UNIVERSITI PUTRA MALAYSIA

CLASSIFICATION AND DERIVATIONS OF LOW-DIMENSIONAL COMPLEX DIALGEBRAS

WITRIANY BINTI BASRI

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CLASSIFICATION AND DERIVATIONS OF LOW-DIMENSIONAL COMPLEX DIALGEBRAS

By

WITRIANY BINTI BASRI

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

December 2014

DEDICATIONS

To

My husband and my children

Luqman Hakim bin Musa, Ihtisyam, Insyirah, Irdina and Ishamina.

For their great patience

My Lovely parents, Basri bin Minsan and Zulherni binti Jamil

and my little sister, Ritayuliana.

For their encouragement

and

My Dear Teachers

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

CLASSIFICATION AND DERIVATIONS OF LOW-DIMENSIONAL COMPLEX DIALGEBRAS

By

WITRIANY BINTI BASRI

December 2014

Chair: Professor Isamiddin S. Rakhimov, Ph.D.

Faculty: Science

The thesis is mainly comprised of two parts. In the first part we consider the classification problem of low-dimensional associative, diassociative and dendriform algebras. Since so far there are no research results dealing with representing diassociative and dendriform algebras in form of precise tables under some basis, it is desirable to have such lists up to isomorphisms. There is no standard approach to the classification problem of algebras. One of the approaches which can be applied is to fix a basis and represent the algebras in terms of structure constants. Due to the identities we have constraints for the structure constants in polynomial form. Solving the system of polynomials we get a redundant list of all the algebras from given class. Then we erase isomorphic copies from the list. It is slightly tedious to perform this procedure by hand. For this case we construct and use several computer programs. They are applied to verify the isomorphism between found algebras, to find automorphism groups and verify the algebra identities.

In conclusion, we give complete lists of isomorphism classes for diassociative and dendriform algebras in low dimensions. We found for diassociative algebras four isomorphism classes (one parametric family and another three are single class) in dimension two, 17 isomorphism classes (one parametric family and others are single classes) in dimension three and for nilpotent diassociative algebras we obtain 16 isomorphism classes (all of them are parametric family) in dimension four. In dendriform algebras case there are twelve isomorphism classes (one parametric family and another eleven are single classes) in dimension two.

The second part of the thesis is devoted to the computation of derivations of low-dimensional associative, diassociative and dendriform algebras. We give the derivations the above mentioned classes of algebras in dimensions two and three.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

PENGELASAN DAN TERBITAN BAGI DIMENSI RENDAH KOMPLEKS DWIALJABAR

Oleh

WITRIANY BINTI BASRI

Disember 2014

Pengerusi: Profesor Isamiddin S. Rakhimov, Ph.D.

Fakulti: Sains

Tesis ini terdiri daripada dua bahagian. Dalam bahagian pertama kita mempertimbangkan masalah pengelasan bagi dimensi rendah aljabar sekutuan, dwisekutuan dan dendriform. Oleh kerana setakat ini tiada hasil penyelidikan berkaitan dengan mewakili aljabar dwisekutuan dan dendriform dalam bentuk jadual di bawah beberapa asas, adalah wajar untuk mempunyai senarai tersebut hingga ke isomorfisma. Tiada pendekatan piawai kepada masalah pengelasan aljabar. Salah satu pendekatan yang boleh digunakan adalah dengan menetapkan asas dan mewakili aljabar dari segi pemalar struktur. Oleh kerana identiti tersebut, kami mempunyai kekangan untuk pemalar struktur dalam bentuk polinomial. Penyelesaian sistem polinomial, kami akan memperoleh senarai berlebihan untuk semua aljabar dari kelas yang diberikan. Kemudian kami memansuhkan salinan isomorfik dari senarai. Ia adalah sedikit merumitkan untuk melakukan prosedur ini secara manual. Untuk kes ini dalam penyelidikan, kami membina dan menggunakan beberapa program komputer. Ia digunakan untuk mengesahkan isomorfisma antara aljabar yang diperolehi, untuk mencari kumpulan automorfisma dan mengesahkan identiti aljabar.

