Graded modules over the simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$

by

© *Abdallah A. Shihadeh*

A thesis submitted to the

School of Graduate Studies

in partial fulfilment of the

requirements for the degree of

Doctor of Philosophy

Department of *Mathematics and Statistics*

Memorial University of Newfoundland

2019

St. John's Newfoundland

Abstract

This thesis provides a contribution to the area of group gradings on the simple modules over simple Lie algebras. A complete classification of gradings on finite-dimensional simple modules over arbitrary finite-dimensional simple Lie algebras over algebraically closed fields of characteristic zero, in terms of graded Brauer groups, has been recently given in the papers of A. Elduque and M. Kochetov. Here we concentrate on infinite-dimensional modules. A complete classification by R. Block of all simple modules over a simple Lie algebra is known only in the case of $\mathfrak{sl}_2(\mathbb{C})$. Thus, we restrict ourselves to the gradings on simple $\mathfrak{sl}_2(\mathbb{C})$-modules. We first give a full description for the $\mathbb{Z}$- and $\mathbb{Z}_2^2$-gradings of all weight modules over $\mathfrak{sl}_2(\mathbb{C})$. Then we show that $\mathbb{Z}$-gradings do not exist on any torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules of finite rank. After this, we treat $\mathbb{Z}_2^2$-gradings on torsion-free modules of various ranks. A construction for these modules was given by V. Bavula, and J. Nilson gave a classification of the torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules of rank 1. After giving some, mostly negative, results about the gradings on these latter modules, we construct the first family of simple $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-modules (of rank 2). We also construct a family of graded-simple torsion-free modules of rank 2. For each of the modules in these families, we give a complete description of their tensor products with simple graded finite-dimensional modules.

Co-Authorship

Some of the results in Chapter 4, Chapter 5, Chapter 6, and Chapter 7 were obtained in collaboration with Dr. Yuri Bahturin.

Acknowledgments

First and foremost, I owe my deepest gratitude to my supervisor, Dr. Yuri Bahturin. I have been extremely fortunate to have a supervisor who cared very much about my work and passed his knowledge on to me with great dedication. I am also very grateful to my co-supervisor Dr. Mikhail Kotchetov, who offered suggestions and guidance to improve the quality of this work. I also acknowledge their partial financial support. Also, I would like to express my most profound appreciation to the Department of Mathematics and Statistics, and the people of the algebra group for their valuable questions and discussions. Further, my thanks go to my supervisory committee, Dr. Tom Baird and Dr. Eduardo Martinez-Pedroza. I also gratefully acknowledge financial support from The Hashemite University.

Above all, I can never express my gratitude enough to my great father, Abdul Karim, and my beloved mother, Enshirah, who have a special place in my heart. They provided me over the years with the highest support any one has ever given to me. Also, I express my appreciation to my wife Rasha for her support, encouragement, compassion, patience, and great love. She has been unfailingly supportive and encouraging throughout this long Ph.D. journey. She accompanied me moment by moment, and gave me power when I felt weakness. My deepest appreciation and love to my wonderful kids, Salam and Mohammad, they are always behind my success. I would not have achieved this work without their love and patience. I would like to thank my brothers, my sisters, my father-in-law, and my mother-in-law for their kind assistance and support. Finally, to my dear friends in St. John's, you are my second family.

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Introduction

Gradings are an important topic in mathematics. A lot of modern literature is devoted to various aspects of this topic. To mention just a few, we list [ABFP08, BK10, BS19, BM04, BSZ01, DEK17, DV16, EK13, EK15, EK15b, EK17, Eld16, Koc09, MZ18, PP70, Smi97]. In [EK15], [EK15b] and [DEK17] the authors gave the classification of graded simple finite-dimensional modules over simple finite-dimensional Lie algebras most types over algebraically closed fields of characteristic zero.

The algebra $\mathfrak{sl}_2(\mathbb{C})$ is the smallest simple Lie algebra, with dim $\mathfrak{sl}_2(\mathbb{C}) = 3$. It is the only semisimple Lie algebra which has a full classification of simple modules, both finite- or infinite-dimensional. The first paper on the classification of $\mathfrak{sl}_2(\mathbb{C})$-modules is due to R. Block in [Blo81]. This paper was followed by a number of papers devoted to various aspects of this classification. V. Bavula reduced the classification of $\mathfrak{sl}_2(\mathbb{C})$-modules to the modules over certain associative algebras, called generalized Weyl algebras. The monograph of V. Mazorchuk [Maz09] is completely devoted to this topic. In [MP16], the authors introduced examples of torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules for any finite rank. The torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules of rank 1 have been classified in [Nil15]. Actually, the author of this latter paper has classified all torsion-free modules of rank 1 over the simple Lie algebras $\mathfrak{sl}_{n+1}(\mathbb{C})$, for any $n \in \mathbb{N}$.

This thesis is concerned with the gradings on the simple $\mathfrak{sl}_2(\mathbb{C})$-modules, both finite- and infinite-dimensional. Chapter 1 recalls the basic concepts of Lie algebras and their gradings. We focus on the universal enveloping algebras and the root space decomposition, which produces the first main grading, namely the Cartan grading. Section 1.2 provides general context about gradings.

In Chapter 2 and Chapter 3 we introduce the weight and torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules. Note that, any weight $\mathfrak{sl}_2(\mathbb{C})$-module has a grading compatible with the Cartan grading on $\mathfrak{sl}_2(\mathbb{C})$, the graded components are the weight subspaces of the module.

Our results start in Chapter 4, where we discuss the $\mathbb{Z}_2^2$-gradings on the weight $\mathfrak{sl}_2(\mathbb{C})$-modules. Furthermore, we will give an explicit $\mathbb{Z}$-grading for each simple weight $\mathfrak{sl}_2(\mathbb{C})$-module. In Chapter 5, we treat torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules. First, we establish a simple, yet significant result about the $\mathbb{Z}$-gradings on simple torsion-free $\mathfrak{sl}_2(\mathbb{C})$-module of finite rank:

Theorem 5.1.1. *No simple $L = \mathfrak{sl}_2(\mathbb{C})$-module which is torsion-free of finite rank with respect to a fixed Cartan subalgebra can be given a $\mathbb{Z}$-grading compatible with the Cartan $\mathbb{Z}$-grading of L, defined by this subalgebra.*

Then we deal with the modules constructed by Bavula, V. in [Bav92]. The last section of this chapter gives the following result:

Theorem 5.2.2. *Torsion-free $L = \mathfrak{sl}_2(\mathbb{C})$-modules of rank 1 cannot be given a $\mathbb{Z}_2^2$-grading, compatible with the respective Pauli $\mathbb{Z}_2^2$-grading of L.*

In Chapter 6, we construct a family of $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-modules. The family consists of the modules M_λ^C where,

$$M_\lambda^C = U(I_\lambda)/U(I_\lambda)C, \qquad \lambda \in \mathbb{C}.$$

The main results of this chapter are:

Theorem 6.1.1. *Let $\lambda \in \mathbb{C}\backslash 2\mathbb{Z}$. Then M_λ^C is a simple $\mathfrak{sl}_2(\mathbb{C})$-module.*

Theorem 6.2.3. *Let $\lambda \in 2\mathbb{Z}$. Then M_λ^C has a unique maximal (graded) submodule N_λ^C such that $N_\lambda^C = P \oplus Q$, where P and Q are simple $\mathfrak{sl}_2(\mathbb{C})$-module of rank 1.*

In Chapter 7, we discuss the tensor products $L_{(\lambda,2n)}$ of the modules M_λ^C with simple finite-dimensional $\mathbb{Z}_2^2$-graded modules $V(2n)$. The main result in this chapter is this:

Theorem 7.2.2. *Let* $\mu \in \mathbb{C}\setminus\{-1,0\}$. *Then*

$$L_{(\lambda,2n)} \cong \bigoplus_{i=0}^{2n} M^C_{\lambda+2n-2i}.$$

Chapter 1

Lie algebras and their gradings

1.1 Lie algebras

In this section we briefly introduce Lie algebras in general, in particular, classical Lie algebras (see e.g. [EW11, Hum78, Jac79, Ser09]). The first section deals with Lie algebras, simple and semisimple Lie algebras, general Lie algebras and classical Lie algebras of types A, B, C, and D. Representations and modules for Lie algebras are presented in Subsection 1.1.3. In Subsection 1.1.4 we recall universal enveloping algebras.

The last section in this section treats root systems of semisimple Lie algebras, the Killing form, Dynkin diagrams, Cartan integers and Cartan matrices for classical simple Lie algebras.

1.1.1 Definitions and basic concepts

Definition 1.1.1. *A* Lie algebra L *over a field* $\mathbb{F}$*is a vector space over this field equipped with a bilinear map* $[\cdot,\cdot] : L \times L \longrightarrow L$ *such that:*

(i) $[x, x] = 0$ *for all* $x \in L$*,*

(ii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ *for all* $x, y, z \in L$*.*

The bilinear operation $[\cdot,\cdot]$ in Definition 1.1.1 is called *Lie bracket (or commutator)*, and the second condition above is called the *Jacobi identity*.

Note that condition (ii) implies that $[x, y] = -[y, x]$ for all $x, y \in L$. This property called the anti-symmetry condition.

Definition 1.1.2. *A subspace* M *of a Lie algebra* L *is called a* Lie subalgebra *of* L *if , given* $x, y \in M$*, one has also has* $[x, y] \in M$*.*

Definition 1.1.3. *Let* L *be a Lie algebra, then the subspace* I *of* L *is an ideal of* L *if for all* $x \in L$*,* $a \in I$*, we have* $[x, a] \in I$*.*

Definition 1.1.4. *A Lie algebra* L *is called* simple *if it is non-abelian and has no proper (that is, different from* L*) nonzero ideals.*

Proposition 1.1.5. *If* I *is an ideal of a Lie algebra* L*, then the quotient space* L/I *equipped with the bracket operation*

$$[x + I, y + I] = [x, y] + I \text{ for all } x, y \in L \tag{1.1}$$

is a Lie algebra, called the quotient Lie algebra *modulo the ideal* I*.*

Definition 1.1.6. *Let L_1 and L_2 be two Lie algebras. A linear map $\varphi : L_1 \longrightarrow L_2$ is said to be* a Lie homomorphism *if $\varphi([x, y]) = [\varphi(x), \varphi(y)]$, for all $x, y \in L_1$.*

In addition, if φ is a bijection, then we call it a Lie isomorphism. *In this case we say that L_1 isomorphic to L_2. If L_1 is isomorphic to L_2, we write $L_1 \cong L_2$.*

Definition 1.1.7. *Let L be a Lie algebra, then the* derived series *of L is the series of ideals*

$$L \supseteq L^{(1)} \supseteq L^{(2)} \supseteq \dots$$

where $L^{(1)} = [L, L]$, and $L^{(k)} = \left[L^{(k-1)}, L^{(k-1)}\right]$ for $k > 1$.

One often writes $L^{(1)} = L'$.

Definition 1.1.8. *A Lie algebra L is called solvable if $L^{(n)} = 0$ for some $n \in \mathbb{N}$.*

Any subalgebra and quotient-algebra of a solvable Lie algebra is itself solvable. If L has a solvable ideal I such that L/I is solvable then L is solvable. By Lie's Theorem, any finite-dimensional solvable Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic zero is isomorphic to a subalgebra of the Lie algebra of upper (or lower) triangular matrices of some order n over $\mathbb{F}$, which is itself solvable.

Definition 1.1.9. *The maximal solvable ideal of L is called the* radical *of L. We denote the radical of L by $\mathrm{Rad}(L)$.*

Definition 1.1.10. *A Lie algebra L is called semisimple if $\mathrm{Rad}(L) = 0$.*

Hence, any simple Lie algebra is semisimple.

1.1.2 Classical Lie algebras

Let A be an associative algebra, then the *commutator*

$$[x, y] := xy - yx \tag{1.2}$$

satisfies the identities in Definition 1.1.1. We denote by $A^{(-)}$ the Lie algebra with the same underlying vector space A, equipped with the commutator operation.

Definition 1.1.11. *Let V be a vector space, and* $\mathrm{End}(V)$ *is the associative algebra of all endomorphisms of V. The Lie algebra* $\mathrm{End}(V)^{(-)}$ *is called the* general linear algebra*. We denote the general linear algebra of V by $\mathfrak{gl}(V)$. Any subalgebra of $\mathfrak{gl}(V)$ is called* linear*.*

Let V be a finite dimensional vector space. Then we can identify $\mathrm{End}(V)$ with the set of all $n \times n$ matrices. Hence we can identify $\mathfrak{gl}(V)$ with the Lie algebra $M_n(\mathbb{F})^{(-)}$. We denote this algebra by $\mathfrak{gl}_n(\mathbb{F})$.

Definition 1.1.12. *The special linear algebra $\mathfrak{sl}_n(\mathbb{F})$ (or $\mathfrak{sl}(V)$) is the subalgebra of $\mathfrak{gl}_n(\mathbb{F})$ consists of all $n \times n$ matrices (endomorphisms) of trace zero.*

Proposition 1.1.13. *The special Lie algebra $\mathfrak{sl}_n(\mathbb{F})$ is an ideal of $\mathfrak{gl}_n(\mathbb{F})$. Hence, $\mathfrak{gl}_n(\mathbb{F})$ is not simple.*

The algebra $\mathfrak{sl}_{r+1}(\mathbb{F})$, $r \geq 1$, is called the Lie algebra of type A_r.

Definition 1.1.14. *The symplectic Lie algebra $\mathfrak{sp}_{2r}(\mathbb{F})$ is a linear Lie algebra identified with the set of all $2r \times 2r$ matrices of the form*

$$\begin{bmatrix} A & B \\ C & -A^t \end{bmatrix}, \tag{1.3}$$

where B and C are symmetric matrices.

The algebra $\mathfrak{sp}_{2r}(\mathbb{F})$, $r \geq 3$, is called the Lie algebra of type C_r.

Definition 1.1.15. *The orthogonal Lie algebra $\mathfrak{o}_{2r+1}(\mathbb{F})$ is the linear Lie algebra identified with the set of all $(2r+1) \times (2r+1)$ matrices of the form*

$$\begin{bmatrix} 0 & a & b \\ -b^t & A & B \\ -a^t & C & -A^t \end{bmatrix}, \tag{1.4}$$

where B and C are skew symmetric matrices of order r.

The algebra $\mathfrak{o}_{2r+1}(\mathbb{F})$, $r \geq 2$, is called the Lie algebra of type B_r.

Definition 1.1.16. *The orthogonal Lie algebra $\mathfrak{o}_{2r}(\mathbb{F})$ is the linear Lie algebra identified with the set of all $2r \times 2r$ matrices of the form*

$$\begin{bmatrix} A & B \\ C & -A^t \end{bmatrix}, \tag{1.5}$$

where B and C are skew symmetric matrices.

The algebra $\mathfrak{o}_{2r}(\mathbb{F})$, $r \geq 4$, is called the Lie algebra of type D_r.

The Lie algebras of types $A_r, B_r, C_r,$ and D_r are called the *classical Lie algebras.*

Note that all of the classical Lie algebras are simple, see e.g. [Hum78, §1.2].

1.1.3 Representations of Lie algebras

Definition 1.1.17. *Let V be a vector space over a field $\mathbb{F}$. A Lie homomorphism $\rho : L \longrightarrow \mathfrak{gl}(V)$ is called a representation of the Lie algebra L by linear transformations of V.*

Example 1.1.18. *Let $V = L$. Consider the representation*

$$\begin{aligned} \mathrm{ad}: \ L &\longrightarrow \mathfrak{gl}(L) \\ x &\mapsto \mathrm{ad}x \end{aligned}$$

defined by $\mathrm{ad}x(y) = [x, y]$. *This representation is called the adjoint representation of L.*

Definition 1.1.19. *A module over the Lie algebra L is a vector space V equipped with an operation*

$$\begin{aligned} L{\times}V &\longrightarrow V \\ (x,v) &\mapsto x.v \end{aligned}$$

such that:

(i) $(ax_1 + bx_2).v = a(x_1.v) + b(x_2.v)$,

(ii) $x.(av_1 + bv_2) = a(x.v_1) + b(x.v_2)$,

(iii) $[x_1, x_2].v = x_1.(x_2.v) - x_2.(x_1.v)$.

Here $a, b \in \mathbb{F}$, $x, x_1, x_2 \in L$, $v, v_1, v_2 \in V$.

Given a representation $\rho : L \to \mathfrak{gl}(V)$, setting $x.v = \rho(x)(v)$ for $x \in L$ and $v \in V$, makes V an L-module. Conversely, if V is an L-module, sending $x \to \rho_x$, where $\rho_x(v) = x.v$ defines a homomorphism from L to $\mathfrak{gl}(V)$, which is a representation of L by linear transformations of V.

Definition 1.1.20. *Let L be a Lie algebra. A representation $\varphi : L \longrightarrow \mathfrak{gl}(V)$ is said to be* faithful *if φ is $1-1$.*

Definition 1.1.21. *A subspace W of an L-module V is called a* submodule *if for all $x \in L$ and all $w \in W$, we have $x.w \in W$.*

Definition 1.1.22. *A module V over a Lie algebra is said to be* simple *if it has no proper nonzero submodules.*

Definition 1.1.23. *A module V over a Lie algebra is called* semisimple *if it is a direct sum of simple submodules.*

Definition 1.1.24. *Let V_1, V_2 be two L-modules, where L is a Lie algebra. The linear map $\varphi : V_1 \longrightarrow V_2$ is said to be a homomorphism of L-modules if for all $x \in L$ and all $v \in V_1$, we have $\varphi(x.v)=x.\varphi(v)$.*

Although some of the results and definitions given here are valid over various fields, in this thesis, we will be concerned with Lie algebras over the field of complex numbers. Hence, from now on, we work over the complex numbers, unless stated otherwise.

Definition 1.1.25. *The* center *of a Lie algebra L is defined by*

$$\mathrm{C}(L) = \{z \in L \mid [z, a] = 0 \text{ for all } a \in L\}. \tag{1.6}$$

Theorem 1.1.26 (Schur's Lemma)**.** *Let V be a simple finite-dimensional L-module. A map $\varphi : V \longrightarrow V$ is a Lie homomorphism if and only if φ is a scalar map.*

Corollary 1.1.27. *Let V be a simple L-module, $z \in \mathrm{C}(L)$. Then, there is $\alpha \in \mathbb{C}$ such that $z.v = \alpha v$ for all $v \in V$.*

Theorem 1.1.28 (Weyl's Theorem)**.** *Any finite dimensional L-module for a semisimple Lie algebra L is semisimple.*

Note that this theorem is not true for infinite-dimensional modules, as will be seen, for example, from some of our results in Chapter 6. In the case of the fields of positive characteristic, this theorem is not true even for finite-dimensional modules.

1.1.4 The universal enveloping algebra

Definition 1.1.29. *A universal enveloping algebra of a Lie algebra L is a unital associative algebra $\mathcal{U}$ together with a Lie homomorphism*

$$\varepsilon : L \longrightarrow \mathcal{U}^{(-)}, \tag{1.7}$$

satisfying the following condition:

For any unital associative algebra A, and any Lie homomorphism

$$\delta : L \longrightarrow A^{(-)},$$

there exists a unique homomorphism of associative algebras

$$\bar{\delta} : \mathcal{U} \longrightarrow A$$

such that $\bar{\delta} \circ \varepsilon = \delta$.

It is well known, that the universal enveloping algebra of L exists and is unique (up to an isomorphism that is identical on L). We denote the universal enveloping algebra of L by $U(L)$.

In order to give an explicit construction of $U(L)$(see e.g. [Hum78, §17.2]), we start with the tensor algebra $\mathcal{T}(L)$.

First we consider the n^{th} tensor power of the vector space L by:

$$\mathcal{T}^n(L) = L^{\otimes n} = \underbrace{L \otimes L \otimes \ldots \otimes L}_{\text{n times}}. \tag{1.8}$$

The tensor algebra $\mathcal{T}(L)$ defined as:

$$\mathcal{T}(L) = \bigoplus_{n=0}^{\infty} \mathcal{T}^n(L) = \mathbb{C} \oplus L \oplus (L \otimes L) \oplus \ldots. \tag{1.9}$$

Now consider the ideal $\mathcal{J}$ of $\mathcal{T}(L)$ generated by the elements

$$[x, y] - x \otimes y + y \otimes x, \text{ for all } x, y \in L.$$

Finally, to obtain the universal enveloping algebra of L one sets

$$U(L) := \mathcal{T}(L)/\mathcal{J}. \tag{1.10}$$

The following theorem is called the Poincaré-Birkhoff-Witt theorem, or PBW-Theorem.

Theorem 1.1.30 (PBW Theorem)**.** *Let $\{x_1, x_2, x_3, \ldots\}$ be an ordered basis of L. Then the element 1 together with all the monomials of the form*

$$\varepsilon(x_{i(1)})\varepsilon(x_{i(2)}) \cdots \varepsilon(x_{i(k)}), k \in \mathbb{N}, \textit{ and } i(1) \leq i(2) \leq \cdots \leq i(k)$$

form a basis of $U(L)$.

It follows from the PBW Theorem that the map ε is injective. Hence we may identify the algebra L with it is image in $U(L)$, using monomorphism ε, and write $\varepsilon(x) = x$. In this case the basis of $U(L)$ takes the form of the set of all monomials

$$x_{i(1)}x_{i(2)} \cdots x_{i(k)}, k \in \mathbb{N}, \text{ and } i(1) \leq i(2) \leq \cdots \leq i(k).$$

The next fact describes the relation between the L-modules and the $U(L)$-modules, (see e.g. [Dix77, EW11]).

Proposition 1.1.31. *Let L be a Lie algebra. Any L-module is a $U(L)$-module, and vice versa.*

The following subalgebra of $U(L)$ will play an important role in our future text.

Definition 1.1.32. *The* center *of the universal enveloping algebra $\mathcal{Z}(L)$ is the set of all elements commuting with all $x \in U(L)$.*

If V is a simple L-module then by Schur's Lemma, for any $z \in \mathcal{Z}(L)$ there exists a complex number $\lambda(z) \in \mathbb{C}$ such that $z.v = \lambda(z)v$, for any $v \in V$.

1.1.5 Root space decomposition

Let L be a Lie algebra. An element $s \in L$ is called *semisimple* if $\operatorname{ad} s : L \to L$ is a semisimple (diagonalizable) linear transformation. An element $n \in L$ is called *nilpotent* if $\operatorname{ad} n : L \to L$ is a nilpotent linear transformation. Every element $x \in L$ can be uniquely written as $x = s+n$, where s is semisimple, n nilpotent and $[s, n] = 0$. This is called the *Jordan decomposition* of x (see [Hum78, §6.4]).

