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Yan LONG

Jingyi XUE
Singapore Management University, JYXUE@smu.edu.sg

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Yan Long, Jingyi Xue

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Parametric rationing with uncertain needs (Preliminary and incomplete)

Yan Long
Huazhong University of Science and Technology
ylong@hust.edu.cn

Jingyi Xue
Singapore Management University
jyxue@smu.edu.sg

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Abstract

We study resource allocation in the face of uncertain needs. We extend Young (1987)'s parametric rules to the uncertain context. We re-establish the axiomatic characterization of parametric rules and show the optimality of the rules.

1 Setting

A single and perfectly divisible resource is to be divided among a group of agents. Agents are denoted by elements in the set $\mathbb{N}$ of natural numbers. Let $\mathcal{N}$ be the set of all finite subsets of $\mathbb{N}$. An agent's uncertain need, called a **claim**, of the resource is modeled as a cumulative distribution function with its support a closed interval on $\mathbb{R}_+$. Let $\mathcal{F}$ be the set of possible claims. For each $i \in \mathbb{N}$, we denote by $F_i \in \mathcal{F}$ a typical claim of agent i, and c_i and C_i the maximal and minimal values in the support of F_i. Note that c_i is the amount of the resource that agent i needs for sure, and is called the **sure need** of agent i; C_i is the maximal amount of the resource that agent i would possibly need, and is called the **maximal need** of agent i.

An **allocation problem** is a pair (F, T), where $F \in \mathcal{F}^I$ is a list of claims that are indexed by some $I \in \mathcal{N}$, and $T \in \mathbb{R}_+$ is the total endowment to be allocated. A **solution** of the problem is a vector $t \in \mathbb{R}^I_+$ such that $\sum t_i \leq T$ and $0 \leq t_i \leq C_i$ for all $i \in I$. An original feature of this model is that we allow the endowment not to be fully allocated. As argued in Long, Sethuraman, Xue (2019), when agents have uncertain satiation points, it may not be desirable to fully allocate the resource. Whether and to what extent agents should be satiated are hence questions to be answered.

We denote by $\mathcal{P}^I$ the set of problems with population I. An **allocation rule** is a function r that assigns to every problem with any finite population a solution. For each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each $i \in I$, we denote by $r_i(F, T)$ agent i's assignment given by r.

2 Parametric rules

In the bankruptcy/rationing literature where agents have deterministic claims, the class of so-called parametric rules is introduced and characterized by Young (1987). This is an important class since it unifies all rules satisfying some basic axioms with a general parametric representation. Replacing deterministic claims with abstract types for agents' characteristics, the same class of rules is formulated and characterized by Kaminski (2000, 2006). The types of our agents are simply their (uncertain) claims. Below we adapt their definition of parametric rules to our setting.

Parametric rules: Let $\underline{\alpha}$, $\overline{\alpha} \in \mathbb{R} \cup \{-\infty, \infty\}$ be such that $\underline{\alpha} < \overline{\alpha}$. A function $f : \mathcal{F} \times [\underline{\alpha}, \overline{\alpha}] \to \mathbb{R}$ is called a **parametric function** if for each $F_i \in \mathcal{F}$, $f(F_i, \cdot)$ is non-decreasing and continuous with $f(F_i, \underline{\alpha}) = 0$ and $f(F_i, \overline{\alpha}) \leq C_i$. For each parametric function f, define r^f as follows. For each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each $j \in I$,

$$r^f_j(F, T) := f(F_j, \alpha^*), \quad \text{where } \alpha^* \in [\underline{\alpha}, \overline{\alpha}] \text{ satisfies } \sum f(F_i, \alpha^*) = \min\{T, \sum f(F_i, \overline{\alpha})\}.$$

We call r^f the **parametric rule with the parametric function** f.

The key difference between our definition and that of Young (1987) and Kaminski (2000, 2006) is that each of our parametric rules determines, in addition to a parametric way of rationing, a maximal assignment for each agent. Specifically, for each $F_i \in \mathcal{F}$, we allow $f(F_i, \overline{\alpha})$, the maximal assignment of agent i, to be any amount no larger than his maximal need C_i. Thus, our parametric rules associated with different parametric functions choose different maximal assignments for agents. In contrast, both Young (1987) and Kaminski (2000, 2006) require the maximal assignment of an agent to be an exogenously fixed amount that depends only on the agent's type, not chosen by rules. Moreover, we further modify their definition by requiring the sum of agents' assignments in a problem to be either the endowment or the sum of agents' maximal assignments, whichever is smaller.

