Alma Mater Studiorum · Università di Bologna

Scuola di Scienze

Dipartimento di Fisica e Astronomia

Corso di Laurea Magistrale in Fisica

Quantum probes for minimum length estimation

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Anno Accademico 2018/2019

Anche quando pare di poche spanne, un viaggio può restare senza ritorno.
– Italo Calvino

Abstract

In this thesis we address the estimation of the minimum length arising from gravitational theories. In particular, we use tools from Quantum Estimation theory to provide bounds on precision and to assess the use of quantum probes to enhance the estimation performances. After a brief introduction to classical estimation theory and to its quantum counterparts, we introduce the concept of minimum length and show how it induces a perturbative term appearing in the Hamiltonian of any quantum system, which is proportional to a parameter depending on the minimum length. We have then systematically studied the effects of this perturbation on different state preparations for several 1-dimensional systems, and evaluated the Quantum Fisher Information in order to find the ultimate bounds to the precision of any estimation procedure. Eventually, we have investigated the role of dimensionality on the precision, studying the 2-dimensional Harmonic oscillator. Our results provide a guideline to design possible future experiments to detect minimal length, and show the potential enhancement achievable using quantum probes.

Acknowledgments

Dal settembre del 2017 in cui ho intrapreso questo percorso, molte cose sono cambiate, sia a livello personale che scientifico. Vorrei ricordare e ringraziare chi è stato presente in questi due anni e ha contribuito direttamente o indirettamente a questo lavoro finale.

Devo ringraziare, in primo luogo, il prof. Paris e il prof. Degli Esposti Boschi per la loro disponibilità nel supportare questo questo lavoro di tesi. Devo dire un "grazie" enorme a Steve, molto più di un educatore, molto più di un amico. Ci tengo a ringraziare tutti i miei amici che mi hanno sopportato e supportato. Ognuno a modo suo mi ha sostenuto e ha fatto sì che questi due anni potessero concludersi al meglio. Voglio anche ringraziare alcuni miei compagni di studi con cui ho condiviso questi ultimi due anni: Matteo, per tutte le discussioni e i confronti, e Roberto, compagno ciclista. Una nota speciale va ai miei ragazzi e ai miei co-educatori, parte fondamentale della mia vita, senza i quali quest'ultima sarebbe sicuramente un po' più grigia.

Infine, un grazie alla mia famiglia, in particolare ai miei nonni e ai miei genitori, che si sono sempre fidati ciecamente di me e che mi hanno sempre sostenuto in quello che facevo. Nonostante in questa tesi si parli di stima, penso sia difficile quantificare il loro contributo.

Contents

Introduction

This thesis consists in the application of Quantum Estimation Theory to a problem arising from gravity considerations about the existence of a minimum length.

The estimation theory is a branch of statistics that studies how parameters describing a probabilistic distribution can be estimated from sets of empirical data [27]. The main goal of estimation theory is to find the best estimator among all the possible functions of the outcomes so that it best approximates the real value of the parameter. Estimation theory plays a crucial role in finding the value of many physical quantities of interest, considering that lots of them are not directly observable (at the classical level too). One of the key theorems in estimation theory is the Cramer Rao Bound [9], which sets a lower bound on the variance of all the possible estimators. In particular, the estimators that saturate the Cramer Rao inequality are named efficient estimators and provide the most accurate estimate from the data sample. The quantum version for the Quantum Cramer Rao bound was proved, which sets an even lower bound on the variance of the estimator [27]. These results have been widely used in many metrological problems in quantum interferometry and quantum optics.

On the other side, the existence of a minimum length is a widely studied subject that has many effects on gravity theories [18]. One of the main consequences is the generalization of the uncertainty principle. These new relations are named Generalized Uncertainty Principle (GUP) and any Hamiltonian with a classical kinetic term is affected by them. In particular, an additional term proportional to this minimum length [10] appears in any Hamiltonian.

This thesis aims to study the achievable accuracy that an estimate can provide about the measurement of this minimum length using the tools of Quantum Estimation Theory.

On the basis that the quantity to be estimated is small, we choose to use perturbation theory throughout the thesis.

The thesis is structured as follows.

In chapter 1, we review the main results of the estimation theory. We start in section 1.2, introducing the reader to the fundamental concepts of statistics, such as the probability distribution of a random variable. Then, in the following session, we address the classical estimation problem, and we prove the Cramer Rao bound, which sets the lower bound on the estimation precision among all the possible estimators. Finally, we define the Fisher Information, and we describe its role in the estimation theory. In the last section 1.3, we move to the Quantum mechanics framework, and we study the differences due to quantum formalism. We notice that, according to the Born rule, the distribution probabilities of the outcomes strongly depends on the measurement implemented. Thus, we prove that there exists a more restrictive lower bound if we choose a suitable measurement procedure. This lower bound is the Quantum Cramer Rao bound, which involves a new quantity called Quantum Fisher Information. We study this quantity, and we find its relation with the geometry of the space state, proving that it is related to the distinguishability of states. Finally, we define the power of an estimate and the signal-noise ratio.

In chapter 2, we apply the estimation theory outlined in the previous chapter to the general problem of estimating the coupling parameter of a small generic perturbation. This study is preparatory for the second part of the thesis. In section 2.1, we study a pure static preparation, and we compute the Quantum Fisher Information, both exactly and perturbatively. We also compare the Fisher Information and the Quantum Fisher Information for several measurement procedures in order to understand which is optimal. Finally, we compute the Symmetric Logarithmic Derivative, from which we can determine the optimal **POVM** for the estimation procedure. In section 2.2 we study the role of time evolution in superpositions of states, computing the Quantum Fisher Information and also searching for the preparation that maximizes it. Eventually, in section 2.3 we study also the Quantum Fisher Information for thermal state preparations.

In chapter 3 we briefly show how the GUP arises from the assumption of a minimum length. We first see it from some thoughts experiments in 3.1 and eventually from a

device-independent argument. In the second section 3.2, we derive the new commutations relations and we prove that the main consequence is a universal perturbation that affects any Hamiltonian. The perturbation is proportional to a parameter strictly related to the minimum length. This fact is the starting point of our investigation in the second part of the thesis.

In chapter 4 we apply the result of chapter 2 to the estimation of the parameter that arises in 3.2. The aim is to find the best state preparation among different systems so that we have the higher Quantum Fisher Information, which means a better accuracy in the estimation process. The systems considered are the free particle, the infinite and finite square well and the 1-dimensional harmonic oscillator. In 4.1 we study perturbed energy eigenstates preparations, while in 4.2 we study superpositions of perturbed energy eigenstates. At the end of each section, we compare the results obtained in the four different systems. Moreover, in the last chapter 5, we investigate how the dimensionality of the systems affects the Quantum Fisher Information, studying the 2-dimensional Harmonic oscillator. Finally, in 5.1.3 we also study the entanglement and the coherence in the perturbed state preparations and how they are related to the Quantum Fisher Information.

Part I

Background Material

Chapter 1

Methods, ideas and tools of Quantum Estimation Theory

The estimation theory [9] deals with the problem of estimating the values of a set of parameters from a data set of empirical values. Differently from a statistical inference problem, where we do not know the probability distribution of the empirical values, in an estimation problem this is well known: what it is not known is the set of the parameter from which the distribution depends on.

The starting point of the estimation theory is the following question: "How should we use the data to form the best estimates?". However, we should define what is the best estimate first. The natural answer should be that the best estimate of the parameter is the estimate which falls nearest to the actual value of the parameter. Nevertheless, we need to keep in mind that our estimate of the parameter is a function of the empirical values and, as a consequence, it must be considered as an observed value of a specific random variable following a particular distribution function, the sampling distribution. Since the estimate is itself a random variable, we can not judge its goodness from a single estimate.

Instead, we should judge the estimation process from the shape of the sampling distribution: if the estimates are concentrated around a specific value, it will be more probable to observe another estimate that is very close to the others already observed. There are different methods we can use to measure the concentration of specific distribution: each

of them focuses on different features of the distribution and gives a distinct characterization of the best estimate. The most relevant concentration measures are the variance, the skewness and the kurtosis.

In our study, we will use the variance as a measure of concentration for the sample distribution. This chapter aims to prove the most relevant theorem of classical estimation theory, which is the Cramer-Rao inequality, that fixes a lower bound on the variance in the set of the estimator of the parameter. The Cramer-Rao inequality shows that the lower bound is the inverse of the Fisher Information, which is a statistical measure of the information carried by the random variable about the parameter. In the next step, we will focus on the quantum mechanics statistical description, studying the estimation theory in the quantum mechanics realm. In this framework, we can maximize the Classical Fisher Information among the set of all possible measurements. Consequently, we will prove the Quantum Cramer-Rao inequality, and we will introduce the Quantum Fisher Information, which provides an even lower bound on the estimation accuracy.

1.1 Introduction and notation

We start considering a probability distribution described by a one dimensional random variable ξ [2, 9]. Associated with ξ we have a set function $P(\mathcal{S})$ called the probability function of the probability distribution: $P(\mathcal{S})$ gives a number representing the probability that the event ξ occurs, considering all the events in S. A fundamental request is the normalization

$$P(\mathcal{R}) = 1, \tag{1.1}$$

where $\mathcal{R}$ is the union of all possible $\mathcal{S}$. An alternative description of the random variable ξ can be given by the distribution function $\mathbf{F}(x)$ defined as

$$\mathbf{F}(x) = P(\xi \leq x). \tag{1.2}$$

The distribution function represents the probability of finding the random variable ξ below the value of x. In the one dimensional case, where ξ is real number, we will use $F(x)$ instead of $P(\mathcal{S})$, given that the probability distribution is uniquely determined by either $P(\mathcal{S})$ or $\mathbf{F}(x)$.

If we are interested in the probability of finding the variable in very small interval $x < \xi < x + \Delta x$, we shall take the derivative of $\mathbf{F}(x)$. We thus find the probability density function (or the frequency function)

$$\mathbf{f}(x) = \frac{d\mathbf{F}(x)}{dx}. \tag{1.3}$$

It is a positive function, but differently from the $\mathbf{F}(x)$, it can assume values greater than one. Moreover, it is normalized as

$$\int_{-\infty}^{+\infty} \mathbf{f}(x) = 1. \tag{1.4}$$

Next, Let us consider a function of the random variable ξ. We call mean value of $g(\xi)$ the integral

$$\mathbf{E}(g(\xi)) = \int_{-\infty}^{+\infty} g(x)\mathbf{f}(x)dx. \tag{1.5}$$

An important properties of $\mathbf{E}$ is that if $g(\xi)$ and $h(\chi)$ are two functions of two random variable the mean value of $g(\xi) + h(\chi)$ is identically to

$$\mathbf{E}(g(\xi) + h(\chi)) = \mathbf{E}(g(\xi)) + \mathbf{E}(h(\chi)), \tag{1.6}$$

thus, independently of the nature of the dependence of ξ and χ, the mean function is linear.

A group of important quantity in statistics is the set of moments, that specify the shape of the distribution. Generally, there are different kind of moments: the general one are the moments of order ν,

$$\alpha_\nu = \mathbf{E}(\xi^\nu) = \int_{-\infty}^{+\infty} x^\nu \mathbf{f}(x)dx. \tag{1.7}$$

In particular, the first moment is equal to the mean value of the random variable ξ and we will be denoted by m

$$\alpha_1 = \mathbf{E}(\xi) = m. \tag{1.8}$$

The second kind of moment are the moments about a point c

$$\mathbf{E}[(\xi - c)^\nu] = \int_{-\infty}^{+\infty} (x - c)^\nu \mathbf{f}(x)dx. \tag{1.9}$$

The moments about the mean m are usually called central moments and are particularly important. We will denote them as

$$\mu_\nu = \mathbf{E}[(\xi - m)^\nu] = \int_{-\infty}^{+\infty} (x - m)^\nu \mathbf{f}(x) dx. \tag{1.10}$$

An interesting property is the following: if we consider the second moment about a generic point c, we have that it has a minimum when it is taken about the mean m

$$\mathbf{E}[(\xi - c)^2] = \mathbf{E}[(\xi - m + m - c)^2] = \mu_2 + (c - m)^2 \geq \mu_2. \tag{1.11}$$

In practical applications it is important to be able to describe the main features of a distribution by a certain number of parameters. They are called typical value of the distribution probability and some of them are the mean, the median and the mode.

At the same time is important to know how much the values of the random variable are spread around a typical value. A parameter describing this feature is usually called a measure of spread or dispersion. It is sometimes also called a measure of concentration, since dispersion and concentration are inverses of each other.

If we choose our typical value to be the mean m, then it is natural to consider the second moment about the mean to be a dispersion measure. In this case we call μ_2 the variance of the variable. In order to have the same dimension of m, it is usually taken the square root of μ_2, that is called the standard deviation $\mathbf{D}(\xi)$

$$\mathbf{D}^2(\xi) = \sigma^2 = \mu_2 = \mathbf{E}[(\xi - \mathbf{E}(\xi))^2]. \tag{1.12}$$

The properties of the mean $\mathbf{E}$ naturally extends to that of the variance $\mathbf{D}^2$.

In a symmetric distribution probability every moment of odd order about the mean is equal to 0. As a result, for a generic distribution, we can regard any odd moment as a measurement of asymmetry in the case they are different from 0. The simplest of these measure is the third central moment μ_3. In order to have a zero dimension measure, we divide it by σ^3, obtaining

$$\zeta_1 = \frac{\mu_3}{\sigma^3} \tag{1.13}$$

that is called the coefficient of skewness. If the asymmetry is given by the fact that there is a longer tail on the positive side, the cubes of the positive deviations will outweigh the

negative cubes, so that $\zeta_1 > 0$. On the contrary, if the tail is longer on the negative side we have that $\zeta_1 < 0$.

Another measure of asymmetry is given by the kurtosis. It is defined as the normalized fourth central moment

$$\kappa = \frac{\mu_4}{\sigma^4} \tag{1.14}$$

and its bounded by below by the skewness

$$\kappa \geq \zeta^2 + 1. \tag{1.15}$$

The interpretation of the kurtosis is the following: since any number less than one raised to the fourth power is almost 0, the μ_4 gives only information about the region outside the peak. Therefore, high values of kurtosis are given in two cases: when the probability is concentrated around the mean, because σ is very small, or, on the contrary, when the probability is concentrated in the tails. A measure of the asymmetry is given by the excess of kurtosis

$$\zeta_2 = \kappa - 3. \tag{1.16}$$

For a normal distribution $\zeta_2 = 0$, while if $\zeta_2 > 0$ the distribution probability has fatter tail and it is usually called super-Gaussian, while if $\zeta_2 < 0$ the distribution probability has thinner tail and it is usually called sub-Gaussian.

The next important notion we introduce is the concept of random sample [9, 35]. If we perform n-times an experiment concerning a random variable ξ, following the distribution function $\mathbf{F}(x)$, we obtain n independent observation values $\{x_1, \ldots, x_n\}$. The set of the possible values that ξ may assume is called the parent population, while the set of observed values $\{x_1, \ldots, x_n\}$ is named the random sample drawn from a population with distribution function $\mathbf{F}(x)$.

The distribution of the sample will then be defined as the distribution obtained by placing a weight equal to $1/n$ for each of the points x_i. The corresponding discrete distribution function is

$$\mathbf{F}^*(x) = \frac{\nu}{n} \tag{1.17}$$

where ν is the number of sample values that are $\leq x$.

From the probability distribution specified by $\mathbf{F}^*$ we can calculate all the characteristics previously defined. This will be a feature *of the sample* and it must be distinguished

from the feature *of the distribution/population* associated with the probability distribution of the random variable ξ.

The n time reiteration of an experiment give rise to a combined random variable $(x_1, \ldots, x_n)$ in a n-dimensional space. Considering that each sample value x_i is a random variable, any function of them is itself a random variable, with a distribution uniquely determined by the distribution of the x_i, i.e. $\mathbf{F}(x)$.

Furthermore, if we have a sequence of random sample $\mathcal{S}_i$ each of them made up of n sample values, for each sample we have a value of the random variable $g(\mathcal{S}_i)$: the sequence of $g(\mathcal{S}_i)$ constitutes a sequence of observed values of the random variable g. The probability density function associated with that random variable is called the sampling distribution of g.

We give a basic result that we will use later: if we take a one dimension sample $\{x_i\}$, where x_i are independent random variables, we obtain that for the arithmetic mean $\bar{x}$

$$\mathbf{E}(\bar{x}) = \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}(x_i) = m, \tag{1.18}$$

$$\mathbf{D}^2(\bar{x}) = \frac{1}{n^2}\sum_{i=1}^{n}\mathbf{D}^2(x_i) = \frac{\mu_2}{n}, \tag{1.19}$$

It follows that the sample mean $\bar{x}$, as random variable, converges to the population mean m as $n \to \infty$ since $\sigma(\bar{x}) = \sigma/\sqrt{n} \to 0$ in this limit.

1.2 Classical Estimation Theory

1.2.1 What is Estimation Theory?

Suppose that we have a data sample which follows a probability distribution depending on a set of unknown parameters. For example, Let us consider the measurement of a voltage a [39]. We assume that the voltage is between $-V$ and $+V$ and the measurement is corrupted by noise which may be modeled as an independent additive zero-mean Gaussian random variable n. Thus, the observed variable is

$$r = a + n, \tag{1.20}$$

and the probability density governing the observation process is

$$p_{r|a}(R|A) = p_n(R-A) = \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left\{-\frac{(R-A)^2}{2\sigma_n^2}\right\}. \tag{1.21}$$

In general, a model of estimation consists of 4 components

1. The Parameter Space: it is the set of possible values of the parameter we want to estimate. A possible value is a point in the parameter space.

2. Probabilistic Mapping from Parameter Space to Observational Space: it is the probability law that governs the effect of a possible observation of the value of the parameter. In the example case, this is the probability density function as a function of a.

3. Observation space, or Sample Space: this is a finite dimensional space and in the example is the set $\mathbf{R}$ of observations of the observable r.

4. Estimation Rule: after observing $\mathbf{R}$ we want to estimate the value of a. The function from the Observation Space to the Parameter Space is the Estimation Rule $a^*(\mathbf{R})$, or an estimator.

There will be always a set of infinite estimation rule of the set $\mathbf{R}$ that might be proposed as an estimation of the parameter a, but which one gives the best Estimation Rule for the parameter? Another question arises: what do we mean by *best estimation*?.

Firstly, we need to answer the last question in order only to consider the first[9]. A natural answer should be that the best estimation is the estimation that falls nearest to the real value of the parameter to be estimated. However, since every Estimation Rule is a function of the observation values, it must be considered as an observation value itself. Therefore we cannot predict how much the individual value of the parameter is near the real value from a single set of observations and moreover, we can not judge the goodness of an estimate by a single sample set. Thus, to answer the question, we need to move from a single set of observations $\mathbf{R}$ to several sets of a sample $\mathbf{R}_i$. Each of them gives an estimate of the parameter and the distribution of these values is called the sampling distribution. An estimate will be better if the Sampling Distribution shows a higher concentration about the real value: thus the question of what is the best estimate can

be rephrased as "What is the best way to use our data to obtain estimation rule with maximum concentration?"

In general, the concentration of a distribution may be measured in various ways, and any measure of dispersion can be associated a definition of best estimate. It means that there is arbitrariness about the choice of the best estimation rule, depending upon the meaning we want to give to the concentration (or dispersion) of a distribution. In the sequel, we will consider only the measure of dispersion associated with the variance.

1.2.2 The Cramer Rao Bound

Having established what we mean by best estimate, we may ask if there exists a lower bound on it. The answer is positive: there exists a inequality that any variance of an estimation must satisfy. Consequently, any estimation procedure that saturates the inequality may be considered as a best estimation. But Let us proceed by grade.

Suppose we have a parameter γ we want to estimate [9, 39]. We assume that γ belongs to an interval $\mathcal{A}$, the parameter space. Furthermore, suppose we have a distribution function $\mathbf{F}(x;\gamma)$ that depends on the parameter γ and we have a data sample $\{x_1,\ldots,x_n\} \in \mathcal{S}_n$ where each sample follows $\mathbf{F}(x;\gamma)$. We want to show that, under some regularities assumption, the variance $\mathbf{Var}(\gamma^*)$ can not fall below a positive value that depends only on the probability distribution function $\mathbf{f}(x;\gamma)$ and on the size n of the sample: this is the so called Cramer Rao Bound. One of the first proof of the Cramer-Rao theorem can be found in [9]. However, here we follow a different line depicted in [25, 22]. We will see the proof only for continuous probability density, since the discrete case is very similar, and we consider n fixed.

Some assumptions must be taken in order to prove the theorem. First, that the derivative of the $\log \mathbf{f}(x|\gamma)$ exists for all x and γ. Secondly, that the variance of $\log \mathbf{f}(x|\gamma)$ is finite for all γ. Finally, that we can exchange the derivation ∂_γ and the integration on the random space where x belongs.[1]

Therefore, we consider our sample $\{x_i\}$, with $i = 1,\ldots n$, following the probability distribution $\mathbf{f}(x|\gamma)$, and we assume that we have an unbiased estimator $\gamma^*(\vec{x})$[2]. Its mean

[1] Indeed, if we relax this hypothesis, we will obtain generalization of the Cramer Rao Bound, see [36].

[2] In order not to create confusion, we restate the distinction between γ and γ^*: the former is an

value is given by

$$\mathbf{E}[\gamma^*(\chi)] = \int d\vec{x}\gamma^*(\vec{x}) \prod_{i=1}^{n} \mathbf{f}(x_i|\gamma) = \gamma. \tag{1.23}$$

Now, if we derive by γ we see that

$$\begin{aligned} 1 = \frac{\partial \mathbf{E}[\gamma^*(\{x_i\})]}{\partial\gamma} &= \int d\vec{x}\gamma^*(\vec{x}) \frac{\partial}{\partial\gamma} \left(\prod_{i=1}^{n} \mathbf{f}(x_i|\gamma) \right) = \\ &= \int d\vec{x}\gamma^* \frac{\partial}{\partial\gamma} \left[\log \prod_{i=1}^{n} \mathbf{f}(x_i|\gamma) \right] \prod_{i=1}^{n} \mathbf{f}(x_i|\gamma) = \\ &= \mathbf{E}\left[\gamma^* \frac{\partial}{\partial\gamma} \left(\log \prod_{i=1}^{n} \mathbf{f}(x_i|\gamma) \right) \right] = \mathbf{E}\Big[\gamma^* s_\gamma\Big], \end{aligned} \tag{1.24}$$

where we have used the following identity

$$\frac{\partial_\gamma \left(\prod_{i=1}^{n} \mathbf{f}(x_i|\gamma)\right)}{\prod_{i=1}^{n} \mathbf{f}(x_i|\gamma)} = \sum_{i=1}^{n} \frac{\partial_\gamma \left[\mathbf{f}(x_i|\gamma)\right]}{\mathbf{f}(x_i|\gamma)} = \frac{\partial}{\partial\gamma} \log \prod_{i=1}^{n} \mathbf{f}(x_i|\gamma). \tag{1.25}$$

The function $s_\gamma(\{x_i\})$ is called score of the distribution and we will study it in the next section.

Considering two random variable X, Y, we can define the inner product in the space of the random variable induced by the mean value $\mathbf{E}$ as $\langle X, Y\rangle = \mathbf{E}(XY)$. Moreover, using the Schwartz inequality, which in this context has the form

$$|\mathbf{E}(XY)|^2 \leq \mathbf{E}(X^2)\mathbf{E}(Y^2), \tag{1.26}$$

we can prove that the covariance between X and Y satisfies the following inequality

$$\begin{aligned} |\mathbf{Cov}(X,Y)|^2 = |\mathbf{E}[(X - \mathbf{E}(X))(Y - \mathbf{E}(Y))]|^2 &\leq \mathbf{E}[(X - \mathbf{E}(X))^2]\mathbf{E}[(Y - \mathbf{E}(Y))^2] = \\ &= \mathbf{D}^2(X)\mathbf{D}^2(Y). \end{aligned} \tag{1.27}$$

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ordinary analytic variable which can assume any value in the parameter space, the latter is a function of the sample values $\gamma^*(\vec{x})$ and is what we have called the estimator, that is any function of the random sample values to the Parameter Space

$$\gamma^* : \mathcal{S}_n \to \mathcal{A}. \tag{1.22}$$

This function is itself a random variable an follows the sample distribution. We say that the estimator is unbiased if its mean value is the value of the parameter $\mathbf{E}[\gamma^*(\chi)] = \gamma$ for all the possible sample χ.

If we now consider as our random variable γ^* and s_γ, the covariance between them is

$$\mathbf{Cov}(\gamma^*, s_\gamma) = \mathbf{E}(\gamma^* s_\gamma) - \mathbf{E}(\gamma^*)\mathbf{E}(s_\gamma) = \mathbf{E}(\gamma^* s_\gamma), \tag{1.28}$$

since it can be easily seen that the expectation value of the score is null. As a result of this last identity and of (1.24), the covariance of γ^* and s_γ is equal to 1. But the inequality (1.27) holds, thus we have that

$$1 \leq \mathbf{D}^2(\gamma^*)\mathbf{D}^2(s_\gamma). \tag{1.29}$$

Computing the variance of the score we obtain that

$$\mathbf{D}^2(s_\gamma) = \mathbf{E}[s_\gamma^2] = \mathbf{E}\left[\left(\partial_\gamma \log \prod_{i=1}^{n} \mathbf{f}(x_i|\gamma)\right)^2\right] = \mathbf{E}\left[\left(\sum_{i=1}^{n} \partial_\gamma \log \mathbf{f}(x_i|\gamma)\right)^2\right] =$$

$$= \sum_i \sum_j \mathbf{E}\left[\left(\partial_\gamma \log \mathbf{f}(x_i|\gamma)\right)\left(\partial_\gamma \log \mathbf{f}(x_j|\gamma)\right)\right] = n\mathbf{E}\left[\left(\partial_\gamma \log \mathbf{f}(x|\gamma)\right)^2\right], \tag{1.30}$$

where we have used the fact that the

$$\mathbf{E}\left[\partial_\gamma \log \mathbf{f}(x|\gamma)\right] = 0, \tag{1.31}$$

and the fact the the outcomes x_i and x_j are independent and we have only the terms when $i = j$, which are exactly n terms. We can finally rewrite (1.29) as

$$\mathbf{D}^2(\gamma^*) \geq \frac{1}{n\displaystyle\int_{-\infty}^{+\infty} \left(\frac{\partial \log \mathbf{f}(x;\gamma)}{\partial \gamma}\right)^2 \mathbf{f}(x;\gamma)dx}, \tag{1.32}$$

that is the celebrated Cramer Rao bound. In particular, we call

$$\mathcal{F}_c(\gamma) = \mathbf{E}\left[\left(\partial_\gamma \log \mathbf{f}(x|\gamma)\right)^2\right] = \int_{-\infty}^{+\infty} \left(\frac{\partial \log \mathbf{f}(x;\gamma)}{\partial \gamma}\right)^2 \mathbf{f}(x;\gamma)dx \tag{1.33}$$

the classical Fisher Information (F.I. later on).

The inequality states that, among all the possible data processing, there exists a lower bound on the variance that the estimator γ^* can achieve [15], which is independent of the estimator used, indicating that it is a universal bound. Thus, if the estimator

reaches the lower bound, it is said that the estimator is fully efficient and it is called the minimum variance unbiased estimator. The saturation of the Cramer Rao bound is only possible if (1.27) becomes an equality. As it is well know, a sufficient condition is that $\gamma^* - \mathbf{E}(\gamma^*)$ is proportional to s_γ, that is

$$\partial_\gamma \log \prod_{i=1}^{n} \mathbf{f}(x_i|\gamma) = K(\gamma, n)(\gamma^*(\vec{x}) - \gamma) \tag{1.34}$$

where K is a proportionality factor. The existence of an efficient estimator, however, is not guaranteed in all the statistical models. Nevertheless, maximum-likelihood estimators, for instance, are efficient estimator in the asymptotic limit of $n \to \infty$ [9].

1.2.3 Fisher Information

We have introduced in the last section the classical Fisher Information $\mathcal{F}_c(\gamma)$ and in this section we would like to deepen this important quantity. Given a probability distributions depending on a parameter γ, the F.I. is a way of measuring the amount of information that the random variable x carries about γ. To see it, we need to consider the likelihood function.

