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Abstract: We study sufficient conditions for the emergence of consensus and flocking in a class of strongly cooperative non-linear multi-agent systems subject to arbitrary communication failures. Our approach is based on a combination of Lyapunov analysis along with the formulation of a novel persistence of excitation condition for cooperative systems. This assumption can be interpreted in terms of average connectedness of the interaction graph of the system, and provides quantitative convergence rates towards consensus and flocking.

Keywords: Multi-agent systems, flocking, persistence of excitation, Lyapunov methods

1. INTRODUCTION

The study of emerging patterns in dynamical systems describing collective behaviour has been the object of an increasing attention in the last decades. There is by now a large literature devoted to the analysis of *consensus formation* in the class of so-called *cooperative systems*, see e.g. Smith (1995). These systems are widely used, for example, to study crowd motion Cristiani et al. (2014), robot swarms Elamvazhuthi and Berman (2015); Berman et al. (2009) and animal groups such as bird flocks or fish schools Albi et al. (2014); Bertozzi and Topaz (2004).

Since the seminal paper of Cucker and Smale (2007), a great deal of interest has been manifested towards the analysis of the so-called *flocking behaviour* (see Definition 4 below), which describes the appearance of alignment patterns in second-order cooperative multi-agent systems. In Ha and Liu (2009), the authors proposed a simpler proof of the emergence of asymptotic flocking based on Lyapunov methods. One of the main strength of the latter approach was that it could be applied to both finite and infinite dimensional multi-agent systems, while providing a strong unifying framework for consensus and flocking analysis with very diverse interaction topologies (see e.g. Caponigro et al. (2013); McQuade et al. (2019)). It also allowed to design efficient control law for key models, see Caponigro et al. (2015, 2017); Piccoli et al. (2015).

When communications between agents are subject to possibly severe failures, it is crucial to verify under which conditions convergence can still be guaranteed. For discrete-time first and second order systems, opinion formation models have been thoroughly investigated in a graph theoretic framework, see for instance the seminal paper Moreau (2005) and subsequent developments in Tanner et al. (2007); Martin et al. (2014). Further results allowed to incorporate asymmetric communication rates and random communication failures e.g. in Dalmao and Mordecki (2011); Ru et al. (2015), as well as stochastic perturbations described by Brownian motions Ha et al. (2009). However, to the best of our knowledge, there does not exist in the literature a proof of convergence towards flocking for general non-linear time-continuous systems subject to arbitrary communication failures.

In this paper, we investigate sufficient conditions for both asymptotic consensus and flocking formation, based on Lyapunov methods. The main novelty of our approach is the introduction of a suitable condition of *persistence of excitation* (see Definition 3 below) for multi-agent systems. This type of condition is quite standard in classical control theory (see e.g. Narendra and Annaswami (1989); Chitour and Sigalotti (2010); Chitour et al. (2012)), and has proven its adaptability in stability theory, in particular in allowing to build *strict Lyapunov functions* for perturbed systems, see Mazenc and Malisoff (2009); Maghenem and Loría (2017); Maghenem et al. (2018). Besides the practical interest of having a strict Lyapunov function (e.g. to recover quantitative convergence rates towards an equilibrium point or for studying input-to-state stability), our notion of persistence of excitation has both a deep and simple meaning in terms of cooperative dynamics. Indeed, it transcribes the fact that on average on a sliding time window, the interaction graph describing the multi-agent system is connected. It also imposes a quantitative lower bound on the intensity of this averaged interaction. This type of average connectedness assumption is standard when studying first-order time-varying interaction topologies (see e.g. Beard and Ren (2008); Blondel et al. (2005);

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Moreau (2005)), and is even proven to be necessary for consensus to arise in a large number of cases in Moreau (2005). In the way we formulate it, this condition further encodes the idea that one only requires the system to be persistently exciting with respect to the agents which have *not reached consensus* yet.

The structure of the paper is the following. In Section 2, we introduce our approach by proving the convergence towards consensus for persistently excited first-order dynamics. We then extend this result in Section 3 to derive the asymptotic flocking for Cucker-Smale type systems with strongly interacting kernels in the sense of Hypothesis **(K)**, which is the main result of this paper. We conclude with some remarks and open perspectives in Section 4.

2. CONSENSUS UNDER PERSISTENT EXCITATION FOR FIRST-ORDER DYNAMICS

In this section, we introduce the main tools used throughout this article in the particular case of consensus formation. We study first-order cooperative systems of the form

$$\begin{cases} \dot{x}_i(t) = \frac{1}{N}\sum_{i=1}^{N} \xi_{ij}(t)\phi(|x_i(t) - x_j(t)|)(x_j(t) - x_i(t)), \\ x_i(0) = x_i^0, \end{cases} \tag{CS$_1$}$$

where $(x_1^0, \dots, x_N^0) \in \mathbb{R}^{dN}$ is a given initial datum. We assume that the interaction kernel $\phi \in \mathrm{Lip}(\mathbb{R}_+, \mathbb{R}_+^*)$ is **strictly positive**.

The functions $\xi_{ij} \in L^\infty(\mathbb{R}_+, [0,1])$ represent **communication rates**, taking into account potential communication failures that can occur in the system (when $\xi_{ij}(t) < 1$, see Example 1 below). We require them to be *symmetric*, i.e. $\xi_{ij}(\cdot) = \xi_{ji}(\cdot)$, which means that the interaction graph of the system is *undirected*. One of the main motivations for this choice of communication rates is to study consensus and flocking when *random interaction failures* occur. This article is the first step towards a more general theory for such systems, in which the $\xi_{ij}(\cdot)$ will be realizations of stochastic processes.

From now on, we use the notation $\boldsymbol{x} = (x_1, \dots, x_N)$ for the state in $\mathbb{R}^{dN}$ and $\bar{\boldsymbol{x}} = \frac{1}{N}\sum_{i=1}^N x_i$ for its mean value. For systems of the form (CS$_1$), we aim at studying the formation of *asymptotic consensus*, defined as follows.

