

RESEARCH

Open Access



An analytical approach to obtain exact solutions of some space-time conformable fractional differential equations

Nematollah Kadkhoda^{1*} and Hossein Jafari²

*Correspondence:

kadkhoda@buqaen.ac.ir¹Department of Mathematics,
Faculty of Basic Sciences,
Bozorgmehr University of Qaenat,
Qaenat, IranFull list of author information is
available at the end of the article**Abstract**

In this paper, the sine-Gordon expansion method is used to obtain analytical solutions of the conformable space-time generalized reaction Duffing model and conformable space-time Eckhaus equation with the aid of symbolic computation. These equations can be reduced into ordinary differential equations (ODEs) using a suitable wave transformation with a predicted polynomial-type solution.

Keywords: Space-time fractional Eckhaus equation; Sine-Gordon expansion method; Space-time fractional generalized reaction Duffing model; Conformable derivative

1 Introduction

Nonlinear fractional differential equations (FDEs) have an important role in studying various areas of engineering, physics, and applied mathematics [1–5]. Investigation of analytical solutions of nonlinear FDEs is very important in the analysis of some physical phenomena, such as plasma physics, solid-state physics, nonlinear optics, and so on [6–9]. In order to understand the mechanisms of these cases, it is necessary to obtain their exact solutions [10, 11]. Thus, many researchers have tried to obtain analytical solutions of these equations. Therefore, many methods and techniques are found to seek exact solutions of nonlinear FDEs, such as separating variables method [12], homotopy analysis method [13], Adomian decomposition method [14], fractional complex transform [15], variational iteration method [16], Hirota's bilinear method [17], homotopy perturbation pade technique [18], $\frac{G'}{G^2}$ -expansion method [19], sub-equation method [20, 21], simplest equation method [22, 23], first integral method [24], and so on. In this letter, the following two FDEs are solved using the sine-Gordon expansion method:

(I) The form of the space-time fractional Eckhaus equation

$$iD_t^\alpha \varphi + D_x^{2\alpha} \varphi + 2D_x^\alpha |\varphi|^2 \varphi + |\varphi|^4 \varphi = 0, \quad (1)$$

where $\varphi = \varphi(x, t)$, $\varphi : \mathbb{R}^2 \rightarrow \mathbb{C}$. This equation was introduced by Wiktor Eckhaus to describe the propagation of waves in dispersive media [25]. Many of the properties of the Eckhaus were studied in [26].

(II) The form of the space-time fractional generalized reaction Duffing model [27]

$$D_t^{2\alpha} \varphi + aD_x^{2\alpha} \varphi + b\varphi + c\varphi^2 + d\varphi^3 = 0. \tag{2}$$

This model illustrates the motion of a damped oscillator. It is a case of a dynamical system which presents chaotic behavior. Here, D^α is the conformable derivative [28, 29]. The paper is organized as follows. Section 2 is given to introduce the definition of the conformable derivative and its properties. Description of the conformable sine-Gordon expansion method and its application to the space-time FDEs are given in Sect. 3. We obtain exact solutions to the space-time fractional Eckhaus equation and the space-time fractional generalized reaction Duffing model with sine-Gordon expansion method in Sect. 4. Conclusions of this paper are summarized in Sect. 5.

2 Definition of the conformable derivative and its properties

In this section, we illustrate the definition of the conformable derivative and some of its important properties of order α with respect to the independent variable z as follows [28].

Definition 1 For a function $g : [0, \infty) \rightarrow \mathbb{R}$, the conformable fractional derivative of g of order α is defined by

$$D^\alpha \{g(z)\} = \lim_{\eta \rightarrow 0} \frac{g(z + \eta z^{1-\alpha}) - g(z)}{\eta}. \tag{3}$$

Some well-known properties to this newly defined fractional derivative are as follows.

