Stable limit theorems on the Poisson space

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Abstract

We prove limit theorems for functionals of a Poisson point process using the Malliavin calculus on the Poisson space. The target distribution is conditionally either a Gaussian vector or a Poisson random variable. The convergence is stable and our conditions are expressed in terms of the Malliavin operators. For conditionally Gaussian limits, we also obtain quantitative bounds, given for the Monge-Kantorovich transport distance in the univariate case; and for another probabilistic variational distance in higher dimension. Our work generalizes several limit theorems on the Poisson space, including the seminal works by Peccati, Solé, Taqqu & Utzet [32] for Gaussian approximations; and by Peccati [33] for Poisson approximations; as well as the recently established fourth-moment theorem on the Poisson space of Döbler & Peccati [8]. We give an application to stochastic processes.

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Introduction

One of the celebrated contributions of Rényi [40, 41] is a refinement of the notion of convergence in law, commonly referred to as *stable convergence*. Stable convergence is tailored for studying conditional limits of sequences of random variables. Thus, a stable limit is, typically, a *mixture*, that is, in our terminology: a random variable whose law depends on a random parameter; for instance, a centered Gaussian random variable with random variance, or a Poisson random variable with random mean. In the setting of semi-martingales, one book by Jacod & Shiryaev [15] summarizes archetypal stable convergence results involving such mixtures. More recently, results by Nourdin & Nualart [25]; Harnett & Nualart [13]; and Nourdin, Nualart & Peccati [26] give sufficient conditions and quantitative bounds for the stable convergence of functionals of an isonormal Gaussian process to a *Gaussian mixture*. Typically, applications of such results study the limit of a sequence of quadratic functionals of a fractional Brownian motion. The three references [25, 13, 26] make a pervasive use of the Malliavin calculus to prove such limit theorems. Earlier works by Nualart & Ortiz-Latorre [29] and by Nourdin & Peccati [28] initiate this approach: they use Malliavin calculus in order to prove central limit theorems for iterated Itô integrals initially obtained by Nualart & Peccati [31] with different tools. These far-reaching contributions form a milestone in the theory of limit theorems and inaugurate an independent field of research, known as the *Malliavin-Stein approach* (see the webpage of Nourdin [24] for a comprehensive list of contributions on the subject).

The trendsetting work of Peccati, Solé, Taqqu & Utzet [32] extends the Malliavin-Stein approach beyond the scope of Gaussian fields to Poisson point processes. Despite being a very

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active field of research, the considered limit distributions are, most of the time, Gaussian [18, 17, 20, 39, 34, 37, 42, 8, 9, 5] or, sometimes, Poisson [33] or Gamma [35]; to the best of our knowledge, prior to the present work, mixtures, have not been considered as limit distributions. The aim of this paper is to tackle this problem, by proving an array of new quantitative and stable limit theorems on the Poisson space, with a target distribution given either by a Gaussian mixture, that is the distribution of a centered Gaussian variable with random covariance; or a Poisson mixture, that is the distribution of a Poisson variable with random mean. We rely on two standard techniques to obtain our limit theorems: the *characteristic functional* method, to obtain qualitative results; and an interpolation approach, known as *smart path*, for the quantitative results. In the two cases, we build upon various tools from stochastic analysis for Poisson point processes, such as the Malliavin calculus, integration by parts for Poisson functionals, and a representation of the carré du champ associated to the generator of the Ornstein-Uhlenbeck semi-group on the Poisson space. Provided mild regularity assumptions on the functional under study, our approach allows us to deal, in Theorems 3.1 and 3.5, with any target distribution of the form SN, where S is a matrix-valued random variable (measurable with respect to the underlying Poisson point process) and N is a Gaussian vector independent of the underlying Poisson point process. In the same way, in Theorem 3.2, we can consider any target distribution of the form of a Poisson mixture, whose precise definition is given below.

Let us now give a more detailed sample of the main results. Throughout the paper, we study the asymptotic behaviour of a sequence $\{F_n = f_n(\eta)\}$ of square-integrable functionals of a Poisson point process η. Here, η is a Poisson point process on an arbitrary σ-finite measured space $(Z, \mathfrak{Z}, \nu)$ (for the moment, we simply recall that η is a random integer-valued measure on Z satisfying some strong independence properties and such that $\mathbb{E}\eta = \nu$). Moreover, we assume that the F_n's are of the form $F_n = \delta u_n$, where δ is the Kabanov stochastic integral and $u_n = \{u_n(z);\ z \in Z\}$ is a random function on Z (for the moment, one can think of the slightly abusive definition of δ as the following pathwise stochastic integral $\delta u = \int u(z)(\eta - \nu)(\mathrm{d}z)$). As we will see, assuming that $F_n = \delta u_n$ is not restrictive, as, provided $\mathbb{E}F_n = 0$, this equation always admits infinitely many solutions. An important object in our study is the *Malliavin derivative* of F_n given by $D_z F_n = f_n(\eta + \delta_z) - f_n(\eta)$. The crucial tool to establish our results is a duality relation (also referred to as integration by parts) between the operators D and δ: $\mathbb{E}F\delta u = \mathbb{E}\nu(uDF)$. This relation is at the heart of the Malliavin-Stein approach to obtain limit theorems both in a Gaussian [27, Chapter 5] and in a Poisson setting [32]. For instance, we have the following result in our Poisson setting.

Theorem 0.1 ([32, Theorem 3.1])**.** *Let the previous notation prevails, and assume that:*

$$\nu(u_n F_n) \xrightarrow[n\to\infty]{\mathscr{L}^1(\mathbb{P})} \sigma^2, \tag{0.1}$$

and

$$\mathbb{E}\int |u_n(z)||D_z^+ F_n|^2 \nu(\mathrm{d}z) \xrightarrow[n\to\infty]{} 0. \tag{0.2}$$

Then[1]*, we have that* $F_n \xrightarrow[n\to\infty]{law} \mathbf{N}(0, \sigma^2)$.

By integration by parts, we see that $\mathbb{E}\nu(u_n DF_n) = \mathbb{E}F_n^2$ and, at the heuristic level, the quantity $\nu(u_n DF_n)$ controls the asymptotic variance of F_n. The condition (0.2) arises from the non-diffusive nature of the Poisson process. Following our heuristic, it is very natural to ask what happens to the conclusions of Theorem 0.1 when $\nu(u_n DF_n)$ converges to a non-negative random variable S^2. Theorem 3.1 states that, in this case, provided (0.2) and a condition of asymptotic independence hold, (F_n) converges stably to the Gaussian mixture $\mathbf{N}(0, S^2)$. In fact, in

[1]To be precise, the theorem of [32] chooses one particular solution of $F_n = \delta u_n$ but we do not enter into too many technical details in this introduction.

Theorem 3.1, we are also able to deal with vector-valued random variables. In the same fashion, Theorem 3.2 gives sufficient conditions involving u_n and DF_n to ensure the convergence of (F_n) to a Poisson mixture (thus generalizing a result by Peccati [33] for convergence to Poisson random variables). When targeting Gaussian mixtures, we are also able to provide quantitative bounds in a variational distance between probability laws (Theorem 3.5 for the multivariate case, and Theorem 3.8 for the univariate case).

Following a recent contribution by Döbler & Peccati [8], we derive from our analysis a stable fourth moment theorem: a sequence of iterated Itô-Poisson integrals converges stably to a Gaussian (with deterministic variance) if and only if its second and fourth moment converge to those of Gaussian (Proposition 4.1). For the limit of a sequence of order 2 Itô-Poisson stochastic integrals to be a Gaussian or Poisson mixture, we obtain sufficient conditions expressed in terms of analytical conditions on the integrands (Theorems 4.2 and 4.3). We also apply our results to study the limit of a sequence of quadratic functionals of a rescaled Poisson process on the line (Theorem 5.2); hence, adapting to the Poisson setting a theorem of Peccati & Yor [36] for a standard Brownian motion (generalized by [26] to the setting of a sufficiently regular fractional Brownian motion using Malliavin-Stein techniques; and generalized to any fractional Brownian motion by [38] using ad-hoc computations).

The paper is organized as follows. Each section starts with its own short introduction that presents its structure and that recalls, if necessary, the context and the definition of the main objects under study. Section 1 fixes the notations for the rest of the paper; recalls the definitions of probabilistic distances and of the Poisson point process; and gives more information on Gaussian and Poisson mixtures that serve as target distributions in our limit theorems. We present, in Section 2, extended material about stochastic analysis for Poisson point processes with a focus on Poisson integrals, Malliavin operators, and Dirichlet forms. We prove several intermediary results of independent interest regarding stochastic analysis for Poisson point processes. In particular, Proposition 2.2 establishes a complete representation of the carré du champ operator on the Poisson space (generalizing the one of [8]). We present in Section 3 the main results of this paper: Theorems 3.1, 3.2 and 3.5; they contain bounds and stable limit theorems for Poisson functionals. In Section 3.2.2, we refine our results when the F_n's are univariate, and we establish, in Theorem 3.8, a bound in the Monge-Kantorovich transport distance. A detailed comparison of these results with the aforementioned works on the Gaussian space of [25, 13, 26], as well as with limit theorems on the Poisson space [18, 17, 32, 33], follows in Section 3.3. A special attention is paid to stochastic integrals in Section 4. From our main results, we deduce: Proposition 4.1, a stable version of the recently proved fourth moment theorem on the Poisson space of [8, 9]; Theorems 4.2 and 4.3, giving analytical criteria for conditionally normal or Poisson limit for order 2 Itô-Wiener stochastic integrals. Section 5 contains the application to quadratic functionals of rescaled Poisson processes on the line.

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Contents

1 Preliminaries

1.1 Notations

Sets

The symbols $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}$, and $\mathbb{N}_{>0}$ always designate the set of real numbers, of non-negative real numbers, of non-negative integers, of and positive integers respectively. For p and $q \in \mathbb{N}_{>0}$ with $p \leq q$, we write: $[q] = \{1, \ldots, q\}$, $[p, q] = \{p, \ldots, q\}$, and $[0] = \emptyset$.

Norms

For $x, y \in \mathbb{R}^d$, we write $\langle x, y\rangle_{\ell^2}$ for the standard scalar product of x and y, and $|x|_{\ell^2}$ for the induced norm. We call *p-tensor* every p-linear form $T\colon (\mathbb{R}^d)^p \to \mathbb{R}$. Recall that a tensor is canonically identified with an element of $(\mathbb{R}^d)^p$ whose coordinate $(i_1, \ldots, i_p) \in [d]^p$ is given by $T(e_1, \ldots, e_p)$, where $\{e_i;\ i \in [d]\}$ is the canonical basis of $\mathbb{R}^d$. 1-tensors are vectors via the identification $x \mapsto \langle x, \cdot\rangle_{\ell^2}$; 2-tensors are square matrices via the identification $A \mapsto \langle \cdot, A\cdot\rangle_{\ell^2}$. Consistently with the notation introduced before, we use $\langle \cdot, \cdot\rangle_{\ell^2}$ for tensors. In particular, given two matrices A and B of size $d \times d$, we write $\langle A, B\rangle_{\ell^2}$ for $\mathrm{tr}(A^T B)$, and $|A|^2_{\ell^2}$ for $\langle A, A\rangle_{\ell^2}$.

Derivatives

For $k \in \mathbb{N} \cup \{\infty\}$, we denote the space of k times continuously differentiable functions from $\mathbb{R}^d$ to $\mathbb{R}$ by $\mathscr{C}^k(\mathbb{R}^d)$. If, moreover, the derivatives up to k are bounded, we write $\mathscr{C}_b^k(\mathbb{R}^d)$. For $\phi \in \mathscr{C}^k(\mathbb{R}^d)$, we write $\nabla^k \phi$ for the k-th derivative that we identify with a k-form over $\mathbb{R}^d$. In particular, for all $x \in \mathbb{R}^d$, $\nabla^k \phi(x)$ is a k-tensor whose coordinates (in the canonical basis) are written $\{\partial^k_{i_1,\dots,i_k}\phi(x),\, i_1, \dots, i_k \in [d]\}$. We write $\nabla = \nabla^1$. We let

$$|\nabla^k \phi|_{\ell^2,\infty} = \sup_{x\in\mathbb{R}^d} |\nabla^k \phi(x)|_{\ell^2} = \sup_{x\in\mathbb{R}^d} \left(\sum_{i_1,\dots,i_k \le d} \left|\partial^k_{i_1,\dots,i_k}\phi(x)\right|^2 \right)^{1/2}. \tag{1.1}$$

Lipschitz functions

We say that a function $\phi\colon \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ is *Lipschitz* if

$$Lip(\phi) := \sup_{x,y} \frac{|\phi(x) - \phi(y)|_{\ell^2}}{|x-y|_{\ell^2}} < \infty. \tag{1.2}$$

Recall that if ϕ is differentiable,

$$Lip(\phi) = \sup_{x\in\mathbb{R}^{d_1}} \frac{|\phi'(x)|_{\ell^2}}{\sqrt{d_1}}. \tag{1.3}$$

The space of Lipschitz functions from $\mathbb{R}^d$ to $\mathbb{R}$ is denoted by $Lip(\mathbb{R}^d)$. The space of bounded Lipschitz functions from $\mathbb{R}^d$ to $\mathbb{R}$ is denoted by $\mathscr{W}^{1,\infty}(\mathbb{R}^d)$; it is a Banach space for the norm

$$|\phi|_{\mathscr{W}^{1,\infty}(\mathbb{R}^d)} = |\phi|_\infty + Lip(\phi), \tag{1.4}$$

where $|\phi|_\infty$ is the supremum norm of ϕ.

Lebesgue spaces

Let Z be a measurable space with its σ-algebra $\mathfrak{Z}$. Given a measure ν and a non-negative (or ν-integrable) function f, we write $\nu(f)$ or $\int_Z f(x)\nu(\mathrm{d}x)$ to designate the Lebesgue integral of f with respect to ν. For $p \in [1,\infty]$, the space of (equivalence classes of ν-almost everywhere equal) measurable functions f such that $\nu(|f|^p) < \infty$ (or $esssup(|f|) < \infty$, when $p = \infty$) is denoted by $\mathscr{L}^p(Z, \mathfrak{Z}, \nu)$ and is equipped with its standard Banach structure. We commonly abbreviate this notation to $\mathscr{L}^p(Z)$ or $\mathscr{L}^p(\mathfrak{Z})$ or $\mathscr{L}^p(\nu)$. Unless otherwise specified, identities between elements of some $\mathscr{L}^p(\nu)$ are always understood in an ν-almost everywhere sense. We simply write ν^q for the q-th tensor power of ν.

Symmetric functions

For $f \in \mathscr{L}^p(\nu^q)$, we denote by f_σ the symmetrization of f, that is:

$$f_\sigma(x_1, \dots, x_q) = q!^{-1} \sum_{\sigma\in\Sigma_q} f(x_{\sigma(1)}, \dots, x_{\sigma(q)}), \tag{1.5}$$

where Σ_q is the group of permutations of $[q]$. We say that $f \in \mathscr{L}^p_\sigma(\nu^q)$ if $f \in \mathscr{L}^p(\nu^q)$ and $f_\sigma = f$.

Probability space

Every random element lives in a sufficiently big probability space $(\Omega, \mathfrak{O}, \mathbb{P})$. Unless otherwise specified, equality between random objects is always understood in an almost sure sense. By convention, we reserve the term *random variable* to designate a random object with value in $\mathbb{R}^d$.

1.2 Probabilistic approximations and limit theorems

Stable convergence

(See [15, § VIII.5c].) Let $\mathfrak{W}$ be a sub-σ-algebra of $\mathfrak{O}$. A sequence of $\mathfrak{W}$-random variables (F_n) is said to *converge stably* to a $\mathfrak{O}$-random variable F_∞ whenever, for all $Z \in \mathscr{L}^\infty(\mathfrak{W})$:

$$(F_n, Z) \xrightarrow[n\to\infty]{law} (F_\infty, Z). \tag{1.6}$$

This convergence is denoted by

$$F_n \xrightarrow[n\to\infty]{stably} F_\infty. \tag{1.7}$$

Of course, stable convergence implies convergence in law but the reverse implication does not hold. In practice, we use the following characterisation of stable convergence.

