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A smooth extension method for transmission problems

Astrid Decoene[1], Sébastien Martin[2], and Fabien Vergnet[*,1,2]

[1]Laboratoire de Mathématiques d'Orsay, Univ. Paris-Sud, CNRS, Université Paris-Saclay, 91405 Orsay, France.
[2]Laboratoire MAP5, Université Paris Descartes, CNRS, 75006 Paris, France.

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Abstract

In this work, we present a numerical method for the resolution of transmission problems with non-conformal meshes which preserves the optimal rates of convergence in space. The smooth extension method is a fictitious domain approach based on a control formulation stated as a minimization problem, that we prove to be equivalent to the initial transmission problem. Formulated as a minimization problem, the transmission problem can be solved with standard finite element function spaces and usual optimization algorithms. The method is applied to different transmission problems (Laplace, Stokes and a fluid-structure interaction problem) and compared to standard finite element methods.

Keywords. *Partial differential equation, Transmission problem, Fictitious domain method, Control and optimization, Fluid-structure interaction*

1 Introduction

The numerical simulation of transmission problems (also known as interface problems or problems with discontinuous coefficients) is of major importance for the mathematical study of many physical and living systems. As examples, we can mention the study of composite materials (see [OK06]), the flow of multiphasic fluids or the interaction of fluid and structure problems (see [DMM18]). There exists a wide variety of numerical methods for this type of problems, including conformal and non-conformal mesh techniques.

Conformal mesh methods are successful when dealing with stationary problems or problems with domains ongoing moderate displacements. In the latter, the mesh is moved using a regular extension of the displacement of the interface, which is only possible in practice if this displacement is small enough. For example, in the case of fluid-structure interaction problems it can be done using an arbitrary Lagrangian-Eulerian (ALE) description of the fluid (see [dS07]). The main advantage of fitted mesh techniques is their optimal convergence rate in space. However, when transmission problems involve large deformations, a much more appropriate method is the use of non-conformal meshes approaches, in which the interface deforms independently of a background fixed mesh.

Non-conformal mesh techniques use a mesh that does not fit the interface. Compared to conformal mesh methods, these approaches have numerous advantages to fasten the numerical resolution of transmission problems. For example, they allow to consider Cartesian meshes in order to use fast solvers or to accurately mesh small included materials without necessarily considering a thin mesh on the whole domain. Among these methods we can reference some classical methods such as fictitious domain methods with Lagrange multipliers (see [Baa01], [Yu05]) or penalization terms (see [JLM05]) and the immersed boundary method (see [Pes02]) which are easy to implement but do not convergence with optimal orders in space (see [GG95], [GGLV01], [Tom97], [Mau09]), because of the discrete treatment of transmission conditions. More recent methods, such as the extended-finite element method (XFEM, see [MB02], [FL+14] and [FL17]) or the Nitsche-XFEM method (see [AFFL16]) circumvent theses difficulties, but they require a specific evaluation of the interface intersections, which can be difficult, especially in three space dimensions (see [BKFG19]). An other fictitious domain approach designed to recover the optimal convergence at any order is the fat boundary method (FBM, see [BIM11]). It is well suited for elliptic stationary problems or problems involving rigid domains, but it is not straightforward to adapt it to deformable moving materials. An other class of non-conformal mesh techniques is the control based approach presented in [ADG+91], initially developed to solve boundary value problems in complex geometries and which is based on an optimal control formulation. In [ADG+91], the authors consider the Poisson problem for the Laplace operator with an included obstacle on which Dirichlet and Neumann boundary conditions are applied. The idea of the method is to extend the problem inside the obstacle and use the right-hand side of the equations as a control to impose the boundary conditions on the frontier of the obstacle. This method has also been used to treat boundary conditions in a fictitious domain approach for the Helmoltz equations in [ADG+91], [AJ93] and [Per98]. In [Fab12] and [FGM13], the authors present the *smooth extension method*, an extension of the control based method to the resolution of fluid-structure interaction problems involving rigid particles and a low-Reynolds number fluid. This method has the advantages to recover optimal

*Corresponding author: fabien.vergnet@math.u-psud.fr

convergence in space and to be rather simple to implement (it does not require any mesh adaptation or local enrichment of the function spaces).

In this method-oriented article, we aim to extend this smooth extension method to general transmission problems; in particular to problems where the behavior of the obstacle is also described by partial differential equations with transmission conditions on the interface. This approach relies on an optimal control formulation of the transmission problem based on a least-squares formulation of the transmission conditions, which has the advantage to be particularly suitable for unfitted mesh techniques. Several transmission problems are considered: a Laplace transmission problem, a Stokes transmission problem and a linear fluid-structure interaction problem between a low-Reynolds number fluid and an elastic structure. For all of them, a constrained minimization problem is introduced, where the cost function is a least-squares formulation of the transmission conditions. We prove that the resolution of this minimization problem, which can be done with standard finite element solvers and usual gradient descent methods, enables to recover the solution of the initial transmission problem. We highlight that this method is intrinsically designed to recover optimal rates of convergence in space, which is not usually the case with fictitious domain techniques using unfitted mesh.

The remaining of the article is organized as follows. In Section 2, we state the smooth extension method for the Laplace transmission problem. Then, Section 3 is devoted to the presentation of numerical experiments using the methodology described in Section 2. Finally, in Section 4, we extend the method to other transmission problems: the Stokes transmission problem and a fluid-structure interaction problem involving a linear elastic structure and a viscous fluid.

2 Presentation of the method

The smooth extension method presented here is devoted to the numerical simulation of transmission problems. It is a fictitious domain method which enables to recover the optimal order of convergence in space by smoothly extending a part of the exact solution to the whole domain. In this section we focus on a toy model, the Laplace transmission problem, in order to properly explain the main steps of the method. This section is divided in four parts. In Subsection 2.1, we state the smooth extension formulation of the problem as a control problem and explain how to recover the exact solution from it. Then, in Subsection 2.2 we rewrite this control problem as a minimization problem and prove the equivalence of the two formulations. Finally, in Subsection2.3 we discuss the advantage of this method compared to the classical finite element methods with both fitted and unfitted meshes.

2.1 The smooth extension formulation

Let $n > 0$ and Ω be a domain of $\mathbb{R}^n$ that satisfies the following set of hypotheses:

$$
\left.\begin{array}{rl}
i) & \text{Domain } \Omega \text{ is a bounded connected Lipschitz domain of } \mathbb{R}^n.\\
ii) & \text{Domain } \Omega \text{ is divided in two subdomains, } \Omega_1 \text{ and } \Omega_2 \text{, which have Lipschitz}\\
 & \text{boundaries.}\\
iii) & \text{The interface } \Gamma = \partial\Omega_1 \cap \partial\Omega_2 \text{ is not empty.}\\
iv) & \text{The remaining boundaries } \Gamma_1 = \partial\Omega_1 \setminus \Gamma \text{ and } \Gamma_2 = \partial\Omega_2 \setminus \Gamma \text{ are not empty.}
\end{array}\right. \qquad (H_1)
$$

The problem we are interested in, for now, is the coupled Laplace problem with homogeneous Dirichlet boundary conditions, also called Laplace transmission problem or Laplace problem with discontinuous coefficients. It is the simplest coupled system of partial differential equations that we can think of and, given two positive real constants μ_1 and μ_2 such that $\mu_1 \neq \mu_2$ and two source terms $f_1 \in L^2(\Omega_1)$ and $f_2 \in L^2(\Omega_2)$, it writes:

$$
\left\{\begin{array}{lll}
\text{find } u_1 \colon \Omega_1 \to \mathbb{R} \text{ and } u_2 \colon \Omega_2 \to \mathbb{R} \text{ such that} & & \\
\qquad -\mu_1 \Delta u_1 = f_1 & \text{in } \Omega_1, & \\
\qquad u_1 = 0 & \text{on } \Gamma_1, & (1a)\\
\qquad -\mu_2 \Delta u_2 = f_2 & \text{in } \Omega_2, & \\
\qquad u_2 = 0 & \text{on } \Gamma_2, & (1b)\\
\qquad u_1 = u_2 & \text{on } \Gamma, & \\
\qquad \mu_1 \nabla u_1 \cdot n_1 = -\mu_2 \nabla u_2 \cdot n_2 & \text{on } \Gamma. & (1c)
\end{array}\right.
$$

This problem is completely equivalent to the more classical formulation of the Laplace problem, written in the whole domain Ω,

$$
\left\{\begin{array}{lcll}
\text{find } u \colon \Omega \to \mathbb{R} \text{ such that} & & & \\
-\operatorname{div}(\mu \nabla u) & = & f & \text{in } \quad \Omega,\\
u & = & 0 & \text{on } \quad \partial\Omega,
\end{array}\right.
$$

where μ and f are defined by

$$
\mu = \left\{\begin{array}{ll} \mu_1 & \text{in } \quad \Omega_1,\\ \mu_2 & \text{in } \quad \Omega_2, \end{array}\right. \qquad f = \left\{\begin{array}{ll} f_1 & \text{in } \quad \Omega_1,\\ f_2 & \text{in } \quad \Omega_2. \end{array}\right.
$$

However, the formulation (1) has the advantage of making the coupling between the two subproblems (1a) and (1b) clear. The so-called coupling conditions are detailed in equations (1c) and physically represent the continuity of the field and the continuity of the normal constraint through the interface Γ. Moreover, n_1 (resp. n_2) is the unit exterior normal vector of Ω_1 (resp. Ω_2). The solution of problem (1) lies in $V_1 \times V_2$, where these two functional spaces are defined by

$$V_1 = \{v_1 \in H^1(\Omega_1); v_{1|_{\Gamma_1}} = 0\}, \quad V_2 = \{v_2 \in H^1(\Omega_2); v_{2|_{\Gamma_2}} = 0\}.$$

In addition, the dual spaces of V_1 and V_2 will be denoted by V_1' and V_2'.

Let v be a distribution in $\mathcal{D}(\Omega)$ and suppose that u_1 and u_2 are sufficiently regular. We can formally multiply the first equation in (1a) by $v_{|\Omega_1}$ and the first equation in (1b) by $v_{|\Omega_2}$ and integrate respectively over Ω_1 and Ω_2. After an integration by part, we obtain

$$\mu_1 \int_{\Omega_1} \nabla u_1 \cdot \nabla v_{|\Omega_1} - \int_\Gamma (\mu_1 \nabla u_1 \cdot n_1) v + \mu_2 \int_{\Omega_2} \nabla u_2 \cdot \nabla v_{|\Omega_2} - \int_\Gamma (\mu_2 \nabla u_2 \cdot n_2) v = \int_{\Omega_1} f_1 v_{|\Omega_1} + \int_{\Omega_2} f_2 v_{|\Omega_2}, \forall v \in \mathcal{D}(\Omega).$$

Using the second transmission condition in (1c) i.e., that $\mu_1 \nabla u_1 \cdot n_1 = -\mu_2 \nabla u_2 \cdot n_2$ on Γ, it follows that

$$\mu_1 \int_{\Omega_1} \nabla u_1 \cdot \nabla v_{|\Omega_1} + \mu_2 \int_{\Omega_2} \nabla u_2 \cdot \nabla v_{|\Omega_2} \quad = \quad \int_{\Omega_1} f_1 v_{|\Omega_1} + \int_{\Omega_2} f_2 v_{|\Omega_2}, \quad \forall v \in \mathcal{D}(\Omega).$$

Then, introducing the space

$$\mathcal{V} \quad = \quad \{(v_1, v_2) \in V_1 \times V_2; v_{1|\Gamma} = v_{2|\Gamma}\},$$

we can define the weak formulation of problem (1):

$$\left\{ \begin{array}{l} \text{find } (u_1, u_2) \text{ in } \mathcal{V} \text{ such that,} \\ \mu_1 \displaystyle\int_{\Omega_1} \nabla u_1 \cdot \nabla v_1 + \mu_2 \int_{\Omega_2} \nabla u_2 \cdot \nabla v_2 \quad = \quad \int_{\Omega_1} f_1 v_1 + \int_{\Omega_2} f_2 v_2, \quad \forall (v_1, v_2) \in \mathcal{V}. \end{array} \right. \tag{2}$$

Problem (2) is well-posed, since $f_1 \in L^2(\Omega_1)$ and $f_2 \in L^2(\Omega_2)$, according to the Lax-Milgram theorem. Furthermore, the unique solution of problem (2) will be denoted by $(\overline{u}_1, \overline{u}_2)$.

At this point, we can precise in what sense the solution of (2) is also solution of the initial problem (1). Taking test functions v_1 in $\mathcal{D}(\Omega_1)$ and v_2 in $\mathcal{D}(\Omega_2)$ in (2) gives that

$$-\mu_1 \Delta \bar{u}_1 = f_1 \text{ and } -\mu_2 \Delta \bar{u}_2 = f_2 \text{ in } L^2.$$

It also implies that $\nabla \bar{u}_1$ belongs to $H_{\mathrm{div}}(\Omega_1)$ and that $\nabla \bar{u}_2$ belongs to $H_{\mathrm{div}}(\Omega_2)$, where $H_{\mathrm{div}}(\Omega_1)$ and $H_{\mathrm{div}}(\Omega_2)$ are the spaces defined by

$$H_{\mathrm{div}}(X) \quad = \quad \left\{\sigma \in (L^2(X))^n; \mathrm{div}(v) \in L^2(X)\right\},$$

and where X stands for either Ω_1 or Ω_2. In particular, we are able to give a weak sense to the normal derivatives of $\bar{u}_1$ and $\bar{u}_2$. Let X stand for either Ω_1 or Ω_2 and let η be the unit exterior normal vector of X. Let $\Lambda = H^{1/2}_{00}(\Gamma)$ be the image of $H^1_{\partial X \setminus \Gamma}(X)$ by the trace operator on the interface Γ, i.e. the space of functions in $H^{1/2}(\Gamma)$ whose extension by zero on $\partial X \setminus \Gamma$ belongs to $H^{1/2}(\partial X)$. Then, For all σ in $H_{\mathrm{div}}(X)$, we have the following Stokes formula:

$$\int_X \sigma \cdot \nabla v + \int_X \mathrm{div}(\sigma) v \quad = \quad \langle \gamma_\eta(\sigma), v \rangle_{\Lambda', \Lambda}, \quad \forall v \in H^1_{\partial X \setminus \Gamma}(X), \tag{3}$$

where $\Lambda' = (H^{1/2}_{00}(\Gamma))'$ is the dual space of Λ and γ_η denotes the normal trace operator on Γ. Then, $\bar{u}_1$ verifies the equality $-\mu_1 \Delta \bar{u}_1 = f_1$ a.e. in Ω_1, $\bar{u}_2$ verifies $-\mu_2 \Delta \bar{u}_2 = f_2$ a.e. in Ω_2, the first transmission condition, $\bar{u}_1 = \bar{u}_2$ on Γ, is included in the functional space $\mathcal{V}$ and the second one is verified in a weak sense:

$$\langle \mu_1 \gamma_{n_1}(\nabla \bar{u}_1), v \rangle_{\Lambda', \Lambda} \quad = \quad - \langle \mu_2 \gamma_{n_2}(\nabla \bar{u}_2), v \rangle_{\Lambda', \Lambda}, \quad \forall v \in \Lambda. \tag{4}$$

Now, we present the smooth extension method applied to problem (1). This method consists in *i)* extending the problem in Ω_1 into a problem defined in the whole domain Ω, *ii)* relaxing the constraint in the functional space $\mathcal{V}$ and *iii)* finding a control term in Ω_2 to enforce the condition of equality on Γ. We denote by g this control term, which should belong to V_2', and we define $\overline{g}^\Omega$, the extension of g in the whole domain Ω such that

$$\langle \overline{g}^\Omega, v \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} = \left\langle g, v_{|\Omega_2} \right\rangle_{V_2', V_2}, \forall v \in H^1_0(\Omega).$$

In a similar way, as f_1 belongs to $L^2(\Omega_1)$, $\overline{f_1}^\Omega$ denotes the extension of f_1 by 0 over the whole space Ω.

Formally, we define the smooth extension problem associated to problem (1) as the problem of finding a suitable control g in V_2', such that the solution of the following problem,

$$
\begin{cases}
\text{find } u_1\colon \Omega \to \mathbb{R} \text{ and } u_2\colon \Omega_2 \to \mathbb{R} \text{ such that} & \\
\qquad -\mu_1 \Delta u_1 = \overline{f_1}^{\Omega} + \overline{g}^{\Omega} \qquad \text{in } \Omega, & \\
\qquad u_1 = 0 \qquad \text{on } \partial\Omega, & \quad (5a) \\
\qquad -\mu_2 \Delta u_2 = f_2 \qquad \text{in } \Omega_2, & \\
\qquad u_2 = 0 \qquad \text{on } \Gamma_2, & \quad (5b) \\
\qquad \mu_2 \nabla u_2 \cdot n_2 = -\mu_1 \nabla u_1 \cdot n_1 \qquad \text{on } \Gamma, &
\end{cases}
$$

verifies the transmission condition

$$u_1 = u_2 \text{ on } \Gamma.$$

In fact, we will show that if the solution of (5a) and (5b) verifies the continuity condition on Γ, then the couple $(u_{1|\Omega_1}, u_2)$ is solution of the initial transmission problem (1), i.e.

$$(u_{1|\Omega_1}, u_2) = (\overline{u}_1, \overline{u}_2).$$

There are numerous reasons to consider this problem instead of the initial transmission problem for the numerical resolution. Subproblems (5a) and (5b) are defined on two independent domains, thus two independent meshes can be constructed to approximate their solutions. Moreover, if the domain Ω has a simple geometry, the use of a Cartesian mesh and a fast solver can drastically fasten the numerical resolution of problem (5a). Since the constraint in the function space $\mathcal{V}$ has been relaxed, problems (5a) and (5b) can be subsequently solved using standard finite element solvers. Finally, both finite element problems will converge with optimal rates in space, using $P1$ finite elements, if the control g is regular enough.

We now write problem (5) in variational formulation. let v be in $\mathcal{D}(\Omega)$, suppose that u_1 and u_2 are sufficiently regular and let us assume for the moment that g belongs to $L^2(\Omega_2)$ in order to do formal computations. We multiply equation (5a) by v and equation (5b) by $v_{|\Omega_2}$ and integrate respectively over Ω and Ω_2. After an integration by part and using the Neumann condition on Γ, we find

$$
\begin{aligned}
\mu_1 \int_\Omega \nabla u_1 \cdot \nabla v &= \int_{\Omega_1} f_1 v_{|\Omega_1} + \int_{\Omega_2} g v_{|\Omega_2}, \\
\mu_2 \int_{\Omega_2} \nabla u_2 \cdot \nabla v_{|\Omega_2} &= \int_{\Omega_2} f_2 v_{|\Omega_2} - \int_\Gamma (\mu_1 \nabla u_1 \cdot n_1) v.
\end{aligned}
$$

Furthermore, we remark that

$$
\begin{aligned}
-\int_\Gamma (\mu_1 \nabla u_1 \cdot n_1) v &= \mu_1 \int_{\Omega_2} \Delta u_{1|\Omega_2} v_{|\Omega_2} + \mu_1 \int_{\Omega_2} \nabla u_{1|\Omega_2} \cdot \nabla v_{|\Omega_2}, \\
&= -\int_{\Omega_2} g v_{|\Omega_2} + \mu_1 \int_{\Omega_2} \nabla u_{1|\Omega_2} \cdot \nabla v_{|\Omega_2}.
\end{aligned}
$$

Thus, we define the weak formulation of the smooth extension problem associated to problem (2), that makes sense for g in V_2', as the problem of finding a suitable control g in V_2', such that the solution of the following problem,

$$
\begin{cases}
\text{find } (u_1, u_2) \text{ in } H_0^1(\Omega) \times V_2 \text{ such that} \\
\mu_1 \int_\Omega \nabla u_1 \cdot \nabla v_1 = \int_{\Omega_1} f_1 v_{1|\Omega_1} + \left\langle g, v_{1_{|\Omega_2}} \right\rangle_{V_2', V_2}, & \forall v_1 \in H_0^1(\Omega), \\
\mu_2 \int_{\Omega_2} \nabla u_2 \cdot \nabla v_2 = \int_{\Omega_2} f_2, v_2 - \langle g, v_2 \rangle_{V_2', V_2} + \mu_1 \int_{\Omega_2} \nabla u_{1|\Omega_2} \cdot \nabla v_2, & \forall v_2 \in V_2.
\end{cases} \tag{6}
$$

verifies the transmission condition

$$u_1 = u_2, \text{ in } \Lambda. \tag{7}$$

By standard arguments and considering the regularity of f_1 and f_2, it is straightforward to prove that problem (6) is well-posed for every g in V_2', using the Lax-Milgram theorem. Consequently, we denote by (u_1^g, u_2^g) its unique solution.

In what follows, we will prove the existence of at least one control g and detail the process of finding it. With these notations we state the following theorem, showing the existence of a control g for which the couple (u_1^g, u_2^g), the unique solution of (6), verifies condition (7).

Theorem 2.1. *Let Ω be a domain that satisfies Assumption (H_1). Consider f_1 in $L^2(\Omega_1)$, f_2 in $L^2(\Omega_2)$ and let $(\overline{u}_1, \overline{u}_2)$ be the unique solution of problem (2). Then, there exists a function g in V_2' such that the solution (u_1^g, u_2^g) of problem (6) verifies (7).*

Proof. We can construct an extension operator E that extend $\overline{u}_1$ into the whole space $H_0^1(\Omega)$. Indeed, consider the operator defined by

$$E\overline{u}_1 \quad = \quad \left\{ \begin{array}{lll} \overline{u}_1 & \text{in} & \Omega_1 \\ \overline{u}_2 & \text{in} & \Omega_2 \end{array} \right. .$$

Since $(\overline{u}_1, \overline{u}_2)$ belongs to $\mathcal{V}$, $E\overline{u}_1$ is an extension of $\overline{u}_1$ which belongs to $H_0^1(\Omega)$. Furthermore, $\Delta(E\overline{u}_1)$ belongs to $L^2(\Omega)$. Then, we construct g in V_2' such that

$$\begin{aligned} \langle g, v\rangle_{V_2',V_2} &= \mu_1 \int_{\Omega_2} \nabla(E\overline{u}_1)_{|\Omega_2} \cdot \nabla v + \langle \mu_1 \gamma_{n_1}(\nabla E\overline{u}_1), v\rangle_{\Lambda',\Lambda}, \\ &= -\mu_1 \int_{\Omega_2} \Delta(E\overline{u}_1)_{|\Omega_2} v + \langle \mu_1 \gamma_{n_2}(\nabla E\overline{u}_1), v\rangle_{\Lambda',\Lambda} + \langle \mu_1 \gamma_{n_1}(\nabla E\overline{u}_1), v\rangle_{\Lambda',\Lambda}, \quad \forall v \in V_2. \end{aligned} \tag{8}$$

Using the Stokes formula (3), it follows that the extension $E\bar{u}_1$ in $H_0^1(\Omega)$ verifies

$$\begin{aligned} \mu_1 \int_\Omega \nabla(E\overline{u}_1)\cdot\nabla v_1 &= \mu_1 \int_{\Omega_1} \nabla\overline{u}_1 \cdot \nabla v_{1|\Omega_1} + \mu_1 \int_{\Omega_2} \nabla(E\overline{u}_1)_{|\Omega_2} \cdot \nabla v_{1|\Omega_2}, \\ &= -\mu_1 \int_{\Omega_1} \Delta\overline{u}_1 v_{1|\Omega_1} + \left\langle \mu_1\gamma_{n_1}(\nabla E\overline{u}_1), v_{1|\Omega_2}\right\rangle_{\Lambda',\Lambda} + \mu_1 \int_{\Omega_2} \nabla(E\overline{u}_1)_{|\Omega_2}\cdot\nabla v_{1|\Omega_2}, \\ &= \int_{\Omega_1} f_1 v_{1|\Omega_1} + \left\langle g, v_{1|\Omega_2}\right\rangle_{V_2',V_2}, \qquad \forall v_1 \in H_0^1(\Omega). \end{aligned}$$

Similarly, the function $\bar{u}_2$ in V_2 verifies, using (3) and (4),

$$\begin{aligned} \mu_2 \int_{\Omega_2} \nabla\bar{u}_2 \cdot \nabla v_2 &= -\mu_2 \int_{\Omega_2} \Delta\bar{u}_2 v_2 + \langle \mu_2\gamma_{n_2}(\nabla\bar{u}_2), v_2\rangle_{\Lambda',\Lambda} \\ &= \int_{\Omega_2} f_2 v_2 - \langle \mu_1\gamma_{n_1}(\nabla\bar{u}_1), v_2\rangle_{\Lambda',\Lambda}, \\ &= \int_{\Omega_2} f_2 v_2 + \mu_1 \int_{\Omega_2} \nabla(E\overline{u}_1)_{|\Omega_2}\cdot\nabla v_2 - \langle g, v_2\rangle_{V_2',V_2}, \quad \forall v_2 \in V_2. \end{aligned}$$

Finally, we can conclude that the couple $(E\overline{u}_1, \overline{u}_2)$ is the solution of problem (6). Thus

$$(u_1^g, u_2^g) = (E\overline{u}_1, \overline{u}_2)$$

and by construction of the operator E, the condition (7) is verified since $(u_{1|\Omega_1}^g, u_2^g) = (\overline{u}_1, \overline{u}_2)$. ☐

Remark 1. The control g is not unique. In fact, the extension operator E constructed in the preceding proof can be defined in different ways, leading to the construction of a different control g from an other extension of $\overline{u}_1$.