If f and $g \neq 0$ are two functions α -differentiable, $\alpha \in (0, 1]$ and $a, b \in \mathbb{R}$, then we have

- (1) $D^\alpha \{af(z) + bg(z)\} = aD^\alpha f(z) + bD^\alpha g(z),$
- (2) $D^\alpha \{f(z)g(z)\} = f(z)D^\alpha g(z) + g(z)D^\alpha f(z),$
- (3) $D^\alpha \left\{ \frac{f(z)}{g(z)} \right\} = \frac{g(z)D^\alpha f(z) - f(z)D^\alpha g(z)}{g^2(z)},$
- (4) $D^\alpha C = 0,$ for all constant functions $f(z) = C,$
- (5) $D^\alpha (f)(z) = z^{1-\alpha} \frac{df}{dz}.$

Also, conformable fractional derivatives of some special functions are as follows:

- (a) $D^\alpha (z^r) = rz^{r-\alpha}$ for all $r \in \mathbb{R},$
- (b) $D^\alpha (1) = 0,$
- (c) $D^\alpha (e^{cz}) = cz^{1-\alpha} e^{cz},$ $c \in \mathbb{R},$
- (d) $D^\alpha (\sin bz) = bz^{1-\alpha} \cos bz,$ $b \in \mathbb{R},$
- (e) $D^\alpha (\cos bz) = -bz^{1-\alpha} \sin bz,$ $b \in \mathbb{R},$
- (f) $D^\alpha \left(\frac{1}{\alpha} z^\alpha \right) = 1.$

The proofs of these properties can be seen in [28].

Definition 2 Let $\alpha \in (n, n + 1]$, and g be α -differentiable at $t > 0$. Then the conformable fractional derivative of g of order α is defined as

$$D^\alpha(g(t)) = \lim_{\eta \rightarrow 0} \frac{g^{([\alpha]-1)}(t + \eta t^{([\alpha]-\alpha)}) - g^{([\alpha]-1)}(t)}{\eta}, \tag{4}$$

where $[\alpha]$ is the smallest integer greater than or equal to α .

3 Conformable sine-Gordon expansion method and its applications to the space-time fractional differential equations

The proper fractional form $\varphi(x, t) = \phi(\xi)$ with

$$\xi = k \left(\frac{x^\alpha}{\alpha} - 2r \frac{t^\alpha}{\alpha} \right) \tag{5}$$

of wave transformation reduces the fractional sine-Gordon equation

$$D_x^{2\alpha} \varphi - D_t^{2\alpha} \varphi = m^2 \sin \varphi, \tag{6}$$

where m is constant, to the ODE

$$\frac{d^2 \phi}{d\xi^2} = \frac{m^2}{k^2(1 - 4r^2)} \sin \phi. \tag{7}$$

Here, r is velocity of the traveling wave. Therefore, we have

$$\left(\frac{d(\frac{\phi}{2})}{d\xi} \right)^2 = \frac{m^2}{k^2(1 - 4r^2)} \sin^2 \frac{\phi}{2}, \tag{8}$$

with $\omega(\xi) = \frac{\phi(\xi)}{2}$ and $b^2 = \frac{m^2}{k^2(1 - 4r^2)} = 1$, equation (8) is changed to

$$\frac{d\omega}{d\xi} = \sin \omega. \tag{9}$$

The following relations can be obtained from (9):

$$\sin \omega(\xi) = \frac{2\gamma e^\xi}{\gamma^2 e^{2\xi} + 1} \Big|_{\gamma=1} = \operatorname{sech} \xi \tag{10}$$

or

$$\cos \omega(\xi) = \frac{\gamma^2 e^{2\xi} - 1}{\gamma^2 e^{2\xi} + 1} \Big|_{\gamma=1} = \tanh \xi, \tag{11}$$

where γ is integral constant and $\gamma \neq 0$. Nonlinear FDEs

$$P(\varphi, D_t^\alpha \varphi, D_x^\alpha \varphi, D_t^{2\alpha} \varphi, D_x^{2\alpha} \varphi, \dots) = 0 \tag{12}$$

with the traveling transformation (5) can be reduced to an ODE

$$\tilde{P}(\phi, \phi', \phi'', \dots) = 0. \tag{13}$$

Here prime denotes the derivative with respect to ξ . Now, the solution to (13) can be written

$$\phi(\xi) = a_0 + \sum_{j=1}^n \tanh^{j-1}(\xi)(a_j \tanh(\xi) + b_j \operatorname{sech}(\xi)), \tag{14}$$

or

$$\phi(\omega) = a_0 + \sum_{j=1}^n \cos^{j-1}(\omega)(a_j \cos(\omega) + b_j \sin(\omega)), \tag{15}$$

where a_0, a_j, b_j ($1 \leq j \leq n$) are constants to be determined later, and n is fixed by balancing between the highest nonlinear term and highest order derivative in (13). It is worth noting the following derivatives of $\phi(\omega)$ in equation (15).