Kesimpulannya, kami memberikan senarai lengkap kelas isomorfisma untuk aljabar dwisekutuan dan dendriform dalam dimensi rendah. Kami dapati dengan aljabar dwisekutuan, empat kelas isomorfisma (satu keluarga parametrik dan tiga lagi adalah kelas tunggal) dalam dimensi dua, 17 kelas isomorfisma (satu keluarga parametrik dan selainnya kelas tunggal) dalam tiga dimensi dan aljabar dwisekutuan nilpoten, kami mendapatkan 16 kelas isomorfisma (semuanya adalah keluarga parametrik) dalam dimensi empat. Dalam kes aljabar dendriform terdapat dua belas kelas isomorfisma (satu keluarga parametrik dan sebelas lagi adalah kelas

tunggal) dalam dimensi dua.

Bahagian kedua tesis ini adalah dikhaskan untuk pengiraan terbitan dimensi rendah aljabar sekutuan, dwisekutuan dan dendriform. Kami memberikan terbitan bagi kelas-kelas aljabar yang dinyatakan di atas dalam dimensi dua dan tiga.

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This acknowledgement will not be complete if I do not include my co-supervisor Prof. Dato' Dr. Kamel Ariffin M. Atan, the director of Institute for Mathematical Research (INSPEM) for all his support and guidance. Also to Dr. Mohammad Alinor bin Abdul Kadir who has generously contributed articles and books rarely found in Malaysia, and Associate Professor Dr. Mohamad Rushdan Md. Said as a member of supervisory committee for his cooperation. this thesis will not be possible without contributions from them.

My PhD study was supported by Universiti Putra Malaysia and therefore, I would like to take this opportunity to thank UPM for the financial support.

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As my final note, everything that I have been doing is intended to uplift the image of muslim and let it be my humble contribution to the ummah.

I certify that a Thesis Examination Committee has met on 12 December 2014 to conduct the final examination of Witriany binti Basri on her thesis entitled "Classification and Derivations of Low-Dimensional Complex Dialgebras" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doctor of Philosophy.

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Name of Chairman of Supervisory Committee: Assoc. Prof. Dr. Muhamad Rushdan Md Said

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CHAPTER 1
INTRODUCTION AND LITERATURE REVIEW

1.1 Introduction

This thesis is concerned with two problems. The first is the classification problem of low-dimensional dialgebras and another one is the description of derivations of these algebras. In classification problem of diassociative algebras, we consider the diassociative algebra as a combination of two associative algebras. The dendriform algebra characterized structure on a vector space is associative in multiplication, i.e., $x * y = x \prec y + x \succ y$. The categories of diassociative and dendriform algebras structures on n-dimensional vector space, we denote by $Dias_n$ and $Dend_n$, respectively. These classes of algebras have been introduced by Loday around 1990.

We begin this chapter by introducing basic concepts of algebra, diassociative and dendriform algebras, followed by literature review and research objectives.

1.2 Basic concepts

Let V be a vector space over a field, K and $\{e_1, e_2, \ldots, e_n\}$ be a basis of V. Then an algebra structure on V is defined by specifiying the products

$$e_ie_j = \sum_{k=1}^{n} \gamma_{ij}^{k} e_k, \quad \gamma_{ij}^{k} \in k, \quad 1 \leq i, j \leq n. \tag{1.1}$$

Indeed, (1.1) extends uniquely to a bilinear product on V by rule

$$(\sum_{i=1}^{n} b_ie_i)(\sum_{j=1}^{n} c_je_j) = \sum_{k=1}^{n}(\sum_{i,j=1}^{n} b_ic_j\gamma_{ij}^{k})e_k.$$

The n^3 elements $\gamma_{ij}^{k} \in K$ are called the *structure constants* of the multiplication that is defined by (1.1).

Every n-dimensional algebra A can be realized (up to isomorphism) by specifying suitable structure constants γ_{ij}^{k}. On the other hand, not all choices of structure constants yield special classes of algebras. Furthermore, different choices of the structure constants can give isomorphic algebras.