Proposition 1.1. *Let (F_n) be a sequence of $\mathfrak{W}$-measurable random variables, and F_∞ be $\mathfrak{O}$-measurable. Let $\mathscr{I} \subset \mathscr{L}^1(\mathfrak{W})$ be a linear space, and $\mathscr{G} \subset \mathscr{L}^\infty(\mathfrak{W})$. Assume that $\sigma(\mathscr{I}) = \sigma(\mathscr{G}) = \mathfrak{W}$. The following are equivalent:*

(i) $F_n \xrightarrow[n\to\infty]{stably} F_\infty$;

(ii) for all $\phi \in \mathscr{C}_b(\mathbb{R}^d)$: $\phi(F_n) \xrightarrow[n\to\infty]{\sigma(\mathscr{L}^1(\mathfrak{W}):\mathscr{L}^\infty(\mathfrak{W}))} \mathbb{E}[\phi(F_\infty)|\mathfrak{W}]$;

(iii) for all $G \in \mathscr{G}$ *and for all* $\lambda \in \mathbb{R}^d$: $\mathbb{E}\,\mathrm{e}^{i\langle\lambda,F_n\rangle_{\ell^2}}\,G \xrightarrow[n\to\infty]{} \mathbb{E}\,\mathrm{e}^{\langle\lambda,F_\infty\rangle_{\ell^2}}\,G$;

(iv) for all $I \in \mathscr{I}^d$ *and for all* $\lambda \in \mathbb{R}^d$: $\mathbb{E}\,\mathrm{e}^{i\langle\lambda,F_n+I\rangle_{\ell^2}} \xrightarrow[n\to\infty]{} \mathbb{E}\,\mathrm{e}^{i\langle\lambda,F_\infty+I\rangle_{\ell^2}}$.

Proof. Stable convergence is equivalent to (ii) by [15, Proposition VIII.5.33.v]. Thus, stable convergence is also equivalent with (iii) since $\mathscr{G}$ generates $\mathfrak{W}$. By linearity of $\mathscr{I}$, (iv) implies that for all $J \in \mathscr{I}$, all $t \in \mathbb{R}$, and all $\lambda \in \mathbb{R}^d$, as $n \to \infty$: $\mathbb{E}\,\mathrm{e}^{itJ}\,\mathrm{e}^{i\langle\lambda,F_n\rangle_{\ell^2}} \to \mathbb{E}\,\mathrm{e}^{itJ}\,\mathrm{e}^{i\langle\lambda,F_\infty\rangle_{\ell^2}}$. Letting $t \to 0$ in $(1-\mathrm{e}^{itJ})t^{-1} \to iJ$, shows that $\mathbb{E}J\,\mathrm{e}^{i\langle\lambda,F_n\rangle_{\ell^2}} \to \mathbb{E}J\,\mathrm{e}^{i\langle\lambda,F_\infty\rangle_{\ell^2}}$, when $n \to \infty$. Since $\mathscr{I}$ generates $\mathfrak{W}$, we conclude that (iv) implies stable convergence. The converse implication is immediate. □

Probabilistic variational distances

The *Monge-Kantorovich distance* between two $\mathbb{R}^d$ random variables X and Y is defined by

$$d_1(X,Y) = \inf\left\{\mathbb{E}|\tilde{X}-\tilde{Y}|_{\ell^2},\ \tilde{X}\sim X,\ \tilde{Y}\sim Y\right\}. \tag{1.8}$$

Due to the Kantorovich duality, the Monge-Kantorovich distance (see [12, Theorem 2.1]) between the laws of two integrable $\mathbb{R}^d$-valued random variables X and Y can be rewritten:

$$d_1(X,Y) = \sup\left\{\mathbb{E}\phi(X)-\mathbb{E}\phi(Y),\ \phi\in Lip(\mathbb{R}^d),\ Lip(\phi)\le 1\right\}. \tag{1.9}$$

In the same spirit, the *Fortet-Mourier distance* between the laws of two $\mathbb{R}^d$-valued random variables X and Y is

$$d_0(X,Y) = \sup\left\{\mathbb{E}\phi(X)-\mathbb{E}\phi(Y),\ \phi\in\mathscr{W}^{1,\infty}(\mathbb{R}^d),\ |\phi|_{\mathscr{W}^{1,\infty}(\mathbb{R}^d)}\le 1\right\}. \tag{1.10}$$

For Poisson functionals, the Malliavin-Stein methods typically yield bounds in a distance weaker than the Monge-Kantorovich or the Fortet-Mourier distance. In this paper, we consider a distance first considered by Peccati & Zheng [37], whose variational formulation for two integrable $\mathbb{R}^d$-valued random variables X and Y is given by

$$d_3(X, Y) = \sup\left\{\mathbb{E}\phi(X) - \mathbb{E}\phi(Y),\ \phi \in \mathscr{C}^3(\mathbb{R}^d),\ \phi \in \mathscr{F}_3\right\}, \tag{1.11}$$

where $\mathscr{F}_3$ is the space of functions $\phi \in \mathscr{C}^3(\mathbb{R}^d)$ with the second and third derivatives bounded by 1.

Link with the convergence in law

These three distances depend on X and Y only through their laws. If $Y \sim \nu$, we sometimes write $d_i(X, \nu)$ for $d_i(X, Y)$ ($i \in \{0, 1, 3\}$). The Monge-Kantorovich distance induces a topology on the space of probability measures that corresponds to the convergence in law together with the convergence of the first moment [44, Theorem 6.9]. The Peccati-Zheng distance induces a topology on the space of probability measures which is strictly stronger than the topology of the convergence in law. The Fortet-Mourier distance induces, on the space of probability measures, the topology of the convergence in law [10, Theorem 11.3.3].

1.3 Definition of Poisson point processes

Given some measurable space $(Z, \mathfrak{Z})$, we define $\mathscr{M}_{\bar{\mathbb{N}}}(Z)$ to be the space of all countable sums of $\mathbb{N}$-valued measures on $(Z, \mathfrak{Z})$. The space $\mathscr{M}_{\bar{\mathbb{N}}}(Z)$ is endowed with the σ-algebra $\mathfrak{M}_{\bar{\mathbb{N}}}(Z)$, generated by the *cylindrical mappings*

$$\xi \in \mathscr{M}_{\bar{\mathbb{N}}}(Z) \mapsto \xi(B) \in \mathbb{N} \cup \{\infty\}, \quad B \in \mathfrak{Z}. \tag{1.12}$$

Let ν be a σ-finite measure on $(Z, \mathfrak{Z})$. A random variable $\eta = \eta_\nu$ with values in $\mathscr{M}_{\bar{\mathbb{N}}}(Z)$ is a *Poisson point process* (or *Poisson random measure*) with intensity ν if the following two properties are satisfied:

1. for all $B_1, \ldots, B_n \in \mathfrak{Z}$ pairwise disjoint, $\eta(B_1), \ldots, \eta(B_n)$ are independent;
2. for $B \in \mathfrak{Z}$ with $\nu(B) < \infty$, $\eta(B)$ is a Poisson random variable with mean $\nu(B)$.

Poisson processes with σ-finite intensity exist [21, Theorem 3.6]. We let $\mathfrak{W}$ be the σ-algebra generated by η. Our definition of η implies that $\mathfrak{W} \subset \mathfrak{O}$, and we often tacitly assume that $(\Omega, \mathfrak{O}, \mathbb{P})$ also supports random objects (such as a Brownian motion) independent of η. We always look at stable convergence with respect to $\mathfrak{W}$. However, for simplicity, unless otherwise specified, we assume that random variables are $\mathfrak{W}$-measurable. In particular, we write $\mathscr{L}^2(\mathbb{P})$ for $\mathscr{L}^2(\Omega, \mathfrak{W}, \mathbb{P})$. The *compensated Poisson measure* is defined as the mapping

$$A \mapsto \hat{\eta}(A) = \eta(A) - \nu(A),\ \forall A \in \mathfrak{Z},\ \text{such that}\ \nu(A) < \infty. \tag{1.13}$$

1.4 Gaussian and Poisson mixtures

As anticipated, we shall be interest in the stable convergence (with respect to $\mathfrak{W}$) of a sequence of Poisson functionals (F_n) to conditionally Gaussian and Poisson random variables. Informally, we refer to such objects as *Gaussian mixture* and *Poisson mixture*. Let N be standard Gaussian vector independent of η and $S \in \mathscr{L}^2(\mathfrak{W})$. We denote by $\mathbf{N}(0, S^2)$ the law of the Gaussian mixture SN. Similarly, for N a Poisson process on $\mathbb{R}_+$ (with intensity the Lebesgue measure) independent of η and $M \in \mathscr{L}^2(\mathfrak{W})$ non-negative, we write $\mathbf{Po}(M)$ for the law of the (compensated) Poisson mixture $N(1_{[0,M]}) - M$. We have a characterisation of these two laws in term of their conditional Fourier transforms: $F \sim \mathbf{N}(0, S^2)$ if and only if

$$\mathbb{E}[\mathrm{e}^{i\lambda F} | \eta] = \exp\left(-S^2 \frac{\lambda^2}{2}\right); \tag{1.14}$$

while $F \sim \mathbf{Po}(M)$ if and only if

$$\mathbb{E}[\mathrm{e}^{i\lambda F} | \eta] = \exp\left(M(\mathrm{e}^{i\lambda} - i\lambda - 1)\right). \tag{1.15}$$

2 Further results on the Poisson space

Outline

In this section, we recall basic definitions regarding Poisson point process on an arbitrary measured space. We then carry out an extensive review of different tools about stochastic analysis on the Poisson space: the Itô-Poisson stochastic integrals; the Malliavin derivative; the Skorokhod-Kabanov divergence; the Ornstein-Uhlenbeck generator; the Dirichlet form on the Poisson space; and the energy bracket, a object that we invented. We establish several lemmas in the process. We end the section by deriving some rules of calculus for the Malliavin derivative and Kabanov-Skorokhod divergence. Using the formalism of the Dirichlet form and the energy bracket, we can derive some integration by parts formula on the Poisson space used in Section 3. In particular, Proposition 2.2 gives a complete description of the carré du champ on the Poisson space. The reader can refer to the three monographs by Kingman [16]; Peccati & Reitzner [34]; and Last & Penrose [21] for more information about Poisson point processes.

2.1 Stochastic analysis for Poisson point processes

The Mecke formula

According to [21, Theorem 4.1], we have for all measurable $f : \mathscr{M}_{\bar{\mathbb{N}}}(Z) \times Z \to [0, \infty]$:

$$\mathbb{E} \int f(\eta, z)\eta(\mathrm{d}z) = \int \mathbb{E} f(\eta + \delta_z, z)\nu(\mathrm{d}z). \tag{2.1}$$

If f is replaced by a measurable function with value in $\mathbb{R}$ the previous formula still holds provided both sides of the identity are finite when we replace f by $|f|$.

Proper measures and their factorial power

A measure $m \in \mathscr{M}_{\bar{\mathbb{N}}}(Z)$ is *proper* whenever there exists $I \subseteq \mathbb{N}$ and $z_i \in Z$ ($i \in I$) such that $m = \sum_{i\in I} \delta_{z_i}$. As we are only concerned with properties in law of η, according to [21, Corollary 3.7], without loss of generality, we assume that our Poisson point process η is almost surely proper. Given $I \subseteq \mathbb{N}$, we define its *factorial power* of order $q \in \mathbb{N}$, noted $I^{(q)}$, by

$$I^{(q)} = \{(i_1, \ldots, i_q) \in I^q, \text{ such that } i_l \neq i_{l'}, \forall l \neq l' \in [q]\}. \tag{2.2}$$

Note that $I^{(q)} = \emptyset$, for all $q > |I|$. Given $m = \sum_{i\in I} \delta_{x_i}$ a proper element of $\mathscr{M}_{\bar{\mathbb{N}}}(Z)$, we define the *q-th factorial power* of m, noted $m^{(q)}$, as

$$m^{(q)} = \sum_{(i_1,\ldots,i_q)\in I^{(q)}} \delta_{(x_{i_1},\ldots,x_{i_q})}. \tag{2.3}$$

In the previous formula and in the rest of this article, summation over the empty set is understood as the zero measure. According to [21, Proposition 4.3], $\eta^{(q)}$ is also a random variable (with respect to $\mathfrak{W}$).

Stochastic integrals

In a seminal contribution, Itô [14] introduces the stochastic integrals with respect to a Poisson measure (see also [43]). We follow here the presentation of [21, Chapter 12]. For $q \in \mathbb{N}$ and a function $f \in \mathscr{L}^1(\nu^q)$, the *multiple Wiener-Itô Poisson stochastic integral* of order q, or, for short, *Poisson integral* of order q, noted $I_q(f)$, is defined pointwise as

$$I_q(f) = \sum_{J \subset [q]} (-1)^{q-|J|} \int f(x) \eta^{(|J|)}(\mathrm{d}x_J) \nu^{q-|J|}(\mathrm{d}x_{[q]\setminus J}). \tag{2.4}$$

Here, x_J is the ordered vector of $\mathbb{R}^{|J|} \subset \mathbb{R}^d$ given by $(x_j)_{j\in J}$, where the order is inherited from $[q]$. We extend the mapping I_q (restricted to $\mathscr{L}^2(\nu^q) \cap \mathscr{L}^1(\nu^q)$) to a mapping (still denoted by I_q) on $\mathscr{L}^2(\nu^q)$ and (2.4) holds on a dense subset of $\mathscr{L}^2(\nu^q)$. Whenever $f \in \mathscr{L}^2(\nu^q)$ and $g \in \mathscr{L}^2(\nu^{q'})$, we have the following isometry property

$$\mathbb{E} I_q(f) I_{q'}(g) = q! \langle f_\sigma, g_\sigma \rangle_{\mathscr{L}^2(\nu^q)} 1_{q=q'}, \tag{2.5}$$

where f_σ is the symmetrized version of f.

Product formulae and contractions

Given $f \in \mathscr{L}^2_\sigma(\nu^p)$ and $g \in \mathscr{L}^2_\sigma(\nu^q)$ such that $I_p(f)$ and $I_q(g)$ belong to $\mathscr{L}^4(\mathbb{P})$, then by [6, Lemma 2.4] there exists $h_r \in \mathscr{L}^2_\sigma(\nu^r)$ such that

$$I_p(f) I_q(g) = \sum_{r=0}^{p+q} I_r(h_r). \tag{2.6}$$

For $f \in \mathscr{L}^2_\sigma(\nu^p)$ and $g \in \mathscr{L}^2_\sigma(\nu^q)$, we define the *star contraction* of order (l, r), $r \in \{0, \ldots, p \wedge q\}$ and $l \in \{0, \ldots, r\}$ by

$$f \star_r^l g(x_1, \ldots, x_{p+q-r-l}) = \int f(y_{[l]}, x_{[p-l]}) g(y_{[l]}, x_{[r-l]}, x_{[p-l+1, p+q-r-l]}) \nu^l(\mathrm{d}y_{[l]}). \tag{2.7}$$

Then ([19, Proposition 5]) for $f \in \mathscr{L}^2_\sigma(\nu^p)$ and $g \in \mathscr{L}^2_\sigma(\nu^q)$ such that $f \star_r^l g \in \mathscr{L}^2(\nu^{p+q-r-l})$,

$$I_p(f) I_q(g) = \sum_{r=0}^{p\wedge q} \sum_{l=0}^{r} r! \binom{p}{r} \binom{q}{r} \binom{r}{l} I_{p+q-r-l}(f \star_r^l g). \tag{2.8}$$

The representative of a functional

For every random variable F measurable with respect to η we can write $F = f(\eta)$, for some measurable $f \colon \mathscr{M}_{\bar{\mathbb{N}}}(Z) \to \mathbb{R}$ uniquely defined $\mathbb{P} \circ \eta^{-1}$-almost surely on $(\mathscr{M}_{\bar{\mathbb{N}}}(Z), \mathfrak{M}_{\bar{\mathbb{N}}}(Z))$. We call such f a *representative* of F. In this section, F denotes a random variable, measurable with respect to $\sigma(\eta)$, and f denotes one of its representatives.

The add and drop operators

Given $z \in Z$, we let

$$D_z^+ F = f(\eta + \delta_z) - f(\eta); \tag{2.9}$$

$$D_z^- F = (f(\eta) - f(\eta - \delta_z)) 1_{z\in\eta}. \tag{2.10}$$

The operator D^+ (resp. D^-) is called the *add operator* (resp. *drop operator*). Due to the Mecke formula (2.1), these operations are well-defined on random variables (that is, D^+ and D^- do not depend on the choice of the representative of F).