Definition 1. A solution $\boldsymbol{x}(t)$ of (CS$_1$) *asymptotically converges to consensus* if

$$\lim_{t\to+\infty} |x_i(t) - \bar{\boldsymbol{x}}(t)| = 0,$$

for all $i \in \{1, \dots, N\}$.

As a consequence of the symmetry of the rates $\xi_{ij}(\cdot)$, the system (CS$_1$) can be rewritten in matrix form as

$$\dot{\boldsymbol{x}}(t) = -\boldsymbol{L}(t, \boldsymbol{x}(t))\boldsymbol{x}(t), \ \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{CSM$_1$}$$

where $\boldsymbol{L} : \mathbb{R}_+ \times \mathbb{R}^{dN} \to \mathcal{L}(\mathbb{R}^{dN})$ is the so-called *graph Laplacian*, defined by

$$(\boldsymbol{L}(t, \boldsymbol{x})\boldsymbol{y})_i := \tfrac{1}{N}\textstyle\sum_{j=1}^N \xi_{ij}(t)\phi(|x_i - x_j|)(y_i - y_j). \tag{1}$$

In the following, we will also use $\boldsymbol{L}_\xi(\cdot)$ defined by

$$(\boldsymbol{L}_\xi(t)\boldsymbol{y})_i := \tfrac{1}{N}\textstyle\sum_{j=1}^N \xi_{ij}(t)(y_i - y_j). \tag{2}$$

Observe that both $\boldsymbol{L}(\cdot,\cdot)$ and $\boldsymbol{L}_\xi(\cdot)$ depend on the time-dependent communication rates $\xi_{ij}(\cdot)$, that are L^∞ functions, thus defined for almost every $t \geq 0$.

The structure displayed in (1) is fairly general and allows for a comprehensive study of both consensus and flocking problems in a unified way via Lyapunov methods. With this goal in mind, we introduce the following bilinear form in the spirit of Caponigro et al. (2013, 2015).

Definition 2. The variance bilinear form $B(\cdot,\cdot)$ is

$$B(\boldsymbol{x}, \boldsymbol{y}) := \tfrac{1}{N}\textstyle\sum_{i=1}^N \langle x_i, y_i\rangle - \langle \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}\rangle. \tag{3}$$

It is symmetric and positive semi-definite.

The evaluation $B(\boldsymbol{x}, \boldsymbol{x})$ of this bilinear form is the distance of a given $\boldsymbol{x} \in \mathbb{R}^{dN}$ from the so-called *consensus manifold* $\mathscr{C} = \{\boldsymbol{x} \in \mathbb{R}^{dN} \text{ s.t. } x_1 = \cdots = x_N\}$. It then follows that $B(\boldsymbol{x}, \boldsymbol{x}) = 0$ if and only if $x_i = \bar{\boldsymbol{x}}$ for any index $i \in \{1, \dots, N\}$, i.e. if $\boldsymbol{x}$ is a consensus.

We now list useful properties linking $B(\cdot,\cdot)$ and $\boldsymbol{L}(\cdot,\cdot)$.

Proposition 1. The graph Laplacian $\boldsymbol{L}(t, \boldsymbol{x})$ is positive-semi definite with respect to $B(\cdot,\cdot)$. Moreover, vectors of the form $\boldsymbol{L}(t, \boldsymbol{x})\boldsymbol{y}$ have zero mean.

Proof : By summing over $i \in \{1, \dots, N\}$ the components in (1), the mean of $\boldsymbol{L}(t, \boldsymbol{x})\boldsymbol{y}$ is zero. As a consequence, and by symmetry of the communication rates $\xi_{ij}(\cdot)$, it holds

$$\begin{aligned} B(\boldsymbol{L}(t, \boldsymbol{x})\boldsymbol{y}, \boldsymbol{y}) &= \tfrac{1}{N^2}\sum_{i,j=1}^N \xi_{ij}(t)\phi(|x_i - x_j|)\langle y_i, y_i - y_j\rangle \\ &= \tfrac{1}{2N^2}\sum_{i,j=1}^N \xi_{ij}(t)\phi(|x_i - x_j|)|y_i - y_j|^2 \geq 0, \end{aligned}$$

which proves our claim □

We now introduce the concept of *persistence of excitation* for multi-agent systems described by (CSM$_1$).

Definition 3. Let $\tau, \mu > 0$ be given parameters. We say that the **persistence of excitation condition** (PE$_{\tau,\mu}$) holds for (CSM$_1$) if

$$B\left(\left(\tfrac{1}{\tau}\textstyle\int_t^{t+\tau} \boldsymbol{L}_\xi(s)\mathrm{d}s\right)\boldsymbol{x}, \boldsymbol{x}\right) \geq \mu B(\boldsymbol{x}, \boldsymbol{x}), \tag{PE$_{\tau,\mu}$}$$

for all $\boldsymbol{x} \in \mathbb{R}^{dN}$.

Remark 1. Condition (PE$_{\tau,\mu}$) only involves the communication weights $\xi_{ij}(\cdot)$ through $\boldsymbol{L}_\xi(\cdot)$ and not the state of the system. Moreover, it is formulated using the bilinear form $B(\cdot,\cdot)$, illustrating the fact that one only needs the persistence to hold along directions which are orthogonal to the consensus manifold $\mathscr{C}$. Finally, (PE$_{\tau,\mu}$) can be interpreted as a connectivity condition of the time-averaged interaction graph of the system, which is fairly common in the literature, see Moreau (2005) and (Beard and Ren, 2008, Chapter 2), see also Example 1 below.