- The first, second, and third derivatives of $\phi(\omega)$ for $n = 1$ in equation (15) can be written as follows:

$$\phi(\omega) = a_0 + a_1 \cos \omega + b_1 \sin \omega, \tag{16}$$

$$\phi'(\omega) = b_1 \cos \omega \sin \omega - a_1 \sin^2 \omega, \tag{17}$$

$$\phi''(\omega) = b_1 [\cos^2 \omega \sin \omega - \sin^3 \omega] - 2a_1 \sin^2 \omega \cos \omega, \tag{18}$$

$$\phi'''(\omega) = b_1 [\cos^3 \omega \sin \omega - 5 \sin^3 \omega \cos \omega] - 2a_1 \sin^2 \omega (2 \cos^2 \omega - \sin^2 \omega). \tag{19}$$

- The first and second derivatives of $\phi(\omega)$ for $n = 2$ in equation (15) can be written as follows:

$$\phi(\omega) = a_0 + a_1 \cos \omega + b_1 \sin \omega + a_2 \cos^2 \omega + b_2 \sin \omega \cos \omega, \tag{20}$$

$$\phi'(\omega) = -a_1 \sin^2 \omega - 2a_2 \cos \omega \sin^2 \omega + b_1 \cos \omega \sin \omega + b_2 (\sin \omega - 2 \sin^3 \omega), \tag{21}$$

$$\begin{aligned} \phi''(\omega) = & -2a_1 \sin^2 \omega \cos \omega + 2a_2 \sin^2 \omega (\sin^2 \omega - 2 \cos^2 \omega) \\ & + b_1 (\cos^2 \omega \sin \omega - \sin^3 \omega) + b_2 \cos \omega (\sin \omega - 6 \sin^3 \omega). \end{aligned} \tag{22}$$

Substituting (15) along (9)–(11) into (13) and collecting all terms with the same powers of $\sin \omega \cos \omega$ together, the left-hand side of (13) is converted into a polynomial. After setting each coefficient of this polynomial to zero, we obtain a set of algebraic equations in terms of a_0, a_1, b_1, \dots, k , and r . Solving the system of algebraic equations and then substituting the results into (13), gives solutions of (12).

4 Applications of the sine-Gordon expansion method to the space-time FDEs

- *Applications of the sine-Gordon expansion method to the space-time fractional Eckhaus equation*

Here, we will demonstrate this method to the space-time fractional Eckhaus equation as follows:

$$iD_t^\alpha \varphi + D_x^{2\alpha} \varphi + 2D_x^\alpha |\varphi|^2 \varphi + |\varphi|^4 \varphi = 0, \tag{23}$$

by transformations

$$\begin{aligned} \varphi(x, t) &= \phi(\xi)e^{i\theta}, \\ \xi &= k\left(\frac{x^\alpha}{\alpha} - 2r\frac{t^\alpha}{\alpha}\right), \\ \theta &= \left(r\frac{x^\alpha}{\alpha} + s\frac{t^\alpha}{\alpha}\right), \end{aligned} \tag{24}$$

where θ illustrates the phase component, r is the velocity, s is the frequency, and k is the width of the soliton, equation (23) becomes:

$$k^2\phi_{\xi\xi} - (s + r^2)\phi + 4k\phi_\xi\phi^2 + \phi^5 = 0. \tag{25}$$

Now, we suppose that

$$\phi = \sqrt{\psi}, \tag{26}$$

therefore, equation(25) can be reduced to the following ODE:

$$2k^2\psi\psi_{\xi\xi} - k^2\psi_\xi^2 - 4(s + r^2)\psi^2 + 8k\psi_\xi\psi^2 + 4\psi^4 = 0. \tag{27}$$

