Triangles in convex distance planes

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Abstract

The article deals with a plane equipped with a convex distance function. We extend the notions of equilateral and acute triangles and consider circumcenters of such triangles.

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1. *Introduction.* A *convex distance plane* is a pair (E, K) where E is a 2-dimensional real vector space and $K \subset E$ is a compact convex set with $0 \in \operatorname{int} K$. The set K induces the *convex distance function* $d = d_K \colon E \times E \to \mathbb{R}$, defined by

$$d_K(a, b) = \inf\{r \geq 0 \colon b \in a + rK\}.$$

We write

$$\bar{B}_K(a, r) = a + rK = \{x \in E \colon d_K(a, x) \leq r\},$$
$$B_K(a, r) = a + r \operatorname{int} K = \{x \in E \colon d_K(a, x) < r\}$$

for the closed and open discs with center a and radius r. The subscript K can be omitted if there is no danger of misunderstanding.

For the basic properties of convex distance functions, see e.g. [Ma, Chapters 1-2], [IKLM], [IKLMS] or [HMW, Chapter 1]. Instead of d_K, many authors make use of the *gauge* $\gamma_K \colon E \to \mathbb{R}$, defined by $\gamma_K(x) = d(0, x)$. If K is symmetric with respect to the origin, then γ_K is a norm, but in the general case $d(a, b)$ need not be equal to $d(b, a)$.

A *triangle* in E is a set $T = \{t_1, t_2, t_3\}$ of three noncollinear points. Suppose first that E is equipped with a norm $\|x\|$. Let $m_i = (t_j + t_k)/2$ be the midpoint of the side opposite to t_i, and let $s_i = \|t_j - t_k\|/2 = \|t_j - m_i\| = \|t_k - m_i\|$. The triangle T was called *norm-acute* in [AMS] and *acute* in [Vä] if $\|t_i - m_i\| > s_i$ for all $i = 1, 2, 3$. It was proved that every such triangle has a *circumcenter* z, which means that $\|z - t_1\| = \|z - t_2\| = \|z - t_3\|$.

In this paper we extend this result to convex distance planes. However, if $J = [a, b]$ is a line segment in E, we may have $d(a, b) \neq d(b, a)$. Therefore we reformulate the definition of acuteness by replacing the length of J by the *radius* of J, defined in Section 2.

As a special case, we obtain the result for equilateral triangles.

The convex set $-K$ defines a convex distance function d_{-K}, and we have

$$d_K(a, b) = d_K(-b, -a) = d_{-K}(b, a) = d_{-K}(-a, -b) \tag{1}$$

for all $a, b \in E$. Hence d_K is a metric iff $K = -K$, which means that K is symmetric with respect to the origin. In this case, $\|x\| = d_K(0, x)$ defines a norm in E.

The plane (E, K) and the function d_K are called *strictly convex* if $S = \partial K$ contains no line segment. We let (e_1, e_2) denote the standard basis $e_1 = (1, 0)$, $e_2 = (0, 1)$ of $\mathbb{R}^2$.

From now on we assume that (E, K) is a given convex distance plane.

2. *The radius of a set.* Suppose that $A \neq \varnothing$ is a compact set in E. The *radius* (called *circumradius* in [Ja1]) of A is the number

$$\operatorname{rad}_K A = \operatorname{rad} A = \inf\{r > 0 \colon A \subset \bar{B}(x, r) \text{ for some } x \in E\}.$$

Clearly $\operatorname{rad} A = 0$ iff A contains only one point. If A contains more points, then an easy compactness argument shows that there is at least one disc $\bar{B}(x, \operatorname{rad} A)$ containing A. Such a disc is called a *minimal enclosing disc* of A. Minimal enclosing discs in normed planes have been extensively studied in [AMS]. A set may have several minimal enclosing discs; see Example 11. However, if K is strictly convex, then the minimal enclosing disc of a compact set A is unique. More generally, we have the following result:

3. Lemma. *Let $A \neq \varnothing$ be a compact set in E and let M be the locus of the centers of all minimal enclosing discs of A. Then M is a (possibly degenerate) line segment. If E is strictly convex, then M is a singleton.*

Proof. We may assume that $\operatorname{rad} A = 1$. We first show that M is convex. Let $x, y \in M$, $x \neq y$, and let $z = \lambda x + \mu y$ where $0 < \lambda, \mu < 1$ and $\lambda + \mu = 1$. If $a \in A$, then $a = x + u = y + v$ for some $u, v \in K$. Hence $a = \lambda a + \mu a = z + w$ where $w = \lambda u + \mu v \in K$. Consequently, $A \subset z + K$ and therefore $z \in M$. If E is strictly convex, the proof gives the contradiction $A \subset z + \operatorname{int} K$, and therefore M is a singleton.