Example 1. Consider a system of $N = 3$ agents with states in $\mathbb{R}$, fix two parameters $(\tau, \mu) \in \mathbb{R}_+^* \times [0,1]$, and suppose that the communication weights $\xi_{ij}(\cdot)$ are defined for almost every $t \geq 0$ by

$$\xi_{13}(t) = 0, \qquad \xi_{12}(t) = \begin{cases} \mu & \text{if } \lfloor t/\tau \rfloor = 1 \bmod[4], \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\xi_{23}(t) = \begin{cases} \mu & \text{if } \lfloor t/\tau \rfloor = 3 \bmod[4], \\ 0 & \text{otherwise.} \end{cases}$$

Then, (PE$_{\tau,\mu}$) holds with parameters $(\tau', \mu') = (4\tau, \frac{\mu}{16})$. Notice that this example can be generalized to systems with N agents in $\mathbb{R}^d$, and therefore that (PE$_{\tau,\mu}$) incorporates undirected interaction topologies such that the averaged interaction graph is connected.

In the following theorem, we prove that solutions of (CSM_1) asymptotically converge to consensus when the persistence assumption $(\mathrm{PE}_{\tau,\mu})$ holds. While this result might be derived from earlier works dealing with consensus in undirected graphs, see e.g. in Moreau (2005); Blondel et al. (2005), we believe that it presents a new point of view on this topic. It also allows for a progressive introduction to some of the concepts that shall be necessary for the establishment of our main result Theorem 3.

Theorem 2. (Consensus). Let $\phi(\cdot)$ be positive and non-increasing. Let $\xi_{ij} \in L^\infty(\mathbb{R}_+,[0,1])$ satisfy $(\mathrm{PE}_{\tau,\mu})$ for some $\tau,\mu > 0$. Then, any solution $\boldsymbol{x}(\cdot)$ of (CSM_1) asymptotically converges to consensus.

Proof : Let $c^2 := \sup_{(t,\boldsymbol{x})} \|\boldsymbol{L}(t,\boldsymbol{x})\|_B$ be the operator norm of $\boldsymbol{L}(\cdot,\cdot)$ with respect to $B(\cdot,\cdot)$, i.e.

$$c^2 = \sup_{(t,\boldsymbol{x},\boldsymbol{y})} \left\{ \sqrt{B(\boldsymbol{L}(t,\boldsymbol{x})\boldsymbol{y},\boldsymbol{L}(t,\boldsymbol{x})\boldsymbol{y})} \text{ s.t. } B(\boldsymbol{y},\boldsymbol{y}) = 1 \right\}.$$

Remark that c is finite since $\phi(\cdot)$ is bounded. Denote by $X(\cdot)$ the *standard deviation* of $\boldsymbol{x}(\cdot)$, given by

$$X(t) := \sqrt{B(\boldsymbol{x}(t),\boldsymbol{x}(t))},$$

By the definition of $B(\cdot,\cdot)$, $\boldsymbol{x}(\cdot)$ asymptotically converges to consensus if and only if $X(\cdot)$ vanishes at infinity.

Define the time-dependent family of linear operators $\psi_\tau : \mathbb{R}_+ \to \mathcal{L}(\mathbb{R}^{dN})$ along the trajectory $\boldsymbol{x}(\cdot)$ by

$$\psi_\tau(t) := (1+c^2)\tau\,\mathbb{I}_{dN} - \tfrac{1}{\tau}\textstyle\int_t^{t+\tau}\int_t^s \boldsymbol{L}(\sigma,\boldsymbol{x}(\sigma))\mathrm{d}\sigma\mathrm{d}s. \quad (4)$$

Then, $\psi_\tau(\cdot)$ is Lipschitz with pointwise derivative

$$\dot\psi_\tau(t) = \boldsymbol{L}(t,\boldsymbol{x}(t)) - \tfrac{1}{\tau}\textstyle\int_t^{t+\tau}\boldsymbol{L}(s,\boldsymbol{x}(s))\mathrm{d}s. \quad (5)$$

By definition, it further holds

$$\sqrt{\tau}X(t) \leq \sqrt{B(\psi_\tau(t)\boldsymbol{x},\boldsymbol{x})} \leq \sqrt{(1+c^2)\tau}X(t). \quad (6)$$

Define the candidate Lyapunov function

$$\mathscr{X}_\tau(t) := \lambda X(t) + \sqrt{B(\psi_\tau(t)\boldsymbol{x}(t),\boldsymbol{x}(t))}, \quad (7)$$

where $\lambda > 0$ is a tuning parameter. This type of construction is rather recent in the design of strict Lyapunov function under persistent excitation assumptions: see e.g. Maghenem and Loría (2017); Maghenem et al. (2018); Mazenc and Malisoff (2009). By (6), it holds that

$$(\lambda+\sqrt{\tau})X(t) \leq \mathscr{X}_\tau(t) \leq (\lambda+\sqrt{(1+c^2)\tau})X(t). \quad (8)$$

By Proposition 1, any solution $\boldsymbol{x}(\cdot)$ of (CSM_1) has constant mean, i.e. $\bar{\boldsymbol{x}}(\cdot) \equiv \bar{\boldsymbol{x}}_0$. By invariance with respect to translation of (CSM_1), we assume without loss of generality that $\bar{\boldsymbol{x}}(\cdot) \equiv 0$ from now on.