Analogously, the diassociative algebra structure on V is defined as follows. Let V be an n-dimensional vector space over a field K equipped with two bilinear associative binary operations, denoted by $\dashv$ and $\vdash$:

$$\dashv : V \times V \to V$$

and

$$\vdash : V \times V \to V.$$

If $\dashv$ and $\vdash$ satisfying the following axioms: $\forall a, b, c \in V$

$$\begin{aligned}(a \dashv b) \dashv c &= a \dashv (b \vdash c),\\ (a \vdash b) \dashv c &= a \vdash (b \dashv c),\\ (a \dashv b) \vdash c &= a \vdash (b \vdash c),\end{aligned} \tag{1.2}$$

then the triple $(V, \dashv, \vdash)$ is called a diassociative algebra. The operations $\dashv$ and $\vdash$ are called the left and right products, respectively.

Due to the axioms (1.2) the set of structure constants γ_{ij}^k and δ_{ij}^k form a closed with respect to Zarisski topology subset of $K^{n^3} \times K^{n^3}$. Thus $Dias_n$ can be considered as a subvariety in $2n^3$-dimensional affine space. This variety is denoted by $Dias_n$. Consider a natural action of $GL_n(V)$ on $Dias_n$ by changing a basis. This action can be expressed as follows:

if $g = [g_i^j] \in GL_n(K)$ and $D = \{\gamma_{ij}^k, \delta_{st}^r\}$, then

$$\{(g * D)_{ij}^k, \, (g * D)_{st}^r\} = \{g_i^p \cdot g_j^q \cdot (g_l^k)^{-1} \cdot \gamma_{pq}^t \, , \, g_s^p \cdot g_t^q \cdot (g_l^r)^{-1} \cdot \gamma_{pq}^t\}.$$

Two algebras D_1 and D_2 are isomorphic if and only if they belong to the same orbit under this action.

Definition 1.1 *A homomorphism of two dialgebras D and D_1 (provided both are given over the same field K) is a K-linear map $\phi : D \to D_1$ such that*

$$\phi(x \dashv y) = \phi(x) \dashv \phi(y) \;\; \text{and} \;\; \phi(x \vdash y) = \phi(x) \vdash \phi(y)$$

for all $x, y \in D$.

Remark 1.1 *As usual, ϕ is an isomorphism if it is a bijective homomorphism and ϕ is an automorphism if ϕ is an isomorphism and $D = D_1$.*

Let $O(D)$ be the set of laws isomorphic to D. It is called the orbit of D. Let fix a basis $\{e_1, e_2, \ldots e_n\}$ of V. Then

$$e_i \dashv e_j = \sum_k \gamma_{ij}^k e_k \quad \text{and} \quad e_i \vdash e_j = \sum_k \delta_{ij}^k e_k \tag{1.3}$$

for $i, j, k = 1, 2, 3, \ldots n$.

Once a basis is fixed, we can identify the law D with its structure constants. These constants γ_{ij}^k and δ_{ij}^k satisfy:

$$\sum_s \gamma_{ij}^t \gamma_{tk}^s = \sum_s \gamma_{it}^s \gamma_{jk}^t,$$
$$\sum_s \gamma_{ij}^t \gamma_{tk}^s = \sum_s \gamma_{it}^s \delta_{jk}^t,$$
$$\sum_s \delta_{ij}^t \gamma_{tk}^s = \sum_s \delta_{it}^s \gamma_{jk}^t,$$
$$\sum_s \gamma_{ij}^t \delta_{tk}^s = \sum_s \delta_{it}^s \delta_{jk}^t,$$
$$\sum_s \delta_{ij}^t \delta_{tk}^s = \sum_s \delta_{it}^s \delta_{jk}^t.$$

where $i, j, k, s, t = 1, 2, \ldots, n$.

Another class of algebras introduced by J.-L.Loday and (co)homologically closely related to this class is called a class of *dendriform algebras*.