Lemma 2.1. *Let $F \in \mathscr{L}^\infty(\mathbb{P})$, then $D^+F \in \mathscr{L}^\infty(\mathbb{P} \otimes \nu)$.*

Proof. First of all, $\delta \colon Z \ni z \mapsto \delta_z \in \mathscr{M}_{\bar{\mathbb{N}}}(Z)$ is measurable (if A is of the form $\{\eta(B) = k\}$ for some $B \in \mathfrak{Z}$, then the pre-image by δ of A is B, if $k = 1$; and the pre-image is empty, if $k > 1$). Hence, D^+F is bi-measurable. Now let

$$U = \{t \in \mathbb{R}, \text{ such that } \mathbb{P}(F \geq t) = 0\}; \tag{2.11}$$

$$V = \{t \in \mathbb{R}, \text{ such that } (\mathbb{P} \otimes \nu)(F + D_z^+F \geq t) = 0\}. \tag{2.12}$$

By assumption $U \neq \emptyset$, and we want to show that $V \neq \emptyset$. Take $t \in U$, by the Mecke formula (2.1), we have that

$$\mathbb{E} \int 1_{\{F+D_z^+F \geq t\}} \nu(\mathrm{d}z) = \mathbb{E} \int 1_{\{F \geq t\}} \eta(\mathrm{d}z) = 0. \tag{2.13}$$

Hence $t \in V$, this concludes the proof. □

Based on (2.4), it is easy to check (see also [19, Theorem 3]) that, for all $h \in \mathscr{L}^2_\sigma(\nu^q)$ and $z \in Z$: $D_z^+ I_q(h) = q I_{q-1}(h(z, \cdot))$.

The Itô-Poisson isometry

In [22, Theorem 1.3], it is proved that, for $F \in \mathscr{L}^2(\mathbb{P})$, the mapping $T_q F \colon Z^q \ni (z_1, \dots, z_q) \mapsto \mathbb{E} D^+_{z_1} \dots D^+_{z_q} F$ belongs to $\mathscr{L}^2(\nu^q)$, and that

$$F = \sum_{q \in \mathbb{N}} \frac{1}{q!} I_q(T_q F). \tag{2.14}$$

Together with Itô's isometry (2.5), this implies the isometric orthogonal decomposition

$$\mathscr{L}^2(\mathbb{P}) \simeq \bigoplus_{q \in \mathbb{N}} \mathscr{L}^2_\sigma(\nu^q). \tag{2.15}$$

Malliavin derivative

For a random variable F, we write $F \in \mathscr{D}\mathrm{om}\, D^q$ whenever: $F \in \mathscr{L}^2(\mathbb{P})$ and

$$|F|_p := \int_{Z^p} \mathbb{E}(D^+_{z_1 \dots z_p} F)^2 \nu^p(\mathrm{d}z) < \infty, \quad \forall p \in [q]. \tag{2.16}$$

In view of what precedes, we have that $F \in \mathscr{D}\mathrm{om}\, D$ if and only if

$$\sum_{q \in \mathbb{N}} q I_{q-1}(T_q F) \in \mathscr{L}^2(\mathbb{P} \otimes \nu). \tag{2.17}$$

The space $\mathscr{D}\mathrm{om}\, D^q$ is Hilbert when endowed with the norm

$$|\cdot|_{\mathscr{D}\mathrm{om}\, D^q} = |\cdot|_{\mathscr{L}^2(\mathbb{P})} + \sum_{i=1}^p |\cdot|_p. \tag{2.18}$$

Given $F \in \mathscr{D}\mathrm{om}\, D^q$, we write $D^q F$ to denote the random mapping $D^q F \colon Z^q \ni (z_1, \dots, z_q) \mapsto D^+_{z_1} \dots D^+_{z_q} F$. We set $D = D^1$ for simplicity. We regard D^q as an unbounded operator $\mathscr{L}^2(\mathbb{P}) \to \mathscr{L}^2(\mathbb{P} \otimes \nu^q)$ with domain $\mathscr{D}\mathrm{om}\, D^q$.

The divergence operator

We consider the *divergence operator* $\delta = D^* \colon \mathscr{L}^2(\mathbb{P}\otimes\nu) \to \mathscr{L}^2(\nu)$, that is the unbounded adjoint of D. Its domain $\mathscr{D}\mathrm{om}\,\delta$ is composed of random functions $u \in \mathscr{L}^2(\mathbb{P}\otimes\nu)$ such that there exists a constant $c > 0$ such that

$$\left|\mathbb{E}\int D_z^+ F u(z)\nu(\mathrm{d}z)\right| \leq c\sqrt{\mathbb{E}F^2}, \quad \forall F \in \mathscr{D}\mathrm{om}\, D. \tag{2.19}$$

For $u \in \mathscr{D}\mathrm{om}\,\delta$, the quantity $\delta u \in \mathscr{L}^2(\mathbb{P})$ is completely characterised by the duality relation

$$\mathbb{E}G\delta u = \mathbb{E}\int u(z) D_z F \nu(\mathrm{d}z), \quad \forall F \in \mathscr{D}\mathrm{om}\, D. \tag{2.20}$$

If $h \in \mathscr{L}^2(\nu)$, then $h \in \mathscr{D}\mathrm{om}\,\delta$ and $\delta h = I_1(h)$. From [19, Theorem 5], we have following Skorokhod isometry. For $u \in \mathscr{L}^2(\mathbb{P}\otimes\nu)$, $u \in \mathscr{D}\mathrm{om}\,\delta$ if and only if $\mathbb{E}\int (D_z^+ u(z'))^2 \nu(\mathrm{d}z)\nu(\mathrm{d}z') < \infty$ and, in that case:

$$\mathbb{E}(\delta u)^2 = \mathbb{E}\int u(z)^2\nu(\mathrm{d}z) + \mathbb{E}\int D_z^+ u(z') D_{z'}^+ u(z)\nu(\mathrm{d}z)\nu(\mathrm{d}z'). \tag{2.21}$$

The Skorokhod isometry implies the following Heisenberg commutation relation. For all $u \in \mathscr{D}\mathrm{om}\,\delta$, and all $z \in Z$ such that $z' \mapsto D_z^+ u(z') \in \mathscr{D}\mathrm{om}\,\delta$:

$$D_z \delta u = u(z) + \delta D_z^+ u. \tag{2.22}$$

From [19, Theorem 6], we have the following pathwise representation of the divergence: if $u \in \mathscr{D}\mathrm{om}\,\delta \cap \mathscr{L}^1(\mathbb{P}\otimes\nu)$, then

$$\delta u = \int (1 - D_z^-) u(z)\eta(\mathrm{d}z) - \int u(z)\nu(\mathrm{d}z). \tag{2.23}$$

Note that $\mathscr{D}\mathrm{om}\,\delta \cap \mathscr{L}^1(\mathbb{P}\otimes\nu)$ is dense in $\mathscr{D}\mathrm{om}\,\delta$.

The Ornstein-Uhlenbeck generator

The *Ornstein-Uhlenbeck generator* L is the unbounded self-adjoint operator on $\mathscr{L}^2(\mathbb{P})$ verifying

$$\mathscr{D}\mathrm{om}\, L = \{F \in \mathscr{D}\mathrm{om}\, D, \text{ such that } DF \in \mathscr{D}\mathrm{om}\,\delta\} \quad \text{and} \quad L = -\delta D. \tag{2.24}$$

Classically, $\mathscr{D}\mathrm{om}\, L$ is endowed with the Hilbert norm $\mathbb{E}F^2 + \mathbb{E}(LF)^2$. The eigenvalues of L are the non-positive integers and for $q \in \mathbb{N}$ the eigenvectors associated to $-q$ are exactly the random variables of the form $I_q(h)$ for some $h \in \mathscr{L}^2(\nu^q)$. This yields for $F \in \mathscr{D}\mathrm{om}\, L$:

$$LF = -\sum_{q\in\mathbb{N}} \frac{q}{q!} I_q(\mathbb{E}D^q F) = -\sum_{q\in\mathbb{N}} \frac{1}{(q-1)!} I_q(\mathbb{E}D^q F). \tag{2.25}$$

The kernel of L coincides with the set of constants and the *pseudo-inverse of* L is defined on the quotient $\mathscr{L}^2(\mathbb{P}) \setminus \ker L$, that is the space of centered square integrable random variables. For such F, we have that

$$L^{-1}F = -\sum_{q\in\mathbb{N}_{>0}} \frac{1}{q!q} I_q(\mathbb{E}D^q F). \tag{2.26}$$

For $F \in \mathscr{L}^2(\mathbb{P})$ with $\mathbb{E}F = 0$, we have $LL^{-1}F = F$. Moreover, if $F \in \mathscr{D}\mathrm{om}\, L$, we have $L^{-1}LF = F$. As a consequence of (2.21), $\mathscr{D}\mathrm{om}\, D^2 = \mathscr{D}\mathrm{om}\, L$. In particular, if F has a vanishing expectation, then $L^{-1}F \in \mathscr{D}\mathrm{om}\, D^2$.

The Dirichlet form

We refer to [3, Chapter 1] for more details about the formalism of Dirichlet forms. The introduction of [1] also provides an overview of the subject. For every $F, G \in \mathscr{D}\mathrm{om}\, D$, we let $\mathcal{E}(F,G) = \mathbb{E} \int D_z^+ F D_z^+ G \nu(\mathrm{d}z)$. Since by [19, Lemma 3], the operator D is closed, $\mathcal{E}$ is a Dirichlet form with domain $\mathscr{D}\mathrm{om}\, \mathcal{E} = \mathscr{D}\mathrm{om}\, D$. Moreover, in view of the integration by parts (2.20), the generator of $\mathcal{E}$ is given by L. By [3, Chapter I Section 3], $\mathscr{A} := \mathscr{D}\mathrm{om}\, D \cap \mathscr{L}^\infty(\mathbb{P})$ is an algebra with respect to the pointwise multiplication; $\mathscr{D}\mathrm{om}\, D$ and $\mathscr{A}$ are stable by composition with Lipschitz functions; $\mathscr{A}$ is stable by composition with $\mathscr{C}^k(\mathbb{R}^d)$ functions ($k \in \bar{\mathbb{N}}$).

The carré du champ operator

For every $F \in \mathscr{A}$, we define the *functional carré du champ* of F as the linear form $\mathbf{\Gamma}(F)$ on $\mathscr{A}$, defined by

$$\mathbf{\Gamma}(F)[\Phi] = \mathcal{E}(F, F\Phi) - \frac{1}{2}\mathcal{E}(F^2, \Phi), \quad \text{for all } \Phi \in \mathscr{A}. \tag{2.27}$$

From [3, Proposition I.4.1.1],

$$0 \leq \mathbf{\Gamma}(F)[\Phi] \leq |\Phi|_{\mathscr{L}^\infty(\mathbb{P})} \mathcal{E}(F), \quad \text{for all } F, \Phi \in \mathscr{A}. \tag{2.28}$$

This allows us to extend the definition of the linear form $\mathbf{\Gamma}(F)$ to all $F \in \mathscr{D}\mathrm{om}\, \mathcal{E}$. For $F \in \mathscr{D}\mathrm{om}\, \mathcal{E}$, we write that $F \in \mathscr{D}\mathrm{om}\, \Gamma$ if the linear form $\mathbf{\Gamma}(F)$ can be represented by a measure absolutely continuous with respect to $\mathbb{P}$; in that case we denote its density by $\Gamma(F)$. In other words, $F \in \mathscr{D}\mathrm{om}\, \Gamma$ if and only if there exists a non-negative $\Gamma(F) \in \mathscr{L}^1(\mathbb{P})$ such that

$$\mathbf{\Gamma}(F)[\Phi] = \mathbb{E}\Gamma(F)\Phi, \quad \text{for all } \Phi \in \mathscr{A}. \tag{2.29}$$

From the general theory, we know that $\mathscr{D}\mathrm{om}\, \Gamma$ is a closed sub-linear space of $\mathscr{D}\mathrm{om}\, \mathcal{E}$. In the Poisson case, the following representation of the carré du champ follows from Lemma 2.7.

Proposition 2.2. *We have that $\mathscr{D}\mathrm{om}\, \Gamma = \mathscr{D}\mathrm{om}\, D$ and, for all, $F \in \mathscr{D}\mathrm{om}\, D$:*

$$\Gamma(F) = \frac{1}{2}\int (D_z^+ F)^2 \nu(\mathrm{d}z) + \frac{1}{2}\int (D_z^- F)^2 \eta(\mathrm{d}z). \tag{2.30}$$

We extend Γ to a bilinear map

$$\Gamma(F,G) = \frac{1}{2}\int D_z^+ F D_z^+ G \nu(\mathrm{d}z) + \frac{1}{2}\int D_z^+ F D_z^+ G \eta(\mathrm{d}z), \quad \forall F, G \in \mathscr{D}\mathrm{om}\, D. \tag{2.31}$$

Remark 1. This representation of Γ using the add-one and drop-one operators is, at the formal level, well-known in the literature: it appears without a proof in the seminal paper [2, p. 191]. One of the main assumption of [2] is the existence of an algebra of functions contained in $\mathscr{D}\mathrm{om}\, L$, the so called *standard algebra*. In the case of a Poisson point process, it is not clear what to choose for the standard algebra (note that $\mathscr{A} = \mathscr{D}\mathrm{om}\, \mathcal{E} \cap \mathscr{L}^\infty(\mathbb{P})$ is *not* included in $\mathscr{D}\mathrm{om}\, L$). [8] derives the formula without relying on the notion of standard algebra. However, since [8] follows the strategy of [2], [8] has to assume a restrictive assumption on F: $F \in \mathscr{D}\mathrm{om}\, L$ and $F^2 \in \mathscr{D}\mathrm{om}\, L$. In particular, the authors of [8] did not obtain that $\mathscr{D}\mathrm{om}\, \Gamma = \mathscr{D}\mathrm{om}\, \mathcal{E}$. This is why, following [3], we use the formalism of Dirichlet forms to compute the carré du champ and obtain a representation for the carré du champ under minimal assumptions.

The energy bracket

Given two elements $u \in \mathscr{L}^2(\nu \otimes \mathbb{P})$ and $v \in \mathscr{L}^2(\nu \otimes \mathbb{P})$ (possibly vector valued), we define the *energy bracket* of u and v: it is the random matrix

$$[u, v]_\Gamma = \frac{1}{2} \int u(z) \otimes v(z) \nu(\mathrm{d}z) + \frac{1}{2} \int (1 - D_z^-) u(z) \otimes (1 - D_z^-) v(z) \eta(\mathrm{d}z). \tag{2.32}$$

In the paper, we also consider the two other related objects:

$$[u, v]_\nu = \int u(z) \otimes v(z) \nu(\mathrm{d}z); \tag{2.33}$$

$$[u, v]_\eta = \int (1 - D_z^-) u(z) \otimes (1 - D_z^-) v(z) \eta(\mathrm{d}z). \tag{2.34}$$

If u and v are real-valued, then $[u, v]_\nu$ is simply the scalar product of u and v in $\mathscr{L}^2(\nu)$. By the Cauchy-Schwarz inequality $[u, v]_\nu \in \mathscr{L}^1(\mathbb{P})$, and by the Mecke formula:

$$\mathbb{E}[u, v]_\Gamma = \mathbb{E}[u, v]_\nu = \mathbb{E}[u, v]_\eta. \tag{2.35}$$

Moreover, if F and $G \in \mathscr{D}\mathrm{om}\, D$, we have that

$$\Gamma(F, G) = [DF, DG]_\Gamma. \tag{2.36}$$

This identity is our main motivation for introducing the energy bracket. We denote by $\widetilde{[u, v]}_\beta$ the symmetrization of the matrix $[u, v]_\beta$ ($\beta \in \{\Gamma, \nu, \eta\}$).

Test functions

We say that a measurable function $\psi \colon Z \to \mathbb{R}_+$ such that $\nu(\psi > 0) < \infty$ is a *test function*. We let $\mathscr{G} \subset \mathscr{L}^\infty(\mathbb{P})$ be the linear span of the random variables of the form $\mathrm{e}^{-\eta(\psi)}$, where ψ is a test function. Observe that $\mathscr{G}$ is a sub-algebra of $\mathscr{A}$ and that $\mathscr{D}\mathrm{om}\, D$ is stable by multiplication by elements of $\mathscr{G}$. In view of [22, Lemma 2.2] and its proof, we have that

Proposition 2.3. *The set $\mathscr{G}$ is dense in $\mathscr{L}^2(\mathbb{P})$ (and in fact in every $\mathscr{L}^p(\mathbb{P})$, $1 \leq p < \infty$). Moreover, the σ-algebra generated by $\mathscr{G}$ coincides with $\mathfrak{W}$.*

Extended Malliavin operators

As mentioned above, we assume that $\mathfrak{O}$ is bigger than $\mathfrak{W}$. However, every $\mathfrak{O}$-random variable F can be written $F = f(\eta, \Xi)$, where Ξ is an additional randomness independent of η. We define for every such F the quantity $D_z^+ F = f(\eta + \delta_z, \Xi) - f(\eta, \Xi)$. It is an (easy) exercise to check that we can accordingly modify all the operators and functional spaces defined above, and that their properties are left unchanged. Remark that our definition implies that, if F is independent of η, then $D^+ F = 0$, and that, if $F = ab$ with a independent of $\mathfrak{W}$ and b measurable with respect to $\mathfrak{W}$, $D^+ F = a D^+ b$.