We now aim at proving a strict-dissipation inequality of the form

$$\dot{\mathscr{X}}_\tau(t) \leq -\alpha\mathscr{X}_\tau(t), \quad (9)$$

for some $\alpha > 0$. With this goal, we first compute

$$\dot{\mathscr{X}}_\tau(t) = -\tfrac{\lambda}{X(t)}B(\boldsymbol{L}(t,\boldsymbol{x}(t)),\boldsymbol{x}(t)) + \tfrac{B(\dot\psi_\tau(t)\boldsymbol{x}(t),\boldsymbol{x}(t))}{2\sqrt{B(\psi_\tau(t)\boldsymbol{x}(t),\boldsymbol{x}(t))}} - \tfrac{B(\boldsymbol{L}(t,\boldsymbol{x}(t))\boldsymbol{x}(t),\psi_\tau(t)\boldsymbol{x}(t))}{\sqrt{B(\psi_\tau(t)\boldsymbol{x}(t),\boldsymbol{x}(t))}}.$$

By (5)-(6), it holds that

$$\begin{aligned}\dot{\mathscr{X}}_\tau(t) \leq & -\tfrac{1}{2\sqrt{(1+c^2)\tau}X(t)}B\big(\big(\tfrac{1}{\tau}\textstyle\int_t^{t+\tau}\boldsymbol{L}(s,\boldsymbol{x}(s))\mathrm{d}s\big)\boldsymbol{y},\boldsymbol{y}\big)\\ & + \tfrac{1}{\sqrt{\tau}X(t)}B\big(\big(\tfrac{1}{\tau}\textstyle\int_t^{t+\tau}\int_t^s\boldsymbol{L}(\sigma,\boldsymbol{x}(\sigma))\mathrm{d}\sigma\mathrm{d}s\big)\boldsymbol{y},\boldsymbol{L}(t,\boldsymbol{y})\boldsymbol{y}\big)\\ & + \tfrac{1}{\sqrt{\tau}X(t)}\big(\tfrac{1}{2} - \sqrt{(1+c^2)\tau} - \sqrt{\tau}\lambda\big)B(\boldsymbol{L}(t,\boldsymbol{y})\boldsymbol{y},\boldsymbol{y}),\end{aligned} \quad (10)$$

where we wrote $\boldsymbol{y} \equiv \boldsymbol{x}(t)$ for conciseness.

To estimate the first line of (10), recall that first-order cooperative systems have uniformly compactly supported trajectories, see e.g. (Piccoli et al., 2015, Lemma 1) . Since $\phi(\cdot)$ is positive and continuous, there exists a positive constant C_0 – depending only on $\boldsymbol{x}_0$ – such that

$$\min_{i,j}\phi(|x_i(t)-x_j(t)|) \geq C_0,$$

for all times $t \geq 0$. By definition of $\boldsymbol{L}(\cdot,\cdot)$, this implies that

$$\begin{aligned}B\big(\boldsymbol{L}(t,\boldsymbol{x}(t))\boldsymbol{y},\boldsymbol{y}\big) &\geq \tfrac{C_0}{2N^2}\sum_{i,j=1}^N \xi_{ij}(t)|y_i-y_j|^2\\ &= C_0 B\big(\boldsymbol{L}_\xi(t)\boldsymbol{y},\boldsymbol{y}\big),\end{aligned}$$

for any $\boldsymbol{y} \in \mathbb{R}^{dN}$. By using $(\mathrm{PE}_{\tau,\mu})$, it further holds

$$B\left(\left(\tfrac{1}{\tau}\textstyle\int_t^{t+\tau}\boldsymbol{L}(s,\boldsymbol{x}(s))\mathrm{d}s\right)\boldsymbol{x}(t),\boldsymbol{x}(t)\right) \geq C_0\mu X^2(t). \quad (11)$$

For the second line of (10), one has that

$$\begin{aligned}& B\big(\tfrac{1}{\tau}\big(\textstyle\int_t^{t+\tau}\int_t^s\boldsymbol{L}(\sigma,\boldsymbol{x}(\sigma))\mathrm{d}\sigma\mathrm{d}s\big)\boldsymbol{x}(s),\boldsymbol{L}(t,\boldsymbol{x}(t))\boldsymbol{x}(t)\big)\\ \leq\ & \tau c^2 X(t)\sqrt{B\big(\boldsymbol{L}(t,\boldsymbol{x}(t))\boldsymbol{x}(t),\boldsymbol{L}(t,\boldsymbol{x}(t))\boldsymbol{x}(t)\big)}\\ \leq\ & \tau c^2 X(t)\ \|\boldsymbol{L}(t,\boldsymbol{x}(t))^{1/2}\|_B\ \sqrt{B\big(\boldsymbol{L}(t,\boldsymbol{x}(t))\boldsymbol{x}(t),\boldsymbol{x}(t)\big)}\\ \leq\ & \tau c^3\left(\tfrac{\epsilon}{2}X(t)^2 + \tfrac{1}{2\epsilon}B(\boldsymbol{L}(t,\boldsymbol{x}(t))\boldsymbol{x}(t),\boldsymbol{x}(t))\right),\end{aligned} \quad (12)$$

for any $\epsilon > 0$, by definition of $\|\cdot\|_B$ and by applying Cauchy-Schwartz and Young's inequality. Merging (10)-(11)-(12) and recalling that $\boldsymbol{L}(\cdot,\cdot)$ is positive semi-definite, we obtain that

$$\begin{aligned}\dot{\mathscr{X}}_\tau(t) \leq & -\left(\tfrac{C_0\mu}{2\sqrt{(1+c^2)\tau}} - \tfrac{c^3\sqrt{\tau}}{2}\epsilon\right)X(t)\\ & + \tfrac{1}{X(t)}\left(\tfrac{1}{2\sqrt{\tau}} + \tfrac{c^3\sqrt{\tau}}{2\epsilon} - \lambda\right)B(\boldsymbol{L}(t,\boldsymbol{x}(t))\boldsymbol{x}(t),\boldsymbol{x}(t)).\end{aligned}$$

Choosing furthermore the parameters

$$\epsilon = \tfrac{C_0\mu}{2c^3\tau\sqrt{(1+c^2)}}, \qquad \lambda = \tfrac{1}{2\sqrt{\tau}} + \tfrac{c^3\sqrt{\tau}}{2\epsilon}$$

and using (8), we recover

$$\dot{\mathscr{X}}_\tau(t) \leq -\tfrac{C_0\,\mu}{2\sqrt{(1+c^2)\tau}(\lambda+\sqrt{(1+c^2)\tau})}\mathscr{X}_\tau(t)$$

so that (9) holds with a given constant $\alpha > 0$. By an application of Grönwall's Lemma, we obtain that $\lim\limits_{t\to+\infty}\mathscr{X}_\tau(t) = 0$, and thus $\lim\limits_{t\to+\infty}X(t) = 0$ by (8).