Therefore, the solution of (27) can be expressed as follows:

$$\psi(\xi) = a_0 + a_1 \tanh(\xi) + b_1 \operatorname{sech}(\xi). \tag{28}$$

Substituting (16)–(18) into (27) and equating all terms with the powers of $\sin \omega \cos \omega$ to zero, the following system can be obtained:

$$\begin{aligned} &4a_1^4 - 4a_0^2r^2 - 8a_1a_0r^2 - 4a_1^2r^2 - 4a_0^2s - 8a_1a_0s - 4a_1^2s + 4a_0^4 + 16a_1a_0^3 \\ &\quad + 24a_1^2a_0^2 + 16a_1^3a_0 = 0, \\ &8a_1a_0r^2 - 4a_0^2r^2 - 4a_1^2r^2 - 4a_0^2s + 8a_1a_0s - 4a_1^2s + 4a_0^4 - 16a_1a_0^3 + 24a_1^2a_0^2 \\ &\quad - 16a_1^3a_0 + 4a_1^4 = 0, \\ &4a_0b_1k^2 + 4a_1b_1k^2 + 16a_0^2b_1k + 32a_1a_0b_1k + 16a_1^2b_1k - 16a_0b_1r^2 - 16a_1b_1r^2 \\ &\quad - 16a_0b_1s - 16a_1b_1s + 32a_0^3b_1 + 96a_1a_0^2b_1 + 96a_1^2a_0b_1 + 32a_1^3b_1 = 0, \\ &64b_1^3k - 20a_0b_1k^2 - 44a_1b_1k^2 + 16a_0^2b_1k - 160a_1a_0b_1k - 176a_1^2b_1k - 48a_0b_1r^2 \\ &\quad - 16a_1b_1r^2 - 48a_0b_1s - 16a_1b_1s + 96a_0^3b_1 + 96a_1a_0^2b_1 + 128a_0b_1^3 - 96a_1^2a_0b_1 \\ &\quad + 128a_1b_1^3 - 96a_1^3b_1 = 0, \\ &192a_0^2b_1^2 - 256a_1b_1^2k - 192a_1^2b_1^2 + 16a_1^2k^2 - 64a_1a_0^2k + 64a_1^3k - 24a_0^2r^2 + 8a_1^2r^2 \\ &\quad - 24a_0^2s + 8a_1^2s + 24a_0^4 - 48a_1^2a_0^2 + 24a_1^4 - 40b_1^2k^2 - 32b_1^2r^2 - 32b_1^2s + 64b_1^4 = 0, \\ &44a_1b_1k^2 - 20a_0b_1k^2 - 16a_0^2b_1k - 160a_1a_0b_1k + 176a_1^2b_1k - 48a_0b_1r^2 + 16a_1b_1r^2 \\ &\quad - 48a_0b_1s + 16a_1b_1s + 96a_0^3b_1 - 96a_1a_0^2b_1 + 128a_0b_1^3 - 96a_1^2a_0b_1 - 128a_1b_1^3 \end{aligned}$$

$$\begin{aligned}
 &+ 96a_1^3b_1 - 64b_1^3k = 0, \\
 &4a_0b_1k^2 - 4a_1b_1k^2 - 16a_0^2b_1k + 32a_1a_0b_1k - 16a_1^2b_1k - 16a_0b_1r^2 + 16a_1b_1r^2 \\
 &- 16a_0b_1s + 16a_1b_1s + 32a_0^3b_1 - 96a_1a_0^2b_1 + 96a_1^2a_0b_1 - 32a_1^3b_1 = 0.
 \end{aligned}$$

Solving this system, we find a_0 , a_1 , and b_1 as follows:

• Case 1:

$$a_0 = -\frac{k}{4}, \quad a_1 = -\frac{k}{4}, \quad b_1 = \pm \frac{\sqrt{-r^2 - s}}{2}; \quad k = 2\sqrt{r^2 + s}. \tag{29}$$

Substituting (29) (case 1) in (28), we obtain solutions of (27) as follows:

$$\psi_1(x, t) = -\frac{\sqrt{r^2 + s}}{2} \left(1 + \tanh \left(k \left(\frac{x^\alpha}{\alpha} - 2r \frac{t^\alpha}{\alpha} \right) \right) \right) \pm \frac{i\sqrt{r^2 + s}}{2} \operatorname{sech} \left(k \left(\frac{x^\alpha}{\alpha} - 2r \frac{t^\alpha}{\alpha} \right) \right).$$

Therefore, solutions of (23) can be written as follows:

$$\begin{aligned}
 \varphi_1(x, t) = &\left\{ -\frac{\sqrt{r^2 + s}}{2} \left(1 + \tanh \left(k \left(\frac{x^\alpha}{\alpha} - 2r \frac{t^\alpha}{\alpha} \right) \right) \right) \right. \\
 &\left. \pm \frac{i\sqrt{r^2 + s}}{2} \operatorname{sech} \left(k \left(\frac{x^\alpha}{\alpha} - 2r \frac{t^\alpha}{\alpha} \right) \right) \right\}^{\frac{1}{2}} e^{i(r\frac{x^\alpha}{\alpha} + s\frac{t^\alpha}{\alpha})}.
 \end{aligned}$$