2.2 Chain rules and integration by parts formulae

Substitute for the chain rule

The Markov generator L is not a diffusion (see [23, Equation 1.3]). Likewise, the add operator D^+ and drop operator D^- are not derivation (see [4, Chapter III Section 10] for details on derivations). In particular, the classical chain rule does not apply. However, writing $D_z^+ \phi(F) = \phi(F + D_z^+ F) - \phi(F)$ and applying the fundamental theorem of calculus we obtain the following substitute for the chain rule.

Lemma 2.4. *Let $\phi \in \mathscr{C}^2(\mathbb{R}^d)$. For $F \in \mathscr{L}^2(\mathbb{P})$ and $z \in Z$, we define for all i and $j \in [d]$:*

$$\hat{R}_{i,j}(F, z, \phi) = \int_0^1 \int_0^1 \alpha \partial_{ij} \phi(F + \alpha\beta D_z^+ F) \mathrm{d}\alpha \mathrm{d}\beta; \tag{2.37}$$

$$\check{R}_{i,j}(F, z, \phi) = \int_0^1 \int_0^1 \alpha \partial_{ij} \phi(F - \alpha\beta D_z^- F) \mathrm{d}\alpha \mathrm{d}\beta. \tag{2.38}$$

We have that

$$D_z^+ \phi(F) = \langle \nabla \phi(F), D_z^+ F \rangle_{\ell^2} + \left\langle \hat{R}(F, z, \phi), (D_z^+ F)^{\otimes 2} \right\rangle_{\ell^2}; \tag{2.39}$$

$$D_z^- \phi(F) = \langle \nabla \phi(F), D_z^- F \rangle_{\ell^2} - \left\langle \check{R}(F, z, \phi), (D_z^- F)^{\otimes 2} \right\rangle_{\ell^2}. \tag{2.40}$$

In particular (taking $\phi \colon \mathbb{R} \ni x \mapsto x^2 \in \mathbb{R}$), we have that

$$D_z^+ F^2 = 2F D_z^+ F + (D_z^+ F)^2; \tag{2.41}$$

$$D_z^- F^2 = 2F D_z^- F - (D_z^- F)^2. \tag{2.42}$$

Proof. Simply write the Taylor expansion up to order 2 of $D_z^+ \phi(F) = \phi(F + D_z^+ F) - \phi(F)$ and similarly for $D_z^- \phi(F)$. Details are left to the reader. □

Taylor formula for difference operators

Another application of Taylor's formula gives us the following discrete counterpart of the chain rule.

Lemma 2.5. *Let $\phi \in \mathscr{C}^2(\mathbb{R})$. For $F \in \mathscr{L}^2(\mathbb{P})$, we have that*

$$D^+ \phi(F) = D^+ F(\phi(F+1) - \phi(F)) + D^+ F(D^+ F - 1) \int_0^1 \int_0^1 \alpha \phi''(F + \alpha\beta(D^+ F - 1)) \mathrm{d}\alpha \mathrm{d}\beta. \tag{2.43}$$

Remark 2. It is to obtain a similar formula for D^- or on $\mathbb{R}^d$ but we have no use for it.

Proof. Apply the fundamental theorem of calculus on $\phi(x+h) - \phi(x) - h(\phi(x+1) - \phi(x))$ and take $x = F$ and $h = D^+ F$. □

A formula for the divergence

Since the operator D is not a derivation, [30, Proposition 1.3.3] (obtained in the setting of Malliavin calculus for Gaussian processes) does not hold. We however have the following Poisson counterpart.

Lemma 2.6. *Let $F \in \mathscr{D}\mathrm{om}\, D$ and $u \in \mathscr{D}\mathrm{om}\, \delta$ such that $Fu \in \mathscr{D}\mathrm{om}\, \delta$. Then,*

$$\delta(Fu) = F\delta u - [DF, u]_\eta. \tag{2.44}$$

Proof. Let $G \in \mathscr{A} = \mathscr{D}\mathrm{om}\, D \cap \mathscr{L}^\infty(\mathbb{P})$, and assume moreover that $u \in \mathscr{L}^1(\mathbb{P} \otimes \nu)$. By integration by parts and the Mecke formula, we find that

$$\mathbb{E} G\delta(Fu) = \mathbb{E} \int F u_z D_z G \nu(\mathrm{d}z) = \mathbb{E} G \int (1 - D_z^-)(F u_z) \eta(\mathrm{d}z) - \mathbb{E} G \int F u_z \nu(\mathrm{d}z). \tag{2.45}$$

Using that $(1 - D_z^-)(F u_z) = F(1 - D_z^-) u_z - D_z^- F(1 - D_z^-) u_z$, we conclude by (2.23) that

$$\mathbb{E} G\delta(Fu) = \mathbb{E} G F \delta u - \mathbb{E} G [DF, u]_\eta. \tag{2.46}$$

We conclude by density. □

An integrated chain rule for the energy

Recall that we write $\mathscr{A}$ for the algebra $\mathscr{D}\mathrm{om}\,\mathcal{E} \cap \mathscr{L}^\infty(\mathbb{P})$. We now remark that even if D is not a derivation, the Dirichlet energy $\mathcal{E}$ acts as a derivation.

Lemma 2.7. *Let F and $G \in \mathscr{A}$, and $u \in \mathscr{L}^2(\mathbb{P} \otimes \nu)$. Then,*

$$\mathbb{E}[D(FG), u]_\Gamma = \mathbb{E}F[DG, u]_\Gamma + \mathbb{E}G[DF, u]_\Gamma. \tag{2.47}$$

In particular, with $H \in \mathscr{D}\mathrm{om}\, D$:

$$\mathcal{E}(FG, H) = \mathbb{E}F[DG, DH]_\Gamma + \mathbb{E}G[DF, DH]_\Gamma. \tag{2.48}$$

This establishes Proposition 2.2.

Remark 3. The formula (2.48) for $\mathcal{E}$ *cannot* be iterated. In particular, consistently with the fact that L is not a diffusion, (2.48) does *not* imply $\mathcal{E}(\phi(F), G) = \mathbb{E}\phi'(F)[DF, DG]_\Gamma$.

Proof. Since $F \in \mathscr{L}^\infty(\mathbb{P})$, by Lemma 2.1, we have that $DF \in \mathscr{L}^\infty(\mathbb{P} \otimes \nu)$; and by assumption, $DF \in \mathscr{L}^2(\mathbb{P} \otimes \nu)$. A similar result holds for G, and we find that $DF \otimes DG$ is square integrable. By the Mecke formula, and (2.41) and (2.42), we can write:

$$\begin{aligned} \mathbb{E}[D(FG), u]_\Gamma &= \mathbb{E}F[DG, u]_\Gamma + \mathbb{E}G[DF, u]_\Gamma \\ &\quad + \frac{1}{2}\mathbb{E}\int D_z^+ F \otimes DG \otimes u(z)\nu(\mathrm{d}z) \\ &\quad - \frac{1}{2}\mathbb{E}\int (1 - D_z^-)F(1 - D_z^-)G(1 - D_z^-)u(z)\eta(\mathrm{d}z). \end{aligned} \tag{2.49}$$

By the Mecke formula, the two terms on the two last lines cancel out. This proves the first part of the claim. To establish Proposition 2.2, we simply write, for F and $\Phi \in \mathscr{A}$:

$$\mathcal{E}(F, F\Phi) - \frac{1}{2}\mathcal{E}(F^2, \Phi) = \mathbb{E}F[DF, D\Phi]_\Gamma + \mathbb{E}\Phi[DF, DF]_\Gamma - \mathbb{E}F[DF, D\Phi]_\Gamma. \tag{2.50}$$

This shows that $\mathscr{D}\mathrm{om}\,\Gamma \supset \mathscr{A}$ and that

$$\mathbf{\Gamma}(F)[\Phi] = \mathbb{E}[DF, DF]_\Gamma \Phi. \tag{2.51}$$

We extend this expression to $\mathscr{D}\mathrm{om}\,\mathcal{E} = \mathscr{D}\mathrm{om}\, D$. This concludes the proof. □

Integration by parts formulae

Most of our analysis relies on integration by parts formulae at the level of the Poisson space based on Malliavin calculus.

Lemma 2.8. *Let $F = (F_1, \dots, F_d) \in \mathscr{D}\mathrm{om}\, D$ and let $u = (u_1, \dots, u_d) \in \mathscr{D}\mathrm{om}\,\delta$. Let $G \in \mathscr{G}$. Let $\phi \in \mathscr{C}_b^3(\mathbb{R}^d)$. We write $\hat{R}(F, z, \nabla\phi)$ for the (non-symmetric) 3-tensor whose coordinate (i, j, k) is given by $\hat{R}_{jk}(F, z, \partial_i\phi)$, and we do the same for $\check{R}$. Assume that, for $l \in \{1, 2\}$, $\int |u(z)|_{\ell^2} |D_z^+ F|_{\ell^2}^l \nu(\mathrm{d}z) < \infty$. Then:*

$$\begin{aligned} \mathbb{E}\langle \nabla\phi(F)G, \delta u\rangle_{\ell^2} &= \frac{1}{2}\mathbb{E}G\left\langle \nabla^2\phi(F), [u, DF]_\Gamma\right\rangle_{\ell^2} \\ &\quad + \frac{1}{2}\mathbb{E}\left\langle \nabla\phi(F), [u, DG]_\Gamma\right\rangle_{\ell^2} \\ &\quad + \frac{1}{4}\mathbb{E}G\int \left\langle u(z) \otimes (D_z^+ F)^{\otimes 2}, \hat{R}(F, z, \nabla\phi)\right\rangle_{\ell^2} \nu(\mathrm{d}z) \\ &\quad - \frac{1}{4}\mathbb{E}G\int \left\langle (1 - D_z^-)u(z) \otimes (D_z^- F)^{\otimes 2}, \check{R}(F, z, \nabla\phi)\right\rangle_{\ell^2} \eta(\mathrm{d}z). \end{aligned} \tag{2.52}$$

Proof. We write $A = \langle \nabla\phi(F), \delta u\rangle_{\ell^2}$, and

$$B = \frac{1}{2} G \left\langle \nabla^2\phi(F), [u, DF]_\Gamma \right\rangle_{\ell^2}. \tag{2.53}$$

Write $C = \langle \nabla\phi(F), [u, DG]_\Gamma\rangle_{\ell^2}$ and

$$R^+ = \frac{1}{4} G \int \left\langle u(z) \otimes (D_z^+ F)^{\otimes 2}, \hat{R}(F, z, \nabla\phi) \right\rangle_{\ell^2} \nu(\mathrm{d}z); \tag{2.54}$$

$$R^- = \frac{1}{4} G \int \left\langle (1 - D_z^-) u(z) \otimes (D_z^- F)^{\otimes 2}, \check{R}(F, z, \nabla\phi) \right\rangle_{\ell^2} \eta(\mathrm{d}z). \tag{2.55}$$

First, let us check that every term is well defined. Since $\phi \in \mathscr{C}_b^3(\mathbb{R}^d)$, $\nabla\phi$ is Lipschitz. Since $F \in \mathscr{D}\mathrm{om}\, D$, we find that $\nabla\phi(F) \in \mathscr{D}\mathrm{om}\, D$ and $G\nabla\phi(F) \in \mathscr{D}\mathrm{om}\, D$. Since $u \in \mathscr{D}\mathrm{om}\, \delta$, we have that $\delta u \in \mathscr{L}^2(\mathbb{P})$ and, then, $A \in \mathscr{L}^1(\mathbb{P})$. Applying the Cauchy-Schwarz inequality and the Mecke formula, we find, in view of the assumptions

$$\mathbb{E}|B| \leq |\nabla^2\phi|_{\ell^2,\infty} |G|_{\mathscr{L}^\infty(\mathbb{P})} \mathbb{E} \int |u(z)|_{\ell^2} |D_z^+ F|_{\ell^2} < \infty; \tag{2.56}$$

$$\mathbb{E}|C| \leq |\nabla\phi|_{\ell^2,\infty} \mathbb{E} \int |u(z)|_{\ell^2} |D_z^+ G|_{\ell^2} \nu(\mathrm{d}z) < \infty; \tag{2.57}$$

$$\mathbb{E}|R^+| + \mathbb{E}|R^-| \leq |\nabla^3\phi(F)|_{\ell^2,\infty} |G|_{\mathscr{L}^\infty(\mathbb{P})} \mathbb{E} \int |u(z)|_{\ell^2} |D_z^+ F|_{\ell^2}^2 \nu(\mathrm{d}z) < \infty. \tag{2.58}$$

These estimates also justify the use of the Mecke formula on non-necessarily non-negative quantities that we do in the rest of the proof. Now, we prove the equality (2.52). Let $D = B + C + R^+ - R^-$. By integration by parts (2.20), we find

$$\mathbb{E}A = \mathbb{E} \int \left\langle D_z^+ (\nabla\phi(F) G), u(z) \right\rangle_{\ell^2} \nu(\mathrm{d}z). \tag{2.59}$$

By the Mecke formula (2.1), we get

$$2\mathbb{E}A = \mathbb{E}[D(\nabla\phi(F)G), u]_\Gamma. \tag{2.60}$$

Applying Lemmas 2.4 and 2.7, in the previous identity immediately yields $\mathbb{E}A = \mathbb{E}B$. This concludes the proof. □

When $G = 1$, we can directly apply Lemma 2.4 in (2.59), this yields the following integration by parts involving $[\cdot,\cdot]_\nu$ rather than $[\cdot,\cdot]_\Gamma$.

Lemma 2.9. *Under the same assumptions as for Lemma 2.8, it holds*

$$\begin{aligned} \mathbb{E}\langle \nabla\phi(F), \delta u\rangle_{\ell^2} &= \mathbb{E} \left\langle \nabla^2\phi(F), [u \otimes DF]_\nu \right\rangle_{\ell^2} \\ &\quad + \frac{1}{2} \mathbb{E} \int \left\langle u(z) \otimes (D_z^+ F)^{\otimes 2}, \hat{R}(F, z, \nabla\phi) \right\rangle_{\ell^2} \nu(\mathrm{d}z). \end{aligned} \tag{2.61}$$

3 Main abstract results

Outline

Theorem 3.1 gives sufficient conditions for the stable convergence of a sequence of Poisson functionals to a Gaussian mixture. While Theorem 3.2 gives sufficient conditions for the stable convergence of a sequence of Poisson functionals to a Poisson mixture. Theorem 3.5 is the quantitative counterpart of Theorem 3.1 and provides bounds on the distance d_3 between the distribution of a Poisson functional and that of a Gaussian mixture. We are not able to obtain a quantitative estimates for the convergence to a Poisson mixture. Theorem 3.8 is an improvement of our bound from the d_3 distance to the d_1 distance, when (F_n) is a sequence of univariate random variables.

3.1 Main qualitative results

Thanks to our integration by parts formulae, we derive sufficient conditions to ensure that a sequence of Kabanov integrals converges to a Gaussian mixture or a Poisson mixture. In Section 3.2, we derive quantitative bounds for the convergence to a Gaussian mixture only. However, in the case of Gaussian mixture, obtaining a quantitative estimates requires to control additional terms. This is why we treat first the simple qualitative bound both for Gaussian and Poisson mixtures.