Remark 2. If $\overline{u}_1$ is of regularity $H^2(\Omega_1)$ and if $E\overline{u}_1$ is an extension which preserves this regularity on the whole domain Ω, then the following weak transmission condition holds:

$$\langle \mu_1\gamma_{n_2}(\nabla E\overline{u}_1), v\rangle_{\Lambda',\Lambda} \quad = \quad -\langle \mu_1\gamma_{n_1}(\nabla E\overline{u}_1), v\rangle_{\Lambda',\Lambda}.$$

Because $\Delta(E\overline{u}_1)_{|\Omega_2}$ belongs to $L^2(\Omega_2)$, we see from the definition of g in (8) that g belongs to $L^2(\Omega_2)$ and can be identified to $-\mu_1\Delta(E\overline{u}_1)_{|\Omega_2}$. Thus, the numerical approximation of the solution of the smooth extension problem (6) will converge in space with optimal rates of convergence, using $P1$ finite elements, whereas the numerical approximation of the solution of the transmission problem (2) will not in the general case. In Subsection 2.3, we will show that such a regular extension can constructed with rather weak assumptions on the regularity of the domains.

Remark 3. The hypothesis that Γ_1 and Γ_2 should not be empty could also be weakened. For example, the case where Ω_2 is strictly included in Ω_1 is particularly interesting to study, since a regular extension of u_1^g is directly obtained with Stein's extension Theorem ([AF03, Theorem 5.24]). In this situation, the main difficulty is that u_2^g is not unique in V_2. However, it can be searched such that it has a zero mean value and the solution of the initial transmission problem can be recovered from the solution of the smooth extension problem.

2.2 Formulation as an optimization problem

We saw in the previous subsection that a suitable control g can be obtained by extending $\overline{u}_1$ in the whole domain. However, the control g can not be constructed in such a direct manner, since the solution $(\overline{u}_1, \overline{u}_2)$ of the initial problem (2) is unknown. The main idea is to write the problem of finding a control g such that the solution (u_1^g, u_2^g) of the problem (6) verifies the condition (7) as a constrained minimization problem. To that matter, we rewrite the transmission condition (7) in a least-squares formulation and introduce the following cost function, defined on V_2' as follows,

$$J(g) = \frac{1}{2}\int_\Gamma |u_1^g - u_2^g|^2, \tag{9}$$

where (u_1^g, u_2^g) is the unique solution of problem (6). Note that in the regular case (see Remark 2), J must be minimized in $L^2(\Omega_2)$ in order to get both solutions u_1^g and u_2^g of regularity H^2, since less regular optimal controls exist *a priori* in $V_2' \setminus L^2(\Omega_2)$. A first link between the minimization of J and the research of a good control g is given by the following theorem.

Theorem 2.2. *Let g be in V_2' be such that the solution (u_1^g, u_2^g) of* (6) *satisfies condition* (7)*. Then, g is a minimizer of J.*

Proof. Let g be such a control, which exists according to Theorem 2.1. We know that the couple $(u^g_{1|\Omega_1}, u^g_2)$ is the unique solution of the weak initial problem (2). In particular, this couple satisfies the constraint of continuity through the interface Γ, which writes $u^g_{1|\Gamma} = u^g_{2|\Gamma}$ and implies that $J(g) = 0$, so that g is in fact a minimizer of J. □

Remark 4. At the end of this section, we will prove the reciprocal statement of Theorem 2.2: for every minimizer g of J, (u_1^g, u_2^g) verifies the condition (7). In other words, finding a minimizer of J enables to obtain the solution of the transmission problem (1).

The problem we face now is the minimization of the quadratic cost function J. To do that, classical methods such as gradient methods or quasi-Newton methods are well suited but they require the computation of the gradient of the cost function with respect to the control g. For that purpose, we will use the adjoint approach (see [Cha10]), a suitable method to compute the gradient of a cost function which depends on the solution of a system of differential equations. The idea is the following: knowing that, for any g in V_2', $J(g)$ is obtained by solving problem (6) and computing the explicit formula $\frac{1}{2}\int_\Gamma |u_1^g - u_2^g|^2$, we will prove that we can also compute the gradient $\nabla J(g)$ with an explicit formula, which depends on the resolution of a system of linear partial differential equations, called *the adjoint equations*. The key here is to remark that the minimization of $J(g)$ can be seen as the minimization of the real-valued function

$$\begin{array}{rcl} H_0^1(\Omega) \times V_2 & \to & \mathbb{R}^+ \\ (v_1, v_2) & \mapsto & \displaystyle\frac{1}{2}\int_\Gamma |v_{1|\Gamma} - v_{2|\Gamma}|^2, \end{array}$$

under the constraint that (v_1, v_2) is solution of (6). Thus, it is indicated to introduce the Lagrangian function associated to this constrained optimization problem, defined from

$$V_2' \times (H_0^1(\Omega) \times V_2) \times (H_0^1(\Omega) \times V_2)$$

to $\mathbb{R}$ by

$$\begin{aligned} \mathcal{L}(g, (v_s, v_2), (\lambda_1, \lambda_2)) \quad = \quad & \frac{1}{2}\|v_1 - v_2\|^2_{L^2(\Gamma)} + \mu_1 \int_\Omega \nabla v_1 \cdot \nabla\lambda_1 - \mu_1 \int_{\Omega_2} \nabla v_{1|\Omega_2} \cdot \nabla\lambda_2 + \mu_2 \int_{\Omega_2} \nabla v_2 \cdot \nabla \lambda_2 \\ & - \int_{\Omega_1} f_1 \lambda_{1|\Omega_1} - \int_{\Omega_2} f_2\lambda_2 - \left\langle g, \lambda_{1|\Omega_2} \right\rangle_{V_2', V_2} + \langle g, \lambda_2 \rangle_{V_2', V_2}, \end{aligned} \tag{10}$$

where the Lagrangian multipliers λ_1 and λ_2 are called the *adjoint variables* of v_1 and v_2, associated to the state equations (5a) and (5b). We also introduce the so-called *adjoint equations*, defined for all g in V_2' by

$$\begin{aligned} \left\langle \frac{\partial \mathcal{L}}{\partial v_1}(g, (u_1^g, u_2^g), (\lambda_1, \lambda_2)), \delta v_1 \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} &= 0, \quad \forall \delta v_1 \in H_0^1(\Omega), \\ \left\langle \frac{\partial \mathcal{L}}{\partial v_2}(g, (u_1^g, u_2^g), (\lambda_1, \lambda_2)), \delta v_2 \right\rangle_{V_2', V_2} &= 0, \quad \forall \delta v_2 \in V_2, \end{aligned} \tag{11}$$

where (u_1^g, u_2^g) is the solution of problem (6). The computation of the differential forms $\frac{\partial \mathcal{L}}{\partial v_1}$ and $\frac{\partial \mathcal{L}}{\partial v_2}$ will be detailed later on.

Remark 5. The Lagrangian $\mathcal{L}$ is differentiable with respect to v_1 and v_2 because all its terms are quadratic or linear with respect to each of them. Thus, equations (11) are well defined.

With all these notations, we state the following theorem adapted from [Cha10], where the use of the Lagrangian function $\mathcal{L}$ is made clear in providing a convenient way to compute the gradient of J.

Theorem 2.3. *The mapping $g \in V_2' \mapsto J(g) \in \mathbb{R}$ is differentiable and its gradient $\nabla J(g) \in V_2''$ is given, by*

$$\langle \nabla J(g), \delta g \rangle_{V_2'', V_2'} \quad = \quad \left\langle \delta g, \lambda_2^g - \lambda^g_{1|\Omega_2} \right\rangle_{V_2', V_2}, \quad \forall \delta g \in V_2', \tag{12}$$

where $(\lambda_1^g, \lambda_2^g)$ verifies the adjoint equations (11)*.*

To prove this result, we first need to prove the following lemma.

Lemma 2.1. *There exists a unique $(\lambda_1^g, \lambda_2^g)$ in $H_0^1(\Omega) \times V_2$, solution of the adjoint equations* (11)*.*

Proof. We start by computing the partial derivatives of $\mathcal{L}$ with respect to v_1 and v_2. Let $\varepsilon > 0$. For all $g \in V_2'$, and for all (v_1, v_2), $(\delta v_1, \delta v_2)$ and (λ_1, λ_2) in $H_0^1(\Omega) \times V_2$,

$$\begin{aligned} \mathcal{L}(g, (v_1 + \varepsilon\delta v_1, v_2), (\lambda_1, \lambda_2)) \quad = \quad & \mathcal{L}(g, (v_1, v_2), (\lambda_1, \lambda_2)) + \varepsilon\mu_1 \int_\Omega \nabla(\delta v_1) \cdot \nabla \lambda_1 - \varepsilon\mu_1 \int_{\Omega_2} \nabla(\delta v_{1|\Omega_2}) \cdot \nabla\lambda_2 \\ & + \varepsilon \int_\Gamma (v_1 - v_2)\delta v_1 + \frac{\varepsilon^2}{2}\|\delta v_1\|^2_{L^2(\Gamma)}. \end{aligned}$$

It leads to define $\frac{\partial \mathcal{L}}{\partial v_1}(g,(v_1,v_2),(\lambda_1,\lambda_2)) \in H^{-1}(\Omega)$ as:

$$\begin{aligned}\left\langle \frac{\partial \mathcal{L}}{\partial v_1}(g,(v_1,v_2),(\lambda_1,\lambda_2)),\delta v_1\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} &= \mu_1\int_\Omega \nabla(\delta v_1)\cdot\nabla\lambda_1 - \mu_1\int_{\Omega_2}\nabla(\delta v_{1|\Omega_2})\cdot\nabla\lambda_2 \\ &+\int_\Gamma (v_1-v_2)\delta v_1,\ \forall \delta v_1\in H_0^1(\Omega).\end{aligned}$$

Similarly, we define the derivative of the Lagrangian function $\mathcal{L}$ with respect to v_2 evaluated at point δv_2, denoted by

$$\frac{\partial \mathcal{L}}{\partial v_2}(g,(v_1,v_2),(\lambda_1,\lambda_2))$$

and which belongs to V_2':

$$\left\langle \frac{\partial \mathcal{L}}{\partial v_2}(g,(v_1,v_2),(\lambda_1,\lambda_2)),\delta v_2\right\rangle_{V_2',V_2} = \mu_2\int_{\Omega_2}\nabla(\delta v_2)\cdot\nabla\lambda_2 - \int_\Gamma (v_1-v_2)\delta v_2,\quad \forall \delta v_2\in V_2.$$

Then, we can deduce the *adjoint problem*:

$$\left\{\begin{array}{lcll}\multicolumn{4}{l}{\text{find } (\lambda_1,\lambda_2) \text{ in } H_0^1(\Omega)\times V_2 \text{ such that,}}\\ \mu_1\displaystyle\int_\Omega \nabla\lambda_1\cdot\nabla v_1 &=& \mu_1\displaystyle\int_{\Omega_2}\nabla\lambda_2\cdot\nabla v_{1|\Omega_2} - \int_\Gamma (u_1^g-u_2^g)v_1, & \forall v_1\in H_0^1(\Omega),\\ \mu_2\displaystyle\int_{\Omega_2}\nabla\lambda_2\cdot\nabla v_2 &=& \displaystyle\int_\Gamma (u_1^g-u_2^g)v_2, & \forall v_2\in V_2.\end{array}\right. \tag{13}$$

Finally, we use the Lax-Milgram Theorem to show that problem (13) admits a unique solution $(\lambda_1^g,\lambda_2^g)$. □

We now have all the tools we need to prove Theorem 2.3.

Proof of Theorem 2.3. The solution (u_1^g,u_2^g) of problem (6) is unique for every control g in V_2'. Thus, we can define the so called *direct mapping*

$$\begin{array}{rccl}\phi: & V_2' & \to & H_0^1(\Omega)\times V_2,\\ & g & \mapsto & (\phi_1(g),\phi_2(g)) = (u_1^g,u_2^g).\end{array} \tag{14}$$

Because of the linearity of the equations in (6), the mapping ϕ is linear, thus differentiable in V_2'. Similarly, the mapping which for every (u_1,u_2) in $H_0^1(\Omega)\times V_2$ associates the quadratic functional $\int_\Gamma |u_1-u_2|^2$ is differentiable. By composition, it follows that the mapping $g\mapsto J(g)$ is differentiable in V_2'.

Taking $(v_1,v_2)=(u_1^g,u_2^g)$ the Lagrangian (10) reduces to

$$\mathcal{L}(g,(u_1^g,u_2^g),(\lambda_1,\lambda_2)) = J(g),\ \forall g\in V_2',\ \forall(\lambda_1,\lambda_2)\in H_0^1(\Omega)\times V_2.$$

Then, we differentiate this previous equality with respect to g for a fixed couple (λ_1,λ_2) and we obtain, using the chain rule, the following equation:

$$\begin{aligned}\langle \nabla J(g),\delta g\rangle_{V_2'',V_2'} &= \left\langle \frac{\partial \mathcal{L}}{\partial g}(g,(u_1^g,u_2^g),(\lambda_1,\lambda_2)),\delta g\right\rangle_{V_2'',V_2} + \left\langle \frac{\partial \mathcal{L}}{\partial v_1}(g,(u_1^g,u_2^g),(\lambda_1,\lambda_2)),\frac{\partial\phi_1}{\partial g}(g)\cdot\delta g\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\\ &+\left\langle \frac{\partial \mathcal{L}}{\partial v_2}(g,(u_1^g,u_2^g),(\lambda_1,\lambda_2)),\frac{\partial\phi_2}{\partial g}(g)\cdot\delta g\right\rangle_{V_2',V_2},\end{aligned}$$

for all $\delta g\in V_2'$. Considering $(\lambda_1,\lambda_2)=(\lambda_1^g,\lambda_2^g)$, the solution of the adjoint equations (13) (see Lemma 2.1), this reduces to

$$< \nabla J(g),\delta g >_{V_2'',V_2'} = \left\langle \frac{\partial \mathcal{L}}{\partial g}(g,(u_1^g,u_2^g),(\lambda_1^g,\lambda_2^g)),\delta g\right\rangle_{V_2'',V_2'},\quad \forall \delta g\in V_2'.$$

It remains to compute the derivative of $\mathcal{L}$ with respect to g at point δg: let $\varepsilon>0$, for all g and δg in V_2', and for all (v_1,v_2) and (λ_1,λ_2) in $H_0^1(\Omega)\times V_2$,

$$\mathcal{L}(g+\varepsilon\delta g,(v_1,v_2),(\lambda_1,\lambda_2)) = \mathcal{L}(g,(v_1,v_2),(\lambda_1,\lambda_2)) - \varepsilon\left\langle \delta g,\lambda_{1|\Omega_2}\right\rangle_{V_2',V_2} + \varepsilon\left\langle \delta g,\lambda_2\right\rangle_{V_2',V_2},$$

which leads to the definition of $\frac{\partial \mathcal{L}}{\partial g}(g,(v_1,v_2),(\lambda_1,\lambda_2))\in V_2''$ as:

$$\left\langle \frac{\partial \mathcal{L}}{\partial g}(g,(v_1,v_2),(\lambda_1,\lambda_2)),\delta g\right\rangle_{V_2'',V_2'} = \left\langle \delta g,\lambda_2-\lambda_{1|\Omega_2}\right\rangle_{V_2',V_2},\quad \forall\delta g\in V_2'.$$

Finally, we obtain an explicit expression for the gradient of J, which writes

$$< \nabla J(g),\delta g >_{V_2'',V_2'} = \left\langle \delta g,\lambda_2^g-\lambda_{1|\Omega_2}^g,\right\rangle_{V_2',V_2},\quad \forall\delta g\in V_2'.$$

□

Now that we have an explicit expression for ∇J, we can prove the reciprocal statement of Theorem 2.2.

Theorem 2.4. *Let g be a minimizer of J in V_2'. Then, the solution (u_1^g, u_2^g) of* (6) *satisfies condition* (7).

Proof. As a minimizer of J, g verifies the equality $< \nabla J(g), \delta g >_{V_2'', V_2'} = 0$, for all δg in V_2'. According to Theorem 2.3, it follows that the couple $(\lambda_1^g, \lambda_2^g)$ verifies

$$\left\langle \delta g, \lambda_2^g - \lambda_{1|\Omega_2}^g \right\rangle_{V_2', V_2} = 0, \quad \forall \delta g \in V_2'.$$

It comes that $\lambda_{1|\Omega_2}^g = \lambda_2^g$ in V_2 and the couple $(\lambda_{1|\Omega_1}^g, \lambda_2^g)$ belongs to the space $\mathcal{V}$. Yet, the couple $(\lambda_1^g, \lambda_2^g)$ is solution of problem (13). In particular, for all (v_1, v_2) in the space $\tilde{\mathcal{V}}$ defined by

$$\tilde{\mathcal{V}} = \{(v_1, v_2) \in H_0^1(\Omega) \times V_2; v_{1|\Omega_1} = v_2\},$$

the two equations in (13) write

$$\begin{aligned} \mu_1 \int_\Omega \nabla \lambda_1^g \cdot \nabla v_1 &= \mu_1 \int_{\Omega_2} \nabla \lambda_2^g \cdot \nabla v_{1|\Omega_2} - \int_\Gamma (u_1^g - u_2^g) v_1, \\ \mu_2 \int_{\Omega_2} \nabla \lambda_2^g \cdot \nabla v_2 &= \int_\Gamma (u_1^g - u_2^g) v_1. \end{aligned} \tag{15}$$

Summing the two equations in (15) and using the fact that $\lambda_{1|\Omega_2}^g = \lambda_2^g$, we find that

$$\mu_1 \int_{\Omega_1} \nabla \lambda_{1|\Omega_1}^g \cdot \nabla v_{1|\Omega_1} + \mu_2 \int_{\Omega_2} \nabla \lambda_2^g \cdot \nabla v_2 = 0, \quad \forall (v_1, v_2) \in \tilde{\mathcal{V}}.$$

Moreover, let (v_1, v_2) be in $\mathcal{V}$. We can construct an extension of v_1 in the whole space $H_0^1(\Omega)$ using v_2, as in the proof of Theorem 2.1 and we still denote Ev_1 this extension. Then, the couple (Ev_1, v_2) belongs to $\tilde{\mathcal{V}}$ and it follows that the couple $(\lambda_{1|\Omega_1}^g, \lambda_2^g)$ verifies the equation

$$\mu_1 \int_{\Omega_1} \nabla \lambda_{1|\Omega_1}^g \cdot \nabla v_1 + \mu_2 \int_{\Omega_2} \nabla \lambda_2^g \cdot \nabla v_2 = 0, \quad \forall (v_1, v_2) \in \mathcal{V}.$$

We conclude that the couple $(\lambda_{1|\Omega_1}^g, \lambda_2^g)$ is solution of a weak problem similar to (2) but with no external force and, thus, is the zero of $\mathcal{V}$. Especially, this implies that $\lambda_2^g = 0$ and, according to the second equation of (15), that

$$\int_\Gamma (u_1^g - u_2^g) v_2 = 0, \quad \forall v_2 \in V_2.$$

Because $u_{1|\Gamma}^g$ and $u_{2|\Gamma}^g$ belong to the space Λ the previous equality implies that $u_{1|\Gamma}^g = u_{2|\Gamma}^g$ in Λ, so that the couple $(u_{1|\Omega_1}^g, u_2^g)$ belongs to the space $\mathcal{V}$. As the unique solution of problem (6), the pair (u_1^g, u_2^g) verifies, in particular, for all (v_1, v_2) in $\tilde{\mathcal{V}}$,

$$\begin{aligned} \mu_1 \int_\Omega \nabla u_1^g \cdot \nabla v_1 &= \int_{\Omega_1} f_1 v_{1|\Omega_1} + \langle g, v_2 \rangle_{V_2', V_2}, \\ \mu_2 \int_{\Omega_2} \nabla u_2^g \cdot \nabla v_2 &= \int_{\Omega_2} f_2 v_2 - \langle g, v_2 \rangle_{V_2', V_2} + \mu_1 \int_{\Omega_2} \nabla u_1^g \cdot \nabla v_{1|\Omega_2}. \end{aligned} \tag{16}$$

Summing the two equations in (16) it follows that the pair $(u_{1|\Omega_1}^g, u_2^g)$, which belongs to $\mathcal{V}$, verifies the equation

$$\mu_1 \int_{\Omega_1} \nabla u_{1|\Omega_1}^g \cdot \nabla v_{1|\Omega_1} + \mu_2 \int_{\Omega_2} \nabla u_2^g \cdot \nabla v_2 = \int_{\Omega_1} f_1 v_{1|\Omega_1} + \int_{\Omega_2} f_2 v_2, \quad \forall (v_1, v_2) \in \tilde{\mathcal{V}}.$$

As before, for all (v_1, v_2) in $\mathcal{V}$, the couple (Ev_1, v_2) belongs to $\tilde{\mathcal{V}}$ and, finally, the couple $(u_{1|\Omega_1}^g, u_2^g)$ verifies the equation

$$\mu_1 \int_{\Omega_1} \nabla u_{1|\Omega_1}^g \cdot \nabla v_1 + \mu_2 \int_{\Omega_2} \nabla u_2^g \cdot \nabla v_2 = \int_{\Omega_1} f_1 v_1 + \int_{\Omega_2} f_2 v_2, \quad \forall (v_1, v_2) \in \mathcal{V}.$$

Thus, $(u_{1|\Omega_1}^g, u_2^g)$ is the unique solution to the initial coupled problem (2) and verifies, *a fortiori*, condition (7). □

The minimization of the function J in V_2' is therefore equivalent to the resolution of the problem of finding a suitable control g such that the solution (u_1^g, u_2^g) verifies (7) and thus, to the resolution of the initial coupled problem (2).