Let V be an $n-$dimensional dendriform algebra. Dendriform algebra is an algebra equipped with two binary operations

$$\succ: V \times V \to V, \quad \text{and} \quad \prec: V \times V \to V$$

satisfying the following axioms:

$$\begin{aligned}(a \prec b) \prec c &= (a \prec c) \prec b + a \prec (b \succ c),\\ (a \succ b) \prec c &= a \succ (b \prec c),\\ (a \prec b) \succ c + (a \succ b) \succ c &= a \succ (b \succ c).\end{aligned}$$

$\forall a, b, c \in V$. The triple $(V, \succ, \prec)$ is called dendriform algebra.

A dendriform algebra in fixed basis $\{e_1, e_2, \ldots e_n\}$ can be written as follows.

$$e_i \prec e_j = \sum_s \alpha_{ij}^s e_s \quad \text{and} \quad e_l \succ e_p = \sum_q \beta_{lp}^t e_t, \tag{1.4}$$

for $1 \leq i, j, s, p, q, t \leq n$.

The structure constants α_{ij}^s and β_{lp}^t of the dendriform algebras satisfies the conditions

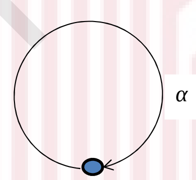
$$\sum_s \alpha_{ij}^s \alpha_{sk}^t = \sum_s (\alpha_{jk}^s \alpha_{is}^t + \alpha_{is}^t \beta_{jk}^s),$$
$$\sum_s \alpha_{sk}^t \beta_{ij}^s = \sum_s \alpha_{is}^t \beta_{jk}^s,$$
$$\sum_s (\alpha_{ij}^s \beta_{sk}^t + \beta_{ij}^s \beta_{sk}^t) = \sum_s \beta_{is}^t \beta_{jk}^s.$$

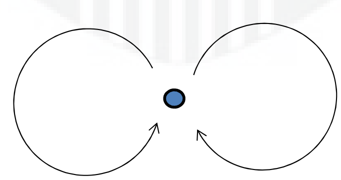
for $1 \leq i, j, k, s, t \leq n$.

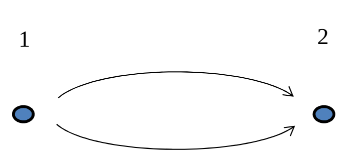
Since a diassociative algebra is a combination of two associative algebras, but an associative algebra is represented by quivers. Let us discuss brief on the quivers first. We assume that K is an algebraically closed field. All the results of this section have appeared elsewhere, particularly in Hazewinkel et al. (2007).

Definition 1.2 *A quiver $Q = (VQ, AQ, s, e)$ is a finite directed graph which consists of finite sets VQ, AQ and two mappings $s, e : AQ \to VQ$. The elements of VQ are called vertices (or points), and those of AQ are called arrows.*

Usually, the set of vertices VQ will be a set 1,2,...,n. We say that each arrow $\sigma \in AQ$ starts at the vertex $s(\sigma)$ and ends at the vertex $e(\sigma)$. The vertex $s(\sigma)$ is called the start (or initial, or source) vertex and the vertex $e(\sigma)$ is called the end (or target) vertex of σ. Some examples of quivers are:







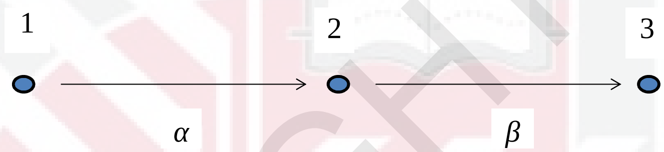
A quiver can be given by its adjacency (or incidence) matrix

$$[Q] = \begin{pmatrix} t_{11} & t_{12} & \dots & t_{1n} \\ t_{21} & t_{22} & \dots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n1} & t_{n2} & \dots & t_{nn} \end{pmatrix}$$

where t_{ij} is the number of arrows from the vertex i to the vertex j.