By definition of $X(\cdot)$, this implies that $\boldsymbol{x}(\cdot)$ exponentially converges to consensus. □

3. FLOCKING FOR CUCKER-SMALE SYSTEMS WITH STRONG INTERACTIONS

In this section, we derive sufficient conditions for the asymptotic convergence to flocking of a class of Cucker-Smale type subject to arbitrary communication failures. These systems are of the form

$$\begin{cases}\dot{x}_i(t) = v_i(t),\\ \dot{v}_i(t) = \frac{1}{N}\sum\limits_{j=1}^N \xi_{ij}(t)\phi(|x_i(t)-x_j(t)|)(v_j(t)-v_i(t)).\end{cases} \quad (\mathrm{CS}_2)$$

Similarly to Section 2, (CS_2) can be rewritten in matrix form using the graph Laplacian defined in (1):

$$\begin{cases}\dot{\boldsymbol{x}}(t) = \boldsymbol{v}(t), & \boldsymbol{x}(0) = \boldsymbol{x}_0,\\ \dot{\boldsymbol{v}}(t) = -\boldsymbol{L}(t,\boldsymbol{x}(t))\boldsymbol{v}(t), & \boldsymbol{v}(0) = \boldsymbol{v}_0.\end{cases} \quad (\mathrm{CSM}_2)$$

We now recall the definition of asymptotic flocking.

Definition 4. A solution $(\boldsymbol{x}(\cdot), \boldsymbol{v}(\cdot))$ of (CSM_2) *asymptotically converges to flocking* if

$$\sup_{t\geq 0} |x_i(t) - \bar{\boldsymbol{x}}(t)| < +\infty \quad \text{and} \quad \lim_{t\to+\infty} |v_i(t) - \bar{\boldsymbol{v}}(t)| = 0,$$

for any $i \in \{1, \ldots, N\}$.

For this problem, we assume that the interaction kernel $\phi(\cdot) \in \mathrm{Lip}(\mathbb{R}_+, \mathbb{R}_+^*)$ satisfies the following *strong interaction* assumption.

Hypothesis (K) There exist positive constants K, σ along with a parameter $\beta \in (0, \frac{1}{2})$ such that

$$\phi(r) \geq \frac{K}{(\sigma + r)^\beta}. \tag{13}$$

In particular, $\phi \notin L^1(\mathbb{R}_+, \mathbb{R}_+^*)$. Up to replacing $\phi(\cdot)$ by this lower estimate, we can assume with no loss of generality that $\phi(\cdot)$ is **non-increasing**.

Remark 2. Hypothesis **(K)** is a strengthened version of the usual strong interaction condition, which requires that $\phi \notin L^1(\mathbb{R}_+, \mathbb{R}_+^*)$. Remark that here, we require that the Cucker-Smale exponent β be less that $\frac{1}{2}$, whereas in the literature the expected critical exponent beyond which unconditional flocking may fail to occur is $\beta = 1$, see Section 4 for more details.

Remark 3. When $\phi(\cdot)$ is uniformly bounded from below by a positive constant, then flocking in the full-communication setting occurs, see e.g. Cucker and Smale (2007); Ha and Liu (2009); Piccoli et al. (2015). In our framework, this result is a simple consequence of Theorem 3. For positive kernels not satisfying (13), one can easily construct examples of initial conditions $(\boldsymbol{x}_0, \boldsymbol{v}_0)$ for which flocking does not occur, see Cucker and Smale (2007).

Solutions of (CSM_2) satisfy

$$\dot{\bar{\boldsymbol{x}}}(t) = \bar{\boldsymbol{v}}(t), \qquad \dot{\bar{\boldsymbol{v}}}(t) = 0.$$

By known invariance properties of multi-agent systems, we can assume with no loss of generality that $\bar{\boldsymbol{x}}(\cdot) = \bar{\boldsymbol{v}}(\cdot) \equiv 0$. We further define the standard deviation maps

$$X(t) := \sqrt{B(\boldsymbol{x}(t), \boldsymbol{x}(t))}, \qquad V(t) := \sqrt{B(\boldsymbol{v}(t), \boldsymbol{v}(t))}.$$

As a consequence of symmetry of $\xi_{ij}(\cdot)$, (CSM_2) is *weakly dissipative* in the sense that

$$\dot{X}(t) \leq V(t), \qquad \dot{V}(t) \leq 0. \tag{14}$$

In the seminal paper Ha and Liu (2009), the authors introduced a concise proof of the Cucker-Smale flocking based on the analysis of a system of *strictly dissipative inequalities.* More precisely, if it holds that

$$\dot{X}(t) \leq V(t), \qquad \dot{V}(t) \leq -\phi(2\sqrt{N}X(t))V(t), \tag{15}$$

with an interaction kernel $\phi \notin L^1(\mathbb{R}_+, \mathbb{R}_+^*)$, then the system converges to flocking. Our aim is to adapt their strategy while taking into account possible communication failures. We prove the following main result of this paper.

Theorem 3. (Main result - Flocking). Let $\phi(\cdot)$ be positive, non-increasing and satisfying **(K)**. Let $\xi_{ij} \in L^\infty(\mathbb{R}_+, [0,1])$ be such that $(\mathrm{PE}_{\tau,\mu})$ holds. Then, any solution of (CSM_2) asymptotically converges to flocking.

The proof of this result relies on the construction of a locally-strict Lyapunov function for (CSM_2), for which a system of inequalities akin to (15) holds **only on a bounded time interval**. This local-in-time strict dissipation allows us to recover the asymptotic flocking of the system by a reparametrization of the time variable. To the best of our knowledge, this combination of strict Lyapunov design and flocking analysis via locally dissipative inequalities is fully new in the context of multi-agent systems.