Remark 6. Theorem 2.4 also shows that all extrema of J correspond to global minimizers. Actually, in Theorem 2.4, it is enough to assume that g verifies $< \nabla J(g), \delta g >_{V_2'', V_2'} = 0$, i.e. that $J(g)$ is a local extremum of J, to conclude that the solution (u_1^g, u_2^g) of (6) verifies the condition (7). According to Theorem 2.2, this implies that g is a global minimizer of J.

2.3 On convergence rates for the numerical method

The main difficulty with the numerical simulation of transmission problems on unfitted meshes, is to recover the optimal rate of convergence when the solution is more regular in each subdomain. Actually, even if f_1 belongs to $L^2(\Omega_1)$, f_2 to $L^2(\Omega_2)$ and the boundaries of Ω_1 and Ω_2 are smooth, the solution of problem (2) is not of regularity H^2 in the whole domain Ω, because of the jump in its gradient across the interface Γ. As a consequence, the standard finite element method with $P1$ elements does not enable to recover the optimal rates of convergence in L^2 and H^1-norms, unless we use a conformal mesh that fully represents the interface Γ. Nevertheless, with such a regularity on the data and the boundaries, one can show that the solution of problem (2) is *partially of regularity* H^2, in the following sense: $\overline{u}_1$ belongs to $H^2(\Omega_1) \cap V_1$ and $\overline{u}_2$ to $H^2(\Omega_2) \cap V_2$ (see [CDN10, Theorem 5.2.1]). Thus, if we can extend $\overline{u}_1$ in the whole domain Ω with regularity H^2, we know (see Remark 2) that there exists a suitable control g in $L^2(\Omega_2)$ such that the solution of the smooth extension problem (6) is also of regularity H^2: u_1^g belongs to $H^2(\Omega) \cap H_0^1(\Omega)$ and u_2^g belongs to $H^2(\Omega_2) \cap V_2$. A first order finite element method will therefore converge with optimal rates.

In the following theorem, we state that such a regular extension exists under rather weak assumptions on the regularity of the domains.

Theorem 2.5. *Let $\Omega \in \mathbb{R}^n$, with $n \in \{2,3\}$, be a bounded open set with Lipschitz boundary $\partial\Omega$, and let Γ be an interface that divides Ω into two bounded open connected subdomains Ω_1 and Ω_2 with Lipschitz boundaries. Consider a function u_1 in $H^2(\Omega_1) \cap V_1$. Moreover, if $n = 3$, we assume that $\Gamma \cap \partial\Omega$ is a curve of regularity C^2. If Γ and $\partial\Omega$ have at least C^2-regularity in a neighborhood of each element of $\Gamma \cap \partial\Omega$ (a curve in three space dimensions and a point in two space dimensions), then there exists a regular extension $u \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $u = u_1$ in Ω_1.*

Remark 7. The case where $\partial\Omega_2 = \Gamma$ is particularly easy to study since the construction of a regular extension is directly given by Stein's theorem ([AF03, Theorem 5.24]).Thus we will only consider the case where $\partial\Omega_1 \cap \partial\Omega$ and $\partial\Omega_2 \cap \partial\Omega$ are both non empty in the following proof.

Proof. Suppose that $\partial\Omega_1 \cap \partial\Omega$ and $\partial\Omega_2 \cap \partial\Omega$ are both non empty. For the sake of simplicity, we consider the two-dimensional case, i.e. $\Omega \subset \mathbb{R}^2$, but the extension to the three-dimensional case can be performed in a similar manner. In two space dimension, the interface Γ intersects the boundary $\partial\Omega$ in two points that we denote by x_1 and x_2, i.e. $\Gamma \cap \partial\Omega = \{x_1, x_2\}$.

For $i \in \{1,2\}$, let us denote by $B_i = B(x_i, \varepsilon_i)$ the open ball of radius ε_i and center x_i. Moreover, we consider two open balls $B_i^+ = B(x_i, \varepsilon_i + \varepsilon_i/4)$ and $B_i^- = B(x_i, \varepsilon_i - \varepsilon_i/4)$, such that $B_i^- \subset B_i \subset B_i^+$. Furthermore, we should ensure that, for $i \in \{1,2\}$, ε_i is small enough such that $\Gamma \cap B_i^+$ and $\partial\Omega \cap B_i^+$ are included in the neighborhood of x_i where both frontiers Γ and $\partial\Omega$ have C^2-regularity. A quite general example of such a configuration is illustrated in Figure 1.

Let $\varphi \in \mathcal{C}^\infty(\mathbb{R}^+)$ be a real-valued function satisfying

$$\varphi(x) = \begin{cases} 1 & \text{if } x < 3/4, \\ 0 & \text{if } x > 1. \end{cases}$$

We define, for $i \in \{1,2\}$, the functions

$$\begin{array}{rcccc} \varphi_i & : & \Omega_1 & \to & [0,1], \\ & & x & \mapsto & \varphi\left(\dfrac{|x - x_i|}{\varepsilon}\right) \end{array} \qquad \begin{array}{rcccc} \xi_i & : & \Omega_1 & \to & \mathbb{R} \\ & & x & \mapsto & u_1(x)\varphi_i(x). \end{array}$$

Thus, we can write the following decomposition for the function u_1 on Ω_1:

$$u_1 = u_1\varphi_1 + u_1\varphi_2 + u_1(1 - \varphi_1 - \varphi_2) = \xi_1 + \xi_2 + \xi_0,$$

where $\xi_0 = u_1(1 - \varphi_1 - \varphi_2)$. The problem to construct a regular extension for u_1 is equivalent to extend the three functions ξ_0, ξ_1 and ξ_2 to the whole domain Ω. Moreover, let us remark that we have

$$\xi_i(x) = \begin{cases} u_1(x), & \forall x \in B_i^- \cap \Omega_1, & \forall i \in \{0,1\}, \\ 0, & \forall x \in \Omega_1 \setminus (B_i \cap \Omega_1), & \forall i \in \{0,1\}, \end{cases}$$

and

$$\xi_0(x) = \begin{cases} 0, & \forall x \in B_i^- \cap \Omega_1, & \forall i \in \{0,1\}, \\ u_1(x), & \forall x \in \Omega_1 \setminus (B_i \cap \Omega_1), & \forall i \in \{0,1\}. \end{cases}$$

We begin with the function ξ_0. For that matter, we make use of Stein's theorem ([AF03, Theorem 5.24]), which states that, for any bounded Lipschitz domain $\mathcal{O} \subset \mathbb{R}^n$, there exists a total extension operator, i.e. an extension operator from $H^m(\mathcal{O})$ into the whole space $H^m(\mathbb{R}^n)$, for all $m \geq 0$. Then, Ω_1 being a bounded Lipschitz domain, there exists an extension operator from $H^2(\Omega_1)$ into the whole space $H^2(\mathbb{R}^2)$. We denote by $E\xi_0$ the restriction to Ω of such a regular extension of ξ_0 to $\mathbb{R}^2$. This extension is regular enough but it is not zero on the boundary $\partial\Omega \cap \partial\Omega_2$. Therefore, we introduce a compact set $\mathfrak{V}$ such that $(\partial\Omega \cap \partial\Omega_2) \subset \mathfrak{V}$ and $\mathfrak{V} \cap \Omega_1 = \emptyset$, and an open neighborhood of $\mathfrak{V}$, denoted by $\mathfrak{U}$, such that $(\mathfrak{U} \cap \Omega_1) \subset (B_1^- \cup B_2^-)$. We also define a smooth cut-off function $\psi : x \in \Omega \to \psi(x)$ such that $\psi(x) = 0$ in $\mathfrak{V}$, and $\psi(x) = 1$ in $\Omega \setminus \mathfrak{U}$. Then, the function $\tilde{\xi_0} = \psi E\xi_0$ is an extension of ξ_0 to the whole domain Ω of regularity H^2, which satisfies $\tilde{\xi_0} = 0$ on $\partial\Omega$. The construction of $\tilde{\xi_0}$ is represented in Figure 1.

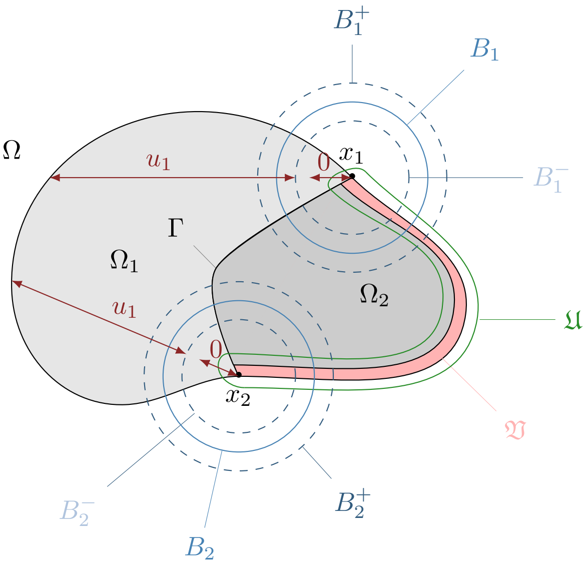


Figure 1: Construction of the function ξ_0 and its regular extension, $\tilde{\xi}_0$.

Let us now proceed with the construction of a smooth extension of ξ_i, for $i \in \{1,2\}$. First, since $\xi_i = 0$ in $\Omega_1 \setminus (B_i \cap \Omega_1)$, we can extend it by 0 in $\Omega_2 \setminus (B_i^+ \cap \Omega_2)$. Then, it remains to construct the extension in $B_i^+ \cap \Omega_2$. Due to the C^2-regularity of the interface Γ and the boundary $\partial\Omega$ inside the balls B_1^+ and B_2^+ (if ε_1 and ε_2 are chosen small enough), there exist two C^2-diffeomorphisms χ_1 and χ_2 that map B_1^+ and B_2^+ respectively into the open unit square Q, and such that

$$\begin{array}{rcll} \chi_i(x_i) & = & 0, & \forall i \in \{1,2\}, \\ \chi_i(\Gamma \cap B_i^+) & = & \{0\} \times \left]0, \frac{1}{2}\right[, & \forall i \in \{1,2\}, \\ \chi_i(\partial\Omega \cap B_i^+) & = & \left]-\frac{1}{2}, \frac{1}{2}\right[\times \{0\}, & \forall i \in \{1,2\}. \end{array}$$

For $i \in \{1,2\}$, we define $\hat{\xi}_i = \xi_i \circ \chi_i^{-1}$ in

$$\chi_i(B_i^+ \cap \Omega_1) = \left]-\frac{1}{2}, 0\right[\times \left]0, \frac{1}{2}\right[,$$

which belongs to $H^2(\chi_i(B_i^+ \cap \Omega_1))$ (see [AF03, Theorem 3.41]) and, recalling that $\xi_i = 0$ in $(B_i^+ \setminus B_i) \cap \Omega_1$, it follows that $\hat{\xi}_i = 0$ in $\chi_i((B_i^+ \setminus B_i) \cap \Omega_1)$. Then, a regular extension of $\hat{\xi}_i$ to the whole square Q can be obtained using Babič's extension (see [Bab53] or [AF03, Theorem 5.19]), defined for $i \in \{1,2\}$ by

$$E\hat{\xi}_i(x,y) = \begin{cases} \hat{\xi}(x,y), & \text{if } x \leq 0, \\ -3\hat{\xi}(-x,y) + 4\hat{\xi}(-\frac{x}{2}, y), & \text{if } x > 0, \end{cases} \qquad \forall (x,y) \in \left]-\tfrac{1}{2}, \tfrac{1}{2}\right[\times \left]0, \tfrac{1}{2}\right[.$$

Clearly, for $i \in \{1,2\}$, the extension $E\hat{\xi}_i$ and its first derivative are continuous through the interface $\{0\}\times]0, \frac{1}{2}[$, such that $E\hat{\xi}_i$ well defines an extension of $\hat{\xi}_i$ in $H^2(\chi_i(B_i^+ \cap \Omega))$. However, the extension $E\hat{\xi}_i$ is not zero on the exterior frontier $\frac{1}{2}\times]0, \frac{1}{2}[$, thus we multiply it by a smooth cut-off function that is zero for $x > 1/2 - \varepsilon$ and equal to 1 for $x < 1/2 - 2\varepsilon$, for some $\varepsilon < 1/4$. Let us denote by $\bar{\xi}_i$ the product of $E\hat{\xi}_i$ by this cut-off function, which belongs to $H^2(\chi_i(B_i^+ \cap \Omega))$ and is zero on $\chi_i(\partial\Omega \cap B_i^+)$. Mapping this extension into B_i^+ we obtain $\bar{\xi}_i \circ \chi_i$, which then is an extension of ξ_i in $B_i^+ \cap \Omega$ of regularity H^2. Now, we can construct the smooth extension of ξ_i to the whole domain Ω, denoted $\tilde{\xi}_i$ and defined by

$$\tilde{\xi}_i = \begin{cases} \xi_i & \text{in } \Omega_1, \\ 0 & \text{in } \Omega_2 \setminus (B_i^+ \cap \Omega_2), \\ \bar{\xi}_i \circ \chi_i & \text{in } B_i^+ \cap \Omega. \end{cases}$$

Thus, for $i \in \{1,2\}$, $\tilde{\xi}_i$ is an extension of ξ_i which belongs to $H^2(\Omega)$ and is zero on $\partial\Omega$.

Finally, we define

$$\tilde{u} = \tilde{\xi}_0 + \tilde{\xi}_1 + \tilde{\xi}_2 \text{ in } \Omega$$

and it follows that $\tilde{u}$ belongs to $H^2(\Omega) \cap H_0^1(\Omega)$ and $\tilde{u} = u_1$ in Ω_1. $\square$

Remark 8. The proof of Theorem 2.5 in three dimensions of space follows the same path. We start by considering three tubular neighborhoods of the intersection $\Gamma \cap \partial\Omega$ with circular cross sections and decompose the function u_1 in two regular functions ξ_0 and ξ_1. The function ξ_0 is zero inside the smallest neighborhood and is equal to u_1 far from the intersection $\Gamma \cap \partial\Omega$, while the function ξ_1 is equal to u_1 inside the smallest neighborhood and is zero far from $\Gamma \cap \partial\Omega$.

On one hand, the extension of the function ξ_0 is done using Stein's extension theorem. On the other hand, the extension of the function ξ_1 is conducted by transforming the largest tubular neighborhood into the unit torus with square cross section with a C^2-diffeomorphism. Then, Babič's extension theorem is used in the unit torus and the resulting extension is brought back to the initial domain. In both cases, the extensions are multiplied by cut-off functions to ensure that they are zero on the exterior frontier $\partial\Omega$.

In the next section, numerical results will serve as a validation of the method on some particular examples. Indeed, they show that the method is effective in practice and has optimal convergence rates (i.e. convergence rates obtained with conformal meshes techniques). However, the numerical analysis of the global numerical method (including the minimization process) remains to be done.

3 Validation of the method

In Subsection 2.2 we have proved the equivalence between the initial coupled problem and the smooth extension formulation stated as a minimization problem. Based on this result, we detail the numerical procedure used to solve the smooth extension formulation of the Laplace transmission problem presented above. In particular, in Subsection 3.1, we explain how to minimise the function J defined in (9) using its gradient. Thereafter, we present in Subsection 3.2 and Subsection 3.3 some numerical experiments obtained through this process.

3.1 Numerical procedure for the smooth extension method

To solve the smooth extension formulation of problem (2), we consider an optimization problem whose solution allows to directly recover the solution of the initial transmission problem. In the case of the Laplace transmission problem, it gives the right control term g for which the solution u_1^g of problem (6) is a smooth extension of $\overline{u}_1$ in the whole domain Ω. Then, instead of directly finding $\overline{u}_1$ and $\overline{u}_2$ by solving a Laplace problem with discontinuous coefficients, one solves a minimization problem on the control g. As already explained, this formulation is advantageous for the numerical resolution because it allows the use of non-conformal meshes on Ω. Thus, the smooth extension method is a fictitious domain method in the sense that the various problems appearing in the numerical resolution process are solved on two meshes, one for Ω and an other for Ω_2 which are not conformal.

An explicit formula is provided for the computation of the gradient of the cost function to minimize (see Theorem 2.3), which enables to treat the minimization problem with a classical descent method. The general algorithm that we use is the following: we choose an initial guess g_0 for the control term. For each iteration k of the gradient algorithm, we first solve problem (6) with $g = g_k$ and obtain the couple $(u_1^{g_k}, u_2^{g_k})$. Then, we solve the adjoint problem (13) with $g = g_k$ to get the adjoint variables $(\lambda_1^{g_k}, \lambda_2^{g_k})$. The gradient $\nabla J(g_k)$ is computed using the explicit formula given by Theorem 2.3. Finally, the control is updated using the chosen optimization algorithm. The general formula for the update can be written

$$g_{k+1} = g_k - \rho_k \nabla J(g_k),$$

where ρ_k is either a real positive parameter or a matrix, depending on the chosen optimization algorithm. This process is summarized in the Algorithm 1.

Algorithm 1 Implementation of the smooth extension method

Choose an initial guess g_0 for the control term.
$k = 0$.
while exit criteria are not satisfied **do**
  Compute the solution $(u_1^{g_k}, u_2^{g_k})$ of problem (6) with $g = g_k$.
  Compute the solution $(\lambda_1^{g_k}, \lambda_2^{g_k})$ of the adjoint problem (13) with $g = g_k$.
  Compute the gradient $\nabla J(g_k) = \lambda_2^{g_k} - \lambda_{1|\Omega_2}^{g_k}$.
  Update the control $g_{k+1} = g_k - \rho_k \nabla J(g_k)$, with ρ_k depending on the chosen gradient method.
  Update the number of iterations $k = k + 1$.
end while

In practice, the exit criteria for a descent method usually concern the norm of the difference between two successive solutions $\|g_{k+1} - g_k\|$ divided by the update coefficient ρ_k. The choice of the initial guess g_0 is of minor importance for the convergence of the algorithm since, as we explained in Remark 6, every extremum of the cost function J corresponds to one of its minimizers.

3.2 Test case 1: the transmission Laplace problem in the unit square

Let Ω be the unit square of $\mathbb{R}^2$, divided in two pieces by a vertical segment which represents the interface Γ (see Figure 2). We denote by x_Γ the position of this interface on the x-axis. We aim to apply the smooth extension method to the Laplace transmission problem (1) with homogeneous Dirichlet boundary conditions, studied in Subsection 2.1. Because the interface Γ meets the boundary $\partial\Omega$ with a right-angled corner and because $f1$ and f_2 are constants, the solution

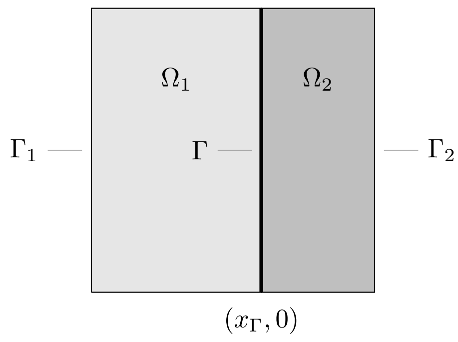


Figure 2: Two dimensional representation of the geometry for the Laplace transmission problems (1).

of (1), denoted by $\overline{u}$, is partially of regularity H^2, i.e. $\overline{u}_{|\Omega_1}$ belongs to $H^2(\Omega_1)$ and $\overline{u}_{|\Omega_2}$ belongs to $H^2(\Omega_2)$. Moreover, the test case lies in the scope of Theorem 2.5, such that there exists a regular extension of $\overline{u}_{|\Omega_1}$ in the whole domain Ω. Thus, all conditions are satisfied so that the smooth extension method applied to this problem converges with optimal rates.

The numerical values of all parameters are $\mu_1 = 1$, $\mu_2 = 10$, $f_1 = 1$, $f_2 = 1$ and $x_\Gamma = 0.57$. An approximation of the solution $\overline{u}$ of (1), obtained with the classical finite element method, is represented in Figure 3. Because of the jump of its gradient through Γ, the entire field $\overline{u}$ does not belong to $H^2(\Omega)$. As a consequence, the use of the classical finite element method to approach the solution of (1) leads to different rates of convergence when refining the space discretization, whether the mesh of the domain fits the interface Γ or not. Indeed, in Figure 4 we can observe that if the mesh fits the interface Γ, the rates of convergence are of order 2 for the L^2-norm and of order 1 for the H^1_0-norm, using $P1$ elements. These are the classical rates of convergence with $P1$ elements for a H^2 solution of a Laplace problem, which is not the case here. Then, this result is only due to the fact that the interface Γ is well represented by the mesh. On the other hand, if the mesh does not fit the interface, we recover degraded rates of convergence: here we find a rate of order 1 in L^2-norm and 0.5 in H^1_0-norm.

For the numerical resolution with the smooth extension method, we follow Algorithm 1. In particular, we choose the initial guess g_0 to be zero and use a classical gradient descent with constant parameter ρ to minimize J. The value of ρ is chosen such that the gradient method convergences and can be different depending on the mesh size. We use unstructured meshes for both Ω and Ω_2, such that the interface Γ is not represented by the mesh on Ω. At each iteration of the method, we solve 4 second-order boundary problems (the direct and adjoint equations) using $P1$ elements, whose solutions enable us to compute the gradient $\nabla J(g_k)$. Then, we update the control such that,

$$g_{k+1} = g_k - \rho \nabla J(g_k).$$

We also compute the residual error,

$$\frac{\|g_{k+1} - g_k\|}{\rho}, \tag{17}$$

and stop the algorithm if it is smaller that a given tolerance ε.

The solution of (1) obtained through the minimization of the function J is represented in Figure 3. It is clear than $u^g_{1|\Omega_1}$ and u^g_2 are approximations of $\overline{u}_{|\Omega_1}$ and $\overline{u}_{|\Omega_2}$. Moreover, superposing these two solutions, we can observe that the condition on the equality of u^g_1 and u^g_2 on Γ is fulfilled. Refining the mesh, we compute the L^2 and H^1_0-errors between the couple (u^g_1, u^g_2) and the reference solution obtained with the classical finite element method and using a fine conformal mesh. We obtain the convergence graph presented in Figure 5 where we observe than optimal rates of convergence are conserved by the smooth extension method in this case.

3.3 Test case 2: the transmission Laplace problem in a L-shape domain

Let Ω be a L-shape domain of $\mathbb{R}^2$ and let Ω_2 be a quadrilateral included in Ω (see Figure 6). As before, we apply the smooth extension method to the Laplace transmission problem with homogeneous Dirichlet boundary condition (1). Because of the geometry of the problem and especially the presence of a reentrant corner, the solution is not regular and only belongs to $H^1_0(\Omega)$. Thus, there do not exists a H^2-extension from Ω_1 to the whole domain and the method will not converge with optimal order. However, the smooth extension method can also be applied in this case and we will see that it is still better than the finite element method with non-conformal mesh (see Table 1). Considering the set of parameters $\mu_1 = 1$, $\mu_2 = 10$, $f_1 = 1$, $f_2 = 1$, $(x_c, y_c) = (0.62, 0.7)$, $x_1 = 0.43$, $x_2 = 0.58$, $(x_3, y_3) = (0.36, 0.38)$ and $(x_4, y_4) = (0.64, 0.49)$, we obtain the convergence graph represented in Figure 6.