3 Characterization

We extend some familiar axioms to our setting. The first axiom, *symmetry*, is a reflection of Aristotle's famous principle "equal treatment of equals".

Symmetry: For each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each $i, j \in I$, if $F_i = F_j$, then $r_i(F, T) = r_j(F, T)$.

The second axiom, *continuity*, requires that small changes in a problem not lead to large changes in the chosen allocation. We adopt the same convergence concept as in Long, Sethuraman, and Xue (2019). For each $i \in \mathbb{N}$,each $F_i \in \mathcal{F}$, and each sequence $\{F_i^n\}_{n=1}^{\infty}$ of elements of $\mathcal{F}$, we say that F_i^n converges to F_i if F_i^n converges weakly to F_i, $\lim c_i^n = c_i$, and $\lim C_i^n = C_i$. For each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each sequence $\{(F^n, T^n)\}_{n=1}^{\infty}$ of elements of $\mathcal{P}^I$, we say that (F^n, T^n) converges to (F, T), denoted by $(F^n, T^n) \to (F, T)$, if for each $i \in I$, F_i^n converges to F_i, and $\lim T^n = T$.

Continuity: For each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each sequence $\{(F^n, T^n)\}_{n=1}^{\infty}$ of elements of $\mathcal{P}^I$, if $(F^n, T^n) \to (F, T)$, then $\lim r(F^n, T^n) = r(F, T)$.

A weakening of continuity, *endowment continuity*, is obtained by considering only small changes of the endowment.

Endowment continuity: For each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each sequence $\{(F, T^n)\}_{n=1}^{\infty}$ of elements of $\mathcal{P}^I$, if $\lim T^n = T$, then $\lim r(F, T^n) = r(F, T)$.

Our last axiom, *consistency*, formulates an invariance principle that has played a central role in resource allocation with a variable population. In our uncertain setting, since endowment is not necessarily exhausted, there are two reasonable ways in specifying the amount to be divided among the remaining agents. The endowment in the reduced problem could be the sum of the amounts initially assigned to the remaining agents, or the difference between the initial endowment and the sum of the assignments to the agents who leave. We require the invariance principle to hold in both cases.

Consistency: For each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each $J \subseteq I$,

$$r_J(F, T) = r(F_J, \sum_{j \in J} r_j(F, T)) = r(F_J, T - \sum_{j \in I \setminus J} r_j(F, T)),$$

where $r_J(F, T)$ and F_J are, respectively, the restriction of $r(F, T)$ and F onto J.

Remember that a question unique to our setting is that when and how an endowment should be used only partially. In defining parametric rules, we impose a maximum assignment for each agent, require each of them to get his maximum when the endowment is greater than the total maximum, and require the resource to be exhausted when the endowment cannot meet all the maximum. The following result, first presented in Long, Sethuraman, and Xue (2019), shows that, surprisingly, a combination of the above axioms implies these same natural properties.

Lemma 1. *(Long, Sethuraman, and Xue (2019)) Let r be a symmetric, endowment continuous, and consistent rule. Then there is a function $M : \mathcal{F} \to \mathbb{R}_+$ such that for each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each $i \in I$, (1) $T < \sum M(F_j) \Longrightarrow \sum r_j(F, T) = T$ and $r_i(F, T) \leq M(F_i)$, and (2) $T \geq \sum M(F_j) \Longrightarrow r_i(F, T) = M(F_i)$.*

A further implication of Lemma 1 is that a rule satisfying the axioms in our setting can be think of as doing two separate things. First it determines maximum, and then it determines how to rationing when the endowment cannot meet all the maximum. Combining Lemma 1 and the result of Kanmiski, 2002, we can characterize the parametric rules.

Theorem 1. *A rule is symmetric, continuous, and consistent if and only if it is a parametric rule with a continuous parametric function.*

In order to prove the theorem, we first define a metric on $\mathcal{F}$ and prove that the corresponding metric space is separable.

Let $\mathbb{Q}_+$ be the set of non-negative rational numbers. A claim is a Borel probability measure on $\mathbb{R}_+$, represented by a CDF. We assume that each claim has a compact support.

Let F_i denote a typical claim, and c_i and C_i the minimal and the maximal values of the support of F_i. Let $\mathcal{F}$ be the set of such claims. Define $d : \mathcal{F} \to \mathbb{R}$ by setting for each pair $F_i, F_j \in \mathcal{F}$,

$$d(F_i, F_j) := d_{LP}(F_i, F_j) + |c_i - c_j| + |C_i - C_j|,$$

where d_{LP} is the Lévy-Prokhorov metric on the set of Borel probability measures on $\mathbb{R}$. It can be readily seen that d is a metric on $\mathcal{F}$.