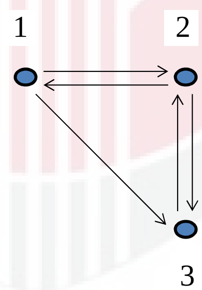
Two quivers Q_1 and Q_2 are called isomorphic if there is a bijective correspondence between their vertices and arrows such that starts and ends of corresponding arrows map into one other. It is not difficult to see that $Q_1 \simeq Q_2$ if and only if the adjacency matrix $[Q_1]$ can be transformed into the adjacency matrix $[Q_2]$ by a simultaneous permutation of rows and columns.

Example 1.1 *1. For the quiver*

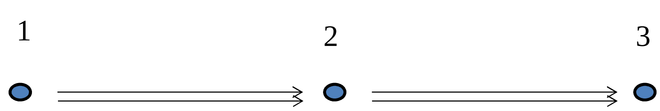


we have $VQ = 1,2,3$ *and* $AQ = \alpha, \beta$. *We also have* $s(\alpha) = 1,\ s(\beta) = 2,\ e(\alpha) = 2$ *and* $e(\beta) = 3$.

2. *A quiver may have several arrows in the same or in opposite direction. For example:*



and



3. A quiver may also have loops. For example:

For a quiver $Q = (VQ, AQ, s, e)$ and a field K one defines the path algebra KQ of Q over K. Recall that a path p of the quiver Q from the vertex i to the vertex j is a sequence of r arrows $\sigma_1\sigma_2\ldots\sigma_r$ such that the start vertex of each arrow σ_m coincides with the end vertex of the previous one σ_{m-1} for $1 < m < r$, and moreover, the vertex i is the start vertex of σ_1, while the vertex j is the end vertex of σ_r. The number r of arrows is called the length of the path p. For such a path p we define $s(p) = s(\sigma_1) = i$ and $e(\sigma_k) = j$. By convention we also include into the set of all paths the trivial path ϵ_i of length zero which connects the vertex i with itself without any arrow and we set $s(\epsilon_i) = e(\epsilon_i) = i$ for each $i \in VQ$, and, also, for any arrow $\sigma \in AQ$ with start at i and end at j we set $\epsilon_i\sigma = \sigma\epsilon_j = \sigma$. A path, connecting a vertex of a quiver with itself and of length not equal to zero, is called an oriented cycle.

Definition 1.3 *The path algebra KQ of a quiver Q over a field K is the (free) vector space with a K-basis consisting of all paths of Q. Multiplication in KQ is defined in the obvious way: the product of two paths is given by composition when possible, and is defined to be 0 otherwise.*

Therefore if the path $\sigma_1\ldots\sigma_m$ connects i and j and the path $\sigma_{m+1}\ldots\sigma_n$ connects j and k, then the product $\sigma_1\ldots\sigma_m\sigma_{m+1}\ldots\sigma_n$ connects i with k. Otherwise, the product of these paths equals 0. Extending the multiplication by distributivity, we obtain a K-algebra KQ (not necessarily finite-dimensional), which is obviously associative.

Remark 1.2 *Note that if a quiver Q has an infinitely many vertices, then KQ has no an identity element. If Q has infinitely many arrows, then KQ is not finitely generated, and so it is not finite-dimensional over K. In future we shall always assume that VQ is finite and $VQ = 1, 2, \ldots, n$.*

In the algebra KQ the set of trivial paths forms a set of pairwise orthogonal idempotents i.e.,

$$\epsilon_i^2 = \epsilon_i \quad \text{for all} \quad i \in VQ$$

$$\epsilon_i\epsilon_i = 0 \quad \text{for all} \quad i, j \in VQ \quad \text{such that} \quad i \neq j.$$

If $VQ = 1, 2, \ldots, n$, the identity of KQ is the element which is equal to the sum of all the trivial paths ϵ_i of length zero, that is, $1 = \epsilon_1 + \epsilon_2 + \ldots + \epsilon_n$. The elements $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ together with the paths of length one generate Q as an algebra. So KQ is a finitely generated algebra.