To summarise the results obtained for all test cases that we have studied, the convergence rates are presented in Table 1. As a result, the smooth extension method applied to the Laplace transmission problem, converges with optimal rates even with non-conformal meshes when the solution of the Laplace transmission problem is partially of regularity H^2. When the solution is less regular (because of the geometry of the domain or the regularity of the right-hand sides)

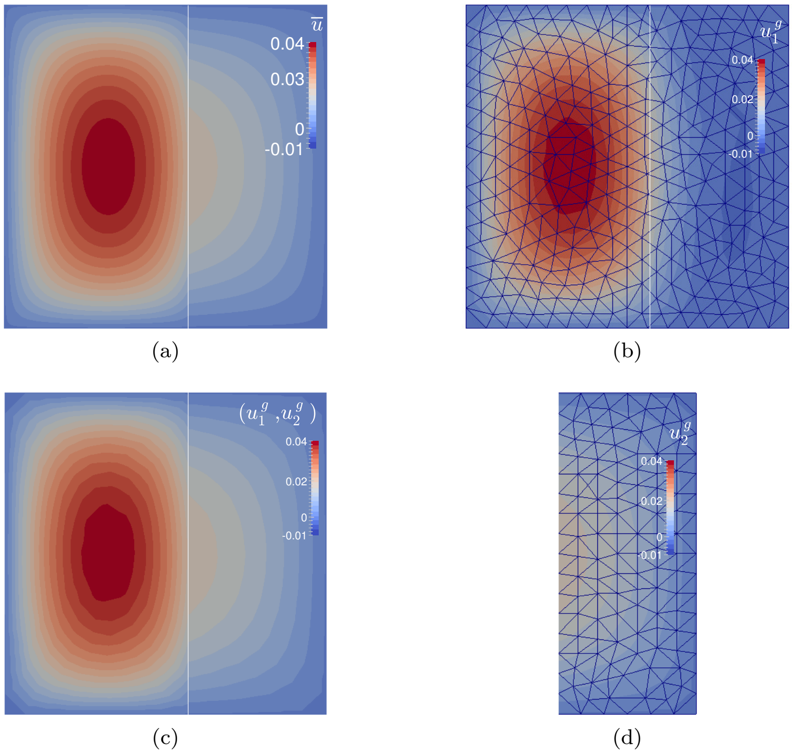


Figure 3: Numerical solution for the Laplace problem (1) with $\mu_1 = 1$, $\mu_2 = 10$, $f_1 = 1$, $f_2 = 1$ and $x_\Gamma = 0.57$. We compare the reference solution $\overline{u}$ obtained with the standard finite element method on a fine mesh (a) to the one obtained through the smooth extension method (c). The fields u_1^g (b) and u_2^g (d) are superposed (c) to show the continuity through the interface Γ despite the use of non-conformal meshes. For the SEM, the mesh is 16×16.

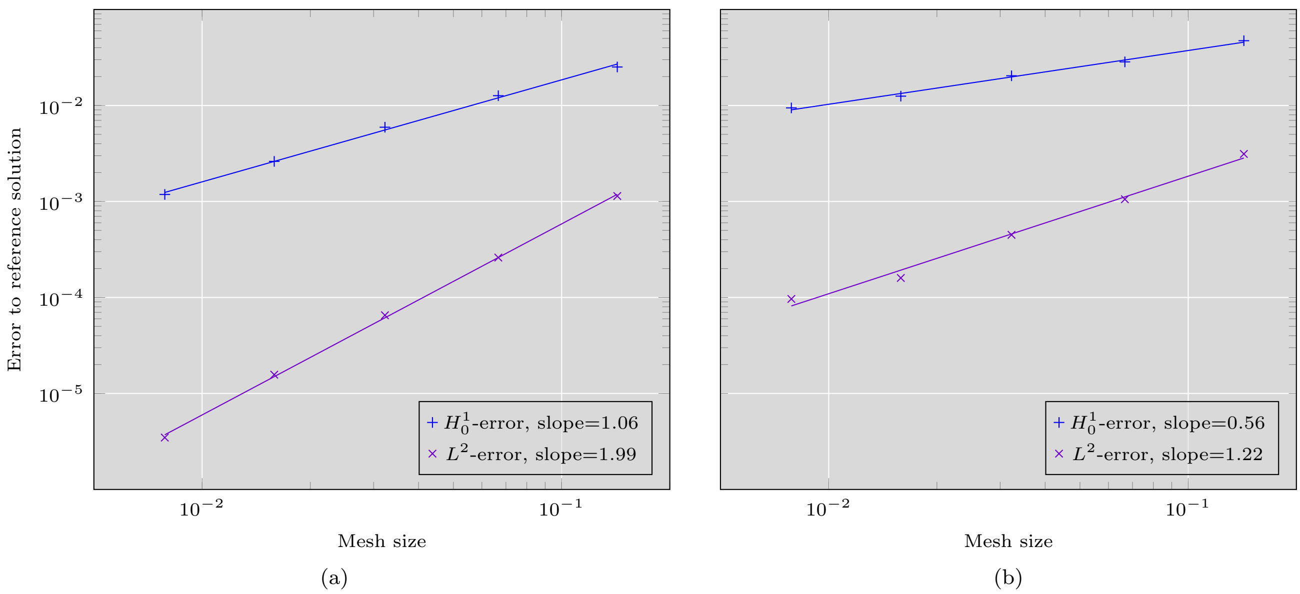


Figure 4: Rates of convergence in space for the finite element method ($P1$ elements) applied to the resolution of the Laplace problem (1) with $\mu_1 = 1$, $\mu_2 = 10$, $f_1 = 1$, $f_2 = 1$ and $x_\Gamma = 0.57$. The solution with a conformal mesh (a) is compared to the one with a non-conformal mesh (b)

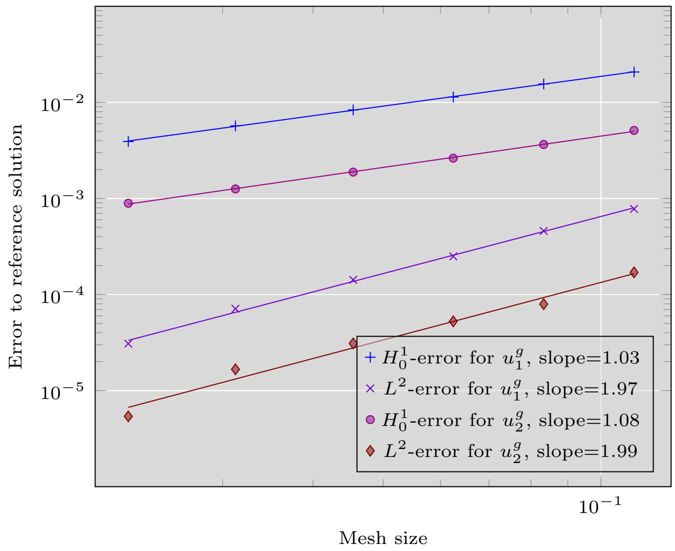
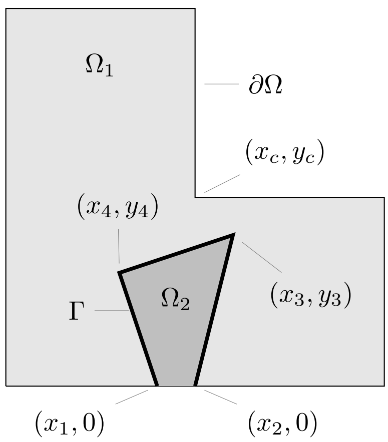
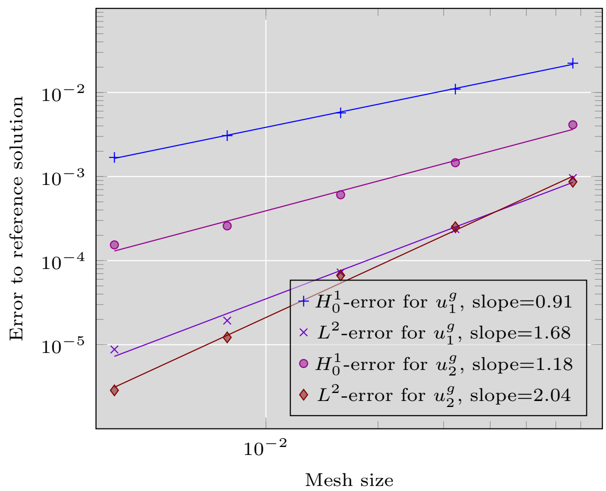


Figure 5: Rates of convergence in space for the smooth extension method applied to the resolution of the Laplace transmission problem with homogeneous boundary conditions (1) , with $\mu_1 = 1$, $\mu_2 = 10$, $f_1 = 1$, $f_2 = 2$ and $x_\Gamma = 0.67$.



(a)



(b)

Figure 6: Two dimensional representation of the geometry for the Laplace transmission problem (1) in a L-shape domain (a) and rates of convergence in space for the smooth extension method applied to the resolution of the Laplace transmission problem (1) in a L-shape domain (b).

Test case	Method	Mesh conformity	conv. rate in H_0^1-norm	conv. rate in L^2-norm
1	FEM	Conformal	1.06	1.99
1	FEM	Non-conformal	0.56	1.22
1	SEM	Non-conformal	1.03 (in Ω_1) / 1.08 (in Ω_2)	1.97 (in Ω_1) / 1.99 (in Ω_2)
2	FEM	Conformal	0.96	1.7
2	FEM	Non-conformal	0.76	1.43
2	SEM	Non-conformal	0.91 (in Ω_1) / 1.18 (in Ω_2)	1.68 (in Ω_1) / 2.04 (in Ω_2)

Table 1: Comparison of the rates of convergence between the finite element method (FEM) with a conformal mesh, the FEM with a non-conformal mesh and the smooth extension method (SEM) with a non-conformal mesh for different test cases.

the smooth extension method can also be applied and showed to converge with the same rates than the Finite Element method with conformal mesh. In the next section we will show that this method can be extended to other kind of coupled problems and enables to treat the numerical resolution of more general transmission problems with non-conformal meshes.

4 Extension to other coupled problems

In this section we extend the smooth extension method to two other coupled problems: the Stokes transmission problem in Subsection 4.1 and a fluid-structure interaction problem in Subsection 4.2.

4.1 The Stokes transmission problem

Let $n > 0$ and Ω be a domain of $\mathbb{R}^n$ that satisfies the following set of hypotheses:

$$\begin{array}{rl} i) & \text{Domain } \Omega \text{ is a bounded connected Lipschitz domain of } \mathbb{R}^n. \\ ii) & \text{Domain } \Omega \text{ is divided into two connected Lipschitz} \\ & \text{subdomains, } \Omega_1 \text{ and } \Omega_2. \\ iii) & \text{The interface } \Gamma = \partial\Omega_1 \cap \partial\Omega_2 \text{ is not empty.} \\ iv) & \text{The remaining boundaries } \Gamma_1 = \partial\Omega_1 \setminus \Gamma \text{ and } \Gamma_2 = \partial\Omega_2 \setminus \Gamma \\ & \text{are not empty.} \end{array} \tag{H_2}$$

Consider two real coefficients, μ_1 and μ_2, and two external forces, f_1 in $(L^2(\Omega_1))^n$ and f_2 in $(L^2(\Omega_2))^n$. Then, the Stokes transmission problem writes:

$$\left\{\begin{array}{ll} \multicolumn{2}{l}{\text{find } u_1 \colon \Omega_1 \to \mathbb{R}^n,\ p_1 \colon \Omega_1 \to \mathbb{R},\ u_2 \colon \Omega_2 \to \mathbb{R}^n \text{ and } p_2 \colon \Omega_2 \to \mathbb{R} \text{ such that}} \\ -\mu_1 \Delta u_1 + \nabla p_1 = f_1 & \text{in } \Omega_1, \\ \mathrm{div}(u_1) = 0 & \text{in } \Omega_1, \\ u_1 = 0 & \text{on } \Gamma_1, \\ -\mu_2 \Delta u_2 + \nabla p_2 = f_2 & \text{in } \Omega_2, \\ \mathrm{div}(u_2) = 0 & \text{in } \Omega_2, \\ u_2 = 0 & \text{on } \Gamma_2, \\ u_1 = u_2 & \text{on } \Gamma, \\ (\mu_1 \nabla u_1 - p_1 I) \cdot n_1 = -(\mu_2 \nabla u_2 - p_2 I) \cdot n_2 & \text{on } \Gamma. \end{array}\right. \quad \begin{array}{r} (18a) \\ (18b) \\ (18c) \end{array}$$

Equations (18a) and (18b) are two sets of Stokes equations coupled at the interface Γ with the coupling conditions (18c). These conditions represent the continuity of the fluid velocity and the continuity of the constraints applied by the fluid on Γ. The vectors n_1 and n_2 still denote the unit exterior normal vector of Ω_1 and Ω_2. Of course, problem (18) is equivalent to the Stokes problem with discontinuous viscosity and external force,

$$\left\{\begin{array}{rcll} \multicolumn{4}{l}{\text{find } u : \Omega \to \mathbb{R}^n \text{ and } p : \Omega \to \mathbb{R} \text{ such that}} \\ -\mathrm{div}(\mu \nabla u - pI) & = & f & \text{in} \quad \Omega, \\ \mathrm{div}(u) & = & 0 & \text{in} \quad \Omega, \\ u & = & 0 & \text{on} \quad \partial\Omega, \end{array}\right. \tag{19}$$

where μ and f are defined by

$$\mu = \left\{\begin{array}{ll} \mu_1 & \text{in} \quad \Omega_1 \\ \mu_2 & \text{in} \quad \Omega_2 \end{array}\right., \quad f = \left\{\begin{array}{ll} f_1 & \text{in } \Omega_1 \\ f_2 & \text{in } \Omega_2 \end{array}\right. .$$

We can define a weak formulation of problem (18). Let us introduce the functional spaces

$$\begin{array}{rcl} W_1 & = & \{v_1 \in (H^1(\Omega_1))^n; v_{1|\Gamma_1} = 0\}, \\ W_2 & = & \{v_2 \in (H^1(\Omega_2))^n; v_{2|\Gamma_2} = 0\}, \\ \mathcal{W} & = & \left\{(v_1, v_2) \in W_1 \times W_2; v_{1|\Gamma} = v_{2|\Gamma}\right\}, \\ Q & = & \left\{(p_1, p_2) \in L^2(\Omega_1) \times L^2(\Omega_2); \dfrac{1}{|\Omega|}\left(\displaystyle\int_{\Omega_1} p_1 + \int_{\Omega_2} p_2\right) = 0\right\}, \\ (H_{\mathrm{div}}(X))^n & = & \left\{\sigma \in (L^2(X))^{n\times n}; \mathrm{div}(\sigma) \in (L^2(X))^n\right\}, \end{array}$$

where X stands for either Ω_1 or Ω_2. Then, the weak problem associated to problem (18) writes:

$$\left\{\begin{array}{l} \text{find } (u_1, u_2) \in \mathcal{W}, \text{ and } (p_1, p_2) \in Q \text{ such that} \\ \mu_1 \displaystyle\int_{\Omega_1} \nabla u_1 \colon \nabla v_1 + \mu_2 \int_{\Omega_2} \nabla u_2 \colon \nabla v_2 - \int_{\Omega_1} p_1 \mathrm{div}(v_1) - \int_{\Omega_2} p_2 \mathrm{div}(v_2) \\ \qquad\qquad\qquad\qquad = \displaystyle\int_{\Omega_1} f_1 \cdot v_1 + \int_{\Omega_2} f_2 \cdot v_2, \quad \forall (v_1, v_2) \in \mathcal{W}, \\ \qquad \displaystyle\int_{\Omega_1} q_1 \mathrm{div}(u_1) + \int_{\Omega_2} q_2 \mathrm{div}(u_2) = 0, \qquad\qquad \forall (q_1, q_2) \in Q. \end{array}\right. \tag{20}$$

The well-posedness of problem (19) in $(H_0^1(\Omega))^n \times L_0^2(\Omega)$ is a particular case of [BF12, Theorem IV.8.1], where μ (positive) needs to be in $L^\infty(\Omega)$ and f belongs to $(L^2(\Omega))^n$. It follows that problem (20) is well-posed and we denote by $((\overline{w}_1, \overline{w}_2), (\overline{p}_1, \overline{p}_2))$ its unique solution. In particular, $\sigma_1 = \mu_1 \nabla u_1 - p_1 I$ belongs to $(H_{\mathrm{div}}(\Omega_1))^n$, $\sigma_2 = \mu_2 \nabla u_2 - p_2 I$ belongs to $(H_{\mathrm{div}}(\Omega_2))^n$ and we are able to give a weak sense to the second transmission condition on Γ in (18c). Let X stand for either Ω_1 or Ω_2 and let η be the unit exterior normal vector to X. Let $\Upsilon = (H_{00}^{1/2}(\Gamma))^n$ be the image of $(H^1_{\partial X \setminus \Gamma}(X))^n$ by the trace operator on Γ, i.e. the space of functions in $(H^{1/2}(\Gamma))^n$ whose extension by zero on $\partial X \setminus \Gamma$ belongs to $(H^{1/2}(\partial X))^n$. Then, for all σ in $(H_{\mathrm{div}}(X))^n$, we have the following Stokes formula:

$$\int_X \sigma : \nabla v + \int_X \mathrm{div}(\sigma) \cdot v \quad = \quad \langle \gamma_\eta(\sigma), v \rangle_{\Upsilon', \Upsilon}, \quad \forall v \in (H^1_{\partial X \setminus \Gamma}(X))^n. \tag{21}$$

where Υ' is the dual space of Υ. Then, the second transmission condition in (18c) is satisfied in the following sense:

$$\langle \gamma_{n_1}(\sigma_1), v \rangle_{\Upsilon', \Upsilon} \quad = \quad - \langle \gamma_{n_2}(\sigma_2), v \rangle_{\Upsilon', \Upsilon}, \quad \forall v \in \Upsilon. \tag{22}$$

Now, we present the smooth extension method applied to problem (18). Formally, it writes: find g such that the solution of the following problem,

$$\left\{\begin{array}{lr} \text{find } u_1 : \Omega_1 \to \mathbb{R}^n,\ p_1 : \Omega_1 \to \mathbb{R},\ u_2 : \Omega_2 \to \mathbb{R}^n \text{ and } p_2 : \Omega_2 \to \mathbb{R} \text{ such that} & \\ \qquad -\mu_1 \Delta u_1 + \nabla p_1 = \overline{f_1}^\Omega + \overline{g}^\Omega \qquad \text{in } \Omega, & \\ \qquad \mathrm{div}(u_1) = 0 \qquad \text{in } \Omega, & (23a) \\ \qquad u_1 = 0 \qquad \text{in } \partial\Omega, & \\ \qquad -\mu_2 \Delta u_2 + \nabla p_2 = f_2 \qquad \text{in } \Omega_2, & \\ \qquad \mathrm{div}(u_2) = 0 \qquad \text{in } \Omega_2, & \\ \qquad u_2 = 0 \qquad \text{on } \Gamma_2, & (23b) \\ \qquad (\mu_2 \nabla u_2 - p_2 I) \cdot n_2 = (\mu_1 \nabla u_1 - p_1 I) \cdot n_2 \qquad \text{on } \Gamma, & \end{array}\right.$$

satisfies the equality

$$u_1 \quad = \quad u_2 \quad \text{ on } \Gamma$$

Let v be in $(\mathcal{D}(\Omega))^n$ and suppose that u_1, p_1, u_2 and p_2 are sufficiently regular. Moreover, we assume for the moment that g belongs to $(L^2(\Omega_2))^n$ in order to do formal computations. Formally, we multiply equation (23a) by v and equation (23b) by $v_{|\Omega_2}$ and integrate respectively over Ω and Ω_2. After an integration by part and using the Neumann condition on Γ, we find

$$\begin{array}{rcl} \mu_1 \displaystyle\int_\Omega \nabla u_1 \colon \nabla v - \int_\Omega p_1 \mathrm{div}(v) & = & \displaystyle\int_{\Omega_1} f_1 \cdot v_{|\Omega_1} + \int_{\Omega_2} g \cdot v_{|\Omega_2}, \\ \mu_2 \displaystyle\int_{\Omega_2} \nabla u_2 \colon \nabla v_{|\Omega_2} - \int_{\Omega_2} p_2 \mathrm{div}(v_{|\Omega_2}) & = & \displaystyle\int_{\Omega_2} f_2 \cdot v_{|\Omega_2} + \int_\Gamma ((\mu_1 \nabla u_1 - p_1 I) n_2) \cdot v. \end{array}$$

Moreover, we remark that

$$\begin{array}{rcl} \displaystyle\int_\Gamma ((\mu_1 \nabla u_1 - p_1 I) n_2) \cdot v & = & \displaystyle\int_{\Omega_2} (\mu_1 \Delta u_1 - \nabla p_1) \cdot v_{|\Omega_2} + \mu_1 \int_{\Omega_2} \nabla u_1 \colon \nabla v_{|\Omega_2} - \int_{\Omega_2} p_1 \mathrm{div}(v_{|\Omega_2}), \\ & = & - \displaystyle\int_{\Omega_2} g \cdot v_{|\Omega_2} + \mu_1 \int_{\Omega_2} \nabla u_1 \colon \nabla v_{|\Omega_2} - \int_{\Omega_2} p_1 \mathrm{div}(v_{|\Omega_2}). \end{array}$$

Thus, we define the weak formulation of the smooth extension problem associated to problem (23), which makes sense for $g \in W_2'$, as the problem of finding a suitable control g in W_2', such that the solution of the following problem,

$$\left\{\begin{array}{l} \text{find } (u_1,p_1) \in (H_0^1(\Omega))^n \times L_0^2(\Omega) \text{ and } (u_2,p_2) \in W_2 \times L^2(\Omega_2) \text{ such that} \\ \qquad \begin{array}{rcll} \mu_1 \displaystyle\int_\Omega \nabla u_1 : \nabla v_1 - \int_\Omega p_1 \mathrm{div}(v_1) & = & \displaystyle\int_{\Omega_1} f_1 \cdot v_{1|\Omega_1} + \langle g, v_{1|\Omega_2}\rangle_{W_2',W_2}, & \forall v_1 \in (H_0^1(\Omega))^n, \\ \displaystyle\int_\Omega q_1 \mathrm{div}(u_1) & = & 0, & \forall q_1 \in L_0^2(\Omega), \end{array} \\ \\ \begin{array}{rcll} \mu_2 \displaystyle\int_{\Omega_2} \nabla u_2 : \nabla v_2 - \int_{\Omega_2} p_2 \mathrm{div}(v_2) & = & \displaystyle\int_{\Omega_2} f_2 \cdot v_2 - \langle g, v_2\rangle_{W_2',W_2} & \\ & & +\mu_1 \displaystyle\int_{\Omega_2} \nabla u_1 : \nabla v_2 - \int_{\Omega_2} p_1 \mathrm{div}(v_2), & \forall v_2 \in W_2, \\ \displaystyle\int_{\Omega_2} q_2 \mathrm{div}(u_2) & = & 0, & \forall q_2 \in L^2(\Omega_2), \end{array} \end{array}\right. \tag{24}$$

satisfies the equality

$$u_1 = u_2 \quad \text{in } \Upsilon. \tag{25}$$

For every f_1 in $(L^2(\Omega_1))^n$, every f_2 in $(L^2(\Omega_2))^n$ and every g in W_2', the two subproblems in problem (24) admit a unique solution. They are denoted by (w_1^g, p_1^g) and (w_2^g, p_2^g). These are well-known results on Stokes equations; we refer to [BF12] for details. In the following theorem, we state the existence of a control g such that the solution of (24) satisfies the condition (25) and explain how to recover the solution of the initial Stokes transmission problem (20).

