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Transverse instability of gravity-capillary solitary waves on deep water in the presence of constant vorticity

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The bifurcation of two-dimensional gravity-capillary waves into solitary waves when the phase velocity and group velocity are nearly equal is investigated in the presence of constant vorticity. We found that gravity-capillary solitary waves with decaying oscillatory tails exist in deep water in the presence of vorticity. Furthermore we found that the presence of vorticity influences strongly (i) the solitary wave properties and (ii) the growth rate of unstable transverse perturbations. The growth rate and bandwidth instability are given numerically and analytically as a function of the vorticity.

1. Introduction

We consider the bifurcation of gravity-capillary waves propagating at the free surface of a vertically sheared current of constant vorticity when the phase and group velocities are nearly equal. In the absence of vorticity, the existence of steadily propagating solitary gravity-capillary waves on deep water was proved numerically by Longuet-Higgins (1989). Iooss & Kirchgässner (1990) investigated the problem mathematically and found it was a 1:1 resonance phenomenon. Later on, Vanden-Broeck & Dias (1992) established the link between the numerical calculation of Longuet-Higgins (1989) and the rigorous analysis of Iooss & Kirchgässner (1990). Dias *et al.* (1996) investigated numerically the behavior of these waves in arbitrary depth away from their bifurcation point. Longuet-Higgins (1993) and Akylas (1993) demonstrated that gravity-capillary solitary waves occur when the phase velocity of the carrier wave is equal to the group velocity of the envelope-soliton of the NLS equation. Note that the linear dispersion relation of gravity-capillary waves has a minimum in phase velocity which is equal to the group velocity. For values of the phase velocity less than this extremum the NLS equation admits envelope-soliton solutions such that the wave crests are stationary in the frame of reference of the wave envelope. Later on, steady three-dimensional gravity-capillary solitary waves have been investigated by several authors, among them Parau *et al.* (2005), Kim & Akylas (2005) and Milewski (2005).

Gravity-capillary solitary waves have been observed experimentally in wind waves generated in a wind wave facility by Zhang & Cox (1994). During laboratory experiments (Zhang (1995)) and field experiments (Zhang (1999)) on wind waves on deep water,

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gravity-capillary solitary waves were observed and found to closely match numerical profiles obtained by Longuet-Higgins (1989) and Vanden-Broeck & Dias (1992). Longuet-Higgins & Zhang (1997) generated experimentally gravity-capillary solitary waves on deep water by the resonant excitation of surface waves at a velocity below the minimum of gravity-capillary waves. The mechanism of generation is due to a jet of air impinging on the surface of steady current. Forced three-dimensional gravity-capillary solitary waves on deep water (called lumps) have been investigated experimentally by several authors (Dorio *et al.* (2011), Masnadi & Duncan (2017*a*), Masnadi & Duncan (2017*b*) and Park & Cho (2016)).

To the best of our knowledge, there is no study regarding the effect of constant vorticity on the transverse instability of two-dimensional envelope solitons. On the contrary, many investigations on the transverse instability have been carried out in the absence of vorticity. Zakharov & Rubenchik (1974) and Saffman & Yuen (1978) demonstrated that plane gravity envelope solitons are unstable to long wave transverse perturbations. Later on, Ablowitz & Segur (1979) extended their analyses to show equivalent results in the case of finite depth. Deconinck *et al.* (2006) obtained a variety of transverse instabilities of gravity envelope solitons in infinite depth. Within the framework of the fully nonlinear potential flow equations Kim & Akylas (2007) found that plane gravity-capillary solitary waves of depression are unstable to transverse perturbations. Kim (2012) verified the long wave transverse instability within the framework of the weakly nonlinear third-order truncation model. The presence of transverse instability suggests the existence of solitary lumps which are the three-dimensional counterpart of plane gravity-capillary solitary waves. Our purpose is not to consider three-dimensional localized solitary gravity-capillary waves. Akers & Milewski (2009), Akers & Milewski (2010), Wang & Milewski (2012) and Milewski & Wang (2014) used several model equations to investigate numerically the stability of two-dimensional and three-dimensional gravity-capillary solitary waves in deep water. Regarding two-dimensional waves they found that both gravity-capillary solitary waves of elevation and depression are unstable to transverse perturbations.

The purpose of the present paper is twofold: (i) to revisit the problem of two-dimensional gravity-capillary waves of solitary type on deep water when the effect of an underlying vertically sheared current is taken into account and (ii) to investigate the stability of one-dimensional gravity-capillary solitons, solution of the NLS equation, to transverse perturbation in the presence of constant vorticity.

Section 2 is devoted to the computation of the bifurcated gravity-capillary solitary wave on deep water in the presence of constant vorticity. In section 3 the stability of its envelope to infinitesimal transverse perturbations is investigated.

2. Gravity-capillary solitary waves with constant vorticity

We consider envelope soliton bifurcation from a uniform gravity-capillary wave train in the presence of vorticity. Our study is confined to two-dimensional water waves propagating in infinite depth. The fluid is considered inviscid and incompressible. The waves move along the x-axis and the z-axis is oriented upwards opposite to the gravity $\mathbf{g} = (0, -g)$. The origin $z = 0$ is the undisturbed free surface. The waves are travelling at the surface of a vertically sheared current of constant vorticity. In the fixed frame the underlying current is given by $U(z) = U_0 + \Omega z$ where Ω is the shear intensity and U_0 is the current velocity at the surface. Note that the vorticity is $-\Omega$. We choose a reference frame moving with the horizontal velocity U_0. Consequently, in this frame of reference the current at the surface vanishes and the underlying current is given by $U(z) = \Omega z$.