The Likelihood function $\mathcal{L}_{\{x\}}$ describes the plausibility of the value of a parameter when we fix a set of observations [9, 35]. In the case of a random sample $\{x_1, \ldots, x_n\}$ it is equal to the joint probability distribution, while if $n = 1$, the likelihood function is nothing but the probability density function seen as a function of the sample space, i.e.

$$\mathcal{L}_x : \mathcal{A} \to \mathbb{R}, \quad \mathcal{L}_x(\gamma) = \mathbf{p}(x|\gamma). \tag{1.35}$$

In the estimation procedure, we want to infer the value of the parameters in such a way that the observed random sample has the maximum probability of occurring. This is the maximum-likelihood estimation, by means of which we select the parameters that make observed data more probable.

Often, it is more convenient to work with the logarithmic form $l_x = \log \mathcal{L}_x$ of the likelihood function. In the problem of maximization, since the logarithmic is a strictly increasing function, maximizing the log-likelihood gives the same result of maximizing the likelihood function. Moreover, since in the likelihood we have products, in the log-likelihood we have sums and performing the derivative for a sum is easier than for a

product. In order to find the maximum of the log-likelihood function, we study its derivative: this function is known as the score [30, 4]

$$s_\gamma(x) = \frac{\partial \log \mathcal{L}_x(\gamma)}{\partial \gamma} = \frac{\partial \log \mathbf{p}(x|\gamma)}{\partial \gamma} = \frac{\partial_\gamma \mathbf{p}(x|\gamma)}{\mathbf{p}(x|\gamma)}. \tag{1.36}$$

We stress out that score depends on the data at which the likelihood function is evaluated. Thus, if we solve the equation $s_\gamma(x) = 0$, we find the parameter that would better reproduce the observed data, and the rule that give us the parameter from the data is what we have previously called the estimator $a^*_{ML}(\{x\})$.

However, we would like to quantify how much information about γ is carried by the random variable x. We observe that, if the likelihood function $\mathcal{L}_x(\gamma)$ is sharply peaked with respects to changes in γ, it will be easier to infer the correct value of γ from a random sample. It results that, if $\mathbf{p}(x|\gamma)$ satisfy this property, then the random variable will provide significant information about the parameter γ. Conversely, if $\mathcal{L}_x(\gamma)$ or $\mathbf{p}(x|\gamma)$ is flat and spread out, it would require a larger number of random samples to estimate the parameter γ better, meaning that the probability distribution provides little information on γ. These observations lead us to the conclusion that a good measure of the information may be given by the score function, since it is the γ derivative [3] of the probability density function: a significant value of this derivative would mean a very peaked function of $\mathbf{p}(x|\gamma)$ in the parameter space, and consequently it would bring much information about the value of γ. However, the derivative of the score still depends on the random sample. To avoid that, we can average the $s_\gamma(x)$ over the set of possible outcomes. However, it can be easily found that the mean is identically null. Therefore, in order to characterize the amount of information, we move to the variance of the score, which is the expectation value of the square of the score. We eventually find that [30]

$$\mathbf{Var}(s(\gamma)) = \mathbf{E}\left[\left(\frac{\partial \log \mathbf{p}(x|\gamma)}{\partial \gamma}\right)^2\right] = -\mathbf{E}\left[\frac{\partial^2 \log \mathbf{p}(x|\gamma)}{\partial \gamma^2}\right] = \mathcal{F}_c(\gamma) \tag{1.37}$$

which is the result found in the previous section. We conclude that the Fisher Information can be considered an informational measure of the amount of information carried by the probability distribution $\mathbf{p}(x|\gamma)$ on the parameter γ, since the higher is the variance of

[3] Or the gradient in the case of multi-parameter estimation

the score (that is, the higher is the Fisher Information), the sharper is the likelihood function around its peak. But a very sharp likelihood function means that from a set of observation we have a larger probability to estimate the true value of the parameter. As a result, we can say that a distribution with a larger Fisher Information brings along with it a larger amount of information on γ, where with information we mean the possibility to have a better inference on the parameter.

1.2.4 Application of the Cramer Rao Bound

To see how the Cramer Rao bound works, we recall the problem of the voltage that we introduced in 1.2.1. Suppose that we have N outcomes $\vec{r}$ of the random variable (1.20). Thus, the score of the function is [39]

$$\frac{\partial \log p_{\vec{r}|a}(\vec{R}|A)}{\partial A} = \frac{N}{\sigma_n^2}\left(\frac{1}{N}\sum_{i=1}^{N} R_i - A\right) \tag{1.38}$$

In order to use the Cramer Rao bound, we need to verify that our estimate is unbiased[4]. Following the previous section, we need to solve the likelihood equation

$$\frac{N}{\sigma_n^2}\left(\frac{1}{N}\sum_{i=1}^{N} R_i - A\right)\Bigg|_{A=\hat{a}_{ML}(\vec{R})} = 0, \tag{1.39}$$

which lead us to the estimator

$$\hat{a}_{ML}(\vec{R}) = \frac{1}{N}\sum_{i=1}^{N} R_i. \tag{1.40}$$

As a result, the mean value of the estimate is

$$\mathbf{E}(\hat{a}_{ML}(\vec{R})) = \frac{1}{N}\sum_{i=1}^{N}\mathbf{E}(R_i) = \frac{1}{N}\sum_{i=1}^{N} A = A, \tag{1.41}$$

and as we expected the estimator is unbiased. Therefore, to determine the Cramer Rao bound, we use the second expression of the Fisher Information provided in (1.37) that is

$$\frac{\partial^2 \log \mathbf{p}_{\vec{r}|a}(\vec{R}|A)}{\partial A^2} = -\frac{N}{\sigma_n^2}, \tag{1.42}$$

[4]There exists a version of the Cramer Rao bound for unbiased estimator, but we are not interested in it in our discussion.

and consequently the Fisher Information is

$$\mathcal{F}_c(A) = \frac{N}{\sigma_n^2}. \tag{1.43}$$

We eventually obtain the Cramer Rao inequality as

$$\mathbf{D}^2(\hat{a}_{ML}(\vec{R})) \geq \frac{\sigma_n^2}{N} \tag{1.44}$$

However, we notice that the equation (1.38) has the same form of (1.34), thus the in this case the inequality is saturated

$$\mathbf{D}^2(\hat{a}_{ML}(\vec{R})) = \frac{\sigma_n^2}{N} \tag{1.45}$$

and we have an efficient estimator.

1.3 Quantum Estimation Theory

When we move to quantum mechanics, the probability density function is determined by both the states and the measurement procedure and it is determined by Born rule. Therefore, we can generalize some statistical concepts to their quantum equivalents. The aim of this section is to study the consequence of this extension in the framework of the estimation theory.

1.3.1 There is more in quantum theory?

Postulates from quantum theory [26, 28, 16] say that a general quantum measurement is described by a collection $\{\mathcal{M}_m\}$ of measurement operators satisfying the following properties

$$\mathbf{p}(m) = \langle\psi|\mathcal{M}_m^\dagger \mathcal{M}_m|\psi\rangle \geq 0, \tag{1.46}$$

$$\sum_m \mathcal{M}_m^\dagger \mathcal{M}_m = \mathbf{1}, \tag{1.47}$$

$$|\psi_{p.m.}\rangle \longrightarrow \frac{\mathcal{M}_m|\psi\rangle}{\sqrt{\mathbf{p}(m)}}. \tag{1.48}$$

The first equation describes the measurement statistics, while the second one assures that the probabilities sums to one. Finally, the third one specifies the post-measurement state.

There are cases where there is no interest in the resulting state after the measurement. In that case we see that the only quantities of interest are the operators $\mathcal{E}_m = \mathcal{M}_m^\dagger \mathcal{M}_m$. From these new operators $\mathcal{E}_m$ it can be recovered all the statistical information about the measurement process since the first two properties (1.46) and (1.47) can be reformulated as

$$\mathbf{p}(m) = \langle\psi|\mathcal{E}_m|\psi\rangle, \tag{1.49}$$

$$\sum \mathcal{E}_m = \mathbf{1}. \tag{1.50}$$

The set $\{\mathcal{E}_m\}$ is called a **POVM**, which means Positive Operator-Valued Measure, and it is completely equivalent to the general quantum measurement formalism, apart from giving no information about the post measurement states.

We can see this fact if we consider a set of **POVM** $\{\mathcal{E}_m\}$ and we try to recover a set of general measurement operator. For every **POVM** $\mathcal{E}_m$ we can set a general measurement operator as $\mathcal{M}_m = \mathcal{U}_m\sqrt{\mathcal{E}_m}$ where $\mathcal{U}_m$ is an arbitrary unitary operator. The set of $\{\mathcal{M}_m\}$ thus defined form a set of general measurement since the measurement statistics of the **POVM** and of $\{\mathcal{M}_m\}$ are the same

$$\langle\psi|\mathcal{E}_m|\psi\rangle = \langle\psi|\mathcal{M}_m^\dagger\mathcal{M}_m|\psi\rangle, \tag{1.51}$$

$$\sum \mathcal{E}_m = \mathbf{1} = \sum \mathcal{M}_m^\dagger\mathcal{M}_m. \tag{1.52}$$

Therefore for any **POVM** there are infinite possible set of general measurement operators that realizes the **POVM** due to the arbitrariness of $\mathcal{U}_m$, and in general the post measurement state is arbitrary too as it is nothing but

$$|\psi_{p.m.}\rangle \longrightarrow \frac{\mathcal{M}_m|\psi\rangle}{\sqrt{\mathbf{p}(m)}} = \mathcal{U}_m\frac{\sqrt{\mathcal{E}_m}|\psi}{\sqrt{\mathbf{p}(m)}}. \tag{1.53}$$

In this way we have proved that the post measurement states is not fixed unless we choose a unitary operation $\mathcal{U}_m$.

In the global set of all possible general measurement there is a special case of interest. A request that it is often but not always satisfied is that the set of general measurement

operators is made up of orthogonal projectors $\{\mathcal{P}_m\}$. It follows that each of the $\mathcal{P}_m$ is Hermitian and satisfy

$$\mathcal{P}_{m'}\mathcal{P}_m = \delta_{m,m'}\mathcal{P}_m \tag{1.54}$$

They describe a projective measurement and satisfy some properties. Firstly, they admit a spectral decomposition

$$\mathcal{P} = \sum_m m\mathcal{P}_m. \tag{1.55}$$

Secondly, the probability of the outcomes m and the post measurement states are

$$\mathbf{p}(m) = \langle\psi|\mathcal{P}_m|\psi\rangle, \tag{1.56}$$

$$|\psi_{p.m.}\rangle \longrightarrow \frac{\mathcal{P}_m|\psi\rangle}{\sqrt{\mathbf{p}(m)}}. \tag{1.57}$$

Thirdly, the average value of the observable $\mathcal{P}$ described by a set o projective measurement is given by

$$\mathbf{E}(\mathcal{P}) = \langle\psi|\mathcal{P}|\psi\rangle = \mathrm{tr}\{\mathcal{P}|\psi\rangle\langle\psi|\}, \tag{1.58}$$

and the standard deviation is

$$(\Delta\mathcal{P})^2 = \langle\mathcal{P}^2\rangle - \langle\mathcal{P}\rangle^2. \tag{1.59}$$

The projective measurement formalism has the important feature of repeatability, that is specific of some kind of measure. Therefore is a subset of general measurement formalism.

The formalism here exposed can be easily reformulated for quantum density matrix ρ, which can describes both pure states and mixed states. We recall the most important properties without deriving them. The Born rule is

$$\mathbf{p}(m) = \mathrm{tr}\big\{\mathcal{M}_m^\dagger\mathcal{M}_m\rho\big\}, \tag{1.60}$$

while the post measurement state is

$$\rho_{p.m.} \longrightarrow \sum_m \mathcal{M}_m\rho\mathcal{M}_m^\dagger. \tag{1.61}$$

After this introduction on measurement in quantum mechanics, we focus on the estimation problem. When we consider the quantum theory, the usual estimation procedure consists of recovering the value of a parameter γ, which is encoded in the state ρ_γ. In

general, ρ_γ lives in a Hilbert space $\mathcal{H}$ while, as usual, γ lives in the parameter space $\mathcal{A}$. This is called a quantum statistical model and it is the quantum generalization of the classical statistical model discussed before. The map $\gamma \to \rho_\gamma$ gives the coordinate map on the statistical manifold.

The issue in estimating a parameter in the quantum theory framework is that generally speaking, γ is not an observable. Since in quantum theory the measurement process of a quantity is fixed by the axiom stated above, the direct measure of a quantity that is not an observable is impossible, and the only way left is to estimate it. This is nothing new since many quantities of interest in classical theories are already unobservable and must be estimated from indirect measurements (we may think about the electric field as the most significant example).

At the same time, the fact that the measurement in quantum mechanics follows the axiom in eqs. (1.46) to (1.48) has some advantages. In fact, in the estimation procedure, we have extra freedom in choosing the measurement strategies that we want to implement. Clearly, among all the possible measurements, we want to select the one that can lead to the most accurate estimation. Since the lower bound on the accuracy of an estimate is encoded in the Fisher Information $\mathcal{F}_c(\gamma)$, we need to maximize it over all the possible measurement apparatus [6, 30, 13, 27]. In this manner, we find the ultimate bound to experimental accuracy.

The strategy here is not to work on the elaboration of the data after the experiment, as we did when we found the Cramer Rao bound, that is, minimizing the error on the set of the estimators. Instead, we focus on the probability distribution itself, which depends on the choice of the measurement, as the Born rule (1.60) indicates. This are the additional features, stated in the title of the section, that Quantum mechanics brings about.

1.3.2 The Quantum Cramer Rao Bound

The axioms of quantum mechanics specifies the rule for computing the probability of measuring values of some observable. In the case of study, our statistical model is parameterized by γ and every measure will depend on its value. In general, a **POVM**

$\{\mathcal{E}_m\}$ is introduced and the probabilities are

$$\mathbf{p}(m|\gamma) = \mathrm{tr}\{\mathcal{E}_m \rho_\gamma\} \tag{1.62}$$

where it has been assumed that the **POVM** is independent from γ.

A significant quantity is the Symmetric Logarithmic Derivative (hereinafter SLD) [27]. It is the operator Λ_γ which satisfy the equation

$$\Lambda_\gamma \rho_\gamma + \rho_\gamma \Lambda_\gamma = 2\frac{\partial \rho_\gamma}{\partial \gamma}. \tag{1.63}$$

In order to express the classical Fisher Information as a function of the state and of the measurement, we need to calculate $\partial_\gamma \mathbf{p}(m|\gamma)$. This is easily computed as

$$\partial_\gamma \mathbf{p}(m|\gamma) = \partial_\gamma \,\mathrm{tr}\{\mathcal{E}_m \rho_\gamma\} = \mathrm{tr}\{\mathcal{E}_m \partial_\gamma \rho_\gamma\} = \mathrm{Re}\{\mathrm{tr}\{\mathcal{E}_m \Lambda_\gamma \rho_\gamma\}\}, \tag{1.64}$$

where we have used the fact that $\partial_\gamma \rho_\gamma = \mathrm{Re}\{\Lambda_\gamma \rho_\gamma\}$. It follows that the classical Fisher Information is

$$\mathcal{F}_c(\gamma) = \int_{-\infty}^{+\infty} dm \frac{(\mathrm{Re}\{\mathrm{tr}\{\mathcal{E}_m \Lambda_\gamma \rho_\gamma\}\})^2}{\mathrm{tr}\{\mathcal{E}_m \rho_\gamma\}}. \tag{1.65}$$

The first minimization step taken in section 1.2.2 was related to the minimization on the set of the estimators. This means that (1.65) establish the classical bound achievable with an efficient data processing, for instance the maximum likelihood.

On the other side, it has been noted that in the quantum world the optimization can be also taken over the set of all quantum measurement. Now, we try to maximize (1.65) over $\{\mathcal{E}_m\}$ [27]. Since for any imaginary number z the following property holds

$$\mathrm{Re}\{z\}^2 \leq \mathrm{Re}\{z\}^2 + \mathrm{Im}\{z\}^2 = |z|^2, \tag{1.66}$$

we have that

$$\mathcal{F}_c(\gamma) \leq \int_{-\infty}^{+\infty} dm \frac{|\mathrm{tr}\{\mathcal{E}_m \Lambda_\gamma \rho_\gamma\}|^2}{\mathrm{tr}\{\mathcal{E}_m \rho_\gamma\}}. \tag{1.67}$$

At the same time the argument of the integral can be written in a more convenient way as

$$\begin{aligned} \frac{|\mathrm{tr}\{\mathcal{E}_m \Lambda_\gamma \rho_\gamma\}|^2}{\mathrm{tr}\{\mathcal{E}_m \rho_\gamma\}} &= \left| \frac{\mathrm{tr}\{\mathcal{E}_m \Lambda_\gamma \rho_\gamma\}}{\sqrt{\mathrm{tr}\{\mathcal{E}_m \rho_\gamma\}}} \right|^2 = \left| \mathrm{tr}\left\{ \frac{\rho_\gamma \mathcal{E}_m \Lambda_\gamma}{\sqrt{\mathrm{tr}\{\mathcal{E}_m \rho_\gamma\}}} \right\} \right|^2 = \\ &= \left| \mathrm{tr}\left\{ \frac{\sqrt{\rho_\gamma}\sqrt{\mathcal{E}_m}}{\sqrt{\mathrm{tr}\{\mathcal{E}_m \rho_\gamma\}}} \sqrt{\mathcal{E}_m} \Lambda_\gamma \sqrt{\rho_\gamma} \right\} \right|^2. \end{aligned} \tag{1.68}$$

Let us consider the scalar product defined as the trace operation $\mathrm{tr}\{\mathcal{X}^\dagger\mathcal{Y}\}$. The Schwartz inequality states that

$$\left|\mathrm{tr}\{\mathcal{X}^\dagger\mathcal{Y}\}\right|^2 \leq \mathrm{tr}\{\mathcal{X}^\dagger\mathcal{X}\}\,\mathrm{tr}\{\mathcal{Y}^\dagger\mathcal{Y}\}, \tag{1.69}$$

and applying it to the (1.68) we obtain

$$\left|\mathrm{tr}\left\{\frac{\sqrt{\rho_\gamma}\sqrt{\mathcal{E}_m}}{\sqrt{\mathrm{tr}\{\mathcal{E}_m\rho_\gamma\}}}\sqrt{\mathcal{E}_m}\Lambda_\gamma\sqrt{\rho_\gamma}\right\}\right|^2 \leq \frac{\mathrm{tr}\{\rho_\gamma\mathcal{E}_m\}}{\mathrm{tr}\{\mathcal{E}_m\rho_\gamma\}}\,\mathrm{tr}\{\rho_\gamma\Lambda_\gamma\mathcal{E}_m\Lambda_\gamma\} = \mathrm{tr}\{\rho_\gamma\Lambda_\gamma\mathcal{E}_m\Lambda_\gamma\}. \tag{1.70}$$

Using this inequality, the measurement dependence $\{\mathcal{E}_m\}$ of the $\mathcal{F}_c$ can be removed, and exploiting the completeness of the **POVM** the optimization leads to

$$\mathcal{F}_c(\gamma) \leq \int_{-\infty}^{+\infty} dm \frac{|\mathrm{tr}\{\mathcal{E}_m\Lambda_\gamma\rho_\gamma\}|^2}{\mathrm{tr}\{\mathcal{E}_m\rho_\gamma\}} \leq \int_{-\infty}^{+\infty} dm\,\mathrm{tr}\{\rho_\gamma\Lambda_\gamma\mathcal{E}_m\Lambda_\gamma\} = \tag{1.71}$$

$$= \mathrm{tr}\{\rho_\gamma\Lambda_\gamma^2\} = \mathcal{F}_q(\gamma) \tag{1.72}$$

The quantity $\mathcal{F}_q(\gamma)$ is the Quantum Fisher Information. It follows that we have a new lower bound for the variance of the estimator of the parameter γ^* which is given by

$$\mathbf{Var}(\gamma^*) \geq \frac{1}{\mathcal{F}_c(\gamma)} \geq \frac{1}{\mathcal{F}_q(\gamma)}. \tag{1.73}$$

This is the Quantum Cramer Rao bound [6, 27]. Consequently, similarly to the Classical Fisher Information, the Quantum Fisher Information represents the reachable accuracy achievable in the quantum statistical model among all possible **POVM** implementations.

If we want to find the optimal quantum measurement for the estimation process, we need to investigate which **POVM** saturates both inequality in (1.71), or more specifically which **POVM** saturates simultaneously (1.67) and (1.70) [27]. For this specific **POVM** we have that the classical Fisher information is equal to its quantum counterpart.

The first inequality (1.67) becomes an equality if and only if the $\mathrm{tr}\{\mathcal{E}_m\Lambda_\gamma\rho_\gamma\}$ is a real number for every $\gamma \in \mathcal{A}$. Instead the second inequality (1.70) is a Schwartz-inequality with

$$\mathcal{X}^\dagger = \frac{\sqrt{\rho_\gamma}\sqrt{\mathcal{E}_m}}{\sqrt{\mathrm{tr}\{\mathcal{E}_m\rho_\gamma\}}} \tag{1.74}$$

$$\mathcal{Y} = \sqrt{\mathcal{E}_m}\Lambda_\gamma\sqrt{\rho_\gamma} \tag{1.75}$$

and we know that we have an equality if and only if $\mathcal{X} \propto \mathcal{Y}$, that is

$$\frac{\sqrt{\mathcal{E}_m}\sqrt{\rho_\gamma}}{\sqrt{\mathrm{tr}\{\mathcal{E}_m\rho_\gamma\}}} \propto \sqrt{\mathcal{E}_m}\Lambda_\gamma\sqrt{\rho_\gamma}. \tag{1.76}$$

We see that the solution of these equations is given by the set $\mathcal{E}_m$ of projectors over the eigenstates of Λ_γ [27], which can be easily seen expanding the state and the operator in their basis.

Therefore, the optimal **POVM** for the estimation of the parameter is given by the projective measurement on the eigenstates of Λ_γ. It should be emphasized that Λ_γ may not represent the optimal observable to be measured since the saturations specify the **POVM**, not the estimator, which is a function of the eigenvalues of Λ_γ and which corresponds to a post-measurement classical data processing. In other words, the **POVM** thus found saturates only the inequality $\mathcal{F}_c(\gamma) \leq \mathcal{F}_q(\gamma)$ but in general it does not saturate the first inequality in the Cramer Rao

$$\mathbf{Var}(\gamma^*) \geq \frac{1}{\mathcal{F}_c} = \frac{1}{\mathcal{F}_q}. \tag{1.77}$$

In fact, it can be easily seen that the variance of the observable defined by Λ_γ is

$$\mathbf{Var}(\Lambda_\gamma) = \langle\Lambda_\gamma^2\rangle - \langle\Lambda_\gamma\rangle^2 = \mathcal{F}_q, \tag{1.78}$$

since $\mathrm{tr}\{\Lambda_\gamma\rho_\gamma\} = 0$.

However, an explicit form of the optimal quantum estimator is be given by

$$\mathcal{O}_\gamma = \gamma\mathbf{1} + \frac{\Lambda_\gamma}{\mathcal{F}_q(\gamma)}. \tag{1.79}$$

This can be easily seen computing the variance of $\mathcal{O}_\gamma$

$$\langle\mathcal{O}_\gamma\rangle = \mathrm{tr}\{\rho_\gamma\mathcal{O}_\gamma\} = \gamma, \tag{1.80}$$

$$\langle\mathcal{O}_\gamma^2\rangle = \mathrm{tr}\big\{\rho_\gamma\mathcal{O}_\gamma^2\big\} = \gamma^2 + \frac{\mathrm{tr}\big\{\rho_\gamma\Lambda_\gamma^2\big\}}{\mathcal{F}_q(\gamma)^2} = \gamma^2 + \frac{1}{\mathcal{F}_q(\gamma)}, \tag{1.81}$$

and we obtain that it saturates the Quantum Cramer Rao inequality

$$\mathbf{Var}(\mathcal{O}_\gamma) = (\Delta\mathcal{O}_\gamma)^2 = \langle\mathcal{O}_\gamma^2\rangle - \langle\mathcal{O}_\gamma\rangle^2 = \frac{1}{\mathcal{F}_q(\gamma)}. \tag{1.82}$$

We point out that in the derivation of the bound we have assumed that the **POVM** does not depend on the parameter, as well as the measure dm. However, for some problems and physical system [31], these assumptions are no longer true, and the inequality needs to be generalized. An attempt in that direction can be found in [36].

1.3.3 Quantum Fisher Information and the geometry of the space state

The quantum statistical model we are studying has a natural geometrical interpretation [2, 12]. In particular, the set of pure states $\mathcal{H}$ has a naturally distance, which name is the Fubini-Study distance [37]

$$D_{FS}(|\phi_A\rangle, |\phi_B\rangle) = 2 \arccos |\langle \phi_A | \phi_B \rangle|. \tag{1.83}$$

We can interpret it as the angle between two pure state: sometimes it is called the quantum angle and it is a real-valued quantity.

This distance can be generalized to the case of mixed states. To do so, we need to define the Fidelity [37, 26] of two states ρ and σ as

$$F(\rho, \sigma) = \mathrm{tr}\left\{ \sqrt{\sqrt{\rho}\sigma\sqrt{\rho}} \right\}^2. \tag{1.84}$$

In the case of two pure states it reduces to their overlap

$$\begin{aligned} F(|\phi_A\rangle, |\phi_B\rangle) &= \mathrm{tr}\left\{ \sqrt{|\phi_A\rangle\langle\phi_A|\phi_B\rangle\langle\phi_B|\phi_A\rangle\langle\phi_A|} \right\}^2 = \\ &= |\langle \phi_A | \phi_B \rangle|^2 \, \mathrm{tr}\left\{ \sqrt{|\phi_A\rangle\langle\phi_A|} \right\}^2 = |\langle \phi_A | \phi_B \rangle|^2. \end{aligned} \tag{1.85}$$

This mean that the fidelity is a good candidate to measure the overlap of generic mixed states. Indeed, the Fidelity can assume value in the interval $[0, 1]$ and is equal to 1 if and only if $\rho = \sigma$. Conversely, $F(\rho, \sigma) = 0$ if and only if the two states have support on two orthogonal subspaces, which provides an indication that two states are distinguishable.

Due to these properties and to the distance definition, however, the Fidelity is not a distance. Nevertheless, we can define a distance as a function of the fidelity. Doing so, we obtain the Bures distance

$$D_B(\rho, \sigma) = \sqrt{2\left(1 - \sqrt{F(\rho, \sigma)}\right)} = \sqrt{2 - 2\,\mathrm{tr}\left\{ \sqrt{\sqrt{\rho}\sigma\sqrt{\rho}} \right\}}. \tag{1.86}$$

This notion of distance is consistent with the Fubini-Study distance when evaluated on pure states.

Taking in consideration the quantum statistical model and the state ρ_γ, an infinitesimal variation in the parameter defines the Bures metric g [27, 37, 32, 33]

$$d_B^2 = D_B^2(\rho_\gamma, \rho_{\gamma+d\gamma}) = g d\gamma^2 \tag{1.87}$$

This definition shows that the Bures metric measure how much two close density matrix ρ_γ and $\rho_{\gamma+d\gamma}$ are distinguishable in quantum statistical model.

In the case of non-singular density matrix[5], g has the form [19, 27]

$$g = \frac{\mathcal{F}_q}{4}. \tag{1.88}$$

We conclude that the Bures metric, which measures the statistical distance between two infinitesimally close density matrices, is only a different name for the Quantum Fisher Information, which gives the ultimate precision bound on the estimation of the parameter encoded in a quantum state. According to that, the more ρ_γ and $\rho_{\gamma+d\gamma}$ are distinguishable, the more they are distant and the higher is the ultimate precision in the estimation procedure using the state ρ_γ.

1.3.4 Quantifying the power of estimate

The possibility to have a more precise estimate for a larger parameter is higher since the measure does not need to be very precise. Conversely, for the estimate of a small parameter, it is necessary to implement a very accurate measure to obtain a good estimate. Thus, the degree of precision of an estimate is proportional to the square of the parameter (in order to have a scalar) and inversely proportional to his variance. The quantity that identify the performance of an estimator is the signal-to-noise ratio [27]

$$R_\gamma = \frac{\gamma^2}{\mathbf{Var}(\gamma)}. \tag{1.89}$$

A very good estimator has a large R_γ. In particular, in the case of very small γ, we need very high precision in order to achieve larger R_γ, as stated before. Recalling the classical Cramer Rao bound (1.32) and the Quantum Cramer Rao bound (1.73) we have that

$$R_c(\gamma) \leq \gamma^2 \mathcal{F}_c(\gamma) \leq \gamma^2 \mathcal{F}_q(\gamma) = R_q(\gamma) \tag{1.90}$$

[5]Some issues appear when the density matrix is not full-rank, see [32] for further information.

where $R_q(\gamma)$ is the quantum signal-to-noise ratio and gives an upper limit in the precision of the estimate. It is said that the parameter γ is really estimable quantum-mechanically where it has a large $R_q(\gamma)$. Naturally, the real power of estimation is achieved when the two inequalities above are saturated, that is when we choose an efficient estimator and an optimal **POVM**, as discussed before.