Theorem 4.1. *Let Ω be a domain that satisfies Assumption (H_2). Consider f_1 in $(L^2(\Omega_1))^n$ and f_2 in $(L^2(\Omega_2))^n$. Then, there exists a function g in W_2' such that the solution $((w_1^g, p_1^g), (w_2^g, p_2^g))$ of (24) satisfies (25). Moreover, we can recover the solution of the Stokes transmission problem (20):*

$$\begin{array}{rcl} (\overline{w}_1, \overline{w}_2) & = & (w^g_{1|\Omega_1}, w_2^g), \\ (\overline{p}_1, \overline{p}_2) & = & (p^g_{1|\Omega_1} - C, p_2^g - C), \end{array}$$

where

$$C = \frac{1}{|\Omega|}\left(\int_{\Omega_1} p^g_{1|\Omega_1} + \int_{\Omega_2} p_2^g\right).$$

The proof of Theorem 4.1 is similar to the proof of Theorem 2.1. It relies on the construction of extensions for the velocity $\overline{w}_1$ and the pressure $\overline{p}_1$ in the whole domain Ω. Details are given in A.

As before, the problem of finding a suitable control g such that the solution of problem (24) satisfies (25), can be formulated as an optimization problem. In practice, it is this minimization problem which is solved in order to obtain a suitable control and recover the solution of the transmission problem (18). The cost function to consider, that we denote by $\tilde{J}$, is now defined from W_2' to $\mathbb{R}^+$ with the formula

$$\tilde{J}(g) = \frac{1}{2}\int_\Gamma |w_1^g - w_2^g|^2, \tag{26}$$

where w_1^g and w_2^g are the velocities of the fluid, solutions of problem (24). Yet, the minimization of this cost function is equivalent to the minimization of the real-valued function

$$\begin{array}{rcl} (H_0^1(\Omega))^n \times W_2 & \to & \mathbb{R}^+ \\ (v_1, v_2) & \mapsto & \displaystyle\frac{1}{2}\int_\Gamma |v_{1|\Gamma} - v_{2|\Gamma}|^2, \end{array}$$

under the constraint that v_1 and v_2 are the velocities that solve the problem (24). Then, to this constrained optimization problem we associate the following Lagrangian function defined from

$$W_2' \times \left[\left((H_0^1(\Omega))^n \times L_0^2(\Omega)\right) \times \left(W_2 \times L^2(\Omega_2)\right)\right] \times \left[\left((H_0^1(\Omega))^n \times L_0^2(\Omega)\right) \times \left(W_2 \times L^2(\Omega_2)\right)\right]$$

to $\mathbb{R}$ by

$$\begin{aligned} \tilde{\mathcal{L}}(g, ((u_1,p_1),(u_2,p_2)), ((\lambda_1,\pi_1),(\lambda_2,\pi_2))) &= \frac{1}{2}\int_\Gamma |u_1 - u_2|^2 + \mu_1 \int_\Omega \nabla u_1 : \nabla\lambda_1 - \mu_1 \int_{\Omega_2} \nabla u_1 : \nabla \lambda_2 \\ +\mu_2 \int_{\Omega_2} \nabla u_2 : \nabla\lambda_2 - \int_\Omega p_1 \mathrm{div}(\lambda_1) - \int_\Omega \pi_1 \mathrm{div}(u_1) &- \int_{\Omega_2} p_2 \mathrm{div}(\lambda_2) - \int_{\Omega_2} \pi_2 \mathrm{div}(u_2) + \int_{\Omega_2} p_1 \mathrm{div}(\lambda_2) \\ - \int_{\Omega_1} f_1 \cdot \lambda_{1|\Omega_1} - \langle g, \lambda_{1|\Omega_2}\rangle_{W_2',W_2} - \int_{\Omega_2} f_2 \cdot \lambda_2 + \langle g, \lambda_2\rangle_{W_2',W_2} & \end{aligned} \tag{27}$$

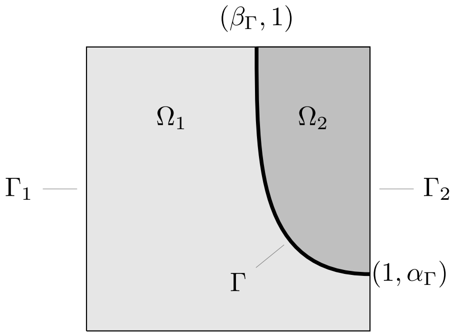


Figure 7: Two dimensional representation of the geometry for the Stokes transmission problem.

Again, this Lagrangian function enables to compute the gradient of $\tilde{J}$ and it is possible to show the equivalence between the minimization of $\tilde{J}$ and the research of a suitable control such that the solution of (24) satisfies the condition (25). For that matter, we introduce the adjoint problem of (24),

$$\left\{\begin{array}{rcll} \multicolumn{4}{l}{\text{find } (\lambda_1, \pi_1) \in (H_0^1(\Omega))^n \times L_0^2(\Omega) \text{ and } (\lambda_2, \pi_2) \in W_2 \times L^2(\Omega_2) \text{ such that,}} \\ \mu_2 \displaystyle\int_{\Omega_2} \nabla\lambda_2 : \nabla v_2 - \int_{\Omega_2} \pi_2 \mathrm{div}(v_2) & = & \displaystyle\int_\Gamma (w_1^g - w_2^g) \cdot v_2, & \forall v_2 \in W_2, \\ \displaystyle\int_{\Omega_2} q_2 \mathrm{div}(\lambda_2) & = & 0, & \forall q_2 \in L^2(\Omega_2), \\ \mu_1 \displaystyle\int_\Omega \nabla\lambda_1 : \nabla v_1 - \int_\Omega \pi_1 \mathrm{div}(v_1) & = & -\displaystyle\int_\Gamma (w_1^g - w_2^g) \cdot v_1 + \mu_1 \int_{\Omega_2} \nabla\lambda_2 : \nabla v_1, & \forall v_1 \in (H_0^1(\Omega))^n, \\ \displaystyle\int_\Omega q_1 \mathrm{div}(\lambda_1) & = & \displaystyle\int_{\Omega_2} q_1 \mathrm{div}(\lambda_2), & \forall q_1 \in L_0^2(\Omega). \end{array}\right. \tag{28}$$

Problem (28) admits a unique solution denoted by $((\omega_1^g, \pi_1^g), (\omega_2^g, \pi_2^g))$ (see Appendix A). Then, the existence and the characterization of the gradient of $\tilde{J}$ is given in the following theorem,

Theorem 4.2. *The mapping $g \in W_2' \mapsto \tilde{J}(g) \in \mathbb{R}$ is differentiable and its gradient $\nabla\tilde{J}(g) \in W_2''$ is given by,*

$$\left\langle \nabla\tilde{J}(g), \delta g \right\rangle_{W_2'', W_2'} = \left\langle \delta g, \omega_2^g - \omega_{1|\Omega_2}^g \right\rangle_{W_2', W_2}, \quad \forall \delta g \in W_2', \tag{29}$$

where ω_1^g and ω_2^g are the unique velocities that verify the adjoint problem (28).

Moreover, the equivalence between the smooth extension problem and its formulation as a minimization problem can now be proved,

Theorem 4.3. *A control g in W_2' is a minimizer of $\tilde{J}$ if and only if the solution of* (24) *satisfies the condition* (25).

For reasons of clarity, the proofs of Theorem 4.2 and Theorem 4.3 are done in Appendix A.

As for the Laplace transmission problem, the formulation of the Stokes transmission problem (18) as a control problem and as a minimization problem on the function $\tilde{J}$ enables us to numerically solve these equations with a fictitious domain approach. Here, we apply this method to the numerical simulation of a two-layer Stokes fluid in the unit square of $\mathbb{R}^2$. The geometric settings are represented in Figure 7: Ω_1 and Ω_2 are separated by a smooth curve which encounters the boundary $\partial\Omega$ at two points, $(1, \alpha_\Gamma)$ and $(\beta_\Gamma, 1)$, where α_Γ and β_Γ are two positive constants. The interface Γ is defined such that a point (x, y) in $[\beta_\Gamma, 1] \times [\alpha_\Gamma, 1]$ belongs to Γ if and only if

$$1 - x \leq \beta_\Gamma \sqrt{\frac{y - \alpha_\Gamma}{y}}.$$

We consider homogeneous Dirichlet boundary conditions on the left and right boundaries and a pressure drop of 1 between the top and the bottom boundaries. No external force are considered and constant viscosities $\mu_1 = 1$ and $\mu_2 = 10$ are chosen. For the interface Γ, we choose $\alpha_\Gamma = 0.25$ and $\beta_\Gamma = 0.5$.

The solution given by the smooth extension method is obtained following Algorithm 1 and using the gradient method with fixed parameter ρ. This parameter is chosen such that the gradient method convergences and the stopping criteria is defined in (17). The resolution of the four different Stokes problems appearing in the minimization process is done using Mini elements. To compare the smooth extension method to the classical finite element method, we use meshes that have about the same numbers of cells: 26×26.

Results are shown in Figure 8, where both the velocity and the pressure of the fluid are plotted. We compare the solution obtained through the standard finite element method with a mesh which is actually conform with the interface Γ, to the one produced by the smooth extension method with unstructured meshes. We observe that both solutions are similar at the difference that the smooth extension method well represents the physical jump in pressure through the interface Γ, while the finite element method with Mini elements does not. Moreover, the former also has the advantage to be computed with a non-conformal mesh.

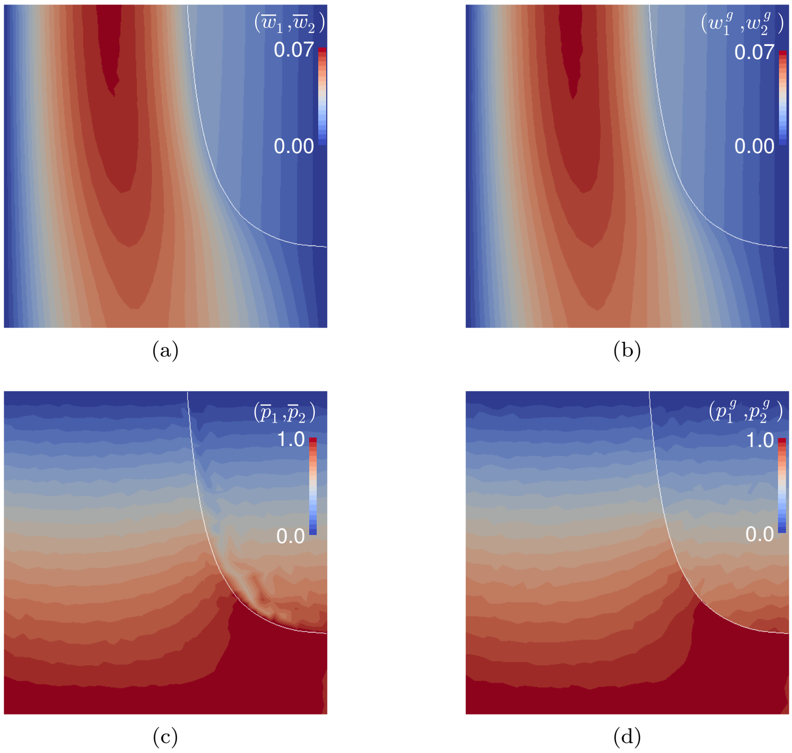


Figure 8: Representation of the solution for the Stokes transmission problem (18) with $\mu_1 = 1$, $\mu_2 = 10$, $f_1 = (1, 0)$, $f_2 = (1, 0)$, $\alpha_\Gamma = 0.25$ and $\beta_\Gamma = 0.5$. One the top, we compare the magnitude of the fluid velocity obtained with a standard finite element method (a) and with the smooth extension method (b). On the bottom, we compare the pressure of the fluid computed with the classical finite element method (c) and the smooth extension method (d). The interface Γ is highlighted with a white curve.

4.2 A fluid-structure interaction problem

Now, we are interested in the resolution of a fluid-structure problem where the fluid is modeled by the Stokes equations and the structure by the stationary equations of linear elasticity. The unknowns for these two systems of equations are the fluid velocity and pressure as well as the displacement of the structure from its reference configuration. The fluid problem will be set in Eulerian coordinates, i.e. in the current configuration, whereas the elastic equations will be written in Lagrangian coordinates, i.e. in the reference configuration. This amounts to consider two configurations for both problems, supposing that, at each time, there exists a smooth enough mapping between the two configurations to ensure that all boundaries in the current configuration are sufficiently regular.

Let $n > 0$ and Ω be a domain of $\mathbb{R}^n$ that satisfies the following set of hypotheses:

$$\left.\begin{array}{rl} i) & \text{Domain } \Omega \text{ is a bounded connected Lipschitz domain of } \mathbb{R}^n. \\ ii) & \text{Domain } \Omega \text{ is divided in two connected Lipschitz subdomains, } \Omega_f \\ & \text{the fluid subdomain and } \Omega_s \text{ the solid subdomain.} \\ iii) & \text{The interface } \Gamma = \partial\Omega_f \cap \partial\Omega_s \text{ is not empty.} \\ iv) & \text{The remaining boundaries } \Gamma_f = \partial\Omega_f \setminus \Gamma \text{ and } \Gamma_s = \partial\Omega_s \setminus \Gamma \text{ are not empty.} \end{array}\right. \qquad (H_3)$$

We assume that, at each time $t \geq 0$, there exists a deformation Φ_t, i.e. a smooth enough injective and orientation-preserving mapping, defined from Ω to $\mathbb{R}^n$, such that the current fluid configuration $\Phi_t(\Omega_f)$ and the current solid configuration $\Phi_t(\Omega_s)$ also are Lipschitz subdomains of Ω. Moreover, $\Phi_t(\Omega_f)$ and $\Phi_t(\Omega_s)$ should be connected. We should precise here that the mapping Φ_t directly depends on the displacement of the structure at time t. For all x in Ω_s, this mapping writes

$$\Phi_t(x) = x + d_s(t)(x),$$

and we can easily extend Φ_t in the whole domain Ω. The existence of a smooth transformation Φ_t is rather complicated to prove and is out of the scope of this study. However, for the numerical resolution of this fluid-structure interaction problem, it is possible to construct Φ_t if the deformation of the structure is reasonable, i.e. if the structure does not enter in contact with itself or the boundary $\partial\Omega \setminus \Gamma_s$ and if the mesh which represents the domain Ω_s is admissible (no overlapping cell).

Then, at each time $t \geq 0$, the subdomain $\Phi_t(\Omega_f)$ is filled with an incompressible Newtonian fluid whose velocity

$u_f(t) : \Phi_t(\Omega_f) \to \mathbb{R}^n$ and pressure $p_f(t) : \Phi_t(\Omega_f) \to \mathbb{R}$ satisfy the Stokes equations in conservative form

$$\begin{array}{rclll} -\mathrm{div}(\sigma_f(u_f(t), p_f(t))) & = & f_f(t) & \text{in} & \Phi_t(\Omega_f), \\ \mathrm{div}(u_f(t)) & = & 0 & \text{in} & \Phi_t(\Omega_f), \\ u_f(t) & = & 0 & \text{on} & \Phi_t(\Gamma_f), \end{array}$$

where σ_f is the fluid tensor defined for all $u\colon \mathbb{R}^n \to \mathbb{R}^n$ and all $p\colon \mathbb{R}^n \to \mathbb{R}$ by

$$\sigma_f(u,p) \quad = \quad 2\mu_f D(u) - pI, \quad D(u) \quad = \quad \frac{1}{2}(\nabla u + \nabla u^T),$$

the constant μ_f is the viscosity of the fluid and $f_f(t) : \Phi_t(\Omega_f) \to \mathbb{R}^n$ is the external force applied to the fluid at time t. Inside the fluid lies an elastic medium, whose displacement at time t, $d_s(t) : \Omega_s \to \mathbb{R}^n$, verifies the following equations of linear elasticity written in the reference solid configuration Ω_s

$$\begin{array}{rclll} -\mathrm{div}(\sigma_s(d_s(t))) & = & f_s(t) & \text{in} & \Omega_s, \\ d_s(t) & = & 0 & \text{on} & \Gamma_s, \end{array}$$

where σ_s is the solid tensor defined for all $u\colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$\sigma_s(u) \quad = \quad 2\mu_s D(u) + \lambda_s \mathrm{div}(u) I,$$

the two positive constants μ_s and λ_s are the Lam coefficients and $f_s(t) : \Omega_s \to \mathbb{R}^n$ is the external force applied to the structure at time t. To complete this system of equations, we consider at each time t the coupling conditions that correspond to the continuity of the velocities and the normal constraints through the fluid-structure interface in the reference configuration Γ. For that matter, we introduce the fluid velocity and pressure written in the reference fluid configuration, denoted by w_f and q_f, and defined at time t by

$$w_f(t) = u_f(t) \circ \Phi_t \quad \text{and} \quad q_f(t) = p_f(t) \circ \Phi_t.$$

Moreover, we introduce the fluid stress tensor written in the fluid reference configuration, denoted by Π_f, and defined at time t by

$$\Pi_f(w_f(t), q_f(t)) = \mu_f(\nabla w_f(t) F(d_s(t)) + \nabla w_f(t)^T F(d_s(t))^T) - q_f(t) G(d_s(t)) \quad \text{in } \Omega_f,$$

where $F(d_s(t))$ and $G(d_s(t))$ are the following matrices:

$$F(d_s(t)) \quad = \quad (\nabla(\Phi(d_s(t))))^{-1} \mathrm{cof}(\nabla(\Phi(d_s(t)))), \quad G(d_s(t)) \quad = \quad \mathrm{cof}(\nabla(\Phi(d_s(t)))). \tag{30}$$

Thus the transmission conditions write

$$\begin{array}{rclll} \dfrac{\partial d_s}{\partial t}(t) & = & u_f(t) \circ \Phi_t & \text{on} & \Gamma, \\ \sigma_s(d_s(t)) n_s & = & \Pi_f(w_f(t), q_f(t)) n_s & \text{on} & \Gamma, \end{array} \tag{31}$$

where the vector n_s denotes the exterior unit normal vector to $\partial\Omega_s$. Similarly we denote n_d the exterior unit normal vector to $\partial\Omega_f$. Furthermore, we suppose that the structure is at rest initially, i.e. that $d_s(0) = 0$.

For the purpose of the numerical resolution we consider a discretization of $\mathbb{R}^+$ for the time variable. Let $\delta t > 0$ be the step size. We construct a sequence $(t_k)_{k \in \mathbb{R}^+}$ such that $t_0 = 0$ and $t_{k+1} = t_k + \delta t$ for $k > 0$. Thus, we define the time-discretizations of u_f, p_f and d_s such that, for all $k \geq 0$,

$$u_f^k = u_f(t_k), \quad p_f^k = p_f(t_k), \quad d_s^k = d_s(t_k).$$

In addition, we also define $f_f^k = f_f(t_k)$ and $f_s^k = f_s(t_k)$ for all $k \geq 0$. The discretization of the first coupling condition (31) is obtained using the implicit Euler scheme:

$$\begin{array}{rcl} d_s^{k+1} & = & d_s^k + \delta t u_f^{k+1}, \quad \forall k \geq 0, \\ d_s^0 & = & \delta t u_f^0. \end{array}$$

At time t_k the current solid domain is given by $\Phi_{t_k}(\Omega_s) = (id + d_s^{k-1})(\Omega_s)$ and the current fluid domain $\Phi_{t_k}(\Omega_f)$ is obtained by extending the mapping Φ_{t_k} in the whole domain Ω. Moreover, because of the homogeneous Dirichlet boundary conditions on the external frontier $\partial\Omega$, it is clear that for all $k \geq 0$, $\Phi_{t_k}(\Omega) = \Omega$. Furthermore, the matrices F_{t_k} and G_{t_k} only depend on the displacement of the structure at time t_{k-1}.

Hence, for all $k \geq 0$, the triplet (u_f^k, p_f^k, d_s^k) is solution of the following problem,

$$
\left\{
\begin{array}{ll}
\text{find } u\colon \Phi_{t_k}(\Omega_f) \to \mathbb{R}^n,\ p\colon \Phi_{t_k}(\Omega_f) \to \mathbb{R} \text{ and } d\colon \Omega_s \to \mathbb{R}^n \text{ such that} & \\
\qquad -\mathrm{div}(\sigma_f(u,p)) = f_f^k & \text{in } \Phi_{t_k}(\Omega_f), \\
\qquad \mathrm{div}(u) = 0 & \text{in } \Phi_{t_k}(\Omega_f), \qquad (32a) \\
\qquad u = 0 & \text{in } \Phi_{t_k}(\Gamma_f), \\
\qquad -\mathrm{div}(\sigma_s(d)) = f_s^k & \text{in } \Omega_s, \qquad (32b) \\
\qquad d = 0 & \text{on } \Gamma_s, \\
\qquad d = d_s^{k-1} + \delta t u \circ \Phi_{t_k} & \text{on } \Gamma, \qquad (32c) \\
\qquad \sigma_s(d) n_s = \Pi_f(u \circ \Phi_{t_k}, p \circ \Phi_{t_k}) n_s & \text{on } \Gamma,
\end{array}
\right.
$$

where by convention $d_s^{-1} = 0$ for $k = 0$. We can define a weak formulation of problem (32). Let X stand for either Ω_f, Ω_s, $\Phi_{t_k}(\Omega_f)$ or $\Phi_{t_k}(\Omega_s)$. We introduce the following functional spaces:

$$
\begin{array}{rcl}
V_f^k & = & \{v \in (H^1(\Phi_{t_k}(\Omega_f)))^n \ ; \ v_{|\Phi_{t_k}(\Gamma_f)} = 0\}, \\
V_s^k & = & \{v \in (H^1(\Phi_{t_k}(\Omega_s)))^n \ ; \ v_{|\Phi_{t_k}(\Gamma_s)} = 0\}, \\
V_s & = & \{v \in (H^1(\Omega_s))^n \ ; \ v_{|\Gamma_s} = 0\}, \\
W_u & = & \left\{(v_f, v_s) \in V_f^k \times V_s \ ; \ (v_f \circ \Phi_{t_k})_{|\Gamma} = v_{s|\Gamma}\right\}, \\
W_d & = & \left\{(v_f, d_s) \in V_f^k \times V_s \ ; \ \delta_t (v_f \circ \Phi_{t_k})_{|\Gamma} + d_s^{k-1} = d_{s|\Gamma}\right\}, \\
(H_{\mathrm{div}}(X))^n & = & \left\{\sigma \in (L^2(X))^{n\times n} \ ; \ \mathrm{div}(\sigma) \in (L^2(X))^n\right\}.
\end{array}
$$

The weak formulation of problem (32) writes:

$$
\left\{
\begin{array}{rcll}
\multicolumn{4}{l}{\text{find } (u,d) \in W_d \text{ and } p \in L^2(\Phi_{t_k}(\Omega_f)) \text{ such that}} \\
\displaystyle\int_{\Phi_{t_k}(\Omega_f)} \sigma_f(u,p)\colon \nabla v_f + \int_{\Omega_s} \sigma_s(d)\colon \nabla v_s & = & \displaystyle\int_{\Phi_{t_k}(\Omega_f)} f_f^k \cdot v_f + \int_{\Omega_s} f_s^k \cdot v_s, & \forall (v_f, v_s) \in W_u, \\
\displaystyle\int_{\Phi_{t_k}(\Omega_f)} q \mathrm{div}(u) & = & 0, & \forall q \in L^2(\Phi_{t_k}(\Omega_f)).
\end{array}
\right. \tag{33}
$$

Problem (33) admits a unique solution that we still denote by (u_f^k, p_f^k, d_s^k) (see Appendix B). Moreover, $\sigma_f(u_f^k, p_f^k)$ belongs to $(H_{\mathrm{div}}(\Phi_{t_k}(\Omega_f)))^n$, $\sigma_s(d_s^k)$ belongs to $(H_{\mathrm{div}}(\Omega_s))^n$ and we can give a weak sense to the second transmission condition in (32c). Let X stands for either Ω_f or Ω_s and let η be the exterior normal vector to X. Let $\Upsilon = (H_{00}^{1/2}(\Gamma))^n$ be the image of $(H^1_{\partial X \setminus \Gamma}(X))^n$ by the trace operator on the interface Γ, i.e. the space of functions in $(H^{1/2}(\Gamma))^n$ whose extension by zero on $\partial X \setminus \Gamma$ belongs to $(H^{1/2}(\partial X))^n$.