Very recently, Hsu *et al.* (2018) derived a nonlinear Schrödinger equation for the envelope of two-dimensional gravity-capillary waves propagating on arbitrary depth at the free surface of a vertically sheared current of constant vorticity $-\Omega$. Herein, we consider the case of infinite depth. This NLS equation for the complex envelope of the elevation is written as follows

$$ia_\tau + \alpha a_{\xi\xi} = \gamma^* \mid a \mid^2 a, \qquad (2.1)$$

where a is the complex envelope of the free surface elevation which depends on slow variables $\xi = \varepsilon(x - c_g t)$ and $\tau = \varepsilon^2 t$. The group velocity of the carrier wave of wavenumber, k, and frequency, ω, is c_g. We consider weakly nonlinear waves and consequently we set $\varepsilon \ll 1$.

The dispersion coefficient α reads

$$\alpha = -\frac{\omega}{k^2(2 + \Omega/\omega)}(\rho^2 - \frac{\kappa}{1+\kappa}\alpha_1),$$

with

$$\alpha_1 = (1 + \Omega/\omega)(1 + 2\rho) + 2(\rho - 2\frac{\kappa}{1+\kappa}(1 + \Omega/\omega)),$$

where

$$\rho = \frac{1 + \Omega/\omega}{2 + \Omega/\omega}(1 + \frac{2\kappa}{1+\kappa}).$$

The inverse of the Bond number is

$$\kappa = \frac{k^2 T}{\rho_w g},$$

where T is the surface tension coefficient and ρ_w the water density.

The nonlinear coefficient γ^* is

$$\gamma^* = \frac{g^2}{4\omega^2}(\frac{1+\kappa}{1+\Omega/\omega})^2\gamma,$$

where

$$\gamma = \frac{k^4}{2\omega(1+\Omega/\omega)(2+\Omega/\omega)}\left[-\frac{3\kappa}{1+\kappa}(1+\Omega/\omega)^2\right.$$
$$+ (1+\Omega/\omega)(8+6\Omega/\omega) + \frac{1+\Omega/\omega}{1-\kappa(2+3\Omega/\omega)}\gamma_1$$
$$\left. + 2\frac{(1+\Omega/\omega)(2+\Omega/\omega)^2}{\rho\Omega/\omega - (1+\Omega/\omega)/(1+\kappa)}(\Omega/\omega)^2\right],$$

and

$$\gamma_1 = 12\kappa + 10(1+4\kappa)\Omega/\omega + 6(3+7\kappa)(\Omega/\omega)^2 + 2(4+7\kappa)(\Omega/\omega)^3 + (1+\kappa)(\Omega/\omega)^4.$$

Equation (2.1) admits the following solution

$$a = \pm a_0 \text{sech}\{a_0(\frac{-\gamma^*}{2\alpha})^{1/2}\xi\} \exp(-\frac{i}{2}\gamma^* a_0^2 \tau),$$

and the elevation of a weakly nonlinear wave train of envelope $a(\xi, \tau)$ writes

$$\zeta = \pm\frac{1}{2}\varepsilon a_0 \text{sech}\{a_0(\frac{-\gamma^*}{2\alpha})^{1/2}\xi\} \exp[ikx - i(\omega + \frac{1}{2}\gamma^* \varepsilon^2 a_0^2)t] + c.c.,$$

where k and ω are the carrier wavenumber and carrier frequency, $\varepsilon \ll 1$ and *c.c.* denotes the complex conjugation. The wave height at the origin is εa_0.

The crests of the wave group travel at phase velocity

$$c = \frac{\omega}{k} + \frac{1}{2}\gamma^* \varepsilon^2 a_0^2/k,$$

whereas the envelope travels at group velocity c_g .

The condition for the envelope-soliton to be a solitary wave is that the group velocity is equal to the phase velocity $c_g = c$

$$c_g = \frac{\omega}{k} + \frac{1}{2}\gamma^* \varepsilon^2 a_0^2/k. \tag{2.2}$$

Whatever Ω, the nonlinear coefficient of the NLS equation is negative (see figure 1), therefore the solitary wave travels at a constant velocity which is a little less than the minimum phase velocity of linear gravity-capillary waves.

Following Longuet-Higgins (1993), we set $g = 1$, $T = 1$ and $\rho_w = 1$ which is equivalent to use $(T/\rho_w g)^{1/2}$ and $(T/\rho_w g^3)^{1/4}$ as unit of length and unit of time, respectively. For the sake of clarity the same symbols are used to define the dimensionless frequency, vorticity and wavenumber. Then, the nonlinear coefficient γ^* becomes in dimensionless form

$$\gamma^*(k) = (\frac{1+k^2}{\omega+\Omega})^2\gamma/4,$$

where

$$\gamma = \frac{k^4}{2\omega(1+\Omega/\omega)(2+\Omega/\omega)}\left[-\frac{3k^2}{1+k^2}(1+\Omega/\omega)^2\right.$$
$$+ (1+\Omega/\omega)(8+6\Omega/\omega) + \frac{1+\Omega/\omega}{1-k^2(2+3\Omega/\omega)}\gamma_1$$
$$\left.+ 2\frac{(1+\Omega/\omega)(2+\Omega/\omega)^2}{\rho\Omega/\omega - (1+\Omega/\omega)/(1+k^2)}(\Omega/\omega)^2\right],$$

with

$$\gamma_1 = 12k^2 + 10(1+4k^2)\Omega/\omega + 6(3+7k^2)(\Omega/\omega)^2 + 2(4+7k^2)(\Omega/\omega)^3 + (1+k^2)(\Omega/\omega)^4,$$

and

$$\rho = \frac{1+\Omega/\omega}{2+\Omega/\omega}(1+\frac{2k^2}{1+k^2}).$$

The dimensionless linear dispersion relation is

$$\omega^2 + \Omega\omega - k(1+k^2) = 0.$$

The dimensionless linear phase velocity and group velocity are equal at the linear phase velocity minimum which occurs when k is solution of the following equation

$$k^2 + \Omega\sqrt{2}\,k^{1/2} - 1 = 0. \tag{2.3}$$

For the case $\Omega = 0$ which was considered by Longuet-Higgins (1993) we find $k = 1$, $\omega = \sqrt{2}$ and $\gamma^*(k) = -11\sqrt{2}/32$. Then, the velocity of the solitary wave is

$$c_{g0} = \sqrt{2}(1 - \frac{11}{64}\varepsilon^2 a_0^2),$$

which is the equation (3.19) of Longuet-Higgins (1993).