It remains to show that $\operatorname{int} M = \varnothing$. This was done by T. Jahn in the recent paper [Ja2, Th. 4.7]. We give a slightly different proof. Assume that M contains a disc $p + sK$ with $0 < s < 1$. Set $\delta = \inf\{d(x, 0) \colon x \in \partial K\} > 0$. Then $d(x, p) \geq s\delta$ for all $x \in p + s\partial K$.

We show that

$$A \subset \bar{B}(p, 1 - s\delta), \tag{2}$$

which gives the contradiction $1 = \operatorname{rad} A \leq 1 - s\delta$. Let $a \in A$ and let $q \in p + s\partial K$ be the point for which $p \in [q, a]$. As $q \in M$, we have $a \in q + K$, and hence

$$1 \geq d(q, a) = d(q, p) + d(p, a) \geq s\delta + d(p, a),$$

which implies (2). □

4. *Remark.* A similar proof in higher dimensions shows (with obvious terminology) that if (E, K) is a convex distance space with $\dim E = n$ and if M is the locus of the centers of all minimal enclosing balls of a compact set $A \subset E$, then M is a convex subset of an $(n-1)$-dimensional affine subspace of E.

From (1) we easily obtain:

5. Lemma. *If $z \in E$ and $r > 0$, then $\bar{B}_{-K}(-z, r) = -\bar{B}_K(z, r)$.* □

6. Lemma. *For every line segment $J = [a, b]$ we have $\operatorname{rad}_K J = \operatorname{rad}_{-K} J$.*

Proof. As rad is invariant in translations, we may assume that $a + b = 0$ and thus $J = -J$. Consequently, if $J \subset \bar{B}_K(z, r)$, then $J \subset B_{-K}(-z, r)$ by Lemma 5. The lemma follows. □

Let $\operatorname{conv} A$ denote the convex hull of A. As discs are convex, each minimal enclosing disc of A is a minimal enclosing disc of $\operatorname{conv} A$, whence

$$\operatorname{rad} \operatorname{conv} A = \operatorname{rad} A$$

for every compact set $A \subset E$. In particular, $\operatorname{rad}\{a,b\} = \operatorname{rad}[a,b]$ for all $a,b \in E$. Since $b \in \bar{B}(a, d(a,b))$, we have

$$\operatorname{rad}[a,b] \le \min\{d(a,b), d(b,a)\}.$$

However, there is no upper bound for $\min\{d(a,b), d(b,a)\}$ in terms of $\operatorname{rad}[a,b]$, as is seen from the following example: Let $R > 1$ and let $K \subset \mathbb{R}^2$ be the solid triangle $\operatorname{conv}\{t_1, t_2, t_3\}$ with Cartesian coordinates $t_1 = (-R, 1-R),\ t_2 = (R, 1-R),\ t_3 = (0,1)$. Then $d(t_1,t_2) = d(t_2,t_1) = 2R$ and $\operatorname{rad}[t_1,t_2] = 1$.

7. Lemma. *Let $a,b,z \in E$ with $a \ne b$ and let $r = \operatorname{rad}\{a,b\}$. Then:*
(i) *If $d(z,a) < d(z,b)$, then $r < d(z,b)$,*
(ii) *if $d(z,a) \le d(z,b) = r$, then $d(z,a) = r$,*
(iii) *if $\bar{B}(z,r)$ is a minimal enclosing disc of $\{a,b\}$, then $d(z,a) = d(z,b) = r$.*

Proof. (i) Assume that $d(z,b) \le r$. Let $0 < \varepsilon < d(b,z)$ and let $y \in [z,b]$ be the point with $d(y,z) = \varepsilon$. Now

$$d(y,a) \le d(y,z) + d(z,a) = \varepsilon + d(z,a) < d(z,b)$$

for small ε. As $d(y,b) < d(z,b)$, we have $\{a,b\} \subset B(y, d(z,b))$. Hence $r < d(z,b) \le r$, a contradiction.