Then, for all σ in $(H_{\mathrm{div}}(X))^n$, we have the following Stokes formula:

$$
\int_X \sigma\colon \nabla v + \int_X \mathrm{div}(\sigma) \cdot v \ = \langle \gamma_\eta(\sigma), v \rangle_{\Upsilon', \Upsilon}, \quad \forall v \in (H^1_{\partial X \setminus \Gamma}(X))^n, \tag{34}
$$

where Υ' is the dual space of Υ and γ_η is the trace normal operator on Γ. Then, the second transmission condition in (32c) is satisfied in the following sense:

$$
\langle \gamma_{n_s}(\sigma_s(d_s^k)), v \rangle_{\Upsilon', \Upsilon} \ = \ - \left\langle \gamma_{n_f}(\Pi_f(u_f^k \circ \Phi_{t_k}, p_f^k \circ \Phi_{t_k})), v \right\rangle_{\Upsilon', \Upsilon}, \quad \forall v \in \Upsilon. \tag{35}
$$

Similarly, we define $\Upsilon^k = (H_{00}^{1/2}(\Phi_{t_k}(\Gamma)))^n$ which enables to also write Stokes formulas for a tensor σ in $\Phi_{t_k}(\Omega_f)$ and in $\Phi_{t_k}(\Omega_s)$.

Now, we present the smooth extension method applied to problem (32). Formally, it writes: find g in $(V_s^k)'$ such that the solution of the following problem,

$$
\left\{
\begin{array}{ll}
\text{find } u : \Omega \to \mathbb{R}^n,\ p : \Omega \to \mathbb{R} \text{ and } d : \Omega_s \to \mathbb{R}^n \text{ such that,} & \\
\qquad -\mathrm{div}(\sigma_f(u,p)) = \overline{f_f^k}^{\Omega} + \overline{g}^{\Omega}, & \text{in } \Omega, \\
\qquad \mathrm{div}(u) = 0, & \text{in } \Omega, \qquad (36a) \\
\qquad u = 0, & \text{in } \partial\Omega, \\
\qquad -\mathrm{div}(\sigma_s(d)) = f_s^k, & \text{in } \Omega_s, \\
\qquad d = 0, & \text{on } \Gamma_s, \qquad (36b) \\
\qquad \sigma_s(d) n_s = \Pi_f(u \circ \Phi_{t_k}, p \circ \Phi_{t_k}) n_s, & \text{on } \Gamma
\end{array}
\right.
$$

satisfies the equality,

$$u\circ\Phi_{t_k} = \frac{1}{\delta t}(d-d_s^{k-1}), \text{ on } \Gamma.$$

Let v be in $(\mathcal{D}(\Omega))^n$ and suppose that u, p and d are sufficiently regular. Moreover, we assume for the moment that g belongs to $(L^2(\Phi_{t_k}(\Omega_s)))^n$ in order to do formal computations. Formally, we multiply the first equation in (36a) by v and the first equation in (36b) by $v_{|\Omega_s}$, and integrate respectively over Ω and Ω_s. After an integration by part and using the Neumann condition on Γ, we find

$$\begin{aligned}
\int_\Omega \sigma_f(u,p)\colon \nabla v &= \int_{\Phi_{t_k}(\Omega_f)} f_f^k\cdot v + \int_{\Phi_{t_k}(\Omega_s)} g\cdot v,\\
\int_{\Omega_s} \sigma_s(d)\colon \nabla v_{|\Omega_s} &= \int_{\Omega_s} f_s^k\cdot v_{|\Omega_s} + \int_\Gamma (\Pi_f(u\circ\Phi_{t_k}, p\circ\Phi_{t_k})n_s)\cdot v.
\end{aligned}$$

Moreover, we remark that

$$\begin{aligned}
\int_\Gamma (\Pi_f(u\circ\Phi_{t_k},p\circ\Phi_{t_k})n_s)\cdot v &= \int_{\Omega_s} \mathrm{div}(\Pi_f(u\circ\Phi_{t_k},p\circ\Phi_{t_k}))\cdot v_{|\Omega_s} + \int_{\Omega_s} \Pi_f(u\circ\Phi_{t_k},p\circ\Phi_{t_k})\colon \nabla v_{|\Omega_s},\\
&= -\int_{\Omega_s} \det(\nabla\Phi_{t_k})(g\circ\Phi_{t_k})\cdot v_{|\Omega_s} + \int_{\Omega_s} \Pi_f(u\circ\Phi_{t_k},p\circ\Phi_{t_k})\colon \nabla v_{|\Omega_s},\\
&= -\int_{\Phi_{t_k}(\Omega_s)} g\cdot (v_{|\Omega_s}\circ\Phi_{t_k}^{-1}) + \int_{\Omega_s} \Pi_f(u\circ\Phi_{t_k},p\circ\Phi_{t_k})\colon \nabla v_{|\Omega_s}.
\end{aligned}$$

Thus, we define the weak formulation of the smooth extension problem (36), which makes sense for g in $(V_s^k)'$, as the problem of finding a suitable control g in $(V_s^k)'$, such that the solution of the following problem,

$$\left\{\begin{aligned}
&\text{find } u\in H_0^1(\Omega),\ p\in L_0^2(\Omega) \text{ and } d\in V_s \text{ such that}\\
&\int_\Omega \sigma_f(u,p)\colon\nabla v_f = \int_{\Phi_{t_k}(\Omega_f)} f_f^k\cdot v_{f|\Phi_{t_k}(\Omega_f)} + \langle g, v_{f|\Phi_{t_k}(\Omega_s)}\rangle_{(V_s^k)',V_s^k}, && \forall v_f\in H_0^1(\Omega),\\
&\int_\Omega q\,\mathrm{div}(u) = 0, && \forall q\in L_0^2(\Omega),\\
&\int_{\Omega_s}\sigma_s(d)\colon\nabla v_s = \int_{\Omega_s} f_s^k\cdot v_s - \langle g, v_s\circ\Phi_{t_k}^{-1}\rangle_{(V_s^k)',V_s^k} + \int_{\Omega_s}\Pi_f(u\circ\Phi_{t_k},p\circ\Phi_{t_k})\colon\nabla v_s, && \forall v_s\in V_s,
\end{aligned}\right. \tag{37}$$

satisfies the equality

$$u\circ\Phi_{t_k} = \frac{1}{\delta t}(d-d_s^{k-1}), \text{ on } \Gamma. \tag{38}$$

For every $f_f^k\in(L^2(\Phi_{t_k}(\Omega_f)))^n$ and every $g\in(V_s^k)'$, there exists a unique solution to the Stokes problem appearing in problem (37), denoted by (u^g,p^g) (see [BF12]). On the other hand, for all $f_s^k\in(L^2(\Omega_s))^n$, the weak problem of linear elasticity that appears in (37) also admits a unique solution in V_s, denoted by d^g (see [Cia88]). In the following theorem, we state the existence of a control g such that the solution of (37) satisfies the equality (38) and explain how to recover the solution of the initial fluid-structure problem (33).

Theorem 4.4. *Let Ω be a domain that satisfies Assumption (H_3). Consider f_f^k in $(L_2(\Phi_{t_k}(\Omega_f)))^n$ and f_s^k in $(L_2(\Omega_s))^n$. Then, there exists a function g in $(V_s^k)'$ such that the solution (u^g,p^g,d^g) of problem (37) satisfies (38). Moreover, we can recover the solution of the fluid-structure problem (33):*

$$\begin{aligned}
(u_f^k,p_f^k) &= (u^g_{|\Phi_{t_k}(\Omega_f)}, p^g_{|\Phi_{t_k}(\Omega_f)}),\\
d_s^k &= d^g.
\end{aligned}$$

The proof of Theorem 4.4 relies on the construction of extensions for the functions u_f^k and p_f^k to the whole domain Ω, which is what we have done for the Stokes transmission problem. Hence, the proof of Theorem 4.4 directly follows from the one of Theorem 4.1. This is detailed in Appendix B.

As before, the problem of finding a control such that the solution of (36) verifies (38) can be formulated as an optimization problem on the following cost function, defined for any $k\geq 0$,

$$\begin{aligned}
J_k\ :\ V'_{s,k} &\to \mathbb{R}^+\\
g &\mapsto \frac12\int_\Gamma |u^g\circ\Phi_{t_k} - \frac{1}{\delta t}(d^g-d_s^{k-1})|^2,
\end{aligned} \tag{39}$$

where u^g and d^g are the velocity of the fluid and the displacement of the structure that solve problem (37) at time t_k. The function d_s^{k-1} is the displacement of the structure that solves problem (32) at time t_{k-1}. Yet, the minimization of this cost function is equivalent to the minimization of the real-valued function,

$$\begin{aligned}
(H_0^1(\Omega))^n\times V_s &\to \mathbb{R}^+\\
(u,d) &\mapsto \frac12\int_\Gamma |u\circ\Phi_{t_k} - \frac{1}{\delta t}(d-d_s^{k-1})|^2,
\end{aligned}$$

under the constraint that u and d are the velocity of the fluid and the displacement of the structure that solve problem (37). Then, to this constrained optimization problem, we can associate the following Lagrangian function, defined from

$$(V_s^k)' \times \left((H_0^1(\Omega))^n \times L_0^2(\Omega) \times V_s\right) \times \left((H_0^1(\Omega))^n \times L_0^2(\Omega) \times V_s\right),$$

to $\mathbb{R}$ by,

$$\begin{aligned}\mathcal{L}_k(g,(u,p,d),(\lambda_f,\pi,\nu_s)) = \frac{1}{2}\int_\Gamma |u\circ\Phi_{t_k} - \frac{1}{\delta t}(d-d_s^{k-1})|^2 + \int_\Omega \sigma_f(u,p)\colon \nabla\lambda_f - \int_\Omega \pi \mathrm{div}(u) + \int_{\Omega_s} \sigma_s(d)\colon \nabla\nu_s \\ - \int_{\Omega_s} \Pi_f(u\circ\Phi_{t_k}, p\circ\Phi_{t_k})\colon \nabla\nu_s - \int_{\Phi_{t_k}(\Omega_f)} f_f^k\cdot\lambda_f - \langle g,\lambda_f\rangle_{(V_s^k)',V_s^k} - \int_{\Omega_s} f_s^k\cdot\nu_s + \langle g, \nu_s\circ\Phi_{t_k}^{-1}\rangle_{(V_s^k)',V_s^k}.\end{aligned} \tag{40}$$

Again, this Lagrangian function can be used to compute the gradient of J_k and show that the minimization of J_k is equivalent to the problem of finding a suitable control g such that the solution of (37) satisfies (38). For that purpose, we introduce the adjoint problem of (37),

$$\left\{\begin{array}{lcll} \multicolumn{4}{l}{\text{find } \nu_s\in V_s,\ \lambda_f\in (H_0^1(\Omega))^n \text{ and } \pi\in L_0^2(\Omega) \text{ such that,}} \\ \displaystyle\int_{\Omega_s}\sigma_s(\nu_s)\colon\nabla v_s &=& \displaystyle\frac{1}{\delta t}\int_\Gamma (u^g\circ\Phi_{t_k} - \frac{1}{\delta t}(d^g - d_s^{k-1})\cdot v_s, & \forall v_s\in V_s, \\ \displaystyle\int_\Omega \sigma_f(\lambda_f,\pi):\nabla v_f &=& \displaystyle 2\mu_f\int_{\Phi_{t_k}(\Omega_s)} D(\nu_s\circ\Phi_{t_k}^{-1}) : D(v_f) & \\ && \displaystyle -\int_\Gamma (u^g\circ\Phi_{t_k} - \frac{1}{\delta t}(d^g-d_s^{k-1}))\cdot v_f\circ\Phi_{t_k}, & \forall v_f\in (H_0^1(\Omega))^n, \\ \displaystyle\int_\Omega q\mathrm{div}(\lambda_f) &=& \displaystyle\int_{\Phi_{t_k}(\Omega_s)} q\mathrm{div}(\nu_s\circ\Phi_{t_k}^{-1}), & \forall q\in L_0^2(\Omega). \end{array}\right. \tag{41}$$

Problem (41) admits a unique solution that we denote (ν^g,λ^g,π^g). Then, the existence and the characterization of the gradient of J_k is given in the following theorem,

Theorem 4.5. *The mapping $g\in (V_s^k)'\mapsto J_k(g)\in\mathbb{R}^+$ is differentiable and its gradient $\nabla J_k(g)$ in $(V_s^k)''$ is given by,*

$$\langle \nabla J_k(g),\delta g\rangle_{(V_s^k)'',(V_s^k)'} = \left\langle \delta g, \nu^g\circ\Phi_{t_k}^{-1} - \lambda^g_{|\Phi_{t_k}(\Omega_s)}\right\rangle_{(V_s^k)',(V_s^k)}, \quad \forall \delta g\in V_s', \tag{42}$$

where λ^g and ν^g satisfy the adjoint problem (41)*.*

Moreover, the equivalence between the smooth extension problem and its formulation as a minimization problem can also be stated in the case of a fluid-structure interaction problem.

Theorem 4.6. *A control g in $(V_s^k)'$ is a minimizer of J_k if and only if the solution of* (37) *satisfies* (38)*.*

Proofs of Theorem 4.5 and Theorem 4.6 can be easily adapted from the ones already done for the Stokes transmission problem. However, the change of domains between the structure in reference configuration and the fluid in current configuration can be confusing. For that matter, all proofs are detailed in Appendix B.

The formulation of the fluid-structure interaction problem as a control problem and as a minimization problem enables us to numerically solve these equations with a fictitious domain approach. Here, we apply this method to the numerical simulation of the bending of an elastic beam in a viscous fluid subjected to shear boundary condition. The initial geometry of the problem is represented in Figure 9: Ω_f is the rectangle $[0,2]\times[0,1]$ in $\mathbb{R}^2$ and Ω_s is a rectangular beam of length L_c and of radius r_c. This beam is anchored at the bottom of Ω_f, at positions $(x_c-r_c,0)$ and $(x_c+r_c,0)$. We consider periodic boundary conditions on the left and right boundaries of the fluid domain and imposed a shear condition on the top, given by

$$u_{bc}(x) \quad = \quad (3,0), \quad \forall x\in[0,1].$$

On the bottom boundaries of the fluid and solid domains, homogeneous Dirichlet boundary conditions are considered. No external force is considered and constant values are chosen for the fluid viscosity, $\mu_f = 1$, the Young's modulus of the solid material, $E_s = 10^5$, and its Poisson's ratio, $\nu_s = 0.49$. The Lam coefficients μ_s and λ_s are then given by

$$\mu_s = \frac{E_s}{2(1+\nu_s)}, \quad \lambda_s = \frac{\nu_s E_s}{((1+\nu_s)(1-2\nu_s))}.$$

For the parameters of the beam, we choose $x_c = 1$, $r_c = 0.05$ and $L_c = 0.65$. The final time of the simulation is set to $T = 0.5$.

The solution given by the smooth extension method is obtained following Algorithm 1 and using the L-BFGS algorithm (see [Noc80]). The resolution of the two different Stokes problems appearing in the minimization process is done using Mini elements, while the resolution of the two elasticity problems is done using $P1$ elements. Fluid problems are solved on a fixed mesh representing the whole domain Ω and elasticity problems are solve on a fixed mesh representing Ω_s, i.e. in the solid reference configuration. Moreover, the fluid mesh does not conform with the solid boundary. We compare this

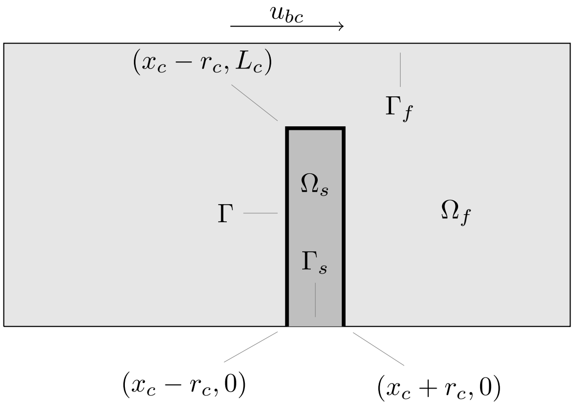
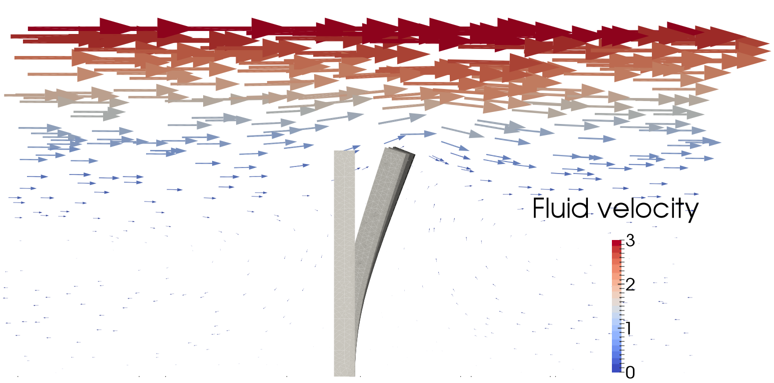
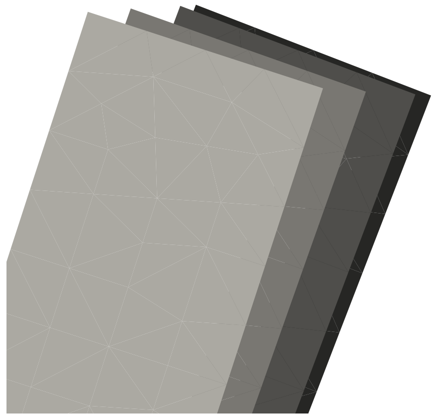


Figure 9: Two dimensional representation of the initial geometry for the fluid-structure problem.



(a)



(b)

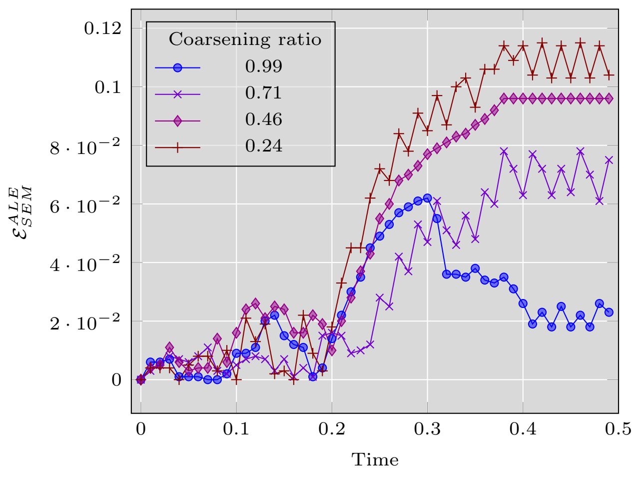
Figure 10: Comparison of the stationary states obtained through the smooth extension method and the ALE method (a). The darkest grey mesh represents the equilibrium position of the beam obtained with the ALE method. The lightest grey mesh represents the initial position of the beam. The other three meshes represent the equilibrium positions of the beam obtained with the smooth extension method for different coarsening ratios (0.99, 0.71, 0.46): the lighter is the colour the coarser is the fluid mesh. Zoom on the tips of the beams (b).

solution to the one obtained with conformal meshes and using a Lagrangian multiplier to ensure the continuity of the fluid and solid velocities through the interface Γ. With this method, an Arbitrary Lagrangian-Eulerian (ALE) methodology (see [BKFG19]) is used to ensure the mesh conformity when the beam bends and prevent cells from overlapping within the fluid mesh. However, if the quality of the fluid mesh is too poor, one needs to remesh it (this is done using the Mmg remeshing software [DDF14]). In the following, this method will be referred as the ALE method.

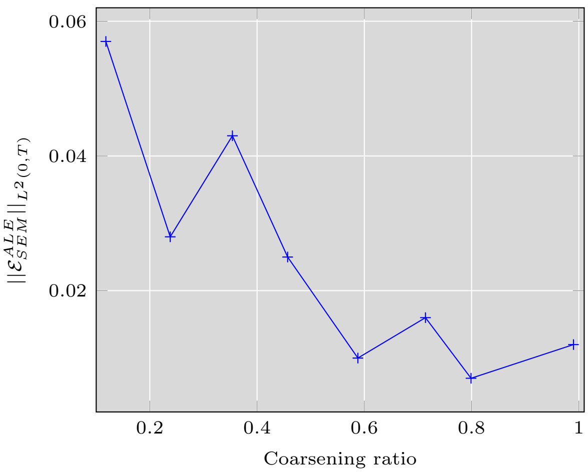
In this test case, the fluid-structure system attains a stationary state, where the beam is at equilibrium in a deformed configuration, which is well catched by both the ALE and the smooth extension methods (see Figure 10). To study the robustness of the smooth extension method, we coarsen the fluid mesh and observe the consequences on the dynamic of the system. To do so we define the *coarsening ratio of the fluid mesh* as the ratio of the number of nodes in the reference fluid mesh used in the ALE method divided by the number of nodes of the fluid mesh used in the smooth extension method. For example, a coarsening ratio of 1 means that the two fluid meshes have the same number of nodes, whereas a ratio of 0.5 means that the fluid mesh used in the smooth extension method has half as many nodes than the reference fluid mesh used for the ALE method. Then, in Figure 10, we observe that the coarser is the fluid mesh, the farther is the stationary state from the equilibrium state obtained with the ALE method. In order to quantify this error, we use the $L^1(\Omega)$ distance between the position of the beam obtained with the ALE method and the one obtained with the smooth extension method at time t, and defined a relative error

$$\mathcal{E}_{SEM}^{ALE}(t) = \frac{\displaystyle\int_\Omega |\chi_{\Omega_s^{SEM}(t)} - \chi_{\Omega_s^{ALE}(t)}|}{\displaystyle\int_\Omega |\chi_{\Omega_s^{SEM}(t)}| + \int_\Omega |\chi_{\Omega_s^{ALE}(t)}|},$$

where$\chi_{\Omega_s^{SEM}(t)}$ is a $L^2(\Omega)$ function which is the characteristic function of the solid current domain $\Omega_s^{SEM}(t)$. Simi-



(a)



(b)

Figure 11: Distance $\mathcal{E}_{SEM}^{ALE}$ between the two solid configuration Ω_s^{SEM} and Ω_s^{ALE} in function of the time (a). Global norm in time $\|\mathcal{E}_{SEM}^{ALE}\|$ in function of the coarsening ratio of the fluid mesh in the smooth extension method (b).

larly, $\chi_{\Omega_s^{ALE}(t)}$ is a $L^2(\Omega)$ function which is the characteristic function of the solid current domain $\Omega_s^{ALE}(t)$. Consequently, this distance is 0 if the two domains are the same and 1 if they do not overlap. Moreover, for a final time $T > 0$, we also consider the global norm in time of $e\mathcal{E}_{SEM}^{ALE}$ defined by

$$\|\mathcal{E}_{SEM}^{ALE}\| = \frac{1}{T}\int_0^T |\mathcal{E}_{SEM}^{ALE}(t)|dt,$$

which is 0 if the two domains $\Omega_s^{ALE}(t)$ and $\Omega_s^{SEM}(t)$ are identical for all t in $[0, T]$ and 1 if they never overlap. Then, we compute the distance $\mathcal{E}_{SEM}^{ALE}$ in function of the time for different coarsening ratios to study their influence on the dynamic of the system when using the smooth extension method. This is represented in Figure 11a, where we observe that, for all coarsening ratios, the distance $\mathcal{E}_{SEM}^{ALE}$ increases in time to reach a constant value when the stationary state is attained. This result corroborates and quantifies what we observed on Figure 10, i.e. that the error on the equilibrium position of the beam seems to increase when the coarsening ratio decreases, but stays relatively low (approximatively 10% with a very coarsening mesh for the fluid domain). To go further, we plot in Figure 11b the global norm in time of $\mathcal{E}_{SEM}^{ALE}$ in function of the coarsening ratio. In addition to the already mentioned fact that the error tends to increase when the coarsening ratio decreases, we remark that the error in time is just above 0.1 for a coarsening ratio of 0.12, which implies that the coarsening of the fluid mesh in the smooth extension method does not drastically change the time dynamic of the bending of the beam in this test case.