3.1.1 Convergence to a Gaussian mixture

Recall that we study asymptotic for (possibly multivariate) random variables of the form $F_n = \delta u_n$. In this setting, let us state the multivariate equivalent of (0.2):

$$\mathbb{E} \int |u_n(z)|_{\ell^2} |D_z^+ F_n|_{\ell^2}^2 \nu(\mathrm{d}z) \xrightarrow[n\to\infty]{} 0. \tag{R$_3$}$$

We also consider

$$\mathbb{E} \int |D_z^+ F_n|_{\ell^2}^4 \nu(\mathrm{d}z) \xrightarrow[n\to\infty]{} 0. \tag{R$_4$}$$

Remark that provided (u_n) is bounded in $\mathscr{L}^2(\mathbb{P}\otimes\nu)$, by the Cauchy-Schwarz inequality (R$_4$) implies (R$_3$). Several works about normal approximation of Poisson functionals (for instance, [32, 18, 17, 39]) also consider conditions such as (R$_3$) and (R$_4$). The random variable $u_n = -DL^{-1}F_n$ is always a solution of the equation $\delta u_n = F_n$ (other choices are possible). Following [32, Theorem 3.1] or [8, Theorem 4.1], let us consider

$$-\nu(DL^{-1}F_n DF_n) = [u_n, DF_n]_\nu \xrightarrow[n\to\infty]{\mathscr{L}^1(\mathbb{P})} \sigma^2; \tag{3.1}$$

or

$$-\Gamma(L^{-1}F_n, F_n) = [u_n, DF_n]_\Gamma \xrightarrow[n\to\infty]{\mathscr{L}^1(\mathbb{P})} \sigma^2. \tag{3.2}$$

Then, we have that (R$_3$) with either (3.1) or (3.2) imply that $F_n \xrightarrow[n\to\infty]{law} \mathbf{N}(0,\sigma^2)$. In our setting of random variance it is thus very natural to consider one of the following condition:

$$[u_n, DF_n]_\nu \xrightarrow[n\to\infty]{\mathscr{L}^1(\mathbb{P})} SS^T; \tag{S$_\nu$}$$

or

$$[u_n, DF_n]_\Gamma \xrightarrow[n\to\infty]{\mathscr{L}^1(\mathbb{P})} SS^T; \tag{S$_\Gamma$}$$

for some $S \in \mathscr{L}^2(\mathbb{P})$. Since we deal with stable convergence we need a condition that provides some form of asymptotic independence. It has one of the following form

$$[u_n, h]_\nu \xrightarrow[n\to\infty]{\mathscr{L}^1(\mathbb{P})} 0, \quad \forall h \in \mathscr{L}^2(\nu); \tag{W$_\nu$}$$

or

$$[u_n, DG]_\Gamma \xrightarrow[n\to\infty]{\mathscr{L}^1(\mathbb{P})} 0, \quad \forall G \in \mathscr{G}. \tag{W$_\Gamma$}$$

Our first statement regarding stable limit theorems on the Poisson space is the following qualitative generalization of the results of [32, 8] to consider Gaussian mixtures in the limit.

Theorem 3.1. *Let $\{F_n = (F_n^{(1)}, \dots, F_n^{(d)});\ n \in \mathbb{N}\} \subset \mathscr{D}\mathrm{om}\, D$. Assume that, for all $n \in \mathbb{N}$, there exists $u_n \in \mathscr{D}\mathrm{om}\, \delta$ such that $F_n = \delta u_n$ and that (R$_3$) hold. Let $S = (S_1, \dots, S_d) \in \mathscr{L}^2(\mathbb{P})$. Assume that either (W$_\nu$) and (S$_\nu$) holds; or (W$_\Gamma$) and (S$_\Gamma$) holds. Then $F_n \xrightarrow[n\to\infty]{stably} \mathbf{N}(0, S^2)$.*

Remark 4. The condition (S$_\Gamma$) is a priori more involved than (S$_\nu$): indeed integrating with respect to η adds some randomness to the object. However, in Section 4.1 we need the result involving $[\cdot,\cdot]_\Gamma$ in order to obtain a stable version of the fourth moment theorem of [8]. On the other hand for the practical applications in Section 5, conditions of type (S$_\nu$) and (W$_\nu$) are easier to manipulate.

3.1.2 Convergence to a Poisson mixture

Here we only consider univariate random variables. Convergence in law of Poisson functionals to a Poisson distribution represents another archetypal limit theorem. In the setting of the Malliavin-Stein method, [33] proves that the two conditions:

$$-\nu(D^+L^{-1}F_nD^+F_n) \xrightarrow[n\to\infty]{} m, \tag{3.3}$$

and

$$\mathbb{E}\int |D_z^+L^{-1}F_nD_z^+F_n(D_z^+F_n - 1)|\nu(\mathrm{d}z) \xrightarrow[n\to\infty]{} 0, \tag{3.4}$$

imply that $F_n \xrightarrow[n\to\infty]{law} \mathbf{Po}(m)$[2]. It is thus very natural to replace (R$_3$) by the following asymptotic conditions for $F_n = \delta u_n$ (here we only considered scalar-valued random variables):

$$\mathbb{E}\int |u_n(z)D_z^+F_n(D_z^+F_n - 1)|\nu(\mathrm{d}z) \xrightarrow[n\to\infty]{} 0. \tag{P$_3$}$$

We also consider the Poisson version of (R$_4$):

$$\mathbb{E}\int |D_z^+F_n|^2|D_z^+F_n - 1|^2\nu(\mathrm{d}z) \xrightarrow[n\to\infty]{} 0. \tag{P$_4$}$$

Again, provided (u_n) is bounded in $\mathscr{L}^2(\mathbb{P})$, we see that (P$_4$) implies (P$_3$). With this notation, we have the following qualitative result for convergence to a Poisson mixture.

Theorem 3.2. *Let $(F_n) \subset \mathscr{D}\mathrm{om}\, D$. Let $M \in \mathscr{L}^1(\mathbb{P})$ with $M \geq 0$. Assume that, for all $n \in \mathbb{N}$, there exists $u_n \in \mathscr{D}\mathrm{om}\, \delta$ such that $F_n = \delta u_n$ and that (P$_3$) and (W$_\nu$) hold, and moreover assume that*

$$[u_n, DF_n]_\nu = \langle u_n, DF_n\rangle_{\mathscr{L}^2(\nu)} \xrightarrow[n\to\infty]{\mathscr{L}^1(\mathbb{P})} M. \tag{M$_\nu$}$$

Then $F_n \xrightarrow[n\to\infty]{stably} \mathbf{Po}(M)$.

Remark 5. (M$_\nu$) is formally equivalent to (S$_\nu$) (we can always write $S^2 = M$). However, it is important to note that our theorem cannot be true if we replace the scalar product by the energy bracket in (M$_\nu$), that is that we work with the condition:

$$[u_n, DF_n]_\Gamma \xrightarrow[n\to\infty]{\mathscr{L}^1(\mathbb{P})} M. \tag{M$_\Gamma$}$$

Indeed take $F = \eta(A) - \nu(A)$, with $A \in \mathfrak{Z}$, $\nu(A) < \infty$. We can write $F = \delta 1_A$, and $DF = 1_A$, hence (P$_3$) is satisfies, since we have

$$\int_A |1_A - 1|\mathrm{d}\nu = 0. \tag{3.5}$$

[2] [33] works with non-centered random variables but the result is equivalent.

On the other hand, we have that

$$[1_A, 1_A]_\Gamma = \frac{1}{2}(\nu(A) + \eta(A)) = M. \tag{3.6}$$

Let $F \sim \mathbf{Po}(M)$, then

$$\begin{aligned} \mathbb{E}\, e^{i\lambda F} &= \mathbb{E}\exp(M(e^{i\lambda} - i\lambda - 1)) \\ &= \exp\left(\frac{1}{2}\nu(A)(e^{i\lambda} - i\lambda - 1)\right)\exp\left(\frac{\nu(A)}{2}\left(\exp(e^{i\lambda} - i\lambda - 1) - 1\right)\right). \end{aligned} \tag{3.7}$$

Hence, we see that the law of F is not the one of $\eta(A) - \nu(A)$. Remark that (S_Γ) and (M_Γ) are also formally equivalent. At a more structural level, [6] proves that if a sequence of Poisson stochastic integrals satisfies a deterministic reinforcement of (S_Γ), that is:

$$[u_n, DF_n]_\Gamma = -\Gamma(L^{-1}F_n, F_n) \xrightarrow[n\to\infty]{\mathscr{L}^2(\mathbb{P})} \sigma^2, \tag{3.8}$$

where $u_n = -DL^{-1}F_n$, then, without further assumptions, the sequence converges in law to a Gaussian.

3.1.3 Proofs

Proof of Theorem 3.1. We first prove the theorem under (W_Γ) and (S_Γ). By (S_Γ), we have that

$$\mathbb{E}F_n F_n^T = \mathbb{E}[u_n, DF_n]_\Gamma \xrightarrow[n\to\infty]{} \mathbb{E}SS^T < \infty. \tag{3.9}$$

So (F_n) is bounded in $\mathscr{L}^2(\mathbb{P})$. Let $G \in \mathscr{G}$. For all $n \in \mathbb{N}$, we let $\xi_n = (F_n, G)$. Since (F_n) is bounded in $\mathscr{L}^2(\mathbb{P})$, (ξ_n) is tight. We can extract a subsequence (still denoted (ξ_n)) such that (ξ_n) converges in law to (F_∞, G). Let $\psi_n(\lambda) = \mathbb{E}G\, e^{i\langle\lambda, F_n\rangle_{\ell^2}}$, and $\psi_\infty(\lambda) = \mathbb{E}G\, e^{i\langle\lambda, F_\infty\rangle_{\ell^2}}$. By convergence in law, we have that $\psi_n \xrightarrow[n\to\infty]{\mathscr{C}^0(0,1)} \psi_\infty$. Since (ξ_n) is bounded in $\mathscr{L}^2(\mathbb{P})$ it is also uniformly integrable, and we find that

$$\nabla\psi_n(\lambda) = i\mathbb{E}F_n G\, e^{i\langle\lambda, F_n\rangle_{\ell^2}} \xrightarrow[n\to\infty]{} i\mathbb{E}F_\infty G\, e^{i\langle\lambda, F_\infty\rangle_{\ell^2}} = \nabla\psi_\infty(\lambda). \tag{3.10}$$

By Lemma 2.8, we also have that

$$\begin{aligned} \nabla\psi_n(\lambda) &= i\mathbb{E}G\Big[u_n, D\left(e^{i\lambda F_n}\right)\Big]_\Gamma + i\mathbb{E}\, e^{i\langle\lambda, F_n\rangle_{\ell^2}}\, [u_n, DG]_\Gamma \\ &= -\lambda\mathbb{E}G\, e^{i\langle\lambda, F_n\rangle_{\ell^2}}\, [u_n, DF_n]_\Gamma + i\mathbb{E}\, e^{i\langle\lambda, F_n\rangle_{\ell^2}}\, [u_n, DG]_\Gamma + R_n, \end{aligned} \tag{3.11}$$

where

$$|R_n| \le \lambda^2\mathbb{E}\int |u_n(z)|_{\ell^2} |D_z^+ F_n|_{\ell^2}^2 \nu(\mathrm{d}z). \tag{3.12}$$

We thus see that (S_Γ), (R_3) and (W_Γ) imply that

$$\nabla\psi_n(\lambda) \xrightarrow[n\to\infty]{} -\lambda\mathbb{E}SS^T\, e^{i\langle\lambda, F_\infty\rangle}. \tag{3.13}$$

All in all, we have proved that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\psi_\infty(\lambda) = i\mathbb{E}GF_\infty\, e^{i\langle\lambda, F_\infty\rangle_{\ell^2}} = -\lambda\mathbb{E}GSS^T\, e^{i\langle\lambda, F_\infty\rangle_{\ell^2}}. \tag{3.14}$$

Thus, we obtain the following differential equation for the conditional characteristic function:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathbb{E}[\mathrm{e}^{i\lambda F_\infty}|\eta] = -\lambda SS^T\mathbb{E}[\mathrm{e}^{i\lambda F_\infty}|\eta]. \tag{3.15}$$

The only solution of this equation with $\psi(0) = 1$ is the one given in (1.14) and this concludes the proof in view of (iii) in Proposition 1.1. For the proof under (W_ν) and (S_ν), we only briefly explain what to modify; the details can be found below, in the proof of Theorem 3.2, where we use this strategy to obtain convergence to a Poisson mixture. To work with (W_ν) and (S_ν), we rather introduce $\psi_n(\lambda) = \mathbb{E}\,\mathrm{e}^{i\langle\lambda, F_n+I_1(h)\rangle_{\ell^2}}$ for some $h \in \mathscr{L}^2(\nu)$. Instead of Lemma 2.8, we have to use Lemma 2.9. The rest of the proof is similar. □

Proof of Theorem 3.2. Let $h \in \mathscr{L}^2(\nu)$. Let $\lambda \in \mathbb{R}$, and consider $\psi_n(\lambda) = \mathbb{E}\,\mathrm{e}^{i\lambda(F_n+I_1(h))}$, and $\psi_\infty(\lambda) = \mathbb{E}\,\mathrm{e}^{i\lambda(F_\infty+I_1(h))}$. Since $\mathbb{E}F_n^2 = \mathbb{E}\langle u_n, DF_n\rangle$, using ($\mathrm{M}_\nu$), we see that $F_n + I_1(h)$ is tight and uniformly integrable. Up to extraction, we can find some F_∞, such that $F_n + I_1(h) \xrightarrow[n\to\infty]{} F_\infty + I_1(h)$, and that

$$\psi_n'(\lambda) = i\mathbb{E}(F_n + I_1(h))\,\mathrm{e}^{i\lambda(F_n+I_1(h))} \xrightarrow[n\to\infty]{} i\mathbb{E}(F_\infty + I_1(h))\,\mathrm{e}^{i\lambda(F_\infty+I_1(h))} = \psi_\infty'(\lambda). \tag{3.16}$$

On the other hand, by Lemma 2.5, we have that

$$\psi_n'(\lambda) = i\mathbb{E}\langle u_n, DF_n + h\rangle\,\mathrm{e}^{i\lambda(F_n+I_1(h))}(\mathrm{e}^{i\lambda} - 1) + i\mathbb{E}I_1(h)\,\mathrm{e}^{i\lambda(F_n+I_1(h))} + R_n, \tag{3.17}$$

where

$$|R_n| \leq \lambda^2 \int |u_n(z)(D_zF_n - 1)D_zF_n|\nu(\mathrm{d}z). \tag{3.18}$$

Thus, under (M_ν), (P_3) and (W_Γ):

$$\lim_{n\to\infty} \psi_n'(\lambda) = i\mathbb{E}(\mathrm{e}^{i\lambda} - 1)M\,\mathrm{e}^{i\lambda F_\infty} + i\mathbb{E}I_1(h)\,\mathrm{e}^{i\lambda(F_\infty+I_1(h))}. \tag{3.19}$$

Equating, (3.16) and (3.19) we obtain that

$$\mathbb{E}\,\mathrm{e}^{i\lambda I_1(h)}\,F_\infty\,\mathrm{e}^{i\lambda F_\infty} = \mathbb{E}\,\mathrm{e}^{i\lambda I_1(h)}\,M(\mathrm{e}^{i\lambda} - 1)\,\mathrm{e}^{i\lambda F_\infty}, \quad \forall\lambda \in \mathbb{R}, \forall h \in \mathscr{L}^2(\nu). \tag{3.20}$$

Arguing, by linearity of I_1, as in the proof of (iv) of Proposition 1.1, we find that:

$$\mathbb{E}I_1(h)F_\infty\,\mathrm{e}^{i\lambda F_\infty} = \mathbb{E}I_1(h)M(\mathrm{e}^{i\lambda} - 1)\,\mathrm{e}^{i\lambda F_\infty}, \quad \forall\lambda \in \mathbb{R}, \forall h \in \mathscr{L}^2(\nu). \tag{3.21}$$

That is to say, we have proved the following differential equation for the conditional characteristic function:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathbb{E}[\mathrm{e}^{i\lambda F_\infty}|\eta] = i(\mathrm{e}^{i\lambda} - 1)M\mathbb{E}[\mathrm{e}^{i\lambda F_\infty}|\eta]. \tag{3.22}$$

The unique solution of this equation satisfying $\psi(0) = 1$ is the function given in (1.15). This concludes the proof by (iv) in Proposition 1.1. □

3.2 Main quantitative results for Gaussian mixtures

3.2.1 General results in any dimension

As [26] on the Gaussian space, we use the integration by parts formulae to obtain quantitative Malliavin-Stein bounds between the law of a Poisson functional and that of a Gaussian mixture. Our results crucially rely on the two following bounds, proved at the end of the section. We obtain these bounds via the so-called *Talagrand's smart path interpolation* method. As for Theorem 3.1 we can either work with $[\cdot,\cdot]_\nu$ or with $[\cdot,\cdot]_\Gamma$ yielding to different bounds. Results involving $[\cdot,\cdot]_\nu$ are a priori easier to handle in applications. However, we state the two bounds for completeness. For short, for $\phi \in \mathscr{C}^k(\mathbb{R}^d)$, let us write $\Phi_k = |\nabla^k \phi|_{\ell^2,\infty}$, and $S \in \mathscr{C}\mathrm{ov}$ whenever $S \in \mathscr{D}\mathrm{om}\, D$ with $SS^T \in \mathscr{D}\mathrm{om}\, D$.