Lemma 2. *The metric space $(\mathcal{F}, d)$ is separable.*

Proof. For each pair $r_1, r_2 \in \mathbb{Q}_+$ with $r_1 < r_2$, let $\mathcal{F}^{[r_1,r_2]} := \{F_i \in \mathcal{F} : [c_i, C_i] \subseteq [r_1, r_2]\}$. Since the set of Borel probability measures on $\mathbb{R}$ equipped with the Lévy-Prokhorov metric d_{LP} is separable, and since each subspace of a separable metric space is separable, for each pair $r_1, r_2 \in \mathbb{Q}_+$ with $r_1 < r_2$, the metric space $(\mathcal{F}^{[r_1,r_2]}, d_{LP})$ is separable. Thus, there is a countable dense subset of $\mathcal{F}^{[r_1,r_2]}$, denoted by $\mathcal{D}^{[r_1,r_2]}$. Let $\mathcal{D} := \bigcup_{r_1,r_2 \in \mathbb{Q}_+ : r_1 < r_2} \mathcal{D}^{[r_1,r_2]}$. It can be readily seen that $\mathcal{D}$ is countable and $\mathcal{D} \subseteq \mathcal{F}$.

We show that $\mathcal{D}$ is dense in $(\mathcal{F}, d)$. Let $F_i \in \mathcal{F}$ and $\epsilon > 0$. We need to show that there is $F_j \in \mathcal{D}$ such that $d(F_i, F_j) < \epsilon$. Let $r_1 \in \mathcal{Q}_+ \cap [c_i - \frac{\epsilon}{4}, c_i]$ and $r_2 \in \mathcal{Q}_+ \cap (C_i, C_i + \frac{\epsilon}{4})$. Then $r_1 \leq c_i \leq C_i < r_2$, and thus, $F_i \in \mathcal{F}^{[r_1,r_2]}$. Since $F_i \in \mathcal{F}^{[r_1,r_2]}$ and $\mathcal{D}^{[r_1,r_2]}$ is dense in $(\mathcal{F}^{[r_1,r_2]}, d_{LP})$, there is a sequence $\{F_n\}_{n=1}^{\infty}$ of elements of $\mathcal{D}^{[r_1,r_2]}$ such that $\lim_{n\to\infty} d_{LP}(F_i, F_n) = 0$. Let $x_i \in (c_i, c_i + \frac{\epsilon}{4})$ and $x'_i \in (C_i - \frac{\epsilon}{4}, C_i)$ be such that F_i is continuous at x_i and x'_i. Since weak convergence of probability measures on $\mathbb{R}$ is equivalent to convergence in the metric d_{LP}, and since $\lim_{n\to\infty} d_{LP}(F_i, F_n) = 0$ and F_i is continuous at x_i and x'_i, $\lim_{n\to\infty} F_n(x_i) = F_i(x_i)$ and $\lim_{n\to\infty} F_n(x'_i) = F_i(x'_i)$. Since $x_i > c_i$ and $x'_i < C_i$, $F_i(x_i) > 0$ and $F_i(x'_i) < 1$. Thus, there is $N \in \mathbb{N}$ such that for each $n \geq N$, $F_n(x_i) > 0$, $F_n(x'_i) < 1$, and thus, $c_n \leq x_i < c_i + \frac{\epsilon}{4}$ and $C_n \geq x'_i > C_i - \frac{\epsilon}{4}$. Since for each $n \in \mathbb{N}$, $F_n \in \mathcal{D}^{[r_1,r_2]}$, $c_n \geq r_1 \geq c_i - \frac{\epsilon}{4}$ and $C_n \leq r_2 < C_i + \frac{\epsilon}{4}$. Thus, for each $n \geq N$, $c_n \in [c_i - \frac{\epsilon}{4}, c_i + \frac{\epsilon}{4})$ and $C_n \in (C_i - \frac{\epsilon}{4}, C_i + \frac{\epsilon}{4})$, and hence, $|c_i - c_n| \leq \frac{\epsilon}{4}$ and $|C_i - C_n| \leq \frac{\epsilon}{4}$. Moreover, since $\lim_{n\to\infty} d_{LP}(F_i, F_n) = 0$, there is $N' \in \mathbb{N}$ such that for each $n \geq N'$, $d_{LP}(F_i, F_n) < \frac{\epsilon}{2}$. Let $j \geq \max\{N, N'\}$. Then $d(F_i, F_j) = d_P(F_i, F_j) + |c_i - c_j| + |C_i - C_j| < \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon$, as desired. □

Now we prove Theorem 1.