Items (ii) and (iii) follow from (i). □

8. *Bisectors and central sets.* Let $A \subset E$ be a set containing at least two points. The *bisector* of A is the set

$$\operatorname{bis}_K A = \operatorname{bis} A = \{x \in E \colon d(a,x) = d(b,x) \text{ for all } a,b \in A\}.$$

The *central set* of A is the set

$$\operatorname{cent}_K A = \operatorname{cent} A = \{x \in E \colon d(x,a) = d(x,b) \text{ for all } a,b \in A\}.$$

Thus $x \in \operatorname{cent} A$ iff $A \subset \partial B(x,r)$ for some $r > 0$. Points of $\operatorname{cent} A$ are called circumcenters of A in the literature. By (1) we have

$$\operatorname{bis}_K A = -\operatorname{bis}_{-K}(-A) = \operatorname{cent}_{-K} A = -\operatorname{cent}_K(-A). \tag{3}$$

If K is symmetric, then $\operatorname{bis} A = \operatorname{cent} A$ for all $A \subset E$. In interesting cases we have $\#A \le 3$. This is because the bisector and the central set are usually empty for larger sets A.

The structure of the bisector $\operatorname{bis}\{a,b\}$ of two points $a \ne b$ in E is well known; see e.g. [Ma, Section 2.1.1]. Indeed, if the unit circle $S = \partial K$ does not contain any line segment parallel to $[a,b]$, then $\operatorname{bis}\{a,b\}$ is homeomorphic to a line. If S contains precisely one segment parallel to $[a,b]$, then $\operatorname{bis}\{a,b\}$ consists of a closed cone C and a curve homeomorphic to a ray starting from the apex of C. If S contains two segments parallel to $[a,b]$, then $\operatorname{bis}\{a,b\}$ consists of two closed cones and an arc joining the apexes of the cones. In view of (3) this implies:

9. Lemma. *The sets* $\operatorname{bis}\{a,b\}$ *and* $\operatorname{cent}\{a,b\}$ *are connected for each pair of points $a \ne b$ in E.* □

By Lemma 7(iii) we have

$$\operatorname{rad}\{a,b\} = \inf\{d(x,a) \colon x \in \operatorname{cent}\{a,b\}\}. \tag{4}$$

for all $a \ne b$ in E.

10. *Triangles.* Recall from the introduction that a *triangle* is a set $T = \{t_1, t_2, t_3\} \subset E$ of three noncollinear points. For $\{i, j, k\} = \{1, 2, 3\}$ we let J_i denote the side $[t_j, t_k]$ of T opposite to t_i. We say that T is *weakly acute* if for each i there is a minimal enclosing disc $\bar{B}(z_i, r_i)$ of J_i such that $d(z_i, t_i) \geq r_i$. If this holds with $d(z_i, t_i) > r_i$, T is *strictly acute.* Trivially, strictly acute implies weakly acute.

In these definitions, the minimal enclosing disc can be clearly replaced by any disk containing J_i.

Recall that in a normed plane E, a triangle T was called *acute* in [Vä] (norm-acute in [AMS]) if for each $i \in \{i, j, k\} = \{1, 2, 3\}$ the midpoint $m_i = (t_j + t_k)/2$ of J_i satisfies the inequality

$$\|t_i - m_i\| > \|t_j - m_i\| = \|t_k - m_i\|. \tag{5}$$

Clearly acute implies weakly acute, but strictly acute does not imply acute in normed planes, because it suffices that (5) holds with m_i replaced by the center z_i of *some* minimal enclosing disc of of J_i. See Example 11 below. To avoid misunderstanding, we do not use the term "acute triangle" in convex distance planes which are not normed.

11. *Example.* See Fig. 1. Let $E = \mathbb{R}^2$ with the l_∞-norm $\|x\| = \max\{|x_1|, |x_2|\}$ for $x = (x_1, x_2)$, and let T be the triangle with vertices $t_1 = -e_1$, $t_2 = e_1$, $t_3 = 3e_2/4$. Then for $z_3 = -e_2/2$, the square $\bar{B}(z_3, 1)$ is a minimal enclosing disc of $J_3 = [t_1, t_2]$, and $\|t_3 - z_3\| > 1$.

For $i = 1, 2$ and $z_1 = (e_1 + e_2)/2$, $z_2 = (-e_1 + e_2)/2$, the disc $\bar{B}(z_i, 1/2)$ is a minimal enclosing disc of J_i, and $\|t_i - z_i\| = 3/2$. It follows that T is strictly acute.

On the other hand, as $m_3 = 0$, we have $\|t_3 - m_3\| = 3/4 < 1 = \|t_1 - m_3\|$, whence T is not acute.