Chapter 2

Quantum Estimation for a generic perturbation in one dimension

The aim of this chapter is to investigate the estimation bound for a generic small perturbation $\gamma\mathbb{H}_1$, where γ quantifies the intensity of the perturbation. In this case, the full Hamiltonian is

$$\mathbb{H} = \mathbb{H}_0 + \gamma\mathbb{H}_1 + \mathcal{O}(\gamma^2), \tag{2.1}$$

where $\mathbb{H}_0$ is the Hamiltonian of the system, namely the kinetic term plus the potential $\mathbb{V}$. This study is preparatory for the problem that we will introduce in the next chapter. In our study, we make the assumption that the parameter γ is very small. Consequently, the use of time-independent perturbation theory is justified [21].

In the this chapter we will use the following notation. It is assumed that the unperturbed Hamiltonian $\mathbb{H}_0$ has a discrete energy spectrum which will be denoted as $E_n^{(0)}$, and the corresponding eigenstates as $|\psi_n\rangle$. The first-order perturbed ket will be

$$\begin{aligned} |\psi_n^\gamma\rangle &= \frac{1}{\sqrt{1+\gamma^2\|\psi_n^{(1)}\|}}\left(|\psi_n\rangle + \gamma\sum_{m\neq n}^{+\infty}\frac{\langle\psi_m|\mathbb{H}_1|\psi_n\rangle}{E_n^{(0)}-E_m^{(0)}}|\psi_m\rangle\right) = \\ &= \frac{1}{\sqrt{N_n(\gamma)}}\left(|\psi_n\rangle + \gamma|\psi_n^{(1)}\rangle\right) \end{aligned} \tag{2.2}$$

and the perturbed energy eigenstates are

$$E_n^\gamma = E_n^{(0)} + \gamma E_n^{(1)} \tag{2.3}$$

with the first correction to the energy given by $E_n^{(1)} = \langle\psi_n|\mathbb{H}_1|\psi_n\rangle$.

For the following calculation, we introduce an orthonormal set of vectors for each energy level $\{|e_n^0\rangle, |e_n^1\rangle\}$ defined as

$$|e_n^0\rangle = |\psi_n\rangle, \tag{2.4}$$

$$|e_n^1\rangle = \frac{|\psi_n^{(1)}\rangle}{\|\psi_n^{(1)}\|}. \tag{2.5}$$

With this set of state, we can recast the perturbed ket as

$$|\psi_n^\gamma\rangle = \frac{1}{\sqrt{N_n(\gamma)}}\left(|e_n^0\rangle + \gamma\|\psi_n^{(1)}\||e_n^1\rangle\right). \tag{2.6}$$

2.1 Pure Static States

We start considering the most elementary kind of states depending on the parameter γ, that is pure states $\rho_\gamma = |\phi^\gamma\rangle\langle\phi^\gamma|$, but without specifying the nature of $|\phi^\gamma\rangle$. In order to compute and compare Fisher, Quantum Fisher Information and the other quantities of interest, the Symmetric Logarithmic Derivative must be evaluated. For a pure state this is done because ρ_γ is idempotent, thus

$$\rho_\gamma^2 = \rho_\gamma. \tag{2.7}$$

Indeed, deriving both sides gives

$$\partial_\gamma\rho_\gamma\rho_\gamma + \rho_\gamma\partial_\gamma = \partial_\gamma\rho_\gamma \tag{2.8}$$

that matches with the definition of the LSD (1.63) if

$$\Lambda_\gamma = 2\partial_\gamma\rho_\gamma = 2\left(|\partial_\gamma\phi^\gamma\rangle\langle\phi^\gamma| + |\phi^\gamma\rangle\langle\partial_\gamma\phi^\gamma|\right), \tag{2.9}$$

where the derived ket is defined as $|\partial_\gamma\phi^\gamma\rangle = \sum_n \partial_\gamma\psi(\gamma)_n|n\rangle$, in which it is used a γ-independent vector base. Now, the computation for the QFI is straightforward and yields to

$$\mathcal{F}_q(\gamma) = \mathrm{Tr}\{\rho_\gamma\Lambda_\gamma^2\} = 4\left[\langle\partial_\gamma\phi^\gamma|\partial_\gamma\phi^\gamma\rangle - |\langle\phi_\gamma|\partial_\gamma\phi^\gamma\rangle|^2\right] \tag{2.10}$$

The derived ket has an interesting property: since the norm of $|\phi^\gamma\rangle$ is $\langle\phi^\gamma|\phi^\gamma\rangle = 1$, deriving both sides lead to

$$\langle\partial_\gamma\phi^\gamma|\phi^\gamma\rangle + \langle\phi^\gamma|\partial_\gamma\phi^\gamma\rangle = 0. \tag{2.11}$$

This last equation means that the scalar product between the vector state and is derived vector is equal to an imaginary number

$$\langle\partial_\gamma\phi^\gamma|\phi^\gamma\rangle = ib. \tag{2.12}$$

Moreover, if the state is time independent, it can be rotated in the complex plane with any phase and this does not change the physical description of the system. This means that, with a suitable rotation, the last scalar product may be set to 0 and consequently the QFI simplifies as

$$\mathcal{F}_q(\gamma) = 4\langle\partial_\gamma\phi^\gamma|\partial_\gamma\phi^\gamma\rangle = 4\|\partial_\gamma\phi^\gamma\|^2 \tag{2.13}$$

In conclusion, for a pure static state the QFI is proportional to the square of the norm of the derived ket.

Now Let us consider system with a perturbed Hamiltonian like in (2.1). In that case, the pure static case is the perturbed one $|\psi_n^\gamma\rangle$ and the derived ket is given by

$$|\partial_\gamma\psi_n^\gamma\rangle = \frac{1}{\sqrt{N_n(\gamma)}}\left[\|\psi_n^{(1)}\||e_n^0\rangle - \frac{\|\psi_n^{(1)}\|^2}{\sqrt{N_n(\gamma)}}|\psi_n^\gamma\rangle\right], \tag{2.14}$$

and its norm is

$$\|\partial_\gamma\psi_n^\gamma\|^2 = \frac{\|\psi_n^{(1)}\|^2}{N_n(\gamma)^2}. \tag{2.15}$$

At the same time, the scalar product $\langle\psi_n^\gamma|\partial_\gamma\psi_n^\gamma\rangle$ is identically null because of the comment above. Then the QFI simplifies as

$$\mathcal{F}_q^n(\gamma) = 4\frac{\|\psi_n^{(1)}\|^2}{N_n(\gamma)^2} = 4\|\psi_n^{(1)}\|^2 + \mathcal{O}(\gamma^2), \tag{2.16}$$

where terms of order higher than second are neglected. The QFI information for a pure static first-order perturbed state is proportional to the norm of the perturbation ket $\|\psi_n^{(1)}\|$ and does not depend on γ. This means that for $\gamma \simeq 0$ the QFI is almost constant for this particular choice of the state.

Furthermore, given the n-th perturbed energy eigenstate, the SLD can be computed explicitly, obtaining

$$\Lambda_{\gamma,n} = 2\|\psi_n^{(1)}\| \left(|e_1^n\rangle\langle e_0^n| + |e_0^n\rangle\langle e_1^n|\right) + 4\gamma\|\psi_n^{(1)}\|^2 \left(|e_1^n\rangle\langle e_1^n| - |e_0^n\rangle\langle e_0^n|\right) \tag{2.17}$$

It is worth noticing that, in the case of $\gamma = 0$, the SLD $\Lambda_{0,n}$ takes the form of an X-Gate, if we consider the basis state as a q-bit basis. This is reasonable since, if the goal is to estimate the parameter γ and the state is of the form $|\psi\rangle \propto |e_0\rangle + \gamma|e_1\rangle$, in some sense, what has to be done is to measure the angle between $|e_0\rangle$ and $|e_1\rangle$.

On the other side, classical Fisher Information can be computed considering different kind of measurements, like position, energy and so on. Comparing it with the QFI, we can find whether a measure is an optimal one or not.

Firstly, let us consider a position measurement. The wave function in the position representation is given by

$$\psi_\gamma(x) = \langle x|\psi^\gamma\rangle, \tag{2.18}$$

and the probability distribution is given by

$$\mathbf{p}_{pos}(x|\gamma) = |\psi_\gamma(x)|^2. \tag{2.19}$$

The classical Fisher Information is given by (1.33) and in that case it becomes

$$\mathcal{F}_c(\gamma) = \int_{-\infty}^{+\infty} dx \frac{1}{|\psi_\gamma(x)|^2} \left[\partial_\gamma |\psi_\gamma(x)|^2\right]^2 = \frac{1}{2}\mathcal{F}_q(\gamma) + \tag{2.20}$$

$$+ \int_{-\infty}^{+\infty} dx \frac{(\psi_\gamma(x)^*)^2}{|\psi_\gamma(x)|^2}(\partial_\gamma\psi_\gamma(x))^2 + \frac{(\psi_\gamma(x))^2}{|\psi_\gamma(x)|^2}(\partial_\gamma\psi_\gamma(x)^*)^2 = \tag{2.21}$$

$$= \frac{1}{2}\mathcal{F}_q(\gamma) + \frac{1}{2}\mathcal{P}(\psi, \psi^*). \tag{2.22}$$

In the case of real wave function $\psi_\gamma(x)^* = \psi_\gamma(x)$ (as prooved before, this assumption can always be satisfied in the pure static case) the $\mathcal{P}(\psi, \psi) = \mathcal{F}_q(\gamma)$ and we arrive a the notable result

$$\mathcal{F}_c(\gamma) = \mathcal{F}_q(\gamma). \tag{2.23}$$

This means that the position measurement is always an optimal measurement for a pure static state which belongs to a generic system depending on a parameter, since in

our derivation it was not necessary to assume any form of the Hamiltonian and of the state $|\psi^\gamma\rangle$ itself.

Secondly, we consider energy measurement. The energy of the system is the expectation value of the full Hamiltonian $\mathbb{H}$. However, we can consider the adjoint operator $\mathbb{H}_0$ as the 0th energy order of the system and we can study the measurement induced only by $\mathbb{H}_0$.

If our system is in the perturbed ground state (the results we are going to derive can be easily generalized to any perturbed excited state), the probability density distribution is given by $\mathbf{p}_{en}(E_n^{(0)}|\gamma) = |\langle\psi_n|\psi_0^\gamma\rangle|^2$. We have two cases: for $n = 0$ we have that

$$\mathbf{p}_{en}(E_0^{(0)}|\gamma) = \left|\frac{1}{\sqrt{N_0(\gamma)}}\right|^2, \tag{2.24}$$

while for $n \neq 0$ we have that

$$\mathbf{p}_{en}(E_n^{(0)}|\gamma) = \left|\frac{\gamma}{\sqrt{N_0(\gamma)}}\frac{\langle\psi_n|\mathbb{H}_1|\psi_0\rangle}{E_0^{(0)} - E_n^{(0)}}\right|^2 = \frac{\gamma^2}{N_0(\gamma)}\left|\frac{\langle\psi_n|\mathbb{H}_1|\psi_0\rangle}{E_0^{(0)} - E_n^{(0)}}\right|^2. \tag{2.25}$$

Eventually, we obtain the Fisher Information

$$\begin{aligned}\mathcal{F}_c(\gamma) &= \frac{1}{\mathbf{p}_{en}(E_0^{(0)}|\gamma)}\left[\partial_\gamma \mathbf{p}_{en}(E_0^{(0)}|\gamma)\right]^2 + \sum_{n\neq 0}\frac{1}{\mathbf{p}_{en}(E_n^{(0)}|\gamma)}\left[\partial_\gamma \mathbf{p}_{en}(E_n^{(0)}|\gamma)\right]^2 = \\ &= \frac{4\gamma^2}{N_0(\gamma)^3}\|\psi_0^{(1)}\|^2 + \|\psi_0^{(1)}\|^2\frac{4}{N_0(\gamma)}\left(1 - \gamma^2\frac{\|\psi_0^{(1)}\|^2}{N_0(\gamma)}\right) \simeq \\ &= 4\|\psi_0^{(1)}\|^2. \end{aligned} \tag{2.26}$$

We see that for the energy measurement on a first-order perturbed energy eigenstate the Fisher Information and the Quantum Fisher Information are the same $\mathcal{F}_q(\gamma) = \mathcal{F}_c(\gamma)$ only at first-order in γ and it follows that the energy measurement is an optimal measurement only at first-order in γ.

Finally, we investigate whether the measurement given by the SLD for the n-th perturbed energy eigenstate in the case of $\gamma = 0$, that is

$$\Lambda_0 = 2\|\psi_n^{(1)}\|\left(|e_1^n\rangle\langle e_0^n| + |e_0^n\rangle\langle e_1^n|\right), \tag{2.27}$$

can be considered as an optimal measurement or not. Since the the projectors of Λ_γ are the optimal **POVM**, it could be interesting to know if this result is valid for Λ_0 too. In

this case we would find that an optimal measurement can be implemented independently of the value of γ.

To understand if this fact is true, we evaluate the projectors of Λ_0. The eigenvalues of Λ_0 are $\{-2\|\psi_n^{(1)}\|, 2\|\psi_n^{(1)}\|\}$ and the related eigenvectors are respectively

$$|\psi_-\rangle = \frac{1}{\sqrt{2}}\left(|e_0^n\rangle - |e_1^n\rangle\right), \tag{2.28}$$

$$|\psi_+\rangle = \frac{1}{\sqrt{2}}\left(|e_0^n\rangle + |e_1^n\rangle\right). \tag{2.29}$$

The SLD Λ_0 can be written as

$$\Lambda_0 = -2\|\psi_n^{(1)}\||\psi_-\rangle\langle\psi_0| + 2\|\psi_n^{(1)}\||\psi_+\rangle\langle\psi_+|. \tag{2.30}$$

Now, we compute the probabilities of finding the perturbed energy eigenstate $|\psi_n^\gamma\rangle$ in one of the two possible eigenstate. We find that

$$\mathbf{p}_{\Lambda_0}(-|\gamma) = |\langle\psi_-|\psi_n^\gamma\rangle|^2 = \frac{1}{2N_n(\gamma)}\left[1 + \gamma^2\|\psi_n^{(1)}\|^2 - 2\gamma\|\psi_n^{(1)}\|\right] \tag{2.31}$$

$$\mathbf{p}_{\Lambda_0}(+|\gamma) = |\langle\psi_+|\psi_n^\gamma\rangle|^2 = \frac{1}{2N_n(\gamma)}\left[1 + \gamma^2\|\psi_n^{(1)}\|^2 + 2\gamma\|\psi_n^{(1)}\|\right]. \tag{2.32}$$

Finally, we compute the Fisher Information

$$\begin{aligned}\mathcal{F}_c^{\Lambda_0}(\gamma) &= \frac{1}{\mathbf{p}_{\Lambda_0}(-|\gamma)}\left(\frac{\partial\mathbf{p}_{\Lambda_0}(-|\gamma)}{\partial\gamma}\right)^2 + \frac{1}{\mathbf{p}_{\Lambda_0}(+|\gamma)}\left(\frac{\partial\mathbf{p}_{\Lambda_0}(+|\gamma)}{\partial\gamma}\right)^2 = \\ &= \frac{4(\gamma^2+1)\|\psi_n^{(1)}\|^2}{N_n(\gamma)^3} \simeq 4\|\psi_n^{(1)}\|^2.\end{aligned} \tag{2.33}$$

We see that, also in this case, the Fisher Information is equal to the Quantum Fisher Information (2.16) only at first-order in γ.

2.2 Pure states evolving in time

Up to now, we have considered only pure state and static state in the case of a complete general perturbation $\mathbb{H}_1$. Now, we want to move forward studying time evolving states.

As first attempt, we start from the first-order perturbed ket $|\psi_n^\gamma\rangle$. This ket is already an eigenstate of the full Hamiltonian $\mathbb{H}$ since we have that at first-order

$$\mathbb{H}|\psi_n^\gamma\rangle = E_n^\gamma|\psi_n^\gamma\rangle. \tag{2.34}$$

It follows that, if we let the system evolving in time, the state becomes

$$|\psi_n^\gamma(t)\rangle = \exp\left\{-\frac{i}{\hbar}\mathbb{H}t\right\}|\psi_n^\gamma\rangle = \exp\left\{-\frac{i}{\hbar}E_n^\gamma t\right\}|\psi_n^\gamma\rangle, \tag{2.35}$$

gaining a phase proportional to the perturbed energy. Likewise, the density matrix of the time evolving state becomes

$$\rho_n^\gamma(t) = |\psi_n^\gamma(t)\rangle\langle\psi_n^\gamma(t)| = |\psi_n^\gamma\rangle\langle\psi_n^\gamma|, \tag{2.36}$$

and we see that the time-dependent density matrix is equal to the time-independent static one. Therefore, for this kind of state the Quantum Fisher Information is equal to the static one given by (2.16).

However, we can consider as time evolving state the unperturbed energy eigenstate $|\psi_n\rangle$, that is not an eigenstate of the perturbed Hamiltonian $\mathbb{H}$. Thereby, the evolution of the state is not trivial. We find out that it can be written as

$$\begin{aligned}|\psi_{n,unp}^\gamma(t)\rangle &= \exp\left\{-\frac{i}{\hbar}\mathbb{H}t\right\}|\psi_n\rangle = \exp\left\{-\frac{i}{\hbar}\mathbb{H}t\right\}\sum_m |\psi_m^\gamma\rangle\langle\psi_m^\gamma|\psi_n\rangle = \\ &= \sum_m \exp\left\{-\frac{i}{\hbar}E_n^\gamma t\right\}|\psi_m^\gamma\rangle\langle\psi_m^\gamma|\psi_n\rangle.\end{aligned} \tag{2.37}$$

In this case we see that we have not a closed formula, since the last expression depends on the value of the bra-ket product $\langle\psi_m^\gamma|\psi_n\rangle$. The closed formula can be obtained only by specifying the perturbation $\mathbb{H}_1$ and the system $\mathbb{H}_0$.

Another interesting preparation we can explore is the superposition of states. We consider at time $t = 0$ the state

$$|\psi^\gamma(0)\rangle = \sum_{n=0}^{+\infty}\psi_n(0)|\psi_n\rangle, \tag{2.38}$$

where $\psi_n(0)$ is the normalized distribution of the unperturbed energy eigenstates of the superposition, $\sum_n |\psi_n(0)|^2 = 1$.

To simplify the problem, we assume that the unperturbed energy eigenstates are eigenstate of $\mathbb{H}_1$ too. Even though it is not the general case, we will see that there are systems for which this condition is realized. The state, subjected to the unitary evolution generated by $\mathbb{H}$, becomes

$$|\psi^\gamma(t)\rangle = \sum_{n=0}^{+\infty} \psi_n(0) \exp\left\{-\frac{i}{\hbar}\mathbb{H}t\right\}|\psi_n\rangle = \sum_{n=0}^{+\infty} \psi_n(0) \exp\left\{-\frac{i}{\hbar}E_n^\gamma t\right\}|\psi_n\rangle \tag{2.39}$$

As always, for a pure state the Quantum Fisher Information is given by

$$\mathcal{F}_q(t,\gamma) = 4\left[\langle\partial_\gamma\psi^\gamma(t)|\partial_\gamma\psi^\gamma(t)\rangle - |\langle\psi^\gamma(t)|\partial_\gamma\psi^\gamma(t)\rangle|^2\right]. \tag{2.40}$$

Since the derived ket is

$$|\partial_\gamma\psi^\gamma(t)\rangle = -\frac{i}{\hbar}t\sum_{n=0}^{+\infty}\psi_n(0)E_n^{(1)}\exp\left\{-\frac{i}{\hbar}E_n^\gamma t\right\}|\psi_n\rangle, \tag{2.41}$$

we have that the Quantum Fisher Information is equal to

$$\mathcal{F}_q(t,\gamma) = 4\frac{t^2}{\hbar^2}\left[\sum_{n=0}^{+\infty}|\psi_n(0)|^2\left[E_n^{(1)}\right]^2 - \left|\sum_{n=0}^{+\infty}|\psi_n(0)|^2E_n^{(1)}\right|^2\right]. \tag{2.42}$$

We note two distinct facts: the first is that the QFI is quadratic in time; the second is that it does not depend on the value of γ, therefore it is constant on γ.

We may ask which finite superposition of states $|\psi_n\rangle$ maximizes the Quantum Fisher Information $\mathcal{F}_q(\gamma)$. The function $f(\{\mathbf{p}_n\})$ to be maximized is defined as

$$f(\{\mathbf{p}_n\}) = \sum_{n=0}^{N}\mathbf{p}_n\left[E_n^{(1)}\right]^2 - \left(\sum_{n=0}^{N}\mathbf{p}_nE_n^{(1)}\right)^2, \tag{2.43}$$

where $\mathbf{p}_n = |\psi_n(0)|^2$. Its domain is the set

$$\mathcal{D} = \left\{0 \leq \mathbf{p}_i \leq 1, \sum_{i=1}^{N}\mathbf{p}_i = 1\right\} \tag{2.44}$$

which is compact. Since the function $f(\{\mathbf{p}_n\})$ is continuous and the domain $\mathcal{D}$ is compact, a global maximum and a global minimum must exist. Considering only the maximum, there are two possibilities: it lives either on the boundary of $\mathcal{D}$ or on the interior of $\mathcal{D}$.

In the last case, we can find the local maximum and local minimum with the Lagrange multiplier. The Lagrange constraint is given by $g(\mathbf{p}_n) = 0$, where

$$g(\{\mathbf{p}_n\}) = \sum_{n=0}^{N} \mathbf{p}_n - 1, \tag{2.45}$$

and N represents the number of state in the superposition. Thus, the Lagrange function is defined as

$$\mathcal{L}(\{\mathbf{p}_n\}, \lambda) = f(\{\mathbf{p}_n\}) - \lambda g(\{\mathbf{p}_n\}). \tag{2.46}$$

and its gradient is

$$\frac{\partial \mathcal{L}(\mathbf{p}_n, \lambda)}{\partial \mathbf{p}_m} = \left[E_m^{(1)}\right]^2 - 2\left(\sum_{n=0}^{+\infty} p_n E_n^{(1)}\right) E_m^{(1)} - \lambda = 0. \tag{2.47}$$

As we know, the stationary points of $\mathcal{L}$ are the extremal points of $f(\mathbf{p}_n)$ given the constraint $g(\{\mathbf{p}_n\}) = 0$ in the interior of $\mathcal{D}$, which are local minimum or local maximum. In addition to them, we must consider also the boundary points if we want to find the global maximum.

We start from the simplest case where $N = 2$, since for $N = 1$ the Q.F.I. is identically null. The functional to be maximized is

$$f(\mathbf{p}_1, \mathbf{p}_2) = \mathbf{p}_1 \left[E_1^{(1)}\right]^2 + \mathbf{p}_2 \left[E_2^{(1)}\right]^2 - \left(\mathbf{p}_1 E_1^{(1)} + \mathbf{p}_2 E_2^{(1)}\right)^2. \tag{2.48}$$

The gradient of the Lagrange function and the constraint leads us to the following linear system

$$\left[E_1^{(1)}\right]^2 - 2E_1^{(1)}\left(\mathbf{p}_1 E_1^{(1)} + \mathbf{p}_2 E_2^{(1)}\right) - \lambda = 0, \tag{2.49}$$

$$\left[E_2^{(1)}\right]^2 - 2E_2^{(1)}\left(\mathbf{p}_1 E_1^{(1)} + \mathbf{p}_2 E_2^{(1)}\right) - \lambda = 0, \tag{2.50}$$

$$\mathbf{p}_1 + \mathbf{p}_2 = 1. \tag{2.51}$$

that can be solved. The solution is

$$\mathbf{p}_1 = 1/2, \tag{2.52}$$

$$\mathbf{p}_2 = 1/2, \tag{2.53}$$

and the functional value in this extremal point is

$$f(1/2, 1/2) = \frac{1}{4}\left(E_1^{(1)} - E_2^{(1)}\right)^2. \tag{2.54}$$

Next, we consider the boundary of the domain, which is formed by two points, $\{1, 0\}$ and $\{0, 1\}$. But the functional evaluated on these points is null. Thus, the global maximum of the functional is given by the pair $\{\mathbf{p}_1 = 1/2, \mathbf{p}_2 = 1/2\}$.

Now, we move to the case $N = 3$. The gradient of the Lagrange functional $\mathcal{L}$ and constraint lead us to the linear system

$$\left[E_1^{(1)}\right]^2 - 2E_1^{(1)}\left(\mathbf{p}_1 E_1^{(1)} + \mathbf{p}_2 E_2^{(1)} + \mathbf{p}_3 E_3^{(1)}\right) - \lambda = 0, \tag{2.55}$$

$$\left[E_2^{(1)}\right]^2 - 2E_2^{(1)}\left(\mathbf{p}_1 E_1^{(1)} + \mathbf{p}_2 E_2^{(1)} + \mathbf{p}_3 E_3^{(1)}\right) - \lambda = 0, \tag{2.56}$$

$$\left[E_3^{(1)}\right]^2 - 2E_3^{(1)}\left(\mathbf{p}_1 E_1^{(1)} + \mathbf{p}_2 E_2^{(1)} + \mathbf{p}_3 E_3^{(1)}\right) - \lambda = 0, \tag{2.57}$$

$$\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = 1. \tag{2.58}$$

In this case, a solution of the system exists only if at least two corrections $E_i^{(1)}$ out of three are equal. However, this scenario is very unlikely and we discard it. As a result, the global maximum is not a stationary point but it is a point on the boundary. In this case the boundary is the union of the sets

$$\{\mathbf{p}_1 = 0 \quad \& \quad \mathbf{p}_2 + \mathbf{p}_3 \leq 1\} \cup \{\mathbf{p}_2 = 0 \quad \& \quad \mathbf{p}_1 + \mathbf{p}_3 \leq 1\} \cup \{\mathbf{p}_3 = 0 \quad \& \quad \mathbf{p}_1 + \mathbf{p}_2 \leq 1\}. \tag{2.59}$$

However, evaluating the maximum on the boundary bring us back to the case $N = 2$, that we have already studied. Thus, the maximum on each of the three set that form the boundary is given by these triplets

$$\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} = \begin{cases} \{0, \frac{1}{2}, \frac{1}{2}\} \\ \{\frac{1}{2}, 0, \frac{1}{2}\} \\ \{\frac{1}{2}, \frac{1}{2}, 0\} \end{cases} \tag{2.60}$$

and the value of the functional is respectively

$$f(\{\mathbf{p}_i\}) = \begin{cases} \frac{1}{4}\left(E_2^{(1)} - E_3^{(1)}\right)^2 \\ \frac{1}{4}\left(E_1^{(1)} - E_3^{(1)}\right)^2 \\ \frac{1}{4}\left(E_1^{(1)} - E_2^{(1)}\right)^2 \end{cases} \tag{2.61}$$

We conclude that the preparation that gives the maximum Q.F.I. is the superposition of the two states whose corrections $E_i^{(1)}$ and $E_j^{(1)}$ have the maximum difference.

Finally, we consider the case with general N. The linear system generated by the derivative of the Lagrange multiplier with the constraint, as always, has no solutions unless some of the corrections $E_i^{(1)}$ are equals. But we assume that is a pretty rare case which probably can arise for degenerate energy spectrum and that probably lead us to the same conclusion with the additional freedom of choosing the state coming from the same degenerate eigenspace. However, the solution must be found on the boundary, that in this case is the union of N different sets defined as

$$\{\mathbf{p}_i \quad \& \quad \sum_{j\neq i} p_j \leq 1\}, \qquad i = 1, 2, \dots N. \tag{2.62}$$

For each of these N sets we have a new extremal problem but now with $N-1$ variable. With the same line of reasoning, each of these $N-1$ problem boils down to $N-2$ extremal problem since the Lagrangian systems with $N-1$ has no solution for all different energy corrections. Repeating this argument, we arrive at the case with only two $\mathbf{p}_i \neq 0$, that has a unique solution found above. If we do it for all the N sets, we will find that all the extremal points stay in the borders and are made of all the set of $\{\mathbf{p}_i\}$ where only two elements are different from 0 and equal to $1/2$. The number of these extremal points found on the border is

$$\frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}, \tag{2.63}$$

and evaluating the function $f(0, \dots, \mathbf{p}_i, \dots, \mathbf{p}_j, \dots, 0)$ for each pair (i, j) we find $n(n-1)/2$ extremal points with values

$$f(0, \dots, \mathbf{p}_i, \dots, \mathbf{p}_j, \dots, 0) = \frac{1}{4}\left(E_i^{(1)} - E_j^{(1)}\right)^2. \tag{2.64}$$

Among them, the maximum is reached for the choice of i and j so that the difference between the respectively corrections is maximum.

