensor products in p-adic Hodge theory. *Duke Math. J.*, 83(1):79–104, 1996.

[25] I. Vollaard and T. Wedhorn. The supersingular locus of the Shimura variety of $\mathrm{GU}(1, n-1)$ II. *Inventiones Math.*, (184):591–627, 2011.

CNRS, Sorbonne Université, Université Paris Diderot, Institut de Mathématiques de Jussieu-Paris Rive Gauche, IMJ-PRG, F-75005, Paris, France.

E-mail address: `christophe.cornut@imj-prg.fr`

URL: `https://www.imj-prg.fr/~christophe.cornut/`