Notation 1. We define the *rescaled interaction kernel* by

$$\boldsymbol{\phi}_\tau(r) := \phi\big(2\sqrt{N}(r + \tau V(0))\big) \tag{16}$$

for any $r \geq 0$, and we denote by $\boldsymbol{\Phi}_\tau(\cdot)$ its uniquely determined primitive which vanishes at $X(0)$, i.e.

$$\boldsymbol{\Phi}_\tau(X) := \int_{X(0)}^{X} \boldsymbol{\phi}_\tau(r)\mathrm{d}r. \tag{17}$$

We start the proof of Theorem 3 by a series of lemmas which will progressively highlight the role of the different assumptions made on the system.

Lemma 4. Let $(\boldsymbol{x}(\cdot), \boldsymbol{v}(\cdot))$ be a solution of (CSM_2). If $(\mathrm{PE}_{\tau,\mu})$ holds, then one has

$$B\left(\left(\tfrac{1}{\tau}\int_t^{t+\tau}\boldsymbol{L}(s, \boldsymbol{x}(s))\mathrm{d}s\right)\boldsymbol{w}, \boldsymbol{w}\right) \geq \mu\boldsymbol{\phi}_\tau(X(t))B(\boldsymbol{w}, \boldsymbol{w}) \tag{18}$$

for any $\boldsymbol{w} \in \mathbb{R}^{dN}$, with $\boldsymbol{\phi}_\tau(\cdot)$ defined as in (16).

Proof : By definition of $\boldsymbol{L}(\cdot,\cdot)$, it holds that

$$\begin{aligned} & B\left(\left(\tfrac{1}{\tau}\int_t^{t+\tau}\boldsymbol{L}(s, \boldsymbol{x}(s))\mathrm{d}s\right)\boldsymbol{w}, \boldsymbol{w}\right) \\ \geq\ & \tfrac{1}{2N^2}\sum_{i,j=1}^{N}\left(\tfrac{1}{\tau}\int_t^{t+\tau}\xi_{ij}(s)\phi(|x_i(s)-x_j(s)|)\mathrm{d}s\right)|w_i - w_j|^2 \\ \geq\ & \tfrac{1}{2N^2}\sum_{i,j=1}^{N}\left(\tfrac{1}{\tau}\int_t^{t+\tau}\xi_{ij}(s)\phi(2\sqrt{N}X(s))\mathrm{d}s\right)|w_i - w_j|^2, \end{aligned} \tag{19}$$

where we used that $\phi(\cdot)$ is non-increasing. As a consequence of the weak dissipation (14), one further has

$$X(s) = X(t) + \textstyle\int_t^s \dot{X}(\sigma)\mathrm{d}\sigma \leq X(t) + \tau V(0).$$

for all $s \in [t, t+\tau]$. By (19), and using again that $\phi(\cdot)$ is non-increasing, it holds

$$\begin{aligned} & B\left(\left(\tfrac{1}{\tau}\int_t^{t+\tau}\boldsymbol{L}(s, \boldsymbol{x}(s))\mathrm{d}s\right)\boldsymbol{w}, \boldsymbol{w}\right) \\ & \geq \tfrac{\phi\left(2\sqrt{N}(X(t)+\tau V(0))\right)}{2N^2}\sum_{i,j=1}^{N}\left(\tfrac{1}{\tau}\int_t^{t+\tau}\xi_{ij}(s)\mathrm{d}s\right)|w_i - w_j|^2 \\ & = \boldsymbol{\phi}_\tau(X(t))B\left(\left(\tfrac{1}{\tau}\int_t^{t+\tau}\boldsymbol{L}_\xi(s)\mathrm{d}s\right)\boldsymbol{w}, \boldsymbol{w}\right) \\ & \geq \mu\boldsymbol{\phi}_\tau(X(t))B\left(\boldsymbol{w}, \boldsymbol{w}\right), \end{aligned}$$

where we used $(\mathrm{PE}_{\tau,\mu})$ in the last inequality. □

We now define the candidate Lyapunov function

$$\mathscr{V}_\tau(t) := \lambda(t)V(t) + \sqrt{B(\psi_\tau(t)\boldsymbol{v}(t), \boldsymbol{v}(t))}, \tag{20}$$

where $\psi_\tau(\cdot)$ is defined in (4) and $\lambda(\cdot)$ is a smooth tuning curve. We have the following lemma.

Lemma 5. For any $\epsilon_0 > 0$, there exists a time horizon $T_{\epsilon_0} > 0$ such that for almost every $t \in [0, 2T_{\epsilon_0})$, it holds

$$\dot{\mathscr{V}}_\tau(t) \leq -\tfrac{\mu\,\boldsymbol{\phi}_\tau(X(t))}{2\sqrt{(1+c^2)\tau}}V(t). \tag{21}$$

Proof : Following the proof of Theorem 2, we have

$$\dot{\mathscr{V}}_\tau(t) \leq \frac{\left(\frac{1}{2\sqrt{\tau}}+\frac{c^3\tau}{2\epsilon(t)}-\sqrt{\tau}\lambda(t)\right)}{\sqrt{\tau}V(t)}B(\boldsymbol{L}(t,\boldsymbol{x}(t))\boldsymbol{v}(t),\boldsymbol{v}(t)) - \left(\frac{\mu\,\boldsymbol{\phi}_\tau(X(t))}{2\sqrt{(1+c^2)\tau}} - \frac{c^3\sqrt{\tau}}{2}\epsilon(t) - \dot{\lambda}(t)\right)V(t). \tag{22}$$

The two differences with respect to the proof of Theorem 2 are the choice of *time-dependent families of parameters* $(\lambda(\cdot),\epsilon(\cdot))$ and the use of (18) instead of $(\mathrm{PE}_{\tau,\mu})$.