1.1.5.1 Cartan subalgebras

Now let L be a semisimple Lie algebra.

Definition 1.1.33. *A* toral *subalgebra of L is a subalgebra consisting of semisimple elements.*

Proposition 1.1.34. *Any toral subalgebra of L is abelian.*

Definition 1.1.35. *A maximal toral subalgebra of L is called a* Cartan subalgebra*.*

Definition 1.1.36. *Let V be an L-module, and fix a Cartan subalgebra H of L. Then, for $\lambda \in H^*$, consider the subspace*

$$V_\lambda := \{v \in V \mid h.v = \lambda(h)v,\ \forall h \in H\}.$$

If $V_\lambda \neq 0$, then it is called the weight space *of V of* weight *λ. The set of all weights is the* support *of V,*

$$\mathrm{Supp}(V) = \{\lambda \in H^* \mid V_\lambda \neq 0\}.$$

V is called a weight module *if*

$$V = \bigoplus_{\lambda \in H^*} V_\lambda.$$

The elements of V_λ are called the weight vectors *of weight λ.*

Simple weight modules over semisimple Lie algebras satisfying $\dim V_\lambda < \infty$ for all λ are completely classified (see [Mat00]).

It is well known that any finite-dimensional L-module is a weight module.

In general, we can mention that any submodule, quotient, direct sum, and finite tensor product of weight modules is a weight module again (see e.g. [Hum78, Hum08, Fer90]).

1.1.5.2 Root systems

In this section, we will consider the case of the adjoint module, that is $V = L$, equipped with the adjoint action defined in Example (1.1.18). The nontrivial weight subspaces

$$L_\alpha = \{x \in L \mid [h, x] = \alpha(h)x \text{ for all } h \in H\}, \tag{1.11}$$

for $\alpha \in H^*$, are called *root spaces*.

The weights $\alpha \in \mathrm{Supp}(L)$ are called *roots*. The set of all nonzero roots Φ is called the *root system* of L with respect to the Cartan subalgebra H. It is well known (see e.g. [Hum78, §8.2]) that $L_0 = H$ and $\dim L_\alpha = 1$ for $\alpha \in \Phi$. Moreover,

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha. \tag{1.12}$$

Definition 1.1.37. *A subset $\Delta \subseteq \Phi$ is a* base *of Φ if*

1. *Δ is linearly independent;*
2. *Any $\alpha \in \Phi$ can be written as*

$$\alpha = \sum_{\beta \in \Delta} z_\beta \beta, \text{ where } \beta \in \Delta \text{ and } z_\beta \in \mathbb{Z}$$

such that the values of the integers z_β are either all non-negative ($\alpha \succ 0$) or all non-positive ($\alpha \prec 0$).

We write $\alpha \prec \beta$ if $\alpha - \beta \prec 0$.

The elements of Δ are called *simple roots*. It is known that Δ is a basis of H^*. The number of elements in Δ is called the *rank* of Φ.

Definition 1.1.38. *Fix a base $\Delta \subseteq \Phi$. Let $N = N(\Delta) := \bigoplus_{\alpha \succ 0} L_\alpha$, and $N^- := \bigoplus_{\alpha \prec 0} L_\alpha$, then $L = N^- \oplus H \oplus N$. This is called the triangular decomposition of L determined by H. The subalgebra $B = B(\Delta) := H \oplus N$ is called the* standard Borel subalgebra *of L.*

Definition 1.1.39. *A vector v of weight λ ($\lambda \in H^*$) is a* maximal vector *if* $L_\alpha.v = 0$ *for all* $\alpha \succ 0$*.* V *is said to be a* highest weight module *of weight* λ *if there is a maximal vector* v *such that* $V = U(L).v_0$*.*

Definition 1.1.40. *Let* $B = B(\Delta) = H \oplus N$ *be the standard Borel subalgebra of a semisimple Lie algebra* L*. For each* $\lambda \in H^*$*, let* D_λ *be a 1-dimensional* B*-module, with trivial* N*-action and* H *acting through* λ*, and set* $Z(\lambda) = U(L) \otimes_{U(B)} D_\lambda$*. Then* $Z(\lambda)$ *is a* $U(L)$*-module called* the Verma module of weight λ.

1.1.5.3 Killing form and classical Lie algebras

Definition 1.1.41. *Let* L *be a Lie algebra. Define the symmetric bilinear form* k *on* L *by:*

$$k(x, y) := tr(\text{ad } x \text{ ad } y). \tag{1.13}$$

k *is called the* Killing form *of the Lie algebra* L*.*

The next result is the *Cartan criterion* of semisimplicity for arbitrary Lie algebras.

Proposition 1.1.42. *[Hum78, Theorem 5.1] The Lie algebra* L *is semisimple if and only if the Killing form of* L *is nondegenerate.*

We will return now to our assumption in this section, that is L is a semisimple Lie algebra.

Proposition 1.1.43. *[Hum78, Corollary 8.2] The restriction of the Killing form to the Cartan subalgebra* H *is nondegenerate.*

It follows from Proposition 1.1.43, that we can identify the Cartan subalgebra H with H^*.

If $\lambda \in H^*$, then there is a unique element $t_\lambda \in H$ such that $k(t_\lambda, h) = \lambda(h)$ for all $h \in H$. Hence we can define a symmetric bilinear form in H^* by:

$$(\alpha, \beta) = k(t_\alpha, t_\beta). \tag{1.14}$$

For any $\alpha \in \Phi$, define the element $h_\alpha \in H$ by:

$$h_\alpha = \frac{2t_\alpha}{k(t_\alpha, t_\alpha)}. \tag{1.15}$$

Fix a base $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$ of the root system Φ. Since Δ is a basis of H^*, the elements $\{h_{\alpha_1}, h_{\alpha_2}, \dots, h_{\alpha_r}\}$ form a basis of H. Note that for any $\alpha \in \Phi$ and any $0 \neq x_\alpha \in L_\alpha$, there exists $y_\alpha \in L_{-\alpha}$ such that $[x_\alpha, y_\alpha] = h_\alpha$ (see e.g. [Hum78, Proposition 8.3]).

The *rank* of the semisimple Lie algebra L is the dimension of the Cartan subalgebra H. Note that, when L is semisimple, the rank of L coincides with the rank of the root system Φ (see e.g. [Hum78, §16.4]). For $\alpha, \beta \in \Phi$, an important number is

$$\langle \alpha, \beta \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}.$$

Note that the number $\langle \alpha, \beta \rangle$ defined above is always integer (see e.g. [Hum78, §8]).

Definition 1.1.44. *Fix a base $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$ of the root system Φ. The matrix $(\langle \alpha_i, \alpha_j \rangle)$ is called the* Cartan matrix *of Φ, and it's entries are called the* Cartan integers*.*

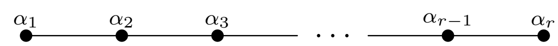
Lemma 1.1.45. *[Hum78, §9.4] For any $\alpha, \beta \in \Phi$, the value $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 0, 1, 2,$ or 3.*

Definition 1.1.46. *Let Φ be a root system of rank r. The Coxeter graph of Φ is the graph having r vertices, the i^{th} vertex joined the j^{th} vertex by $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ edges.*

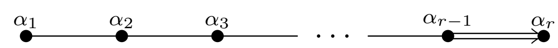
In the case when we have more than one edge, we can add an arrow pointing towards the longer root (based to the Definition 1.1.37). This graph called the *Dynkin diagram* of the root system Φ.

The following graphs are the Dynkin diagrams of the classical Lie algebras of rank r.

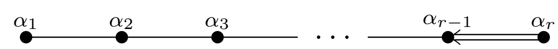
$A_r \quad (r \geq 1)$



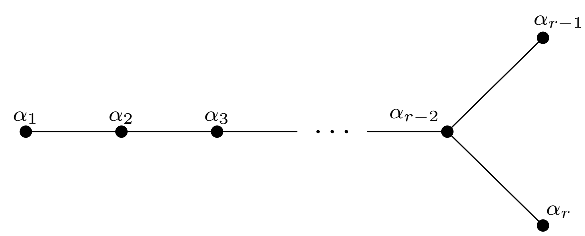
$B_r \quad (r \geq 2)$



$C_r \quad (r \geq 3)$



$D_r \quad (r \geq 4)$



1.1.5.4 Root system of $\mathfrak{sl}_n(\mathbb{C})$

Through this section, L will stand for the Lie algebra $L = \mathfrak{sl}_{n+1}(\mathbb{C})$, the Lie algebra of type A_n. Consider the subalgebra H of L consisting of all diagonal matrices in L with trace zero. Then, H is a Cartan subalgebra of L. Define the linear function $\varepsilon_i \in H^*$ where $\varepsilon_i(h)$ is the i^{th} entry of h. Then,

$$L = H \oplus \bigoplus_{i \neq j} L_{\varepsilon_i - \varepsilon_j}, \tag{1.16}$$

where $1 \leq i, j \leq n+1$. Hence,

$$\Phi = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}. \tag{1.17}$$

The root system Φ has a base

$$\Delta = \{\alpha_i \mid 1 \leq i \leq n\}, \tag{1.18}$$

where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$.

For $\alpha = \varepsilon_i - \varepsilon_j$, set

$$x_\alpha = E_{ij} \qquad i \neq j, \tag{1.19}$$

where Eij the standard matrix units. The set of all elements of the form

$$h_i = E_{ii} - E_{i+1,i+1} \tag{1.20}$$

forms a basis of the Cartan subalgebra H.

Let $y_{\alpha_i} = x_{-\alpha_i}$, and denote $x_i = x_{\alpha_i}$ and $y_i = y_{\alpha_i}$. Then

$$\begin{aligned} [x_i, y_i] &= h_i, \\ [h_i, x_i] &= 2x_i, \\ [h_i, y_i] &= -2y_i. \end{aligned} \tag{1.21}$$

Example 1.1.47. *Consider the simple Lie algebra $L = \mathfrak{sl}_2(\mathbb{C})$. The standard basis of $\mathfrak{sl}_2(\mathbb{C})$ consists of the elements*

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{1.22}$$

where

$$[x, y] = h, \ [h, x] = 2x, \ [h, y] = -2y.$$

Through this thesis we will fix the Cartan subalgebra of $\mathfrak{sl}_2(\mathbb{C})$ *with* $H = \langle h \rangle$.

Since $\dim(H) = 1$, *we can think of* H^* *as* $\mathbb{C}$. *Hence*

$$L = L_2 \oplus H \oplus L_{-2}, \tag{1.23}$$

where $L_2 = \langle x \rangle$ *and* $L_{-2} = \langle y \rangle$.

Later we will need another basis of $\mathfrak{sl}_2(\mathbb{C})$, *consists of*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = h, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{1.24}$$

Note that the subalgebras $\langle B \rangle$ *and* $\langle C \rangle$, *are some other Cartan subalgebras of* $\mathfrak{sl}_2(\mathbb{C})$. *Actually, this is true for the linear span of any semisimple element of* $\mathfrak{sl}_2(\mathbb{C})$.

Let $L = \mathfrak{sl}_n(\mathbb{C})$, then the subalgebra spanned by the elements x_i, h_i, y_i in (1.21) is isomorphic to the algebra $\mathfrak{sl}_2(\mathbb{C})$, see e.g. [Hum78, Proposition 8.3].

1.2 Gradings

In this section, we recall the basic concepts of gradings of and modules Lie algebras (see e.g. [ABFP08, BK10, BSZ01, DEK17, EK13, EK15, MZ18, Koc09]). The main goal is to establish the gradings of the simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ in Section 1.2.3. The main reference in this section is [EK13].

1.2.1 Gradings on simple Lie algebras

Definition 1.2.1. *Let* G *be a non-empty set. A* G-grading *on a vector space* V *over a field* $\mathbb{F}$ *is a direct sum decomposition of the form*

$$V = \bigoplus_{g \in G} V_g. \tag{1.25}$$

We will sometimes use Greek letters to refer to gradings, for example, we may write $\Gamma : V = \bigoplus_{g\in G} V_g$. If such a grading is fixed, V is called *G-graded*.

Note that the subspaces V_g are allowed to be zero.

Definition 1.2.2. *The subset $S \subseteq G$ consisting of those $g \in G$ for which $V_g \neq \{0\}$ is called the* support *of the grading Γ and denoted by* $\operatorname{Supp}\Gamma$ *or* $\operatorname{Supp} V$.

The subspaces V_g are called the homogeneous components *of Γ, and the non-zero elements in V_g are called* homogeneous of degree g *(with respect to Γ).*

Definition 1.2.3. *A* graded subspace $U \subseteq V$ *is an $\mathbb{F}$-subspace satisfying*

$$U = \bigoplus_{g\in G} U \cap V_g,$$

so U itself becomes G-graded.

Definition 1.2.4. *Let $\Gamma : V = \bigoplus_{g\in G} V_g$, and $\Gamma' : V = \bigoplus_{g'\in G'} V'_{g'}$ be two gradings on V with supports S and S', respectively. We say that Γ is a* refinement *of Γ' (or Γ' is a* coarsening *of Γ), if for any $s \in S$ there exists $s' \in S'$ such that $V_s \subseteq V'_{s'}$. The refinement is* proper *if this inclusion is strict for at least one $s \in S$.*

Definition 1.2.5. *An $\mathbb{F}$-algebra A (not necessarily associative) is said to be* graded by a set G, *or* G-graded *if A is a G-graded vector space and for any $g, h \in G$ such that $A_g A_h \neq \{0\}$ there is $k \in G$ (automatically unique) such that*

$$A_g A_h \subseteq A_k. \tag{1.26}$$

In this thesis, we will always assume that G is an *abelian group* and k in Equation (1.26) is determined by the operation of G. Thus, if G is written additively, then Equation (1.26) becomes $A_g A_h \subseteq A_{g+h}$. If G is written multiplicatively, then it becomes $A_g A_h \subseteq A_{gh}$. In particular, Definition 1.2.5 applies to Lie algebras: L is a G-graded Lie algebra if $L = \bigoplus_{g \in G} L_g$ such that $[L_g, L_h] \subseteq L_{g+h}$ for all $g, h \in G$.

Let V be a G-graded vector space, U a G-graded subspace of V, then V/U is *G-graded* with $V/U = \bigoplus_{g \in G} (V/U)_g$, where $(V/U)_g = (V_g + U)/U$.

Definition 1.2.6. *A grading on A is called* fine *if it does not have a proper refinement.*

Note that this concept depends on the class of gradings under consideration: by sets, groups, abelian groups, etc. It is well known that the latter two classes coincide for simple Lie algebras, see e.g [EK13].

Definition 1.2.7. *Given a grading $\Gamma : A = \bigoplus_{g \in G} A_g$ with support S, the* universal group *of Γ, denoted by G^u, is the group given in terms of generators and defining relations as follows: $G^u = \langle S \mid R \rangle$, where R consists of all relations of the form $gh = k$ with $\{0\} \neq A_g A_h \subseteq A_k$.*

If Γ is a group grading, then S is embedded in G^u and the identity map id_S extends to a homomorphism $G^u \to G$ so that Γ can be viewed as a G^u-grading Γ^u.

In fact, any group grading $\Gamma' : A = \bigoplus_{g' \in G'} A'_{g'}$ that is a coarsening of Γ can be induced from Γ^u by a (unique) homomorphism $\nu : G^u \to G'$ in the sense that $A'_{g'} = \bigoplus_{g \in \nu^{-1}(g')} A_g$ for all $g' \in G'$.

In this situation, one may say that Γ' is a *quotient* of Γ^u. In the above considerations, we can replace "group" by "abelian group" and, in general, this leads to a different G^u. However, there is no difference for gradings on simple Lie algebras.

Example 1.2.8. *Choose the elements*

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

as a basis of $L = \mathfrak{sl}_2(\mathbb{C})$ *and consider the following grading by* $G = \mathbb{Z}_3$*:*

$$\Gamma : \ L_1 = \langle x \rangle, \ L_0 = \langle h \rangle, \ L_2 = \langle y \rangle.$$

The support of Γ *is* G *itself, the universal group is* $\mathbb{Z}$*, and*

$$\Gamma^u : \ L^u_{-1} = \langle x \rangle, \ L^u_0 = \langle h \rangle, \ L^u_1 = \langle y \rangle.$$

The following grading by $G' = \mathbb{Z}_2$ *is a coarsening of* Γ*:*

$$\Gamma' : \ L^{'}_1 = \langle x, y \rangle, \ L^{'}_0 = \langle h \rangle.$$

Both Γ *and* Γ' *are quotients of* Γ^u*, while* Γ' *is a coarsening but not a quotient of* Γ*.*

Definition 1.2.9. *Let* G, G' *be two abelian groups. Let* L *be a* G*-graded algebra, and* M *a* G'*-graded algebra:*

$$\Gamma : L = \bigoplus_{g \in G} L_g \quad \textit{and} \quad \Gamma' : M = \bigoplus_{h \in G'} M_h$$

A linear map $\varphi : L \longrightarrow M$ *is* graded *if for any* $g \in G$ *there is* $h \in G'$ *such that* $\varphi(L_g) \subseteq M_h$*. An isomorphism* φ *is called an* equivalence of graded algebras *if* φ *and* φ^{-1} *are graded maps. If such* φ *exists then we say that* Γ *and* Γ' *are* equivalent *and that* φ *is an* equivalence *of* Γ *and* Γ'*.*

Definition 1.2.10. G*-graded algebras* L *and* M *are said to be* isomorphic *if there exists an isomorphism* $\varphi : L \longrightarrow M$ *such that* $\varphi(L_g) = M_g$ *for all* $g \in G$*.*

1.2.2 Graded modules

Definition 1.2.11. *If* $\Gamma : L = \bigoplus_{g\in G} L_g$ *is a* G*-grading on* L*, and* $\alpha : G \longrightarrow G'$ *is a group homomorphism, then the decomposition* ${}^{\alpha}\Gamma : L = \bigoplus_{h\in G'} L'_h$*, where*

$$L'_h = \bigoplus_{g\in G:\ \alpha(g)=h} L_g,$$

is a G'*-grading on* L*. We say that* ${}^{\alpha}\Gamma$ *is the grading* induced from Γ by the homomorphism α.

Definition 1.2.12. *A left module* M *over a* G*-graded associative algebra* A *is called* G-graded *if* M *is a* G*-graded vector space and*

$$A_g M_h \subseteq M_{g+h} \text{ for all } g, h \in G.$$

Definition 1.2.13. *A* G*-graded left* A*-module* M *is called graded-simple if* M *has no graded submodules different from* $\{0\}$ *and* M*.*

Graded modules and graded-simple modules over a graded Lie algebra L are defined in the same way.

If a Lie algebra L is graded by an abelian group G, then its universal enveloping algebra $U(L)$ is also G-graded. Every graded L-module is a graded left $U(L)$-module and *vice versa*. The same is true for graded-simple modules.

Definition 1.2.14. *A module* M *over a* G*-graded Lie (or associative) algebra* L *is called a* G-graded module *if* $M = \bigoplus_{g\in G} M_g$ *such that* $L_g.M_h \subseteq M_{g+h}$ *for all* $g, h \in G$*. An* L*-module homomorphism* $\varphi : M \longrightarrow N$ *is said to be* homogeneous of degree $h \in G$ *if* $\varphi(M_g) \subseteq N_{g+h}$ *for all* $g \in G$*. The* L*-modules are* graded isomorphic *if there is an isomorphism of degree* 0 *between them.*

In the case of gradings by finitely generated abelian groups, there is a close connection between the gradings of algebras by groups and the actions on algebras by automorphisms (see [EK13, §1.4]).

Let L be a G-graded finite-dimensional Lie algebra over $\mathbb{C}$, and $\mathrm{Aut}(L)$ be the group of all automorphisms of L. Let G be a finitely generated abelian group. We denote by $\widehat{G}$ the group of multiplicative characters of G, with values in the multiplicative group of $\mathbb{C}$. We can define a $\widehat{G}$-action on L by:

$$\chi \circ z = \chi(g)z, \text{ for all } z \in L_g, \ \chi \in \widehat{G}. \tag{1.27}$$

This action defines a homomorphism $\widehat{G} \to \mathrm{Aut}(L)$ sending $\chi \mapsto \alpha_\chi$ where $\alpha_\chi(z) := \chi \circ z$ for all $z \in L$. This action extends uniquely to an automorphism of $U(L)$. Conversely, consider the $\widehat{G}$-action on L, sending $(\chi, z) \mapsto \chi \circ z$. The grading induced by this action is given by:

$$L = \bigoplus_{g \in G} L_g,$$

where

$$L_g = \left\{ z \in L \mid \chi \circ z = \chi(g)z \text{ for all } \chi \in \widehat{G} \right\}. \tag{1.28}$$

There is a correspondence between the G-gradings on L and the set of all homomorphisms of algebraic groups $\widehat{G} \longrightarrow \mathrm{Aut}(L)$, see [EK13, Proposition 1.28].

$$\{G\text{-}grading \text{ on } L\} \Longleftrightarrow \left\{ \widehat{G}\text{-}action \text{ on } L \right\} \tag{1.29}$$

Let V be a G-graded L-module, and $\chi \in \widehat{G}$. Then $\widehat{G}$ also acts on V. If $\chi \in \widehat{G}$, then we can define the action on V by:

$$\chi * v = \chi(g)v \text{ for all } v \in V_g, \ \chi \in \widehat{G}. \tag{1.30}$$

If $\chi \in \widehat{G}$, let φ_χ denote the linear transformation of V defined by

$$\varphi_\chi(v) = \chi * v. \tag{1.31}$$

Since V is a G-graded L-module, it follows that

$$\varphi_\chi(z.v) = \alpha_\chi(z)\varphi_\chi(v). \tag{1.32}$$

There is a correspondence between graded subspaces of a module and submodules invariant under all linear transformations φ_χ, $\chi \in \widehat{G}$, that satisfy the condition (1.32). That is, if U is a submodule of the graded module V, which is invariant under φ_χ, for all $\chi \in \widehat{G}$, then U is G-graded. The converse is also true: any graded submodule is invariant under φ_χ, for all $\chi \in \widehat{G}$. In what follows, we will need the following result.