The subspace $\epsilon_1 A$ has as basis all paths starting at i, and the subspace $A\epsilon_j$ has as basis all paths ending at j. The subspace $\epsilon_i A \epsilon_j$ has as basis all paths starting at i and ending at j.

Since $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$ is a set of pairwise orthogonal idempotents for $A = KQ$ with sum equal to 1, we have the following decomposition of A into a direct sum:

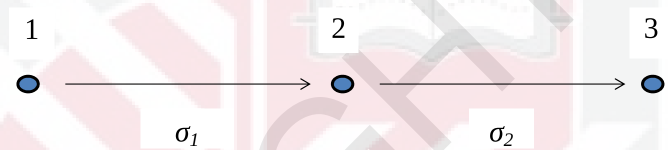
$$A = \epsilon_1 A \oplus \epsilon_2 A \oplus \ldots \oplus \epsilon_n A.$$

So each $\epsilon_i A$ is a projective right A-module. Analogously, each $A\epsilon_i$ is a projective left A-module.

Lemma 1.1 *Each ϵ_i for $i \in VQ$, is a primitive idempotent, and $\epsilon_i A$ is an indecomposable projective right A-module.*

Lemma 1.2 *$\epsilon_i A \not\simeq \epsilon_j A$, for $i, j \in VQ$ and $i \neq j$.*

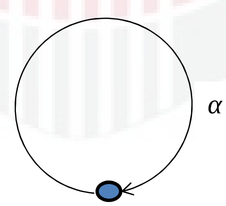
Example 1.2 *1. Let Q be the quiver*



i.e., $VQ = \{1, 2, 3\}$, $AQ = \{\sigma_1, \sigma_2\}$.

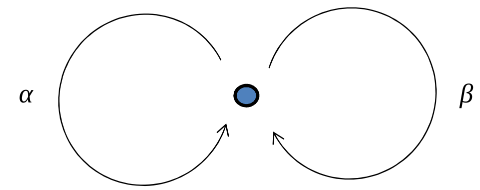
Then KQ has a basis $\{\epsilon_1, \epsilon_2, \epsilon_3, \sigma_1, \sigma_2, \sigma_1\sigma_2\}$ and $KQ \simeq T_3(K) = \begin{pmatrix} K & K & K \\ 0 & K & K \\ 0 & 0 & K \end{pmatrix} \subset$ $M_3(K)$. *So the algebra KQ is finite-dimensional over K.*

2. Let Q be the quiver with one vertex and one loop:



Then KQ has a basis $\{\epsilon, \alpha, \alpha^2, \ldots, \alpha^n, \ldots\}$. Therefore $KQ \simeq K[x]$, the polynomial algebra in one variable x. Obviously, this algebra is finitely generated but it is not finite-dimensional.

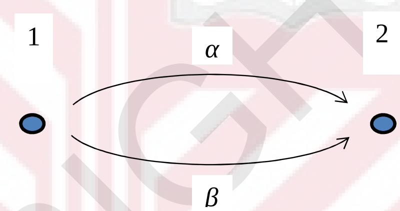
3. *Let Q be the quiver with one vertex and two loops:*



Then KQ has two generators α, β and a path in KQ is any word in α, β. Therefore $KQ \simeq K\langle\alpha, \beta\rangle$, the free associative algebra generated by α, β, which is non-commutative and infinite-dimensional over K.

If Q is a quiver with one vertex and $n \geqslant 2$ loops $\alpha_1, \alpha_2, \ldots, \alpha_n$, then $KQ \simeq K\langle\alpha_1, \alpha_2, \ldots, \alpha_n\rangle$, the free associative algebra generated by $\alpha_1, \alpha_2, \ldots, \alpha_n$, which is also non-commutative and infinite-dimensional over K.

4. *Let Q be the quiver with two vertices and two arrows:*



i.e., $VQ = \{1, 2\}$ and $AQ = \{\alpha, \beta\}$. The algebra KQ has a basis $\{\epsilon_1, \epsilon_2, \alpha, \beta\}$. This algebra is isomorphic to the Kronecker algebra $A = \begin{pmatrix} K & K \oplus K \\ 0 & K \end{pmatrix}$, which is four-dimensional over K.