Equation (2.3) admits the following aproximate solution

$$k = 1 - \frac{\Omega}{\sqrt{2}} + \mathcal{O}(\Omega^2). \tag{2.4}$$

Then

$$\omega = \sqrt{2} - \frac{3}{2}\Omega + \mathcal{O}(\Omega^2).$$

Negative vorticity ($\Omega > 0$) reduces k and ω whereas positive vorticity ($\Omega < 0$) increases k and ω.

An approximate solution of equation (2.2) is

$$c_g = \sqrt{2} - \frac{\Omega}{2} - (\frac{11\sqrt{2}}{64} + \frac{31}{64}\Omega)\varepsilon^2 a_0^2 + \mathcal{O}(\Omega^2, \varepsilon^3 a_0^3),$$

which can be rewritten as follows

$$c_g = c_{g0} - \frac{\Omega}{2}(1 + \frac{31}{32}\varepsilon^2 a_0^2) + \mathcal{O}(\Omega^2, \varepsilon^3 a_0^3). \tag{2.5}$$

Solitary wave velocity is increased when the vorticity is positive ($\Omega < 0$) and reduced when the vorticity is negative ($\Omega > 0$).

To extend the work of Akylas (1993), we investigate the bifurcation of a train of periodic gravity-capillary waves into a solitary wave with decaying oscillatory tail in the presence of constant vorticity. Now, we use $T/(\rho_w c^2)$ and $T/(\rho_w c^3)$ as unit length and unit time where c is the phase velocity of the envelope-soliton, which means that c is the reference phase velocity scale. Note that these time and length units were used by Vanden-Broeck & Dias (1992) and Akylas (1993). Consequently, with these new scaled variables the dimensionless form of the NLS equation (2.1) becomes

$$i\tilde{a}_{\tilde{\tau}} + \tilde{\alpha}\tilde{a}_{\tilde{\xi}\tilde{\xi}} = \beta^2 \tilde{\gamma}^* \mid \tilde{a} \mid^2 \tilde{a}, \tag{2.6}$$

where $\beta = gT/(\rho_w c^4)$, $\tilde{\tau} = \rho_w c^3 \tau/T$, $\tilde{\xi} = \rho_w c^2 \xi/T$, $\tilde{a} = \rho_w c^2 a/T$, $\tilde{\alpha} = \rho_w c\alpha/T$ and $\tilde{\gamma}^* = Tc\gamma^*/(\rho_w g^2)$.

Using $\tilde{\omega} = T\omega/(\rho_w c^3)$, $\tilde{k} = Tk/(\rho_w c^2)$, $\tilde{\Omega} = T\Omega/(\rho_w c^3)$ and $\kappa = \tilde{k}^2/\beta$ the coefficient $\tilde{\gamma}^*$ is

$$\tilde{\gamma}^* = \frac{\tilde{\omega}\tilde{k}^4(1 + \tilde{k}^2/\beta)^2}{8(\tilde{\omega} + \tilde{\Omega})^3(2\tilde{\omega} + \tilde{\Omega})}\Big\{-\frac{3\tilde{k}^2(\tilde{\omega} + \tilde{\Omega})^2}{\tilde{\omega}^2(\beta + \tilde{k}^2)} + \frac{(\tilde{\omega} + \tilde{\Omega})(8\tilde{\omega} + 6\tilde{\Omega})}{\tilde{\omega}^2} + \frac{(\tilde{\omega} + \tilde{\Omega})}{\beta\tilde{\omega} - \tilde{k}^2(2\tilde{\omega} + 3\tilde{\Omega})}(\beta\tilde{\gamma_1})$$

$$+ \frac{2(\tilde{\omega} + \tilde{\Omega})(2\tilde{\omega} + \tilde{\Omega})^2\tilde{\Omega}^2}{\tilde{\omega}^4(\tilde{\rho}\tilde{\Omega} - \beta(\tilde{\omega} + \tilde{\Omega})/(\beta + \tilde{k}^2))}\Big\},$$

where

$$\beta\tilde{\gamma_1} = \beta(10\Delta + 18\Delta^2 + 8\Delta^3 + \Delta^4) + \tilde{k}^2(12 + 40\Delta + 42\Delta^2 + 14\Delta^3 + \Delta^4), \qquad \Delta = \frac{\tilde{\Omega}}{\tilde{\omega}},$$

and

$$\tilde{\rho} = \frac{(\tilde{\omega} + \tilde{\Omega})(1 + 2\kappa/(1 + \kappa))}{2\tilde{\omega} + \tilde{\Omega}}.$$

Let $\tilde{\mu} = \beta^2 \tilde{\gamma}^*$. For $\tilde{\Omega} = 0$, we have checked that $-\tilde{\mu}$ is equal to the expression derived by Akylas (1993). In figure 1 is plotted the nonlinear coefficient $\tilde{\mu}$ as a function of Ω. An enlargement is inserted in the figure in the vicinity of $\Omega \approx -0.2$ to show that this
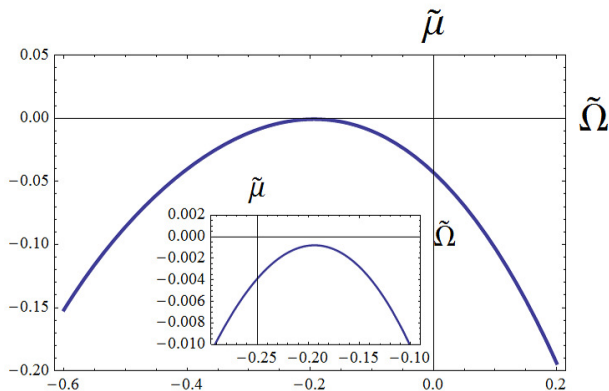
FIGURE 1. Dimensionless nonlinear coefficient, $\tilde{\mu}$, of the NLS equation as a function of the vorticity.

coefficient is always negative.