In the end, the Q.F.I for the superposition of pure states reaches its maximum in the case of the superposition of the two states whose corresponding energy corrections are the most distant, and the value is

$$\mathcal{F}_q(\mathbf{p}_i, \mathbf{p}_j) = \frac{t^2}{\hbar^2}\left(\max_i E_i^{(1)} - \min_j E_j^{(1)}\right)^2. \tag{2.65}$$

2.3 Thermal States

In statistical mechanics, an ensemble is an idealization for the state of the system which represents a huge number of different possible states each of which can be a real state of the system [29]. In our case, we are interested in what is called the canonical ensemble: the canonical ensemble describes a quantum system that is in equilibrium with a heat bath at a fixed temperature. The canonical ensemble is isolated to ensure it does not exchange energy with any external object besides the heat bath. Consequently, it is assumed that the temperature T, the volume V and the number of particle N are fixed.

Generally speaking, in quantum mechanics, a statistical ensemble in equilibrium is represented by the canonical ensemble

$$\rho_\gamma = \frac{1}{Z(\gamma)} \exp\{-\beta \mathbb{H}\}. \tag{2.66}$$

where $Z(\gamma)$ is the canonical partition function. Here, the Hamiltonian is $\mathbb{H} = \mathbb{H}_0 + \gamma \mathbb{H}_1$ and we assume that the eigenstates of $\mathbb{H}_0$ are the same of $\mathbb{H}_1$. Clearly, this assumption is not always true. Thereafter, we will see some systems in which it holds and some systems in which it does not. However, in this case we can expand ρ_γ in the unperturbed eigenbasis $\{|\psi_n\rangle\}$ of $\mathbb{H}_0$ as

$$\rho_\gamma = \sum_n \rho_n(\gamma) |\psi_n\rangle\langle\psi_n|, \tag{2.67}$$

where the canonical ensemble matrix is

$$\rho_n(\gamma) = \frac{\exp\{-\beta E_n(\gamma)\}}{Z(\gamma)}, \tag{2.68}$$

and $Z(\gamma)$ is the partition

$$Z = \sum_n \exp\{-\beta E_n(\gamma)\}. \tag{2.69}$$

For these states, we have that the Quantum Fisher Information is given by [27]

$$\mathcal{F}_q(\gamma) = 2 \sum_{nm} \frac{|\langle\psi_m|\partial_\gamma \rho_\gamma|\psi_n\rangle|^2}{\rho_n + \rho_m}. \tag{2.70}$$

Because the dependency on γ is only in the $\rho_n(\gamma)$, the $\mathcal{F}_q$ simplify as

$$\mathcal{F}_q(\gamma) = \sum_n \frac{\partial_\gamma \rho_n(\gamma)^2}{\rho_n(\gamma)}. \tag{2.71}$$

Considering that the energy in this case is exactly $E_n^\gamma = E_n^{(0)} + \gamma E_n^{(1)}$, we find that

$$\partial_\gamma \rho_n(\gamma) = \rho_n(\gamma) \left(-\beta E_n^{(1)} + \frac{\beta}{Z} \sum_m \exp\{-\beta E_m^\gamma\} E_m^{(1)} \right) = \tag{2.72}$$

$$= \rho_n(\gamma)\beta \left(\langle \mathbb{H}_1 \rangle - E_n^{(1)} \right), \tag{2.73}$$

and subsequently the Quantum Fisher Information is

$$\mathcal{F}_q(\gamma) = \beta^2 \sum_n \rho_n(\gamma)(\langle \mathbb{H}_1 \rangle - E_n^{(1)})^2. \tag{2.74}$$

Since the energy eigenstates are also eigenstates of $\mathbb{H}_1$ we can write that

$$\sum_n \rho_n(\gamma)(E_n^{(1)})^2 = \langle \mathbb{H}_1^2 \rangle, \tag{2.75}$$

and consequently the Quantum Fisher Information becomes

$$\mathcal{F}_q(\gamma) = \beta^2(\langle \mathbb{H}_1^2 \rangle - \langle \mathbb{H}_1 \rangle^2) = \beta^2 \mathbf{D}^2(\mathbb{H}_1) \tag{2.76}$$

In this case, the Cramer-Rao inequality (1.32) can be written as

$$\mathbf{D}^2(\gamma^*) \geq \frac{1}{M\beta^2 \mathbf{D}^2(\mathbb{H}_1)}, \tag{2.77}$$

which can be reformulated in a similar shape of the Heisenberg indetermination principle

$$\mathbf{D}^2(\gamma^*)\mathbf{D}^2(\mathbb{H}_1) \geq \frac{1}{M\beta^2}. \tag{2.78}$$

In any case, we see that if we want a small variance for our estimator γ^* we need very low temperature or alternatively a large variance for the operator $\mathbb{H}_1$, which means that we have to choose states that are more scattered by $\mathbb{H}_1$.

We are interested also int the SLD Λ_γ. A general solution for Λ_γ may be written as [27]

$$\Lambda_\gamma = \int_0^{+\infty} dt \exp\{-\rho_\gamma t\} \partial_\gamma \rho_\gamma \exp\{-\rho_\gamma t\}. \tag{2.79}$$

In the easy case where the state is diagonal and only the weights $\rho_n(\gamma)$ depend on γ, we have that the solution for Λ_γ can be written as

$$\Lambda_\gamma = \sum_n \frac{\partial_\gamma \rho_n(\gamma)}{\rho_n(\gamma)} |\psi_n\rangle\langle\psi_n|, \tag{2.80}$$

and the previous calculation (2.73) lead us to

$$\Lambda_\gamma = \sum_n \frac{\rho_n(\gamma)\beta(\langle \mathbb{H}_1 \rangle - E_n^{(1)})}{\rho_n(\gamma)} |\psi_n\rangle\langle\psi_n| = \beta\langle \mathbb{H}_1 \rangle \left(\mathbb{1} - \sum_n \frac{E_n^{(1)}}{\langle \mathbb{H}_1 \rangle} |\psi_n\rangle\langle\psi_n| \right). \tag{2.81}$$

We deduce that the eigenstate of Λ_γ that realize the optimal **POVM** are the eigenstates of the Hamiltonian $\mathbb{H}$, which are the same of the unperturbed Hamiltonian $\mathbb{H}_0$, meaning that an energy measurement is an optimal one.

Chapter 3

Gravity Corrections

In this chapter we are going to describe how the Generalized Uncertainty Principle (for the sake of brevity GUP) arises. These results can be obtained in several different ways. We start considering some thought experiments that lead us to the GUP. Next, we will consider the consequence on the commutation relations, and we will see that, in order to be consistent with the GUP, we need to modify them. Finally, we will see the consequences of the new commutation relations on quantum systems.

3.1 Thought experiments

Thought experiments had been consistently used by physicists or to underline the essential consequences of physical theories that could not be tested or when the experiments were not possible to conduct. They can lead to two different principal considerations: a theoretical one, challenging theory and its self-consistency; an experimental one, trying to predict new kind of experimental observation. In our case, we are interested mainly in the second case, without forgetting the first one.

3.1.1 The Heisenberg microscope with gravitational effects

The Heisenberg microscope is a thought experiment that was first proposed by Werner Heisenberg in order to explain the principle which bears his name [14]. We briefly review the main idea.

Suppose we have an object on the x axis whose position we want to measure. We consider an optical microscope and a photon with wavelength λ. To measure the position of the object we scatter the photon with to object itself, and, after the interaction, the photon will have a scattering angle θ. According to classical optics, the attainable precision of the measurement is

$$\Delta x \gtrapprox \frac{\lambda}{2\pi \sin\theta}, \tag{3.1}$$

where we have that the less is the wavelength, the more is the precision of the measurement. Moreover, at the momentum of the scattering, the object has a recoil due to the interaction and consequently we have a transfer of the momentum from the photon to the object. This is the Compton recoil: the object acquires a momentum of h/λ, where h is the Planck constant. But the crucial point here is that we do not know the exact direction of the scattered object. As a result, the uncertainty in the momentum of the object in the x direction is

$$\Delta p_x \gtrapprox \frac{h}{\lambda} \sin\theta. \tag{3.2}$$

Combining the relations we obtain the Heisenberg's uncertainty relation

$$\Delta x \Delta p_x \gtrapprox \frac{h}{2\pi} = \hbar. \tag{3.3}$$

This is a heuristic derivation of the Heisenberg's relation. Nowadays, we know that this relation is fundamental and can be derived directly from the axiom of quantum mechanics, thus we do not need any special experimental apparatus to obtain it.

At this point, we include gravity in the experimental picture. Firstly, with heuristic arguments, we derive the modifications of the Heisenberg's relations. Next, we will see that these changes do not depend on the particular experimental apparatus we consider if we assume a discrete space-time.

As first approximation we can consider gravity as a Newtonian force [18, 1, 24]. In order to treat the photon as a classical particle, we assign to it an effective mass equal to its energy divided by c^2. Suppose that we have an electron in a region with size R. When the photon approaches the region, the electron interacts both electromagnetically (the effects are considered in the previous section) and gravitationally, since the photon has an effective mass. The acceleration due to this effect is

$$a_x = -\frac{G(E/c^2)}{x^2}, \tag{3.4}$$

where x is the relative distance between the electron and the photon. Assuming that the electron is non-relativistic, the interaction lasts the time the photon is in the region of size R and therefore the characteristic time of interaction is R/c. During this time the electron acquires a velocity and cover a distance of

$$\Delta v_x \sim \frac{GE}{c^2 R^2} = \frac{GE}{c^3 R}, \tag{3.5}$$

$$\Delta x_G \sim \frac{GE}{c^2 R^2}\left(\frac{R}{c}\right)^2 = \frac{GE}{c^4} = \frac{Gp}{c^3}, \tag{3.6}$$

where in the last one we have used the energy momentum relation for a photon.

But since the photon momentum uncertainty has the same order of the electron momentum uncertainty we can substitute Δp of the electron to p of the photon. Using the Planck length

$$L_{Pl} = \sqrt{\frac{G\hbar}{c^3}}, \tag{3.7}$$

we obtain that the uncertainty due to the gravitational interaction is

$$\Delta x_G \sim \frac{G\Delta p}{c^3} = L_{Pl}^2 \frac{\Delta p}{\hbar}. \tag{3.8}$$

Assuming that the uncertainty given by the electromagnetic interaction is independent from the one caused by the gravitational interaction, we can add the two contributions to obtain the generalized uncertainty principle

$$\Delta x \sim \frac{\hbar}{\Delta p} + L_{Pl}^2 \frac{\Delta p}{\hbar}. \tag{3.9}$$

We note that the GUP is invariant under the transformation

$$L_{Pl}\Delta p \longleftrightarrow \frac{1}{L_{Pl}\Delta p}, \tag{3.10}$$

which relates long to short distances and high to low energies.

Far from being a formal derivation (since at high energies the scattering is not elastic, for example), the generalized Heisenberg microscope gives us the fundamental idea that gravity, also at the Newtonian level, give rise to a modification of uncertainty relation. Thus, if we want to implement the gravitational effect, we need to take care of this modification.

To improve our understanding of GUP, we consider now the effects due to General Relativity. We prefer to give a general idea of the derivation and we do not enter the technical calculations that can be found in [24, 1, 18]. We can follow two distinct but both correct ways. The first, as usual, consider a target particle whose position we want to measure by the help of a test particle. We do not specify the nature of the test particle yet: it can be both a massive particle with mass μ or a photon with $\mu \to 0$. In both cases, the test particle has a velocity v and produces a gravitational field to which it is subjected the target particle. We assume that this gravitational field can be described by the Schwarzschild metric.

The main point of the consideration is the following: if we have a particle with energy E, its size must be at least the Compton length $R_p = hc/E$. At the same time, if the extension of the particle is lower than $4\pi GE$, according to general relativity, we will have a black-hole. Since we want to avoid a horizon in the rest frame that would limit the precision of our measurement, we will use this constraint in our following consideration.

If we evaluate how much the target particle moves due to the test particle vicinity, it can be seen that the shorter the interaction, the less the influence of the target particle: therefore we conclude that the photon is the best test particle we can use. Considering the test particle time to move across the region of size R (where the target particle is), we obtain, as before, that the uncertainty given by the fact that the photon direction is not known is

$$\Delta x \gtrapprox \frac{GE}{c^4}, \tag{3.11}$$

and following the same line of reasoning of (3.1) we find the GUP (3.9), as we expected. Adler e Santiago in [1] obtain the same heuristic result but considering a linear approximation of Einstein's field equation.

We see from (3.9) that the two effects are significant for different values of Δp: if the momentum of the photon is very small, the precision of the momentum of the electron is very small too, and consequently the measurement of the electron position is very inaccurate due to quantum mechanics terms in the GUP, namely because the photon wavelength is so small that the microscope has a terrible resolution. On the other side, if the momentum of the photon is large, then the momentum that the electron acquires is considerable: in this case, the responsibility for the inaccuracy of the position measure-

ment is due to the gravitational field created by the energy of the photon. Between the two regimes, we can find a value of Δp for which the position uncertainty is minimum. We find that the minimum of the right-hand side of (3.9) is given by

$$\Delta x_{min} \simeq 2\sqrt{\frac{G\hbar}{c^3}} = 2L_{Pl}, \tag{3.12}$$

for the value of

$$\Delta p \simeq \sqrt{\frac{\hbar c^3}{G}}\frac{E_p}{c}. \tag{3.13}$$

This mean that we can never localize the position of a particle with a better precision than approximately twice the Planck length: according to this result, any distance lower than L_{Pl} has no sense.

An alternative method to derive the GUP is considering a Black-Hole and investigating how precisely we can measure its radius [23, 18].

3.1.2 Device-independent limit to distance measurements

Up to now, we have considered thought experiments, and each of them leads us to the same inequality, the generalized uncertainty principle. Now we see how the GUP can be derived in a general device-independent way [18, 7, 8], thus it must be true for every system we consider.

The main idea of the proof is to consider a discrete space-time: this means that the position operator $\hat{x}$ must have a discrete spectrum $\{x_i\}$. The separation of the eigenvalues is not fixed: it can be of the order of L_{Pl} or less. We are going to see that it is not possible to measure any distance, even in principle, with precision less than L_{Pl}.

We consider an object of mass M whose position we want to measure. The displacement operator of the position of the object is

$$\hat{x}(t) - \hat{x}(0) = \hat{p}(0)\frac{t}{M}. \tag{3.14}$$

This operator may not have discrete eigenvalues, thus, to exclude a minimum length, we would have to measure a position eigenvalue x and a nearby eigenvalue x' with

$$|x - x'| \ll L_{Pl}. \tag{3.15}$$

From the axiom of quantum mechanics we know that any two hermitian operators H_1 and H_2 must satisfy the inequality

$$\Delta H_1 \Delta H_2 \geq \frac{1}{2i}[H_1, H_2]. \tag{3.16}$$

Given that the commutator between $\hat{x}(0)$ and $\hat{x}(t)$ is

$$[\hat{x}(0), \hat{x}(t)] = \frac{it\hbar}{M}, \tag{3.17}$$

we have that

$$\Delta(x(0))\Delta(x(t)) \geq \frac{t\hbar}{2M}. \tag{3.18}$$

The measurement of the discreteness of the position operator requires at least two position measurements and its precision is limited by the greater of $\Delta(x(0))$ or $\Delta(x(t))$

$$\Delta x = \max[\Delta(x(0)), \Delta(x(t))] \geq \sqrt{\frac{t\hbar}{2M}}. \tag{3.19}$$

We see that the more the massive is the test particle the more is the device precision. But if the mass exceeds a particular value we will run into the black-hole regime. Therefore, to avoid the formation of an horizon we require that

$$R > \frac{2GM}{c^2} \tag{3.20}$$

where R is the size of the apparatus. As a result, we have a lower limit in the dimension of the apparatus. At the same time, we can not make it arbitrarily large since its components must be in causal contact during the measuring process, otherwise they can not affect the measurement. Consequently, we need that

$$t > R/c. \tag{3.21}$$

It follows that the measurement precision is bounded from below by

$$\Delta x \geq \sqrt{\frac{t\hbar}{2M}} > \sqrt{\frac{R\hbar}{2cM}} > \sqrt{\frac{G\hbar}{c^3}} = L_{Pl} \tag{3.22}$$

that is the usual relation found above that assures the existence of a minimum length. We stress here that we have not specified any device in our line of reasoning, thus the result is device-independent. It follows only from the axiom of quantum mechanics and from general relativity arguments.

3.2 Modified commutation relations

These new uncertainty relations have been discussed with the use of thoughts experiments. It can be seen that many quantum theories of gravity, such as String Theory, Loop Quantum Gravity, Asymptotic Safe Gravity etc., predict a fundamental limit to the resolution of physical structure. [18].

At this point, we may ask how we can build in the notion of minimal length into the already in place theory of quantum mechanics. This approach is basically different from the one that explains the minimum length from a fundamental description of space and time: here we want to justify the arising of a minimum length scale with a tailor-made modification of the standard quantum mechanics. The aim is to provide an effective description in order to reproduce the results we obtained in the previous section. The main intention is to make contact between the requirement of a minimum length and the quantum phenomenology.

The main idea to reproduce the GUP is to modify the commutation relation for position and momentum operators. To modify them, we start considering the standard commutation relations between position and momentum

$$[x^i, x^j] = 0, \quad [p_0^i, p_0^j] = 0, \quad [x^i, p_0^j] = i\hbar\delta^{ij}. \tag{3.23}$$

If now we define a new momentum as a function of $\vec{p}_0$

$$\vec{p} = \vec{f}(\vec{p}_0), \tag{3.24}$$

we obtain a new set of commutation relations

$$[x^i, x^j] = 0, \quad [p^i, p^j] = 0, \quad [x^i, p^j] = i\hbar\frac{\partial f^i}{\partial p_{0j}}. \tag{3.25}$$

The redefinition of the momentum as a function of the momentum at low energy p_0 can be interpreted as the effects of gravity. Thus, the uncertainty relations given by (3.16) becomes

$$\Delta x_i \Delta p_j \geq \frac{\hbar}{2}\left\langle \frac{\partial f_i}{\partial p_0^j} \right\rangle. \tag{3.26}$$

At this point we can guess the function $f(\vec{p}_0)$. Since we expect that changes due to gravity are little, we can imagine the new momentum as an expansion of the form [18, 10]

$$\vec{p} \simeq \vec{p}_0\left(1 + \gamma\frac{\vec{p}_0^2}{(m_{Pl}c)^2}\right), \tag{3.27}$$

where γ is a dimensionless parameter that quantify the modifications due to gravity. We point out that these relations are at first-order in γ. We can compute the inverse relation, that is

$$\vec{p}_0 \simeq \vec{p}\left(1 - \gamma\frac{p^2}{(m_{Pl}c)^2}\right). \tag{3.28}$$

We can easy calculate the derivative of $\vec{f}(\vec{p}_0)$ now

$$\frac{\partial f_i}{\partial p_0^j} = \delta_{ij}\left(1 + \gamma\frac{p^2}{(m_{Pl}c)^2}\right) + 2\gamma\frac{p_i p_j}{(m_{Pl}c)^2}, \tag{3.29}$$

and consequently we obtain the new uncertainty relations

$$\Delta x_i \Delta p_i \geq \frac{\hbar}{2}\left(1 + \gamma\frac{\langle p^2\rangle}{(m_{Pl}c)^2} + 2\gamma\frac{\langle p_i^2\rangle}{(m_{Pl}c)^2}\right). \tag{3.30}$$

Considering that the variance of an operator H is $\Delta H^2 = \langle H^2\rangle - \langle H\rangle^2$ we obtain that

$$\Delta x_i \Delta p_i \geq \frac{\hbar}{2}\left(1 + \gamma\frac{\Delta p^2 + \langle p\rangle^2}{(m_{Pl}c)^2} + 2\gamma\frac{\Delta p_i^2 + \langle p_i\rangle^2}{(m_{Pl}c)^2}\right) \geq \frac{\hbar}{2}\left(1 + 3\gamma\frac{\Delta p_i^2}{(m_{Pl}c)^2}\right), \tag{3.31}$$

which can be reformulated as

$$\Delta x_i \geq \frac{\hbar}{2}\left(\frac{1}{\Delta p_i} + 3\gamma\frac{\Delta p_i}{(m_{Pl}c)^2}\right). \tag{3.32}$$

This is nothing but the GUP we have found in the prior section and which is now depicted in 3.1. The new modified Heisenberg algebra [10] we obtain is given by

$$[x^i, p^j] = i\hbar\left(\delta^{ij} + \gamma\delta^{ij}\frac{p^2}{(m_{Pl}c)^2} + 2\gamma\frac{p^i p^j}{(m_{Pl}c)^2}\right). \tag{3.33}$$

If now we consider any Hamiltonian of the standard form

$$\mathbb{H} = \frac{\hat{p}^2}{2m} + \mathbb{V}(\hat{\vec{x}}) \tag{3.34}$$

we can rewrite it as [5]

$$\mathbb{H} = \frac{\hat{p}_0^2}{2m} + \gamma\frac{\hat{p}_0^4}{m(m_{Pl}c)^2} + \mathbb{V}(\hat{\vec{x}}) + \mathcal{O}(\gamma^2). \tag{3.35}$$

Thus we see that any system with a defined quantum Hamiltonian $\mathbb{H}_0 = \hat{p}_0^2/2m + \mathbb{V}(\hat{\vec{x}})$ that depends only on the canonical momentum $\vec{p}_0$ is perturbed by

$$\gamma\mathbb{H}_1 = \gamma\frac{\hat{p}_0^4}{m(m_{Pl}c)^2} \tag{3.36}$$

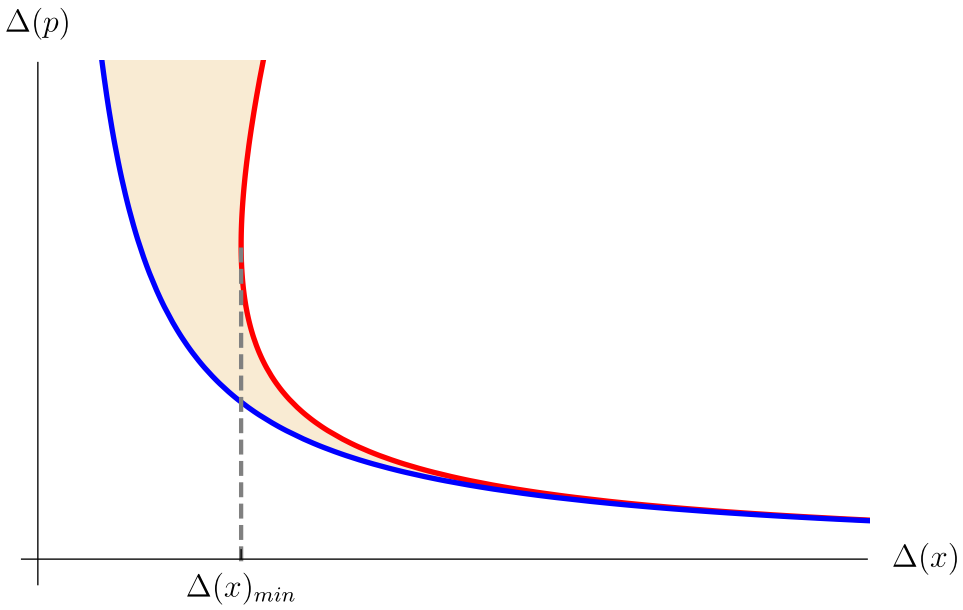

Figure 3.1: In blue the standard uncertainty relations. In red the GUP, we see that there is a minimum and a forbidden area in the latter case.

Since this perturbation affects any system, we can say that gravitational effects are universal and we call $\mathbb{H}_1$ universal gravity correction [10]. In the next chapter we will study the consequences of this kind of perturbation and the possibility of estimate the parameter γ.

Part II

Quantum Estimation for Universal Quantum Gravity Corrections

Chapter 4

Quantum estimation for gravity perturbation in one dimension

Starting from the results we derived in chapter 5, we consider the perturbation that arises in the section 3.2, which we rewrite here

$$\mathbb{H}_1 = \frac{\hat{p}_0^4}{m(M_{Pl}c)^2}. \tag{4.1}$$

What we are going to do is to study systematically different preparations of the state introduced in 5 for different systems in order to understand which can be the best one to use for the estimate of the parameter γ. Therefore, we will discuss pure perturbed energy eigenstates and their superpositions for the free particle, the infinite square well, the finite potential well and the Harmonic oscillator in one dimension and we will compare the different results.

We point out that in the sections where we consider a single specific system we set $\hbar = m = M_{Pl} = c = 1$ in the graphics regarding the Quantum Fisher Information, in order to simplify numerical calculations. This choice does not affect our considerations since for a fixed system the aim is only to compare the different values of $\mathcal{F}_q$ for different states to understand which ones give the higher Quantum Fisher. Conversely, when we compare different systems, we use S.I. values for the physical constants. Moreover, in the comparison sections, we choose the systems' parameters (such as the width of the well potential, or angular frequency of the Harmonic Oscillator) in a range corresponding to

a real physical system.

4.1 Perturbed Energy eigenstates

In this section we will consider only pure states given by the perturbed energy eigenstates $|\psi_n^\gamma\rangle$ as we studied in 2.1.

4.1.1 Free Particle

We start our investigation with the most elementary system we know: the free particle in one dimension. The free Hamiltonian of the system is given by

$$\mathbb{H}_0 = \frac{\hat{p}_0}{2m} \tag{4.2}$$

where $\hat{p}_0$ is the momentum of the particle and m is the mass. If we consider the ket-basis in the momentum space $\{|p\rangle\}$ with the generalized orthonormality condition $\langle p'|p\rangle = \delta(p'-p)$, we find that this basis is a set of eigenstate of the operator $\mathbb{H}_0$ with energy eigenvalues

$$E_p^{(0)} = \frac{p^2}{2m}. \tag{4.3}$$

A similar way to see this is solving the Schroedinger equation for the energy eigenfunction

$$-\frac{\hbar^2}{2m}\frac{d^2\phi_k(x)}{dx^2} = E_k\phi_k(x) \tag{4.4}$$

$$\frac{d^2\phi_k(x)}{dx^2} = -k^2\phi_k(x) \tag{4.5}$$

where we have defined the wave number $k = \sqrt{2mE_k}/\hbar = p/\hbar$. The most general solution is

$$\phi_k(x) = A\exp\{ikx\} \tag{4.6}$$

where we have considered only the progressive component. From the generalized orthonormality condition

$$\int_{-\infty}^{+\infty} dx\phi_{k'}^*(x)\phi_k(x) = \delta(k'-k) \tag{4.7}$$

we find out that $A = 1/\sqrt{2\pi}$ and eventually we obtain that

$$\phi_p(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{ \frac{i}{\hbar} px \right\} \tag{4.8}$$

If we move to the momentum space representation, we have that

$$\tilde{\phi}_p(p') = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} dx \phi_p(x) \exp\left\{ -\frac{i}{\hbar} p'x \right\} = \delta(p - p'). \tag{4.9}$$

Finally, we obtain the eigenstate with momentum p as we expect it, that is

$$|\psi_p\rangle = \int_{-\infty}^{+\infty} dp' |p'\rangle\langle p'|\psi_p\rangle = \int_{-\infty}^{+\infty} dp' |p'\rangle \tilde{\phi}_p(p') = |p\rangle. \tag{4.10}$$

At this point we take into account the perturbation

$$\mathbb{H} = \mathbb{H}_0 + \gamma \mathbb{H}_1. \tag{4.11}$$

We immediately notice that both $\mathbb{H}_0$ and $\mathbb{H}_1$ are proportional to a power of $\hat{p}_0$ and thus commute, meaning that the eigenstate of $\mathbb{H}_0$ are the same of $\mathbb{H}_1$. Therefore the perturbation affects only the energy spectrum, which becomes

$$E_p^\gamma = \frac{p^2}{2m} + \gamma \frac{p^4}{m(M_{Pl}c)^2}. \tag{4.12}$$

Turning to the states, there is no sense talking about the first-order perturbed ket since $\mathbb{H}$ is already diagonal in the unperturbed basis. This means that a pure eigenstate of $\mathbb{H}$ is given by $\rho_p = |p\rangle\langle p|$ and it does not depend on γ. Subsequently, the SLD Λ_γ is null as well as the Quantum Fisher Information $\mathcal{F}_q(\gamma) = 0$. We conclude that with a pure static eigenstate of the free particle we can not realize any estimate of the parameter γ.