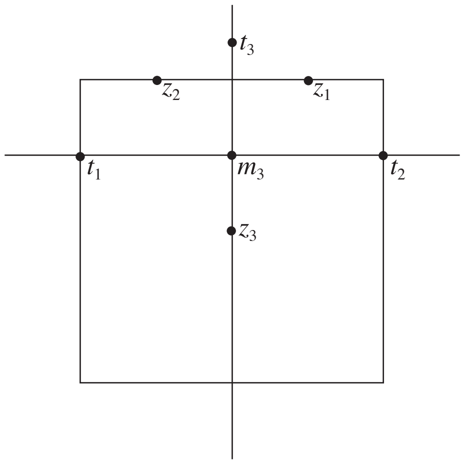

Fig. 1. Example 11

12. *Definition.* A triangle T is called *equilateral* if all sides of T have equal radius.

In a normed space this means that all sides have equal length. By Lemma 6 we obtain:

13. Lemma. *If a triangle is equilateral in* (E, K)*, it is equilateral also in* $(E, -K)$. □

14. Theorem. *Every equilateral triangle in a convex distance plane is weakly acute.*

Proof. Assume that the triangle $T = \{t_1, t_2, t_3\}$ is equilateral and let r be the common radius of the sides of T. Let $\bar{B}(z, r)$ be a minimal enclosing disc of the side $J_3 = [t_1, t_2]$. Then $d(z, t_1) = d(z, t_2) = r$ by Lemma 7(iii). If $d(z, t_3) < r$, then we can apply Lemma 7(i) with $a = t_3$, $b = t_1$ and obtain the contradiction $r < d(z, t_1)$. Hence $d(z, t_3) \geq r$, and the lemma follows. □

15. *Example.* Let $E = \mathbb{R}^2$ with the l_1-norm $\|x\| = |x_1| + |x_2|$. Then the triangle $T = \{-e_1, e_1, e_2\}$ is equilateral but not acute. Observe that $\operatorname{bis} T = \{(0, s) : s \le 0\}$.

The following result was in normed planes given in [AMS, Th. 6.1] for acute triangles. Other proofs were given in [Vä].

16. Theorem. *If a triangle* $T = \{t_1, t_2, t_3\}$ *is weakly acute, then* $\operatorname{cent} T \neq \varnothing$*.*

Proof. For each $i = 1, 2, 3$ there is a minimal enclosing disc $\bar{B}(z_i, r_i)$ of the side $[t_j, t_k]$ of T opposite to t_i such that

$$d(z_i, t_i) \ge r_i \tag{6}$$

We may assume that $r_3 \ge \max\{r_1, r_2\}$. By Lemma 7(iii) we have

$$d(z_i, t_j) = d(z_i, t_k) = r_i \tag{7}$$

for all $\{i, j, k\} = \{1, 2, 3\}$. Hence $z_3 \in \gamma := \operatorname{cent}\{t_1, t_2\}$ and

$$d(x, t_1) = d(x, t_2) \ge r_3 \tag{8}$$

for all $x \in \gamma$. See Fig. 2.

Let A be the broken line with successive vertices t_1, z_2, t_3, z_1, t_2. Define continuous functions $f : \gamma \to \mathbb{R}$ and $g : A \to \mathbb{R}$ by

$$f(x) = d(x, t_3) - d(x, t_1), \quad g(x) = d(x, t_1) - d(x, t_2).$$

As $g(t_1) < 0$, $g(t_2) > 0$ and A is connected, there is a point $y \in A$ with $g(y) = 0$. Then $y \in \gamma$. By (6) and (7) we have $f(z_3) \ge r_3 - r_3 = 0$. As γ is connected by Lemma 9, it suffices to show that $f(y) \le 0$.

We may assume that $y \in [t_3, z_1] \cup [z_1, t_2]$. If $y \in (z_1, t_2]$, then (7) implies that $d(y, t_2) < d(z_1, t_2) = r_1 \le r_3$, which is impossible by (8). Hence $y \in [t_3, z_1]$. Now $d(y, t_3) \le d(z_1, t_3) = r_1 \le r_3$. As $d(y, t_1) \ge r_3$ by (8), we obtain $f(y) \le 0$, and the theorem is proved. □

By (3) we obtain:

17. Corollary. *If a triangle* T *is weakly acute in* $(E, -K)$*, then* $\operatorname{bis}_K T \neq \varnothing$. □

18. Theorem. *If* T *is an equilateral triangle, then* $\operatorname{cent} T \neq \varnothing \neq \operatorname{bis} T$. □

Proof. By Theorem 14 the triangle T is weakly acute and hence $\operatorname{cent}_K T \neq \varnothing$ by Theorem 16. By Lemma 13, T is equilateral also in $(E, -K)$, and hence $\operatorname{bis}_K T \neq \varnothing$ by Corollary 17. □

19. *Remark.* For normed spaces, Theorem 18 was proved in [MSp, Lemma 2.4] for strictly convex spaces and in [Ko, Prop. 1.2] for arbitrary spaces.