The dimensionless form of the dispersive coefficient is

$$\tilde{\alpha} = -\frac{\tilde{\omega}^2}{\tilde{k}^2(2\tilde{\omega}+\tilde{\Omega})}(\tilde{\rho}^2 - \frac{\tilde{k}^2}{\beta+\tilde{k}^2}\tilde{\alpha_1}),$$

where

$$\tilde{\rho} = \frac{\tilde{\omega}+\tilde{\Omega}}{2\tilde{\omega}+\tilde{\Omega}}(1+\frac{2\tilde{k}^2}{\beta+\tilde{k}^2}),$$

and

$$\tilde{\alpha_1} = \frac{(\tilde{\omega}+\tilde{\Omega})(1+2\tilde{\rho})}{\tilde{\omega}} + 2(\tilde{\rho} - \frac{2\tilde{k}^2(\tilde{\omega}+\tilde{\Omega})}{\tilde{\omega}(\beta+\tilde{k}^2)}).$$

The dimensionless NLS equation (2.6) becomes, dropping the tildes

$$ia_\tau + \alpha a_{\xi\xi} = \mu \mid a \mid^2 a, \tag{2.7}$$

and the free surface elevation writes

$$\zeta = \pm\frac{1}{2}\varepsilon a_0 \text{sech}\{a_0(\frac{-\mu}{2\alpha})^{1/2}\xi\}\exp[ikx - i(\omega + \frac{1}{2}\mu\varepsilon^2 a_0^2)t] + c.c.$$

The condition for the envelope-soliton to be a solitary wave is, due to the chosen normalisation, $c_g = c = 1$

$$1 = \frac{\omega}{k} + \frac{1}{2}\mu\varepsilon^2 a_0^2/k.$$

Hence,

$$k = \omega(k, \Omega, \beta) + \frac{1}{2}\mu\varepsilon^2 a_0^2. \tag{2.8}$$

For infinite depth, the dimensionless linear gravity-capillary dispersion relation in the presence of a shear of intensity Ω is

$$\omega^2 + \Omega\omega - \beta k - k^3 = 0, \tag{2.9}$$

which has a minimum in phase velocity at $k = k_0$. At this minimum the phase velocity is equal to the group velocity

$$k_0 = \omega(k_0, \Omega, \beta_0) \qquad \text{and} \qquad \frac{\partial \omega}{\partial k}(k_0, \Omega, \beta_0) = 1.$$

Solving these two equations gives

$$k_0 = \frac{1}{2}, \tag{2.10}$$

and

$$\beta_0 = \frac{1}{4} + \Omega.$$

For $\Omega = 0$ the values of Akylas (1993) are rediscovered. Note that the shear has no effect on the dimensionless value of k_0. Nevertheless, with the scaled variables used by Longuet-Higgins (1993) equation (2.10) reduces to equation (2.4) at $\mathcal{O}(\Omega)$.

Herein, $\omega > 0$ and consequently $\omega_0 = 1/2$. From the dispersion relation (2.9), it can be demonstrated that we always have $-\Omega < \omega$ or $-\omega < \Omega$. In our case the carrier wave frequency is close to ω_0 and as a result the considered values of Ω are larger than a value close to $-1/2$.

The wavenumber k and β are expanded about (k_0, β_0) as follows

$$k = k_0 + k_1 \varepsilon^2 + \mathcal{O}(\varepsilon^3),$$
$$\beta = \beta_0 + \beta_1 \varepsilon^2 + \mathcal{O}(\varepsilon^3).$$

Expanding $\omega(k, \Omega, \beta)$ about (k_0, β_0) and using equation (2.8) with $\partial \omega / \partial k (k_0, \Omega, \beta_0) = 1$ gives

$$\beta_1 = -\frac{1}{2} \frac{\mu(k_0, \Omega, \beta_0)}{\partial \omega / \partial \beta (k_0, \Omega, \beta_0)} a_0^2,$$

and

$$k_1 = -\frac{\frac{\partial^2 \omega}{\partial \beta \partial k}(k_0, \Omega, \beta_0)}{\frac{\partial^2 \omega}{\partial k^2}(k_0, \Omega, \beta_0)} \beta_1.$$

Using

$$\frac{\partial^2 \omega}{\partial k^2}(k_0, \Omega, \beta_0) = 2\alpha(k_0, \Omega, \beta_0),$$

we obtain

$$k_1 = -\frac{\frac{\partial^2 \omega}{\partial \beta \partial k}(k_0, \Omega, \beta_0)}{2\alpha(k_0, \Omega, \beta_0)} \beta_1.$$

Let $\delta\beta$ be the difference between β and the critical value β_0

$$\delta\beta = \beta_1 \varepsilon^2 + \mathcal{O}(\varepsilon^3),$$

$$\delta\beta = -\frac{1}{2} \frac{\mu(k_0, \Omega, \beta_0)}{\partial \omega / \partial \beta (k_0, \Omega, \beta_0)} a_0^2 \varepsilon^2.$$

Knowing that

$$\partial \omega / \partial \beta (k_0, \Omega, \beta_0) = \frac{1}{2(1 + \Omega)},$$

then

$$a_0^2 \varepsilon^2 = -\frac{\delta\beta}{(1+\Omega)\mu(k_0, \Omega, \beta_0)}. \tag{2.11}$$

From equation (2.11) and figure 1 we can see that the wave height decreases as positive values of Ω increases. For negative values of Ω the wave height presents a maximum in the vicinity of $\Omega \approx -0.2$. Finally,

$$a_0^2 \varepsilon^2 = \frac{16^2 \delta\beta}{11 + 128\Omega + 464\Omega^2 + 512\Omega^3 + 192\Omega^4}. \tag{2.12}$$