In the next section we will see that all is not lost, because we can take a superposition of state, that in a continuous scenario is a wave packet.

4.1.2 Infinite Square Well

The next system we consider is the infinite square well. In this case the quantum system is subject to a potential defined as

$$\mathbb{V}(x) = \begin{cases} +\infty & |x| > a \\ 0 & |x| < a \end{cases}, \tag{4.13}$$

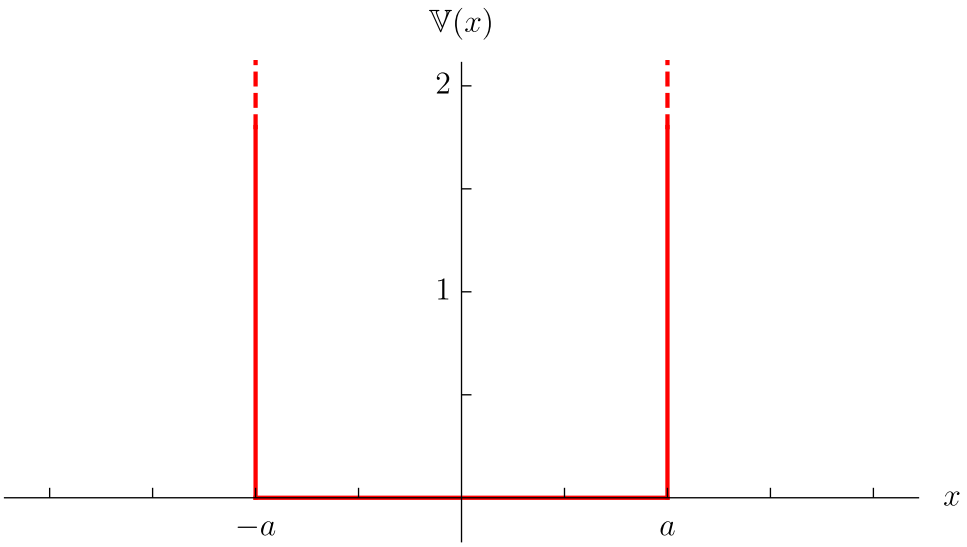

Figure 4.1: Infinite square well with width of $2a$ and centered in the origin of the x-axis.

where the width of the well is $2a$. The Hamiltonian of the unperturbed system is given by

$$\mathbb{H}_0 = \frac{\hat{p}_0^2}{2m} + \mathbb{V}(x). \tag{4.14}$$

This system admits a close solution in terms of energy eigenfunctions. Since the potential is symmetric, the set of energy eigenfunctions splits up in two subset where functions have definite parity. These solutions are

$$\phi^{(+)}(x) = A^{(+)} \cos(kx), \tag{4.15}$$

$$\phi^{(-)}(x) = A^{(-)} \sin(kx). \tag{4.16}$$

In order to find the available energy we have to impose the boundary condition $\phi(\pm a) = 0$. Since the solutions are parity-defined, we have that the boundary lead us to two kinds of solutions. The first one is the even parity solution (boundary condition $\cos(ka) = 0$) with solution

$$k_n^{(+)} = \frac{(n - 1/2)\pi}{a}, \qquad n = 1, 2, 3, \ldots. \tag{4.17}$$

The second one is the odd parity solution (boundary condition $\sin(ka) = 0$) with solution

$$k_n^{(-)} = \frac{n\pi}{a}, \qquad n = 1, 2, 3, \ldots, \tag{4.18}$$

where we have discarded the solution $n = 0$ because it would lead to a null wave function, which we know is forbidden in quantum mechanics.

Then we have the energy of each eigenstates given by

$$E_{n^{(+)}} = \frac{\hbar^2\pi^2}{2ma^2}\left(n - \frac{1}{2}\right)^2 \tag{4.19}$$

$$E_{n^{(-)}} = \frac{\hbar^2\pi^2 n^2}{2ma^2} \tag{4.20}$$

with the normalized energy eigenfunctions

$$\phi_{n^{(+)}}(x) = \sqrt{\frac{1}{a}}\cos\left[\left(n - \frac{1}{2}\right)\frac{\pi x}{a}\right] \tag{4.21}$$

$$\phi_{n^{(-)}}(x) = \sqrt{\frac{1}{a}}\sin\left[\frac{n\pi x}{a}\right]. \tag{4.22}$$

We see that the ground state is given by the $n = 1$ even parity energy eigenfunction.

In this case, we have difficulties to evaluate the value of the commutator $[\mathbb{H}_0, \mathbb{H}_1]$ since the potential is not bounded, and we can not deduce if the eigenstates of $\mathbb{H}_0$ are eigenstate of $\mathbb{H}_1$ too. Therefore, we guess that we need perturbation theory. As a first case of study, we take the perturbed ket of the ground state and we evaluate the first-order perturbation ket

$$|\psi^{(1)}_{1^+}\rangle = \sum_{n^\pm \neq 1^+} |n^\pm\rangle \frac{\langle n^\pm|\mathbb{H}_1|1^+\rangle}{E^{(0)}_{1^+} - E^{(0)}_{n^\pm}}. \tag{4.23}$$

Computing the matrix element of $\mathbb{H}_1$

$$\langle n^\pm|\mathbb{H}_1|q^\pm\rangle = \int_0^a dx \phi^*_{n^{(\pm)}}(x)\left(\frac{\hbar^4}{m(M_{Pl}c)^2}\partial_x^4\right)\phi_{q^{(\pm)}}(x) = \frac{1}{m(M_{Pl}c)^2}\left(\eta_\pm(n)\frac{\pi\hbar}{a}\right)^4 \delta_{n^\pm - q^\pm}, \tag{4.24}$$

where $\eta_+(n) = n - 1/2$ and $\eta_-(n) = n$. We immediately see that the matrix elements are diagonal. As a result, in (4.23) there are no elements in the sum and the ket $|\psi^{(1)}_{1^+}\rangle$ is identically 0. From this observation we can infer that, since any element at any order in the perturbation expansion is of the form $\sum_{n^\pm} \ldots$, where the dots are products of matrix element of $\mathbb{H}_1$ in which at least one element of the product is of the form $\langle n^\pm|\mathbb{H}_1|1^+\rangle$, each order of the perturbation is 0 and that the perturbed ket is equal to the unperturbed energy eigenstate

$$|\psi^\gamma_{1^+}\rangle = |1^+\rangle. \tag{4.25}$$

We can use the same line of reasoning to prove that any perturbed ket of any excited state is equal to the corresponding unperturbed energy eigenstates, i.e.

$$|\psi^{\gamma}_{n^{\pm}}\rangle = |n, \pm\rangle. \tag{4.26}$$

Consequently, the energy eigenfunctions of the perturbed Hamiltonian $\mathbb{H}$ are equal to the unperturbed one. For this reason, as we have seen in the subsection of the free particle, we conclude that the Quantum Fisher Information for a pure energy eigenstate is null

$$\mathcal{F}_q(\gamma) = 0. \tag{4.27}$$

In this case too, however, all is not lost, since we can consider the case where we prepare the state in a superposition of state and subsequently let the system evolve in time. In this case, the state will depend on γ because the energy of the system itself depends on it.

4.1.3 Finite Potential Well

A similar potential we are interested in is the finite potential well, given by

$$\mathbb{V}(x) = \begin{cases} 0 & |x| < a \\ V_0 & |x| > a \end{cases}, \tag{4.28}$$

where $2a$ is the width of the well like the infinite square well. We have three different regions and we have three different Schroedinger equation in in each of them, that is

$$\frac{d^2\phi(x)}{dx^2} + \frac{2mE}{\hbar^2}\phi(x) = 0 \qquad |x| < a, \tag{4.29}$$

$$\frac{d^2\phi(x)}{dx^2} + \frac{2m(E - V_0)}{\hbar^2}\phi(x) = 0 \qquad |x| > a. \tag{4.30}$$

It is natural to define two wave numbers from these equations, obtaining respectively

$$k^2 = \frac{2m(E - V_0)}{\hbar^2}, \tag{4.31}$$

$$k'^2 = \frac{2mE}{\hbar^2}. \tag{4.32}$$

Like in the case of infinite square well, the potential is an even function. It follows that the energy eigenfunctions have defined parity. Thus, we can write the general solution of (4.29) and of (4.30) in the following way [42]

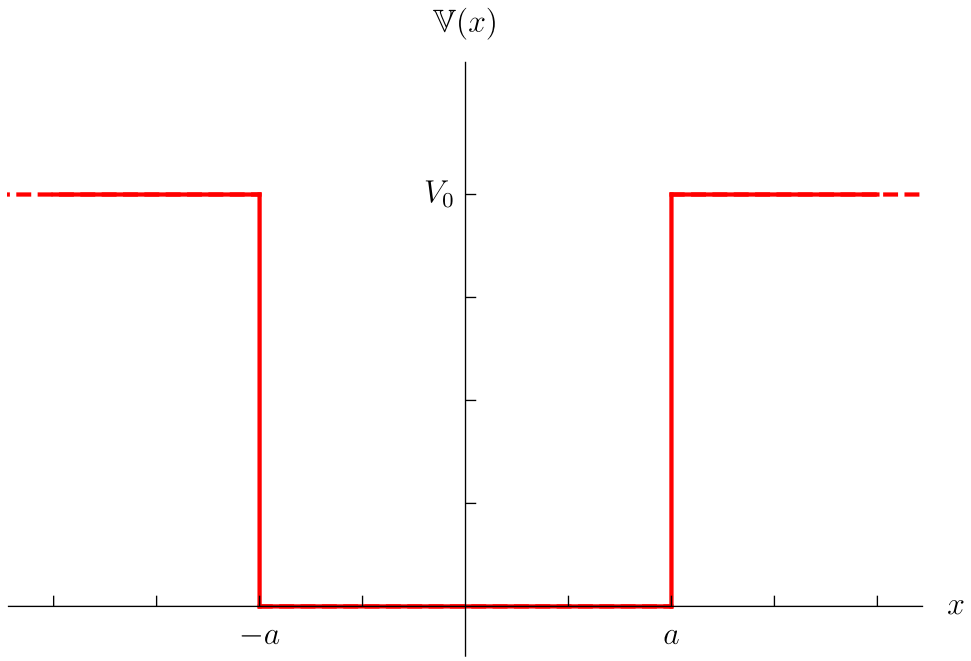

Figure 4.2: Finite square well with width $2a$, centered in the origin and depth V_0.

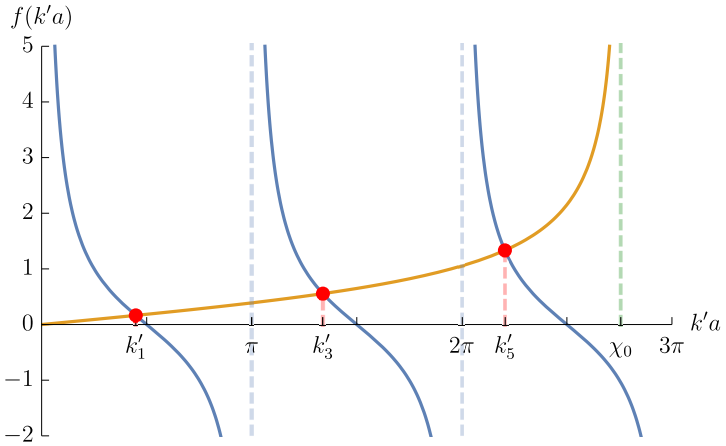

Figure 4.3: Numerical solutions of the transcendent equation (4.37) (red points), representing the intersection between $\cot(k'a)$ (blue line) with $k'/\sqrt{\chi_0^2 - k'^2}$ (orange line), with $a = 1$ and $\chi_0 = \sqrt{75}$. We see that we have three even solutions.

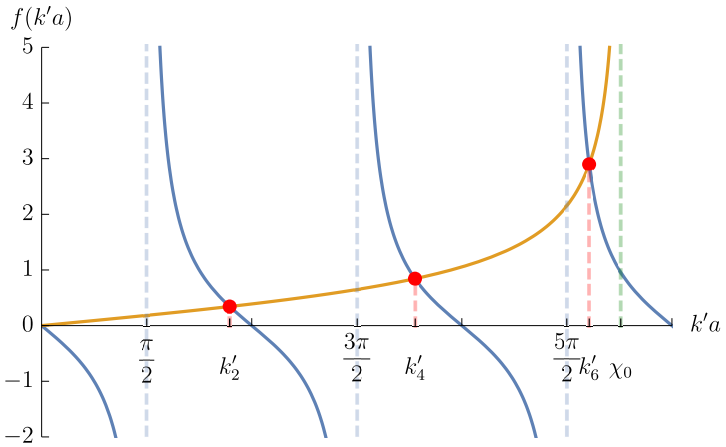

Figure 4.4: Numerical solutions of the transcendent equation (4.38). The solutions are the red points, representing the intersection between $-\tan(k'a)$ (blue line) with $k'/\sqrt{\chi_0^2 - k'^2}$ (orange line), with $a = 1$ and $\chi_0 = \sqrt{75}$. We see that we have three odd solutions. Similarly to the infinite square well, we exclude from the solutions the case $k' = 0$, since it would lead to a null wave function.

$$\phi^{\pm}(x) = A^{\pm}[\exp\{ik'x\} \pm \exp\{-ik'x\}] \qquad |x| < a, \tag{4.33}$$

$$\begin{aligned}\phi^{\pm}(x) = &\frac{A^{\pm}}{2}\sigma_{\pm}(x)\left\{\left[\left(1+\frac{k'}{k}\right)\exp\{ik'a\} \pm \left(1-\frac{k'}{k}\right)\exp\{-ik'a\}\right]\exp\{ik(|x|-a)\}\right. \\ &\left. + \left[\left(1-\frac{k'}{k}\right)\exp\{ik'a\} \pm \left(1+\frac{k'}{k}\right)\exp\{-ik'a\}\right]\exp\{-ik(|x|-a)\}\right\} \qquad |x| > a,\end{aligned} \tag{4.34}$$

where $\sigma_{\pm}(x)$ is equal to $+1$ for $x > 0$ and to ± 1 per $x \leq 0$. There are two kind of solutions: the first one is the continuous solution for energy E ranging from V_0 to infinity. Anyway, we do not take into consideration it because we focus on the discrete energy spectrum. The second one is the discrete spectrum for $E \leq V_0$. To deal only with real quantities, we write the imaginary wave number as

$$k = i\tilde{k} = i\frac{\sqrt{2m(V_0 - E)}}{\hbar}. \tag{4.35}$$

Moreover, we request to have bounded solution: in the case of the discrete spectrum, $|\exp\{\pm ik(|x| - a)\}|$ is not bounded as $|x| \to \infty$ and we need that the coefficient of

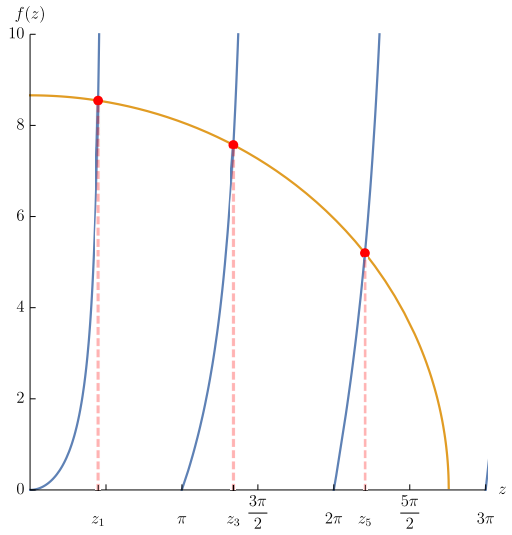

Figure 4.5: Numerical solutions of the transcendent equation (4.40) (red points), representing the intersection between $z_n \tan z_n$ (blue line) with $\sqrt{z_0^2 - z_n^2}$ (orange line), with $a = 1$, $\chi_0 = \sqrt{75}$ and consequently $z_0 = \sqrt{75}$.

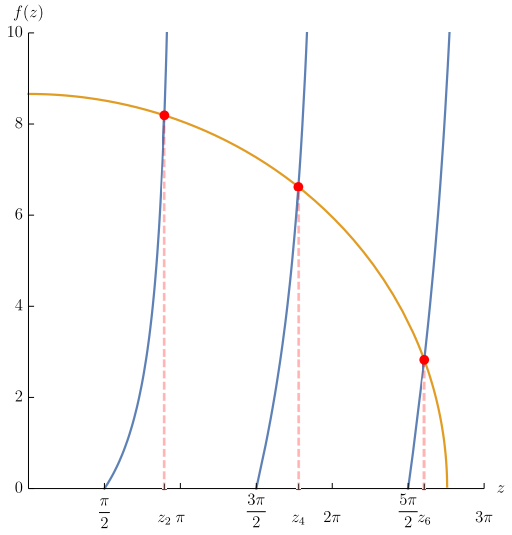

Figure 4.6: Numerical solutions of the transcendent equation (4.41) (red points), representing the intersection between $-z_n \cot z_n$ (blue line) with $\sqrt{z_0^2 - z_n^2}$ (orange line), with $a = 1$, $\chi_0 = \sqrt{75}$ and consequently $z_0 = \sqrt{75}$.

$\exp\{-ik(|x| - a)\}$ vanishes. This yields the eigenvalue relation [42]

$$\left(1 - \frac{k'}{i\tilde{k}}\right) \exp\{ik'a\} \pm \left(1 + \frac{k'}{i\tilde{k}}\right) \exp\{-ik'a\} = 0, \tag{4.36}$$

and determines the discrete value of the energy. We have that respectively for the even and odd solutions the constraints on k' are

$$\cot k'a = \frac{k'}{\tilde{k}}, \tag{4.37}$$

$$\tan k'a = -\frac{k'}{\tilde{k}}. \tag{4.38}$$

Both equations are transcendent and there are no closed formula for k'. However, we can solve it graphically following [11], see figs. 4.3 and 4.4 . Defining two new variables

$$z_n = k'_n a, \quad z_0^2 = \frac{2mV_0a^2}{\hbar^2}, \tag{4.39}$$

we have that $\tilde{k}^2 = z_0^2/a^2 - k'^2$ and the eigenvalues relations becomes respectively

$$z_n \tan z_n = \sqrt{z_0^2 - z_n^2} \quad n = 1, 3, 5, 7, \ldots, \tag{4.40}$$

$$z_n \cot z_n = -\sqrt{z_0^2 - z_n^2} \quad n = 2, 4, 6, 8, \ldots. \tag{4.41}$$

We interpret these equations graphically too [11] and we see that represents the intercept from a semicircle of radius z_0 and two curves, $z_n \tan z_n$ and $z_n \cot z_n$. From the figs. 4.5 and 4.6 we see that we get a solution every $\pi/2$: therefore the total number of solution is given by

$$N_{sol} = \left\lfloor \frac{2z_0 + \pi}{\pi} \right\rfloor, \tag{4.42}$$

Similarly, we have that the number of positive solution is given by

$$N_{sol}^{(+)} = \left\lfloor \frac{z_0 + \pi}{\pi} \right\rfloor \tag{4.43}$$

while the number of negative solution is

$$N_{sol}^{(-)} = \left\lfloor \frac{z_0 + \pi/2}{\pi} \right\rfloor. \tag{4.44}$$

Although we have no closed form for the transcendent equations, we can recast them in a more convenient form in order to find approximate solution. If we square (4.40) and (4.41) we obtain

$$\tan^2 z_n = \frac{z_0^2}{z_n^2} - 1 \quad n = 1, 3, 5, 7, \ldots, \tag{4.45}$$

$$\cot^2 z_n = \frac{z_0^2}{z_n^2} - 1 \quad n = 2, 4, 6, 8, \ldots, \tag{4.46}$$

and using trigonometric identities we can see that these are the same as

$$|\cos z_n| = \frac{z_n}{z_0} \quad n = 1, 3, 5, 7, \ldots, \tag{4.47}$$

$$|\sin z_n| = \frac{z_n}{z_0} \quad n = 2, 4, 6, 8, \ldots. \tag{4.48}$$

with $n\pi/2 - \pi/2 \leq z \leq n\pi/2$, see figs. 4.7 and 4.7. Despite the equations are still transcendent, in this form we can manipulate them in a more convenient way since here we are dealing with bounded functions in the interval considered. Moreover, we have

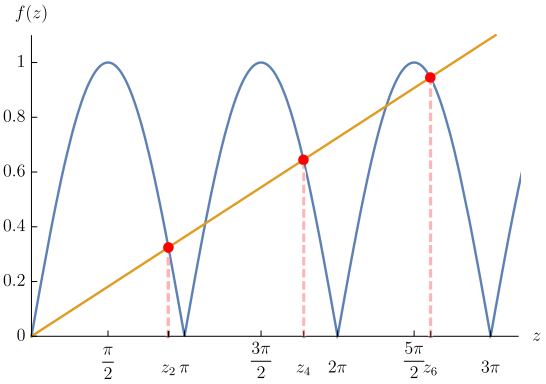

Figure 4.7: Solution to the transcendent equation (4.47): the blue line is $|\cos z_n|$, while the orange one is z_n/z_0. In this case too $z_0 = \sqrt{75}$.

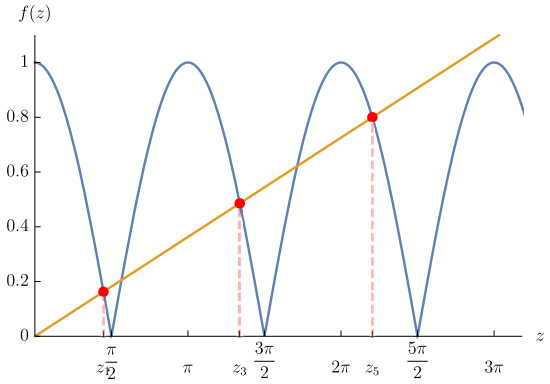

Figure 4.8: Solution to the transcendent equation (4.48): the blue line is $|\sin z_n|$, while the orange one is z_n/z_0. In this case too $z_0 = \sqrt{75}$.

also the possibility of approximating the cosine and the sine with polynomial functions in order to find a closed even though approximate solution for z.

In our case we choose to approximate the cosine with the parabolic approximation

$$\cos(x) \simeq 1 - \left(\frac{2x}{\pi}\right)^2, \quad 0 \leq x \leq \frac{\pi}{2}. \tag{4.49}$$

Since this relation is valid only in the interval $[0, \pi/2]$ we shift the variable $z_n = x_n + (n-1)\pi/2$ in (4.47) and in (4.48). Thus, we get

$$|\cos x_n| = \frac{z_n}{z_0}, \quad n = 1, 3, 5, 7 \ldots, \tag{4.50}$$

$$|\cos x_n| = \frac{z_n}{z_0}, \quad n = 2, 4, 6, 8 \ldots. \tag{4.51}$$

In this way the equations for the even and the odd parity eigenvalue equations have the same form, but we have to remember that we have shifted the variable. Expanding the cosine as (4.49), we find that the solution for x_n is given by

$$x_n = -\frac{\pi}{8z_0}\left[\pi - \sqrt{(4z_0 + \pi)^2 - 8\pi n z_0}\right]. \tag{4.52}$$

It follows that

$$z_n \simeq \frac{\pi}{8z_0}\left[4(n-1)z_0 - \pi + \sqrt{(4z_0 + \pi)^2 - 8\pi n z_0}\right], \tag{4.53}$$

where for n odd we have the even solutions and for even n we have odd solutions. We can expand z_n both for $z_0 \to 0$, getting

$$z_n \simeq n(2-n)z_0 + O(z_0)^2, \tag{4.54}$$

and for $z_0 \to \infty$, obtaining

$$z_n \simeq n\frac{\pi}{2} - \frac{\pi^2}{8}n\frac{1}{z_0}. \tag{4.55}$$

Given that the parabolic approximation is a series expansion around 0, it is a good approximation for $x_n \simeq 0$ and consequently for $z_n \simeq (n-1)\pi/2$. This is the case when $z_0 \to 0$. In addition, it is also a pretty good approximation for $x_n \simeq \pi/2$, that is the case where $z_0 \to \infty$.

Having solved the eigenvalue problem, even if in an approximate way, we can find the energy as a function of z_0

$$E_n^{(0)} \simeq \frac{\hbar^2 z_n^2}{2ma^2} = \frac{\hbar^2\pi^2}{128ma^2 z_0^2}\left[4(n-1)z_0 - \pi + \sqrt{(4z_0+\pi)^2 - 8\pi n z_0}\right]^2, \tag{4.56}$$

and in the limit of $z_0 \to \infty$ we came back with the result of the infinite square well

$$E_n^{(0)} = \frac{n^2\hbar^2\pi^2}{8ma^2}. \tag{4.57}$$

Furthermore, we can compute also the normalization constants for the discrete spectrum eigenfunctions

$$\int_{-\infty}^{+\infty} dx \phi_n^{(\pm)*}(x)\phi_n^{(\pm)}(x) = 1, \tag{4.58}$$

obtaining for even parity

$$\begin{aligned}\left|N_n^{(+)}\right|^2 &= \left(\frac{\cos\left(k_n'^{(+)}a\right)^2}{\tilde{k}_n^{(+)}} + \frac{\sin\left(2k_n'^{(+)}a\right)}{2k_n'^{(+)}} + a\right)^{-1} = \\ &= \left(\frac{\cos\left(k_n'^{(+)}a\right)^3}{\sin\left(k_n'^{(+)}a\right)k_n'^{(+)}} + \frac{\sin\left(k_n'^{(+)}a\right)\cos\left(k_n'^{(+)}a\right)}{k_n'^{(+)}} + a\right)^{-1},\end{aligned} \tag{4.59}$$

and for odd parity

$$\left|N_n^{(-)}\right|^2 = \left(\frac{\sin^2(k_n'^{(-)}a)}{\tilde{k}_n^{(-)}} - \frac{\sin\left(2ak_n'^{(-)}\right)}{2k_n'^{(-)}} + a\right)^{-1}. \tag{4.60}$$

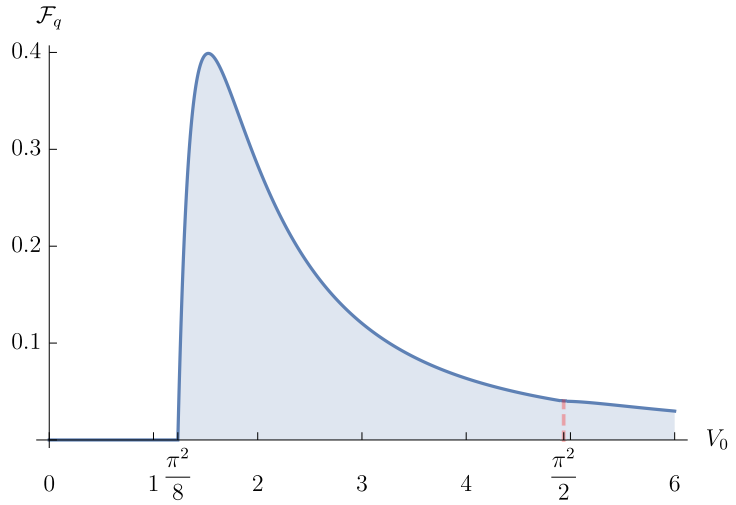

Figure 4.9: Quantum Fisher Information for $a = 1$ as a function of V_0. We see that, till the value of $\pi^2/8$, the $\mathcal{F}_q$ is null, due to the fact that $N_{sol} = 1$. After that value, the $\mathcal{F}_q$ rise and reach the maximum and then slowly decrease to 0. Furthermore, we can see that, for the value $V_0 = \pi^2/2$, we have a singular point due to the fact that here N_{sol} switches from 2 to 3.