• Case 2:

$$a_0 = \frac{k}{4}, \quad a_1 = -\frac{k}{4}, \quad b_1 = \pm \frac{\sqrt{-r^2 - s}}{2}; \quad k = 2\sqrt{r^2 + s}. \tag{30}$$

Substituting (30) (case 2) in (28), we obtain solutions of (27) as follows:

$$\psi_2(x, t) = \frac{\sqrt{r^2 + s}}{2} \left(1 - \tanh \left(k \left(\frac{x^\alpha}{\alpha} - 2r \frac{t^\alpha}{\alpha} \right) \right) \right) \pm \frac{i\sqrt{r^2 + s}}{2} \operatorname{sech} \left(k \left(\frac{x^\alpha}{\alpha} - 2r \frac{t^\alpha}{\alpha} \right) \right).$$

Now, solutions of (23) can be written as follows:

$$\begin{aligned}
 \varphi_2(x, t) = &\left\{ \frac{\sqrt{r^2 + s}}{2} \left(1 - \tanh \left(k \left(\frac{x^\alpha}{\alpha} - 2r \frac{t^\alpha}{\alpha} \right) \right) \right) \right. \\
 &\left. \pm \frac{i\sqrt{r^2 + s}}{2} \operatorname{sech} \left(k \left(\frac{x^\alpha}{\alpha} - 2r \frac{t^\alpha}{\alpha} \right) \right) \right\}^{\frac{1}{2}} e^{i(r\frac{x^\alpha}{\alpha} + s\frac{t^\alpha}{\alpha})}.
 \end{aligned}$$

• Case 3:

$$a_0 = -\frac{k}{2}, \quad a_1 = -\frac{k}{2}, \quad b_1 = 0; \quad k = -\sqrt{r^2 + s}. \tag{31}$$

Substituting (31) (case 3) in (28), we obtain solutions of (27) as follows:

$$\psi_3(x, t) = \frac{\sqrt{r^2 + s} e^{2k(\frac{x^\alpha}{\alpha} - 2r\frac{t^\alpha}{\alpha})}}{e^{2k(\frac{x^\alpha}{\alpha} - 2r\frac{t^\alpha}{\alpha})} + 1}.$$

Also, solutions of (23) can be obtained as follows:

$$\varphi_3(x, t) = \left\{ \frac{\sqrt{r^2 + s} e^{2k(\frac{x^\alpha}{\alpha} - 2r\frac{t^\alpha}{\alpha})}}{e^{2k(\frac{x^\alpha}{\alpha} - 2r\frac{t^\alpha}{\alpha})} + 1} \right\}^{\frac{1}{2}} e^{i(r\frac{x^\alpha}{\alpha} + s\frac{t^\alpha}{\alpha})}.$$

• Case 4:

$$a_0 = \frac{k}{2}, \quad a_1 = -\frac{k}{2}, \quad b_1 = 0; \quad k = \sqrt{r^2 + s}. \tag{32}$$

Finally, substituting (32) (case 4) in (28), solutions of (27) can be calculated as follows:

$$\psi_4(x, t) = \frac{\sqrt{r^2 + s}}{e^{2k(\frac{x^\alpha}{\alpha} - 2r\frac{t^\alpha}{\alpha})} + 1}.$$

Furthermore, we obtain solutions of (27) as follows:

$$\varphi_4(x, t) = \left\{ \frac{\sqrt{r^2 + s}}{e^{2k(\frac{x^\alpha}{\alpha} - 2r\frac{t^\alpha}{\alpha})} + 1} \right\}^{\frac{1}{2}} e^{i(r\frac{x^\alpha}{\alpha} + s\frac{t^\alpha}{\alpha})}.$$

• *Applications of the sine-Gordon expansion method to the space-time fractional generalized reaction Duffing model*

Now, we will demonstrate the sine-Gordon expansion method to the space-time fractional generalized reaction Duffing model as follows:

$${}_0^A D_t^{2\alpha} \varphi + a_0^A D_x^{2\alpha} \varphi + b\varphi + c\varphi^2 + d\varphi^3 = 0 \tag{33}$$

by transformations

$$\xi = k \left(\frac{x^\alpha}{\alpha} - 2r \frac{t^\alpha}{\alpha} \right). \tag{34}$$