Knowing that

$$\frac{\partial^2 \omega}{\partial\beta\partial k}(k_0, \Omega, \beta_0) = \frac{\Omega}{(1+\Omega)^2},$$

and

$$\frac{\partial^2 \omega}{\partial k^2}(k_0, \Omega, \beta_0) = 2\alpha(k_0, \Omega, \beta_0) = \frac{1}{1+\Omega},$$

then

$$k_1 = -\frac{\Omega}{1+\Omega}\beta_1,$$

and

$$k = k_0 + k_1\varepsilon^2 = k_0 - \frac{\Omega}{1+\Omega}\beta_1\varepsilon^2 = k_0 - \frac{\Omega}{1+\Omega}\delta\beta. \tag{2.13}$$

Note that in the absence of vorticity the dimensionless carrier wavenumber is unchanged at $\mathcal{O}(\varepsilon^2)$. Negative vorticity ($\Omega > 0$) reduces k whereas positive vorticity ($\Omega < 0$) increases k.

The expression of the solitary wave is

$$\zeta(x,t) = \pm 16 \left(\frac{\delta\beta}{11 + 128\Omega + 464\Omega^2 + 512\Omega^3 + 192\Omega^4}\right)^{1/2} \text{sech}\{(\delta\beta)^{1/2}(x-t)\} \cos\{k(x-t)\}. \tag{2.14}$$

The profile of the solitary wave is known once the bifurcation parameter $\delta\beta$ is fixed. Equations (2.12) and (2.13) allows the determination of the parameter εa_0 and k, respectively. Profiles of solitary waves of elevation and depression which bifurcate at $\beta = \beta_0$ from a linear uniform periodic wave train of frequency ω_0 and wavenumber k_0, are plotted in figures 2, 3, 4 and 5. For negative vorticity ($\Omega > 0$), an increase of Ω leads to a decrease of the wave height (see figures 2 and 4). For positive vorticity ($\Omega < 0$), an increase of $|\Omega|$ may lead to an increase (see figures 3, 5, 6 and 7) of wave height.

3. Transverse instability of the envelope of gravity-capillary solitary waves with constant vorticity

The two-dimensional version of the NLS equation in deep water (2.1) is

$$ia_\tau + \alpha a_{\xi\xi} + \lambda a_{\eta\eta} + 2\Gamma \mid a \mid^2 a = 0, \tag{3.1}$$

where η is the transverse coordinate and $\Gamma = -\gamma^*/2$.

The transverse dispersion coefficient λ reads

$$\lambda = \frac{\omega'}{2k} = \frac{g(1+3\kappa)}{4k(\omega + \Omega/2)}.$$
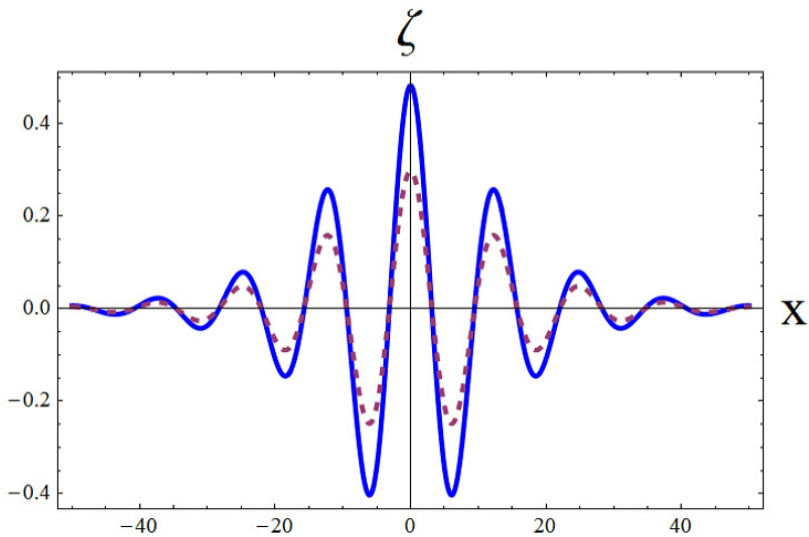
FIGURE 2. Dimensionless profiles of solitary waves of elevation: Solid line ($\Omega = 0$) and dashed line ($\Omega = 0.10$).
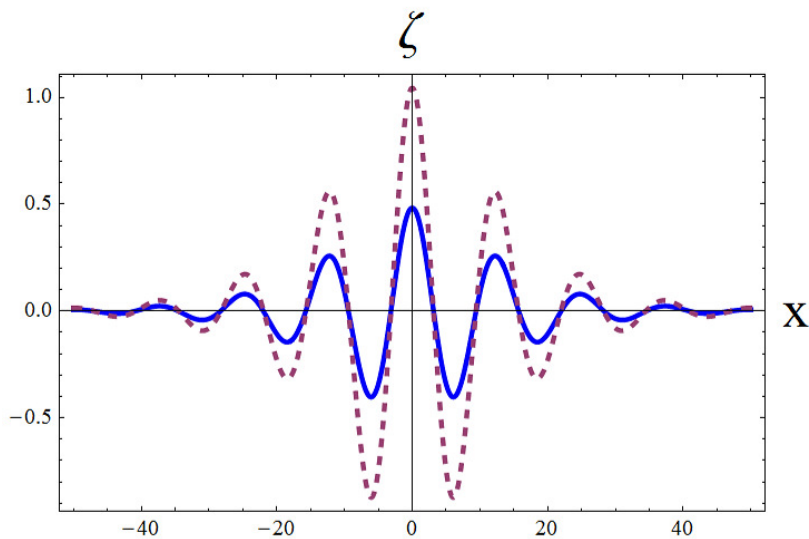

FIGURE 3. Dimensionless profiles of solitary waves of elevation: Solid line ($\Omega = 0$) and dashed line ($\Omega = -0.1$).