An object to be considered in the Gröbner bases theory is an ideal $I = \langle g_1, \ldots, g_r\rangle$ in the algebra $\mathbb{K}[x_1, x_2, \ldots, x_n]$ of commutative polynomials over a field $\mathbb{K}$, in other words, we deal with polynomial generators in several variables.

1.3 Literature Review

In 1993, Loday (1993) introduced the notion of Leibniz algebra, which is a generalization of Lie algebra. Such generalization is appeared when the skew-symmetricity of the bracket is dropped and the Jacobi identity is changed by the Leibniz identity. Loday et al. (2001) also showed that the relationship between Lie algebras and associative algebras can be extended to an analogous relationship between Leibniz algebras and the so-called dialgebras which are a generalization of associative algebras possessing two products denoted by $\dashv$ and $\vdash$.

A dissociative algebra (or dialgebra) is a vector space with two bilinear operations $\vdash$, $\dashv$, satisfying five conditions (Loday et al., 2001). Diassociative algebras are associative when the two operations coincide. The main motivation of Loday to introduce this class of algebras was the search of an "obstruction" to the periodicity in algebraic K-theory. Besides this purely algebraic motivation some relationships them with classical geometry, non-commutative geometry and physics have been recently discovered.

The classification of associative algebras is an old and often recurring problem. The first investigation into it was perhaps done by Peirce (1881). Many other publications related to the problem have appeared. Without any claim of completeness, we mention work by Hazlett (1916), (nilpotent algebras of dimension ≤ 4 over $\mathbb{C}$), Mazolla (1979) - associative unitary algebras of dimension 5 over algebraically closed fields of characteristic not 2, Mazzola (1980) - nilpotent commutative associative algebras of dimension ≤ 5, over algebraically closed fields of characteristic not 2,3, and recently, Poonen (2008) - nilpotent commutative associative algebras of dimension ≤ 5, over algebraically closed fields.

A new era in the development of the theory of finite-dimensional associative algebras begun due to works of Wedderburn (1907), who obtained the fundamental results of this theory: description of the structure of semisimple algebras over a field, a theorem on the lifting of the quotient by the radical, the theorem on the commutativity of finite division rings, and others.

Further development of the theory of associative algebras was in the 80-s of the last century, when many open problems, remaining unsolved since 30-s, have been solved.

The next two theorems are basis of the structural theory of associative algebras (see Hazewinkel et al. (2007)).

Theorem 1.1 *(Wedderburn - Artin) Any finite-dimensional semisimple associative algebra A is uniquely decomposed into a direct sum of a number of simple algebra:*

$$A = B_1 \oplus B_2 \oplus \ldots \oplus B_k.$$

Recall that an algebra is simple if it has no nontrivial two-sided ideals.

Theorem 1.2 *Any finite-dimensional simple associative algebra A is isomorphic to the algebra of matrices $M_n(D)$ over a division ring D, the number n and the division ring D are uniquely determined by the algebra A.*

These theorems give a complete description of semisimple algebras. At the same time on the structure of nonsemisimple algebras, not much is known, even for an algebraically closed field.

Complex associative algebras in dimensions up to 5 were first classified by B. Pierce back in 1870, initially in the form of manuscripts, which appeared later in Peirce (1881). There are classifications of unital 3, 4 and 5-dimensional associative algebra by Scorza (1938), Gabriel (1975) and Mazolla (1979), respectively.