Equation (33) becomes:

$$(ak^2 + 4k^2r^2)\phi_{\xi\xi\xi} + b\phi + c\phi^2 + d\phi^3 = 0. \tag{35}$$

With balancing, we get $n = 1$. Therefore, the solution of (35) can be expressed as follows:

$$\varphi(\xi) = a_0 + a_1 \tanh(\xi) + b_1 \operatorname{sech}(\xi). \tag{36}$$

Substituting (36) along (17) and (18) into (35) and equating all terms with the powers of $\sin \omega \cos \omega$ to zero, the following system can be obtained:

$$\begin{aligned} a_0b + a_1b + a_0^2c + 2a_1a_0c + a_1^2c + a_0^3d + 3a_1a_0^2d + 3a_1^2a_0d + a_1^3d &= 0, \\ 4a_0b_1c + 4a_1b_1c + 6a_0^2b_1d + 6a_1^2b_1d + 12a_0a_1b_1d + 2ab_1k^2 + 8b_1k^2r^2 + 2bb_1 &= 0, \\ 8a_0b_1c + 12a_0^2b_1d - 12a_1^2b_1d - 12ab_1k^2 + 8b_1^3d - 48b_1k^2r^2 + 4bb_1 &= 0, \\ 4a_0b_1c - 4a_1b_1c + 6a_0^2b_1d + 6a_1^2b_1d - 12a_0a_1b_1d + 2ab_1k^2 + 8b_1k^2r^2 + 2bb_1 &= 0, \end{aligned}$$

$$\begin{aligned}
 &a_0b - a_1b + a_0^2c - 2a_1a_0c + a_1^2c + a_0^3d - 3a_1a_0^2d + 3a_1^2a_0d - a_1^3d = 0, \\
 &12a_0b_1^2d - 12a_1b_1^2d + 3a_0b - a_1b + 3a_0^2c - 2a_1a_0c - a_1^2c + 3a_0^3d - 3a_1a_0^2d - 3a_1^2a_0d \\
 &\quad + 3a_1^3d + 32a_1k^2r^2 + 8aa_1k^2 + 4b_1^2c = 0, \\
 &12a_0b_1^2d + 12a_1b_1^2d + 3a_0b + a_1b + 3a_0^2c + 2a_1a_0c - a_1^2c + 3a_0^3d + 3a_1a_0^2d - 3a_1^2a_0d \\
 &\quad - 3a_1^3d - 32a_1k^2r^2 - 8aa_1k^2 + 4b_1^2c = 0.
 \end{aligned}$$

Solving this system, we obtain $a_0, a_1,$ and b_1 as follows:

• Case 1:

$$a_0 = -\frac{3b}{2c}, \quad a_1 = \pm \frac{3b}{2c}, \quad b_1 = 0; \quad a = \frac{-b - 16k^2r^2}{4k^2}, \quad b = \frac{2c^2}{9d}. \tag{37}$$

Substituting (37) (case 1) in (35), we obtain solutions of (33) as follows:

$$\varphi_1(x, t) = \frac{3b}{2c} \left(-1 \pm \tanh \left(k \left(\frac{x^\alpha}{\alpha} - 2r \frac{t^\alpha}{\alpha} \right) \right) \right).$$

• Case 2:

$$a_0 = -\frac{3b}{2c}, \quad a_1 = 0, \quad b_1 = \pm \frac{3\sqrt{2}b}{2c}; \quad a = \frac{b - 8k^2r^2}{2k^2}, \quad b = \frac{2c^2}{9d}. \tag{38}$$

Now, substituting (38) (case 2) in (35), we obtain solutions of (33) as follows:

$$\varphi_2(x, t) = \frac{3b}{2c} \left(-1 \pm \sqrt{2} \operatorname{sech} \left(k \left(\frac{x^\alpha}{\alpha} - 2r \frac{t^\alpha}{\alpha} \right) \right) \right).$$

• Case 3

$$a_0 = -\frac{3b}{2c}, \quad a_1 = \frac{3b}{2c}, \quad b_1 = \pm \frac{3ib}{2c}; \quad a = \frac{-b - 4k^2r^2}{k^2}, \quad b = \frac{2c^2}{9d}. \tag{39}$$