Proposition 3.3. *Let $F = (F_1, \dots, F_d) \in \mathscr{D}\mathrm{om}\, D$, $S \in \mathscr{C}\mathrm{ov}$, and N be a standard d-dimensional Gaussian vector independent of η. Assume that there exists $u \in \mathscr{D}\mathrm{om}\, \delta$ such that $F = \delta u$. Then, for all $\phi \in \mathscr{C}_b^3(\mathbb{R}^d)$ and all $I = I_1(h)$, $h \in \mathscr{L}^2(\nu)$:*

$$
\begin{aligned}
|\mathbb{E}\phi(F+I) - \mathbb{E}\phi(SN+I)| \leq & \frac{1}{4}\Phi_2 \mathbb{E}\left|\widetilde{[u, DF]}_\nu - SS^T\right|_{\ell^2} \\
& + \frac{1}{3}\Phi_3 \mathbb{E}\left|\left[u, (DS)S^T\right]_\nu\right|_{\ell^2} \\
& + \frac{1}{6}\Phi_3 \mathbb{E}\int |h(z) + u(z)|_{\ell^2} \left(|D_z^+ F|_{\ell^2}^2 + |DS|_{\ell^2}^2\right) \nu(\mathrm{d}z).
\end{aligned} \tag{3.23}
$$

Proposition 3.4. *Let $F = (F_1, \dots, F_d) \in \mathscr{D}\mathrm{om}\, D$, $S \in \mathscr{C}\mathrm{ov}$, and N be a standard d-dimensional Gaussian vector independent of η. Assume that there exists $u \in \mathscr{D}\mathrm{om}\, \delta$ such that $F = \delta u$. Then, for all $\phi \in \mathscr{C}_b^3(\mathbb{R}^d)$ and all $G \in \mathscr{G}$:*

$$
\begin{aligned}
|\mathbb{E}\phi(F)G - \mathbb{E}\phi(SN)G| \leq & \frac{1}{4}\Phi_2 |G|_\infty \mathbb{E}\left|\widetilde{[u, DF]}_\Gamma - SS^T\right|_{\ell^2} \\
& + \frac{1}{3}\Phi_3 |G|_\infty \mathbb{E}|[u, (DS)S^T]_\Gamma|_{\ell^2} \\
& + \frac{1}{6}\Phi_3 |G|_\infty \mathbb{E}\int |u(z)|_{\ell^2} \left(|D_z^+ F|_{\ell^2}^2 + |DS|_{\ell^2}^2\right) \nu(\mathrm{d}z) \\
& + \frac{1}{2}\Phi_1 \mathbb{E}|[u, DG]_\Gamma|_{\ell^2},
\end{aligned} \tag{3.24}
$$

where for short, we write $|G|_\infty = |G|_{\mathscr{L}^\infty(\mathbb{P})}$.

We are now in position to state our bound in the d_3 distance of a Poisson functional to a Gaussian mixture.

Theorem 3.5. . *Let $\beta \in \{\nu, \Gamma\}$. Let $F \in \mathscr{D}\mathrm{om}\, D$, and $S \in \mathscr{C}\mathrm{ov}$. Then,*

$$
\begin{aligned}
d_3(F, \mathbf{N}(0, S^2)) \leq & \frac{1}{4}\mathbb{E}\left|\widetilde{[u, DF]}_\beta - SS^T\right|_{\ell^2} \\
& + \frac{1}{3}\mathbb{E}\left|[u, (DS)S^T]_\beta\right|_{\ell^2} \\
& + \frac{1}{6}\mathbb{E}\int |u(z)|(|D_z F|_{\ell^2}^2 + |DS|_{\ell^2}^2)\nu(\mathrm{d}z).
\end{aligned} \tag{3.25}
$$

Proof. We simply use either Proposition 3.3 with $h = 0$ and $\phi \in \mathscr{F}_3$; or Proposition 3.4 with $G = 1$ and $\phi \in \mathscr{F}_3$. □

In Theorem 3.1, (S_ν) enforces that the asymptotic covariance S is measurable with respect to η. Thanks to Proposition 3.3, when $S_n^2 = \widetilde{[u_n, DF_n]}_\nu$ is non-negative, we can deduce sufficient conditions for the stable convergence of a Poisson functional that involves stable convergence of S_n to some S (not necessarily measurable with respect to η). This weaker form of convergence can allow, for instance, S to be independent of η.

Theorem 3.6. *Let $(F_n)_{n\in\mathbb{N}} \subset \mathscr{D}\mathrm{om}\, D$, and $S \in \mathscr{L}^2(\Omega)$ (not necessarily measurable with respect to η). Let $(u_n)_{n\in\mathbb{N}} \subset \mathscr{D}\mathrm{om}\,\delta$ such that $F_n = \delta u_n$ for all $n \in \mathbb{N}$, and (W$_\nu$) and (R$_3$) holds. Assume, moreover, that for n sufficiently big $\widetilde{[u_n, DF_n]}_\nu = C_n + \epsilon_n$, where $C_n = S_n S_n^T$ is a symmetric non-negative random matrix, and:*

(C.st) $$C_n \xrightarrow[n\to\infty]{stably} SS^T;$$

(ϵ) $$\epsilon_n \xrightarrow[n\to\infty]{\mathscr{L}^1(\mathbb{P})} 0;$$

(RS) $$[u_n, (DS_n)S_n]_\nu \xrightarrow[n\to\infty]{\mathscr{L}^1(\mathbb{P})} 0;$$

(Rh) $$\int |h(z)|_{\ell^2} |D_z^+ F_n|^2_{\ell^2} \nu(\mathrm{d}z) \xrightarrow[n\to\infty]{\mathscr{L}^1(\mathbb{P})} 0;$$

(S$_3$) $$\int |u_n(z)|_{\ell^2} |D_z^+ S_n|^2_{\ell^2} \nu(\mathrm{d}z) \xrightarrow[n\to\infty]{\mathscr{L}^1(\mathbb{P})} 0;$$

(Sh) $$\int |h(z)|_{\ell^2} |D_z^+ S_n|^2_{\ell^2} \nu(\mathrm{d}z) \xrightarrow[n\to\infty]{\mathscr{L}^1(\mathbb{P})} 0.$$

Then $F_n \xrightarrow[n\to\infty]{stably} \mathbf{N}(0, S^2)$.

Remark 6. We can also consider

(S$_4$) $$\int |D_z^+ S_n|^4_{\ell^2} \nu(\mathrm{d}z) \xrightarrow[n\to\infty]{\mathscr{L}^1(\mathbb{P})} 0.$$

Provided (u_n) is bounded in $\mathscr{L}^2(\mathbb{P}\otimes\nu)$ then (R$_4$) implies (R$_3$) and (Rh), and (S$_4$) implies (S$_3$) and (Sh).

Remark 7. We formulated our result with $[\cdot,\cdot]_\nu$; we could do the same for $[\cdot,\cdot]_\Gamma$. Details are left to the reader.

Proof. Let $h \in \mathscr{L}^2(\nu)$, and $N \sim \mathbf{N}(0, id_{\mathbb{R}^d})$. For $n \in \mathbb{N}$, we write:

$$\begin{aligned}\mathbb{E}\phi(F_n + I_1(h)) - \mathbb{E}\phi(SN + I_1(h)) &= \mathbb{E}\phi(F_n + I_1(h)) - \mathbb{E}\phi(S_nN + I_1(h)) \\ &\quad + \mathbb{E}\phi(S_nN + I_1(h)) - \mathbb{E}\phi(SN + I_1(h)).\end{aligned} \tag{3.26}$$

From Proposition 3.3, we have that under (W$_\nu$), (R$_3$), (ϵ), (RS), (Rh), (S$_3$) and (Sh):

$$\mathbb{E}\phi(F_n + I_1(h)) - \mathbb{E}\phi(S_nN + I_1(h)) \xrightarrow[n\to\infty]{} 0. \tag{3.27}$$

On the other hand, under (C.st), $S_nN \xrightarrow[n\to\infty]{stably} SN$, consequently

$$\mathbb{E}\phi(S_nN + I_1(h)) - \mathbb{E}\phi(SN + I_1(h)) \xrightarrow[n\to\infty]{} 0. \tag{3.28}$$

We conclude using Proposition 1.1. □

3.2.2 Bounds in the Monge-Kantorovich distance for the one-dimensional case

The results of the previous section are stated in the rather abstract distance d_3. When F is univariate, one can use the following regularization lemma in order to turn the estimate of Proposition 3.4 into a quantitative bound for the Monge-Kantorovich distance d_1. In this section, all the random variables are implicitly univariate.

Lemma 3.7. *Let F and $F' \in \mathscr{L}^1(\mathbb{P})$ such that there exists a, b, and $c \geq 0$ such that for all $\phi \in \mathscr{C}_b^3(\mathbb{R})$:*

$$\mathbb{E}\phi(F) - \mathbb{E}\phi(F') \leq a|\phi'|_\infty + b|\phi''|_\infty + c|\phi'''|_\infty. \tag{3.29}$$

Then,

$$d_1(F, F') \leq a + \max\left(\left(2^{\frac{1}{3}} + 2^{-\frac{2}{3}}\right)\Delta_1^{\frac{2}{3}}\Delta_2^{\frac{1}{3}}, \Delta_1\Delta_2\right) \tag{3.30}$$

where

$$\Delta_1 = 2\left(\frac{2}{\pi}\right)^{\frac{1}{2}} + \mathbb{E}|F| + \mathbb{E}|G|; \tag{3.31}$$

$$\Delta_2 = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} b + 2^{\frac{1}{2}} c. \tag{3.32}$$

Proof. This result is well-known at different levels of generality. We follow here the proof of [26, Theorem 3.4] (where the reader is referred to for details). For $t \in (0,1)$, we define $\phi_t(x) = \int \phi(t^{\frac{1}{2}}y + (1-t)^{\frac{1}{2}}x)\gamma(\mathrm{d}y)$, with $\gamma = \mathbf{N}(0,1)$. Then, we have that

$$|\phi_t'|_\infty \leq |\phi'|_\infty; \tag{3.33}$$

$$|\phi_t''|_\infty \leq \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{|\phi'|_\infty}{t}; \tag{3.34}$$

$$|\phi_t'''|_\infty \leq 2^{\frac{1}{2}} \frac{|\phi'|_\infty}{t}. \tag{3.35}$$

On the other hand, we have that

$$\mathbb{E}\phi(F) - \mathbb{E}\phi_t(F) \leq t^{\frac{1}{2}}|\phi'|_\infty \left(\left(\frac{2}{\pi}\right)^{\frac{1}{2}} + \mathbb{E}|F|\right). \tag{3.36}$$

Combining all the estimates and optimizing in t yields the desired result. □

We can now state our main quantitative result for univariate random variables.

Theorem 3.8. *Let $F \in \mathscr{D}\mathrm{om}\, D$ such that $F = \delta u$ for some $u \in \mathscr{D}\mathrm{om}\,\delta$, and let $S \in \mathscr{C}\mathrm{ov}$. Consider*

$$\Delta_1 = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}(2 + \mathbb{E}|S|) + \mathbb{E}|F|; \tag{3.37}$$

$$\Delta_2 = \frac{1}{4}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\mathbb{E}|\nu(uDF) - S^2| + 2^{\frac{1}{2}}\left(\frac{1}{3}\mathbb{E}|S\nu(uDS)| + \frac{1}{6}\mathbb{E}\nu\left(|u|\left(|DF|^2 + |DS|^2\right)\right)\right). \tag{3.38}$$

Then, we have that

$$d_1(F, \mathbf{N}(0, S^2)) \leq \max\left(\left(2^{\frac{1}{3}} + 2^{-\frac{2}{3}}\right)\Delta_1^{\frac{2}{3}}\Delta_2^{\frac{1}{3}}, \Delta_1\Delta_2\right). \tag{3.39}$$

Proof. We just combine Proposition 3.3 (with $h = 0$) and Lemma 3.7. □

This theorem allows us to prove a quantitative version of Theorem 3.6 in the univariate case.

Theorem 3.9. *Let the assumptions and the notations of Theorem 3.6 prevail. Consider*

$$\Delta_{1,n} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (2 + \mathbb{E}|S_n|) + \mathbb{E}|F_n|; \tag{3.40}$$

$$\Delta_{2,n} = \frac{1}{4}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \mathbb{E}|\epsilon_n| + 2^{\frac{1}{2}} \left(\frac{1}{3}\mathbb{E}|S_n \nu(u_n DS_n)| + \frac{1}{6}\mathbb{E}\nu\left(|u_n|\left(|DF_n|^2 + |DS_n|^2\right)\right)\right). \tag{3.41}$$

Then,

$$d_1(F_n, \mathbf{N}(0, S^2)) \leq \max\left(\left(2^{\frac{1}{3}} + 2^{-\frac{2}{3}}\right)\Delta_{1,n}^{\frac{2}{3}}\Delta_{2,n}^{\frac{1}{3}}, \Delta_{1,n}\Delta_{2,n}\right) + \left(\frac{2}{\pi}\right)^{\frac{1}{2}} d_1(S_n, S). \tag{3.42}$$

Proof. By the triangle inequality, we write

$$d_1(F_n, \mathbf{N}(0, S^2)) \leq d_1(F_n, \mathbf{N}(0, S_n^2)) + d_1(\mathbf{N}(0, S_n^2), \mathbf{N}(0, S^2)). \tag{3.43}$$

By Theorem 3.8, we have that

$$d_1(F_n, \mathbf{N}(0, S_n^2)) \leq \max\left(\left(2^{\frac{1}{3}} + 2^{-\frac{2}{3}}\right)\Delta_{1,n}^{\frac{2}{3}}\Delta_{2,n}^{\frac{1}{3}}, \Delta_{1,n}\Delta_{2,n}\right). \tag{3.44}$$

Thus, to conclude the proof we need to prove that

$$a_n := d_1(\mathbf{N}(0, S_n^2), \mathbf{N}(0, S^2)) \leq \left(\frac{2}{\pi}\right)^{\frac{1}{2}} d_1(S_n, S). \tag{3.45}$$

Let $A_n \sim S_n$ and $A \sim S$. Let $N \sim \mathbf{N}(0,1)$ independent of A and A_n. Then $(A_n N, AN)$ is a coupling of $(\mathbf{N}(0, S_n^2), \mathbf{N}(0, S^2))$. Hence, by the formulation of the Monge-Kantorovich distance as an infimum over couplings, we find that:

$$a_n \leq \mathbb{E}|(A - A_n)N| = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \mathbb{E}|A - A_n|. \tag{3.46}$$

Minimizing over all couplings (A, A_n) proves the claim. This completes the proof. ☐

Remark 8. From the proof, we see that working with the Monge-Kantorovich distance is crucial. For instance, we do not know if $d_3(\mathbf{N}(0, S^2), \mathbf{N}(0, T^2)) \leq c d_3(S, T)$, for some $c > 0$.

3.2.3 Proofs of the technical bounds

We start by proving in details the bounds involving $[\cdot,\cdot]_\Gamma$ that is more involved, then we explain how to adapt the proof for $[\cdot,\cdot]_\nu$.