Proposition 1.2.15. *Let U be a unique maximal submodule of a G-graded L-module V. Then U is a G-graded L-submodule.*

Proof. Consider that $\chi \in \widehat{G}$ and $u \in \varphi_\chi(U)$. Let $z \in L$. Without loss of generality, assume that z is a homogeneous element of degree g. Since $u \in \varphi_\chi(U)$, there exists $v \in U$ such that $u = \varphi_\chi(v)$. Now

$$\begin{aligned} z.u &= z.\varphi_\chi(v) \\ &= \frac{1}{\chi(g)}\chi(g)z.\varphi_\chi(v) \\ &= \frac{1}{\chi(g)}\alpha_\chi(z).\varphi_\chi(v) \\ &= \frac{1}{\chi(g)}\varphi_\chi(\underbrace{z.v}_{\substack{\cap \\ U}}) \in \varphi_\chi(U). \end{aligned}$$

Hence $\varphi_\chi(U)$ is a submodule of V, which is automatically maximal. Since the maximal submodule of V is unique, it follows that $\varphi_\chi(U) = U$. Hence U is graded. $\square$

Definition 1.2.16. *For any $g \in G$, the* shift ${}^{[g]}V$ *of a G-graded module V equals V is the same as the G-graded space V where the homogeneous components V_h have degree $g+h$, that is, ${}^{[g]}V_{g+h} := V_h$, for all $h \in G$.*

Definition 1.2.17. *Let $L = \bigoplus_{g \in G} L_g$ be a G-graded algebra. An ideal I of L is a G-graded ideal if*

$$I = \bigoplus_{g \in G} (A_g \cap I). \tag{1.33}$$

It follows from the above definition, that the ideal I is G-graded if it is generated by homogeneous elements. In other words, I is graded if and only if for any $z \in I$ with $z = z_{g_1} + z_{g_1} + \cdots + z_{g_n}$, for some homogeneous elements $z_{g_i} \in L_{g_i}$, it follows that $z_{g_i} \in I$.

Let I be a G-graded ideal of the G-graded algebra L. Then L/I is G-graded, where

$$L/I = \bigoplus_{g \in G} (L_g + I)/I. \tag{1.34}$$

We now consider some examples of gradings on the matrix algebra $M_n(\mathbb{C})$ and its Lie subalgebra $L = \mathfrak{sl}_n(\mathbb{C})$.

Example 1.2.18. *Let $\mathcal{H} = \{x \in M_n(\mathbb{C}) \mid x^t = x\}$, $\mathcal{K} = \{x \in M_n(\mathbb{C}) \mid x^t = -x\}$, and let $G = \mathbb{Z}_2$. $L_0 = \mathcal{K} \cap L$, and $L_1 = \mathcal{H} \cap L$. Then $L = L_0 \oplus L_1$ is a G-grading on L. Note that, although $M_n(\mathbb{C}) = \mathcal{K} \oplus \mathcal{H}$, this is not a G-grading on the associative algebra $M_n(\mathbb{C})$.*

Example 1.2.19. *For any abelian group G and any n-tuple $(g_1, g_2, \ldots, g_n) \in G^n$, we obtain a G-grading on the matrix algebra $R = M_n(\mathbb{C})$ by setting*

$$R_g = \mathrm{Span}\,\{E_{ij} \mid g_i - g_j = g\},$$

where E_{ij} are the matrix units.

Definition 1.2.20. *A grading $\Gamma : R = \bigoplus_{g \in G} R_g$ on the matrix algebra $R = M_n(\mathbb{C})$ is called*

- elementary *if it is isomorphic to a grading as in Example 1.2.19;*
- *a* division *if every non-zero homogeneous element is invertible. This is equivalent to the condition* $\dim R_g \leqslant 1$ *for all $g \in G$ (see e.g. [EK13, p.38]).*

Example 1.2.21. *If $\varepsilon \in \mathbb{C}$ is a primitive n-th root of 1, consider the $n \times n$ matrices:*

$$X = \begin{bmatrix} \varepsilon^{n-1} & 0 & 0 & \dots & 0 & 0 \\ 0 & \varepsilon^{n-2} & 0 & \dots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & \varepsilon & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Note that $XY = \varepsilon YX$ and $X^n = Y^n = I$, so that

$$R = \bigoplus_{0 \leq k,l < n} R_{(k,l)} \text{ where } R_{(k,l)} = \operatorname{Span}\left\{X^k Y^l\right\}$$

is a $\mathbb{Z}_n^2$-grading on $R = M_n(\mathbb{C})$, called the Sylvester grading. *If $n = 2$ this grading is also called* Pauli *grading.*

Example 1.2.22. *Let L be a semisimple Lie algebra over $\mathbb{C}$, let H be a Cartan subalgebra, and let Φ the root system of L with respect to H. Then the root space decomposition* (1.12) *is a grading over the free abelian group $G = \langle \Delta \rangle$. This grading is called the* Cartan grading.

The classification, up to isomorphism, of abelian group gradings on $M_n(\mathbb{C})$, $\mathfrak{sl}_n(\mathbb{C})$ and other classical simple Lie algebras are obtained in (see e.g. [BSZ01, BK10, EK13, EK15b, Koc09]).

For the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$, we have two types of gradings, Type I and Type II.

Type I grading on $\mathfrak{sl}_n(\mathbb{C})$ comes as the restriction of a grading on the matrix algebra $M_n(\mathbb{C})$. Type II grading of $\mathfrak{sl}_n(\mathbb{C})$ by a group G comes as a refinement of a grading of $M_n(\mathbb{C})$ by a proper factor group $\overline{G}$ of G, see e.g. [ABFP08, EK13].

1.2.3 Group gradings of $\mathfrak{sl}_2(\mathbb{C})$

All group gradings on $\mathfrak{sl}_2(\mathbb{C})$ are well known, see e.g [EK13]. We will use the bases in of $\mathfrak{sl}_2(\mathbb{C})$ mentioned in (1.22) and (1.24).

Up to equivalence, there are precisely two *fine* gradings on $\mathfrak{sl}_2(\mathbb{C})$ (see [EK13, Theorem 3.55]):

- *The Cartan grading* with the universal group $\mathbb{Z}$,

$$\Gamma^1_{\mathfrak{sl}_2} : \mathfrak{sl}_2(\mathbb{C}) = L_{-1} \oplus L_0 \oplus L_1, \tag{1.35}$$

  where $L_0 = \langle h \rangle$, $L_1 = \langle x \rangle$, $L_{-1} = \langle y \rangle$.

- *The Pauli grading* with the universal group $\mathbb{Z}_2^2$,

$$\Gamma^2_{\mathfrak{sl}_2} : \mathfrak{sl}_2(\mathbb{C}) = L_{(1,0)} \oplus L_{(0,1)} \oplus L_{(1,1)}, \tag{1.36}$$

  where $L_{(1,0)} = \langle A \rangle$, $L_{(0,1)} = \langle B \rangle$, $L_{(1,1)} = \langle C \rangle$.

Hence, up to isomorphism, any G-grading on $\mathfrak{sl}_2(\mathbb{C})$ is a coarsening of one of the two gradings: Cartan or Pauli.

Note that any grading Γ of a Lie algebra L uniquely extends to a grading $U(\Gamma)$ of its universal enveloping algebra $U(L)$. The grading $U(\Gamma)$ is a grading in the sense of associative algebras but also as L-modules where $U(L)$ is either a (left) regular L-module or an adjoint L-module. In our study of gradings on $\mathfrak{sl}_2(\mathbb{C})$-modules we will often consider a $\mathbb{Z}_2$-coarsening of $U(\Gamma^2_{\mathfrak{sl}_2})$, in which the component of the coarsening labeled by 0 is the sum of components of the original grading labeled by $(0,0)$ and $(1,0)$ while the component labeled by 1 is the sum of components labeled by $(0,1)$ and $(1,1)$. Moreover, we will consider a $\mathbb{Z}_2$-coarsening of $U(\Gamma^1_{\mathfrak{sl}_2})$, in which the component of the coarsening labeled by 0 is the same component labeled by 0 while the component labeled by 1 is the sum of components labeled by 1 and -1.

Chapter 2

Weight $\mathfrak{sl}_2(\mathbb{C})$-modules

In this chapter we recall the weight $\mathfrak{sl}_2(\mathbb{C})$-modules (see [Bav90, Blo81, EW11, Fer90, Hum78, Hum08, Jac79, Mat00, Maz09]). We treat finite dimensional $\mathfrak{sl}_2(\mathbb{C})$-modules, Verma modules, anti-Verma modules, and Dense modules.

Moreover, we recall the classification of simple weight $\mathfrak{sl}_2(\mathbb{C})$-modules.

2.1 Simple weight $\mathfrak{sl}_2(\mathbb{C})$-modules

In this section we will describe all simple weight $\mathfrak{sl}_2(\mathbb{C})$-modules. Thus, we customize Definition 1.1.5.1 to the $\mathfrak{sl}_2(\mathbb{C})$ case.

Let V be an $\mathfrak{sl}_2(\mathbb{C})$-module, $H = \langle h \rangle$ be the Cartan subalgebra of $\mathfrak{sl}_2(\mathbb{C})$. We call

$$V_\mu = \{v \in V \mid h.v = \mu v\}, \text{ for } \mu \in \mathbb{C},$$

the *weight spaces* for V, and if V_μ is nontrivial we call $\mu \in \mathbb{C}$ the *weight* of V. If V is the direct sum of these weight spaces, we say that V is *a weight module*. The set of all weights is called the *support* of V. We denote the support of V by $\mathrm{Supp}(V)$.

In the case of a weight module, if $\lambda \in \mathrm{Supp}(V)$ and $\lambda + 2 \notin \mathrm{Supp}(V)$, λ is called the *highest weight* of V, and the elements of the space V_λ are called *highest weight vectors*. Similarly, if $\lambda \in \mathrm{Supp}(V)$ and $\lambda - 2 \notin \mathrm{Supp}(V)$, then λ is called the *lowest weight* and the elements of the space V_λ are called *lowest weight vectors*. If the weight module is generated by v_λ, where v_λ is a highest (resp., lowest) weight vector, then V is called *highest (resp., lowest) weight module* of weight λ.

Not all weight modules are highest or lowest weight modules, for example the weight dense modules (in Section 2.5) are neither highest nor lowest weight modules.

The proof of the next lemma will follow the steps of [Maz09, Proposition 3.8].

Lemma 2.1.1. *Any h-invariant subspace of a weight $\mathfrak{sl}_2(\mathbb{C})$-module is spanned by weight vectors.*

Proof. Let V be a weight $\mathfrak{sl}_2(\mathbb{C})$-module and let W be an h-invariant subspace of V. Let $w \in W \subset V$, and write $w = v_1 + v_2 + \cdots + v_k$, where v_i is a nonzero weight vector of weight $\mu_i \in \mathbb{C}$, for all $i = 1, 2, \ldots, k$. We may assume that $\mu_1, \mu_2, \ldots, \mu_k$ are distinct. Define the elements $h_i \in U(H)$, $i = 1, \ldots, k$, by

$$h_i = \prod_{l \neq i} (h - \mu_l).$$

Then

$$h_i.v_j = \begin{cases} 0 & \text{if } i \neq j; \\ \prod_{l \neq i} (\mu_i - \mu_l) v_i & \text{if } i = j. \end{cases}$$

Hence,

$$W \ni h_i.w = \sum_{j=1}^{k} h_i.v_j = h_i.v_i = \prod_{l \neq i} (\mu_i - \mu_l) v_i,$$

which means that $v_i \in W$. $\square$

2.2 Simple finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-modules

In this section we will describe all simple finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-modules (see e.g. [Hum78, §7.2]).

Let V be any simple finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-module. Since h is diagonalizable, then

$$V = \bigoplus_{\mu \in \mathbb{C}} V_\mu. \tag{2.1}$$

The following lemma describes the behavior of the action by the basis elements of $\mathfrak{sl}_2(\mathbb{C})$. The proof of this lemma can be found in (e.g. [Hum78, Lemma 7.1] or [Maz09, Lemma 1.15]).

Lemma 2.2.1. *Let $V = \bigoplus_{\mu \in \mathbb{C}} V_\mu$ be any simple finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-module. Let $\mu \in \mathbb{C}$, then*

1. $x.V_\mu \subseteq V_{\mu+2}$,
2. $y.V_\mu \subseteq V_{\mu-2}$,
3. $h.V_\mu \subseteq V_\mu$.

Moreover, there exists $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $V_{\lambda_1} \neq 0$ but $V_{\lambda_1+2i} = 0$ for all $i \in \mathbb{N}$, and $V_{\lambda_2} \neq 0$ but $V_{\lambda_2-2i} = 0$ for all $i \in \mathbb{N}$. □

Consider V be a simple finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-module, and λ_1 as in Lemma (2.2.1). Let $v_0 \in V_{\lambda_1}$, and let n be the minimal natural number where $y^{n+1}.v_0 = 0$. Set $v_i = \frac{1}{i!} y^i.v_0$ for $i = 1, 2, \ldots, n$. Since the sum in (2.1) is direct, the vectors v_i's are linearly independent. Consider $V(n)$ be the subspace with the basis $\{v_i \mid i = 0, 1, \ldots n\}$.

Lemma 2.2.2. *[Hum78, Lemma 7.2] Let $v_i \in V(n)$, then*

$$\begin{aligned} h.v_i &= (n-2i)v_i; \\ y.v_i &= (i+1)v_{i+1}; \\ x.v_i &= \begin{cases} (n-(i-1))v_{i-1} & \textit{if} \quad i \geq 1; \\ 0 & \textit{if} \quad i = 0. \end{cases} \end{aligned} \tag{2.2}$$

□

It follows from the above lemma that $V(n)$ is a non-zero submodule of the simple module V, hence $V = V(n)$.

Theorem 2.2.3. *Let $n \in \mathbb{N}_0$, then*

1. *The module $V(n)$ is a simple $\mathfrak{sl}_2(\mathbb{C})$-module of dimension $n+1$. Moreover,*

$$V(n) = V_n \oplus V_{n-2} \oplus \cdots \oplus V_{-(n-2)} \oplus V_{-n}.$$

   where $\dim(V_i) = 1$ *for $i = n, n-2, \ldots, -n$.*

2. *Any simple finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-module of dimension $n+1$ is isomorphic to the module $V(n)$.* □

For the proof of Theorem 2.2.3, see e.g. [Hum78, §7.2] or [Maz09, Theorem 1.22].

2.3 Verma $\mathfrak{sl}_2(\mathbb{C})$-modules

Now we will turn to the first family of infinite-dimensional weight $\mathfrak{sl}_2(\mathbb{C})$-modules.

The general construction for the Verma modules over a semisimple Lie algebra L is given by Definition 1.1.40. In the case of $L = \mathfrak{sl}_2(\mathbb{C})$, we have $B = B(\Delta) = \langle h, x\rangle$ and

$N = \langle x \rangle$. In view of the general definition of the Verma module, Verma $\mathfrak{sl}_2(\mathbb{C})$-module of highest weight $\lambda \in \mathbb{C}$, is

$$Z(\lambda) = U(\mathfrak{sl}_2(\mathbb{C})) \otimes_{U(B)} D_\lambda, \tag{2.3}$$

where D_λ is a one-dimensional B-module, such that $x.v = 0$ and $h.v = \lambda v$, for all $v \in D_\lambda$.

Proposition 2.3.1. *[Hum78, §20.3] Let $\lambda \in \mathbb{C}$, and consider the left ideal $I(\lambda)$ of $U(\mathfrak{sl}_2(\mathbb{C}))$, generated by x and $h - \lambda$. Then*

$$Z(\lambda) \cong U(\mathfrak{sl}_2(\mathbb{C}))/I(\lambda). \tag{2.4}$$

In order to give an explicit construction of $Z(\lambda)$(see e.g. [Hum78, Exercise 7.7, Exercise 20.4] or [Maz09, §3.2]), we use mathematical induction to generalize the case of simple finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-modules to the Verma $\mathfrak{sl}_2(\mathbb{C})$-modules. We set $v_i = \frac{1}{i!} y^i.v_0$, for $i \in \mathbb{N}_0$. Then

$$Z(\lambda) = \langle v_0, v_1, v_2, \ldots \rangle$$

and the action is given by the formulas (2.2) with n replaced by λ. Thus,

$$Z(\lambda) = \bigoplus_{i \in \mathbb{N}_0} V_{\lambda - 2i},$$

where $V_{\lambda-2i} = \mathbb{C} v_i$.

Theorem 2.3.2. *[Maz09, Theorem 3.16]*

The module $Z(\lambda)$ is a simple $\mathfrak{sl}_2(\mathbb{C})$-module if and only if $\lambda \notin \mathbb{N}_0$. If n is a non-negative integer, then $Z(n)$ has a unique nontrivial submodule $Z(-n-2)$, with $Z(n)/Z(-n-2) \cong V(n)$.

Theorem 2.3.3. *[Maz09, Corollary 3.18] Let $\lambda \in \mathbb{C}$. Then up to isomorphism, there is a unique simple highest weight $\mathfrak{sl}_2(\mathbb{C})$-module $L(\lambda)$ with highest weight λ. Moreover,*

$$L(\lambda) \cong \begin{cases} Z(\lambda) & \text{if } \lambda \in \mathbb{C}\backslash\mathbb{N}_0; \\ V(\lambda) & \text{if } \lambda \in \mathbb{N}_0. \end{cases} \tag{2.5}$$

2.4 Anti-Verma $\mathfrak{sl}_2(\mathbb{C})$-modules

In this section, we deal with one of the lowest weight $\mathfrak{sl}_2(\mathbb{C})$-module. This module is very close to the Verma module, with some differences in the actions.

Proposition 2.4.1. *Let V be a vector space with the basis $\{v_i \mid i \in \mathbb{N}_0\}$. Define the $\mathfrak{sl}_2$-action on V for $\lambda \in \mathbb{C}$ as:*

$$\begin{aligned} h.v_i &= (\lambda + 2i)v_i; \\ x.v_i &= v_{i+1}; \\ y.v_i &= \begin{cases} -i(\lambda + i - 1)v_{i-1} & \text{if} \quad i \geq 1 \\ 0 & \text{if} \quad i = 0. \end{cases} \end{aligned} \tag{2.6}$$

Then, the formulas in (2.6) define a lowest weight $\mathfrak{sl}_2(\mathbb{C})$-module with lowest weight λ. We denote this module by $\overline{Z}(\lambda)$ and call it the anti-Verma module.

The support of the anti-Verma module is

$$\mathrm{Supp}(\overline{Z}(\lambda)) = \{\lambda + 2i \mid i \in \mathbb{N}_0\}.$$

Proposition 2.4.2. *The module $\overline{Z}(\lambda)$ is a simple $\mathfrak{sl}_2(\mathbb{C})$-module if and only if $-\lambda \notin \mathbb{N}_0$. If n is a negative integer, then $\overline{Z}(n)$ has a unique maximal submodule $\overline{Z}(-n+2)$, with $\overline{Z}(n)/\overline{Z}(-n+2) \cong V(n)$.*

Theorem 2.4.3. *[Maz09, §3.2] Let $\lambda \in \mathbb{C}$, then up to isomorphism, there is a unique simple lowest weight $\mathfrak{sl}_2(\mathbb{C})$-module with lowest weight λ, say $\overline{L}(\lambda)$. Moreover,*

$$\overline{L}(\lambda) \cong \begin{cases} \overline{Z}(\lambda) & \text{if } -\lambda \in \mathbb{C}\backslash\mathbb{N}_0; \\ V(\lambda) & \text{if } -\lambda \in \mathbb{N}_0. \end{cases} \tag{2.7}$$

2.5 Simple dense $\mathfrak{sl}_2(\mathbb{C})$-modules

Definition 2.5.1. *A weight $\mathfrak{sl}_2(\mathbb{C})$-module is called a* dense module *if* $\mathrm{Supp}(V) = \lambda + 2\mathbb{Z}$ *for some $\lambda \in \mathbb{C}$.*

It is clear that no dense module has a highest or a lowest weight. For the existence of these modules, consider the following module:

Let $Z(\lambda) = \langle v_0, v_1, \ldots \rangle$ be the Verma module with highest weight λ. $\overline{Z}(\lambda+2) = \langle w_0, w_1, \ldots \rangle$ be the anti-Verma module of lowest weight $\lambda+2$. Let $V = Z(\lambda) \oplus \overline{Z}(\lambda+2)$ be the vector space of the external direct sum of $Z(\lambda)$ and $\overline{Z}(\lambda+2)$. Consider the $\mathfrak{sl}_2$-action on V by $y.(0, w_0) = (v_0, 0)$, and otherwise, use the usual action of the Lie algebra on an external direct sum of Lie modules. This action defines a dense $\mathfrak{sl}_2(\mathbb{C})$-module with $\mathrm{Supp}(V) = \lambda + 2\mathbb{Z}$.

Now we will study a large class of dense modules.

For $\xi \in \mathbb{C}/2\mathbb{Z}$ and $\tau \in \mathbb{C}$, consider V to be a vector space with the basis

$\{v_\mu \mid \mu \in \xi\}$. Define the action on V as:

$$\begin{aligned} h.v_\mu &= \mu v_\mu, \\ x.v_\mu &= \frac{1}{4}(\tau - (\mu+1)^2)v_{\mu+2}, \\ y.v_\mu &= v_{\mu-2}, \end{aligned} \tag{2.8}$$

This action makes V a dense weight $\mathfrak{sl}_2(\mathbb{C})$-module with $\mathrm{Supp}(V) = \xi$. We denote this module by $V(\xi, \tau)$.

Proposition 2.5.2. *The module $V(\xi,\tau)$ is simple if and only if $\tau \neq (\lambda+1)^2$ for all $\lambda \in \xi$. If the module $V(\xi,\tau)$ is not simple, then it contains a unique maximal submodule isomorphic to a Verma module for some highest weight.*

Now we are ready to give a classification of simple weight $\mathfrak{sl}_2(\mathbb{C})$-modules.

Theorem 2.5.3. *[Maz09, Theorem 3.32] Up to isomorphism, any simple weight $\mathfrak{sl}_2(\mathbb{C})$-module is one of the following modules:*

1. *$V(n)$ for some $n \in \mathbb{N}$.*
2. *$Z(\lambda)$ for some $\lambda \in \mathbb{C}\backslash\mathbb{N}_0$.*
3. *$\overline{Z}(\lambda)$ for some $-\lambda \in \mathbb{C}\backslash\mathbb{N}_0$*
4. *$V(\xi,\tau)$ for some $\xi \in \mathbb{C}/2\mathbb{Z}$ and $\tau \in \mathbb{C}$, with $\tau \neq (\lambda+1)^2$, for all $\lambda \in \xi$.*

□

Chapter 3

Torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules

In chapter, we recall torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules. Thus, we have to recall the concept of the algebra $U(I_\lambda)$, which plays an important role in the torsion-free modules. See [AP73, BS19, BM04, Bav90, Bav92, Blo81, Cat98, Dix77, Hum78, MP16, Maz09, Nil15].

3.1 The algebra $U(I_\lambda)$

The goal of this section is to study the algebra $U(I_\lambda)$ in (3.7). We recall the central character, the Casimir element and it's action, the primitive ideal, the center of the universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$, the algebra $U(I_\lambda)$ and it's basis.

3.1.1 The Casimir element

Recall that $\mathcal{Z}(L)$ denotes the center of universal enveloping algebra $U(L)$ of a Lie algebra L.