Furthermore, substituting (39) (case 3) in (35), we obtain solutions of (33) as follows:

$$\varphi_3(x, t) = \frac{3b}{2c} \left(-1 + \tanh \left(k \left(\frac{x^\alpha}{\alpha} - 2r \frac{t^\alpha}{\alpha} \right) \right) \pm i \operatorname{sech} \left(k \left(\frac{x^\alpha}{\alpha} - 2r \frac{t^\alpha}{\alpha} \right) \right) \right).$$

• Case 4:

$$\begin{aligned}
 a_0 &= \frac{-2b + ak^2 + 4k^2r^2}{c}, \quad a_1 = 0, \quad b_1 = \pm \frac{2k\sqrt{a + 4r^2}(-2b + ak^2 + 4k^2r^2)}{c\sqrt{b}}; \\
 a &= -4r^2, \quad b = \frac{c^2}{4d}.
 \end{aligned} \tag{40}$$

Finally, substituting (40) (case 4) in (35), we obtain solutions of (33) as follows:

$$\varphi_4(x, t) = \frac{k^2(a + 4r^2) - 2b}{c} \left(1 \pm \frac{2k\sqrt{a + 4r^2}}{\sqrt{b}} \operatorname{sech} \left(k \left(\frac{x^\alpha}{\alpha} - 2r \frac{t^\alpha}{\alpha} \right) \right) \right).$$

5 Conclusion

In this paper, the sine-Gordon expansion method has been successfully used to obtain exact solutions of the space-time fractional Eckhaus equation and the space-time fractional generalized reaction Duffing model. For this method, by means of balance equations, we obtained exact solutions of the studied class nonlinear FDEs. These solutions can be useful to describe some physical phenomena. The results show that the sine-Gordon expansion method is accurate and effective. Mathematica has been used for computations and programming in this paper.

Funding

The authors would like to acknowledge the financial support of Bozorgmehr University of Qaenat for this research under contract number 39151.

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this manuscript. The authors declare that they have no competing interests.

Authors' contributions

The authors read and approved the final version of the manuscript.

Author details

¹Department of Mathematics, Faculty of Basic Sciences, Bozorgmehr University of Qaenat, Qaenat, Iran. ²Department of Mathematical Sciences, University of South Africa, Pretoria, South Africa.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 5 June 2019 Accepted: 24 September 2019 Published online: 11 October 2019

References

1. Kilbas, A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
2. Podlubny, I.: Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications. Mathematics in Science and Engineering, vol. 198. Academic Press, San Diego (1999)
3. Qi, F., Rahman, G., Hussain, S., Du, W.S., Nisar, K.: Some inequalities of Chebyshev type for conformable k -fractional integral operators. *Symmetry* **10**(11), 614 (2018)
4. Rahman, G., Ullah, Z., Khan, A., Set, E., Nisar, K.S.: Certain Chebyshev-type inequalities involving fractional conformable integral operators. *Mathematics* **7**(4), 364 (2019)
5. Sooppy, N.S.K., Huang, C.J., Rahman, G., Ghaffar, A., Qi, F.: Some inequalities of the Hermite–Hadamard type for k -fractional conformable integrals. *Aust. J. Math. Anal. Appl.* **16**(1), 1–9 (2019)
6. Kalamani, P., Baleanu, D., Mallika Arjunan, M.: Local existence for an impulsive fractional neutral integro-differential system with Riemann–Liouville fractional derivatives in a Banach space. *Adv. Differ. Equ.* **2018**, 416 (2018)
7. Veerasha, P., Prakasha, D.G., Baleanu, D.: An efficient numerical technique for the nonlinear fractional Kolmogorov–Petrovskii–Piskunov equation. *Mathematics* **7**(3), 265 (2019)
8. Kadkhoda, N.: Exact solutions of $(3 + 1)$ -dimensional nonlinear evolution equations. *Casp. J. Math. Sci.* **4**(2), 189–195 (2015)
9. Jafari, H., Kadkhoda, N., Khaliq, C.M.: Exact solutions of equation using Lie symmetry approach along with the simplest equation and exp-function methods. *Abstr. Appl. Anal.* **2012**, Article ID 350287 (2012)
10. Ali, M., Alquran, M., Jaradat, I., Baleanu, D.: Stationary wave solutions for new developed two-waves' fifth-order Korteweg–de Vries equation. *Adv. Differ. Equ.* **2019**, 263 (2019)
11. Baleanu, D., Ghafarnezhad, K., Rezapour, S.: On a three step crisis integro-differential equation. *Adv. Differ. Equ.* **2019**, 153 (2019)
12. Chen, J., Liu, F., Anh, V.: Analytical solution for the time-fractional telegraph equation by the method of separating variables. *J. Math. Anal. Appl.* **338**(2), 1364–1377 (2008)
13. Dehghan, M., Manafian, J., Saadatmandi, A.: Solving nonlinear fractional partial differential equations using the homotopy analysis method. *Numer. Methods Partial Differ. Equ.* **26**(2), 448–479 (2010)
14. El-Wakil, S.A., Elhanbaly, A., Abdou, M.A.: Adomian decomposition method for solving fractional nonlinear differential equations. *Appl. Math. Comput.* **182**(1), 313–324 (2006)
15. Su, W.-H., Yang, X.-J., Jafari, H., Baleanu, D.: Fractional complex transform method for wave equations on Cantor sets within local fractional differential operator. *Adv. Differ. Equ.* **2013**, 97 (2013)
16. He, J.H., Wu, G.C., Austin, F.: The variational iteration method which should be followed. *Nonlinear Sci. Lett. A, Math. Phys. Mech.* **1**, 1–30 (2010)
17. Hirota, R.: Exact solution of the Korteweg–de Vries equation for multiple collisions of solitons. *Phys. Rev. Lett.* **27**, 1192–1194 (1971)