At this point we can take care of the perturbation $\mathbb{H}_1$. For the moment we consider only the ground state of the system $|1^+\rangle$ and the effects of $\mathbb{H}_1$ on it. The first-order perturbation ket is given by

$$|\psi_0^{(1)}\rangle = \sum_{n\neq 1}^{N_{sol}} |n\rangle \frac{\langle n|\mathbb{H}_1|1\rangle}{E_1^{(0)} - E_n^{(0)}} + 2\pi \int_{k_0}^{+\infty} dk |\phi_k^{(\pm)}\rangle \frac{\langle \phi_k^{(\pm)}|\mathbb{H}_1|1\rangle}{E_1^{(0)} - E_k^{(0)}}, \tag{4.61}$$

in which we have also considered the term of the continuous spectrum although we do not know how to compute it because of normalization problems. Thus, we will consider only a finite square well with a large V_0 so that we can neglect its contribution. The discrete matrix elements are

$$\langle n^{(\pm)}|\mathbb{H}_1|1^{(+)}\rangle = \int_{-\infty}^{+\infty} dx \phi_n^{(\pm)*}(x)\mathbb{H}_1\phi_1^{(+)}(x), \tag{4.62}$$

which have a quite complicated expression. In any case, from the evaluation of the matrix element we obtain that there are two terms that contribute to the Quantum

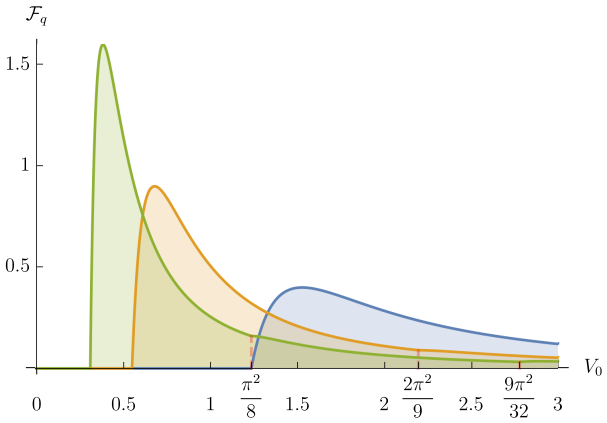

Figure 4.10: Quantum Fisher Information as a function of V_0 for different value of a. Blue Line $a = 1$, Orange Line $a = 1,5$, Green Line $a = 2$. The red-dot-line represents the singular point where there is a discontinuity in N_{sol}.

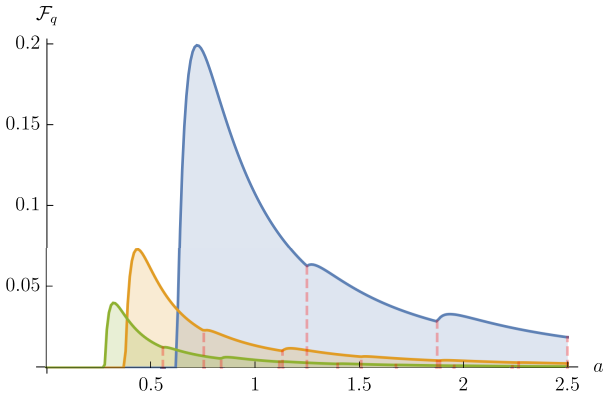

Figure 4.11: Quantum Fisher Information as a function of a for different value of V_0. Blue Line $V_0 = \sqrt{10}$, Orange Line $V_0 = \sqrt{75}$, Green Line $V_0 = \sqrt{250}$. The red-dot-line represents the singular point where there is a discontinuity in N_{sol}.

Fisher Information

$$\mathcal{F}_q(\gamma; V_0, a) = 4\left[\sum_{n^+\neq 1^+}^{N_{sol}^{(+)}} \frac{|\langle n^+|\mathbb{H}_1|1^+\rangle|^2}{\left(E_{1^+}^{(0)} - E_{n^{(+)}}^{(0)}\right)^2} + \sum_{n^-=1^-}^{N_{sol}^{(-)}} \frac{|\langle n^-|\mathbb{H}_1|1^+\rangle|^2}{\left(E_{1^+}^{(0)} - E_{n^{(-)}}^{(0)}\right)^2}\right]. \tag{4.63}$$

Given that the system can not be solved analytically, the value of the Quantum Fisher Information must be computed numerically too. The results are depicted in figs. 4.9 to 4.11.

These results, however, must be taken with the opportune cautions: as we have already stated, in the computation of the Quantum Fisher Information we have entirely discarded the continuous energy spectrum since the continuous energy eigenfunctions are not normalizable. The main consequence is that, in the figs. 4.9 to 4.11, for small values of V_0 we would have had a contribution from the continuous part which, unfortunately, we can not quantify.

Nonetheless, we can do some observation. In 4.10 we can see that for larger values of a the $\mathcal{F}_q$ peaks at smaller values of V_0 (but here we enter the regime where we do not know how much the continuous terms contribute). At the same time, for $V_0 \to \infty$,

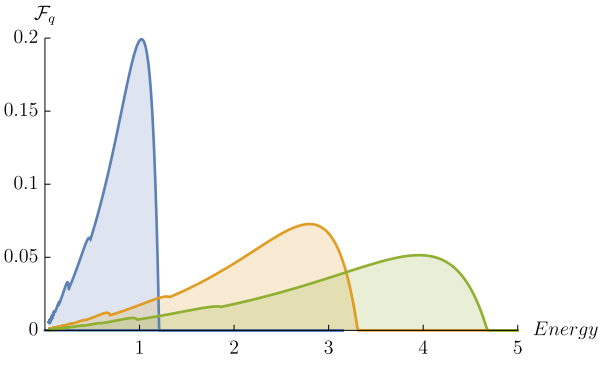

Figure 4.12: Parametric plot of the ground state energy versus Quantum Fisher Information. Blue line $V_0 = \sqrt{10}$, Orange Line $V_0 = \sqrt{75}$, Green Line $V_0 = \sqrt{150}$.

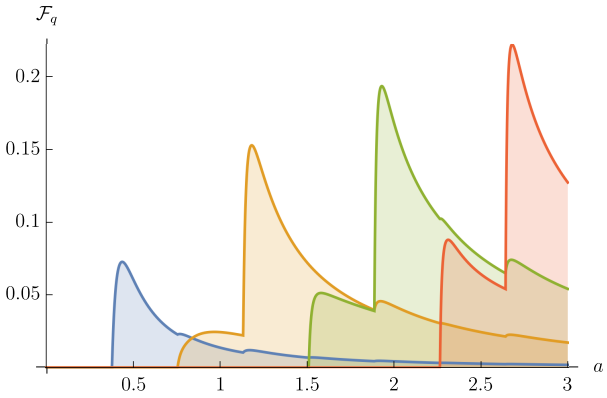

Figure 4.13: Quantum Fisher Information for even excited states as a function of a, with $V_0 = \sqrt{75}$. In blue $n^+ = 1$, in orange $n^+ = 2$, in green $n^+ = 3$, in red $n^+ = 4$.

the $\mathcal{F}_q$ decrease with increasing a, and as we expect, in the limit of $a \to \infty$, we find the result of the free particle, $\mathcal{F}_q = 0$. In a similar way, in 4.11 we see that, as V_0 grows, the $\mathcal{F}_q$ has the peaks for smaller values of the width a and their absolute value decrease too. To conclude, better estimates can be achieved for smaller values of the width a.

Moreover, we can see from 4.12 how the Quantum Fisher Information depends on the energy of the ground state. In this figure, the plot is taken by varying the width of the well a. As we see, for high energy the $\mathcal{F}_q$ is null. The reason is that the higher the energy, the smaller is a. But at the same time, for small a the N_{sol} is null and consequently $\mathcal{F}_q = 0$.

Finally, we compute also the $\mathcal{F}_q$ for the case of even excited states, generalizing the (4.63). The results can be seen in figure 4.13. We see that for higher excited state we have a higher peak in the $\mathcal{F}_q$. However, this peak can be reached only for larger width, since if a is too small there will not be enough excitations.

4.1.4 Harmonic Oscillator in 1D

The last system we consider in this section is the 1-dimensional massive Harmonic Oscillator. Several systems can be modeled by the Harmonic Oscillator: the Hooke's atom or a charge q of mass m in a magnetic field. Generally speaking, the Hamiltonian

of the Quantum Harmonic Oscillator is

$$\mathbb{H}_0 = \frac{\hat{p}_0^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2. \tag{4.64}$$

We are going to solve it with the ladder operator method, thus we introduce the annihilation operator $\hat{a}$ and creation operator $a^\dagger$

$$\hat{a} = \frac{1}{(2m\omega\hbar)^{1/2}}(\hat{p}_0 - im\omega\hat{q}), \tag{4.65}$$

$$\hat{a}^\dagger = \frac{1}{(2m\omega\hbar)^{1/2}}(\hat{p}_0 + im\omega\hat{q}). \tag{4.66}$$

The Hamiltonian can be easily diagonalized and we find that

$$\mathbb{H}_0 = \hbar\omega\left(\mathbb{N} + \frac{\mathbb{1}}{2}\right). \tag{4.67}$$

where the energy eigenstates $|n\rangle$ are labeled by the integer number n and the energy eigenvalues are

$$E_n^{(0)} = \hbar\omega\left(n + \frac{1}{2}\right). \tag{4.68}$$

The operator $\mathbb{N} = \hat{a}^\dagger\hat{a}$ is the number operator because

$$\mathbb{N}|n\rangle = n|n\rangle. \tag{4.69}$$

After this brief introduction to the Quantum Harmonic Oscillator, as usual, we consider now the full system described by the Hamiltonian

$$\mathbb{H} = \mathbb{H}_0 + \gamma\mathbb{H}_1. \tag{4.70}$$

In this case, however, we do not have the useful property we had in the free particle case, namely $[\mathbb{H}_0, \mathbb{H}_1] = 0$, since the perturbation expressed as function of $\hat{a}$ and $\hat{a}^\dagger$ has the form

$$\mathbb{H}_1 = \frac{\hat{p}_0^4}{m(M_P c)^2} = \frac{1}{m(M_P c)^2}\left(\frac{m\omega\hbar}{2}\right)^2\left(\hat{a} + \hat{a}^\dagger\right)^4, \tag{4.71}$$

whose commutator with $\mathbb{N}$ is different from 0. For that reason, $\mathbb{H}$ is not diagonalizable and we must consider the perturbation theory, at least at first-order in γ.

If we consider, as first case of study, a perturbation of the ground state $|0\rangle$, we see that the effect due to $\mathbb{H}_1$ is anything else but a quartic and a quadratic excitation

$$\left(\hat{a}+\hat{a}^\dagger\right)^4|0\rangle = \left(3|0\rangle + 2\sqrt{6}|4\rangle + 6\sqrt{2}|2\rangle\right). \tag{4.72}$$

As a result, the first-order perturbation ket for the ground state is

$$|\psi_0^{(1)}\rangle = \sum_{n>0}|n\rangle\frac{\langle n|\mathbb{H}_1|0\rangle}{E_0^{(0)}-E_n^{(0)}} = -\frac{m\omega\hbar}{4(M_P c)^2}\left(\frac{\sqrt{6}}{2}|4\rangle + 3\sqrt{2}|2\rangle\right), \tag{4.73}$$

and its norm is

$$\|\psi_0^{(1)}\|^2 = \frac{39(m\omega\hbar)^2}{32(M_P c)^4}. \tag{4.74}$$

Finally, we obtain the perturbed ket

$$|\psi_0^\gamma\rangle = \frac{1}{\sqrt{1+\|\psi_0^{(1)}\|^2}}\left(|0\rangle + \gamma|\psi_0^{(1)}\rangle\right). \tag{4.75}$$

In the case of a pure static case we know that the Quantum Fisher Information is easily obtained from the norm of $|\psi_0^{(1)}\rangle$, see equation (2.16), and lead us to

$$\mathcal{F}_q(\gamma) = \frac{39(m\omega\hbar)^2}{8(M_P c)^4}. \tag{4.76}$$

We conclude that, at first-order in γ, the Quantum Fisher Information does not depend on the parameter γ and it is a constant different from 0. It is quadratic in the Plank constant $\hbar$, meaning that its value is really small, and it does not depend on the value of γ. Moreover, the factor $(M_P c)^{-4}$ make the Quantum Fisher Information almost zero.

We can easily evaluate the Quantum Fisher Information for any excited state. Firstly, the perturbed ket is

$$|\psi_n^\gamma\rangle = \frac{1}{\sqrt{1+\|\psi_n^{(1)}\|^2}}\left(|n\rangle + \gamma|\psi_n^{(1)}\rangle\right). \tag{4.77}$$

with the first-order perturbation ket given by

$$|\psi_n^{(1)}\rangle = \sum_{l\neq n}|l\rangle\frac{\langle l|\mathbb{H}_1|n\rangle}{E_m^{(0)}-E_n^{(0)}}. \tag{4.78}$$

We can compute the generic matrix element

$$\begin{aligned}\langle l|\mathbb{H}_1|n\rangle =&\frac{\hbar^2 m\omega^2}{4(M_P c)^2}\Bigg[\left(6n^2+6n+3\right)\delta_{l-n}+2\sqrt{n-1}\sqrt{n}(2n-1)\delta_{l-n+2}+\\&+\sqrt{n-3}\sqrt{n-2}\sqrt{n-1}+\sqrt{n}\delta_{l-n+4}+4n\sqrt{n+1}\sqrt{n+2}\delta_{-l+n+2}+\\&+6\sqrt{n+1}\sqrt{n+2}\delta_{-l+n+2}+\sqrt{n+1}\sqrt{n+2}\sqrt{n+3}\sqrt{n+4}\delta_{-l+n+4}\Bigg],\end{aligned}\tag{4.79}$$

and the computation of Quantum Fisher Information is pretty straightforward, leading to

$$\mathcal{F}_q(\gamma)=\frac{(\hbar m\omega)^2}{32(M_P c)^4}\left(65n^4+130n^3+487n^2+422n+156\right).\tag{4.80}$$

We see that the dependence on the constants of the system, such as m, ω and $\hbar$, does not change compared to the Quantum Fisher Information of the ground state, but we have a different factor, quartic in n, as we would have expected given that the perturbation is quartic in the creation and annihilation operator.

4.1.5 Comparison of different systems

In this last section, we would like to compare the different results obtained for the pure energy eigenstates, in order to rank the best systems for the estimation of γ. Differently from the previous chapter, here we choose a range of the parameter corresponding to specific real systems in order to have a proper comparison. We set the value of particle mass at $m = 10^{-27}$ Kg, which is in the order of magnitude of the Hydrogen mass.

As we have seen, for pure energy eigenstates preparations, the only systems with $\mathcal{F}_q \neq 0$ are the Finite Potential Well and the Harmonic Oscillator. To have a clearer comparison, we choose to use a logarithmic plot for the Quantum Fisher Information. For the Finite Well, we set $V_0 = 50$eV and we vary the width in a range that goes from 0.001 nm to 1nm, which is the typical scale of quantum dots. The parametric plot can be seen in 4.15, where $\log \mathcal{F}_q$ is approximately 10^{-378}. Similarly, for a Harmonic Oscillator with the same mass, we vary the angular frequency ω from 10^{13} Hz to 10^{14} Hz, the typical range of frequencies of diatomic molecules, and we depict the result in 4.14. In this case, the $\log \mathcal{F}_q$ increases with the energy of the ground state. We see that for the

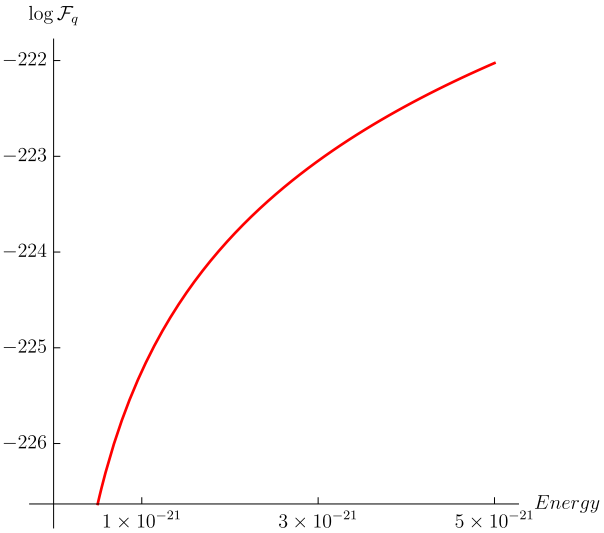

Figure 4.14: Parametric plot of the logarithm of Quantum Fisher Information versus the Energy of the system in the 1-dimensional Harmonic Oscillator for a pure perturbed ground energy eigenstate. Here we varied the angular frequency of the system in the range $10^{13} - 10^{14}$ Hz.

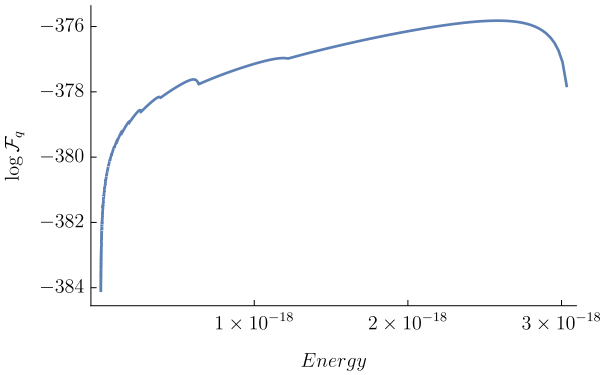

Figure 4.15: Parametric plot of the logarithm of Quantum Fisher Information versus the Energy of the system in the Finite Square Well for a pure perturbed ground energy eigenstate. Here we varied the width of the well in the range $0.01^{-10} - 10^{-10}$ m.

values under consideration, the power of estimate in the Finite Square Well outweighs that of the Quantum Harmonic Oscillator of many orders of magnitude.

4.2 Superposition of perturbed energy eigenstates

4.2.1 Free Particle

As we have seen in the subsection 4.1.1, a pure energy eigenstate of the free particle is not a suitable choice for the estimation of the parameter, given that it does not depend on the parameter γ. In these cases, where the effect of the perturbation is restricted only to modification on the energy spectrum, considering only pure eigenstates or their superpositions leads nowhere since the states have no dependence on γ. On the contrary, superpositions of states evolving in time acquire a phase proportional to their energy

which instead depends on γ leading to nontrivial effects.

In the case of the free particle the superposition is given by a wave packet evolving in time

$$|\psi^\gamma(t)\rangle = \exp\left\{-\frac{i}{\hbar}\mathbb{H}t\right\}\int_{-\infty}^{+\infty} dp\psi_0(p)|p\rangle = \tag{4.81}$$

$$= \int_{-\infty}^{+\infty} dp\psi_0(p)\exp\left\{-\frac{i}{\hbar}E_p^\gamma t\right\}|p\rangle, \tag{4.82}$$

where the function $\psi_0(p)$ describes the shape of the wave packet in the momentum space. As usual, the density matrix $|\psi^\gamma(t)\rangle$ is

$$\rho_\gamma(t) = |\psi^\gamma(t)\rangle\langle\psi^\gamma(t)| = \int_{-\infty}^{+\infty} dp\int_{-\infty}^{+\infty} dp'\exp\left\{-\frac{i}{\hbar}(E_p^\gamma - E_{p'}^\gamma)t\right\}\psi_0(p)\psi_0(p')|p\rangle\langle p'|. \tag{4.83}$$

As always, for a pure state, the Quantum Fisher Information is (2.10). To evaluate $\mathcal{F}_q(\gamma)$ we need the derived ket, i.e.

$$|\partial_\gamma\psi_\gamma(t)\rangle = \int dp\psi_0(p)\exp\{-iE_p^\gamma t\}|p\rangle\left(-it\frac{p^4}{m(M_{Pl}c)^2}\right) = \tag{4.84}$$

$$= \left(-\frac{it}{m(M_{Pl}c)^2}\right)\int dp\psi_0(p)\exp\{-iE_p^\gamma t\}p^4|p\rangle. \tag{4.85}$$

and it follows that the two bra-ket product are

$$\langle\psi_\gamma(t)|\partial_\gamma\psi_\gamma(t)\rangle = \int dp|\psi_0(p)|^2p^4\left(-\frac{it}{m(M_{Pl}c)^2}\right), \tag{4.86}$$

$$\langle\partial_\gamma\psi_\gamma(t)|\partial_\gamma\psi_\gamma(t)\rangle = \int dp|\psi_0(p)|^2p^8\left(\frac{t^2}{m^2(M_{Pl}c)^4}\right). \tag{4.87}$$

Eventually, we obtain the Quantum Fisher Information

$$\mathcal{F}_q(\gamma) = 4\frac{t^2}{m^2(M_{Pl}c)^4}\left[\int dp|\psi_0(p)|^2p^8 - \left|\int dp|\psi_0(p)|^2p^4\right|^2\right]. \tag{4.88}$$

As usual, we see the quadratic dependence on time. To understand the consequence of the result so found, we study the Quantum Fisher Information for a wave packet with width σ and mean value p_m, i.e.

$$|\psi_0(p)|^2 = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\{-(p-p_m)^2/(2\sigma^2)\}. \tag{4.89}$$

Evaluating the integrals in (4.88), we find that

$$\mathcal{F}_q(\gamma;\sigma,p_m) = 32\frac{t^2}{m^2(M_Pc)^4}\sigma^2\left[2p_m^6 + 21p_m^4\sigma^2 + 48p_m^2\sigma^4 + 12\sigma^6\right] \tag{4.90}$$

We see that in the limit $\sigma \to 0$ of a very peaked wave packet we recover the result found for the pure energy eigenstate, that is $\mathcal{F}_q(\gamma;0,p_m) = 0$. Studying the QFI as a function of σ and p_m we find the following. At fixed variance σ, the minimum of the QFI is given at $p_m = 0$. We have the same result if we fix p_m and we search the minimum for σ: we find out that the minimum (as we would expect) is for $\sigma = 0$.

4.2.2 Infinite Square Well

As we have seen in section 4.1.2, the eigenstates of $\mathbb{H}$ are the unperturbed eigenstates $|n,\pm\rangle$ with defined parity. We recall that the energy is

$$E_{n,\pm} = \frac{\hbar^2\pi^2}{2ma^2}\eta_\pm^2(n) + \gamma\frac{1}{m(M_{Pl}c)^2}\left(\eta_\pm(n)\frac{\pi\hbar}{a}\right)^4, \tag{4.91}$$

which is an exact expression due to the fact that any order higher than first is identically null.

Here, we consider the case discussed at the end of section 2.2, where we have examined a generic superposition of unperturbed eigenstates. We have seen that the state maximizing the Quantum Fisher Information is the superposition of the two energy eigenstates whose corrections have the largest difference, leading to the Quantum Fisher Information given by (2.65). In the system into account, the energy correction is

$$E^{(1)}_{n,\pm} = \frac{1}{m(M_{Pl}c)^2}\left(\eta_\pm(n)\frac{\pi\hbar}{a}\right)^4, \tag{4.92}$$

recalling that $\eta_+(n) = n - 1/2$ and $\eta_-(n) = n$. The minimum energy correction is obviously given by the ground state

$$E^{(1)}_{1,+} = \frac{1}{16m(M_{Pl}c)^2}\left(\frac{\pi\hbar}{a}\right)^4. \tag{4.93}$$

Since the energy levels are not limited from above, we can consider two different superposition, namely

$$|\psi^{\gamma}(0)\rangle = \frac{1}{\sqrt{2}}\Big(|\psi_{1^+}\rangle + |\psi_{n^+}\rangle\Big), \tag{4.94}$$

$$|\psi^{\gamma}(0)\rangle = \frac{1}{\sqrt{2}}\Big(|\psi_{1^+}\rangle + |\psi_{n^-}\rangle\Big), \tag{4.95}$$

to which corresponds the Quantum Fisher Informations

$$\mathcal{F}_q(1^+, n^+) = \frac{t^2\pi^8\hbar^6}{m^2a^8(M_{Pl}c)^4}\left(\eta_+(n)^4 - \eta_+(1)^4\right)^2 = \frac{t^2\pi^8\hbar^6}{4m^2a^8(M_{Pl}c)^4}n^2\left(2n^3 - 3n^2 + 3n - 1\right)^2, \tag{4.96}$$

$$\mathcal{F}_q(1^+, n^-) = \frac{t^2\pi^8\hbar^6}{m^2a^8(M_{Pl}c)^4}\left(\eta_-(n)^4 - \eta_+(1)^4\right)^2 = \frac{t^2\pi^8\hbar^6}{m^2a^8(M_{Pl}c)^4}\left(n^4 - \frac{1}{16}\right)^2. \tag{4.97}$$

We note that the relative phase between the two superposed state is not important since it does not enter the QFI. Moreover, in both the results, we have that $\mathcal{F}_q$ is proportional to n^8.

4.2.3 Finite Potential Well

Similarly to the previous subsection, we can study the evolving-in-time superpositions for the finite potential well, although not analytically but numerically. Also in this case the energy corrections are

$$E^{(1)}_{n^{(\pm)}} = \langle n^{(\pm)}|\mathbb{H}_1|n^{\pm}\rangle, \tag{4.98}$$

and, among them, the minimum is given by the ground state correction $E^{(1)}_{+}$. Thus, we consider the superposition of the ground state with a generic excited state

$$|\psi^{\gamma}_{n^{\pm}}(0)\rangle = \frac{1}{\sqrt{2}}\Big(|\psi_{1^+}\rangle + |\psi_{n^{\pm}}\rangle\Big), \tag{4.99}$$

and we study how the Quantum Fisher Information depends on n and on $(\pm)$. In the plot 4.16 we can see the results given by the (2.65). We first notice that for smaller value of a we have a significant increase in the Quantum Fisher Information, approximately of two order of magnitude. Moreover, the value for the n^- is slightly larger than the value for n^+. The reason for this is that for the same n, the odd energies are larger than the even.

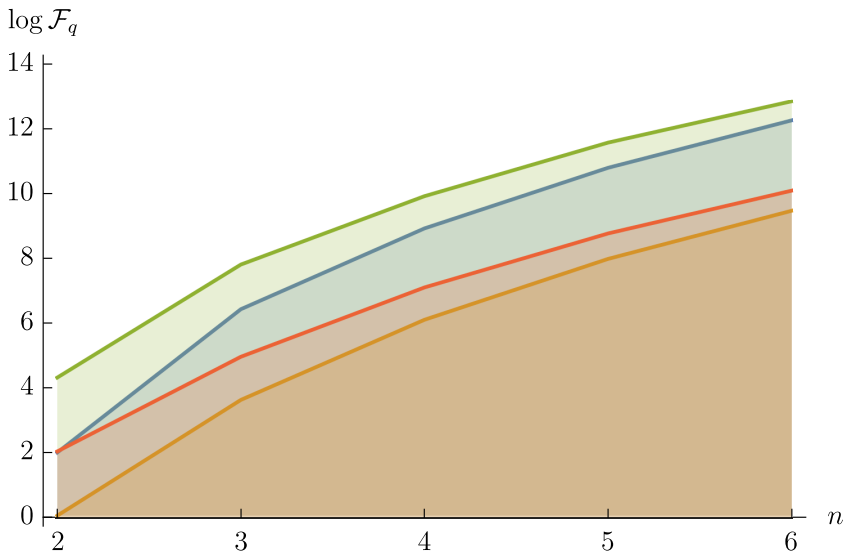

Figure 4.16: Logarithmic Plot of the Quantum Fisher Information in the finite potential well for the state $|\psi^{\gamma}_{n^{\pm}}(0)\rangle$ as a function of the excitation number n at first-order in γ. In Blue and Orange we have the n^{+}, respectively with $a = 3.5, V_0 = \sqrt{250}$ and $a = 5, V_0 = \sqrt{75}$, in Green and Red we have the n^{-}, respectively with $a = 3.5, V_0 = \sqrt{250}$ and $a = 5, V_0 = \sqrt{75}$. In all the results, we choose $t = 1$.

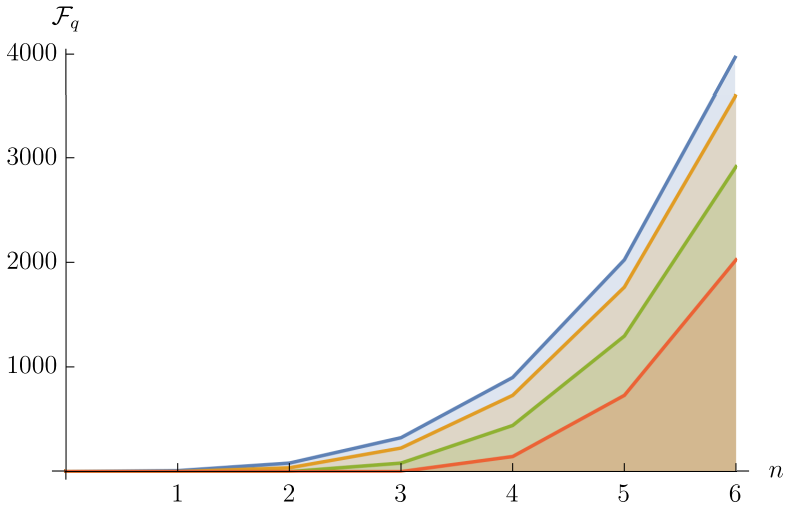

Figure 4.17: Quantum Fisher Information $\mathcal{F}_q$ of superposition of unperturbed energy eigenstates $|\psi^\gamma(0)\rangle = 1/\sqrt{2}\,(|m\rangle + |n\rangle)$ as a function of n in the 1-dimensional harmonic oscillator. In the plots we fixed $\omega = 1$. With the blue line is represented $\mathcal{F}_q(0, n)$, with the orange line $\mathcal{F}_q(1, n)$, with the green line $\mathcal{F}_q(2, n)$, with the red line $\mathcal{F}_q(3, n)$. In all the plot, $t = 1$. We clearly see the result found in 2.1 for the superposition that maximizes the Quantum Fisher.