Definition 3.1.1. *Let L be a semisimple Lie algebra, let $\{x_1, \ldots, x_n\}$ a basis of L, and let $\{y_1, \ldots, y_n\}$ be its dual basis with respect to the nondegenerate bilinear form $\beta(x, y) = \mathrm{tr}(\varphi(x)\varphi(y))$, where $\varphi : L \longrightarrow \mathfrak{gl}(V)$ is a faithful representation of L. The Casimir element of the representation φ is the element c_φ defined by:*

$$c_\varphi = \sum_{i=1}^{n} \varphi(x_i)\varphi(y_i). \tag{3.1}$$

If the representation (module) is fixed, we always use the notation c instead of c_φ. It is known that the element c_φ is an endomorphism commuting with $\varphi(L)$ (see e.g. [Hum78, 6.2]).

Theorem 3.1.2. *[Hum78, §6.2] Let L be a semisimple Lie algebra. Then for any simple L-module V, the Casimir element c acts as a scalar on V.*

We will use the term *Casimir constant* to refer to the scaler in Theorem 3.1.2.

Proposition 3.1.3. *[Maz09, §3]*

Let V be one of the weight $\mathfrak{sl}_2(\mathbb{C})$-modules, which we studied in Chapter 2. Then the Casimir constant α_c of V is given by:

$$\alpha_c = \begin{cases} (n+1)^2 & \text{if } V = V(n),\ n \in \mathbb{N} \\ (\lambda+1)^2 & \text{if } V = Z(\lambda),\ \lambda \in \mathbb{C}, \\ (\lambda-1)^2 & \text{if } V = \overline{Z}(\lambda),\ \lambda \in \mathbb{C}, \\ \tau & \text{if } V = V(\varepsilon, \tau),\ \tau \in \mathbb{C},\ \varepsilon \in \mathbb{C}/2\mathbb{Z}. \end{cases} \tag{3.2}$$

The Casimir element for $U(\mathfrak{sl}_2(\mathbb{C}))$ with respect to the $\mathbb{Z}$-homogeneous basis $\{h, x, y\}$ can be written as

$$c = (h+1)^2 + 4yx = h^2 + 1 + 2xy + 2yx. \tag{3.3}$$

This Casimir element with respect to the $\mathbb{Z}_2^2$-homogeneous basis $\{h, B, C\}$ can be calculated by

$$\begin{aligned} c &= 2xy + 2yx + h^2 + 1 = 2\left(\frac{B+C}{2}\right)\left(\frac{B-C}{2}\right) \\ &+ 2\left(\frac{B-C}{2}\right)\left(\frac{B+C}{2}\right) + h^2 + 1 \\ &= \frac{1}{2}(B^2 + CB - BC - C^2) + \frac{1}{2}(B^2 + BC - CB - C^2) + h^2 + 1, \end{aligned}$$

thus

$$c = B^2 - C^2 + h^2 + 1. \tag{3.4}$$

3.1.2 The algebra $U(I_\lambda)$

Let R be an associative algebra (or just an associative ring), and V be a left R-module. The annihilator of V, denoted by $\mathrm{Ann}_R(V)$, is a two-sided ideal of R defined as the set of all elements r in R such that, for all v in V, $r.v = 0$:

$$\mathrm{Ann}_R(V) = \{r \in R \mid r.v = 0 \text{ for all } v \in V\}.$$

Lemma 3.1.4. *Let R be a graded algebra and M be a graded R-module, then* $\mathrm{Ann}_R(M)$ *is graded.*

Proof. Let $I = \mathrm{Ann}_R(M) = \{x \in R \mid x.M = 0\}$, and $0 \neq x \in I \subseteq R$, then $x = x_1 + x_2 + \cdots + x_k$, where x_i are homogeneous elements in R (belonging to different homogeneous components). Let $v \in M$ be an arbitrary homogeneous element, then $0 = x.v = x_1.v + x_2.v + \cdots + x_k.v$. Since the components $x_i.v$ belong to different homogeneous subspaces, it follows that $x_i.v = 0$ for all i. Since v is an arbitrary homogeneous element, it follows that $x_i \in I$ for all i. □

Definition 3.1.5. *An ideal I of the universal enveloping algebra is said to be* primitive *if I is the annihilator of some simple $\mathfrak{sl}_2(\mathbb{C})$-module. If I is a primitive ideal, then the quotient algebra*

$$U(I) := U(\mathfrak{sl}_2(\mathbb{C}))/I, \tag{3.5}$$

is called the primitive quotient of the primitive ideal I.

The center of the universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$ is described in the following way (see e.g [Maz09, Theorem 2.32]).

Theorem 3.1.6. *$\mathcal{Z}(\mathfrak{sl}_2(\mathbb{C}))$ is the polynomial algebra in one variable c, the Casimir element, that is*

$$\mathcal{Z}(\mathfrak{sl}_2(\mathbb{C})) = \mathbb{C}[c]. \tag{3.6}$$

□

Given $\lambda \in \mathbb{C}$, Consider the two-sided ideal I_λ of $U(\mathfrak{sl}_2(\mathbb{C}))$, generated by the central element $c - (\lambda + 1)^2$.

Theorem 3.1.7. *[Maz09, Theorem 4.2] The annihilator of the Verma module with highest weight λ is the ideal I_λ.* □

Note that, if $\lambda_1 \neq \lambda_2$, then the ideals I_{λ_1} and I_{λ_2} are equal if and only if $\lambda_1 = -\lambda_2 - 2$.

Proposition 3.1.8. *The ideal I_λ is both $\mathbb{Z}$ - and $\mathbb{Z}_2^2$-graded ideal.*

Proof. Since $c-(\lambda+1)^2$ is homogeneous of degree 0 (resp., (0,0)) with respect to the $\mathbb{Z}$-grading (resp., $\mathbb{Z}_2^2$- grading), it follows that I_λ is graded. □

The next results (see e.g. [Maz09]) provide a strong relationship between the primitive ideals I_λ and the annihilators of simple $\mathfrak{sl}_2(\mathbb{C})$-modules.

Theorem 3.1.9. *For any simple $U(\mathfrak{sl}_2(\mathbb{C}))$-module M, there exists $\lambda \in \mathbb{C}$ such that $I_\lambda \subset \mathrm{Ann}_{U(\mathfrak{sl}_2(\mathbb{C}))}(M)$.* □

Hence the ideals I_λ, $\lambda \in \mathbb{C}$, are the minimal primitive ideals of $U(\mathfrak{sl}_2(\mathbb{C}))$.

Proposition 3.1.10. *[Maz09, §4.1] Let $J_n := \mathrm{Ann}_{U(\mathfrak{sl}_2(\mathbb{C}))}(V(n))$, where $V(n)$ is a finite-dimensional simple $\mathfrak{sl}_2(\mathbb{C})$-module. Then*

1. *$I_n \subset J_n$.*

2. *$\mathrm{Ann}_{U(\mathfrak{sl}_2(\mathbb{C}))}(\overline{Z}(\lambda)) = I_{\lambda-2}$.*

3. *Let $\xi \in \mathbb{C}/2\mathbb{Z}$ and $\tau = (\lambda+1)^2 \in \mathbb{C}$, then $\mathrm{Ann}_{U(\mathfrak{sl}_2(\mathbb{C}))}(V(\xi,\tau)) = I_\lambda$.*

   Moreover, any primitive ideal of $U(\mathfrak{sl}_2(\mathbb{C}))$ coincides with I_λ or J_n, for some $\lambda \in \mathbb{C}$ or $n \in \mathbb{N}$ □

Now for any $\lambda \in \mathbb{C}$, the primitive quotient of the primitive ideal I_λ, is

$$U(I_\lambda) := U(\mathfrak{sl}_2(\mathbb{C}))/I_\lambda. \tag{3.7}$$

This quotient algebra plays an important role in the theory of torsion-free modules.

Using Proposition 3.1.8, $U(I_\lambda)$ is a $\mathbb{Z}$-graded algebra and $\mathbb{Z}_2^2$-graded algebra.

It is well known (see e.g. [Maz09, Theorem 4.15]) that the algebra $U(I_\lambda)$ is a free $\mathbb{C}[h]$-module with basis

$$\mathcal{B}_0 = \left\{1, x, y, x^2, y^2, , x^3, y^3, \dots\right\}, \tag{3.8}$$

and so it is free over $\mathbb{C}$ with basis

$$\mathcal{B} = \left\{1, h, h^2, \ldots\right\}.\mathcal{B}_0. \tag{3.9}$$

In the next result, we construct a basis of $U(I_\lambda)$ with respect to the basis $\{h, B, C\}$, that is a basis of $\mathbb{Z}_2^2$-homogeneous elements.

Proposition 3.1.11. *Set* $\widehat{B}_0 = \{1, B, C, BC, B^2, B^2C, B^3, B^3C, \ldots\}$. *Then* $\widehat{B} = \{1, h, h^2, \ldots\} \cdot \widehat{B}_0$ *is a* $\mathbb{Z}_2^2$*-homogeneous basis of* $U(I_\lambda)$.

Proof. Consider the natural ascending filtration $\{U^{(n)} \mid n = 0, 1, 2, \ldots\}$ of $U(\mathfrak{sl}_2(\mathbb{C}))$, where $U^{(n)}$ is the linear span of monomials of the form $e_1 e_2 \cdots e_k$ where $e_i \in \mathfrak{sl}_2(\mathbb{C})$ for all i and $k \leq n$. Using PBW Theorem, if $\{f_1, f_2, f_3\}$ is an ordered basis of $\mathfrak{sl}_2(\mathbb{C})$, then this filtration is a linear span of the standard monomials of degree less than or equal n, that is the monomials of the form $f_1^i f_2^j f_3^l$ where the degree of a monomial is the sum of the exponents of its variables. Consider now the filtration $\{I^{(n)} \mid n = 0, 1, 2, \ldots\}$ of $U(I_\lambda)$ induced by the natural filtration of $U(\mathfrak{sl}_2(\mathbb{C}))$. Using the above basis $\mathcal{B}$, we can write that the basis of $I^{(n)}$ as the set

$$\{h^k x^\ell \mid 0 \leq k, \ell \leq n,\ k + \ell \leq n\} \cup \{h^k y^\ell \mid 0 \leq k, \ell \leq n\ k + \ell \leq n\}. \tag{3.10}$$

Now using (1.24), we may assume that a basis of $I^{(n)}$ is formed by some of the monomials in the set $\{h^k B^l C^m \mid k + l + m \leq n\}$.

We prove that $I^{(n)}$ is spanned by those monomials with $m = 0$ or 1. Let us use induction on n, with an obvious basis, and use the relation $C^2 = h^2 + B^2 - \mu$, where $\mu = \lambda^2 + 2\lambda$, holding in $U(I_\lambda)$. As a result, it is sufficient to deal with the elements $h^k B^l C^2$ with $k + \ell + 2 = n$. We write

$$h^k B^l C^2 = h^k B^l (h^2 + B^2 - \mu) = h^k (B^l h^2 + B^{l+2} - \mu B^l).$$

Since

$$h^k B^l h^2 = h^{k+2} B^l + h^k [B^l, h^2]$$

where the second term belongs to $I^{(s)}$ for some $s < n$, we can use the inductive step and write $h^k B^l C^2$ as a sum of terms of the desired form. Now it follows that $I^{(n)}$ is spanned by the set of monomials

$$\left\langle h^k B^l C^m \mid k + l + m \leq n \text{ and } m = 0, 1 \right\rangle . \tag{3.11}$$

Using the basis (3.10), we find that the dimension of $I^{(n)}$ equals $\sum_{i=0}^{n}(2i + 1)$. Clearly, the number of elements in (3.11) is the same and hence a basis of $U(I_\lambda)$ is formed by the monomials $\{h^k B^\ell\} \cup \{h^k B^\ell C\}$ where $k, \ell = 0, 1, 2, \ldots$ $\square$

In the following relations, we follow [AP74]. Let $p(t) = \frac{1}{4}((\lambda^2+2\lambda)-2t-t^2) \in \mathbb{C}[t]$. Then, inside $U(I_\lambda)$, for any $q(t) \in \mathbb{C}[t]$, we have the following relations:

$$x^k q(h) = q(h - 2k)x^k,$$
$$y^j q(h) = q(h + 2j)y^j.$$

If $k \geq j$ then

$$x^k y^j = p(h - 2k) \cdots p(h - 2(k - j + 1))x^{k-j},$$
$$y^j x^k = p(h + 2(j - 1)) \cdots p(h)x^{k-j}.$$

If $j \geq k$ then

$$x^k y^j = p(h - 2k) \cdots p(h - 2)y^{j-k},$$
$$y^j x^k = p(h + 2(j - 1)) \cdots p(h + 2(j - k))y^{j-k}.$$

Moreover, $U(I_\lambda)$ is a generalized Weyl algebra (see e.g [Bav92]) and has the following properties, (see e.g. [Maz09, Theoram 4.15]).

Theorem 3.1.12. *1. $U(I_\lambda)$ is both left and right Noetherian.*

2. $U(I_\lambda)$ is a domain.

3. The algebra $U(I_\lambda)$ is simple for all $\lambda \in \mathbb{C} \setminus \mathbb{Z}$.

$\square$

One more property that is important for us is the following.

Theorem 3.1.13. *[Maz09, Theorem 4.26] For any non-zero left ideal $I \subset U(I_\lambda)$, the $U(I_\lambda)$-module $U(I_\lambda)/I$ has finite length.* $\square$

3.1.3 Central characters and graded modules

In this section L denotes a semisimple Lie algebra.

Definition 3.1.14. *An associative homomorphism $\chi : \mathcal{Z}(L) \longrightarrow \mathbb{C}$ is called a central character of $U(L)$.*

Definition 3.1.15. *Let χ be a central character. One says that an L-module M* admits a central character χ *if*

$$z.m = \chi(z)m \text{ for all } m \in M \text{ and } z \in \mathcal{Z}(L).$$

In this case, χ is called the central character of M. We denote the central character of M by χ_M.

Definition 3.1.16. *Let χ be a central character, the two-sided ideal I_χ of $U(L)$ is the ideal generated by the elements $z - \chi(z),\ z \in \mathcal{Z}(L)$. For an ideal I_χ, one can define the quotient algebra*

$$U(I_\chi) := U(L)/I_\chi$$

Recall Definition 1.1.40 of the Verma module $M(\lambda)$ for any semisimple Lie algebra L. Then for any L-module V admitting a central character χ_V, there is $\lambda \in H^*$ such that $\chi_V = \chi_{M(\lambda)}$.

Thus, the study of simple modules over a semisimple Lie algebra L reduces to the study of the simple modules over an associative algebra $U(I_\chi)$, for an appropriate central character of $U(L)$. Thus, it is important to know, given a grading by a group G on a semisimple Lie algebra L, whether this grading induces a grading of the algebra $U(I_\chi)$.

In [BM04], a family of central elements have been constructed for certain simple Lie algebras admitting a division grading, that is, a grading that comes as a restriction of the division grading on the respective matrix algebra. In particular, this is possible for the Lie algebras $\mathfrak{sl}_n, \mathfrak{o}_{2^n}$, and $\mathfrak{sp}_{2^n}$. Each of these algebras can be considered as a Lie subalgebra of a suitable matrix algebra $M_k(\mathbb{C})$, $k = n$ in the first case and 2^n in the last two cases.

Let L be a Lie algebra admitting a division grading by a finite abelian group G. Hence, each nontrivial homogeneous component is 1-dimensional, see Definition 1.2.20. For each $g \in S := \operatorname{Supp}(L)$, choose a non-zero element $X_g \in L_g$. Hence, for any $g_1, g_2 \in S$, there is a number $\sigma(g_1, g_2) \in \mathbb{C}$ such that

$$X_{g_1} X_{g_2} = \sigma(g_1, g_2) X_{g_1+g_2}. \tag{3.12}$$

Hence,

$$[X_{g_1} X_{g_2}] = (\sigma(g_1, g_2) - \sigma(g_2, g_1)) X_{g_1+g_2}. \tag{3.13}$$

Note that, the map $\sigma : S \times S \longrightarrow \mathbb{C}^*$ is a 2-cocycle, that is

$$\sigma(g_1, g_2)\sigma(g_1 + g_2, g_3) = \sigma(g_2, g_3)\sigma(g_1, g_2 + g_3), \tag{3.14}$$

for all $g_1, g_2, g_3 \in S$.

Set $B_1 = X_e$, and for $k \geq 2$ write

$$B_k = \sum_{g_1+g_2+\cdots g_k=e} \frac{1}{\prod_{i=1}^{k-1} \sigma(g_i, g_{i+1} + \cdots + g_k)} X_{g_1} \cdots X_{g_k}. \tag{3.15}$$

Clearly, our Casimir element for $\mathfrak{sl}_2(\mathbb{C})$ in the form (3.4) is one of the elements (3.15).

Theorem 3.1.17. *[BM04, Theorem 4.1] The elements B_k are central elements of $U(L)$, for all $k \in \mathbb{N}$.*

Moreover, the authors indicate that, in the case when $L = \mathfrak{sl}_n$, these central elements generate $\mathcal{Z}(\mathfrak{sl}_n)$.

It is clear that the central elements above are homogeneous of degree e. Hence, if χ is a central character of $U(L)$, then all the elements $B_k - \chi(B_k)$ are homogeneous of degree e.

Proposition 3.1.18. *Suppose $L = \mathfrak{sl}_n(\mathbb{C})$ admits a division grading by a finite abelian group G, and assume that χ is a central character of $U(L)$. Then the ideal I_χ and the quotient algebra $U(I_\chi)$ are G-graded.*

As a consequence, if a is a graded elements of $U(I\chi)$, then $U(I\chi)/U(I\chi)a$ is a graded $\mathfrak{sl}_n$-module. We will use this idea in the particular case $n = 2$ to construct simple graded torsion-free modules for the Pauli grading of $\mathfrak{sl}_2(\mathbb{C})$.

In conclusion, note the following important classical result.

Proposition 3.1.19. *If L is a complex finite-dimensional semisimple Lie algebra, then*

$$\mathcal{Z}(L) = \mathbb{C}[z_1, z_2, \ldots, z_r],$$

where the z_i are algebraically independent central elements, and r is the rank of L.

For the proof of the above theorem, see e.g. [Kos75].

3.2 Torsion-free modules over $\mathfrak{sl}_2(\mathbb{C})$

In this section we will recall the torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules. In particular, we discuss the classification of torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules of rank 1, see [Nil15].

Definition 3.2.1. *Let M be an $\mathfrak{sl}_2(\mathbb{C})$-module. The module M is called torsion if for any $m \in M$ there exists non-zero $p(t) \in \mathbb{C}[t]$ such that $p(h).m = 0$. The module M is* torsion-free *if $M \neq 0$ and $p(h).m \neq 0$ for all $0 \neq m \in M$ and all non-zero $p(t) \in \mathbb{C}[t]$. If M is a free $\mathbb{C}[h]$-module of rank n, we say that M is a torsion-free $\mathfrak{sl}_2(\mathbb{C})$ module of rank n.*

Theorem 3.2.2. *[Maz09, Theorem 6.3] A simple $\mathfrak{sl}_2(\mathbb{C})$-module is either a weight or a torsion-free module.* □

It follows from Theorem 3.2.2 that if h has at least one eigenvector on a simple module M, then M is a weight module. As a consequence of Theorem 3.1.9, the description of simple $\mathfrak{sl}_2(\mathbb{C})$-modules reduces to the description of simple torsion-free $U(I_\lambda)$-modules (see e.g [Maz09, §6.1]).

A further reduction can be achieved as follows. Let us consider the field of rational functions in h, $\mathbb{K} = \mathbb{C}(h)$, and set $\mathbb{A}$ to be the *algebra of skew Laurent polynomials* over $\mathbb{K}$, that is

$$\mathbb{A} = \mathbb{K}[X, X^{-1}, \sigma] = \left\{ \sum_{i \in \mathbb{Z}} q_i(h) X^i \mid q_i(h) \in \mathbb{K}, \text{ almost all } q_i(h) = 0 \right\},$$

with the usual addition and scalar multiplication, and the product

$$(\sum_{i\in\mathbb{Z}} p_i(h)X^i)(\sum_{j\in\mathbb{Z}} q_j(h)X^j) = \sum_{i,j\in\mathbb{Z}} p_i(h)\sigma^i(q_j(h))X^{i+j},$$

where $\sigma(h) = h - 2$. Note that $\mathbb{A}$ is an Euclidean domain and it is isomorphic to $S^{-1}U(I_\lambda)$, the localization of the generalized Weyl algebra $U(I_\lambda)$, where $S = \mathbb{C}[h] \setminus \{0\}$. An embedding of $\Phi_\lambda : U(I_\lambda) \to \mathbb{A}$ is the unique extension of the following map:

$$\Phi_\lambda(h) = h,\ \Phi_\lambda(x) = X,\ \Phi_\lambda(y) = \frac{(\lambda+1)^2 - (h+1)^2}{4}X^{-1}.$$

Thanks to this embedding, $\mathbb{A}$ becomes a $\mathbb{A} - U(I_\lambda)$-bimodule and given an $U(I_\lambda)$-module M, one can define an $\mathbb{A}$-module $\mathcal{F}(M)$ by

$$\mathcal{F}(M) = \mathbb{A} \otimes_{U(I_\lambda)} M.$$

Theorem 3.2.3. *[Maz09, Theorem 6.24] The following are true.*

(i) The functor $\mathcal{F}$ induces a bijection $\widehat{\mathcal{F}}$ from the isomorphism classes of simple torsion-free $U(I_\lambda)$-modules to the set of isomorphism classes of simple $\mathbb{A}$-modules;

(ii) The inverse of the bijection from (i) is the map that sends a simple $\mathbb{A}$-module N to its $U(I_\lambda)$-socle $\mathrm{soc}_{U(I_\lambda)}(N)$.

Theorem 3.2.4. *[Blo81, Corollary 4.4.1] Let M be a simple torsion-free $U(I_\lambda)$-module. Then $M \cong U(I_\lambda)/(U(I_\lambda) \cap \mathbb{A}\alpha)$, for some $\alpha \in U(I_\lambda)$ which is irreducible as an element of $\mathbb{A}$.*

Many examples of torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules have been introduced, see e.g [Maz09, MP16, Nil15].

In [MP16], the author produced torsion-free modules of arbitrary finite rank.

Theorem 3.2.5. *[MP16, Theoram 6.1] Let $\alpha = x^n - p(h)x^{n-1} - a_0 \in U(I_\lambda)$, with $a_0 \in \mathbb{C}\setminus\{0\}$, and $p(h)$ is a non-constant polynomial in h. Then the module $U(I_\lambda)/U(I_\lambda)\alpha$ is a simple torsion-free $\mathfrak{sl}_2(\mathbb{C})$-module of rank n.*

Now we will highlight the torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules for which we can decide if those modules are graded or not.