For $\partial/\partial\eta = 0$, equation (3.1) admits the following solution

$$\bar{a} = A(\xi)\exp(i\Gamma a_0^2 \tau), \tag{3.2}$$

where

$$A(\xi) = a_0 \mathrm{sech}\{a_0(\frac{\Gamma}{\alpha})^{1/2}\xi\}.$$
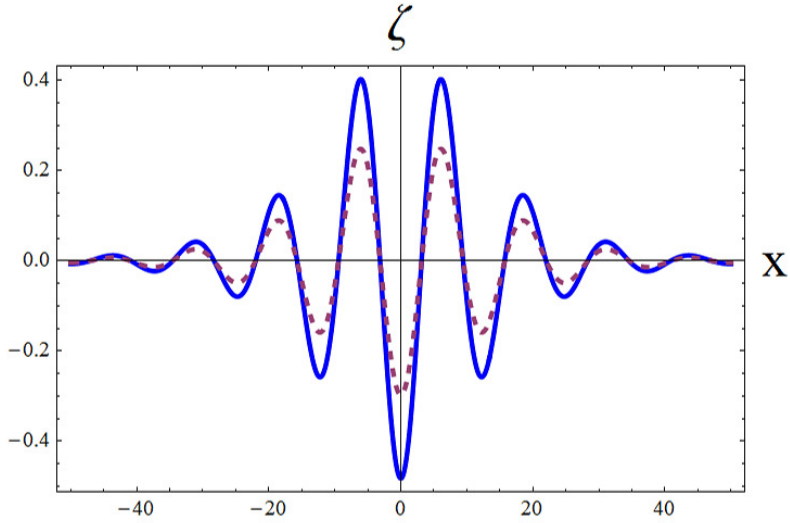
FIGURE 4. Dimensionless profiles of solitary waves of depression: Solid line ($\Omega = 0$) and dashed line ($\Omega = 0.10$).
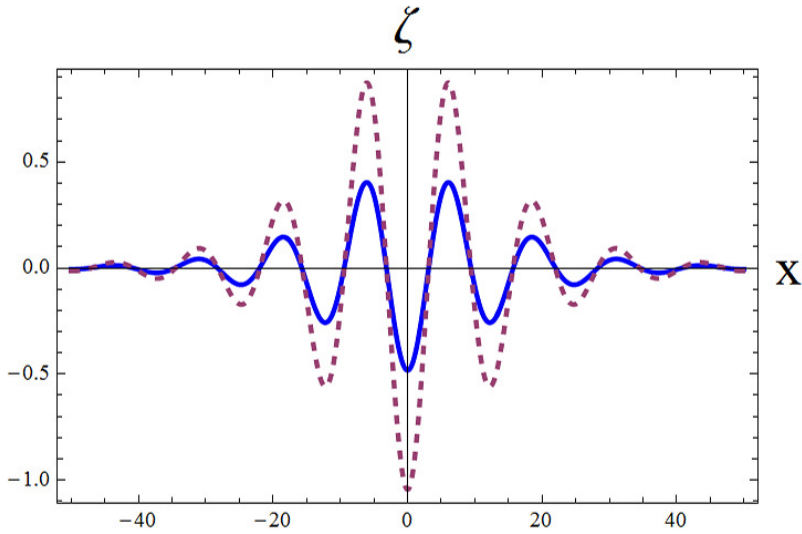

FIGURE 5. Dimensionless profiles of solitary waves of depression: Solid line ($\Omega = 0$) and dashed line ($\Omega = -0.1$).

Let us perturb with an infinitesimal perturbation the solution (3.2)

$$a = \bar{a} + a' \exp(i\Gamma a_0^2 \tau).$$

Linearisation of equation (3.1) about $\bar{a}$ gives

$$i a'_\tau - \Gamma a_0^2 a' + \alpha a'_{\xi\xi} + \lambda a'_{\eta\eta} = -2\Gamma A^2 a'^* - 4\Gamma A^2 a'. \tag{3.3}$$
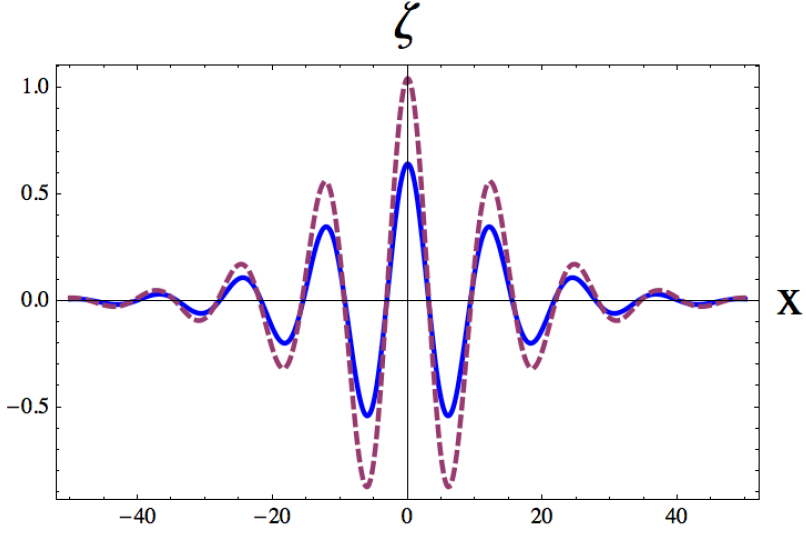
FIGURE 6. Dimensionless profiles of solitary waves of elevation: Solid line ($\Omega = -0.10$) and dashed line ($\Omega = -0.4$).
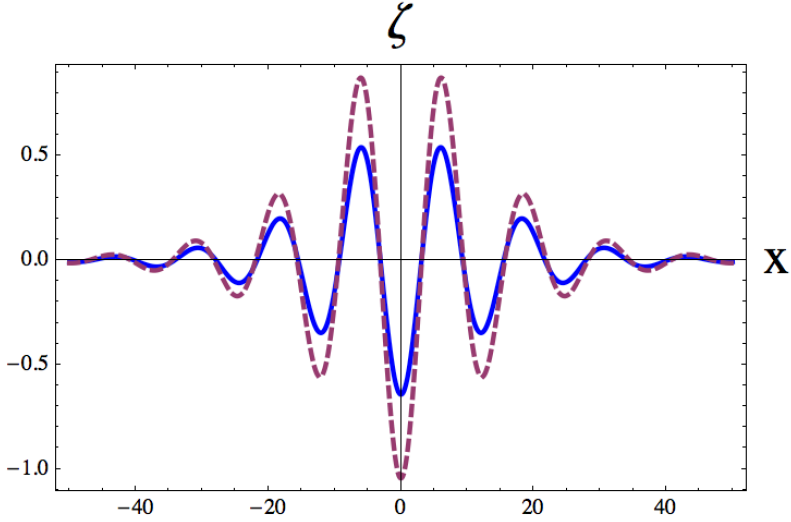