Start by fixing

$$\lambda(t) = \tfrac{1}{2\sqrt{\tau}} + \tfrac{c^3\sqrt{\tau}}{2\epsilon(t)}. \tag{23}$$

This implies in particular that $\dot{\lambda}(t) = -\frac{c^3\sqrt{\tau}}{2\epsilon^2(t)}\dot{\epsilon}(t)$. Choose now $\epsilon(\cdot)$ as the solution of

$$\dot{\epsilon}(t) = \epsilon^3(t),\ \epsilon(0) = \epsilon_0,$$

for a given constant $\epsilon_0 > 0$, i.e. $\epsilon(\cdot)$ defined by

$$\epsilon(t) = \tfrac{\epsilon_0}{\sqrt{1-2\epsilon_0^2 t}}, \tag{24}$$

for $t \in [0, 1/2\epsilon_0^2)$. Then, (22) reads as

$$\dot{\mathscr{V}}_\tau(t) \leq -\tfrac{\mu\,\boldsymbol{\phi}_\tau(X(t))}{2\sqrt{(1+c^2)\tau}}V(t),$$

and (21) holds with $T_{\epsilon_0} = 1/4\epsilon_0^2$. □

Observe that (21) involves both $\mathscr{V}_\tau(\cdot)$ and $V(\cdot)$. We now aim at finding an estimate involving solely $V(\cdot)$.

Lemma 6. There exists a function $\epsilon_0 \in \mathbb{R}_+^* \mapsto X_M(\epsilon_0)$ such that $X(t) \leq X_M(\epsilon_0)$ for all $t \in [0, T_{\epsilon_0}]$. Moreover, for any $\epsilon_0 \in \mathbb{R}_+^*$ one has that

$$V(T_{\epsilon_0}) \leq \left(\tfrac{\alpha_1+\beta_1\epsilon_0}{\alpha_2+\beta_2\epsilon_0}\right)V(0)\exp\left(-\tfrac{\mu\,\boldsymbol{\phi}_\tau(X_M(\epsilon_0))}{4(\alpha_3+\beta_3\epsilon_0)\epsilon_0}\right), \tag{25}$$

where $\{\alpha_k,\beta_k\}_{k=1}^3$ are constants depending on (c,τ).

Proof : Choose $\epsilon_0 > 0$ and denote by $(\lambda(\cdot),\epsilon(\cdot))$ the corresponding functions given by (23)-(24) respectively.

Similarly to (6), it holds that

$$\sqrt{\tau}V(t) \leq \sqrt{B(\psi_\tau(t)\boldsymbol{v}(t),\boldsymbol{v}(t))} \leq \sqrt{(1+c^2)\tau}V(t).$$

By definition of $\mathscr{V}_\tau(\cdot)$ in (20), it then holds that

$$\begin{cases} \mathscr{V}_\tau(t) \leq \left(\sqrt{(1+c^2)\tau} + \frac{1}{2\sqrt{\tau}} + \frac{c^3\sqrt{\tau}}{2\epsilon_0}\right)V(t), \\ \mathscr{V}_\tau(t) \geq \left(\sqrt{\tau} + \frac{1}{2\sqrt{\tau}} + \frac{c^3\sqrt{2\tau}}{4\epsilon_0}\right)V(t), \end{cases}$$

for any $t \in [0, T_{\epsilon_0}]$, where we used the fact that $\epsilon(t) \leq \sqrt{2}\epsilon_0$ on this time interval. By simple identification of the coefficients, these estimates can be rewritten as

$$\left(\tfrac{\alpha_2}{\epsilon_0} + \beta_2\right)V(t) \leq \mathscr{V}_\tau(t) \leq \left(\tfrac{\alpha_1}{\epsilon_0} + \beta_1\right)V(t), \tag{26}$$

for positive constants $\{\alpha_k,\beta_k\}_{k=1}^2$ depending on (c,τ).

We can further integrate (21) on $[0,t]$ to recover

$$\mathscr{V}_\tau(t) \leq \mathscr{V}_\tau(0) - \tfrac{\mu}{2\sqrt{(1+c^2)\tau}}\int_0^t \boldsymbol{\phi}_\tau(X(s))V(s)\mathrm{d}s.$$

which in turn implies that

$$V(t) \leq \left(\tfrac{\alpha_1+\beta_1\epsilon_0}{\alpha_2+\beta_2\epsilon_0}\right)V(0) - \tfrac{\mu\epsilon_0}{\alpha_2'+\beta_2'\epsilon_0}\int_0^t \boldsymbol{\phi}_\tau(X(s))V(s)\mathrm{d}s, \tag{27}$$

where $(\alpha_2',\beta_2') = 2\sqrt{(1+c^2)\tau}(\alpha_2,\beta_2)$. Recall now that $\dot{X}(s) \leq V(s)$ by (14) and apply the change of variable $r = X(s)$ in (27) to obtain

$$V(t) \leq \left(\tfrac{\alpha_1+\beta_1\epsilon_0}{\alpha_2+\beta_2\epsilon_0}\right)V(0) - \tfrac{\mu\epsilon_0}{\alpha_2'+\beta_2'\epsilon_0}\int_{X(0)}^{X(t)} \boldsymbol{\phi}_\tau(r)\mathrm{d}r. \tag{28}$$

Since $\boldsymbol{\phi}_\tau \notin L^1(\mathbb{R}_+,\mathbb{R}_+^*)$, its primitive $\boldsymbol{\Phi}_\tau(\cdot)$ is a strictly increasing map which image continuously spans $\mathbb{R}_+$. It is therefore invertible, and for any $\epsilon_0 > 0$ there exists a critical radius $X_M(\epsilon_0)$ such that

$$X_M(\epsilon_0) = \boldsymbol{\Phi}_\tau^{-1}\left(\tfrac{2\sqrt{(1+c^2)\tau}(\alpha_1+\beta_1\epsilon_0)}{\mu\epsilon_0}V(0)\right). \tag{29}$$

Since $V(\cdot)$ is a non-negative quantity by definition, it necessarily follows by plugging (29) into (28) that $X(t) \leq X_M(\epsilon_0)$ on $[0, T_{\epsilon_0}]$.