Let us define a family of $U(I_\lambda)$-modules modules, as follows. Given two polynomials $p(t), g(t) \in \mathbb{C}[t]$, we set

$$M(p(t), g(t), \lambda) := U(I_\lambda)/U(I_\lambda)(g(h)x + p(h))$$

and

$$M'(p(t), g(t), \lambda) := U(I_\lambda)/U(I_\lambda)(g(h)y + p(h)).$$

Theorem 3.2.6. *[Maz09, Theorem 6.50] Let $\lambda \in \mathbb{C}$, and $g(t), p(t)$ be non-zero polynomials in $\mathbb{C}[t]$, such that if $r \in \mathbb{C}$ is a root of $p(t)$. If*

1. *$r + n$ is not a root for $g(t)$ for all $n \in \mathbb{Z}$.*

2. *$(\lambda + 1)^2 \neq (r + n + 1)^2$ for all $n \in \mathbb{Z}$.*

Then the $U(I_\lambda)$-modules $M(p(t), g(t), \lambda)$ and $M'(p(t), g(t), \lambda)$ are simple. □

The so called Whittaker modules are a special case of Theorem 3.2.6. They are defined as follows:

Definition 3.2.7. *Let $\alpha \in \mathbb{C}\setminus\{0\}$ and $\lambda \in \mathbb{C}$. The* Whittaker modules *are the modules $M_\alpha = U(I_\lambda)/U(I_\lambda)(1-\alpha x) = U(I_\lambda)/U(I_\lambda)(1-\frac{\alpha}{2}B-\frac{\alpha}{2}C)$.*

A full description of torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules of rank 1 (over $\mathbb{C}[h]$) was given in [Nil15].

Definition 3.2.8. *Let $0 \neq \alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$. Let us define an $\mathfrak{sl}_2(\mathbb{C})$-module $N(\alpha,\beta)$ as the vector space $\mathbb{C}[h]$ equipped with the following action: for $f(h) \in \mathbb{C}[h]$*

$$\begin{aligned} h.f(h) &= hf(h), \\ x.f(h) &= \alpha(\frac{h}{2}+\beta)f(h-2), \\ y.f(h) &= -\frac{1}{\alpha}(\frac{h}{2}-\beta)f(h+2). \end{aligned} \tag{3.16}$$

Proposition 3.2.9. *[Nil15, Lemma 12] Let $0 \neq \alpha \in \mathbb{C}$. Then the module $N(\alpha,\beta)$ is simple if and only if $2\beta \notin \mathbb{N}_0$.*

Definition 3.2.10. *Let $0 \neq \alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$. Let us define an $\mathfrak{sl}_2(\mathbb{C})$-module $N'(\alpha,\beta)$ as the vector space $\mathbb{C}[h]$ equipped with the following action: for $f(h) \in \mathbb{C}[h]$*

$$\begin{aligned} h.f(h) &= hf(h), \\ x.f(h) &= \alpha f(h-2), \\ y.f(h) &= -\frac{1}{\alpha}(\frac{h}{2}+\beta+1)(\frac{h}{2}-\beta)f(h+2). \end{aligned} \tag{3.17}$$

Proposition 3.2.11. *[Nil15, Lemma 11] Let $0 \neq \alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$. Then the module $N'(\alpha,\beta)$ is simple.*

To have pairwise non-isomorphic simple modules $N'(\alpha,\beta)$, consider the condition that $\mathrm{Re}(\beta) \geq -\frac{1}{2}$.

Definition 3.2.12. *Let $0 \neq \alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$. Let us define an $\mathfrak{sl}_2(\mathbb{C})$-module $\overline{N}(\alpha,\beta)$ as a vector space $\mathbb{C}[h]$ equipped with the following action: for $f(h) \in \mathbb{C}[h]$*

$$
\begin{aligned}
h.f(h) &= -hf(h),\\
x.f(h) &= \frac{1}{\alpha}(\frac{h}{2}+\beta+1)(\frac{h}{2}-\beta)f(h+2), \qquad (3.18)\\
y.f(h) &= -\alpha f(h-2).
\end{aligned}
$$

Proposition 3.2.13. *[Nil15, Lemma 11] Let $0 \neq \alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$. Then the module $\overline{N}(\alpha,\beta)$ is simple.*

Similarly, to have pairwise non-isomorphic simple modules $N'(\alpha,\beta)$, consider the condition that $\mathrm{Re}(\beta) \geq -\frac{1}{2}$. Note that the Whittaker modules are torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules of rank 1 with type $N'(\frac{1}{\alpha},\frac{\lambda}{2})$.

Theorem 3.2.14. *[Nil15, Theorem 9, Lemma 11, Lemma 12] Any simple torsion-free $\mathfrak{sl}_2(\mathbb{C})$-module of rank 1 is isomorphic to one of the following (pairwise non-isomorphic) modules:*

1. *$N(\alpha,\beta)$ for some $0 \neq \alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$ with $2\beta \notin \mathbb{N}_0$.*
2. *$N'(\alpha,\beta)$ for some $0 \neq \alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$ with $\mathrm{Re}(\beta) \geq -\frac{1}{2}$.*
3. *$\overline{N}(\alpha,\beta)$ for some $0 \neq \alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$ with $\mathrm{Re}(\beta) \geq -\frac{1}{2}$.*

Chapter 4

Gradings on the weight $\mathfrak{sl}_2(\mathbb{C})$-modules

In this chapter we study the gradings on the simple weight $\mathfrak{sl}_2(\mathbb{C})$-modules.

4.1 Gradings on simple finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-modules

Every simple finite-dimensional module of $\mathfrak{sl}_2(\mathbb{C})$ is a weight module, that is, decomposes as the direct sum of weight spaces and this decomposition is a grading compatible with the Cartan grading on $\mathfrak{sl}_2(\mathbb{C})$. In [EK15], the authors gave a complete classification of the highest weights λ for any graded finite-dimensional simple Lie algebra L such that the simple highest weight module $V(\lambda)$ admits a grading compatible with the grading of L. In particular, it was shown that the simple module $V(n)$ for $L = \mathfrak{sl}_2(\mathbb{C})$, equipped with the Pauli grading, can be made a graded module if and only if n is an even number. The proof in [EK15] does not provide the explicit form of the grading on $V(n)$. This is our goal in this section.

Let $V = V(n)$ be a simple $\mathfrak{sl}_2(\mathbb{C})$-module with an even highest weight n and the canonical basis $\{v_0, v_1, \ldots, v_n\}$ (see Section 2.2).

To construct a $\mathbb{Z}_2^2$-grading on V, we first define a new basis of V as follows.

Set $e_i = v_i + v_{n-i}$, for $i = 0, 1, \ldots, \frac{n}{2}$, and $d_i = v_i - v_{n-i}$ for $i = 0, 1, \ldots, \frac{n}{2} - 1$. Then $\left\{e_0, e_1, \ldots, e_{\frac{n}{2}}, d_0, d_1, \ldots, d_{\frac{n}{2}-1}\right\}$ is a basis of V and the module action is given by:

$$h.e_i = (n-2i)d_i, \text{ for } i = 0, 1, \ldots, \tfrac{n}{2};$$

$$B.e_i = \begin{cases} (n-i+1)e_{i-1} + (i+1)e_{i+1}, & \text{if } i = 0, 1, \ldots, \frac{n}{2} - 1; \\ 2(\frac{n}{2}+1)e_{\frac{n}{2}-1}, & \text{if } i = \frac{n}{2}; \end{cases}$$

$$C.e_i = \begin{cases} (n-i+1)d_{i-1} - (i+1)d_{i+1}, & \text{if } i = 0, 1, \ldots, \frac{n}{2} - 1; \\ 2(\frac{n}{2}+1)d_{\frac{n}{2}-1}, & \text{if } i = \frac{n}{2}; \end{cases}$$

$$h.d_i = (n-2i)e_i, \text{ for } i = 0, 1, \ldots, \tfrac{n}{2} - 1;$$

$$B.d_i = (n-i+1)d_{i-1} + (i+1)d_{i+1}, \text{ if } i = 0, 1, \ldots, \tfrac{n}{2} - 1;$$

$$C.d_i = (n-i+1)e_{i-1} - (i+1)e_{i+1}, \text{ if } i = 0, 1, \ldots, \tfrac{n}{2} - 1.$$

Let $V_{(0,0)} = \langle e_i \mid i \text{ is even}\rangle$, $V_{(0,1)} = \langle e_i \mid i \text{ is odd}\rangle$, $V_{(1,0)} = \langle d_i \mid i \text{ is even }\rangle$, and $V_{(1,1)} = \langle d_i \mid i \text{ is odd}\rangle$. One now easily checks the following.

Proposition 4.1.1. *The above formulas provide a $\mathbb{Z}_2^2$-grading $\Gamma : V = \bigoplus_{g \in \mathbb{Z}_2^2} V_g$ on V compatible with the Pauli grading on $\mathfrak{sl}_2(\mathbb{C})$.* □

4.2 Gradings on Verma $\mathfrak{sl}_2(\mathbb{C})$-modules

As we mentioned before, any weight $\mathfrak{sl}_2(\mathbb{C})$-module has a grading compatible with the Cartan grading on $\mathfrak{sl}_2(\mathbb{C})$ via the decomposition into weight spaces of this module. As a special case, we will explicitly describe the Cartan gradings on the Verma modules.

Let $\{v_0, v_1, \ldots, v_k, \ldots\}$ be a basis of $V(\lambda)$, as described in Section 2.3. Consider the canonical basis $\{x, y, h\}$ of $\mathfrak{sl}_2(\mathbb{C})$ with the Cartan grading by $\mathbb{Z}$, that is, $\deg(x) = 1,\ \deg(y) = -1,\ \deg(h) = 0$. The action of $\mathfrak{sl}_2(\mathbb{C})$ on V is the following:

$\cdot$	v_0	v_1	v_2	$\ldots$	v_k	$\ldots$
h	λv_0	$(\lambda-2)v_1$	$(\lambda-4)v_2$	$\ldots$	$(\lambda-2k)v_k$	$\ldots$
x	0	λv_0	$(\lambda-1)v_1$	$\ldots$	$(\lambda-k+1)v_{k-1}$	$\ldots$
y	v_1	$2v_2$	$3v_3$	$\ldots$	$(k+1)v_{k+1}$	$\ldots$

Let $V_{-k} = \langle v_k \rangle$ for $k = 0, 1, 2, ...,$ and $V_k = 0$ for $k = 1, 2, \ldots$. Then the grading $V = \bigoplus_{k=0}^{\infty} V_{-k}$ makes V a graded $\mathfrak{sl}_2(\mathbb{C})$-module.

Theorem 4.2.1. *Let V be a Verma $\mathfrak{sl}_2(\mathbb{C})$-module with highest weight $\lambda \in \mathbb{C}\backslash 2\mathbb{N}_0$. Then V is not a $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.*

Proof. Let $V = \bigoplus_{\mu\in\mathbb{C}} V_\mu$, with a maximal vector $v_0 \in V_\lambda$. Then V has a basis $\{v_0, v_1, v_2, \ldots\}$ given in Section 2.3. Assume that V has a grading compatible with the Pauli grading on $\mathfrak{sl}_2(\mathbb{C})$, so it can written as $V = V_{(0,0)} \oplus V_{(1,0)} \oplus V_{(0,1)} \oplus V_{(1,1)}$. Now let $V^0 = V_{(0,0)} \oplus V_{(1,0)}$, and $V^1 = V_{(0,1)} \oplus V_{(1,1)}$. V^0 and V^1 are thus h-invariant, with the action of B sending V^0 to V^1 and vice versa. By Lemma 2.1.1, V^0 and V^1 are spanned by weight vectors. Since $V_\lambda = \mathbb{C}v_0$, we must have $v_0 \in V^0$ or $v_0 \in V^1$. Without loss of generality, suppose $v_0 \in V^0$ (otherwise apply the shift of grading).

Then $V^1 \ni B.v_0 = v_1$, so $v_1 \in V^1$. Hence $V^0 \ni B.v_1 = \lambda v_0 + 2v_2$. Since $v_0 \in V^0$ we get $v_2 \in V^0$. Again $V^1 \ni B.v_2 = (\lambda-1)v_1+3v_3$, which implies $v_3 \in V^1$, and so on. We have shown that V^0 is spanned by the set $\{v_0, v_2, v_4, \ldots\}$ and V^1 by $\{v_1, v_3, v_5, \ldots\}$. Now let $0 \neq v \in V_{(0,0)} \subseteq V^0$. Then v can be written as $v = \alpha_0 v_0 + \alpha_2 v_2 + \cdots + \alpha_{2k} v_{2k}$, for some non-negative integer k, and some non-zero $\alpha_i \in \mathbb{C}$. Since $V_{(0,0)}$ is h^2-invariant, the elements

$$\begin{aligned}
h^2.v &= \alpha_0\lambda^2 v_0 + \alpha_2(\lambda-4)^2 v_2 + \cdots + \alpha_{2k}(\lambda-4k)^2 v_{2k},\\
h^4.v &= \alpha_0\lambda^4 v_0 + \alpha_2(\lambda-4)^4 v_2 + \cdots + \alpha_{2k}(\lambda-4k)^4 v_{2k},\\
&\cdots\\
h^{2k}.v &= \alpha_0\lambda^{2k} v_0 + \alpha_2(\lambda-4)^{2k} v_2 + \cdots + \alpha_{2k}(\lambda-4k)^{2k} v_{2k}
\end{aligned}$$

all belong to $V_{(0,0)}$. In order to use the Vandermonde's argument, we have to show that $\lambda^2, (\lambda-4)^2, \ldots, (\lambda-4k)^2$ are all distinct. Assume that we have two different weights, $(\lambda-4n)$ and $(\lambda-4m)$ such that $(\lambda-4n)^2 = (\lambda-4m)^2$. Then $|\lambda-4n| = |\lambda-4m|$. Hence either $\lambda - 4n = \lambda - 4m$ or $\lambda - 4n = 4m - \lambda$, the first case being obviously impossible. This means that $\lambda = 2(n+m) \in 2\mathbb{N}_0$, which is a contradiction. Hence,

$$\begin{vmatrix}
1 & 1 & \ldots & 1\\
\lambda^2 & (\lambda-4)^2 & \ldots & (\lambda-4k)^2\\
\vdots & \vdots & \ldots & \vdots\\
\lambda^{2k} & (\lambda-4)^{2k} & \ldots & (\lambda-4k)^{2k}
\end{vmatrix} \neq 0.$$

As a result, $V_{(0,0)}$ is spanned by weight vectors, which means that there is $v_s \in V_{(0,0)}$ for some s. Then $h.v_s = (\lambda-2s)v_s \in V_{(1,0)}$, a contradiction. $\square$

Corollary 4.2.2. *Let V be a Verma $\mathfrak{sl}_2(\mathbb{C})$-module with a non-negative even integer highest weight n. Then V cannot be a $\mathbb{Z}_2^2$-graded module.*

Proof. Assume that V is $\mathbb{Z}_2^2$-graded module. Since the highest weight is an integer number, it follows that V is not simple and has a unique maximal submodule $Z(-n-2)$. By Proposition 1.2.15, this must be a graded submodule. But $(-n-2)$ is a negative number, a contradiction with Theorem 4.2.1. $\square$

4.3 Gradings on anti-Verma $\mathfrak{sl}_2(\mathbb{C})$-modules

As we mentioned about the grading of weight modules over $\mathbb{Z}$. The anti-Verma module $V = \overline{Z}(\lambda)$ with basis $\{v_0, V_1, v_2, \ldots\}$ introduced in section 2.4, is a $\mathbb{Z}$-graded $\mathfrak{sl}_2(\mathbb{C})$-module. Let $V = \overline{Z}(\lambda)$ with the basis $\{v_0, v_1, \ldots, v_k, \ldots\}$, and consider the basis $\{x, y, h\}$ of $\mathfrak{sl}_2(\mathbb{C})$ with a Cartan grading over $\mathbb{Z}$, that is, $\deg(x) = 1$, $\deg(y) = -1$, $\deg(h) = 0$. The action of $\mathfrak{sl}_2(\mathbb{C})$ on V is the following:

$\cdot$	v_0	v_1	v_2	$\ldots$	v_k	$\ldots$
h	λv_0	$(\lambda+2)v_1$	$(\lambda+4)v_2$	$\ldots$	$(\lambda+2k)v_k$	$\ldots$
x	v_1	v_2	v_3	$\ldots$	v_{k+1}	$\ldots$
y	0	$-\lambda v_0$	$-2(\lambda+1)v_1$	$\ldots$	$-k(\lambda+k-1)v_{k-1}$	$\ldots$

Taking $V_k = \mathbb{C}v_k$ for $k = 0, 1, 2, \ldots$, and $V_k = 0$ for $k = -1, -2, \ldots$ makes $V = \bigoplus_{k=0}^{\infty} V_k$ into a $\mathbb{Z}$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.

Theorem 4.3.1. *Let V be an anti-Verma $\mathfrak{sl}_2(\mathbb{C})$-module with the lowest weight $\lambda \in \mathbb{C}$. Then V cannot be a $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.*

Proof. Write $V = \bigoplus_{k=0}^{\infty} V_k$ where $V_k = \mathbb{C}v_k$ for $k = 0, 1, 2, \ldots$, as above, and assume that $\{v_0, v_1, v_2, \ldots\}$ be the basis of V. Assume that V has a grading compatible with the

Pauli grading on $\mathfrak{sl}_2(\mathbb{C})$, so it can be written as $V = V_{(0,0)} \oplus V_{(1,0)} \oplus V_{(0,1)} \oplus V_{(1,1)}$. Denote $V^0 = V_{(0,0)} \oplus V_{(1,0)}$, and $V^1 = V_{(0,1)} \oplus V_{(1,1)}$. Then V^0 and V^1 are V h-invariant, with the action of B sending V^0 to V^1 and C sending V^1 to V^0. By Lemma 2.1.1, V^0 and V^1 are spanned by weight vectors. Since $V_0 = \mathbb{C}v_0$, we must have $v_0 \in V^0$ or $v_0 \in V^1$.

Without loss of generality, suppose $v_0 \in V^0$ (otherwise apply the shift of grading). Then $V^1 \ni B.v_0 = v_1$, so $v_1 \in V^1$. Hence $V^0 \ni B.v_1 = v_2 - \lambda v_0$. Since $v_0 \in V^0$ we get $v_2 \in V^0$. Again $V^1 \ni B.v_2 = v_3 - 2(\lambda+1)v_1$, which implies $v_3 \in V^1$, and so on. We have shown that V^0 is spanned by the set $\{v_0, v_2, v_4, \ldots\}$ and V^1 by $\{v_1, v_3, v_5, \ldots\}$.

Now let $0 \neq v \in V_{(0,0)} \subseteq V^0$. Then v can be written as $v = \alpha_0 v_0 + \alpha_2 v_2 + \cdots + \alpha_{2k} v_{2k}$ for some non-negative integer k and some non-zero $\alpha_i \in \mathbb{C}$. But since $V_{(0,0)}$ is h^2-invariant, the elements

$$\begin{aligned}
h^2.v &= \alpha_0 \lambda^2 v_0 + \alpha_2(\lambda+4)^2 v_2 + \cdots + \alpha_{2k}(\lambda+4k)^2 v_{2k}, \\
h^4.v &= \alpha_0 \lambda^4 v_0 + \alpha_2(\lambda+4)^4 v_2 + \cdots + \alpha_{2k}(\lambda+4k)^4 v_{2k}, \\
&\cdots \\
h^{2k}.v &= \alpha_0 \lambda^{2k} v_0 + \alpha_2(\lambda+4)^{2k} v_2 + \cdots + \alpha_{2k}(\lambda+4k)^{2k} v_{2k},
\end{aligned}$$

all belong to $V_{(0,0)}$. Now we have two cases:

Case 1 Assume that $-\lambda \notin 2\mathbb{N}_0$. In order to use the Vandermonde's argument, we need to show that $\lambda^2, (\lambda+4)^2, \ldots, (\lambda+4k)^2$ are all distinct. Assume that we have two different weights, $(\lambda+4n)$ and $(\lambda+4m)$ such that $(\lambda+4n)^2 = (\lambda+4m)^2$. Then $|\lambda+4n| = |\lambda+4m|$. Hence either $\lambda+4n = \lambda+4m$ or $\lambda+4n = -4m-\lambda$, but the first case is impossible. Therefore $-\lambda = 2(n+m) \in 2\mathbb{N}_0$, a contradiction. Hence,

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda^2 & (\lambda+4)^2 & \dots & (\lambda+4k)^2 \\ \vdots & \vdots & \dots & \vdots \\ \lambda^{2k} & (\lambda+4)^{2k} & \dots & (\lambda+4k)^{2k} \end{vmatrix} \neq 0.$$

It follows that $V_{(0,0)}$ is spanned by weight vectors, which means that $v_s \in V_{(0,0)}$ for some s. Then $h.v_s = (\lambda + 2s)v_s \in V_{(1,0)}$, a contradiction.

Case 2 Assume that $-\lambda \in 2\mathbb{N}_0$. Then V is not simple and has a unique maximal submodule $\overline{Z}(-\lambda + 2)$. If V is graded by $\mathbb{Z}_2^2$, then the unique maximal submodule of V must be graded. However, this contradicts Case 1 since $-(-\lambda + 2) \notin 2\mathbb{N}_0$. $\square$

4.4 Gradings on dense $\mathfrak{sl}_2(\mathbb{C})$-modules

As usual, the weight modules are graded by $\mathbb{Z}$, which is compatible with the Cartan grading of $\mathfrak{sl}_2(\mathbb{C})$. Let $\xi \in \mathbb{C}/2\mathbb{Z}$ and $\tau \in \mathbb{C}$, and let $\{v_\mu \mid \mu \in \xi\}$ be the basis of $V = V(\xi, \tau)$, as in Definition 2.5. As usual, the basis $\{x, y, h\}$ of $\mathfrak{sl}_2(\mathbb{C})$ is endowed with a Cartan grading by $\mathbb{Z}$ by setting $\deg(x) = 1,\ \deg(y) = -1,\ \deg(h) = 0$.

Since $\xi \in \mathbb{C}/2\mathbb{Z}$ it follows that $\xi = \lambda + 2\mathbb{Z}$ for some $\lambda \in \mathbb{C}$ and hence, for any $\mu \in \xi$, $\mu = \lambda + 2i$ for some $i \in \mathbb{Z}$. Let $V_i = \mathbb{C}v_{\lambda+2i}$, $i \in \mathbb{Z}$. Then $V = \bigoplus_{i \in \mathbb{Z}} V_i$ is a $\mathbb{Z}$-graded $\mathfrak{sl}_2(\mathbb{C})$-module with $\deg(V_i) = i$.