The Rota-Baxter algebra was introduced by Baxter (1960) in his probability study, and was popularized mainly by the investigations of Rota (1969) and his colleagues. Loday has introduced dendriform algebra notion in connection with dialgebra structure (Loday, 1993). Besides of Loday's motivations, the key point from our perspective is the intimate relation between the Rota-Baxter algebras and such dendriform algebras. In 2002, Ebrahimi has explored the relationship between Rota-Baxter operators and Loday-type algebras, i.e. dendriform di- and tri-algebras (see Ebrahimi-F, 2002). It is shown that associative algebras equipped with a Rota-Baxter operator of arbitrary weight always give such dendriform structures. Discussion more detail the relationship Between Rota-Baxter algebras and dendriform dialgebras and continue the research to study the adjoint functors between the category of Rota-Baxter algebras and the categories of dendriform di- and trialgebras were considered in the works of Ebrahimi and Guo (2005 and 2007). Leroux (2006) proposed a reformulation of the free dendriform algebra over the generator via a parenthesis setting and brief survey on planar binary trees. Ebrahimi and his colleagues showed some new combinatorial identities in dendriform dialgebras and investigate solutions for a particular class of linear equations in dendriform algebras (see Ebrahimi-F., K., Manchon, D. and Patras, F., 2007 and Ebrahimi-F. and Manchon, 2009).

Dialgebra cohomology with coefficients was studied by Frabetti ((1997) and (2001)) and deformations of dialgebras were developed in Majumdar and Mukherjee (2002). Dialgebras appear in different context such as dialgebra can be related to triple product as in (Pozhidaev, 2008). Lin and Zhang (2010) defined a new associative dialgebra over a polynomial algebra $F[x, y]$ with two indeterminates x and y. Left derivations, right derivations, derivations and automorphism of $F[x, y]$ are determined too. Bokut et al. (2010) used the Gröbner-Shirshov basis for a dialgebra. The concept of left-symmetric dialgebras was introduced by Felipe (2011). In 2012, Bremner has explored some recent developments in the theory of associative and nonassociative dialgebras, with an emphasis on polynomial identities and multilinear operations. González (2013) has desccribed the class of zero-cubed algebras and applied its study to two-dimensional associative dialgebras. The problem of finding special identities for dialgebras was studied by Kolesnikov and Voronin (2013). Zhang et al. (2014) has introduced the concepts of a totally compatible dialgebra and a totally compatible Lie dialgebra.

1.4 Research Objectives

In this thesis, we consider the classification problem of low-dimensional diassociative algebras and dendriform algebras. The classes of diassociative and dendriform algebras in dimension n are denoted by $Dias_n$ and $Dend_n$, respectively. We investigate the classification of these two classes of algebras for dimensions up to 4 and 2, respectively. Then we discuss on finding derivations of these classes of algebras.

Firstly, we classify associative algebras in low dimensions. Then by using this result we give classification of low-dimensional diassociative algebras and dendriform algebras.

For both classifications, we apply the same approaches. It is as follows, we fix a basis and then represent the algebras in term of structure constants. Due to the identities we get constraints for the structure constants in polynomial equations form. Solving the system of the polynomial equations gives a redundant list of all the algebras from given class. We break up the set of algebras into several disjoint subsets. For each of these subsets, we consider the classification problem separately. As the result, some of them are represented as a single orbit and others as a union of infinitely many orbits. Finally, we give the list of non-isomorphic classes of complex diassociative and dendriform algebras with the tables of multiplications.

We study the derivations of complex associative, diassociative and dendriform algebras. Simple properties of the right and left multiplication operators in diassociative algebras are also considered. Derivations of two, three-dimensional associative, diassociative algebras and dimension two in dendriform algebras are presented.

1.5 Outline of Contents

The thesis consists of five chapters. Chapter 1 summarises basic knowledge about algebras, dialgebras.

In Chapter 2, we describe a relationship between associative, diassociative and dendriform algebras by Loday diagram and some preliminaries of diassociative and dendriform algebras. Here we introduce the concepts of nilpotency and solvability for dialgebras.

The main results of the thesis are presented in Chapters 3 and 4. In Chapter 3, we present a complete lists of isomorphism classes of $As_n, Dias_n$ and $Dend_n$ in dimension 2 up to 4 (associative and diassociative algebras, while dimension 4, considered nilpotent case only denoted by $Dian_4$ in dimension four) and dimension 2, respectively. In Chapter 4, we construct all possible list of derivations.

Some conclusions and suggestions for further research are given in Chapter 5.

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