Proof of Proposition 3.4. By the assumptions on u and F, we have that $\int |u(z)|_{\ell^2} |D_z^+ F|_{\ell^2} \nu(\mathrm{d}z) < \infty$, and we can assume that $\int |u(z)|_{\ell^2} |D_z^+ F|_{\ell^2}^2 \nu(\mathrm{d}z) < \infty$ (otherwise there is nothing to prove). Let $(s_t)_{t \in [0,1]}$ be a smooth $[0,1]$-valued path such that $s_0 = 0$ and $s_1 = 1$, and define

$$F_t = s_t F + s_{1-t} SN. \tag{3.47}$$

Let $g(t) = \mathbb{E}\phi(F_t)G$. Then,

$$\mathbb{E}\phi(F)G - \mathbb{E}\phi(SN)G = \int_0^1 \dot{g}_t \mathrm{d}t. \tag{3.48}$$

An explicit computation yields

$$\dot{g}_t = \mathbb{E}[\langle \nabla\phi(F_t), (\dot{s}_t F - \dot{s}_{1-t} SN)G\rangle_{\ell^2}]. \tag{3.49}$$

Since $\mathscr{D}\mathrm{om}\, D$ is a linear space, in view of the assumptions, $F_t \in \mathscr{D}\mathrm{om}\, D$. Since $\nabla\phi$ is Lipschitz, $\nabla\phi(F_t) \in \mathscr{D}\mathrm{om}\, D$. Using the integration by part formula Lemma 2.8, we find that

$$\begin{aligned}\mathbb{E}\langle \nabla\phi(F_t)G, F\rangle_{\ell^2} &= \frac{1}{2}\mathbb{E}G\left\langle \nabla^2\phi(F_t), s_t[u, DF]_\Gamma + s_{1-t}[u,(DS)N]_\Gamma\right\rangle_{\ell^2} \\ &+ \frac{1}{2}\mathbb{E}\left\langle \nabla\phi(F_t), [u, DG]_\Gamma\right\rangle_{\ell^2} \\ &+ \frac{1}{8}\mathbb{E}G\int \left\langle u(z)\otimes (D_z^+F_t)^{\otimes 2}, \hat{R}(F_t, z, \nabla\phi)\right\rangle_{\ell^2}\nu(\mathrm{d}z) \\ &- \frac{1}{8}\mathbb{E}G\int \left\langle (1-D_z^-)u(z)\otimes (D_z^-F_t)^{\otimes 2}, \check{R}(F_t, z, \nabla\phi)\right\rangle_{\ell^2}\eta(\mathrm{d}z).\end{aligned} \tag{3.50}$$

Recall that, by integration by parts, $\mathbb{E}N\psi(N) = \mathbb{E}\nabla\psi(N)$, for all smooth ψ. Let

$$\psi(x) = G\partial_{ij}\phi(s_t F + s_{1-t}Sx). \tag{3.51}$$

Then,

$$\partial_k\psi(x) = s_{1-t}G\sum_l S_{lk}\partial_{ijl}(s_t F + s_{1-t}Sx). \tag{3.52}$$

As a consequence, by the previous Gaussian integration by parts:

$$\mathbb{E}G\langle \nabla^2\phi(F_t), [u,(DS)N]_\Gamma\rangle_{\ell^2} = s_{1-t}\mathbb{E}G\langle \nabla^3\phi(F_t), [u,(DS)S^T]_\Gamma\rangle_{\ell^2}. \tag{3.53}$$

Furthermore, by Gaussian integration by parts, we obtain that

$$\mathbb{E}\langle \nabla\phi(F_t), SN\rangle_{\ell^2} = s_{1-t}\mathbb{E}\langle \nabla^2\phi(F_t), SS^T\rangle_{\ell^2}. \tag{3.54}$$

Combining (3.49), (3.50), (3.53) and (3.54), we find that

$$\begin{aligned}\dot{g}_t &= \frac{1}{2}\mathbb{E}G\left\langle \nabla^2\phi(F_t), \left(s_t\dot{s}_t\widetilde{[u, DF]}_\Gamma - s_{1-t}\dot{s}_{1-t}SS^T\right)\right\rangle_{\ell^2} \\ &+ \frac{1}{2}\dot{s}_t s_{1-t}^2\mathbb{E}G\langle \nabla^3\phi(F_t), [u(DS)S^T]_\Gamma\rangle_{\ell^2} \\ &+ \frac{1}{2}\dot{s}_t\mathbb{E}\left\langle \nabla\phi(F_t), [u, DG]_\Gamma\right\rangle_{\ell^2} \\ &+ \dot{s}_t\frac{1}{8}\mathbb{E}G\int \left\langle u(z)\otimes (D_z^+F_t)^{\otimes 2}, \hat{R}(F_t, z, \nabla\phi)\right\rangle_{\ell^2}\nu(\mathrm{d}z) \\ &- \dot{s}_t\frac{1}{8}\mathbb{E}G\int \left\langle (1-D_z^-)u(z)\otimes (D_z^-F_t)^{\otimes 2}, \check{R}(F_t, z, \nabla\phi)\right\rangle_{\ell^2}\eta(\mathrm{d}z).\end{aligned} \tag{3.55}$$

Observe that $|\hat{R}+\check{R}|_{\ell^2} \leq \Phi_3$, and that, if we take the absolute value of the integrands in the two last lines then the expectations become equal (by the Mecke formula (2.1)). Hence, by the Cauchy-Schwarz inequality, we find that

$$\begin{aligned}|\dot{g}_t| &\leq \frac{1}{2}|G|_\infty\Phi_2 s_t\dot{s}_t\left\langle \nabla^2\phi(F_t), \left(s_t\dot{s}_t\widetilde{[u, DF]}_\Gamma - s_{1-t}\dot{s}_{1-t}SS^T\right)\right\rangle_{\ell^2} \\ &+ \frac{1}{2}\Phi_3\dot{s}_t s_{1-t}^2|G|_\infty\mathbb{E}|[u,(DS)S^T]_\Gamma|_{\ell^2} \\ &+ \frac{1}{2}\Phi_1\dot{s}_t\mathbb{E}|[u, DG]_\Gamma|_{\ell^2} \\ &+ \frac{1}{4}\dot{s}_t|G|_\infty\Phi_3\mathbb{E}\int |u(z)|_{\ell^2}|D_z^+F_t|_{\ell^2}^2\nu(\mathrm{d}z).\end{aligned} \tag{3.56}$$

By expending the square in $|D_z^+ F_t|_{\ell^2}^2$, the cross term vanishes (by the fact that N is centered and independent of η). By the fact that N is a normal vector independent of η, we also find that $\mathbb{E}[|(D_z^+ S)N|_{\ell^2}^2|\eta] = |(D_z^+ S)|_{\ell^2}^2$. Following these observations, the results is obtained by selecting $s_t = t^{\frac{1}{2}}$ (other choices of s could possibly yield better constants). The reader can immediately verify that with this choice for s, we have that

$$\int_0^1 \dot{s}_t \mathrm{d}t = 1; \tag{3.57}$$

$$\int_0^1 s_t \dot{s}_t \mathrm{d}t = \int_0^1 s_{1-t} \dot{s}_{1-t} = \frac{1}{2}; \tag{3.58}$$

$$\int_0^1 \dot{s}_t s_{1-t}^2 = \int_0^1 s_t^2 \dot{s}_t = \frac{2}{3}. \tag{3.59}$$

This concludes the proof. □

Proof of Proposition 3.3. The strategy of proof is the same and we simply highlight the differences with the previous proof. We have to consider instead $g(t) = \mathbb{E}\phi(F_t + I_1(h))$ for some $h \in \mathscr{L}^2(\nu)$. Then, using Lemma 2.9, we find that

$$\begin{aligned} \mathbb{E}\langle \nabla\phi(F_t + I_1(h)), F\rangle_{\ell^2} &= \frac{1}{2}\mathbb{E}G \left\langle \nabla^2\phi(F_t), s_t[u, DF]_\nu + s_{1-t}[u, (DS)N]_\nu \right\rangle_{\ell^2} \\ &+ \frac{1}{4}\mathbb{E}G \int \left\langle (u(z) + h(z)) \otimes (D_z^+ F_t)^{\otimes 2}, \hat{R}(F_t, z, \nabla\phi) \right\rangle_{\ell^2} \nu(\mathrm{d}z) \end{aligned} \tag{3.60}$$

The rest of the proof is identical to the previous one. □

3.3 Comparison with existing results

First, on the Gaussian space, the authors of [25, 13, 26] work with iterated Skorokhod integrals of any order $q \in \mathbb{N}$. That is, given a Gaussian functional F and given u such that $F = \delta^q u$, they give probabilistic conditions in terms of u and F for stable convergence of F to a Gaussian mixture. Theorems 3.1 and 3.5 are the Poisson version of their results for the case $q = 1$. Due to the lack of diffusiveness on the Poisson space, it does not seem possible to reach a result involving iterated Kabanov integrals, via our method of proof, that is, via integration by parts.

Second, (S_Γ) enforces that the convergence of $C_\Gamma = [u, DF]$ (or its symmetrized version) determines the asymptotic covariance. The comparison of C_Γ and SS^T is similar in the Gaussian case [25]: the quantity $\langle DF, u\rangle$ (where D is the Malliavin derivative on the Gaussian space) controls the asymptotic variance of the functional $F = \delta u$. In this respect, let us refer to [27, Theorem 5.3.1] for deterministic variance (for the choice $u = -DL^{-1}F$), to [25, Theorem 3.1], to [13, Theorem 3.2] and to [26, Theorem 5.1] for random asymptotic variances. However, we see from (S_ν) that another relevant quantity to consider is $C_\nu = [DF_n, u_n]_\nu$. The matrix C_ν would also correspond in the Gaussian setting to $\langle u, DF\rangle$ since $\Gamma(F)$ and $|DF|^2$ coincide on the Gaussian space. As already observed by [8], working with C_Γ rather than C_ν is critical in obtaining a fourth moment theorem. We also work with C_Γ to obtain our stable version of their fourth moment theorem. When working with deterministic covariances one can choose C_ν and still obtain sufficient conditions for convergence of Poisson functionals to a Gaussian (see, for instance [18, 17, 39]).

Our condition (W_ν) is the exact counterpart of the condition $\langle u_n, h\rangle \to 0$ (see [25, Remark 3.2]) in the Gaussian setting, enforcing some asymptotic independence. When working with the energy bracket, we have (W_Γ) that we can also regard as an asymptotic independence condition. (RS) plays the same role, in our setting, as $\langle u, DS^2\rangle \to 0$ in [26]. On the Gaussian space, by the chain rule, $DS^2 = 2SDS$. In our case we cannot have this simplification, which implies that we have to formulate our condition in terms of SDS. This adds an extra difficulty since, in practice, the convergence of C_ν or C_Γ only provides information on SS^T but not on S. As the condition with DS^2 is already present in the Gaussian setting [26], we do not expect that the condition (RS) could disappear in general. The condition (R_3) is specific to the Poisson setting. Controlling quantities of the form $\int |D_z^+ L^{-1}F||D_z^+ F|^2 \nu(\mathrm{d}z)$ is standard in the theory of limit theorems for Poisson functionals and already appeared in the first result on the Malliavin-Stein method on the Poisson space [32, Theorem 3.1], as well as in the proof of the fourth moment theorem on the Poisson space [8, Equation 4.2]. These correspond to the choice $u = -DL^{-1}F$ in (R_3). In our case we have an extra term of the form $\int |u(z)||D_z^+ S|^2 \nu(\mathrm{d}z)$. This term is also the result of the lack of a chain rule and we do not expect we could remove it.

Furthermore, the authors of [25, 13, 26] only consider results involving the convergence in $\mathscr{L}^1(\mathbb{P})$ of the Stein matrix C_ν, thus imposing measurability with respect to the underlying Gaussian process on the limit covariance. In our case, when the limiting covariance is non-negative, we can replace the condition of convergence in $\mathscr{L}^1(\mathbb{P})$ by the weaker form of stable convergence to obtain Theorem 3.6. This modification relies on our quantitative bounds, which is why, in this case we need to check (RS) while Theorem 3.1 does not need to enforce this condition. Being quantitative, the results of [26] could also be modified in order to obtain a result similar to Theorem 3.6 with the same proof as the one we gave in the Poisson setting.

Lastly, in the multidimensional case, our bound in Theorem 3.5 holds for every symmetric covariance random matrix $C = SS^T$, while the results of [26] are limited to the case of a diagonal matrix. [13] also deals with generic matrices but relies on the so-called method of the characteristic function that is not known to provide quantitative bounds.

On the other hand, the convergence to Poisson mixtures was not considered for Gaussian functionals (recall that by [27, Theorem 2.10.1] random variables in a fixed Wiener chaos are absolutely continuous with respect to the Lebesgue measure). Several authors have applied the Malliavin-Stein approach on the Poisson space to consider convergence to a Poisson random variable with deterministic mean. The work of Peccati [33] is the first result in that direction. Selecting $u_n = -DL^{-1}F_n$ and $M = \mathbb{E}M = c$ in (M_ν) exactly yields the condition of [33, Proposition 3.3]: $\langle -DL^{-1}F_n, DF_n\rangle_{\mathscr{L}^2(\nu)} \to c$ (remark that [33] works with non-centered random variables). For Poisson approximation, the above discussion on the difference between S_D and S_Γ does not apply as we only obtain a condition involving S_D (see Remark 5). Our condition (P_3) is similar to the one in [33].

Contrary to [33], we cannot obtain quantitative bounds for Poisson approximation. In fact, we do not know how to adapt the methods in Section 3.2 to reach estimates for the distance of a Poisson functional to a Poisson mixture. Indeed, our approach towards quantitative estimates relies on the computability of the Malliavin derivative of a Gaussian mixture, since they always can be written SN with N independent of η, and in this case $D(SN) = (DS)N$. However, if $N(M)$ is a Poisson mixture directed by M, we have:

$$D_z N(M) = N(1_{[0,M+D_zM]}) - N(1_{[0,M]}) - D_z M. \tag{3.61}$$

The computations with this quantity seem not tractable, and we need new techniques to tackle this problem; we reserve exploring this direction of research for future works.

4 Convergence of stochastic integrals

Outline

We apply the results of Section 3 to stochastic integrals. In particular, we deduce Proposition 4.1, that is a stable version of the fourth moment theorem of Döbler & Peccati [8], and

Döbler, Vidotto & Zheng [9]; and Theorems 4.2 and 4.3 that give sufficient conditions for a sequence of Itô-Poisson integrals of order 2 to converge to a Gaussian or Poisson mixture.

4.1 A stable fourth-moment theorem for normal approximation

In a recent reference, [9] proves a multidimensional fourth-moment theorem on the Poisson space, thus refining and generalizing the previous findings of [8]. It is worth noting that taking $G = 0$ and S deterministic in (3.24) yields the same bound as [8, Equation 4.2]. In fact, as a first application of Theorem 3.6, we deduce a stable fourth-moment theorem on the Poisson space.

Proposition 4.1 (Stable fourth-moment theorem)**.** *Let $q_1, \dots, q_d \in \mathbb{N}$. For $n \geq 1$, let $(f_n^i) \subset \mathscr{L}^2(\mu^{q_i})$ and let $F_n = (I_{q_1}(f_n^1), \dots, I_{q_d}(f_n^d))$. Assume that (F_n) is bounded in $\mathscr{L}^2(\mathbb{P})$. Then, the following are equivalent:*

(i) F_n converges stably to a Gaussian vector.

(ii) For all $i \in [d]$, F_n^i converges in law to a Gaussian random variable.

(iii) For all $i \in [d]$, $\mathbb{E}\big(F_n^i\big)^4 - 3\Big(\mathbb{E}\big(F_n^i\big)^2\Big)^2 \xrightarrow[n\to\infty]{} 0.$

(iv) $\mathbb{V}\mathrm{ar}\,\Gamma(L^{-1}F_n, F_n) \xrightarrow[n\to\infty]{} 0.$

Remark 9. If either of the conditions of the theorem is satisfied then, as $n \to \infty$, $\Gamma(L^{-1}F_n, F_n) \to \sigma\sigma^T$ in $\mathscr{L}^2(\mathbb{P})$, where σ is some deterministic matrix. The covariance of the limit Gaussian vector is $\sigma\sigma^T$.

Remark 10. Proposition 4.1 is very close to [5, Theorem 2.22]. However, one condition of their theorem requires that the norms of each of the individual star-contractions vanish. This is strictly stronger than a vanishing fourth-moment as, by the product formula, this condition translates in vanishing properly chosen linear combinations of the star-contractions (see [6]).