As for the grading by $\mathbb{Z}_2^2$, some of the dense modules can be graded and some others can not. Let us study the case where $\xi = \bar{0}$.

Proposition 4.4.1. *Let $\tau \in \mathbb{C}$ be such that the module $V = V(\bar{0}, \tau)$ is simple. Then V admits a $\mathbb{Z}_2^2$-grading compatible with the $\mathbb{Z}_2^2$-grading of $\mathfrak{sl}_2(\mathbb{C})$.*

Proof. Since $\xi = \bar{0}$, we can choose $\lambda = 0 \in \xi$. Then $V = \bigoplus_{i \in \mathbb{Z}} V_i$, where $V_i = \mathbb{C}v_{2i}$, and $\{v_{2i} \mid i \in \mathbb{Z} \dots\}$ being the basis of V. We set $e_0 = v_0, e_{-1} = 0$ and $e_k = \frac{1}{4^k}(\prod_{j=0}^{k}(\tau - (2j-1)^2))v_{2k} + v_{-2k}$ We also set $d_0 = 0$, $d_{-1} = 0$ and $d_k = \frac{1}{4^k}(\prod_{j=1}^{k}(\tau-(2j-1)^2))v_{2k} - v_{-2k}$, for $k \in \mathbb{N}$. Since V is simple, the set $\{e_0, e_1, \dots, d_1, d_2, \dots\}$ is a basis for V with a module action given by:

$$\begin{aligned} h.e_k &= 2kd_k, \\ h.d_k &= 2ke_k, \\ B.e_k &= e_{k+1} + \tfrac{1}{4}(\tau - (2k-1)^2)e_{k-1}, \\ B.d_k &= d_{k+1} + \tfrac{1}{4}(\tau - (2k-1)^2)d_{k-1}, \\ C.e_k &= d_{k+1} - \tfrac{1}{4}(\tau - (2k-1)^2)d_{k-1}, \\ C.d_k &= e_{k+1} - \tfrac{1}{4}(\tau - (2k-1)^2)e_{k-1}, \end{aligned} \tag{4.1}$$

Let $V_{(0,0)} = \langle e_i \mid i \text{ is even}\rangle$, $V_{(0,1)} = \langle e_i \mid i \text{ is odd}\rangle$, $V_{(1,0)} = \langle d_i \mid i \text{ is even }\rangle$, and $V_{(1,1)} = \langle d_i \mid i \text{ is odd}\rangle$. Then $\Gamma : V = \bigoplus_{g \in \mathbb{Z}_2^2} V_g$ is a $\mathbb{Z}_2^2$-grading of V. $\square$

Theorem 4.4.2. *Let $\bar{0} \neq \xi \in \mathbb{C}/2\mathbb{Z}$ and $\tau \in \mathbb{C}$ be such that the module $V = V(\xi, \tau)$ is simple. Then V is not a $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.*

Proof. If $\lambda \in \xi$, then $V = \bigoplus_{k \in \mathbb{Z}} V_k$, where $V_k = \mathbb{C}v_{\lambda+2k}$. Here $\{v_{\lambda+2i} \mid i \in \mathbb{Z}\}$ is the basis of V given in Definition 2.5. Assume that V has a grading compatible with the Pauli grading on $\mathfrak{sl}_2(\mathbb{C})$, and write $V = V_{(0,0)} \oplus V_{(1,0)} \oplus V_{(0,1)} \oplus V_{(1,1)}$. Denote $V^0 = V_{(0,0)} \oplus V_{(1,0)}$, and $V^1 = V_{(0,1)} \oplus V_{(1,1)}$. Then V^0 and V^1 are h-invariant, with the action of B sending V^0 to V^1 and the action of C sending V^1 to V^0. By Lemma 2.1.1, V^0 and V^1 are spanned by weight vectors. Since $V_\lambda = \mathbb{C}v_\lambda$, we must have $v_\lambda \in V^0$ or $v_\lambda \in V^1$.

Without loss of generality, suppose $v_\lambda \in V^0$ (otherwise apply the shift of grading). Then $V^1 \ni B.v_\lambda = \frac{1}{4}(\tau - (\lambda+1)^2)v_{\lambda+2} + v_{\lambda-2}$ and $V^1 \ni C.v_\lambda = \frac{1}{4}(\tau - (\lambda+1)^2)v_{\lambda+2} - v_{\lambda-2}$. Since V is simple, it follows that $(\tau - (\lambda+1)^2 \neq 0$ and hence $v_{\lambda+2}, v_{\lambda-2} \in V^1$. Now $B.v_{\lambda+2} = \frac{1}{4}(\tau - (\lambda+3)^2)v_{\lambda+4} + v_\lambda$ and $B.v_{\lambda-2} = \frac{1}{4}(\tau - (\lambda-1)^2)v_\lambda + v_{\lambda-4}$ are both in V^0. Since V is simple and $v_\lambda \in V^0$, it follows that $v_{\lambda+4}, v_{\lambda-4} \in V^0$. Apply B again to $v_{\lambda+4}, v_{\lambda-4}$ to get that $v_{\lambda+6}, v_{\lambda-6} \in V^1$, and so on. We have shown that V^0 is spanned by the set $\{\ldots, v_{\lambda-8}, v_{\lambda-4}, v_\lambda, v_{\lambda+4}, v_{\lambda+8}, \ldots\}$ and V^1 is spanned by $\{\ldots, v_{\lambda-6}, v_{\lambda-2}, v_{\lambda+2}, v_{\lambda+6}, \ldots\}$.

Now let $0 \neq v \in V_{(0,0)} \subseteq V^0$. Then v can be written as $v = \alpha_{-m}v_{\lambda-4m} + \cdots + \alpha_{-1}v_{\lambda-4} + \alpha_0 v_\lambda + \cdots + \alpha_n v_{\lambda+4n}$ for some non-negative integers m, n and some non-zero $\alpha_i \in \mathbb{C}$. But since $V_{(0,0)}$ is h^2-invariant, the following elements are in $V_{(0,0)}$:

$$\begin{aligned}
h^2.v &= \alpha_{-m}(\lambda-4m)^2 v_{\lambda-4m} + \cdots + \alpha_0\lambda^2 v_\lambda + \cdots + \alpha_n(\lambda+4n)^2 v_{\lambda+4n},\\
h^4.v &= \alpha_{-m}(\lambda-4m)^4 v_{\lambda-4m} + \cdots + \alpha_0\lambda^4 v_\lambda + \cdots + \alpha_n(\lambda+4n)^4 v_{\lambda+4n},\\
&\cdots\\
h^{2(m+n)}.v &= \alpha_{-m}(\lambda-4m)^{2(m+n)} v_{\lambda-4m} + \cdots + \alpha_0\lambda^{2(m+n)} v_\lambda + \cdots\\
&\quad + \alpha_n(\lambda+4n)^{2(m+n)} v_{\lambda+4n}.
\end{aligned}$$

Now, to use the Vandermonde's determinant we have to show that $(\lambda-4m)^2, \ldots, \lambda^2, (\lambda+4)^2, \ldots, (\lambda+4n)^2$ are all distinct. Assume that we have two different weights, $(\lambda+4k_1)$ and $(\lambda+4k_2)$, such that $(\lambda+4k_1)^2 = (\lambda+4k_2)^2$. Then $|\lambda+4k_1| = |\lambda+4k_2|$. Hence either $\lambda+4k_1 = \lambda+4k_2$ or $\lambda+4k_1 = -4k_2-\lambda$, but the first one is impossible. This means that $\lambda = -2(k_1+k_2) \in 2\mathbb{Z}$, which is not the case since $\xi \neq \bar{0}$. Hence,

$$\begin{vmatrix} 1 & \dots & 1 & \dots & 1 \\ (\lambda-4m)^2 & \dots & \lambda^2 & \dots & (\lambda+4n)^2 \\ \vdots & \vdots & \dots & \vdots & \\ (\lambda-4m)^{2(m+n)} & \dots & \lambda^{2(m+n)} & \dots & (\lambda+4n)^{2(m+n)} \end{vmatrix} \neq 0.$$

It follows that $V_{(0,0)}$ is spanned by weight vectors, which means that there is $v_{\lambda+4s} \in V_{(0,0)}$ for some $s \in \mathbb{Z}$. Then $h.v_s = (\lambda + 4s)v_s \in V_{(1,0)}$, a contradiction. □

Corollary 4.4.3. *Let* $\bar{0} \neq \xi \in \mathbb{C}/2\mathbb{Z}$ *and* $\tau \in \mathbb{C}$. *Then the module* $V = V(\xi, \tau)$ *cannot be a* $\mathbb{Z}_2^2$*-graded* $\mathfrak{sl}_2(\mathbb{C})$*-module.*

Proof. Theorem 4.4 covers the case where V is simple, so it is enough to prove this fact when V is non-simple. Suppose that V is a non-simple $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module. Then V has a unique maximal Verma submodule (see e.g. [Maz09, Theorem 3.29]), which has to be graded; this is a contradiction since Verma modules cannot be a $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-modules. □

Chapter 5

Gradings on torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules

In this chapter, we study the gradings on the simple torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules introduced in Chapter 3.

5.1 $\mathbb{Z}$-gradings of torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules of finite rank.

In this section, we prove the following.

Theorem 5.1.1. *For $L = \mathfrak{sl}_2(\mathbb{C})$, no simple L-module which is torsion-free of finite rank with respect to a fixed Cartan subalgebra can be given a $\mathbb{Z}$-grading compatible with the Cartan $\mathbb{Z}$-grading of L, defined by this subalgebra.*

Proof. Let M be any simple torsion-free $\mathfrak{sl}_2(\mathbb{C})$-module of finite rank n. Assume that

M is a $\mathbb{Z}$-graded $\mathfrak{sl}_2(\mathbb{C})$-module, that is, $M = \bigoplus_{i\in\mathbb{Z}} M_i$. Since h has degree 0 in $U(\mathfrak{sl}_2(\mathbb{C}))$, it follows that $\mathbb{C}[h]M_i \subseteq M_i$ for all $i \in \mathbb{Z}$. Hence M_i is also $\mathbb{C}[h]$-submodule of some finite rank $n_i \leq n$, which implies that $M = \bigoplus_{i\in I} M_i$, where $I = \{i \in \mathbb{Z} \mid M_i \neq \{0\}\}$, where $|I| = r$ for some positive integer $r \leq n$. Let $t \in I$ be maximal. Then $x.M_t \subseteq M_{t+1} = 0$. For any $0 \neq v \in M_t$, we have that $x.v = 0$, that is $yx.v = 0$. Since M is simple, it follows that the Casimir element c act as a scalar on M. Hence $c.v = ((h+1)^2 + 4yx).v = (h+1)^2.v = \alpha v$, for some $\alpha \in \mathbb{C}$, which implies that M is a torsion module. A contradiction. $\square$

5.2 Gradings of torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules of rank 1

Now we will study the gradings of the torsion-free modules of rank 1.

Lemma 5.2.1. *Let M be a G-graded torsion-free $\mathfrak{sl}_2(\mathbb{C})$-module, $p(h) \in \mathbb{C}[h]$ a homogeneous element in $U(\mathfrak{sl}_2(\mathbb{C}))$, and $v \in M$ a non-homogeneous element. Then the element $p(h).v \in M$ is not homogeneous.*

Proof. Since $p(h)$ is homogeneous, it follows $p(h) \in (U(\mathfrak{sl}_2(\mathbb{C})))_g$ for some $g \in G$. Since v is non-homogeneous, it follows that $v = v_{g_1} + v_{g_2} + \cdots + v_{g_k}$ for some $k > 1$ and $g_1, g_2, \ldots, g_k$ are distinct in G, where $v_{g_i} \in M_{g_i}$, at least two of them non-zero (say v_{g_1}, v_{g_2} are non-zero). Now $p(h).v = p(h).v_{g_1} + p(h).v_{g_2} + \cdots + p(h).v_{g_k}$, where $p(h).v_{g_i} \in M_{g_i+g}$. But $g_1 + g, g_2 + g, \ldots, g_k + g$ are distinct in G. Since M is torsion-free, $p(h).v_{g_1}, p(h).v_{g_2}$ are non-zero, which means that $p(h).v$ is non-homogeneous. $\square$

Theorem 5.2.2. *For $L = \mathfrak{sl}_2(\mathbb{C})$, no torsion-free L-module of rank 1 can be given a $\mathbb{Z}_2^2$-grading compatible with the Pauli $\mathbb{Z}_2^2$-grading of L.*

We prove this theorem on a case-by-case analysis for each kind of torsion-free modules of rank 1. A useful property is the following.

Lemma 5.2.3. *Let M be a torsion-free $\mathfrak{sl}_2(\mathbb{C})$-module, and $0 \neq v \in M$. Then one of $x.v$ or $y.v$ is non-zero.*

Proof. Assume that $x.v = 0$ and $y.v = 0$, then $0 = (xy - yx).v = h.v$, which means that $h.v = 0$, a contradiction. □

Proposition 5.2.4. *The module $N(\alpha, \beta)$, as in Definition 3.2.8, is not a Pauli $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.*

Proof. Assume that $N = N(\alpha, \beta)$ is a $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module, so that $N = N_{(0,0)} \oplus N_{(1,0)} \oplus N_{(0,1)} \oplus N_{(1,1)}$. Given a non-zero homogeneous element $f(h) \in N$, we define $\overline{f}(h)$ to be the same as $f(h)$ but computed in the algebra $U(\mathfrak{sl}_2(\mathbb{C}))$. Now $\overline{f}(h)$ can be written as the sum of a combination of monomials in h^{2k+1}, for $k = 0, 1, 2, \ldots$, and a linear combination of the monomials h^{2k}, for $k = 0, 1, 2, \ldots$, of degrees $(1, 0)$ and $(0, 0)$, respectively. As a result, $\overline{f}(h)$ is a homogeneous element in $U(\mathfrak{sl}_2)$ of degree 0 with respect to the $\mathbb{Z}_2$-grading on $U(\mathfrak{sl}_2)$ given by

$$U(\mathfrak{sl}_2) = (U(\mathfrak{sl}_2))^0 \oplus (U(\mathfrak{sl}_2))^1,$$

where

$$(U(\mathfrak{sl}_2))^0 = (U(\mathfrak{sl}_2))_{(0,0)} \oplus (U(\mathfrak{sl}_2))_{(1,0)}$$

and

$$(U(\mathfrak{sl}_2))^1 = (U(\mathfrak{sl}_2))_{(0,1)} \oplus (U(\mathfrak{sl}_2))_{(1,1)}.$$

Since $f(h)$ is homogeneous with respect to the $\mathbb{Z}_2^2$-grading, it will remain homogeneous relative to the coarser$\mathbb{Z}_2$-grading $N = N^0 \oplus N^1$, where $N^0 = N_{(0,0)} + N_{(1,0)}$ and

$N^1 = N_{(0,1)} + N_{(1,1)}$. Thus either $f(h) \in N^0$ or $f(h) \in N^1$. But $\overline{f}(h).1 = f(h)$. Since $\overline{f}(h)$ is homogeneous in $U(\mathfrak{sl}_2(\mathbb{C}))$, with respect to the $\mathbb{Z}_2$-grading, and $f(h)$ is homogeneous in N with respect to the $\mathbb{Z}_2$-grading, using Lemma 5.2.1 we conclude that 1 is homogeneous in N with respect to the $\mathbb{Z}_2$-grading. Now either $1 \in N^0$ or $1 \in N^1$. Without loss of generality assume that $1 \in N^0$, which means that $N = N^0$ and N^1 is trivial. Using Lemma 5.2.3, we have either $B.1 \neq 0$ or $C.1 \neq 0$. These elements belong to N^1, which provides the desired contradiction. $\square$

Proposition 5.2.5. *The module $N'(\alpha, \beta)$ as in Definition 3.2.10 is not a Pauli $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.*

Proof. Assume that $N = N'(\alpha, \beta)$ is a $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module. Let $f(h) \in N$ be a non-zero homogeneous element, and define $\overline{f}(h)$ to be the same as $f(h)$ but computed in the algebra $U(\mathfrak{sl}_2)$. It follows that $\overline{f}(h)$ is a homogeneous element in $U(\mathfrak{sl}_2(\mathbb{C}))$ of degree 0 with respect to the $\mathbb{Z}_2$-grading on $U(\mathfrak{sl}_2(\mathbb{C}))$. We know that $f(h)$ is homogeneous with respect to the coarsening grading by $\mathbb{Z}_2$. Now either $f(h) \in N^0$ or $f(h) \in N^1$, and $\overline{f}(h).1 = f(h)$. But $\overline{f}(h)$ is homogeneous in $U(\mathfrak{sl}_2(\mathbb{C}))$ with respect to the $\mathbb{Z}_2$-grading, and $f(h)$ is homogeneous in N with respect to the $\mathbb{Z}_2$-grading. Using Theorem 5.2.1, it follows that 1 is homogeneous in N with respect to the $\mathbb{Z}_2$-grading. Hence either $1 \in N^0$ or $1 \in N^1$. Without loss of generality assume that $1 \in N^0$, which means that $N = N^0$ and N^1 is trivial. Using Lemma 5.2.3, we have either $0 \neq B.1 \in N^1$ or $0 \neq C.1 \in N^1$, a contradiction in both cases. $\square$

Proposition 5.2.6. *The module $\overline{N}(\alpha, \beta)$ is not a Pauli $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.*

Proof. Use the argument from the proof of Proposition 5.2.4 and Proposition 5.2.5. $\square$

Using the above results and Theorem 3.2.14 we can conclude that the proof of Theorem 5.2.2 is complete.

As a consequence of Theorem 5.2.2 we can derive the following corollary about the gradings of the Whittaker modules.

Corollary 5.2.7. *The Whittaker modules cannot be $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-modules.*

Chapter 6

Graded $\mathfrak{sl}_2(\mathbb{C})$-modules of rank 2

In this chapter we will deal with the graded torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules of rank 2. We will construct a family of simple $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-modules of rank 2, and another family of $\mathbb{Z}_2^2$-graded simple $\mathfrak{sl}_2(\mathbb{C})$-modules.

6.1 New family of torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules of rank 2

In this section we will construct the first family of simple $\mathbb{Z}_2^2$-graded torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules of rank 2.

Given any $\lambda \in \mathbb{C}$, we consider the $U(I_\lambda)$-module $M_\lambda^C = U(I_\lambda)/U(I_\lambda)C$. For $u, v \in U(\mathfrak{sl}_2(\mathbb{C}))$, we say that u is equivalent to v, and write $u \equiv v$, when $u + U(I_\lambda)C = v + U(I_\lambda)C$. All elements of the form $h^k B^l C$ are equivalent to 0. Moreover $B^2 \equiv \mu - h^2$ where $\mu = \lambda^2 + 2\lambda$. Hence

$$h^k B^2 \equiv h^k(\mu - h^2) = \mu h^k - h^{k+2},$$

which implies that any element of M_λ^C can be written as a linear combination of elements of the form $h^k B^m$ where $m = 0, 1$. This means that M_λ^C is a torsion-free $\mathfrak{sl}_2(\mathbb{C})$-module of rank 2, with basis $\{1, B\}$ as a $\mathbb{C}[h]$-module. One can identify M_λ^C with the $\mathbb{C}[h]$-module $M_\lambda^C = \mathbb{C}[h] \oplus \mathbb{C}[h]B$. Note that M_λ^C is $\mathbb{Z}_2^2$-graded module since $U(I_\lambda)C$ is a $\mathbb{Z}_2^2$-graded left ideal.

Theorem 6.1.1 (Main Theorem 1)**.** *Let $\lambda \in \mathbb{C}\backslash 2\mathbb{Z}$. Then M_λ^C is a simple $\mathfrak{sl}_2(\mathbb{C})$-module.*

The proof of Theorem 6.1.1 is given at the end of this subsection. Before that we need to perform some necessary calculations and establish certain relations.

Let $f(h) \in \mathbb{C}[h] \subset M_\lambda^C$. Then

$$\begin{aligned} C.f(h) &= (x-y)f(h) \\ &= f(h-2)x - f(h+2)y \\ &\equiv \frac{1}{2}(f(h-2) - f(h+2))B. \end{aligned} \tag{6.1}$$

$$\begin{aligned} B.f(h) &= (x+y)f(h) \\ &= f(h-2)x + f(h+2)y \\ &\equiv \frac{1}{2}(f(h-2) + f(h+2))B. \end{aligned} \tag{6.2}$$

Note that

$$xB = \frac{1}{2}(B^2 + CB)$$
$$= \frac{1}{2}(B^2 + BC + 2h)$$
$$\equiv \frac{1}{2}(\mu - h^2 + 2h), \tag{6.3}$$

and

$$yB = \frac{1}{2}(B^2 - CB)$$
$$= \frac{1}{2}(B^2 - BC - 2h)$$
$$\equiv \frac{1}{2}(\mu - h^2 - 2h). \tag{6.4}$$

Using (6.1), (6.3), and (6.4), we write

$$\begin{aligned} C^2.f(h) =& \frac{1}{2}(x-y)(f(h-2) - f(h+2))B \\ =& \frac{1}{2}\left(f(h-4)xB - f(h)xB - f(h)yB + f(h+4)yB\right) \\ \equiv& \frac{1}{4}\big((-h^2+2h+\mu)f(h-4) - 2(\mu - h^2)f(h) \\ &+ (-h^2 - 2h + \mu)f(h+4)\big)\,. \end{aligned}$$

Hence

$$C^2.f(h) \equiv -\frac{1}{4}\,(\,(h^2 - 2h - \mu)f(h-4) \tag{6.5}$$
$$+2(\mu - h^2)f(h) + (h^2 + 2h - \mu)f(h+4)\,)\,.$$

In particular, if $f(h) = h^n$, then

$$\begin{aligned}(h^2{-}2h - \mu)(h-4)^n &= h^{n+2} - 2(2n+1)h^{n+1} \\ &+ \sum_{k=0}^{n-2} \Big(-\mu\binom{n}{k}(-4)^k - 2\binom{n}{k+1}(-4)^{k+1} \\ &+ \binom{n}{k+2}(-4)^{k+2} \Big)\, h^{n-k} \\ &+ (-\mu n(-4)^{n-1} - 2(-4)^n)h - \mu(-4)^n.\end{aligned}$$

And,

$$\begin{aligned}(h^2{+}2h - \mu)(h+4)^n &= h^{n+2} - 2(2n+1)h^{n+1} \\ &+ \sum_{k=0}^{n-2} \Big(-\mu\binom{n}{k}(4)^k + 2\binom{n}{k+1}(4)^{k+1} \\ &+ \binom{n}{k+2}(4)^{k+2} \Big)\, h^{n-k} \\ &+ (-\mu n(4)^{n-1} + 2(4)^n)h - \mu(4)^n.\end{aligned}$$

Hence

$$C^2.h^n \equiv a_n h^n + a_{n-2}h^{n-2} + \cdots + a_0 \tag{6.6}$$

where

$$a_n = -4n^2.$$

Also, for $k = 2, 4, \ldots, n-3$ (or $n-2$) we have,

$$a_{n-k} = -\frac{1}{2}\left(-\mu\binom{n}{k}4^k + 2\binom{n}{k+1}4^{k+1} + \binom{n}{k+2}4^{k+2}\right).$$

Except a_1 (or a_0), all other coefficients equal zero.