Proof. Let r be a *symmetric*, *continuous*, and *consistent* rule. By Lemma 1, there is a function $M : \mathcal{F} \to \mathbb{R}_+$ such that for each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each $i \in I$, (1)

$T < \sum M(F_j) \Rightarrow \sum r_j(F,T) = T$ and $r_i(F,T) \le M(F_i)$, and (2) $T \ge \sum M(F_j) \Rightarrow r_i(F,T) = M(F_i)$. Since $(\mathcal{F}, d)$ is a separable metric space, by Theorem 1 of Kaminski (2006), when restricted to $\bigcup_{I \in \mathcal{N}} \{(F,T) \in \mathcal{P}^I : T \le \sum M(F_i)\}$, r is representable by a continuous parametric function. Therefore, by our definition of a parametric rule, on the entire domain $\bigcup_{I \in \mathcal{N}} \mathcal{P}^I$, r is a parametric rule with a continuous parametric function. □

In the deterministic claims problems, each parametric rule is shown by Young (1987a) to minimize a particular social cost function, and the same result is obtained by Stovall (2013) for asymmetric parametric rules. The optimality of parametric rules remains true in the uncertain context. But different from Young (1987a) and Stovall (2013), the objective function that rationalizes a parametric rule has to admit optimal allocations that do not necessarily fully allocate a resource when agents have uncertain satiation points. This is achieved by constructing individual cost function that consists of two parts: one part increases with the assignment, and the other part decreases with the assignment. Moreover, when agent i's assignment is no more than the maximal amount determined by the parametric rule, an increase of his assignment leads to more decrease in the second part of cost than the increase in the first. Once agent i receives his maximal assignment, a further increase of his assignment leads to more increase in the first part of cost than the decrease in the second. Therefore, it is optimal to assign to agent i no more than the maximal amount specified by the parametric rule. (See the proof of the next theorem for details.)

Theorem 2. *A rule r is symmetric, continuous, and consistent if and only if there is a continuous function $H : \{(F_i, t_i) : F_i \in \mathcal{F}, t_i \in [0, C_i]\} \to \mathbb{R}$ that is strictly convex in the second variable such that for each $I \in \mathcal{N}$ and each $(F,T) \in \mathcal{P}^I$, $r(F,T)$ is the unique solution to the following optimization problem*

$$\min \sum H(F_i, t_i) \text{ subject to } \sum t_i \le T \text{ and for each } i \in I,\ 0 \le t_i \le C_i. \tag{1}$$

Proof. The "if" direction is readily verified, so we omit the proof. To show the "only if" direction, let r be a *symmetric*, *continuous*, and *consistent* rule. By Theorem 1, r is a parametric rule with a continuous parametric function, denoted by f. Without loss of generality, we can assume that the second variable of f takes values between 0 and 1.[1] Thus, $f : \mathcal{F} \times [0,1] \to \mathbb{R}$ is non-decreasing in the second variable satisfying that for each

[1] See e.g., Remark 1 of Kaminski (2006).

$F_i \in \mathcal{F}$, $f(F_i, 0) = 0$ and $f(F_i, 1) \leq C_i$. Moreover, for each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each $j \in I$,

$$r_j(F, T) := f(F_j, \alpha^*), \text{ where } \alpha^* \in [0, 1] \text{ satisfies } \sum f(F_i, \alpha^*) = \min\left\{T, \sum f(F_i, 1)\right\}.$$

For each $F_i \in \mathcal{F}$ and each $t_i \in [0, f(F_i, 1)]$, let $f^{-1}(F_i, t_i) := \{\alpha \in [0, 1] : f(F_i, \alpha) = t_i\}$, and since $f(F_i, \cdot)$ is non-decreasing and continuous on $[0, 1]$, $f^{-1}(F_i, t_i)$ is a non-empty closed interval. Define $h : \{(F_i, t_i) : F_i \in \mathcal{F}, t_i \in [0, C_i]\} \to [0, 1]$ by setting for each $F_i \in \mathcal{F}$ and each $t_i \in [0, C_i]$,

$$h(F_i, t_i) := \begin{cases} \min f^{-1}(F_i, t_i) & t_i \in [0, f(F_i, 1)] \\ 1 + \frac{t_i - f(F_i, 1)}{C_i - f(F_i, 1)} & t_i \in (f(F_i, 1), C_i] \end{cases}.$$

It can be readily seen that for each $F_i \in \mathcal{F}$, $h(F_i, \cdot)$ is increasing.