All these results suggest that the smooth extension method is well suited for time dependent problems involving a moving structure in a viscous fluid, where the fluid mesh is fixed, possibly Cartesian and coarser that the structure mesh.

5 Conclusion and discussion

In this article we have presented a numerical strategy for the resolution of transmission problems with non-conformal meshes and which preserves optimal rates of convergence in space. It is based on a control formulation of the transmission problem, namely the smooth extension formulation, whose numerical resolution can be done by minimizing a particular objective function. This method allows the use of standard finite element functional spaces along with fixed structured or unstructured meshes and pre-existing finite element solvers and optimization algorithms.

This smooth extension method has been derived in the particular case of the transmission Laplace problem with only two subdomains and considering Dirichlet boundary conditions. Other boundary conditions could also be considered with no additional difficulty, provided that the initial transmission problem is well-posed. The same methodology should also work for transmission problems with more than two subdomains.

In addition, we have shown that the smooth extension method can be applied to a wide variety of transmission problems, even the ones with totally different operators, such as the fluid-structure interaction problem studied in Subsection 4.2. For each type of transmission problems considered in the present article, the smooth extension method has been compared to a standard numerical method and has shown to give good approximations of the solutions.

A Proofs of theorems related to the Stokes transmission problem

This appendix is dedicated to the proofs of all results stated in Subsection 4.1. In particular, we are interested in showing the existence of the control g, in giving an explicit formula for the gradient of $\tilde{J}$ and, finally, in proving the equivalence between the minimization of $\tilde{J}$ and the resolution of the Stokes transmission problem.

We follow the order of the previous enumeration and start with the proof of Theorem 4.1.

Proof of Theorem 4.1. We can construct two extension operators E_u and E_p that extend $\overline{w}_1$ into the whole space $(H_0^1(\Omega))^n$ (and such that $E_u\overline{w}_1$ is divergence-free) and $\overline{p}_1$ into the whole space $L_0^2(\Omega)$. Indeed, consider the operators defined by

$$E_u\overline{w}_1 = \left\{ \begin{array}{ccc} \overline{w}_1 & \text{in} & \Omega_1 \\ \overline{w}_2 & \text{in} & \Omega_2 \end{array}\right. , \quad E_p\overline{p}_1 = \left\{ \begin{array}{ccc} \overline{p}_1 & \text{in} & \Omega_1 \\ \overline{p}_2 & \text{in} & \Omega_2 \end{array}\right. .$$

Since $(\overline{w}_1, \overline{w}_2)$ belongs to $\mathcal{W}$, the function $E_u\overline{w}_1$ is an extension of $\overline{w}_1$ which belongs to $(H_0^1(\Omega))^n$. Moreover, it satisfies $\mathrm{div}(E_u\overline{w}_1) = 0$ because both $\overline{w}_1$ and $\overline{w}_2$ are divergence-free. Similarly, since $(\overline{p}_1, \overline{p}_2)$ belongs to Q, the function $E_p\overline{p}_1$ is an extension of $\overline{p}_1$ which belongs to $L_0^2(\Omega)$. Furthermore, $\mu_1\nabla(E_u\overline{w}_1) - (E_p\overline{p}_1)I$ belongs to $(H_{\mathrm{div}}(\Omega))^n$.

Then, we construct a suitable control g in W_2' such that

$$\langle g, v\rangle_{W_2', W_2} = \mu_1 \int_{\Omega_2} \nabla(E_u\overline{w}_1)_{|\Omega_2} : \nabla v - \int_{\Omega_2} \overline{p}_{1|\Omega_2} \mathrm{div}(v) + \langle \gamma_{n_1}(\sigma_E), v\rangle_{\Upsilon', \Upsilon}, \quad \forall v \in W_2, \tag{43}$$

where we define $\sigma_E = \mu_1\nabla(E_u\overline{w}_1) - (E_p\overline{p}_1)I$. Using the Stokes formula (21) and the definition (43), it follows that the extensions $E_u\overline{w}_1$ and $E_p\overline{p}_1$ satisfy

$$\begin{aligned} \mu_1 \int_\Omega \nabla(E_u\overline{w}_1) \colon \nabla v_1 - \int_\Omega (E_p\overline{p}_1)\mathrm{div}(v_1) &= \int_{\Omega_1} f_1 \cdot v_{1|\Omega_1} + \left\langle g, v_{1|\Omega_2}\right\rangle_{W_2', W_2}, \quad \forall v_1 \in H_0^1(\Omega) \\ \int_\Omega q_1 \mathrm{div}(E_u\overline{w}_1) &= 0, \quad \forall q_1 \in L_0^2(\Omega). \end{aligned}$$

Similarly, using (21), (43) and the weak transmission condition (22), $(\overline{w}_2, \overline{p}_2)$ satisfies

$$\begin{aligned} &\mu_2 \int_{\Omega_2} \nabla\overline{w}_2 \colon \nabla v_2 - \int_{\Omega_2} \overline{p}_2 \mathrm{div}(v_2) \\ &= \int_{\Omega_2} f_2 \cdot v_2 + \mu_1 \int_{\Omega_2} \nabla(E_u u_1)_{|\Omega_2} \colon \nabla v_2 - \langle g, v_2\rangle_{W_2', W_2} - \int_{\Omega_2} (E_p\overline{p}_{1|\Omega_2})\mathrm{div}(v_2), \forall v_2 \in W_2, \end{aligned}$$

Finally, we conclude that $((E_u\overline{w}_1, E_p\overline{p}_1), (\overline{w}_2, \overline{p}_2))$ is the solution of problem (24). Thus,

$$((w_1^g, p_1^g), (w_2^g, p_2^g)) = ((E_u\overline{w}_1, E_p\overline{p}_1), (\overline{w}_2, \overline{p}_2))$$

and, by construction, the equality (25) is satisfied. This proves the first part of the theorem.

Now, suppose that g is a control such that the equality (25) is satisfied. In particular, the equality $w^g_{1|\Gamma} = w^g_{2|\Gamma}$ implies that $(w^g_{1|\Omega_1}, w_2^g)$ belongs to the space $\mathcal{W}$. Let us define the following Hilbert space:

$$\tilde{\mathcal{W}} = \left\{(v1, v2) \in (H_0^1(\Omega))^n \times W_2; v_{1|\Omega_1} = v_2\right\}.$$

As the unique solution of problem (24), the couples (w_1^g, p_1^g) and (w_2^g, p_2^g) satisfy, in particular, for all (v_1, v_2) in $\tilde{\mathcal{W}}$ and for all (q_1, q_2) in $L_0^2(\Omega) \times L^2(\Omega_2)$, the equations

$$\begin{aligned} \mu_1 \int_\Omega \nabla w_1^g : \nabla v_1 - \int_\Omega p_1^g \mathrm{div}(v_1) &= \int f_1 \cdot v_{1|\Omega_1} + \left\langle g, v_{1|\Omega_2}\right\rangle_{W_2', W_2}, \\ \int_\Omega q \mathrm{div}(w_1^g) &= 0, \\ \mu_2 \int_{\Omega_2} \nabla w_2^g : \nabla v_2 - \int_{\Omega_2} p_2^g \mathrm{div}(v_2) &= \int f_2 \cdot v_2 - \langle g, v_2\rangle_{W_2', W_2} + \mu_1 \int_{\Omega_2} \nabla w_1^g : \nabla v_2 - \int_{\Omega_2} p_1^g \mathrm{div}(v_2), \\ \int_{\Omega_2} q_{|\Omega_2} \mathrm{div}(w_2^g) &= 0. \end{aligned} \tag{44}$$

Then, summing equations in (44), it follows that $((w^g_{1|\Omega_1}, w_2^g), (p^g_{1|\Omega_1}, p_2^g))$ satisfies

$$\begin{aligned} \mu_1 \int_{\Omega_1} \nabla w^g_{1|\Omega_1} : \nabla v_{1|\Omega_1} - \int_{\Omega_1} p^g_{1|\Omega_1} \mathrm{div}(v_{1|\Omega_1}) + \mu_2 \int_{\Omega_2} \nabla w_2^g : \nabla v_2 - \int_{\Omega_2} p_2^g \mathrm{div}(v_2) \\ = \int_{\Omega_1} f_1 \cdot v_1 + \int_{\Omega_2} f_2 \cdot v_2, \quad \forall (v_1, v_2) \in \tilde{\mathcal{W}}, \\ \int_{\Omega_1} q_{1|\Omega_1} \mathrm{div}(w^g_{1|\Omega_1}) + \int_{\Omega_2} q_2 \mathrm{div}(w_2^g) = 0, \quad \forall (q_1, q_2) \in L_0^2(\Omega) \times L^2(\Omega_2). \end{aligned}$$

Because the test function v_1 belongs to $H_0^1(\Omega)$ and because $v_2 = v_{1|\Omega_1}$, we can redefine the pressures p_1^g and p_2^g up to a constant, by

$$\tilde{p_1^g} = p_1^g - \frac{1}{|\Omega|}\left(\int_{\Omega_2} p_2^g + \int_{\Omega_1} p^g_{1|\Omega_1}\right), \quad \tilde{p_2^g} = p_2^g - \frac{1}{|\Omega|}\left(\int_{\Omega_2} p_2^g + \int_{\Omega_1} p^g_{1|\Omega_1}\right),$$

such that $(\tilde{p}^g_{1|\Omega_1}, \tilde{p}^g_2)$ belongs to the space Q. Moreover, for all (v_1, v_2) in $\mathcal{W}$ we can extend v_1 in the whole space $H^1_0(\Omega)$ using v_2, as we did it for $\overline{w}_1$ and $\overline{w}_2$. We still denote $E_u v_1$ this extension. Similarly, for all (q_1, q_2) in Q, we can extend q_1 in the whole space $L^2_0(\Omega)$ and we still denote $E_p q_1$ this extension. Then, the couple $(E_u v_1, v_2)$ belongs to $\tilde{\mathcal{W}}$, the couple $(E_p q_1, q_2)$ belongs to $L^2_0(\Omega) \times L^2(\Omega_2)$ and, finally, $((w^g_{1|\Omega_1}, w^g_2), (\tilde{p}^g_{1|\Omega_1}, \tilde{p}^g_2))$, satisfies the equations

$$\begin{aligned}
\mu_1 \int_{\Omega_1} \nabla w^g_{1|\Omega_1} : \nabla v_1 - \int_{\Omega_1} \tilde{p}^g_{1|\Omega_1} \mathrm{div}(v_1) + \mu_2 \int_{\Omega_2} \nabla w^g_2 : \nabla v_2 - \int_{\Omega_2} \tilde{p}^g_2 \mathrm{div}(v_2) & \\
= \int_{\Omega_1} f_1 \cdot v_1 + \int_{\Omega_2} f_2 \cdot v_2, \quad & \forall (v_1, v_2) \in \mathcal{W}, \\
\int_{\Omega_1} q_{1|\Omega_1} \mathrm{div}(w^g_{1|\Omega_1}) + \int_{\Omega_2} q_2 \mathrm{div}(w^g_2) = 0, \quad & \forall (q_1, q_2) \in Q.
\end{aligned}$$

Thus, $((w^g_{1|\Omega_1}, w^g_2), (\tilde{p}^g_{1|\Omega_1}, \tilde{p}^g_2))$ is the unique solution of the initial Stokes transmission problem (20), which proves that we can recover the solution of the Stokes transmission problem from the solution of the smooth extension problem. □

We continue with the proof of Theorem 4.2, i.e. we show the existence of the gradient of $\tilde{J}$ and give an explicit formula to compute it.

Proof of Theorem 4.2. We start by computing the derivatives of the Lagrangian function $\tilde{\mathcal{L}}$ with respect to u_1, p_1, u_2 and p_2, to justify the adjoint equations written in (28). These computations are similar to the ones made in the proof of Lemma 2.1 and we have

$$\begin{aligned}
\left\langle \frac{\partial \tilde{\mathcal{L}}}{\partial u_1}, \delta u_1 \right\rangle_{(H^{-1}(\Omega))^n, (H^1_0(\Omega))^n} &= \int_\Gamma (u_1 - u_2) \cdot \delta u_1 + \mu_1 \int_\Omega \nabla \lambda_1 : \nabla \delta u_1 - \int_\Omega \pi_1 \mathrm{div}(\delta u_1) \\
&\quad - \mu_1 \int_{\Omega_2} \nabla \lambda_2 : \nabla \delta u_1, \qquad \forall \delta u_1 \in (H^1_0(\Omega))^n,
\end{aligned}$$

$$\left\langle \frac{\partial \tilde{\mathcal{L}}}{\partial u_2}, \delta u_2 \right\rangle_{W_2', W_2} = -\int_\Gamma (u_1 - u_2) \cdot \delta u_2 + \mu_2 \int_{\Omega_2} \nabla \lambda_2 : \nabla \delta u_2 - \int_{\Omega_2} \pi_2 \mathrm{div}(\delta u_2), \quad \forall \delta u_2 \in W_2,$$

$$\left\langle \frac{\partial \tilde{\mathcal{L}}}{\partial p_1}, \delta p_1 \right\rangle_{L^2(\Omega), L^2(\Omega)} = -\int_\Omega \delta p_1 \mathrm{div}(\lambda_1) + \int_{\Omega_2} \delta p_1 \mathrm{div}(\lambda_2), \quad \forall \delta p_1 \in L^2_0(\Omega),$$

$$\left\langle \frac{\partial \tilde{\mathcal{L}}}{\partial p_2}, \delta p_2 \right\rangle_{L^2(\Omega_2), L^2(\Omega)} = -\int_{\Omega_2} \delta p_2 \mathrm{div}(\lambda_2), \quad \forall \delta p_2 \in L^2(\Omega_2).$$

Then, taking $(u_1, p_1) = (w^g_1, p^g_1)$ and $(u_2, p_2) = (w^g_2, p^g_2)$, we can deduce that the adjoint problem associated to the direct problem (24) is,

$$\left\{ \begin{array}{lll}
\text{find } (\lambda_1, \pi_1) \in (H^1_0(\Omega))^n \times L^2_0(\Omega) \text{ and } (\lambda_2, \pi_2) \in W_2 \times L^2(\Omega) \text{ such that} & & \\
\left\langle \frac{\partial \tilde{\mathcal{L}}}{\partial u_1}, v_1 \right\rangle_{(H^{-1}(\Omega))^n, (H^1_0(\Omega))^n} & = & 0, \quad \forall v_1 \in (H^1_0(\Omega))^n, \\
\left\langle \frac{\partial \tilde{\mathcal{L}}}{\partial p_1}, q_1 \right\rangle_{L^2(\Omega), L^2(\Omega)} & = & 0, \quad \forall q_1 \in L^2_0(\Omega), \\
\left\langle \frac{\partial \tilde{\mathcal{L}}}{\partial u_2}, v_2 \right\rangle_{W_2', W_2} & = & 0, \quad \forall v_2 \in W_2, \\
\left\langle \frac{\partial \tilde{\mathcal{L}}}{\partial p_2}, q_2 \right\rangle_{L^2(\Omega_2), L^2(\Omega)} & = & 0, \quad \forall q_2 \in L^2(\Omega_2).
\end{array} \right. \tag{45}$$

Thus, the adjoint problem for the Stokes transmission problem is indeed the weak problem written in (28). Problem (28) consists in two Stokes problems, whose well-posedness derives from well-known results about the Stokes equations (see [BF12]). We denote $((\omega^g_1, \pi^g_1), (\omega^g_2, \pi^g_2))$ its unique solution in $\big((H^1_0(\Omega))^n \times L^2_0(\Omega)\big) \times \big(W_2 \times L^2(\Omega_2)\big)$.

The differentiability of $\tilde{J}$ relies on the same arguments that the ones used in the proof of Theorem 2.3. Likewise, taking $(u_1, p_1) = (w^g_1, p^g_1)$ and $(u_2, p_2) = (w^g_2, p^g_2)$, the Lagrangian (27) reduces to

$$\tilde{\mathcal{L}}(g, ((w^g_1, p^g_1), (w^g_2, p^g_2)), ((\lambda_1, \pi_1), (\lambda_2, \pi_2))) = \tilde{J}(g), \quad \forall g \in W_2'.$$

Differentiating this previous inequality with respect to g using the chain rule and taking

$$(\lambda_1, \pi_1) = (\omega^g_1, \pi^g_1) \text{ and } (\lambda_2, \pi_2) = (\omega^g_2, \pi^g_2),$$

we find that, for all δg in W_2',

$$\left\langle \nabla \tilde{J}, \delta g \right\rangle_{W_2'', W_2} = \left\langle \frac{\partial \tilde{\mathcal{L}}}{\partial g}(g, ((w_1^g, p_1^g), (w_2^g, p_2^g)), ((\omega_1^g, \pi_1^g), (\omega_2^g, \pi_2^g))), \delta g \right\rangle_{W_2'', W_2'}.$$

Moreover, the differentiate of $\tilde{\mathcal{L}}$ with respect to g, $\frac{\partial \tilde{\mathcal{L}}}{\partial g} \in W_2''$, is defined such that, for all δg in W_2',

$$\left\langle \frac{\partial \tilde{\mathcal{L}}}{\partial g}(g, ((u_1, p_1), (u_2, p_2)), ((\lambda_1, \pi_1), (\lambda_2, \pi_2))), \delta g \right\rangle_{W_2'', W_2'} = \langle \delta g, \lambda_2 - \lambda_{1|\Omega_2} \rangle_{W_2', W_2}$$

Finally, the gradient of $\tilde{J}$ is given by,

$$\left\langle \nabla \tilde{J}, \delta g \right\rangle_{W_2'', W_2} = \langle \delta g, \lambda_2 - \lambda_{1|\Omega_2} \rangle_{W_2', W_2}, \quad \forall \delta g \in W_2'.$$

□

Now, let us show the equivalence between the minimization of $\tilde{J}$ and the research of a suitable control such that the solution of (24) satisfies the conditions (25).

Proof of Theorem 4.3. The reciprocal statement is straightforward. If, for a given g, the solution of (24) satisfies the condition (25) then, $\tilde{J}(g) = 0$.

Now, let g be a minimizer of $\tilde{J}$. It follows that one has

$$\left\langle \delta g, \omega_2^g - \omega_{1|\Omega_2}^g \right\rangle_{W_2', W_2} = 0, \quad \forall \delta g \in W_2',$$

which means that $\omega_{1|\Omega_2}^g = \omega_2^g$ and the couple $(\omega_{1|\Omega_1}^g, \omega_2^g)$ belongs to $\mathcal{W}$. Yet, the functions ω_1^g, ω_2^g, π_1^g and π_2^g satisfy the equations in problem (28). In particular, for all (v_1, v_2) in $\tilde{\mathcal{W}}$ and for all (q_1, q_2) in $L_0^2(\Omega) \times L^2(\Omega_2)$, where

$$\tilde{\mathcal{W}} = \{(v_1, v_2) \in (H_0^1(\Omega))^n \times W_2; v_{1|\Omega_2} = v_2\},$$

these equations write,

$$\begin{aligned} \mu_2 \int_{\Omega_2} \nabla \omega_2^g : \nabla v_2 - \int_{\Omega_2} \pi_2^g \mathrm{div}(v_2) &= \int_\Gamma (w_1^g - w_2^g) \cdot v_1, \\ \int_{\Omega_2} q_2 \mathrm{div}(\omega_2^g) &= 0, \\ \mu_1 \int_\Omega \nabla \omega_1^g : \nabla v_1 - \int_\Omega \pi_1^g \mathrm{div}(v_1) &= -\int_\Gamma (w_1^g - w_2^g) \cdot v_1 + \mu_1 \int_{\Omega_2} \nabla \omega_2^g : \nabla v_{1|\Omega_2}, \\ \int_\Omega q_1 \mathrm{div}(\omega_1^g) &= \int_{\Omega_2} q_{1|\Omega_2} \mathrm{div}(\omega_2^g). \end{aligned} \tag{46}$$

Summing the first equation in (46) with the third one and summing the second equation with the fourth one and using the fact that $\omega_{1|\Omega_2}^g = \omega_2^g$, we find that

$$\mu_1 \int_{\Omega_1} \nabla \omega_{1|\Omega_1}^g : \nabla v_{1|\Omega_1} - \int_{\Omega_1} \pi_{1|\Omega_1}^g \mathrm{div}(v_{1|\Omega_1}) + \mu_2 \int_{\Omega_2} \nabla \omega_2^g : \nabla v_2 - \int_{\Omega_2} (\pi_2^g + \pi_{1|\Omega_2}^g) \mathrm{div}(v_2) = 0, \quad \forall (v_1, v_2) \in \tilde{\mathcal{W}},$$
$$\int_{\Omega_1} q_{1|\Omega_1} \mathrm{div}(\omega_{1|\Omega_1}^g) + \int_{\Omega_2} q_2 \mathrm{div}(\omega_2^g) = 0, \quad \forall (q_1, q_2) \in L_0^2(\Omega) \times L^2(\Omega_2).$$

Moreover, because the test function v_1 belongs to $(H_0^1(\Omega))^n$, we can redefine the pressure π_1^g up to a constant by

$$\tilde{\pi}_1^g = \pi_1^g - \frac{1}{|\Omega|} \int_{\Omega_2} \pi_2^g$$

such that $(\tilde{\pi}_{1|\Omega_1}^g, \pi_2^g + \tilde{\pi}_{1|\Omega_2}^g)$ belongs to the space Q.