4.2.4 Harmonic Oscillator 1D

The case of the Harmonic Oscillator can be addressed in two different ways. The first one is the regular one we have used in the previous sections, which consists in considering the superposition of two unperturbed energy eigenstates. However, the Harmonic Oscillator admits first-order energy perturbed ket different from 0, as we have seen in the section 4.1.4, and we can study superpositions of perturbed energy eigenstates too.

Firstly, we compute the energy corrections given by the perturbation $\mathbb{H}_1$ from (4.79), obtaining

$$E_n^{(1)} = \frac{3m\hbar^2\omega^2}{4(M_{Pl}c)^2}\left(1 + 2n + 2n^2\right), \tag{4.100}$$

and we notice that it is quadratic in the energy level and, as we expected, the minimum is given by $n = 0$. Therefore, following the notation of section 4.1.4, we take into account the superposition of unperturbed eigenstates at time $t = 0$, i.e.

$$|\psi^\gamma(0)\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + |n\rangle\Big), \tag{4.101}$$

to which corresponds the Quantum Fisher Information

$$\mathcal{F}_q(0,n) = \frac{t^2}{\hbar^2}\left(E_n^{(1)} - E_0^{(1)}\right)^2 = \frac{9t^2m^2\hbar^2\omega^4n^2(1+n)^2}{4(cM_{Pl})^4} \tag{4.102}$$

which is quartic in n. In figure 4.17 we see the plot for a generic superposition of two energy eigenstates. It can be clearly seen that the superposition (4.101) gives the larger Quantum Fisher Information for all the n, as we expect.

Now, we consider statics superpositions of the perturbed states

$$|\psi_n^\gamma\rangle = \frac{1}{\sqrt{1+\|\psi_n^{(1)}\|^2}}\left(|n\rangle + \gamma|\psi_n^{(1)}\rangle\right). \tag{4.103}$$

with the first-order perturbation ket given by

$$|\psi_n^{(1)}\rangle = \sum_{l\neq n}|l\rangle\frac{\langle l|\mathbb{H}_1|n\rangle}{E_m^{(0)} - E_n^{(0)}}. \tag{4.104}$$

In this case we have no closed formula for the Quantum Fisher Information, therefore we computed it numerically. As first case of study, we compute the $\mathcal{F}_q$ for static superposition at second order in γ.

The main results are depicted in figs. 4.18 and 4.19. In the first one, we see two different state preparations: the first one (thick line) is

$$|\psi_{0,n}^{\gamma,sup}\rangle = |\psi_0^\gamma\rangle + |\psi_n^\gamma\rangle, \tag{4.105}$$

while the second one (dot line) is

$$|\psi_{1,n}^{\gamma,sup}\rangle = |\psi_1^\gamma\rangle + |\psi_n^\gamma\rangle. \tag{4.106}$$

We see that for the same value of n, the second superposition (4.106) outweigh the first one for almost all the values of γ, which is a different trend if compared to the time-evolving state of unperturbed eigenstates preparations, where the best estimate is achieved for the state with larger separations between n and m. On the contrary, we deduce that when we consider a perturbed state, a better estimation can be reached with higher energy perturbed states. This fact can also be seen in 4.19, where we have considered three superpositions with equally spaced excitations. It can be clearly seen

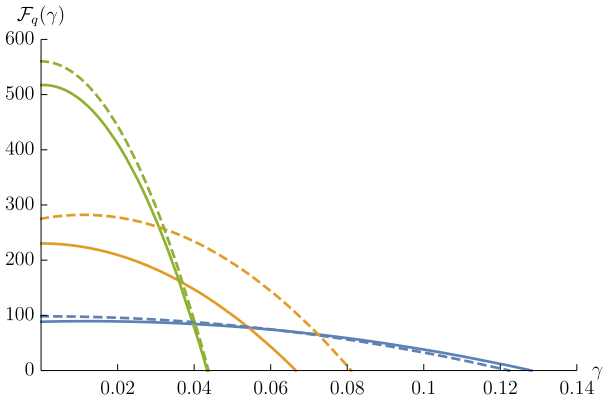

Figure 4.18: Quantum Fisher Information of the superposition of two perturbed energy eigenstates at second order in γ. We fixed $\omega = 1$. Thick Line: first superposed state is $|\psi_0^\gamma\rangle$. Dash Line: the first superposed state is $|\psi_1^\gamma\rangle$. Blue line: second state is $|\psi_2^\gamma\rangle$, Orange line: second state is $|\psi_3^\gamma\rangle$, Green line: second state is $|\psi_4^\gamma\rangle$.

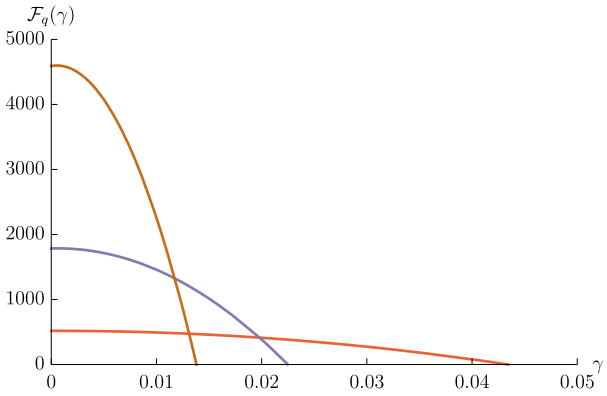

Figure 4.19: Quantum Fisher Information of the superposition of two perturbed energy eigenstates with equally separated excitation, at second order in γ. We fixed $\omega = 1$. Red line: superposition of $|\psi_0^\gamma\rangle$ and $|\psi_4^\gamma\rangle$, Purple line: superposition of $|\psi_2^\gamma\rangle$ and $|\psi_6^\gamma\rangle$. Orange line: superposition of $|\psi_4^\gamma\rangle$ and $|\psi_8^\gamma\rangle$.

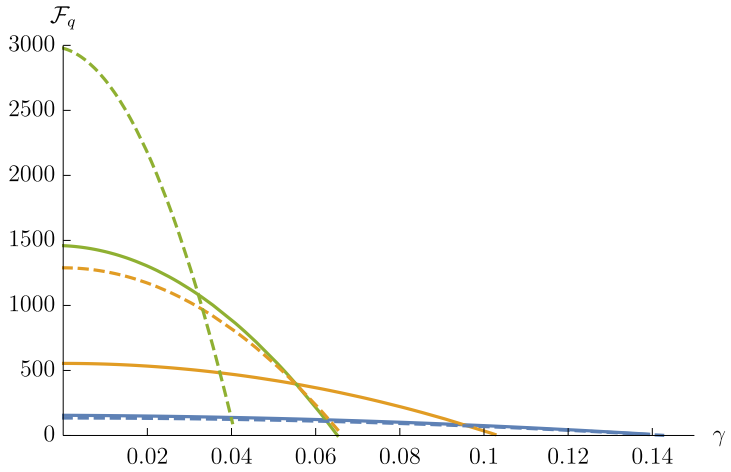

Figure 4.20: Quantum Fisher Information as function of γ for time-evolving superpositions at second order in γ. Same state preparations of 4.18, with $t = 1$ and $\omega = 1$.

that higher excitations have a higher Quantum Fisher Information. Moreover, we observe that in both figures, the state preparations give a better estimation for smaller values of γ than for higher. In addition, the states with higher peak also have a quicker decline if considered as a function of γ.

We now study superpositions of time-evolving states. Also in this case we have no closed formula, and we compute the value of Quantum Fisher Information numerically. In particular, we study the same preparations we used in 4.18, to compare the different results. We see there are some differences. First of all, the value of $\mathcal{F}_q$ is significantly larger in the time-evolving states. This is something expected, given that the Quantum Fisher Information for the unperturbed case is proportional to t^2. Secondly, we see that the trend previously observed for the states (4.105) and (4.106) here is even more evident: indeed, the state $|\psi_{1,2}^{\gamma,sup}\rangle$ has a very similar $\mathcal{F}_q$ of the state $|\psi_{0,3}^{\gamma,sup}\rangle$ and at the same time the Q.F.I. for $|\psi_{1,3}^{\gamma,sup}\rangle$ it is approximately twice the Q.F.I. of the $|\psi_{0,3}^{\gamma,sup}\rangle$, inverting the results we found for the unperturbed superpositions. It seems that what matters in this case is the total energy of the state. Clearly, these considerations are exact for small values of γ, due to the approximations we did. Moreover, as we similarly observed before, the larger is the Quantum Fisher for small values of γ, the quicker is the decline as γ increases.

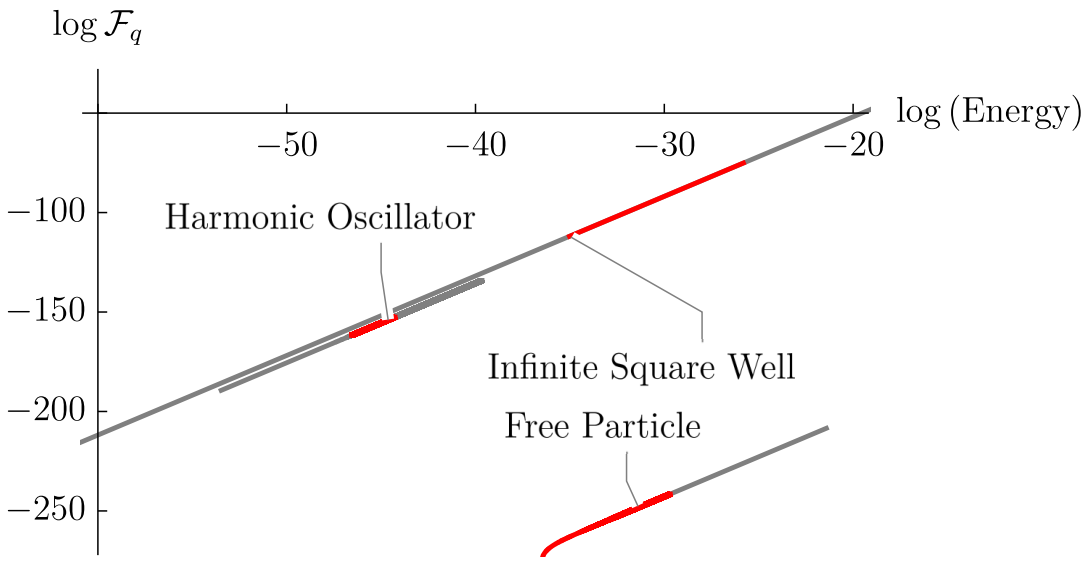

Figure 4.21: Logarithmic parametric plot of the Quantum Fisher Information as a function of the logarithm of the Energy. For a free particle we set $p_m = 1$ MeV/c; for the Harmonic oscillator we consider the Quantum Fisher of the time-evolving state $|\psi_{1,4}^{\gamma,sup}\rangle$; for the infinite square well we use the superposition of $|\psi_{1^+}\rangle$ and $|\psi_{4^+}\rangle$.

4.2.5 Comparison of different systems

In this final section we want to compare the results we have obtained in the previous sections for the superpositions of states using a choice of the range parameter corresponding to real specific systems, as we did in 4.1.5. In figure 4.21 we can see the results. The gray lines represents the logarithm of the Quantum Fisher Information as a function of the logarithm of the energy. We have marked with a red line the interval given by a specific platforms of the parameters. These have been chosen as follows: for the Infinite Square Well we take the same interval of a we used for the Finite Square Well in 4.1.5; for the Harmonic Oscillator we used the same range of frequencies used in 4.1.5; instead, for the free particle, we set $p_m = 1$ MeV/c and we varied the width in the interval that goes from 0 (very peaked wave packet) to 30 MeV/c. For all the systems $m = 10^{-27}$ Kg.

In 4.21 we observe that free particle has the lowest Fisher Information for the same energy levels. Differently, for the Quantum Harmonic oscillator the Quantum Fisher Information as a function of the energy of the systems ranges between 10^{-180} and 10^{-160} and for the same energy levels is just a little less than Infinite Square Well. Instead,

the Infinite Square well in the width range considered shows a larger Quantum Fisher Information, more than 50 orders of magnitude if compared with the Harmonic Oscillator.

Consequently, we conclude that for these values of the systems' parameters, we can have a better estimation with the infinite square well.

Chapter 5

Quantum estimation for gravity perturbation for H.O. in dimension higher than one

In the first chapter we have seen that the perturbation $\mathbb{H}_1$ arises from a modification of the canonical momentum (3.27). The main consequence is the new commutation relation, that we rewrite here for the sake of clarity

$$[x_i, p_j] = i\hbar \left(\delta_{ij} + \gamma \delta_{ij} \frac{(\vec{p})^2}{(M_{Pl}c)^2} + 2\gamma \frac{p_i p_j}{(M_{Pl}c)^2} \right), \tag{5.1}$$

where $\vec{p}$ is the redefined momentum. Differently from the classical commutator $[x^i, p_0^j] = i\hbar\delta^{ij}$, we note that orthogonal directions do not commute anymore, and for $i \neq j$ we have

$$[x_i, p_j] = 2i\hbar\gamma \frac{p_i p_j}{(M_{Pl}c)^2}. \tag{5.2}$$

This lack of commutativity indicates that different directions are not independent anymore. However, this fact arises only for systems with dimension higher than one, which has not been considered yet. In this chapter we want to study this characteristic and the possible enhancement we can have for system with dimension higher than one.

5.1 Harmonic Oscillator in two dimensions

We start our analysis from the two-dimensional Quantum Harmonic Oscillator. In this case the the Hamiltonian of the system is given by

$$\mathbb{H}_0 = \frac{\hat{p}_{x0}^2}{2m} + \frac{\hat{p}_{y0}^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \frac{1}{2}m\omega^2 y^2. \tag{5.3}$$

The form of the Hamiltonian makes clear that the two-dimensional Quantum Harmonic Oscillator is identical to two independent Quantum Harmonic Oscillator. Thus, we can easily diagonalize $\mathbb{H}_0$, introducing for each dimension the creation and annihilation operator as we did in the section 4.1.4. We obtain

$$\mathbb{H}_0 = (\mathbb{N}_x + \mathbb{N}_y + \mathbb{1})\hbar\omega, \tag{5.4}$$

where we have defined the number operator as

$$\mathbb{N}_x = \hat{a}_x^\dagger \hat{a}_x, \tag{5.5}$$

$$\mathbb{N}_y = \hat{a}_y^\dagger \hat{a}_y. \tag{5.6}$$

The eigenstates of the Hamiltonian $\mathbb{H}_0$ are all the states of the form

$$|n_x, n_y\rangle = |n_x\rangle \otimes |n_y\rangle, \tag{5.7}$$

where $|n_x\rangle$ represents the eigenstates of a 1-dimensional quantum Harmonic oscillator. The energy eigenvalue associated to $|n_x, n_y\rangle$ is given by

$$E^{(0)}_{n_x,n_y} = \hbar\omega(n_x + n_y + 1) \tag{5.8}$$

and it is labeled by two quantum numbers.

Here, there is a fundamental difference with the one-dimensional case: with the exception of the ground state, the energy levels are degenerate and we can easily compute the degree of degeneracy. If we fix $n = n_x + n_y$, thus we select an energy level, we have that the possible way to choose n_x and n_y are exactly

$$g_n = n + 1. \tag{5.9}$$

From this we recover the fact that for the ground state $n = 0$ we have $g_0 = 1$, so no degeneracy. This fact quite complicates the problem, because for the perturbation will be

necessary to use the degenerate perturbation theory (see the appendix 7.2 for a complete reference of the degenerate perturbation theory).

Now, we introduce the perturbation $\mathbb{H}_1$ that can be reformulated with the ladder operators as follows

$$\frac{\left(p_{x0}^2+p_{y0}^2\right)^2}{m(M_{Pl}c)^2}=\frac{1}{m(M_{Pl}c)^2}\left(\frac{m\omega\hbar}{2}\right)^2\left((\hat{a}_x+\hat{a}_x^\dagger)^4+(\hat{a}_y+\hat{a}_y^\dagger)^4+2(\hat{a}_x+\hat{a}_x^\dagger)^2(\hat{a}_y+\hat{a}_y^\dagger)^2\right), \tag{5.10}$$

and we see a coupling between the two different directions that does not allow us to consider the two-dimensional system as two independent systems anymore. The full Hamiltonian becomes

$$\mathbb{H}=(\mathbb{N}_x+\mathbb{N}_y+\mathbb{1})\hbar\omega+\frac{\gamma}{m(M_{Pl}c)^2}\left(\frac{m\omega\hbar}{2}\right)^2\left((\hat{a}_x+\hat{a}_x^\dagger)^4+(\hat{a}_y+\hat{a}_y^\dagger)^4+2(\hat{a}_x+\hat{a}_x^\dagger)^2(\hat{a}_y+\hat{a}_y^\dagger)^2\right), \tag{5.11}$$

and the first-order energy corrections are

$$E^{(1)}_{n_x,n_y}=\langle n_x,n_y|\mathbb{H}_1|n_x,n_y\rangle=\frac{1}{2}m\omega^2\hbar^2\left(3n_x^2+3n_y^2+4n_xn_y+5n_x+5n_y+4\right). \tag{5.12}$$

We are now ready to study different preparations of the states to quantify and compare the possibility of estimate the parameter γ.

5.1.1 Perturbed Energy eigenstates

Firstly, Let us start studying the simplest case: the effects of the perturbation on the non-degenerate ground state $|0_x,0_y\rangle$. In the same way of the 1-dimensional case, we find out that the first-order perturbation ket is

$$\begin{aligned}|\psi^{(1)}_{0,0}\rangle&=\sum_{(n_x,n_y)\neq(0,0)}^{+\infty}|n_x,n_y\rangle\frac{\langle n_x,n_y|\mathbb{H}_1|0_x,0_y\rangle}{E^{(0)}_{0,0}-E^{(0)}_{nx,ny}}=\\&=-\frac{\hbar m\omega}{4(M_{Pl}c)^2}\left[4\sqrt{2}(|0_x,2_y\rangle+|2_x,0_y\rangle)+\frac{\sqrt{6}}{2}(|4_x,0_y\rangle+|0_x,4_y\rangle)+|2_x,2_y\rangle\right]\end{aligned} \tag{5.13}$$

and the Quantum Fisher Information is obtained computing from the norm of $|\psi^{(1)}_{0,0}\rangle$, resulting at second order

$$\mathcal{F}_q^{2HO}(\gamma,0,0) = 17\frac{(\hbar m\omega)^2}{(M_{Pl}c)^4} - \frac{289}{2}\frac{(\hbar m\omega)^4}{(M_{Pl}c)^8}\gamma^2 + \mathcal{O}(\gamma^3). \tag{5.14}$$

We can compare the 0 order value with the analogous Quantum Fisher Information of the 1-dimensional Harmonic Oscillator for the corresponding ground state and we find that

$$\frac{\mathcal{F}_q^{2HO}(\gamma,0,0)}{2\cdot\mathcal{F}_q^{1HO}(\gamma,0)} = \frac{17}{2}\cdot\frac{8}{39} \simeq 1.74, \tag{5.15}$$

which is greater than one. We deduce that we have an enhancement for the two-dimensional Quantum Harmonic Oscillator, which is caused by the coupling term between the x and y direction.

After that, we can consider also the first excited state $|1_x,0_y\rangle$. This state belongs to the degenerate energy level $E^{(0)}_{1,0}$, which is doubly degenerate. But fortunately, the matrix element $\langle 0_x,1_y|\mathbb{H}_1|1_x,0_y\rangle$ that would generate the divergence in the perturbed ket

$$|\psi^{(1)}_{1,0}\rangle = \sum_{(n_x,n_y)\neq(1,0)}^{+\infty} |n_x,n_y\rangle\frac{\langle n_x,n_y|\mathbb{H}_1|1_x,0_y\rangle}{E^{(0)}_{1,0}-E^{(0)}_{nx,ny}} \tag{5.16}$$

is identically null. Therefore, we can avoid the use of degenerate perturbation theory and proceed as we have done in the previous case obtaining the first oder perturbation ket

$$|\psi^{(1)}_{1,0}\rangle = -m\hbar\omega\left(\frac{3\sqrt{2}}{2}|1_x,2_y\rangle + \frac{\sqrt{3}}{4}|3_x,2_y\rangle + \frac{1}{4}\sqrt{\frac{3}{2}}|1_x,4_y\rangle + \frac{3\sqrt{6}}{2}|3_x,0_y\rangle + \frac{1}{4}\sqrt{\frac{15}{2}}|5_x,0_y\rangle\right). \tag{5.17}$$

Eventually, we get the Quantum Fisher Information at second order

$$\mathcal{F}_q^{2HO}(\gamma,1,0) = 75\frac{(m\hbar\omega)^2}{(M_{Pl}c)^4} - \frac{5625}{2}\frac{(m\hbar\omega)^4}{(M_{Pl}c)^8}\gamma^2 + \mathcal{O}(\gamma^3). \tag{5.18}$$

which can be compared with the sum of$\mathcal{F}_q^{1HO}(\gamma,1)$ and $\mathcal{F}_q^{1HO}(\gamma,0)$ that is

$$\frac{\mathcal{F}_q^{2HO}(\gamma)}{\mathcal{F}_q^{1HO}(\gamma,1)+\mathcal{F}_q^{1HO}(\gamma,0)} = \frac{75}{\frac{39}{8}+\frac{315}{8}} \simeq 1.69, \tag{5.19}$$

that is slightly lower than the ratio for the ground state.

If we move to the second energy level with degeneracy $g = 3$, we see that we need the full degenerate perturbation theory described in 7.2. In this case we have three states, namely $|\psi_{2,0}\rangle, |\psi_{1,1}\rangle$ and $|\psi_{1,1}\rangle$ and the matrix element of $\mathbb{H}_1$ in the degenerate subspace spanned by the three vectors is

$$\begin{pmatrix} 13m\omega^2\hbar^2 & 0 & m\omega^2\hbar^2 \\ 0 & 12m\omega^2\hbar^2 & 0 \\ m\omega^2\hbar^2 & 0 & 13m\omega^2\hbar^2 \end{pmatrix} \tag{5.20}$$

Following the procedure, we diagonalize the matrix

$$\begin{pmatrix} 14m\omega^2\hbar^2 & 0 & 0 \\ 0 & 12m\omega^2\hbar^2 & 0 \\ 0 & 0 & 12m\omega^2\hbar^2 \end{pmatrix} \tag{5.21}$$

that lead to three new eigenstates

$$|\phi_1\rangle = \frac{|0_x, 2_y\rangle + |2_x, 0_y\rangle}{\sqrt{2}}, \tag{5.22}$$

$$|\phi_2\rangle = \frac{-|0_x, 2_y\rangle + |2_x, 0_y\rangle}{\sqrt{2}}, \tag{5.23}$$

$$|\phi_3\rangle = |1_x, 1_y\rangle\,. \tag{5.24}$$

The first two new energy eigenstates broke the degeneracy, at least at first-order. Therefore, we study the perturbation for these new states, especially for the $|\phi_1\rangle$, first computing the first-order perturbation ket as outlined in section 7.2, and finally obtaining

$$\mathcal{F}_q^{2HO)}(\gamma) = 281m^2\frac{\omega^2\hbar^2}{(M_{Pl}c)^4} - \frac{78961}{2}\gamma^2\frac{m^4\omega^4\hbar^4}{(M_{Pl}c)^8} + \mathcal{O}(\gamma^3). \tag{5.25}$$

In this case too we can study the enhancement due to the dimensionality of the system, studying the ratio

$$\frac{\mathcal{F}_q^{2HO)}(\gamma, |\phi_1\rangle)}{\mathcal{F}_q^{1H0}(\gamma, 0) + \mathcal{F}_q^{1H0}(\gamma, 2)} = \frac{281}{\frac{39}{8} + \frac{1257}{8}} \simeq 1.73, \tag{5.26}$$

which is very similar to the ratio already found.

By the way, we see that for all the state we have studied the enhancement is approximately 1.70, which is lower than the dimensionality $d = 2$ of the Harmonic Oscillator, but greater than 1. Therefore we deduce that for systems with $d = 2$ we have a better power of estimate.

5.1.2 Superposition of perturbed energy eigenstates

Other cases that can be studied are the superposition of energy eigenstates. For instance, we can study the superposition

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|0_x, 0_y\rangle + |0_x, 1_y\rangle\Big) \tag{5.27}$$

and the first-order perturbed ket is

$$|\psi^\gamma\rangle = \frac{1}{\sqrt{2}}\left(|\psi^\gamma_{00}\rangle + |\psi^\gamma_{10}\rangle\right) \tag{5.28}$$

where the ket are defined in the preceding section. Developing the calculations as already done above, we obtain Quantum Fisher Information

$$\mathcal{F}_q(\gamma) = 46\frac{(m\omega\hbar)^2}{(M_{Pl}c)^2} - \gamma^2\frac{2957}{2}\frac{(m\omega\hbar)^4}{(M_{Pl}c)^8} + \mathcal{O}(\gamma^4). \tag{5.29}$$

As usual, we compare this result with the Quantum Fisher Information for the 1-dim Harmonic Oscillator, thus finding that

$$\frac{\mathcal{F}_q^{2HO}(\gamma, |\psi\rangle)}{\frac{3}{2}\mathcal{F}_q^{1HO}(\gamma, 0) + \frac{1}{2}\mathcal{F}_q^{1HO}(\gamma, 1)} = \frac{46}{27} \simeq 1.70 \tag{5.30}$$

which value is very similar to the previous found.

5.1.3 Entanglement and Coherence

At this point in the study, we would like to study the entanglement and coherence for the states in the 2-dim H.O. and how they are related to the Quantum Fisher Information. Let us review what entanglement and coherence are.

Entanglement

The word entanglement was introduced by Schroedinger [34], that together with Einstein, Podolski and Rosen first noticed that some global states of compound systems could not be written as the product of states of the subsystems, and as a result the global description of the state does not provide an accurate description of the subsystems as well.

Generally speaking, the concept of separability fixes the criteria to establish whether a state is entangled or not [17]. In the case of pure bipartite systems, we can easily determine if a state is entangled or not. Because all separable states have the form of the product of pure state of the subsystem, namely $|\psi_A\rangle|\psi_B\rangle$, a state is separable if and only if the squared of the partial trace of ρ_{AB} is equal to 1 [26]. Otherwise, the state can not be written as a product of the pure state, hence it is entangled. Another way to see that is considering the Schmidt decomposition of the bipartite state. If the Schmidt rank $r(|\psi_{AB}\rangle$ is greater than one, then the state is entangled, because its Schmidt decomposition

$$|\psi_{AB}\rangle = \sum_{i=0}^{r(|\psi_{AB}\rangle)} c_i|\tilde{a}_i\rangle|\tilde{b}_i\rangle \tag{5.31}$$

is not separable.

In case of entangled states, we would like to quantify how much the state is entangled. For a generic state there is not a unique way to do that though [17, 41], since entanglement can be characterized in many different ways. However, many entanglement measure evaluated on pure bipartite state reduce to the Von Neumann entropy [40]

$$E(|\psi\rangle) = S(\rho_A) = S(\rho_B) = -\sum_{i=0}^{r(|\psi_{AB}\rangle)} c_i \log c_i, \tag{5.32}$$

where $\vec{c}$ is called the Schmidt vector, which determines the Schmidt decomposition (5.31). As a consequence, the Von Neumann entropy of the reduced state can be used as a proper entanglement measure for pure states.

Coherence

Coherence is another crucial feature of quantum mechanics, due to the wave-particle description of the quantum objects and the interference phenomenon that characterizes many quantum effects, we need only think of the double-slit experiment. In the quantum realm, coherence arises from the superposition principle [20] and consequently can be identified with the presence of off-diagonal terms in the density matrix. As a result, coherence strongly depends on the basis we use to describe our system.