Define $H : \{(F_i, t_i) : F_i \in \mathcal{F}, t_i \in [0, C_i]\} \to \mathbb{R}$ by setting for each $F_i \in \mathcal{F}$ and each $t_i \in [0, C_i]$,

$$H(F_i, t_i) := \int_0^{t_i} h(F_i, t_i')dt_i' + \int_{t_i}^{C_i} [2 - h(F_i, t_i')]dt_i'. \tag{2}$$

For each $F_i \in \mathcal{F}$, since $h(F_i, \cdot)$ is increasing, $H(F_i, \cdot)$ is strictly convex. Moreover,

$$H(F_i, t_i) = 2\int_0^{t_i} [h(F_i, t_i') - 1]dt_i' + \int_0^{C_i} [2 - h(F_i, t_i')]dt_i'. \tag{3}$$

For each $F_i \in \mathcal{F}$, since $h(F_i, t_i) \leq 1$ when $t_i \leq f(F_i, 1)$ and $h(F_i, t_i) > 1$ when $t_i > f(F_i, 1)$, and since $h(F_i, \cdot)$ is increasing, $H(F_i, \cdot)$ is decreasing on $[0, f(F_i, 1)]$ and increasing on $[f(F_i, 1), C_i]$.

Let $I \in \mathcal{N}$ and $(F, T) \in \mathcal{P}^I$. Assume that $T \geq \sum f(F_i, 1)$. Then $r(F, T) = (f(F_i, 1))_{i \in I}$. Since for each $i \in I$, $H(F_i, \cdot)$ is decreasing on $[0, f(F_i, 1)]$ and increasing on $[f(F_i, 1), C_i]$, $r(F, T) = (f(F_i, 1))_{i \in I}$ is the unique solution to the optimization problem (1).

Assume that $T < \sum f(F_i, 1)$. Then there is $\alpha^* \in [0, 1]$ such that $r(F, T) = (f(F_i, \alpha^*))_{i \in I}$ with $\sum f(F_i, \alpha^*) = T$. First, assume that t^* solves the optimization problem (1). We show that $\sum t_i^* = T$ and there is $\alpha \in [0, 1]$ such that for each $i \in I$, $t_i^* = f(F_i, \alpha)$. To see that $\sum t_i^* = T$, suppose to the contrary that $\sum t_i^* < T$. Since $\sum t_i^* < T < \sum f(F_i, 1)$, there is $j \in I$ such that $t_j^* < f(F_j, 1)$. Then there is $t_j \in (t_j^*, f(F_j, 1))$ such that $\sum\limits_{i \in I \setminus \{j\}} t_i^* + t_j \leq T$. Since $t_j \in (t_j^*, f(F_j, 1))$ and $H(F_j, \cdot)$ is decreasing on $[0, f(F_j, 1)]$, $\sum\limits_{i \in I \setminus \{j\}} H(F_i, t_i^*) + H(F_j, t_j) < \sum H(F_i, t_i^*)$, violating the optimality of t^*, as desired.