FIGURE 7. Dimensionless profiles of solitary waves of depression: Solid line ($\Omega = -0.10$) and dashed line ($\Omega = -0.4$).

Let us consider $a' = u + iv$ (with u and v reals) and substitute this expression into (3.3)

$$v_\tau = \alpha u_{\xi\xi} + \lambda u_{\eta\eta} - (-6\Gamma A^2 + \Gamma a_0^2)u, \tag{3.4}$$

$$u_\tau = -\alpha v_{\xi\xi} - \lambda v_{\eta\eta} + (-2\Gamma A^2 + \Gamma a_0^2)v. \tag{3.5}$$

The system (3.4)-(3.5) of linear partial differential equations with constant coefficients admits solutions of the following forms (normal modes)

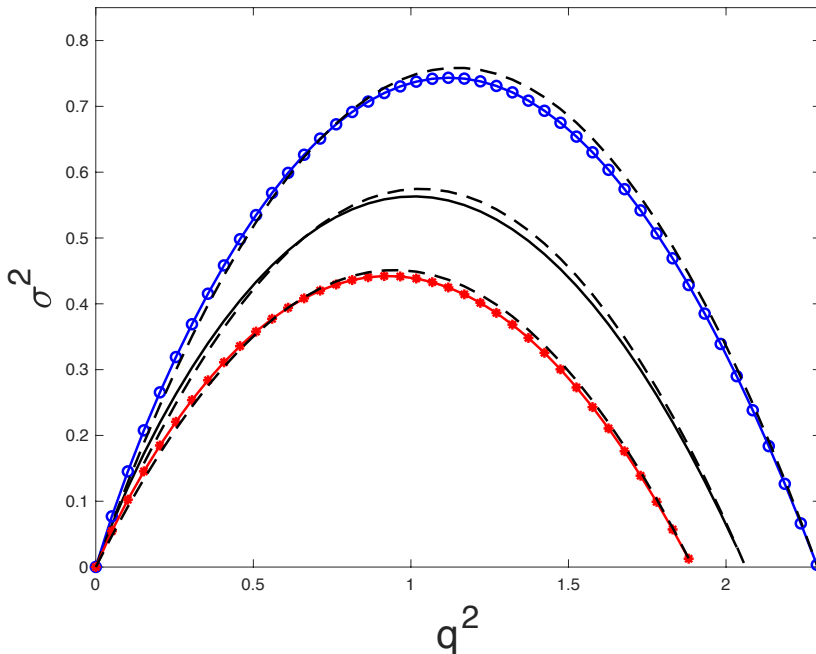$$u = \hat{u}(\xi) \exp(iq\eta + \sigma\tau), \tag{3.6}$$

FIGURE 8. Square of the dimensionless growth rate of the transverse unstable mode of the envelope of gravity-capillary solitary waves in deep water as a function of the square of the dimensionless wavenumber for $\Omega = 0$ (solid line), $\Omega = 0.1$ (∘) and $\Omega = -0.1$ (⋆). The dashed lines correspond to the analytic expression given by equation (3.9). $a_0 = \sqrt{2}$.

$$v = \hat{v}(\xi)\exp(iq\eta + \sigma\tau). \tag{3.7}$$

Herein we can use the complex forms for u and v because we deal with linear equations. Substitution of (3.6) and (3.7) into (3.5) and (3.4) gives

$$\sigma\hat{v} = \alpha\frac{d^2\hat{u}}{d\xi^2} - \lambda q^2\hat{u} - (-6\Gamma A^2 + \Gamma a_0^2)\hat{u},$$

$$\sigma\hat{u} = -\alpha\frac{d^2\hat{v}}{d\xi^2} + \lambda q^2\hat{v} + (-2\Gamma A^2 + \Gamma a_0^2)\hat{v}.$$

The above system can be written as follows

$$\sigma^2\hat{u} = -(L_0 - \lambda q^2)(L_1 - \lambda q^2)\hat{u}, \tag{3.8}$$

with

$$L_0 = \alpha\frac{d^2}{d\xi^2} + 2\Gamma A^2 - \Gamma a_0^2,$$

and

$$L_1 = \alpha\frac{d^2}{d\xi^2} + 6\Gamma A^2 - \Gamma a_0^2,$$

and the eigenfunction $\hat{u}$ vanishes as $|\xi| \to \infty$.

The numerical method to solve equation (3.8) is outlined in the Appendix.