There are many way to quantify coherence [38, 3]. One of them that is strongly related to the off-diagonal elements is the measure $\mathcal{C}$ induced by matrix norms l_1. It can

be defined as

$$\mathcal{C}(\rho) = \sum_{i \neq k} |\rho_{ik}| = \sum_{i,k} \rho_{ik} - \sum_{i} \rho_{ii} = \sum_{i,k} \rho_{ik} - 1, \tag{5.33}$$

where we have considered that the trace of a matrix density is equal to 1. We observe that any diagonal density matrix has null coherence $\mathcal{C} = 0$ and this state, in that basis, represents a classical mixtures.

Quantum Fisher Information relation with Entanglement and Coherence

After a brief review we are ready to study entanglement and coherence for the states of the 2-dim Harmonic Oscillator.

We start with the unperturbed state $|0_x, 0_y\rangle$, which is not entangled yet. However, due to the 2-dimensional perturbation and given that in the Hamiltonian $\mathbb{H}$ we have an extra term that couples the two different space direction, the perturbed state is entangled. Therefore, we take $\rho_{00}(\gamma) = |\psi^{\gamma}_{0,0}\rangle\langle\psi^{\gamma}_{0,0}|$ and we calculate the Schmidt vector, firstly tracing out the x subsystem. In this way we get $\rho_y = \mathrm{Tr}_x(\rho_{0,0})$, and then we diagonalize ρ_y. The eigenvalues of ρ_y are the elements of the Schmidt vector, which enters the Entanglement calculation. In this case the Schmidt number is equal to three, and expanding the Entropy Entanglement at the second order in γ (which is the first-order different from 0), we find

$$E(|\psi^{\gamma}_{0,0}\rangle) = \frac{\gamma^2 m^2 \omega^2 \hbar^2}{16(M_{Pl}c)^4} \left(1 - 2\log_2\left(\frac{\gamma m \omega \hbar}{4(M_{Pl}c)^2}\right)\right) + \mathcal{O}(\gamma^3). \tag{5.34}$$

In a completely analogous way, for the state $|1_x, 0_y\rangle$ we obtain the Entropy Entanglement as

$$E(|\psi^{\gamma}_{1,0}\rangle) = \frac{3\gamma^2 m^2 \omega^2 \hbar^2}{16(M_{Pl}c)^4} \left(1 - \log_2\left[\left(\frac{\gamma m \omega \hbar}{(M_{Pl}c)^2}\right)^2 \frac{3}{16}\right]\right) + \mathcal{O}(\gamma^3). \tag{5.35}$$

Finally, we study the case of the degenerate energy eigenstate

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}(|0_x, 2_y\rangle + |2_x, 0_y\rangle), \tag{5.36}$$

which is already entangled. In order to study the dependency of γ, we take the first-order different from 0 obtaining

$$E(|\phi_1\rangle) = 1 - \frac{4m^2\omega^2\hbar^2}{(M_{Pl}c)^4}\gamma^2 + \mathcal{O}(\gamma^3). \tag{5.37}$$

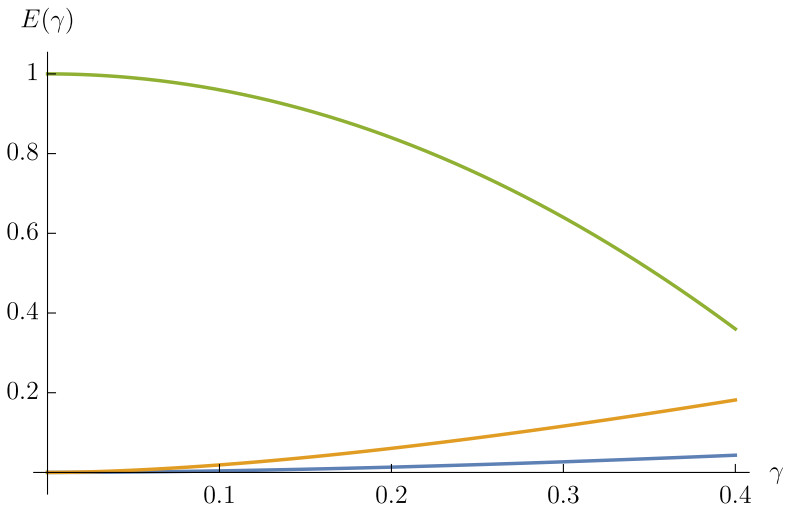

Figure 5.1: The value of the entanglement for the three perturbed states: in blue is $|\psi_{0,0}\rangle$, orange is $|\psi_{1,0}\rangle$ and green is $|\psi_1\rangle$. In order to achieve consistent results, that is, positive Quantum Fisher Information, we choose as a reliable range for the parameter γ the interval $[0, 0.4]$.

In the plots included in this section we set all the physical constants equal to one, with the same philosophy outlined in the introduction of chapter 2. In addition, we set $\omega = 1$ too. We can now see the three entanglements as a function of γ in figure 5.1. For the first two states, the entanglement grows very slowly with γ. The opposite happens for the last state already entangled, which has a moderate decline. In other words, for states not entangled, the stronger is the perturbation, the more the state became entangled. However, the perturbation has an opposite effect on states which already exhibit a form of entanglement, since a stronger perturbation tends to break the entanglement of the states.

Another interesting thing we can study is how entanglement and quantum fisher depends on each other. This is depicted by the parametric plots in figs. 5.2 and 5.3.

From the first figure we deduce that the Quantum Fisher Information does not grow with the entanglement, in fact it is the opposite: the more the state is entangled, meaning we have a larger γ, the less is the precision in the estimation of the parameter itself. This is related to the fact that $\mathcal{F}_q$ decreases with larger γ for the state we considered.

From the second graph, instead, it seems that quantum fisher information grows

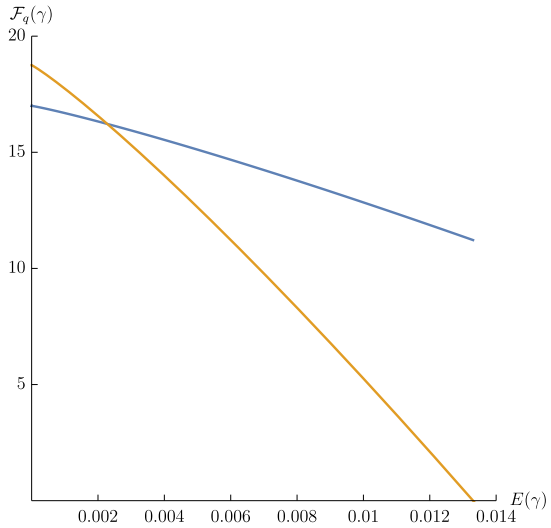

Figure 5.2: Parametric plot of Entanglement ($x-$axis) versus Quantum Fisher Information ($y-$axis). In blue we have the state $|\psi_{0,0}\rangle$, in orange the state $|\psi_{1,0}\rangle$.

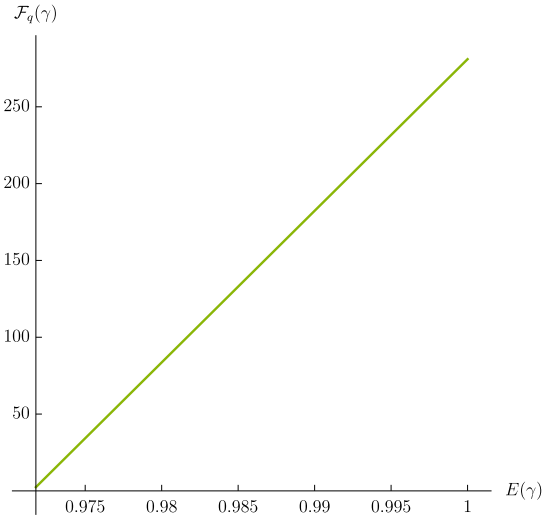

Figure 5.3: Parametric plot of Entanglement ($x-$axis) versus Quantum Fisher Information ($y-$axis). In this figure we have the energy degenerate state $|\phi_1\rangle$.

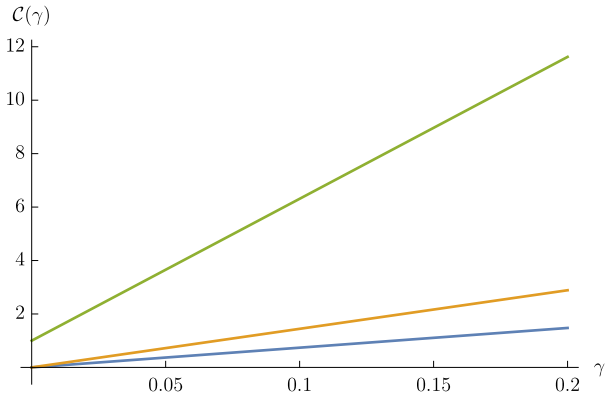

Figure 5.4: Coherence $\mathcal{C}$ as a function of γ. The state ρ_{00} is represented by the blue line, the state ρ_{10} by the orange line, the state $\eta(\gamma)$ by the green line.

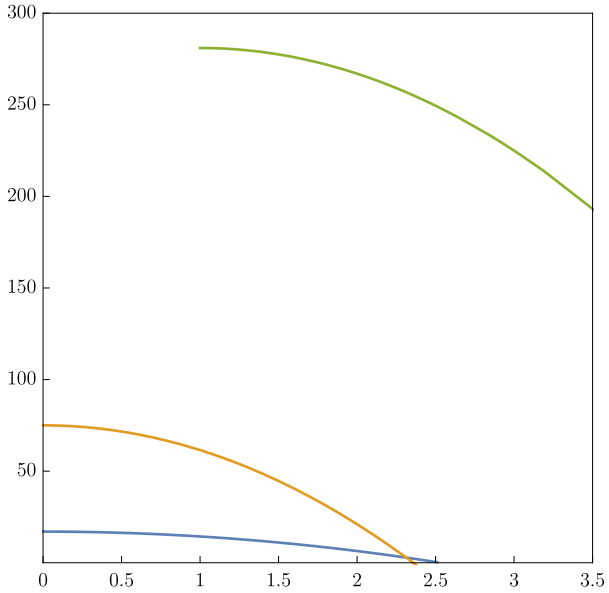

Figure 5.5: Parametric plot of the $\mathcal{F}_q$ as a function of the coherence $\mathcal{C}$. Just as 5.4, the state ρ_{00} is represented by the blue line, the state ρ_{10} by the orange line, the state $\eta(\gamma)$ by the green line.

with the entanglement. A closer look tell us that the larger the $\mathcal{F}_q$, and consequently the large the Entanglement, the smaller γ. This means that the state $|\phi_1\rangle$ gives a better estimability for smaller value of the parameter. We underline that in this figures we have used small value for the parameter γ so that the approximations we did are reliable.

The second step is to take into account the coherence. Using the coherence measure defined above, we can find that for the state $\rho_{00}(\gamma)$ at first-order different from 0 we have

$$\mathcal{C}(|\psi^{\gamma}_{0,0}\rangle) = \frac{\left(1 + 8\sqrt{2} + \sqrt{6}\right)\gamma m\omega\hbar}{2(M_{Pl}c)^2}. \tag{5.38}$$

Similarly we can do the same computation for the state $\rho_{10} = |\psi^{\gamma}_{1,0}\rangle\langle\psi^{\gamma}_{1,0}|$, obtaining

$$\mathcal{C}(|\psi^{\gamma}_{1,0}\rangle) = \frac{\left(12\sqrt{2} + 2\sqrt{3} + 13\sqrt{6} + \sqrt{30}\right)\gamma m\omega\hbar}{4(M_{Pl}c)^2}, \tag{5.39}$$

and for the state $\eta(\gamma) = |\phi_1\rangle\langle\phi_1|$, obtaining

$$\mathcal{C}(|\phi_1\rangle) = 1 + \frac{\left(24\sqrt{2} + 32\sqrt{3} + 3\sqrt{6} + 3\sqrt{10}\right)\gamma m\omega\hbar}{2(M_{Pl}c)^2}. \tag{5.40}$$

The coherence as a function of γ can be found in 5.4. In this case, the first-order different from 0 is the first, thus we see that the coherence is linear in γ. Moreover, we see that the first excited state grows faster than the ground state, while $\mathcal{C}$ for the state $|\phi_1\rangle$ is even more steeper. Similarly to figure 5.2, in 5.5 we see the dependence of the $\mathcal{F}_q$ as a function of the coherence. In this case we have a similar trend for each of the three states under consideration. However, the values of the quantum Fisher for the pure energy eigenstates are significantly smaller if compared to the superposition given by $|\phi_1\rangle$.

Chapter 6

Comments and Conclusion

In this thesis, we have investigated the possible estimation of a minimum length with the help of the Quantum Estimation Theory. In this final section, we sum up the results we have found.

Firstly, in chapter 2 we have studied the estimation problem of a parameter γ for a generic perturbation of the form $\gamma\mathbb{H}_1$. Having assumed that the perturbation is small, we used quantum perturbation theory, and we found that, for pure states, the Quantum Fisher Information is proportional to the square of the norm of the first-order perturbation ket and is independent of the parameter we want to measure, at least at first-order in γ. Moreover, we found explicitly the Symmetric Logarithmic Derivative, which is related to the optimal measurement we can implement for the estimation of γ. We have also proved that position measurement is an optimal measurement, as well as energy measurement. Then, we studied the time evolution, and for a first-order perturbed ket we found that the Quantum Fisher Information is equal to Quantum Fisher of the corresponding static state. Subsequently, we considered superpositions, and in 2.2 we showed that for superpositions of unperturbed energy eigenstates evolving in time, the Quantum Fisher Information is quadratic in t and does not depend on γ. Furthermore, we found that the Quantum Fisher Information has a maximum for a distinct superposition, which is given by the superpositions of the two states whose corresponding energy corrections have the most significant difference. We have also studied the thermal states and found that, for systems where $\mathbb{H}_0$ and $\mathbb{H}_1$ have the same eigenstates, the optimal measurement

is the energy measurement.

Secondly, we investigate the specific form of the perturbation $\mathbb{H}_1$ introduced in chapter 3 in four different systems, the free particle, the infinite square well, the finite square well and the 1-dimensional Harmonic oscillator. We started considering perturbed energy eigenstates 4.1, and we found that, for the Free Particle and the Infinite Square Well, the perturbation affects only the energy spectrum. Consequently, the perturbed states do not depend on γ, and the Quantum Fisher Information is identically null. Conversely, for the Harmonic Oscillator and the Finite Square Well, we have a non-null Quantum Fisher Information. For the former system, the $\mathcal{F}_q$ is proportional to the square of ω and the fourth-power of the energy level of the eigenstate. For the latter, we computed it numerically, and we found that it depends on the number of discrete levels that are available in the well.

Furthermore, in 4.2, we studied superposition preparations. For the free particle, we studied a Gaussian wave packet preparation and found an exact solution for the Quantum Fisher Information. In the Infinite Square Well, given that the perturbations affected only the energy spectrum, we used the result found in 2.2 for the superpositions of unperturbed eigenstates, and we found that the Quantum Fisher is proportional to a^{-8} and to n^8, where n is the energy level of the second state in the superpositions that maximize $\mathcal{F}_q$. For the Finite Square Well, instead, we computed the Quantum Fisher numerically. Finally, for the Harmonic Oscillator, we calculated the Quantum Fisher Information for different cases. Firstly, we checked the results found in 2.2. Secondly, we moved to superpositions of perturbed energy eigenstates, and we showed with numerical calculations that the hierarchy found for unperturbed superpositions does not hold anymore. Instead, we showed that the Quantum Fisher grows with the excitation levels of both the states in the superposition.

Eventually, in 4.2.5, we compared the results so obtained, and we found that for a suitable choice of parameters corresponding to real physical systems, the one with the highest Quantum Fisher Information is the Infinite Square Well.

In chapter 5, we derive the last result we found in this thesis, which is linked to the dimensionality of the systems. We observed that in 2-dimensional systems the perturbation $\mathbb{H}_1$ defined in3 has an additional term that couples the two orthogonal directions.

Thus, we speculated that we could have an enhancement. Therefore, we studied the 2-dimensional Harmonic oscillator, and we compared the value of the Quantum Fisher Information with the one computed in the 1-dimensional Harmonic Oscillator, founding that for analogous states preparations we have a gain of a factor approximately equal to 1.7, that is less than the dimensionality but higher than 1. For these reasons, we deduce that an enhancement in the estimation is possible for higher dimensional systems.

Following this last idea and from the conclusion found in 4.2.5, a possible further study may investigate the role of dimensionality in the Infinite Square Well and may quantify the possible enhancements.

Moreover, the role of the environment could be studied. Actually, we developed our thesis for closed systems considering only unitary evolution, and in these cases we were able to find both the ultimate bound given by the Quantum Fisher Information, and the optimal measurements. A further step we could take should consider the coupling of the system with an environment and should study the role of the induced noise to understand its effect on the results we found.

Part III

Appendix

Chapter 7

Non-relativistic Perturbation theory

Schroedinger equation can be solved exactly only for a limited number of physical systems. In many cases, the equations are too complex. However, for some physical systems, the Hamiltonian is the sum of an unperturbed operator $\mathbb{H}_0$ and of a perturbation operator $\mathbb{V}$:

$$\mathbb{H} = \mathbb{H}_0 + \mathbb{V}, \tag{7.1}$$

where we consider $\mathbb{V}$ a very small correction compared to $\mathbb{H}_0$(we will quantify this statement below). In this case, rather than solving exactly the problem, we can neglect infinitesimal quantities to simplify the problem: this method is called perturbation theory [21] and it finds the first-order corrections of energy eigenstates and energy eigenvalue.

In the following we assume that the energy spectrum of $\mathbb{H}_0$ is already known and it is discrete. We identify the eigenstate and eigenvalue of $\mathbb{H}_0$ respectively with $|\psi_n^{(0)}\rangle$ and $E_n^{(0)}$, namely

$$\mathbb{H}_0|\psi_n^{(0)}\rangle = |\psi_n^{(0)}\rangle. \tag{7.2}$$

The perturbation theory's goal is to find approximate solutions of the equation

$$\mathbb{H}|\psi\rangle = E|\psi\rangle, \tag{7.3}$$

where the approximations are made regarding to $|\psi_n^{(0)}\rangle$ and $E_n^{(0)}$.

7.1 Non-degenerate perturbation theory

In this section we shall assume that the discrete energy spectrum has no degeneracy. We will see the case of degeneracy in the next section.

Given that $|\psi_n^{(0)}\rangle$ are an eigenbasis in the Hilbert Space, we can expand the perturbed vector as follows

$$|\psi\rangle = \sum_m c_m |\psi_m^{(0)}\rangle. \tag{7.4}$$

If we substitute this expansion in (7.3) we find that

$$\sum_m c_m (E_m^{(0)} + \mathbb{V})|\psi_m^{(0)}\rangle = \sum_m c_m E |\psi_m^{(0)}\rangle \tag{7.5}$$

and multiplying both sides with $\langle\psi_n^{(0)}|$ we find that

$$(E - E_k^{(0)})c_k = \sum_m \langle\psi_k^{(0)}|\mathbb{V}|\psi_m^{(0)}\rangle c_m = \sum_m V_{km} c_m \tag{7.6}$$

where we have introduced the matrix element of the perturbation operator $V_{km} = \langle\psi_k^{(0)}|\mathbb{V}|\psi_m^{(0)}\rangle$.

At this point, to better understand what we mean with small perturbation, we introduce a parameter γ which quantifies the perturbation, that is

$$\mathbb{V} = \gamma \mathbb{H}_1, \tag{7.7}$$

and we consider γ a small parameter. Therefore, we can expand the coefficients c_m, the energy eigenvalues and the energy eigenstates in terms of the parameter γ, i.e.

$$E = E^{(0)} + \gamma E^{(1)} + \gamma^2 E^{(2)} + \mathcal{O}(\gamma^3) \tag{7.8}$$

$$c_m = c_m^{(0)} + \gamma c_m^{(1)} + \gamma^2 c_m^{(2)} + \mathcal{O}(\gamma^3) \tag{7.9}$$

$$|\psi\rangle = |\psi^{(0)}\rangle + \gamma|\psi^{(1)}\rangle + \gamma^2|\psi^{(2)}\rangle + \mathcal{O}(\gamma^3) \tag{7.10}$$

Let us study the first-order corrections to the nth eigenvalue and eigenfunction, which means we put $c_n^{(0)} = 1$, $c_m^{(0)} = 0$ for $m \neq n$ and higher corrections than one are negligible. Substituting in (7.6), we find out that

$$E_n^{(1)} = V_{nn} = \gamma \langle\psi_n^{(0)}|\mathbb{H}_1|\psi_n^{(0)}\rangle \tag{7.11}$$

which is the mean value of the perturbation in the state $|\psi_n^{(0)}\rangle$. In a similar way, we can solve the (7.6) for the coefficient $c_k^{(1)}$ in the case $k \neq n$, finding that

$$c_k^{(1)} = \frac{V_k n}{E_n^{(0)} - E_k^{(0)}}, \qquad k \neq n. \tag{7.12}$$

while $c_n^{(1)}$ can not be deduced from (7.6) but it must be chosen so that the function $|\psi_n\rangle$ is normalized. It can be found that $c_n^{(1)} = 0$.

These results can be easily generalized to the continuous spectrum case if the perturbation is applied to a state of the discrete spectrum. Then the first-order perturbed eigenstate is given by

$$|\psi_n^{(1)}\rangle = \sum_{m \neq n} \frac{V_{mn}}{E_n^{(0)} - E_m^{(0)}} |\psi_m^{(0)}\rangle + \int \frac{V_{\nu n}}{E_n^{(0)} - E_\nu} |\psi_\nu^{(0)}\rangle d\nu \tag{7.13}$$

and similarly for the other formulas.

7.2 Degenerate perturbation theory

After a brief review of the non-degenerate scenario, in this section we are going to deal with the degenerate case. For instance, we came across this problem in the 2-dimensional or 3-dimensional H.O systems.

Let us consider the degenerate eigenvalue $E_n^{(0)}$ with eigenspace

$$\mathcal{D} = \{|\psi_{n1}^{(0)}\rangle, |\psi_{n2}^{(0)}\rangle, \dots\}. \tag{7.14}$$

We assume that the energy level has finite degree of degeneracy d, which is also the dimension of $\mathcal{D}$. The choice of these functions is not unique, since we can choose any linear combination of them. Nevertheless, if we want to use the perturbation theory, the arbitrariness will be fixed by the requirement that the change in them under the action of the small perturbation is small.

Therefore, we start considering a general superposition of zeroth therms

$$|\psi\rangle = \sum_{k \in \mathcal{I}(\mathcal{D})} c_k |\psi_k^{(0)}\rangle, \tag{7.15}$$

where, for the sake of brevity, we indicate with $\mathcal{I}(\mathcal{D}) = \{n1, n2, \dots\}$, the set of index of the degenerate states. Taking equation (7.6) with both k and m in $\mathcal{I}(\mathcal{D})$ and substituting $E = E_m^{(0)} + E^{(1)}$ we obtain

$$E^{(1)} c_m = \sum_{k \in \mathcal{I}(\mathcal{D})} V_{mk} c_k. \tag{7.16}$$

Given that we are at first-order, the coefficients can be taken at zero-order, that lead to the homogeneous linear system

$$\sum_{k \in \mathcal{I}(\mathcal{D})} \left(V_{mk} - E^{(1)} \delta_{mk} \right) c_k^{(0)} = 0, \tag{7.17}$$

which is an eigenvalue problem. The results give us the eigenvalues $E^{(1)}$, that is, the first-order corrections to the energy of the degenerate eigenstates. Instead, the eigenvectors represent the zero-order state for which the full Hamiltonian $\mathbb{H}$ is diagonalized. We must stress out that the Hamiltonian can be diagonalized only in the subspace $\mathcal{D}$ and only if the eigensystem has a solution. These new eigenvectors represent new eigenstates which reduce the degeneracy of the subspace $\mathcal{D}$, due to the corrections $E^{(1)}$. However, it is possible that the removal of the degeneracy is total or partial, depending on the degeneracy of $E^{(1)}$ (see 2-dim Harmonic Oscillator).

Having studied the zero-order problem, now we can move to study the eigenfunctions' corrections at first-order. We assume that we have already performed the procedure described in the previous paragraphs and that the $|\psi_{ni}^{(0)}\rangle$ are already the zero-order eigenvectors of $\mathbb{H}$ on $\mathcal{D}$. As usual, we start our investigation from the (7.6), that is

$$(E - E_k^{(0)}) c_k = \sum_m V_{km} c_m. \tag{7.18}$$

where the coefficients at first-order are $c_k = \delta_{k,k} + c_k^{(1)}$ in the case of $k \in \mathcal{I}(\mathcal{D})$, while $c_k = c_k^{(1)}$ for $k \notin \mathcal{I}(\mathcal{D})$. Similarly, the Energy is the energy of the degenerate level, thus at first-order it is equal to $E = E_m^{(0)} + V_{mm}$ with $m \in \mathcal{I}(\mathcal{D})$.

Firstly, we start considering $k \notin \mathcal{I}(\mathcal{D})$ and we take the first-order of (7.18) . Therefore, the state must be at zero order and we have that

$$|\psi\rangle = \sum_m c_m |\psi_m^{(0)}\rangle = \sum_{m \in \mathcal{I}(\mathcal{D})} c_m^{(0)} \delta_{m,m} |\psi_m^{(0)}\rangle = |\psi_m^{(0)}\rangle \tag{7.19}$$

and as a result it follows that (7.18) became at first-order

$$(E_m^{(0)} - E_k^{(0)})c_k^{(1)} = \sum_{m\in\mathcal{I}(\mathcal{D})} c_m^{(0)}\delta_{m,m}V_{km} = V_{km} \tag{7.20}$$

and consequently we have

$$c_k^{(1)} = \frac{V_{km}}{E_m^{(0)} - E_k^{(0)}}. \tag{7.21}$$

Secondly, we consdier $k \in \mathcal{I}(\mathcal{D})$ and we need the second order of (7.18). Thus, we take the first-order of the state

$$\begin{aligned} |\psi\rangle = \sum_m c_m|\psi_m^{(0)}\rangle = \sum_{m\in\mathcal{I}(\mathcal{D})} (c_m^{(0)}\delta_{m,m} + c_m^{(1)})|\psi_m^{(0)}\rangle + \sum_{l\notin\mathcal{I}(\mathcal{D})} c_l^{(1)}|\psi_l^{(0)}\rangle = \\ = |\psi_m^{(0)}\rangle + \sum_{n\in\mathcal{I}(\mathcal{D})} c_n^{(1)}|\psi_n^{(0)}\rangle + \sum_{l\notin\mathcal{I}(\mathcal{D})} c_l^{(1)}|\psi_l^{(0)}\rangle \end{aligned} \tag{7.22}$$

and we obtain that

$$(E_m^{(0)} + E_m^{(1)} - E_m^{(0)})(\delta_{k,k} + c_k^{(1)}) = V_{km} + \sum_{n\in\mathcal{I}(\mathcal{D})} c_n^{(1)}V_{kn} + \sum_{l\notin\mathcal{I}(\mathcal{D})} c_l^{(1)}V_{kl} \tag{7.23}$$

Now, since in the subspace $\mathcal{D}$ the matrix V_{ij} is diagonal, we must have $V_{km} = V_{mm}$ and $V_{kn} = V_{kk}\delta_{k,n}$, and eventually we get

$$V_{mm} + V_{mm}c_k^{(1)} = V_{mm} + c_k^{(1)}V_{kk} + \sum_{l\notin\mathcal{I}(\mathcal{D})} \frac{V_{lm}V_{kl}}{E_m^{(0)} - E_l^{(0)}} \tag{7.24}$$

It follows that the first-order correction is

$$c_k^{(1)} = \frac{1}{V_{mm} - V_{kk}} \sum_{l\notin\mathcal{I}(\mathcal{D})} \frac{V_{lm}V_{kl}}{E_m^{(0)} - E_l^{(0)}}. \tag{7.25}$$

To conclude, the first-order perturbed ket for degenerate energy state is

$$|\psi\rangle = |\psi_m^{(0)}\rangle + \sum_{n\in\mathcal{I}(\mathcal{D})} \sum_{l\notin\mathcal{I}(\mathcal{D})} \frac{V_{nl}V_{lm}}{(V_{mm} - V_{nn})(E_m^{(0)} - E_l^{(0)})}|\psi_n^{(0)}\rangle + \sum_{l\notin\mathcal{I}(\mathcal{D})} \frac{V_{lm}}{E_m^{(0)} - E_l^{(0)}}|\psi_l^{(0)}\rangle. \tag{7.26}$$

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