20. *Example.* The following example shows that Th. 16 is not true if $\operatorname{cent} T$ is replaced by $\operatorname{bis} T$. Let $K \subset \mathbb{R}^2$ be the solid triangle $\operatorname{conv}\{y_1, y_2, y_3\}$ with

$$y_1 = -2e_1 + e_2, \; y_2 = 2e_1 + e_2, \; y_3 = -3e_2,$$

see Fig. 3. The triangle $T = \{t_1, t_2, t_3\}$ with

$$t_1 = -3e_1/2, \; t_2 = 3e_1/2, \; t_3 = e_2$$

is strictly acute. Indeed, the shaded triangles are minimal enclosing discs of the sides J_1 and J_2, and $\operatorname{conv}\{t_1, t_2, y_3\}$ is a minimal enclosing disc of J_3. However, $\operatorname{bis}_K T = \operatorname{cent}_{-K} T = \varnothing$.

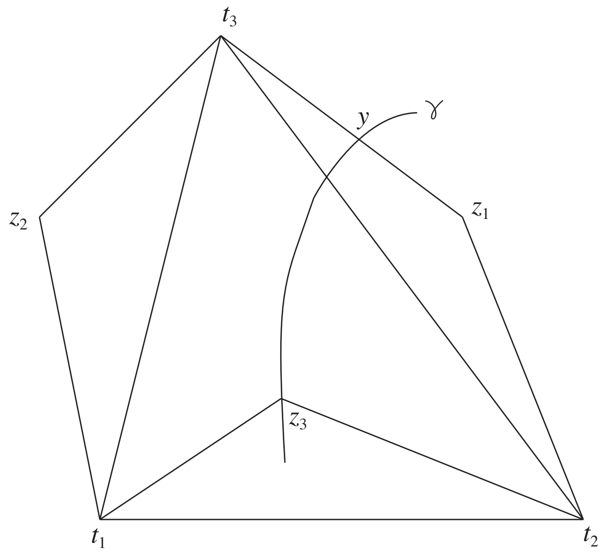

Fig. 2. Proof of Th. 16

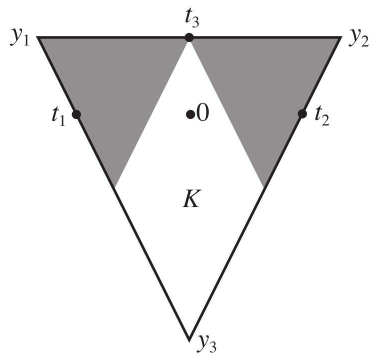

Fig. 3. Example 20

References

[AMS] J. Alonso, H. Martini, M. Spirova, Minimal enclosing discs, circumcircles, and circumcenters on normed planes (Part II), Comput. Geom. 45, 2012, 350-369.

[IKLM] C. Icking, R. Klein, N.-M. Lê, L. Ma, Convex distance functions in 3-space are different, Fund. Inform. 22, 1995, 331-352.

[IKLMS] C. Icking, R. Klein, N.-M. Lê, L. Ma, F. Santos, On bisectors for convex distance functions in 3-space, Proc. 11th Canadian Conf. Comp. Geom. Vancouver, 1999, 291-299.

[HMW] C. He, H. Martini, S. Wu, On bisectors for convex distance functions, Extracta Math. 28, 2013, 57-76.

[Ja1] T. Jahn, Extremal radii, diameter and minimum width in generalized Minkowski spaces, Rocky Mountain J. Math. 47, 2017, 825-848.

[Ja2] T. Jahn, Successive radii and ball operators in generalized Minkowski spaces, Adv. Geom. 17, 2017, 347-354.

[Ko] T. Kobos, An alternative proof of Petty's theorem for equilateral sets, Ann. Polon. Math 109, 2013, 165-175.

[Ma] L. Ma, Bisectors and Voronoi diagrams for convex distance functions, Dissertation, Fernuniversität Hagen, 1999.

[MSp] H. Martini, M. Spirova, Covering disks in Minkowski planes, Canad. Math. Bull. 52, 2009, 424-434.

[MSW] H. Martini, K.J. Swanepoel, G.Weiss, The geometry of Minkowski spaces - a survey, Part I, Expo. Math. 19, 2001, 97-142.

[Vä] J. Väisälä, Observations on circumcenters in normed planes, Beitr. Algebra Geom. 68, 2017, 607-615.

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