18. Jafari, H., Kadkhoda, N., Tajadodi, H., Hosseini Matikolaj, S.A.: Homotopy perturbation Pade technique for solving fractional Riccati differential equations. *Int. J. Nonlinear Sci. Numer. Simul.* **11**, 271–275 (2010)
19. Kadkhoda, N.: Application of $\frac{G}{G^2}$ -expansion method for solving fractional differential equations. *Int. J. Appl. Comput. Math.* **3**, 1415–1424 (2017)
20. Kadkhoda, N., Jafari, H.: Application of fractional sub-equation method to the space-time fractional differential equations. *Int. J. Adv. Appl. Math. Mech.* **4**, 1–6 (2017)
21. Jafari, H., Tajadodi, H., Kadkhoda, N., Baleanu, D.: Fractional subequation method for Cahn–Hilliard and Klein–Gordon equations. *Abstr. Appl. Anal.* **2013**, Article ID 587179 (2013)
22. Kudryashov, N.A.: Simplest equation method to look for exact solutions of nonlinear differential equations. *Chaos Solitons Fractals* **24**, 1217–1231 (2005)
23. Kadkhoda, N., Jafari, H.: Kudryashov method for exact solutions of isothermal magnetostatic atmospheres. *Iran. J. Numer. Anal. Optim.* **6**(1), 43–52 (2016)
24. Lu, B.: The first integral method for some time fractional differential equations. *J. Math. Anal. Appl.* **395**, 684–693 (2012)
25. Taghizadeh, N., Mirzazadeh, M., Tascan, F.: The first integral method applied to the Eckhaus equation. *Appl. Math. Lett.* **25**(5), 798–802 (2012)
26. Calogero, F., Lillo, S.D.: The Eckhaus PDE $i\varphi_t + \varphi_{xx} + 2|\varphi|_x^2\varphi + |\varphi|^4\varphi = 0$. *Inverse Probl.* **3**, 633–681 (1987)
27. Jafari, H., Tajadodi, H., Baleanu, D., Al-Zahrani, A.A., Alhamed, Y.A., Zahid, A.H.: Fractional sub-equation method for the fractional generalized reaction Duffing model and nonlinear fractional Sharma–Tasso–Olver equation. *Cent. Eur. J. Phys.* **11**(10), 1482–1486 (2013)
28. Atangana, A., Baleanu, D., Alsaedi, A.: Analysis of time-fractional Hunter–Saxton equation: a model of neumatic liquid crystal. *Open Phys.* **14**(1), 145–149 (2016)
29. Rahman, G., Nisar, S., Qi, F.: Some new inequalities of the Gruss type for conformable fractional integrals. *AIMS Math.* **3**(4), 575–583 (2018)

Submit your manuscript to a SpringerOpen[®] journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

Submit your next manuscript at ► [springeropen.com](https://www.springeropen.com)