One more relation that we will need refers to $CB.h^n$.

Let $f(h) \in \mathbb{C}[h] \subset M_\lambda^C$. Using (6.2), (6.3), and (6.4), we can write

$$\begin{aligned} CB.f(h) =& \frac{1}{2}(x-y)\left(f(h-2)+f(h+2)\right)B \\ \equiv& \frac{1}{2}\left(f(h-4)xB + f(h)xB - f(h)yB - f(h+4)yB\right). \end{aligned}$$

Hence

$$CB.f(h) \equiv \frac{1}{4}\left(\ (-h^2+2h+\mu)f(h-4) + 4hf(h) + (h^2+2h-\mu)f(h+4)\ \right). \quad (6.7)$$

In particular, if $f(h) = h^n$, then

$$\begin{aligned} (-h^2+&2h+\mu)(h-4)^n = -h^{n+2} + 2(2n+1)h^{n+1} \\ &+ \sum_{k=0}^{n-2}\left(\mu\binom{n}{k}(-4)^k + 2\binom{n}{k+1}(-4)^{k+1}\right. \\ &\left.-\binom{n}{k+2}(-4)^{k+2}\right)h^{n-k} \\ &+ (\ \mu n(-4)^{n-1} + 2(-4)^n\ )\,h + \mu(-4)^n, \end{aligned}$$

and

$$\begin{aligned} (h^2+&2h-\mu)(h+4)^n = h^{n+2} + 2(2n+1)h^{n+1} + \\ &\sum_{k=0}^{n-2}(\ -\mu\binom{n}{k}(4)^k + 2\binom{n}{k+1}(4)^{k+1} + \binom{n}{k+2}(4)^{k+2}\ )\,h^{n-k} \\ &+ (\ -\mu n(4)^{n-1} + 2(4)^n\ )\,h - \mu(4)^n. \end{aligned}$$

Hence

$$CB.h^n \equiv a_{n+1}h^{n+1} + a_{n-1}h^{n-1} + \cdots + a_1 h \quad (6.8)$$

where

$$a_{n+1} = 2(n+1)$$

for $k = 1, 3, \ldots, n-2$ (or $n-1$) we have,

$$a_{n-k} = \frac{1}{2}\left(-\mu\binom{n}{k}4^k + 2\binom{n}{k+1}4^{k+1} + \binom{n}{k+2}4^{k+2}\right).$$

Except a_1 (or a_0), all other coefficients equal zero.

Lemma 6.1.2. *Let $\lambda \in \mathbb{C}\backslash 2\mathbb{Z}$. Consider a non-zero submodule N of M_λ^C. If there is a non-zero element $0 \neq f(h) \in N$, then $1 \in N$.*

Proof. Let $0 \neq v = f(h) \in N$; our claim is to show that $1 \in N$.

We prove this by induction on the degree of $f(h)$, assumed monic. When $i = 1$, we have $0 \neq v = h + a \in N$, where $a \in \mathbb{C}$. If $a \neq 0$, then

$$C^2.v \equiv -4h \in N$$

which implies that $0 \neq a \in N$. If $a = 0$, then $v = h \in N$ and

$$CB.v \equiv 4h^2 - 2\mu \in N,$$

and since $h \in N$ it follows that $h^2 \in N$. Thus, $0 \neq \mu \in N$, hence $1 \in N$.

For the inductive step, suppose $n > 1$ and let

$$v = f(h) = h^n + b_{n-1}h^{n-1} + b_{n-2}h^{n-2} + \cdots + b_0 \in N.$$

Using (6.6), for $n > 3$, we have

$$C^2.v \equiv -\frac{1}{2}\Big(\, 8n^2h^n + \big(\, -\mu\binom{n}{2}4^2 + 2\binom{n}{3}4^3 + \binom{n}{4}4^4\,\big)\, h^{n-2}$$

$$+ \cdots + 8b_{n-1}(n-1)^2h^{n-1} + \cdots + 8b_{n-2}(n-2)^2h^{n-2} + \cdots\Big).$$

For $n = 2, 3$ we have

$$C^2.(h^2 + b_1h + b_0) \equiv -16h^2 - 4b_1h + 8\mu,$$

and

$$C^2.(h^3 + b_2h^2 + b_1h + b_0) \equiv -36h^3 + (24\mu - 64 - 4b_1)h + 8b_2\mu$$

Now consider $v_1 = f_1(h) := C^2.v + 4n^2.v \in N$. Then $f_1(h)$ is a polynomial of degree less than n. If $C^2.v \neq -4n^2.v$, then the result follows from the inductive hypothesis applied to $f_1(h)$. Otherwise, we have

$$-4n^2 b_{n-1} = -4(n-1)^2 b_{n-1},$$

which implies that $b_{n-1} = 0$. For $n > 3$ we have

$$-\mu\binom{n}{2}4^2 + 2\binom{n}{3}4^3 + \binom{n}{4}4^4 + 8b_{n-2}(n-2)^2 = 8b_{n-2}n^2$$

which implies that

$$b_{n-2} = \frac{-\mu\binom{n}{2} + 8\binom{n}{3} + 16\binom{n}{4}}{2(n-1)}. \tag{6.9}$$

For $n = 2$ we have

$$b_0 = -\frac{1}{2}\mu.$$

For $n = 3$ we have

$$b_1 = \frac{8 - 3\mu}{4}.$$

If we have failed to produce a non-zero polynomial of degree less than n using v_1, we can use another action to get such element. Using (6.8), for $n > 2$, we have

$$\begin{aligned} CB.v \equiv & \frac{1}{2}\Bigg(4(n+1)h^{n+1} + \Big(-4n\mu + 2\binom{n}{2}4^2 + \binom{n}{3}4^3 \Big)\, h^{n-1} \\ & + \cdots + 4b_{n-2}(n-1)h^{n-1} + \cdots\Bigg). \end{aligned}$$

For $n = 2$

$$CB.(h^2 + b_0) \equiv 6h^3 + (16 - 4\mu + 2b_0)h.$$

Now consider the element $v_2 = f_2(h) := CB.v - 2(n+1)h.v$. Then $f_2(h)$ is a polynomial of degree less than n. If $CB.v \neq 2(n+1)h.v$, then we have found a non-zero polynomial of degree less than n in N, so induction applied to $f_2(h)$ yield the result. Otherwise, for $n > 2$

$$-4n\mu + 2\binom{n}{2}4^2 + \binom{n}{3}4^3 + 4b_{n-2}(n-1) = 4(n+1)b_{n-2},$$

which implies that

$$b_{n-2} = \frac{-n\mu + 8\binom{n}{2} + 16\binom{n}{3}}{2}. \tag{6.10}$$

For $n = 2$ we have

$$b_0 = 4 - \mu. \tag{6.11}$$

In fact, v_1 and v_2 cannot be zero at the same time. To see this, assume that v_1 and v_2 are both zero. Using (6.9) and (6.10), for $n > 3$ we have

$$\frac{-\mu\binom{n}{2} + 8\binom{n}{3} + 16\binom{n}{4}}{2(n-1)} = \frac{-n\mu + 8\binom{n}{2} + 16\binom{n}{3}}{2}.$$

Multiplying both side by 12, we have

$$\begin{aligned} &-3n\mu + 8n(n-2) + 4n(n-2)(n-3) \\ &= -6n\mu + 24n(n-1) + 16n(n-1)(n-2) \end{aligned}$$

and so

$$-3n\mu + 4n^3 - 12n^2 + 8n = -6n\mu + 16n^3 - 24n^2 + 8n.$$

Hence

$$3n\mu = 12n^3 - 12n.$$

Since $n > 3$, we have

$$\mu = 4(n^2 - n),$$

and so

$$\lambda = -2n, 2n - 2, \tag{6.12}$$

which is not the case.

For $n = 2$ we have

$$-\frac{1}{2}\mu = 4 - \mu \text{ hence } \mu = 8.$$

Then $\lambda = -4, 2$ which is not the case.

For $n = 3$ we have

$$\frac{8 - 3\mu}{4} = \frac{-3\mu + 40}{2} \text{ hence } \mu = 24.$$

Then $\lambda = -6, 4$, which is again not the case. Thus $1 \in N$ $\square$

Corollary 6.1.3. *Let $\lambda \in 2\mathbb{Z}$. Then there is a uniquely determined monic non-constant polynomial of degree n (as in (6.12)), say $r(h) \in M_\lambda^C$, such that*

$$C^2.r(h) \equiv -4n^2 r(h),$$

$$CB.r(h) \equiv 2(n+1)hr(h),$$

$$BC.r(h) \equiv 2nhr(h), \tag{6.13}$$

$$B^2.r(h) \equiv (-h^2 - 4n)r(h).$$

Proof. Using the relations (6.5) and (6.7), it easy to see that the element

$$r(h) = \prod_{i=1}^{n}(h + 2n - 4i + 2), \tag{6.14}$$

satisfies the above conditions, which is uniquely determined. $\square$

Corollary 6.1.4. *Let $\lambda \in 2\mathbb{Z}$, and $r(h)$ as in Corollary 6.1.3. Let N be any non-zero submodule of M_λ^C. If $f(h) \in N$ with $f(h) \neq \gamma r(h)$ for all $\gamma \in \mathbb{C}$, then*

1. *If $\deg(f) \leq n$, then $N = M_\lambda^C$.*

2. *If $\deg(f) > n$, then either $r(h) \in N$ or $N = M_\lambda^C$.* □

Note that, if $\lambda \in 2\mathbb{Z}$, then the polynomial $r(h)$ (as in (6.1.3)) has degree n, where where

$$n = \begin{cases} \frac{-\lambda}{2} & \text{if } \lambda < 0 \\ \\ \frac{\lambda+2}{2} & \text{if } \lambda \geq 0, \end{cases}$$

Note that, $\lambda \in 2\mathbb{Z}$ if and only if $\mu = 4(n^2 - n)$.

Next, we can argue in the same way as above, to evaluate the action on the element $g(h)B \in M_\lambda^C$.

Let $g(h)B \in \mathbb{C}[h]B \subset M_\lambda^C$. Then

$$\begin{aligned} C.g(h)B &= (x-y)g(h)B \\ &= g(h-2)xB - g(h+2)yB \\ &\equiv \frac{1}{2}\left(\, (\mu - h^2 + 2h)g(h-2) + (-\mu + h^2 + 2h)g(h+2)\,\right). \end{aligned} \tag{6.15}$$

Using (6.15), we can write

$$\begin{aligned} C^2.g(h)B =& \frac{1}{2}(x-y)\left(\, (\mu - h^2 + 2h)g(h-2) + (-\mu + h^2 + 2h)g(h+2)\,\right) \\ =& \frac{1}{2}\left(\, (\mu - 8 - h^2 + 6h)g(h-4)x + (-\mu + h^2 - 2h)g(h)x \right. \\ & \left. - (\mu - h^2 - 2h)g(h)y - (-\mu + 8 + h^2 + 6h)g(h+4)y\,\right) \\ \equiv& \frac{1}{4}\left(\, (-h^2 + 6h + \mu - 8)g(h-4) + 2(h^2 - \mu)g(h) \right. \qquad (6.16) \\ & \left. - (h^2 + 6h - \mu + 8)g(h+4)\,\right) B. \end{aligned}$$

In particular, if $g(h) = h^l$, then

$$C^2.h^lB \equiv (a_lh^l + a_{l-2}h^{l-2} + \cdots + a_0)B \tag{6.17}$$

where

$$a_l = -4(l+1)^2.$$

Also, for $k = 2, 4, \ldots, l-3$ (or $l-2$) we have,

$$a_{l-k} = -\frac{1}{2}\left(\, (8-\mu)\binom{l}{k}4^k + 6\binom{l}{k+1}4^{k+1} + \binom{l}{k+2}4^{k+2}\,\right).$$

Another relation that we will need refers to $CB.(h^lB)$. Let $g(h)B \in \mathbb{C}[h]B \subset M_\lambda^C$. Then

$$B.g(h)B \equiv \frac{1}{2}\left(\, (\mu - h^2 + 2h)g(h-2) - (-\mu + h^2 + 2h)g(h+2)\,\right). \tag{6.18}$$

Using (6.3), (6.4), and (6.18), we can write

$$\begin{aligned} CB.g(h)B &= \frac{1}{2}(x-y)\left(\, (\mu - h^2 + 2h)g(h-2) - (-\mu + h^2 + 2h)g(h+2)\,\right) \\ &\equiv \frac{1}{2}\big(\, (\mu - h^2 + 6h - 8)g(h-4)x - (-\mu + h^2 - 2h)g(h)x \\ &\quad -(-\mu - h^2 - 2h)g(h)y - (-\mu + h^2 + 6h + 8)g(h+4)y\,\big). \end{aligned}$$

Hence

$$\begin{aligned} CB.g(h)B \equiv\ & \frac{1}{4}\big(\, (\, -h^2 + 6h + \mu - 8\,)\, g(h-4) + 4hg(h) \\ & + (\, h^2 + 6h - \mu + 8\,)\, g(h+4)\,\big)\, B. \end{aligned} \tag{6.19}$$

In particular, if $g(h) = h^l$, then

$$CB.(h^lB) \equiv (a_{l+1}h^{l+1} + a_{l-1}h^{l-1} + \cdots + a_1h)B \tag{6.20}$$

where

$$a_{l+1} = 2(l+2),$$

for $k = 1, 3, \ldots, l-2$ (or $l-1$), we have,

$$a_{l-k} = \frac{1}{2}\,(\ (8-\mu)\binom{l}{k}4^k + 6\binom{l}{k+1}4^{k+1} + \binom{l}{k+2}4^{k+2}\ ).$$

Lemma 6.1.5. *Let $\lambda \in \mathbb{C}\backslash 2\mathbb{Z}$. Suppose N is a non-zero submodule of M_λ^C. If there is a non-zero element $0 \neq g(h)B \in N$, then $B \in N$.*

Proof. Let $\lambda \in \mathbb{C}\backslash 2\mathbb{Z}$ and assume that N is a non-zero submodule of M_λ^C. If $0 \neq u = g(h)B \in N$, then we are going to prove $B \in N$. We will prove this fact by induction on the degree l of $g(h)$. For the base of induction, we will consider $l = 1$. Let $0 \neq u = (h+a)B \in N$, where $a \in \mathbb{C}$. If $a \neq 0$ then

$$C^2.u = (-16h - 4a)B \in N,$$

which implies that $0 \neq (12a)B \in N$. If $a = 0$, then $u = hB \in N$ and

$$CB.u \equiv (\ 6h^2 + 4(8-\mu)\ )\,B.$$

Since $hB \in N$ it follows that $h^2B \in N$. But λ is not an even integer, so that $0 \neq 4(8-\mu)B \in N$, hence $B \in N$.

For the inductive step, suppose $l > 1$ and let $u = g(h)B = (\ h^l + b_1h^{l-1} + b_{l-2}h^{l-2} + \cdots + b_0\ )\,B \in N$. Using (6.17), for $l > 3$, we will have

$$C^2.u \equiv -\frac{1}{2}\,(\ 8(l+1)^2h^l + (\ (8-\mu)\binom{l}{2}4^2 + 6\binom{l}{3}4^3 + \binom{l}{4}4^4\ )\,h^{l-2}$$
$$+ \cdots + 8b_{l-1}(l)^2h^{l-1} + \cdots + 8b_{l-2}(l-1)^2h^{l-2} + \cdots\ )\,B.$$

For $l = 2, 3$, we will have

$$C^2.(h^2 + b_1 h + b_0)B \equiv (\,-36h^2 - 16b_1 h - (8(8-\mu) + 4b_0)\,)\,B$$

$$C^2(h^3 + b_2 h^2 + b_1 h + b_0)B \equiv (\,-64h^3 - 36b_2 h^2$$

$$-\,(24(8-\mu) + 192 + 16b_1)h - (8(8-\mu)b_2 + 4b_0)\,)\,B$$

Let $u_1 = f_1(h)B := C^2.u - (-4(l+1)^2.u) \in N$. Then $f_1(h)$ is a polynomial of degree less than l. If $C^2.u \neq -4(l+1)^2.u$, then we have found a non-zero polynomial of degree less than l, so induction applied to u_1 yields the result.

Otherwise

$$-4(l+1)^2 b_{l-1} = -4(l)^2 b_{l-1},$$

which implies that $b_{l-1} = 0$, and for $l > 3$ we have

$$(8-\mu)\binom{l}{2}4^2 + 6\binom{l}{3}4^3 + \binom{l}{4}4^4 + 8b_{l-2}(l-1)^2 = 8b_{l-2}(l+1)^2,$$

which implies that

$$b_{l-2} = \frac{(8-\mu)\binom{l}{2} + 24\binom{l}{3} + 16\binom{l}{4}}{2(l)}. \tag{6.21}$$

For $l = 2$ we have

$$b_0 = \frac{8-\mu}{4}.$$

For $l = 3$ we have

$$b_1 = \frac{16-\mu}{2}.$$

Now if we failed to produce a non-zero element with polynomial has a degree less than l using u_1, we can use another action to get such element. Using (6.20), for

$l > 2$, we have

$$CB.u \equiv \frac{1}{2} \left(\, 4(l+2)h^{l+1} + \left(\, 4n(8-\mu) + 6\binom{l}{2}4^2 + \binom{l}{3}4^3 \, \right) h^{l-1} \right.$$
$$\left. + \cdots + 4b_{l-2}nh^{l-1} + \cdots \, \right) B.$$

If $l = 2$, we have

$$CB.(h^2 + b_0)B \equiv \left(8h^3 + (48 + 4(8-\mu) + 4b_0)h\right) B.$$

Now we can consider an element $u_2 = f_2(h)B := CB.u - 2(l+2)h.u$. We have that $f_2(h)$ is a polynomial of degree less than l. If $CB.u \neq 2(l+2)h.u$, then we have found a non-zero element in N with polynomial of degree less than l, so our induction applied to u_2 yields the result.

Otherwise,

$$4n(8-\mu) + 6\binom{l}{2}4^2 + \binom{l}{3}4^3 + 4b_{l-2}(l) = 4(l+2)b_{l-2},$$

which implies that

$$b_{l-2} = \frac{(8-\mu)l + 24\binom{l}{2} + 16\binom{l}{3}}{2}. \tag{6.22}$$

For $l = 2$

$$b_0 = 20 - \mu. \tag{6.23}$$

In fact u_1 and u_2 cannot be both zero. To see this, assume that v_1 and v_2 are both zero. Using (6.21) and (6.22), for $l > 3$ we have

$$\frac{(8-\mu)\binom{l}{2} + 24\binom{l}{3} + 16\binom{l}{4}}{2(l)} = \frac{(8-\mu)l + 24\binom{l}{2} + 16\binom{l}{3}}{2}.,$$

Multiplying both sides by 12, we will have

$$16l^3 + 24l^2 + 8l - 6l\mu = 4l^3 - 4l - 3l\mu + 3\mu$$

which implies that

$$(l+1)\mu = 4l(l+1)^2.$$

Since $l > 0$, we have $\mu = 4(l^2+l)$, which means that λ is an even integer, and this is not the case.

For $l = 2$ we have

$$\frac{8-\mu}{4} = 20-\mu, \text{ hence } \mu = 24.$$

It follows that $\lambda = 4, -6$, which is not the case. For $l = 3$ we have

$$\frac{16-\mu}{2} = \frac{-3\mu+112}{2}, \text{ hence } \mu = 48,$$

which means that $\lambda = 6, -8$, which is again not the case. Hence if $0 \neq v = g(h)B \in N$, it follows that $B \in N$. $\square$

We can see from the previous proof that the system of equations $u_1 = 0$ and $u_2 = 0$ has a unique solution (up to the scalar multiplication) when $\lambda \in 2\mathbb{Z}$. Moreover, if $r = r(h) + g(h)B$ such that $C^2.r = -4n^2r$ and $CB.r = 2(n+1)hr$, then $g(h)$ has the degree $l = n-1$.

Corollary 6.1.6. *Let $\lambda \in 2\mathbb{Z}$. Then there is a uniquely determined monic polynomial $r^*(h) \in M_\lambda^C$ of degree $l = n-1$ such that*

$$\begin{aligned} C^2.r^*(h)B &= -4(l+1)^2 r^*(h)B, \\ CB.r^*(h)B &= 2(l+2)hr^*(h)B, \\ BC.r^*(h)B &= 2(l+1)hr^*(h)B, \\ B^2.r^*(h)B &= (-h^2-4(l+1))r^*(h)B. \end{aligned} \tag{6.24}$$

Proof. The element $C.r(h)$, where $r(h)$ is the polynomial defined Corollary 6.1.3, satisfies the above conditions, where $l = n-1$, and since $r^*(h)$ is uniquely determined, so we can take $r^*(h) = C.(r(h))$. $\square$

Corollary 6.1.7. *Let $\lambda \in 2\mathbb{Z}$, $r(h)$ as in Corollary 6.1.3, and let N be any non-zero submodule of M_λ^C. If $g(h)B \in N$ with $g(h) \neq \gamma C.r(h)$ for all $\gamma \in \mathbb{C}$, then*

1. *If $\deg(g(h)) \leq n-1$, then $N = M_\lambda^C$.*

2. *If $\deg(g(h)) > n-1$, then either $C.r(h) \in N$ or $N = M_\lambda^C$.* $\square$

Corollary 6.1.8. *Let $\lambda \in 2\mathbb{Z}$. Then the elements $u_{(\alpha_1,\alpha_2)} = \alpha_1 r(h) + \alpha_2(C.r(h))$, where $\alpha_1, \alpha_2 \in \mathbb{C}$, are the only elements in M_λ^C, such that*

$$\begin{aligned} C^2.u_{(\alpha_1,\alpha_2)} &\equiv -4n^2 u_{(\alpha_1,\alpha_2)}, \\ CB.u_{(\alpha_1,\alpha_2)} &\equiv 2(n+1)hu_{(\alpha_1,\alpha_2)}, \\ BC.u_{(\alpha_1,\alpha_2)} &\equiv 2nhu_{(\alpha_1,\alpha_2)}, \\ B^2.u_{(\alpha_1,\alpha_2)} &\equiv (-h^2-4n)u_{(\alpha_1,\alpha_2)}. \end{aligned} \tag{6.25}$$

where $r(h)$ and n as in (6.1.3). $\square$

Now we are ready to prove Theorem 6.1.1.