We check that there is $\alpha \in [0,1]$ such that for each $i \in I$, $t_i^* = f(F_i, \alpha)$. Since for each $i \in I$, $H(F_i, \cdot)$ is increasing on $[f(F_i, 1), C_i]$, and since t^* solves the optimization problem (1), for each $i \in I$, $t_i^* \leq f(F_i, 1)$, and thus, $f^{-1}(F_i, t_i^*)$ is well-defined. Since for each $i \in I$, $f^{-1}(F_i, t_i^*)$ is a non-empty closed interval, to show that there is $\alpha \in [0,1]$ such that for each $i \in I$, $t_i^* = f(F_i, \alpha)$, it is sufficient to show that $\max_{i \in I} \min f^{-1}(F_i, t_i^*) \leq \min_{i \in I} \max f^{-1}(F_i, t_i^*)$. Suppose to the contrary that $\max_{i \in I} \min f^{-1}(F_i, t_i^*) > \min_{i \in I} \max f^{-1}(F_i, t_i^*)$. Let $j, k \in I$ be such that $\min f^{-1}(F_j, t_j^*) = \max_{i \in I} \min f^{-1}(F_i, t_i^*)$ and $\max f^{-1}(F_k, t_k^*) = \min_{i \in I} \max f^{-1}(F_i, t_i^*)$. Since $\min f^{-1}(F_j, t_j^*) > \max f^{-1}(F_k, t_k^*)$, $j \neq k$. Let $\alpha' \in [0,1]$ be such that $\min f^{-1}(F_j, t_j^*) > \alpha' > \max f^{-1}(F_k, t_k^*)$. Thus, $t_j^* > f(F_j, \alpha')$ and $f(F_k, \alpha') > t_k^*$. Let $\epsilon > 0$ be such that $t_j^* - \epsilon > f(F_j, \alpha')$ and $f(F_k, \alpha') > t_k^* + \epsilon$. Note that $f(F_j, 1) \geq t_j^* > t_j^* - \epsilon > f(F_j, \alpha') \geq f(F_j, 0) = 0$ and $f(F_k, 1) \geq f(F_k, \alpha') > t_k^* + \epsilon > 0$. Thus, $f^{-1}(F_j, t_j^* - \epsilon)$ and $f^{-1}(F_k, t_k^* + \epsilon)$ are well-defined. Since $t_j^* - \epsilon > f(F_j, \alpha')$ and $f(F_j, \cdot)$ is non-decreasing, $h(F_j, t_j^* - \epsilon) = \min f^{-1}(F_j, t_j^* - \epsilon) > \alpha'$. Since $f(F_k, \alpha') > t_k^* + \epsilon$ and $f(F_k, \cdot)$ is non-decreasing, $h(F_k, t_k^* + \epsilon) = \min f^{-1}(F_k, t_k^* + \epsilon) < \alpha'$. Since $h(F_j, t_j^* - \epsilon) > \alpha' > h(F_k, t_k^* + \epsilon) \geq 0$ and $\epsilon > 0$, and since $h(F_j, \cdot)$ and $h(F_k, \cdot)$ are increasing, $\int_{t_j^*-\epsilon}^{t_j^*} h(F_j, t_j')dt_j' > \alpha'\epsilon > \int_{t_k^*}^{t_k^*+\epsilon} h(F_k, t_k')dt_k'$ Thus,

$$
\begin{aligned}
&H(F_j, t_j^*) + H(F_k, t_k^*) - H(F_j, t_j^* - \epsilon) - H(F_k, t_k^* + \epsilon) \\
=&2 \int_{t_j^*-\epsilon}^{t_j^*} [h(F_j, t_j') - 1]dt_j' - 2 \int_{t_k^*}^{t_k^*+\epsilon} [h(F_k, t_k') - 1]dt_k' \\
=&2 \int_{t_j^*-\epsilon}^{t_j^*} h(F_j, t_j')dt_j' - 2 \int_{t_k^*}^{t_k^*+\epsilon} h(F_k, t_k')dt_k' > 0.
\end{aligned}
$$

Therefore, $\sum_{i \in I \setminus j,k} H(F_i, t_i^*) + H(F_j, t_j^* - \epsilon) + H(F_k, t_k^* + \epsilon) > \sum H(F_i, t_i^*)$, violating the optimality of t^*, as desired.

Recall that $\sum r_i(F, T) = T$ and there is $\alpha^* \in [0,1]$ such that for each $i \in I$, $r_i(F, T) = f(F_i, \alpha^*)$. Note that $\sum H(F_i, \cdot)$ is continuous on $\{t \in \mathbb{R}^I : \sum t_i \leq T \text{ and for each } i \in I, 0 \leq t_i \leq C_i\}$. Thus, there is a solution to the optimization problem (1). Let t' denote the solution. By the previous arguments, $\sum t_i' = T$ and there is $\alpha' \in [0,1]$ such that for each $i \in I$, $t_i' = f(F_i, \alpha')$. Since for each $i \in I$, $f(F_i, \cdot)$ is non-decreasing and $\sum r_i(F, T) = \sum t_i' = T$, $r(F, T) = t'$. Therefore, $r(F, T)$ solves the optimization problem (1). Since $\sum H(F_i, \cdot)$ is strictly convex on $\{t \in \mathbb{R}^I : \sum t_i \leq T \text{ and for each } i \in I, 0 \leq t_i \leq C_i\}$, $r(F, T)$ is the unique solution to the optimization problem (1).

Lastly, to show that H is continuous, let $\{(F_i^n, t_i^n)\}_{n=1}^{\infty}$ be a sequence of elements in $\{(F_i, t_i) : F_i \in \mathcal{F}, t_i \in [0, C_i]\}$ such that it converges, in the product topology, to (F_i^*, t_i^*)

where $F_i^* \in \mathcal{F}$ and $t_i^* \in [0, C_i^*]$. Thus, F_i^n converges weakly to F_i^*, $\lim c_i^n = c_i^*$, $\lim C_i^n = C_i^*$, and $\lim t_i^n = t_i^*$. We want to show that $\lim H(F_i^n, t_i^n) = H(F_i^*, t_i^*)$. Note that for each $F_i \in \mathcal{F}$ and each pair $t_i, t_i' \in [0, C_i]$, $h(F_i, t_i), h(F_i, t_i') \in [0, 2]$, and thus, by (3), $|H(F_i, t_i) - H(F_i, t_i')| \leq 2|t_i - t_i'|$. We divide the proof into the following three cases.