Going back to the dissipative differential inequality (21) combined with (26), we can again use the fact that $\boldsymbol{\phi}_\tau(\cdot)$ is non-increasing to obtain

$$\dot{\mathscr{V}}_\tau(t) \leq -\tfrac{\mu\epsilon_0\,\boldsymbol{\phi}_\tau(X_M(\epsilon_0))}{(\alpha_3+\beta_3\epsilon_0)}\mathscr{V}_\tau(t)$$

for almost every $t \in [0, T_{\epsilon_0}]$, where we denoted $(\alpha_3,\beta_3) = 2\sqrt{(1+c^2)\tau}(\alpha_1,\beta_1)$. By an application of Grönwall Lemma to $\mathscr{V}_\tau(\cdot)$ along with (26), we conclude that

$$V(T_{\epsilon_0}) \leq \left(\tfrac{\alpha_1+\beta_1\epsilon_0}{\alpha_2+\beta_2\epsilon_0}\right)V(0)\exp\left(-\tfrac{\mu\,\boldsymbol{\phi}_\tau(X_M(\epsilon_0))}{4(\alpha_3+\beta_3\epsilon_0)\epsilon_0}\right)$$

where we used the fact that $T_{\epsilon_0} = 1/4\epsilon_0^2$. □

Building on the estimate (25) obtained in Lemma 6, we now conclude the proof of our main result Theorem 3. To lighten the computations, most of the argument will be carried out in terms of asymptotic estimates.

Notation 2. We will use the notations

$$f(x) \underset{x\to a}{\gtrsim} g(x) \quad \text{and} \quad f(x) \underset{x\to a}{\lesssim} g(x)$$

to mean that a map $f(\cdot)$ is bounded from below (resp. from above) by a map which is equivalent to $g(\cdot)$ as $x \to a$.

Proof (Theorem 3): In order to recover the emergence of flocking in (CSM_2), we look into the asymptotic behaviour of our estimates as $\epsilon_0 \to 0^+$, or equivalently as $T_{\epsilon_0} \to +\infty$. Using the analytical expression (29) of $X_M(\epsilon_0)$, we have that

$$\boldsymbol{\phi}_\tau(X_M(\epsilon_0)) = \boldsymbol{\phi}_\tau \circ \boldsymbol{\Phi}_\tau^{-1}\left(C_1 + \frac{C_2}{\epsilon_0}\right),$$

where C_1, C_2 are positive constants depending on the data of the problem. Moreover by integrating (13), we obtain

$$\Phi(X) \geq \tfrac{K}{1-\beta}\Big((\sigma+X)^{1-\beta} - (\sigma+X(0))^{1-\beta}\Big),$$

which along with standard monotonicity properties of inverse functions and the fact that $\boldsymbol{\phi}_\tau(\cdot)$ is non-increasing, yields the existence of a positive constants C such that

$$\boldsymbol{\phi}_\tau \circ \boldsymbol{\Phi}_\tau^{-1}\left(C_1 + \frac{C_2}{\epsilon_0}\right) \underset{\epsilon_0\to 0^+}{\gtrsim} C\epsilon_0^{\frac{\beta}{1-\beta}}. \tag{30}$$

Combining the latter expression (30) with (25) and recalling that $T_{\epsilon_0} = 1/4\epsilon_0^2$, we recover that

$$V(T_{\epsilon_0}) \underset{T_{\epsilon_0}\to+\infty}{\lesssim} \tfrac{\alpha_1}{\alpha_2}V(0)\exp\left(-\tfrac{C\mu}{8\alpha_3}T_{\epsilon_0}^{\frac{1-2\beta}{2(1-\beta)}}\right). \tag{31}$$

Since $\epsilon_0 \in \mathbb{R}_+^* \mapsto T_{\epsilon_0}$ continuously span the whole of $\mathbb{R}_+$, we can reparametrize time using $T \equiv T_{\epsilon_0}$. As we assumed in **(K)** that $\beta \in (0, \frac{1}{2})$, estimate (31) implies that

$$V(T) \underset{T\to+\infty}{\longrightarrow} 0.$$

We now turn our attention to the uniform boundedness of the position radius $X(\cdot)$. The weak-dissipativity (14) of (CSM_1) expressed in terms of the new time variable $T \equiv T_{\epsilon_0}$ writes

$$\sup_{T\geq 0} X(T) \leq X(0) + \int_0^{+\infty} V(T)\mathrm{d}T.$$

This implies that $\sup_{T\geq 0} X(T) < +\infty$ as a consequence of (31) and of the fact that $\beta \in (0, \frac{1}{2})$, which concludes the proof of Theorem 3. □

4. CONCLUSION AND PERSPECTIVES

In this article, we proved two main results of convergence of multi-agent systems under arbitrary communication failures. If communication rates satisfy a persistence of excitation condition, then one has both convergence to consensus for first-order systems (Theorem 2) and convergence to flocking for Cucker-Smale systems under an additional strong interaction condition (Theorem 3). For the sake of conciseness and readability, we assumed that the initial time of the non-stationary dynamics was fixed. Yet, it could be checked by repeating our argument that both convergence results are uniform with respect to the initial time.

In the future, we aim to improve our main result Theorem 3 in two directions. First, we will investigate whether the rather surprising exponent range $\beta \in (0, \frac{1}{2})$ – which is currently necessary in order to ensure that asymptotic flocking occurs – has an intrinsic meaning, or if it is just appearing as a limit of our current choice of Lyapunov function. Answering this question might also pave the way for flocking results with weaker interactions, involving confinement conditions linking the initial state and velocity mean-deviations and the persistence parameters. Second, we will study communication failures defined as the realizations of stochastic processes and try to see under which assumptions and in what sense the convergence towards consensus and flocking can occur (almost surely, in probability, etc...). In this setting, one of the main difficulty will most likely lie in the identificaton of proper generalizations of ($PE_{\tau,\mu}$) to the stochastic setting.

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