Moreover, let (v_1, v_2) be in $\mathcal{W}$. We can construct an extension of v_1 in the whole space $(H_0^1(\Omega))^n$ using v_2, as in the proof of Theorem 4.1 and we still denote $E_u v_1$ this extension. Likewise, for all (q_1, q_2) in Q, we can extend q_1 in the whole space $L_0^2(\Omega)$ and we still denote $E_p q_1$ this extension. Then, the couple $(E_u v_1, v_2)$ belongs to $\tilde{\mathcal{W}}$, the couple $(E_p q_1, q_2)$ belongs to $L_0^2(\Omega) \times L^2(\Omega_2)$ and it follows that $((\omega_{1|\Omega_1}^g, \omega_2^g), (\tilde{\pi}_{1|\Omega_1}^g, \pi_2^g + \tilde{\pi}_{1|\Omega_2}^g))$ satisfies the equations

$$\begin{aligned} \mu_1 \int_{\Omega_1} \nabla \omega_{1|\Omega_1}^g : \nabla v_1 - \int_{\Omega_1} \tilde{\pi}_{1|\Omega_1}^g \mathrm{div}(v_1) + \mu_2 \int_{\Omega_2} \nabla \omega_2^g : \nabla v_2 - \int_{\Omega_2} (\pi_2^g + \tilde{\pi}_{1|\Omega_2}^g) \mathrm{div}(v_2) &= 0, \quad \forall (v_1, v_2) \in \mathcal{W}, \\ \int_{\Omega_1} q_1 \mathrm{div}(\omega_{1|\Omega_1}^g) + \int_{\Omega_2} q_2 \mathrm{div}(\omega_2^g) &= 0, \quad \forall (q_1, q_2) \in Q. \end{aligned}$$

We conclude that $((\omega^g_{1|\Omega_1}, \omega^g_2), (\tilde{\pi}^g_{1|\Omega_1}, \pi^g_2 + \tilde{\pi}^g_{1|\Omega_2}))$ is solution of a Stokes problem similar to (20) but with no external force and, thus, is the zero of $\mathcal{W} \times Q$. Then, the first equation in (28) becomes

$$\int_\Gamma (w^g_1 - w^g_2) \cdot v_2 \quad = \quad 0, \quad \forall v_2 \in \tilde{W}_2,$$

where

$$\tilde{W}_2 = \{v \in W_2; \mathrm{div}(v) = 0\}.$$

Besides, $w^g_{1|\Gamma}$ and $w^g_{2|\Gamma}$ belongs to the space

$$W_\Gamma = \{v \in H^{1/2}_{00}(\Gamma); \int_\Gamma v = 0\},$$

because $\mathrm{div}(w^g_{1|\Omega_1}) = 0$ and $\mathrm{div}(w^g_2) = 0$, and for all v_2 in W_Γ we can construct an extension of v_2 in the whole space $\tilde{W}_2$, according to Bogovskii's result in [Bog79]. In particular, taking $v_2 = w^g_{1|\Omega_1} - w^g_2$ in W_Γ, it follows that

$$\int_\Gamma |w^g_1 - w^g_2|^2 \quad = \quad 0,$$

which means that $w^g_{1|\Gamma} = w^g_{2|\Gamma}$ and the equality (25) is satisfied. $\square$

B Proofs of theorems related to the fluid-structure interaction problem

This appendix is dedicated to the proofs of all results stated in Subsection 4.2. In particular, we are interested in proving the well-posedness of Problem (33), in showing the existence of the control g, in giving an explicit formula for the gradient of J_k and, finally, in proving the equivalence between the minimization of J_k and the resolution of the fluid-structure interaction problem.

We follow the order of the previous enumeration and start with the well-posedness of Problem (33). To study Problem (33), we do a change in variable on the displacement d^k_s, in order to work on a velocity-velocity formulation of the fluid-structure problem. We introduce the velocity of the structure at time t_k,

$$u^k_s = \frac{1}{\delta t}(d^k_s - d^{k-1}_s).$$

Because Problem (33) is linear, it is completely equivalent to the problem where d^k_s has been replaced with $\delta u^k_s + d^{k-1}_s$:

$$\left\{\begin{array}{l} \text{find } (u_f, u_s) \in W_u \text{ and } p \in L^2(\Phi_{t_k}(\Omega_f)) \text{ such that} \\ \quad a((u_f, u_s), (v_f, v_s)) - (B(v_f, v_s), p)_{L^2(\Phi_{t_k}(\Omega_f))} = L(v_f, v_s), \quad \forall (v_f, v_s) \in W_u, \\ \qquad\qquad\qquad (B(u_f, u_s), q)_{L^2(\Phi_{t_k}(\Omega_f))} = 0, \qquad\qquad\quad \forall q \in L^2(\Phi_{t_k}(\Omega_f)), \end{array}\right. \tag{47}$$

where $(\cdot, \cdot)_{L^2(\Phi_{t_k}(\Omega_f))}$ denotes the scalar product in $L^2(\Phi_{t_k}(\Omega_f))$ and a, L and B are defined by

$$\begin{array}{rcll} a((u_f, u_s), (v_f, v_s)) & = & 2\mu_f \displaystyle\int_{\Phi_{t_k}(\Omega_f)} D(u_f) : D(v_f) + 2\delta_t \mu_s \int_{\Omega_s} D(u_s) : D(v_s) & \\ & & +\delta_t \lambda_s \displaystyle\int_{\Omega_s} \mathrm{div}(u_s)\mathrm{div}(v_s), & \forall (u_f, u_s), (v_f, f_s) \in W_u, \end{array} \tag{48}$$

$$\begin{array}{rcll} L(v_f, v_s) & = & -2\mu_s \displaystyle\int_{\Omega_s} D(d^{k-1}_s) : D(v_s) - \lambda_s \int_{\Omega_s} \mathrm{div}(d^{k-1}_s)\mathrm{div}(v_s) + \int_{\Phi_{t_k}(\Omega_f)} f_f \cdot v_f + \int_{\Omega_s} f_s \cdot v_s, & \forall (v_f, v_s) \in W_u, \\ B(v_f, v_s) & = & \mathrm{div}(v_f), & \forall (v_f, v_s) \in W_u. \end{array}$$

This is a saddle-point problem, whose well-posedness can be proved using results from [Bre74]. Using the well-known Cauchy-Schwartz and Korn inequalities, we show that the bilinear form a is continuous and coercive and that the linear form L is continuous. The operator B is also linear and continuous. It remains to show that B is surjective form W_u to $L^2(\Phi_{t_k}(\Omega_f))$. Indeed, let q be in $L^2(\Phi_{t_k}(\Omega_f))$. We can easily extend q in the whole space $L^2_0(\Omega)$; it is sufficient to take

$$E_p q = \left\{\begin{array}{rll} q & \text{in} & \Phi_{t_k}(\Omega_f), \\ \dfrac{-1}{|\Phi_{t_k}(\Omega_s)|} \displaystyle\int_{\Phi_{t_k}(\Omega_f)} q & \text{in} & \Phi_{t_k}(\Omega_s). \end{array}\right. \tag{49}$$

Then, $E_p q$ belongs to $L^2_0(\Omega)$ and Bogovskii's result in [Bog79] implies that there exists $\tilde{u}$ in $H^1_0(\Omega)$ such that $\mathrm{div}(\tilde{u}) = E_p q$. Finally, we define $v_f = \tilde{u}_{|\Phi_{t_k}(\Omega_f)}$ and $v_s = \tilde{u}_{|\Phi_{t_k}(\Omega_s)}$ and it follows that the couple $(v_f, v_s \circ \Phi^{-1}_{t_k})$ belongs to W_u and satisfies $B(v_f, v_s \circ \Phi^{-1}_{t_k}) = \mathrm{div}(v_f) = q$. This proves the surjectivity of the operator B. Thus, problem (47), and consequently problem (33), are well-posed.

We now give a proof of Theorem 4.4.

Proof of Theorem 4.4. We can construct two extension operators, still denoted E_u and E_p, that extend u_f^k into the whole space $(H_0^1(\Omega))^n$ (and stays divergence-free) and p_f^k into the whole space $L_0^2(\Omega)$. Indeed, for the pressure, consider the operator defined in (49). Then, the extension of the pressure $E_p p_f^k$ belongs to $L_0^2(\Omega)$. Now, because the fluid is incompressible, we have that

$$0 = \int_{\Phi_{t_k}(\Omega_f)} \mathrm{div}(u_f^k) = \int_{\Phi_{t_k}(\Gamma)} u_f^k \cdot n_f^k.$$

Thus, using one more time Bogovskii's result, there exists $\tilde{u}_f$ in $H^1(\Phi_{t_k}(\Omega_s))$ such that

$$\mathrm{div}(\tilde{u}_f) = 0,\ \tilde{u}_{f|\Phi_{t_k}(\Gamma)} = u^k_{f|\Phi_{t_k}(\Gamma)} \text{ and } \tilde{u}_{f|\Phi_{t_k}(\Gamma_s)} = 0.$$

Then, the extension of the fluid velocity, $E_u u_f^k$, is defined such that

$$E_u u_f^k \quad = \quad \left\{ \begin{array}{lll} u_f^k & \text{in} & \Phi_{t_k}(\Omega_f) \\ \tilde{u}_f & \text{in} & \Phi_{t_k}(\Omega_s) \end{array} \right. .$$

The extension $E_u u_f^k$ belongs to $(H_0^1(\Omega))^n$, is divergence-free and satisfies the equality

$$E_u u_f^k \circ \Phi_{t_k} = \frac{1}{\delta t}(d_s^k - d_s^{k-1}) \text{ on } \Gamma.$$

Furthermore, $\sigma_f(E_u u_f^k, E_p p_f^k)$ belongs to $(L^2(\Omega))^{n\times n}$.

Now, we construct a suitable control g in $(V_s^k)'$ such that

$$\langle g, v\rangle_{(V_s^k)',V_s^k} \quad = \quad \int_{\Phi_{t_k}(\Omega_s)} \sigma_f(E_u u_f^k, E_p p_f^k) \colon \nabla v + \left\langle \gamma_{n_f^k}(\sigma_f(E_u u_f^k, E_p p_f^k)), v\right\rangle_{(\Upsilon^k)',\Upsilon^k}, \quad \forall v \in V_s^k, \tag{50}$$

where n_f^k is the unit exterior normal vector to $\partial\Phi_{t_k}(\Omega_f)$. Using the Stokes formula (34) and the definition (50), it follows that the extensions $E_u u_f^k$ and $E_p p_f^k$ satisfy

$$\begin{aligned} \int_\Omega \sigma_f(E_u u_f^k, E_p p_f^k) \colon \nabla v_f \quad &= \quad \int_{\Phi_{t_k}(\Omega_f)} f_f^k \cdot v_{f|\Phi_{t_k}(\Omega_f)} + \left\langle g, v_{f|\Phi_{t_k}(\Omega_s)}\right\rangle_{(V_s^k)',V_s^k}, \quad && \forall v_f \in (H_0^1(\Omega))^n, \\ \int_\Omega q\mathrm{div}(E_u u_f^k) \quad &= \quad 0, && \forall q \in L_0^2(\Omega). \end{aligned}$$

Similarly, using (34), (50) and the weak transmission condition (35), d_s^k satisfies

$$\int_{\Omega_s} \sigma_s(d_s^k) \colon \nabla v_s \quad = \quad \int_{\Omega_s} f_s^k \cdot v_s - \langle g, v_s \circ \Phi_{t_k}^{-1}\rangle_{(V_s^k)',V_s^k} + \int_{\Omega_s} \Pi_f(E_u u_f^k \circ \Phi_{t_k}, E_p p_f^k \circ \Phi_{t_k}) \colon \nabla v_s, \quad \forall v_s \in V_s.$$

Finally, we conclude that $(E_u u_f^k, E_p p_f^k, d_s^k)$ is the solution of problem (37). Thus,

$$(u^g, p^g, d^g) = (E_u u_f^k, E_p p_f^k, d_s^k)$$

and condition (38) is satisfied by construction. This proves the first part of the theorem.

Now, suppose that g is a control such that the equality (38) is satisfied. In particular, this implies that the couple $(u^g_{|\Phi_{t_k}(\Omega_f)}, d^g)$ belongs to the space W_d. Let us define the following Hilbert space:

$$\tilde{W}_u = \left\{(v_f, v_s) \in (H_0^1(\Omega))^n \times V_s; (v_f \circ \Phi_{t_k})_{|\Omega_s} = v_s\right\}.$$

As the unique solution of problem (37), the triplet (u^g, p^g, d^g) satisfies, in particular for all (v_f, v_s) in $\tilde{W}_u$ and for all q in $L_0^2(\Omega)$, the equations

$$\begin{aligned} \int_\Omega \sigma_f(u^g, p^g) : \nabla v_f \quad &= \quad \int_{\Phi_{t_k}(\Omega_f)} f_f \cdot v_f + \langle g, v_f\rangle_{(V_s^k)',V_s^k}, \\ \int_\Omega q\mathrm{div}(u^g) \quad &= \quad 0, \\ \int_{\Omega_s} \sigma_s(d^g) : \nabla v_s \quad &= \quad \int_{\Omega_s} f_s \cdot v_s - \langle g, v_f\rangle_{(V_s^k)',V_s^k} + \int_{\Phi_{t_k}(\Omega_s)} \sigma_f(u^g, p^g) : \nabla v_f. \end{aligned} \tag{51}$$

Then adding the first and the third equations in (51) and using the fact that u^g is divergence-free, it follows that the triplet $(u^g_{|\Phi_{t_k}(\Omega_f)}, p^g_{|\Phi_{t_k}(\Omega_f)}, d^g)$ satisfies

$$\begin{aligned} \int_{\Phi_{t_k}(\Omega_f)} \sigma_f(u^g, p^g) : \nabla v_f + \int_{\Omega_s} \sigma_s(d^g) : \nabla v_s \quad &= \quad \int_{\Phi_{t_k}(\Omega_f)} f_f \cdot v_f + \int_{\Omega_s} f_s \cdot v_s, \quad && \forall (v_f, v_s) \in \tilde{W}_u, \\ \int_{\Phi_{t_k}(\Omega_f)} q\mathrm{div}(u^g) \quad &= \quad 0, && \forall q \in L_0^2(\Omega). \end{aligned}$$

Moreover, for all (v_f, v_s) in W_u, we can extend v_f in the whole space $(H_0^1(\Omega))^n$ as we did it for u_f^k and we denote $E_u v_f$ this extension. Similarly, for all q in $L_0^2(\Phi_{t_k}(\Omega_f))$, we extend q in the whole space $L_0^2(\Omega)$ and we denote $E_p q$ this extension. Then, the couple $(E_u v_f, v_s)$ belongs to $\tilde{W}_u$, $E_p q$ belongs to $L_0^2(\Omega)$ and, finally, the triplet $(u^g_{|\Phi_{t_k}(\Omega_f)}, p^g_{|\Phi_{t_k}(\Omega_f)}, d^g)$ satisfies the equations

$$\begin{aligned}
\int_{\Phi_{t_k}(\Omega_f)} \sigma_f(u^g_{|\Phi_{t_k}(\Omega_f)}, p^g_{|\Phi_{t_k}(\Omega_f)}) : \nabla v_f + \int_{\Omega_s} \sigma_s(d^g) : \nabla v_s &= \int_{\Phi_{t_k}(\Omega_f)} f_f \cdot v_f + \int_{\Omega_s} f_s \cdot v_s, \quad \forall (v_f, v_s) \in W_u, \\
\int_{\Phi_{t_k}(\Omega_f)} q \mathrm{div}(u^g_{|\Phi_{t_k}(\Omega_f)}) &= 0, \qquad \forall q \in L^2(\Phi_{t_k}(\Omega_f)).
\end{aligned}$$

Thus, the triplet $(u^g_{|\Phi_{t_k}(\Omega_f)}, p^g_{|\Phi_{t_k}(\Omega_f)}, d^g)$ is the unique solution of the initial fluid-structure problem (33), which proves that we can recover the solution of the fluid-structure interaction problem from the solution of its smooth extension formulation. □

Since there exists at least one suitable control g for the smooth extension problem, we can hope to obtain it by a minimization process on the function J_k. This is actually possible according to Theorem 4.5, which states that J_k is differentiable with respect to g and gives a characterization of its gradient. Here, we prove this result.

Proof of Theorem 4.5. On the first hand, the differentiation of the Lagrangian function $\mathcal{L}_k$, defined in (40), with respect to u, p, d and g follows the same process that the one explained for the Laplace and Stokes transmission problems. Here, we only give the expressions of these different derivatives:

$$\begin{aligned}
\left\langle \frac{\mathcal{L}_k}{\partial u}, \delta u \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} &= \int_{\Phi_{t_k}(\Omega_f)} \sigma_f(\lambda_f, \pi) : \nabla \delta u - 2\mu_f \int_{\Phi_{t_k}(\Omega_s)} D(\nu_s \circ \Phi_{t_k}^{-1}) : D(\delta u) \\
&\quad + \int_\Gamma (u \circ \Phi_{t_k} - \frac{1}{\delta t}(d - d_d^{k-1})) \cdot (\delta u \circ \Phi_{t_k}), \qquad \forall \delta u \in (H_0^1(\Omega))^n, \\
\left\langle \frac{\mathcal{L}_k}{\partial p}, \delta p \right\rangle_{L^2(\Omega), L^2(\Omega)} &= -\int_\Omega \delta p \mathrm{div}(\lambda_f) + \int_{\Omega_s} \delta p \mathrm{div}(\nu_s \circ \Phi_{t_k}^{-1}), \qquad \forall \delta p \in L_0^2(\Omega), \\
\left\langle \frac{\mathcal{L}_k}{\partial d}, \delta d \right\rangle_{V_s', V_s} &= \int_{\Omega_s} \sigma_s(\nu_s) \colon \nabla \delta d - \frac{1}{\delta t} \int_\Gamma (u \circ \Phi_{t_k} - \frac{1}{\delta t}(d - d_s^{k-1})) \cdot \delta d, \quad \forall \delta d \in V_s, \\
\left\langle \frac{\mathcal{L}_k}{\partial g}, \delta g \right\rangle_{V_{s,k}'', V_{s,k}'} &= \left\langle \delta g, \nu_s \circ \Phi_{t_k}^{-1} - \lambda_{f|\Phi_{t_k}(\Omega_s)} \right\rangle_{(V_s^k)', V_s^k}, \qquad \forall \delta g \in V_{s,k}'.
\end{aligned}$$

They enable us, in particular, to recover the adjoint equations, written in (41). Problem (41) consists in a linear elasticity problem and a Stokes problem, whose well-posedness derives from the same arguments that we already used. We denote by $(\nu^g, \lambda^g, \pi^g)$ its unique solution.

On the other hand, the differentiability of J_k relies on the same arguments that the ones used in the proof of Theorem 2.3 and the fact that the transformation Φ_{t_k} is sufficiently regular. Then, replacing (u, p, d) in the Lagrangian function (40), by the solution (u^g, p^g, d^g) of the smooth extension problem (37), the Lagrangian (40) reduces to

$$\mathcal{L}_k(g, (u^g, p^g, d^g), (\lambda_f, \pi, \nu_s)) = J_k(g), \quad \forall g \in (V_s^k)'.$$

Differentiating with respect to g using the chain rule and replacing the triplet (λ_f, π, ν_s) by the solution $(\lambda^g, \pi^g, \nu^g)$ of the adjoint problem (41), the gradient of J_k is finally given by,

$$\langle \nabla J_k(g), \delta g \rangle_{(V_s^k)'', (V_s^k)'} = \left\langle \delta g, \nu_k^g \circ \Phi_{t_k}^{-1} - \lambda^g_{k|\Phi_{t_k}(\Omega_s)} \right\rangle_{(V_s^k)', V_s^k}, \quad \forall \delta g \in (V_s^k)'.$$

□

With this explicit expression of the gradient of J_k, we can now state the equivalence between the research of a suitable control such that the solution of the smooth extension problem (37) satisfies the condition (38) and the minimization of J_k. This is the result of Theorem 4.6, that we prove in the following.

Proof of Theorem 4.6. On the one hand, if for a given g in $(V_s^k)'$, the solution (u^g, p^g, d^g) of the smooth extension problem (37) satisfies the condition (38), then

$$J_k(g) = 0.$$

On the other hand, suppose that g is a minimizer of J_k. Then, the adjoints λ^g and ν^g satisfy the equality,

$$\left\langle \delta g, \nu^g \circ \Phi_{t_k}^{-1} - \lambda^g_{|\Phi_{t_k}(\Omega_s)} \right\rangle_{(V_s^k)', V_s^k} = 0, \quad \forall \delta g \in (V_s^k)',$$

which corresponds to the fact that g is a zero for the gradient of J_k. It follows, in particular, that,

$$\nu^g \circ \Phi_{t_k}^{-1} \quad = \quad \lambda^g_{|\Phi_{t_k}(\Omega_s)}, \tag{52}$$

and the couple $(\lambda^g_{|\Phi_{t_k}(\Omega_f)}, \nu^g)$ belongs to the space W_u. Yet, λ^g, π^g and ν^g satisfy the adjoint equations in (41). In particular, for all (v_f, v_s) in $\hat{W}_u$ and for all q in $L^2_0(\Omega)$, where

$$\hat{W}_u = \left\{(v_f, v_s) \in (H^1_0(\Omega))^n \times V_s; \mathrm{div}(v_f) = 0, (v_f \circ \Phi_{t_k})_{|\Omega_s} = v_s\right\},$$

these equations write

$$\begin{aligned}
\mu_s \int_{\Omega_s} \sigma_s(\nu^g) \colon \nabla v_s \quad &= \quad \frac{1}{\delta t}\int_\Gamma (u^g \circ \Phi_{t_k} - \frac{1}{\delta t}(d^g - d_s^{k-1})) \cdot v_s, \\
2\mu_f \int_{\Phi_{t_k}(\Omega_f)} D(\lambda^g) \colon D(v_f) - \int_\Omega \pi^g \mathrm{div}(v_f) \quad &= \quad -\int_\Gamma (u^g \circ \Phi_{t_k} - \frac{1}{\delta t}(d^g - d_s^{k-1})) \cdot v_s, \\
\int_{\Phi_{t_k}(\Omega_f)} q\mathrm{div}(\lambda^g) \quad &= \quad 0,
\end{aligned} \tag{53}$$

Multiplying the first equation in (53) by δt and summing it with the second one, we obtain that,

$$2\mu_f \int_{\Phi_{t_k}(\Omega_f)} D(\lambda^g_{|\Phi_{t_k}(\Omega_f)}) : D(v_f) + \delta t \mu_s \int_{\Omega_s} \sigma_s(\nu^g) \colon \nabla v_s \quad = \quad 0, \quad \forall (v_f, v_s) \in \hat{W}_u. \tag{54}$$

Moreover, let (v_f, v_s) be in W_u. We can construct an extension of v_f in the whole space $(H^1_0(\Omega))^n$, denoted $E_u v_f$, such that $\mathrm{div}(E_u v_f) = 0$ and $((E_u v_f) \circ \Phi_{t_k})_{|\Gamma} = v_{s|\Gamma}$, as we did it in the proof of Theorem 4.4. Then, the couple $(E_u v_f, v_s)$ belongs to $\hat{W}_u$ and it follows that the couple $(\lambda^g_{|\Phi_{t_k}(\Omega_f)}, \nu^g)$ satisfies the equation

$$2\mu_f \int_{\Phi_{t_k}(\Omega_f)} D(\lambda^g_{|\Phi_{t_k}(\Omega_f)}) : D(v_f) + \delta t \mu_s \int_{\Omega_s} \sigma_s(\nu^g) \colon \nabla v_s \quad = \quad 0, \quad \forall (v_f, v_s) \in W_u.$$

We conclude that $(\lambda^g_{|\Phi_{t_k}(\Omega_f)}, \nu^g)$ is solution of the problem

$$\left\{ \begin{array}{l} \text{find } (\lambda, \nu) \text{ in } W_u \text{ such that,} \\ a((\lambda, \nu), (v_f, v_s)) \quad = \quad 0, \quad \forall (v_f, v_s) \in W_u, \end{array} \right. \tag{55}$$

where the bilinear continuous and coercive form a has been defined in (48), which admits the zero of W_u as unique solution. Then, the first equation in (41) becomes

$$\int_\Gamma \left(u^g \circ \Phi_{t_k} - \frac{1}{\delta_t}(d^g - d_s^{k-1})\right) \cdot v_s \quad = \quad 0, \quad \forall v_s \in V_s.$$

Moreover, for all v_s in $(H^{1/2}_{00}(\Gamma))^n$ we can construct an extension of v_s in the whole space V_s. In particular, taking $v_s = (u^g \circ \Phi_{t_k})_{|\Gamma} - \dfrac{1}{\delta_t}(d^g_{|\Gamma} - d^{k-1}_{s|\Gamma})$, it follows that

$$\int_\Gamma |(u^g \circ \Phi_{t_k})_{|\Gamma} - \frac{1}{\delta_t}(d^g_{|\Gamma} - d^{k-1}_{s|\Gamma})|^2 \quad = \quad 0,$$

which means that $(u^g \circ \Phi_{t_k})_{|\Gamma} = \dfrac{1}{\delta_t}(d^g_{|\Gamma} - d^{k-1}_{s|\Gamma})$ and the equality (38) is satisfied. □

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