Case 1: $C_i^* = 0$. Then $\lim C_i^n = 0$ and $\lim t_i^n = t_i^* = 0$. Thus, $H(F_i^*, t_i^*) = 0$. Since $\lim t_i^n = 0$ and $\lim C_i^n = 0$, by (2), $\lim H(F_i^n, t_i^n) = \lim \int_0^{t_i^n} h(F_i^n, t_i)dt_i + \lim \int_{t_i^n}^{C_i^n} [2 - h(F_i^n, t_i)]dt_i = 0$. Hence, $\lim H(F_i^n, t_i^n) = H(F_i^*, t_i^*)$.

Case 2: $t_i^* < C_i^*$. Let $\delta > 0$ be such that $t_i^* < C_i^* - \delta$. Since $\lim C_i^n = C_i^*$, for sufficiently large $n \in \mathbb{N}$, $C_i^* - \delta < C_i^n$. Thus, for sufficiently large $n \in \mathbb{N}$, $h(F_i^n, \cdot)$ is well-defined on $[0, C_i^* - \delta]$. We claim that $\{h(F_i^n, \cdot)\}$ converges pointwise almost everywhere to $h(F_i^*, \cdot)$ on $[0, C_i^* - \delta]$. To see this, let $t_i \in [0, C_i^* - \delta]$. Assume that $t_i > f(F_i^*, 1)$. Then $h(F_i^*, t_i) = 1 + \frac{t_i - f(F_i^*, 1)}{C_i^* - f(F_i^*, 1)}$. Since f is continuous and $t_i > f(F_i^*, 1)$, for sufficiently large n, $t_i > f(F_i^n, 1)$. Since for sufficiently large n, $t_i \in (f(F_i^n, 1), C_i^n)$, $h(F_i^n, t_i) = 1 + \frac{t_i - f(F_i^n, 1)}{C_i^n - f(F_i^n, 1)}$. Since f is continuous and $\lim C_i^n = C_i^*$, it can be readily seen that $\lim h(F_i^n, t_i) = h(F_i^*, t_i)$, as desired. Assume that $t_i < f(F_i^*, 1)$ and $h(F_i^*, \cdot)$ is continuous at t_i. Then $h(F_i^*, t_i) = \min f^{-1}(F_i^*, t_i)$. Since f is continuous and $t_i < f(F_i^*, 1)$, for sufficiently large n, $t_i < f(F_i^n, 1)$, so that $h(F_i^n, t_i) = \min f^{-1}(F_i^n, t_i) \in [0, 1]$. Let α be the limit of a convergent subsequence $\{h(F_i^{n_m}, t_i)\}_{m=1}^{\infty}$ of the sequence $\{h(F_i^n, t_i)\}$. For each $m \in \mathbb{N}$, by the definition of h, $f(F_i^{n_m}, h(F_i^{n_m}, t_i)) = t_i$. Since $\lim h(F_i^{n_m}, t_i) = \alpha$ and f is continuous, $f(F_i^*, \alpha) = t_i$. Since $h(F_i^*, \cdot)$ is continuous at t_i, $f^{-1}(F_i^*, t_i)$ is a singleton and $h(F_i^*, t_i) = \alpha = \lim h(F_i^{n_m}, t_i)$. Since each convergent subsequence of $\{h(F_i^n, t_i)\}$ has the same limit $h(F_i^*, t_i)$, $\lim h(F_i^n, t_i) = h(F_i^*, t_i)$. Since $h(F_i^*, \cdot)$ is increasing, there are at most countably many points in $[0, f(F_i^*, 1))$ at which $h(F_i^*, \cdot)$ is not continuous. Since $\{h(F_i^n, \cdot)\}$ converges pointwise to $h(F_i^*, \cdot)$ on $(f(F_i^*, 1), C_i^* - \delta]$ and on $[0, f(F_i^*, 1))$ except for at most countably many points, $\{h(F_i^n, \cdot)\}$ converges pointwise almost everywhere to $h(F_i^*, \cdot)$ on $[0, C_i^* - \delta]$.