We set $g = 1$, $T = 1$ and $\rho_w = 1$. For moderate and small values of the vorticity the coefficients of the NLS equation (3.1) are positive, $\Gamma > 0$, $\alpha > 0$ and $\lambda > 0$. For comparison we use herein the analytic relation of the instability growth rate given by

Rypdal & Rasmussen (1989) which writes in dimensionless form

$$\sigma^2 = \frac{64}{\pi^2}\lambda\Gamma q^2(1 - \frac{\lambda}{6\Gamma}q^2). \tag{3.9}$$

The maximum growth rate is

$$\sigma_{\max}^2 = \frac{96\Gamma^2}{\pi^2},$$

and

$$q_{\max} = \sqrt{\frac{3\Gamma}{\lambda}}.$$

The gravity-capillary solitary wave is marginally stable for $q = q_c$ with

$$q_c^2 = \frac{6\Gamma}{\lambda}. \tag{3.10}$$

For small values of the vorticity the instability growth rate and marginal wavenumber are

$$\sigma^2 = q^2(\frac{11}{\pi^2} - \frac{16}{3\pi^2}q^2) + q^2(\frac{153 + 32q^2}{6\sqrt{2}\,\pi^2})\Omega + \mathcal{O}(\Omega^2),$$

and

$$q_c = \frac{33}{16} + \frac{87}{33\sqrt{2}}\Omega + \mathcal{O}(\Omega^2).$$

In figure 8 is shown the numerical and analytic instability growth rates for several values of the vorticity. The solid lines correspond to numerical solutions of equation (3.8). Negative vorticity amplifies the rate of growth whereas positive vorticity reduces its value. Surface elevation of the envelope of gravity-capillary solitary waves perturbed by their most unstable modes corresponding to $\Omega = 0$, $q = 1.01$ and $\Omega = 0.10$, $q = 1.08$ are shown in figure 9. These waves which are localized in the direction of propagation and periodic in the transverse direction have been investigated by Milewski & Wang (2014). This new solution found by the latter authors was called transversally periodic plane solitary wave.

4. Conclusion

Within the framework of the $1D-$ and $2D-$NLS equations, the effect of constant vorticity on the properties and stability of gravity-capillary solitary waves in deep water has been investigated. These two-dimensional solitary waves of elevation and depression types bifurcate from linear uniform wavetrains at the phase speed minimum where the phase velocity equals the group velocity. We found that negative vorticity ($\Omega > 0$) reduces the wave height whereas positive vorticity ($\Omega < 0$) increases the wave height. Numerical and analytic instability growth rates of transverse perturbations have been computed and derived, respectively. We have shown that negative vorticity amplifies the rate of growth and bandwidth of transverse instability whereas positive vorticity diminishes both the rate of growth and bandwidth.

Appendix. Numerical method to solve equation (3.8)

The eigenfunctions $\hat{u}$ are computed numerically using the interpolant function (Tang (1993); Boyd (1989); Weideman & Reddy (2000)).
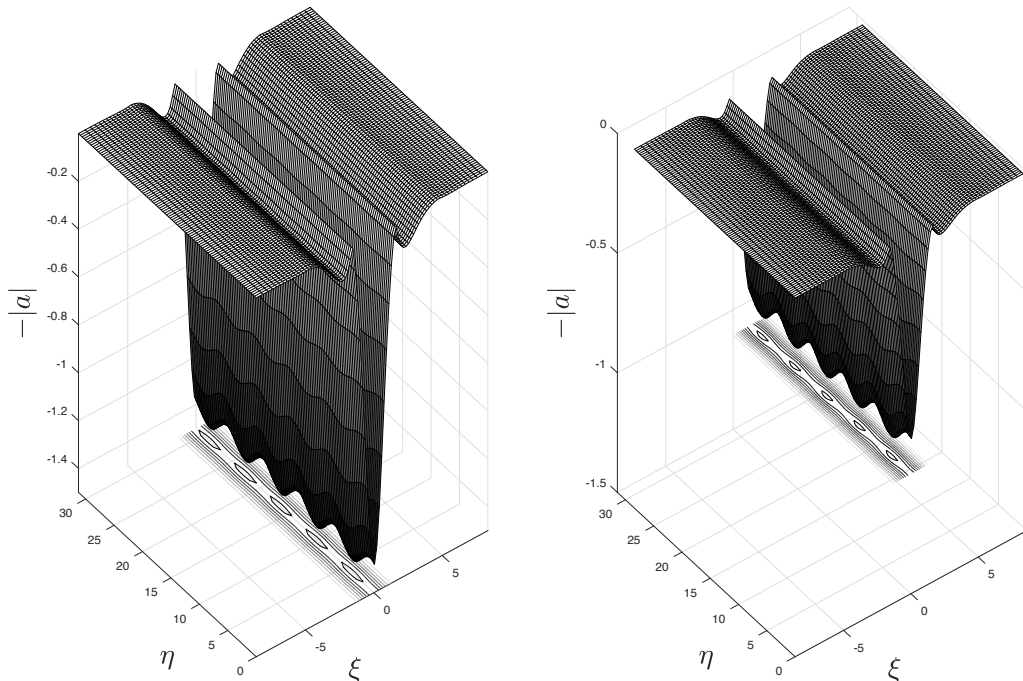
FIGURE 9. Surface elevations of perturbed envelopes of gravity-capillary solitary waves of depression in deep water for $\Omega = 0$, $\kappa = 1$, $q_{max} = 1.01$ (left) and $\Omega = 0.10$, $\kappa = 0.86$, $q_{max} = 1.08$ (right)

$$\hat{u}_N(x) = \sum_{j=1}^{N} \frac{e^{(-x^2/2)}}{e^{(-x_j^2/2)}} \phi_j(x) \hat{u}_j,$$

where

$$\phi_j(x) = \frac{H_N(x)}{H_N'(x_j)(x - x_j)},$$

and $H_N(x)$ is the Hermite polynomial of degree N. The grid points x_1, ..., x_N are the roots of $H_N(x)$. Note that for continuous eigenfunctions on $\mathbb{R}$ vanishing at infinity they can be uniformly approximated by functions of the form $\exp(-x^2)p(x)$ where p is a polynomial (see Lang (1991) page 62).

This spectral interpolant automatically enforces the vanishing of the eigenfunctions at $\pm\infty$. We used $N = 160$ grid points giving a domain half size $L = 8.61$ ($x \in [-L, L]$). With this choice, we obtained $|\hat{u}_N(\pm L)| = O(10^{-8})$.

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