Proof. It is clear that (i) implies (ii). That (ii) implies (and in fact is equivalent to) (iii) is the main finding of [8]. That (iii) implies (iv) is a consequence of [9, Equation 4.3, Lemma 4.1]. Let us prove (iv) implies (i). Under (iv), $-\Gamma(L^{-1}F_n, F_n) \to \sigma\sigma^T$, in $\mathscr{L}^2(\mathbb{P})$, as $n \to \infty$. We apply Theorem 3.1 with $S = \sigma$ and

$$u_n(z) = -\Big(I_{q_1-1}(f_n^1(z,\cdot)), \dots, I_{q_d-1}(f_n^d(z,\cdot))\Big) = -D_z L^{-1} F_n. \tag{4.1}$$

We have that $\delta u_n = -\delta D L^{-1} F_n = F_n$, and that $[u_n, F_n]_\Gamma = -\Gamma(L^{-1}F_n, F_n)$. Thus, ($\mathrm{S}_\Gamma$) is satisfied. From [20, Lemma 3.4], we have that

$$\mathbb{E}\int |D_z L^{-1} F_n|_{\ell^2}^2 \nu(\mathrm{d}z) \leq \mathbb{E}\int |D_z F_n|_{\ell^2}^2 \nu(\mathrm{d}z). \tag{4.2}$$

Hence, applying the Cauchy-Schwarz inequality, we find that

$$\mathbb{E}\int |u_n(z)|_{\ell^2} |D_z F_n|_{\ell^2}^2 \nu(\mathrm{d}z) \leq \sqrt{\mathbb{E}\int |D_z F_n|_{\ell^2}^2 \nu(\mathrm{d}z) \mathbb{E}\int |D_z F_n|_{\ell^2}^4 \nu(\mathrm{d}z)}. \tag{4.3}$$

By Hölder's inequality, we find that (recall $DG \in \mathscr{L}^\infty(\mathbb{P} \otimes \nu)$ by Lemma 2.1):

$$\mathbb{E}|[u_n, DG]| \leq |DG|_\infty \left(\mathbb{E}\int |DF_n|_{\ell^2}^4\right)^{1/4}. \tag{4.4}$$

The quantity

$$\mathbb{E}\int |D_z F_n|^2_{\ell^2}\nu(\mathrm{d}z) = \sum_{i\in[d]} q_i!\nu^{q_i}(|f_n^i|^2), \tag{4.5}$$

is bounded by assumption. Hence it is sufficient to show that under (iv),

$$\mathbb{E}\int |D_z F_n|^4\nu(\mathrm{d}z) \xrightarrow[n\to\infty]{} 0. \tag{4.6}$$

This follows from [8, Lemma 3.2] and [9, Remark 5.2]. The proof is complete. □

Remark 11. Let $\Sigma = (\Sigma_{ij})$ be a random matrix sufficiently integrable. If we assume that

$$-\Gamma(L^{-1}F_n, F_n) \xrightarrow[n\to\infty]{\mathscr{L}^1(\mathbb{P})} \Sigma\Sigma^T; \tag{4.7}$$

$$\mathbb{E}(F_n^i)^4 \xrightarrow[n\to\infty]{} 3\big(\mathbb{E}(\Sigma\Sigma^T)_{ii}\big)^2, \ \forall i = 1,\dots,d. \tag{4.8}$$

Then, from the previous computations, $\Sigma = \sigma$ is deterministic. This shows that fourth-moment theorems cannot capture phenomena with asymptotic random variances.

4.2 Convergence of order 2 Poisson-Wiener integrals to a mixture

We derive an analytic statement for the convergence of a sequence of random variables of the form $F = I_2(g)$ for some $g \in \mathscr{L}^2_\sigma(\nu^2)$. In this case, with $u_0(z) = -D_z L^{-1}F = I_1(g(z,\cdot))$, we have $F = \delta u_0$ However, for every $\hat{g} \in \mathscr{L}^2(\nu^2)$ such that the symmetrization of $\hat{g}$ is g, we also have that $u(z) = I_1(\hat{g}(z,\cdot))$ is a solution to $\delta u = F$. Having made this observation, we can thus specify our Theorems 3.1 and 3.2 to the particular case where F is an Poisson-Wiener stochastic integral of order 2.

Theorem 4.2. *Consider the sequence of random variables $\{F_n = I_2(g_n); n \in \mathbb{N}\}$ for some $(g_n) \subset \mathscr{L}^2_\sigma(\nu^2)$. Suppose that there exists $(\hat{g}_n) \subset \mathscr{L}^2(\nu^2)$ such that, for all $n \in \mathbb{N}$ the symmetrization of $\hat{g}_n$ is g_n. Assume, moreover, that:*

$$\begin{cases} g_n \star_1^1 \hat{g}_n \xrightarrow[n\to\infty]{\mathscr{L}^2(\nu^2)} g_{2,\infty}, \\ \nu^2(g_n\hat{g}_n) \xrightarrow[n\to\infty]{} g_{0,\infty}; \end{cases} \tag{KS}$$

$$g_n \xrightarrow[n\to\infty]{\mathscr{L}^4(\nu^2)} 0; \tag{KR$_4$}$$

$$g_n \star_2^1 g_n \xrightarrow[n\to\infty]{\mathscr{L}^2(\nu)} 0; \tag{KR$_\star$}$$

$$\hat{g}_n \star_1^1 h \xrightarrow[n\to\infty]{\mathscr{L}^2(\nu)} 0. \tag{KW}$$

Assume that $S^2 = I_2(g_{2,\infty}) + g_{0,\infty} \geq 0$. Then $F_n \xrightarrow[n\to\infty]{stably} \mathbf{N}(0, S^2)$.

Remark 12. Following Theorem 3.8, it is of course possible to write conditions for the convergence in the Monge-Kantorovich distance d_1 in terms of the norms of the kernels. However, this task seems tedious and not particularly useful in this abstract setting.

Theorem 4.3. *Consider the sequence of random variables $\{F_n = I_2(g_n); n \in \mathbb{N}\}$ for some $(g_n) \subset \mathscr{L}^2_\sigma(\nu^2)$. Suppose that there exists $(\hat{g}_n) \subset \mathscr{L}^2(\nu^2)$ such that, for all $n \in \mathbb{N}$ the symmetrization of $\hat{g}_n$ is g_n. Assume that (KS), (KW) and (KR$_\star$) hold and that*

$$\nu^2\left(g_n^2\left(g_n - \frac{1}{2}\right)^2\right) \xrightarrow[n\to\infty]{} 0. \tag{KP$_4$}$$

Assume that $M = I_2(g_{2,\infty}) + g_{0,\infty} \geq 0$. Then $F_n \xrightarrow[n\to\infty]{stably} \mathbf{Po}(M)$.

Proof of Theorems 4.2 and 4.3. We prove the two theorems at once. We simply apply Theorems 3.1 and 3.2 to our data. For simplicity, we drop the dependence in n. Let $u = I_1(\hat{g})$. Let us compute $[DF, u]_\nu = \nu(DFu)$ in that case. By the product formula (2.8), we have that

$$\begin{aligned}[DF, u]_\nu = \nu(uDF) &= \int I_1(g(z,\cdot))I_1(\hat{g}(z,\cdot))\nu(\mathrm{d}z) \\ &= \int I_2(g(z,\cdot)\otimes\hat{g}(z,\cdot)) + I_1(g(z,\cdot)\hat{g}(z,\cdot)) + \nu(g(z,\cdot)\hat{g}(z,\cdot)).\end{aligned} \tag{4.9}$$

By linearity of I_1 and I_2, we thus find

$$\nu(uDF) = I_2(g \star_1^1 \hat{g}) + I_1(g \star_2^1 \hat{g}) + \nu^2(g\hat{g}). \tag{4.10}$$

By [7, Lemma 2.4 (vi)] (which according to the proof holds for any σ-finite measure ν), we have that $\nu(|g \star_2^1 \hat{g}|^2) \leq \nu(|g \star_2^1 g|^2)^{\frac{1}{2}}\nu(|\hat{g} \star_2^1 \hat{g}|^2)^{\frac{1}{2}}$. Hence, we see that (KS) and (KR$_\star$) implies either (S$_\nu$) or (M$_\nu$) with S^2 or M as given in the statement. On the one hand, we have that

$$\begin{aligned}\frac{1}{16}\mathbb{E}\int |D_zF|^4\nu(\mathrm{d}z) &= \int \mathbb{E}I_1(g(z,\cdot))^4\nu(\mathrm{d}z) \\ &= 3\int\left(\int g(y,z)^2\nu(\mathrm{d}y)\right)^2\nu(\mathrm{d}z) + \int\int g(y,z)^4\nu(\mathrm{d}y)\nu(\mathrm{d}z) \\ &= 3\nu\left((g \star_2^1 g)^2\right) + \nu^2(g^4).\end{aligned} \tag{4.11}$$

So that (KR$_4$) and (KR$_\star$) readily implies (R$_4$). On the other hand:

$$\frac{1}{16}\mathbb{E}\int |D_zF(D_zF-1)|^2\nu(\mathrm{d}z) = \nu^2\left(g^2\left(g - \frac{1}{2}\right)^2\right) + 3\nu\left((g \star_2^1 g)^2\right). \tag{4.12}$$

We thus see that (KP$_4$) and (KR$_\star$) implies (P$_4$). Finally, we find that

$$\nu(uh) = I_1(\hat{g} \star_1^1 h). \tag{4.13}$$

Consequently, by Itô's isometry (2.5), we find that (KW) implies (W$_\nu$). □

5 Convergence of a quadratic functional of a Poisson process on the line

In this section, we apply our abstract result to study the asymptotic of a particular quadratic functional. Let us recall one of the main applications of [25, 26], refining a result of [36].

Theorem 5.1 ([25, Example 4.2] and [26, Theorem 3.7]). *Let W be a standard Brownian motion on $[0,1]$ and let*

$$Q_n = \frac{n^{\frac{3}{2}}}{\sqrt{2}}\int_0^1 t^{n-1}(W_1^2 - W_t^2)\mathrm{d}t, \quad n \in \mathbb{N}. \tag{5.1}$$

Then,

$$Q_n \xrightarrow[n\to\infty]{stably} \mathbf{N}\left(0, W_1^2\right). \tag{5.2}$$

Moreover, there exists $c > 0$ such that, for all $n \in \mathbb{N}$:

$$d_1\left(Q_n, \mathbf{N}\left(0, W_1^2\right)\right) \leq cn^{-\frac{1}{6}}. \tag{5.3}$$

Let η be a Poisson point process on $\mathbb{R}_+$ with intensity the Lebesgue measure; and $\hat{N}_t = \eta([0,t]) - t$, for $t \in \mathbb{R}$. The process $\hat{N}$ is a martingale called a *compensated Poisson process on the line*. Recall that from Dynkin & Mandelbaum [11], we have that

$$\left\{n^{-\frac{1}{2}}\hat{N}_{nt};\ t \geq 0\right\} \xrightarrow[n\to\infty]{} W, \tag{5.4}$$

where the convergence holds in the sense of finite-dimensional distributions and in a stronger sense that we do not detail here. Having made this remark the following thermo-dynamical limit appears as a natural generalization of Theorem 5.1.

Theorem 5.2. *Let*

$$Q_n = \frac{n^{\frac{3}{2}}}{\sqrt{2}} \int_0^1 t^{n-1} \left(\left(n^{-\frac{1}{2}}\hat{N}_n\right)^2 - \left(n^{-\frac{1}{2}}\hat{N}_{nt}\right)^2 \right) \mathrm{d}t, \quad n \in \mathbb{N}. \tag{5.5}$$

Then,

$$Q_n \xrightarrow[n\to\infty]{stably} \mathbf{N}\left(0, W_1^2\right). \tag{5.6}$$

Proof. By Itô's formula (see, for instance [15, Chapter I Theorem 4.57]), we have that

$$\hat{N}_t^2 = 2\int_0^t \hat{N}_{s-}\mathrm{d}\hat{N}_s + \sum_{s\leq t}\left(\hat{N}_s - \hat{N}_{s-}\right)^2. \tag{5.7}$$

Since, a Poisson process only has jumps of size 1, we find that

$$\hat{N}_t^2 = 2\int_0^t \hat{N}_{s-}\mathrm{d}\hat{N}_s + N_t. \tag{5.8}$$

Hence, we can write

$$Q_n = \sqrt{2}F_n + H_n, \tag{5.9}$$

where

$$F_n = n^{\frac{1}{2}} \int_0^1 t^{n-1} \int_{nt}^n \hat{N}_{s-}\mathrm{d}\hat{N}_s \mathrm{d}t; \tag{5.10}$$

$$H_n = \left(\frac{n}{2}\right)^{\frac{1}{2}} \int_0^1 t^{n-1}(N_n - N_{nt})\mathrm{d}t. \tag{5.11}$$

Recalling that N is a non-decreasing process and that $\mathbb{E}N_t = t$, we find that

$$\begin{aligned}
\mathbb{E}|H_n| &= 2^{-\frac{1}{2}}n^{\frac{3}{2}} \int_0^1 t^{n-1}(1-t)\mathrm{d}t \\
&= 2^{-\frac{1}{2}}n^{\frac{3}{2}}\left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= 2^{-\frac{1}{2}}\frac{n^{\frac{1}{2}}}{n+1} = O(n^{-\frac{1}{2}}).
\end{aligned} \tag{5.12}$$

Consequently, in order to obtain the conclusions of the theorem for (Q_n) it suffices to obtain them for (F_n). By inverting the order of integration, we find:

$$F_n = n^{-\frac{1}{2}} \int_0^n \hat{N}_{s-}\left(\frac{s}{n}\right)^n \mathrm{d}\hat{N}_s = n^{-\frac{1}{2}-n}\int_0^n \hat{N}_{s-}s^n\mathrm{d}\hat{N}_s = \delta u_n, \tag{5.13}$$

where

$$u_n(s) = n^{-\frac{1}{2}}\hat{N}_s 1_{[0,n]}(s)\left(\frac{s}{n}\right)^n. \tag{5.14}$$

We have that

$$F_n = n^{-\frac{1}{2}}\hat{N}_n n^{-n}\int_0^n s^n \mathrm{d}\hat{N}_s + n^{-\frac{1}{2}-n}\int_0^n (\hat{N}_{s^-} - \hat{N}_n)s^n \mathrm{d}\hat{N}_s. \tag{5.15}$$

Now observe that, by Skorokhod's isometry:

$$\begin{aligned}\mathbb{E}\left(n^{-\frac{1}{2}-n}\int_0^n (\hat{N}_{s^-} - \hat{N}_n)s^n \mathrm{d}\hat{N}_s\right)^2 &= n^{-2n-1}\int_0^n (s-n)^2 s^{2n}\mathrm{d}s \\ &= \frac{n}{(2n+1)(2n+2)} = O(n^{-1}).\end{aligned} \tag{5.16}$$

We have that

$$\mathbb{E}\left(n^{-\frac{1}{2}}\hat{N}_n\right)^2 = 1; \tag{5.17}$$

$$\mathbb{E}\left(n^{-\frac{1}{2}}\hat{N}_n\right)^4 = 3 + n^{-1}; \tag{5.18}$$

$$\mathbb{E}\left(n^{-n}\int_0^n s^n \mathrm{d}\hat{N}_s\right)^2 = \frac{n}{2n+1}; \tag{5.19}$$

$$\mathbb{E}\left(n^{-n}\int_0^n s^n \mathrm{d}\hat{N}_s\right)^4 = \frac{n}{4n+1} + 3\frac{n^2}{(2n+1)^2}. \tag{5.20}$$

By our stable fourth moment theorem Proposition 4.1, we immediately find that:

$$\left(n^{-\frac{1}{2}}\hat{N}_n, n^{-n}\int_0^n s^n \mathrm{d}\hat{N}_s\right) \xrightarrow[n\to\infty]{stably} \mathbf{N}\left(0, \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2}\end{pmatrix}\right). \tag{5.21}$$

Thus proving that $\sqrt{2}F_n \xrightarrow[n\to\infty]{stably} \mathbf{N}(0, W_1^2)$, and hence that $Q_n \xrightarrow[n\to\infty]{stably} \mathbf{N}(0, W_1^2)$. □

Remark 13. Rather than studying δu_n with Theorem 3.6, we simplify the problem by studying the convergence of two Itô-Wiener integrals. In fact, in our example, (R_4) is not satisfied. With the notations of the proof, we have that $D_s F_n = n^{-n-\frac{1}{2}}\int_0^n (s\vee t)^n \mathrm{d}\hat{N}_t$. An easy computation yields that

$$\int_0^n \mathbb{E}(D_s F_n)^4 \mathrm{d}s \xrightarrow[n\to\infty]{} \frac{1}{4}.$$

We do not know if (R_3) holds, so we do not know if we could use Theorem 3.6 directly (or even invoke Theorem 3.9 to get a quantitative estimate).

6 Some open questions

- As already mentioned, we are interested in understanding which techniques we should consider to reach quantitative estimates for the convergence to a Poisson mixture.
- According to [26, Remark 3.3 (b)], the results of [25] can be understood as a variant of the *asymptotic Knight theorem* about the convergence of Brownian martingales. In the Poisson setting, it would be interesting to know if our results can be put in contrast with a corresponding martingale result.

- Very commonly, quantitative limit theorems in stochastic geometry rely on Malliavin-Stein bounds on the Poisson space (see among others [39, 18, 17, 33]). In particular, counting statistics of a nice class of rescaled geometric random graphs construct from a Poisson point process exhibit a Gaussian or Poisson asymptotic behaviour depending on the regime of the rescaling. In view of our results, we ask whether it is possible to consider a wider class of geometric random graphs (including the previous one) whose counting statistics exhibit a convergence to a mixture.

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