Since $\{h(F_i^n, \cdot)\}$ converges pointwise almost everywhere to $h(F_i^*, \cdot)$ on $[0, C_i^* - \delta]$, and since for sufficiently large n and for each $t_i \in [0, C_i^* - \delta]$, $h(F_i^n, t_i) \in [0, 2]$, by Lebesgue's dominated convergence theorem, $\lim \int_0^{t_i^*} h(F_i^n, t_i)dt_i = \int_0^{t_i^*} h(F_i^*, t_i)dt_i$ and $\lim \int_{t_i^*}^{C_i^* - \delta} [2 - h(F_i^n, t_i)]dt_i = \int_{t_i^*}^{C_i^* - \delta} [2 - h(F_i^*, t_i)]dt_i$. For each $\epsilon > 0$, let $\delta \in (0, \frac{\epsilon}{12})$ be such that $t_i^* < C_i^* - \delta$. Let $N \in \mathbb{N}$ be such that for each $n \geq N$, $C_i^* - \delta < C_i^n$, $|t_i^n - t_i^*| \leq \frac{\epsilon}{8}$, $|\int_0^{t_i^*} h(F_i^n, t_i)dt_i - \int_0^{t_i^*} h(F_i^*, t_i)dt_i| \leq \frac{\epsilon}{8}$, $|\int_{t_i^*}^{C_i^* - \delta} [2 - h(F_i^n, t_i)]dt_i - \int_{t_i^*}^{C_i^* - \delta} [2 - h(F_i^*, t_i)]dt_i| \leq \frac{\epsilon}{8}$, and $|C_i^n - C_i^*| \leq \delta$. Then for each $n \geq N$,

$$\begin{aligned}
&|H(F_i^n, t_i^n) - H(F_i^*, t_i^*)| \leq |H(F_i^n, t_i^n) - H(F_i^n, t_i^*)| + |H(F_i^n, t_i^*) - H(F_i^*, t_i^*)| \\
\leq& 2|t_i^n - t_i^*| + \left| \int_0^{t_i^*} h(F_i^n, t_i) dt_i - \int_0^{t_i^*} h(F_i^*, t_i) dt_i \right| + \left| \int_{t_i^*}^{C_i^*-\delta} [2 - h(F_i^n, t_i)] dt_i \right. \\
&\left. - \int_{t_i^*}^{C_i^*-\delta} [2 - h(F_i^*, t_i)] dt_i \right| + \left| \int_{C_i^*-\delta}^{C_i^n} [2 - h(F_i^n, t_i)] dt_i \right| + \left| \int_{C_i^*-\delta}^{C_i^*} [2 - h(F_i^*, t_i)] dt_i \right| \\
\leq& 2\frac{\epsilon}{8} + \frac{\epsilon}{8} + \frac{\epsilon}{8} + 2|C_i^n - C_i^* + \delta| + 2\delta \leq \frac{\epsilon}{2} + 2(|C_i^n - C_i^*| + \delta) + 2\delta = \frac{\epsilon}{2} + 6\delta < \epsilon.
\end{aligned}$$

Hence, $\lim H(F_i^n, t_i^n) = H(F_i^*, t_i^*)$.

Case 3: $t_i^* = C_i^* > 0$. For each $\epsilon > 0$, let $\delta \in (0, \frac{\epsilon}{8})$ be such that $t_i^* - \delta > 0$. Since $\lim t_i^n = t_i^*$ and $t_i^* - \delta > 0$, for sufficiently large $n \in \mathbb{N}$, $t_i^n - \delta \geq 0$. Since $t_i^* - \delta < C_i^*$, by the result in Case 2, $\lim H(F_i^n, t_i^n - \delta) = H(F_i^*, t_i^* - \delta)$. Let $N \in \mathbb{N}$ be such that for each $n \geq N$, $t_i^n - \delta \geq 0$ and $|H(F_i^n, t_i^n - \delta) - H(F_i^*, t_i^* - \delta)| \leq \frac{\epsilon}{2}$. Then for each $n \geq N$,

$$\begin{aligned}
&|H(F_i^n, t_i^n) - H(F_i^*, t_i^*)| \\
\leq& |H(F_i^n, t_i^n) - H(F_i^n, t_i^n - \delta)| + |H(F_i^n, t_i^n - \delta) - H(F_i^*, t_i^* - \delta)| + |H(F_i^*, t_i^* - \delta) - H(F_i^*, t_i^*)| \\
\leq& 2\delta + \frac{\epsilon}{2} + 2\delta < 2\frac{\epsilon}{8} + \frac{\epsilon}{2} + 2\frac{\epsilon}{8} = \epsilon.
\end{aligned}$$

Hence, $\lim H(F_i^n, t_i^n) = H(F_i^*, t_i^*)$. □

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