Proof of Theorem 6.1.1. Assume that N is a non-zero submodule of M_λ^C. Choose a non-zero element $v \in N$

Case 1: If $0 \neq v = f(h) \in N$ then by Lemma 6.1.2, $1 \in N$, hence $N = M_\lambda^C$.

Case 2: For the remaining case, write $v = f(h) + g(h)B \in N$ with $g(h) \neq 0$ and the degree of $f(h)$ is minimal. If $\deg(f(h)) \neq$, we can apply the actions by C^2 or CB

and reduce the degree of the polynomial $f(h)$. Hence, $f(h)$ must be constant and we may assume that $v = \gamma + g(h)B$, where $\gamma \in \mathbb{C}$ and $g(h) \neq 0$.

We have

$$B.v = \gamma B + q(h)$$

where,

$$q(h) = B.g(h)B \equiv \frac{1}{2}(g(h-2)(\mu - h^2 + 2h) + g(h+2)(\mu - h^2 - 2h)), \tag{6.26}$$

and $\deg(q(h)) \geq 2$. If $\gamma = 0$ then $0 \neq B.v = q(h) \in N$, leading to Case 1. If $0 \neq \gamma$, then

$$0 \neq w(h) = \frac{1}{\gamma} g(h)B.v - v = \frac{1}{\gamma} g(h)q(h) - \gamma \in N, \tag{6.27}$$

which also reduces to the first case. As a result, M_λ^C is a simple module. □

Corollary 6.1.9. *Let $\lambda \in \mathbb{C} \backslash 2\mathbb{Z}$. Then M_λ^C is a simple $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module of rank* 2. □

6.2 The submodule N_λ^C

Now assume $\lambda \in 2\mathbb{Z}$, and consider the subspace

$$\begin{aligned} N_\lambda^C &= \mathbb{C}[h]r(h) \oplus \mathbb{C}[h](C.r(h)) \\ &= \mathbb{C}[h]r(h) \oplus \mathbb{C}[h](r(h-2) - r(h+2))B. \end{aligned}$$

Lemma 6.2.1. *Let $\lambda \in 2\mathbb{Z}$. Then N_λ^C is a submodule of the module M_λ^C.*

Proof. Let $0 \neq u = f(h)r(h) + g(h)(C.r(h)) \in N_\lambda^C$. It is clear that the action of h leaves N_λ^C invariant. Now

$$\begin{aligned} B.u =& Bf(h)r(h) + Bg(h)(C.r(h)) \\ \equiv& (f'(h)B + f''(h)C)r(h) + (g'(h)B + g''(h)C)(C.r(h)) \\ \equiv& f'(h)B(r(h)) + f''(h)C(r(h)) + g'(h)B(C.r(h)) \\ &+ g''(h)C(C.r(h)) \end{aligned}$$

$$\begin{aligned} &\equiv f'(h)B(r(h)) + f''(h)C(r(h)) + g'(h)BC.r(h) + g''(h)C^2r(h) \\ \text{by (6.13)} \qquad &\equiv f'(h)\,(\,B.r(h)\,) + f''(h)\,(\,C.r(h)\,) \\ &\quad + (\,2nhg'(h) - 4n^2g''(h)\,)\,r(h), \end{aligned} \tag{6.28}$$

where

$$\begin{aligned} f'(h) &= \frac{1}{2}(f(h-2) + f(h+2)) \\ f''(h) &= \frac{1}{2}(f(h-2) - f(h+2)). \end{aligned} \tag{6.29}$$

Similar equations hold for $g'(h)$ and $g''(h)$.

Note that the second and third terms in (6.28), are in N_λ^C. Hence, it is enough to show that $B.r(h) \equiv \frac{1}{2}\,(\,r(h-2) + r(h+2)\,)\,B$ is in N_λ^C.

Using (6.5), (6.13), and $\mu = 4(n^2 - n)$, we find that

$$\begin{aligned} C^2.r(h) \equiv& -\frac{1}{4}((h^2 - 2h - \mu)r(h-4) + 2(\mu - h^2)r(h) \\ &+ (h^2 + 2h - \mu)r(h+4)) \\ \equiv& -4n^2r(h). \end{aligned}$$

It follows that

$$
\begin{aligned}
&(h^2 - 2h - \mu)r(h-4) + (h^2 + 2h - \mu)r(h+4) \\
&\equiv (8n^2 + 8n + 2h^2)r(h). \qquad (6.30)
\end{aligned}
$$

Similarly, using (6.7), (6.13), and $\mu = 4(n^2 - n)$, we get

$$
\begin{aligned}
CB.r(h) \equiv &\frac{1}{4}((-h^2 + 2h + \mu)r(h-4) + 4hr(h) \\
&+ (h^2 + 2h - \mu)r(h+4)) \\
\equiv &2(n+1)hr(h).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&(h^2 - 2h - \mu)r(h-4) - (h^2 + 2h - \mu)r(h+4)) \qquad (6.31) \\
&= (-8nh - 4h)r(h).
\end{aligned}
$$

Adding (6.30) and (6.31), and canceling out 2, we get

$$
\left(h^2 - 2h - 4(n^2 - n)\right) r(h-4) = \left(h^2 - (4n+2)h + (4n^2 + 4n)\right) r(h)
$$

or

$$
(h - 2n)(h + (2n-2))r(h-4) = (h - 2n)(h - (2n+2))r(h),
$$

which implies that

$$
(h + (2n-2))r(h-4) = (h - (2n+2))r(h). \qquad (6.32)
$$

If we replace h by $h+2$ in (6.32), we get the relation

$$
(h + 2n)r(h-2) = (h - 2n)r(h+2) \qquad (6.33)
$$

Using (6.33), we obtain

$$\begin{aligned}
-\frac{1}{4n}(h-2n)\,(\,r(h-2)-r(h+2)\,) &= -\frac{1}{4n}((h-2n)r(h-2)-(h-2n)r(h+2)) \\
\text{by (6.33)} \qquad &= -\frac{1}{4n}((h-2n)r(h-2)-(h+2n)r(h-2)) \\
&= r(h-2). \qquad (6.34)
\end{aligned}$$

Similarly,

$$\begin{aligned}
-\frac{1}{4n}(h+2n)(r(h-2)-r(h+2)) &= -\frac{1}{4n}((h+2n)r(h-2)-(h+2n)r(h+2)) \\
\text{by (6.33)} \qquad &= -\frac{1}{4n}((h-2n)r(h+2)-(h+2n)r(h+2)) \\
&= r(h+2). \qquad (6.35)
\end{aligned}$$

As a result,

$$\begin{aligned}
B.r(h) =&\frac{1}{2}\,(\,r(h-2)+r(h+2)\,)\,B \\
\equiv&\frac{1}{2}\,(\,\frac{(-1)}{4n}(h-2n)\,(\,r(h-2)-r(h+2)\,)\,B \\
&+\frac{(-1)}{4n}(h+2n)(r(h-2)-r(h+2))B\,)
\end{aligned}$$

$$\begin{aligned}
\text{by (6.1)} \qquad &\equiv \frac{1}{2}\,(\,\frac{(-1)}{4n}(h-2n)(2C.r(h))+\frac{(-1)}{4n}(h+2n)(2C.r(h))\,) \\
&\equiv \frac{(-1)}{2n}h(C.r(h)) \in N_\lambda^C. \qquad (6.36)
\end{aligned}$$

A similar calculation shows that $C.u \in N_\lambda^C$. Hence N_λ^C is a submodule of M_λ^C. □

Remark 6.2.2. *If the first two relations in Corollaries (6.1.3), (6.1.6), and (6.1.8) hold true, then so do the last two.*

Theorem 6.2.3 (Main Theorem 2)**.** *Let $\lambda \in 2\mathbb{Z}$. Then M_λ^C has a unique maximal (graded) submodule N_λ^C such that $N_\lambda^C = P \oplus Q$, where P and Q are simple $\mathfrak{sl}_2(\mathbb{C})$-modules of rank* 1*.*

Proof. Let W be a non-zero submodule of M_λ^C. Choose a non-zero element $u = f(h) + g(h)B \in W$. We can apply the actions by C^2 or CB and reduce the degree of the polynomial $f(h)$ until we get either a constant or $r(h)$ plus $\bar{g}(h)B$ in W, for some polynomial $\bar{g}(h)$. Hence we can reduce the cases for the element u, up to the scalar multiplication, to the following cases: $u = f(h)$, $u = g(h)B$, $u = 1 + g(h)B$, and $u = r(h) + g(h)B$, where $g(h) \neq 0$ in Cases 2,3 and 4.

Case 1: If $0 \neq u = f(h) \in W$, then by Corollary 6.1.4, either $W = M_\lambda^C$ or $W = N_\lambda^C$.

Case 2: If $u = g(h)B$ where $g(h) \neq 0$, then, by (6.26), we have that $B.u$ is a non-zero element in $\mathbb{C}[h]$ in W, which returns us to Case 1.

Case 3: If $u = 1+g(h)B$ with $g(h) \neq 0$, then by (6.27), we have that $g(h)B.u-u$ is a non-zero element in $\mathbb{C}[h]$ in W, which also means that either $W = M_\lambda^C$ or $W = N_\lambda^C$.

Case 4: If $u = r(h) + g(h)B$ and $g(h) \neq 0$, then

$$C^2.u - (-4n^2u) = g^*(h)B \in W,$$

for some polynomial $g^*(h)$. If $g^*(h) \neq 0$, then by Case 2, either $W = M_\lambda^C$ or $W = N_\lambda^C$. Otherwise, $C^2u \equiv -4n^2u$, so that we can try again the element

$$CB.u - 2(n+1)h.u = g^{**}(h)B \in W,$$

for some polynomial $g^{**}(h)$. If $g^{**}(h) \neq 0$, then by Case 2, either $W = M_\lambda^C$ or $W = N_\lambda^C$. Otherwise, $CB.u \equiv 2(n+1)hu$, which means by Corollary (6.1.8), that $u = r(h) + \alpha(C(r(h)))$ where $\alpha \in \mathbb{C}^*$. In this case, $\alpha C.u = \alpha C(r(h)) - 4n^2\alpha^2 r(h) \in W$, which implies that $\alpha C.u + u = (4n^2\alpha^2 + 1)r(h) \in W$. If $(4n^2\alpha^2+1) \neq 0$, then $r(h) \in W$, which implies that $W = N_\lambda^C$ or $W = M_\lambda^C$. Hence, the only case remaining is $u = r(h) + \alpha(C(r(h)))$ and $(4n^2\alpha^2 + 1) = 0$, in other words, when $u = r(h) \pm \frac{i}{2n}(C.r(h))$.

Lemma 6.2.4. *Let $\lambda \in 2\mathbb{Z}$, and $r(h)$ as in Corollary 6.1.3. Then*

$P = \mathbb{C}[h]\left(r(h) + \frac{i}{2n}(C.r(h))\right)$ and $Q = \mathbb{C}[h]\left(r(h) - \frac{i}{2n}(C.r(h))\right)$ are submodules of M_λ^C, which are torsion-free modules of rank one.

Proof. Using Corollary (6.1.3) and (6.36) we have,

$$\begin{aligned} B.(\ r(h) + \frac{i}{2n}(C.r(h))\ ) =& B.r(h) + \frac{i}{2n}(BC.r(h)) \\ \equiv& \frac{(-1)}{2n}hC.r(h) + ihr(h) \\ \equiv& ih\ (\ \frac{(-1)}{2ni}C.r(h) + r(h)\ ) \\ =& ih\ (\ r(h) + \frac{i}{2n}C.r(h)\ ) \end{aligned}$$

and

$$\begin{aligned} C.(r(h) + \frac{i}{2n}(C.r(h))) =& C.r(h) + \frac{i}{2n}(C^2 r(h)) \\ \equiv& C.r(h) + \frac{i}{2n}(-4n^2 r(h)) \\ \\ \equiv& (C.r(h) - i(2nr(h))) \\ \equiv& -2ni\ (\ r(h) + \frac{i}{2n}C.r(h)\ ) \end{aligned}$$

For arbitrary elements, let $u = f(h)(r(h) + \frac{i}{2n}(C.r(h)))$. It is clear that the action by h is invariant. Now

$$\begin{aligned}
B.u =& B\,(\ f(h)\,(\ r(h)+\frac{i}{2n}(C.r(h))\ )\ ) \\
\equiv& (\ f'(h)B+f''(h)C\ )\,(\ r(h)+\frac{i}{2n}(C.r(h))\ ) \\
\equiv& ihf'(h)(r(h)+\frac{i}{2n}C.r(h))-2nif''(h)(r(h)+\frac{i}{2n}C.r(h)) \\
\equiv& (\ ihf'(h)-2nif''(h)\ )\,(\ r(h)+\frac{i}{2n}C.r(h)\ )
\end{aligned}$$

where $f'(h)$ and $f''(h)$ as in 6.29. Also

$$\begin{aligned}
C.u =& C\,(\ f(h)(r(h)+\frac{i}{2n}(C.r(h)))\ ) \\
\equiv& (\ f''(h)B+f'(h)C\ )\,(\ r(h)+\frac{i}{2n}(C.r(h))\ )
\end{aligned}$$

$$\begin{aligned}
\equiv& ihf''(h)(r(h)+\frac{i}{2n}C.r(h))-2nif'(h)(r(h)+\frac{i}{2n}C.r(h)) \\
\equiv& (\ ihf''(h)-2nif'(h)\ )\,(\ r(h)+\frac{i}{2n}C.r(h)\ )\,,
\end{aligned}$$

which belongs to P. Hence P is a submodule of N_λ^C. A similar calculation shows that Q is also another submodule of N_λ^C.

□

Hence M_λ^C has exactly 3 proper submodules, $N_\lambda^C, P,$ and Q. Moreover, N_λ^C is the unique maximal submodule of the module M_λ^C. It is clear that $N_\lambda^C = P+Q$; suppose that $0 \neq v \in P\cap Q$. Then v can be written as $v = f(h)\left(r(h)+\frac{i}{2n}(C.r(h))\right)$ for some $0 \neq f(h) \in \mathbb{C}[h]$, and $v = g(h)\left(r(h)-\frac{i}{2n}(C.r(h))\right)$. It follows that $f(h)g(h)r(h) \in P\cap Q$, which means that $P\cap Q$ is either M_λ^C or N_λ^C. This is a contradiction and so $N_\lambda^C = P\oplus Q$.

□

Corollary 6.2.5. *Let $\lambda \in 2\mathbb{Z}$. Then N_λ^C is a $\mathbb{Z}_2^2$-graded simple $\mathfrak{sl}_2(\mathbb{C})$-module.*

Proof. Since P and Q are of rank 1, it follows that they are not graded submodules, see [BS19], which are the only submodules of N_λ^C. Since N_λ^C is the unique maximal submodule of the graded module M_λ^C, it follows that N_λ^C is a graded module which has no graded proper submodules. Hence N_λ^C is graded-simple. $\square$

Remark 6.2.6. *Given $\lambda \in 2\mathbb{Z}$, let us consider the quotient module $V = M_\lambda^C/N_\lambda^C = \mathbb{C}[h] \oplus \mathbb{C}[h]B/\mathbb{C}[h]r(h) \oplus \mathbb{C}[h]r^*(h)B$, where $r^*(h)B = C.r(h)$. Since the polynomials $r(h)$, $r^*(h)$ have degrees n, $n-1$, respectively, the module V is a finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-module, with $\dim(V) = 2n-1$, hence a weight module. Moreover, since the module N_λ^C is maximal in M_λ^C, V is simple. Using [Maz09, Theorem 3.32], we have*

$$V \cong V(2n-2) = \begin{cases} V(\lambda) & \text{If } \lambda \geq 0, \\ V(-(\lambda+2)) & \text{If } \lambda < 0. \end{cases}$$

Remark 6.2.7. *Note that, the classification of the $\mathfrak{sl}_2(\mathbb{C})$-modules, depends on the choice of the Cartan subalgebra. That is, if we change the Cartan subalgebra, a torsion-free $\mathfrak{sl}_2(\mathbb{C})$-module may become a weight $\mathfrak{sl}_2(\mathbb{C})$-module, and vice versa. Two examples of this phenomenon are given below.*

Example 6.2.8. *If we change our regular Cartan subalgebra $H = \langle h \rangle$, to the Cartan subalgebra H_1 spanned by the element $h + 4y$, then the simple module $N(2, -\frac{1}{2})$ in* 3.2.8*, becomes a weight $\mathfrak{sl}_2(\mathbb{C})$-module, see [Nil15, Remark 10].*

Another example is our module M_λ^C.

Example 6.2.9. *Assume $V = M_\lambda^C$. Let us choose $\lambda \notin 2\mathbb{Z}$ to ensure the simplicity of M_λ^C. Consider the Cartan subalgebra H_2 spanned by the element C. Since $C.1 = 0$,* *it follows that* 1 *is an eigenvector for* C. *By Theorem 3.2.2, we conclude that* M_λ^C *is a weight* $\mathfrak{sl}_2(\mathbb{C})$*-module, with respect to the new Cartan subalgebra. Furthermore,*

$$V = \bigoplus_{k \in \mathbb{Z}} V_{2k} \tag{6.37}$$

where the weight space V_{2k} *is*

$$V_{2k} = \begin{cases} \left\langle (B + ih)^k \right\rangle & \text{if } k \geq 0, \\ \left\langle (B - ih)^k \right\rangle & \text{if } k < 0. \end{cases} \tag{6.38}$$

Thus, $\operatorname{Supp}(V) = 2\mathbb{Z}$, *which implies that* V *is a simple dense module, that is* $V \cong V(\varepsilon, \tau)$, *where* $\varepsilon = \overline{0}$ *and* $\tau = (\lambda + 1)^2$.

Chapter 7

Tensor products of M_λ^C with simple graded finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-modules

In this Chapter, we study tensor products of M_λ^C with simple graded finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-modules. We first decompose $M_\lambda^C \otimes V(2)$, as the sum of simple graded submodules. Then we use this result to find the decompositions of $M_\lambda^C \otimes V(n)$.

7.1 Tensor products of M_λ^C and $V(2)$

We start with studying the tensor product of the module M_λ^C and $V(2)$. We chose $V(2)$ because the simple decomposition $M_\lambda^C \otimes V(2)$ could provide us with new simple graded $\mathfrak{sl}_2(\mathbb{C})$-modules, in addition to M_λ^C. Without loss of generality, we will consider the module $V(2)$ as the adjoint module of $\mathfrak{sl}_2(\mathbb{C})$. Clearly, this inherits a $\mathbb{Z}_2^2$-grading from $\mathfrak{sl}_2(\mathbb{C})$ itself. Actually, each $V(2n)$ can be endowed by a $\mathbb{Z}_2$-grading, as shown in [EK15]. In Section 4.1 we gave an explicit description of these gradings.

Recall that we use the term "Casimir constants" to refer to the Casimir eigenvalues.

We will use the standard basis $\{x, h, y\}$ of $\mathfrak{sl}_2(\mathbb{C})$ as the basis of $V(2)$. If $\lambda \in \mathbb{C}$, then we define the module

$$L_{(\lambda,2)} := M_\lambda^C \otimes V(2). \tag{7.1}$$

Hence, any element $z \in L_{(\lambda,2)}$, z can be written as

$$\begin{aligned} z = f_1(h) \otimes x + & f_2(h)B \otimes x + f_3(h) \otimes h + f_4(h)B \otimes h \\ & + f_5(h) \otimes y + f_6(h)B \otimes y. \end{aligned} \tag{7.2}$$

For simplicity, we will write the element z in (7.2) as

$$z = (\ f_1(h), f_2(h), f_3(h), f_4(h), f_5(h), f_6(h)\ ).$$

Occasionally, we will use both notations, as needed.

An important fact which we will be using is the following theorem due to

B. Kostant [Kos75, Theorem 5.1].

Theorem 7.1.1. *Let c be the Casimir element of $U(\mathfrak{sl}_2(\mathbb{C}))$, and let M be an $\mathfrak{sl}_2(\mathbb{C})$-module on which c acts as the scalar ρ. Then for any finite dimensional module $V(n)$, the element*

$$\prod_{\mu_i \in \{n, n-2, \ldots, -n\}} (c - (\sqrt{\rho} + \mu_i)^2)$$

annihilates the module $M \otimes V(n)$. □

7.1.1 Actions on $L_{(\lambda,2)}$

The action of the elements $h, x, y \in \mathfrak{sl}_2(\mathbb{C})$ on the module $L_{(\lambda,2)}$ are given by the following :

$$\begin{aligned}
& h.(f_1(h), f_2(h), f_3(h), f_4(h), f_5(h), f_6(h)) = \\
& h.(f_1(h) \otimes x + f_2(h)B \otimes x + f_3(h) \otimes h + f_4(h)B \otimes h \\
+ \quad & f_5(h) \otimes y + f_6(h)B \otimes y) \\
= \quad & hf_1(h) \otimes x + 2f_1(h) \otimes x + hf_2(h)B \otimes x \\
+ \quad & 2f_2(h)B \otimes x + hf_3(h) \otimes h + hf_4(h)B \otimes h \\
+ \quad & hf_5(h) \otimes y - 2f_5(h) \otimes y + hf_6(h)B \otimes y - 2f_6(h)B \otimes y \\
= \quad & (h+2)f_1(h) \otimes x + (h+2)f_2(h)B \otimes x + hf_3(h) \otimes h \\
+ \quad & hf_4(h)B \otimes h + (h-2)f_5(h) \otimes y + (h-2)f_6(h)B \otimes y \\
= \quad & ((h+2)f_1(h), (h+2)f_2(h), hf_3(h), hf_4(h), \\
& (h-2)f_5(h), (h-2)f_6(h)).
\end{aligned}$$

Next,

$$\begin{aligned}
& x.\,(f_1(h), f_2(h), f_3(h), f_4(h), f_5(h), f_6(h)) \\
= \quad & x.\,(f_1(h) \otimes x + f_2(h)B \otimes x + f_3(h) \otimes h + f_4(h)B \otimes h \\
+ \quad & f_5(h) \otimes y + f_6(h)B \otimes y)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{2}(\mu - h^2 + 2h)f_2(h-2) - 2f_3(h)\right) \otimes x \\
&+ \left(\frac{1}{2}f_1(h-2) - 2f_4(h)\right) B \otimes x \\
&+ \left(\frac{1}{2}(\mu h^2 + 2h)f_3(h-2)\right. \\
&\quad \left. + f_5(h)) \otimes h + (\frac{1}{2}f_3(h-2) + f_6(h)\right) B \otimes h \\
&+ \left(\frac{1}{2}(\mu - h^2 + 2h)f_6(h-2)\right) \otimes y + \left(\frac{1}{2}f_5(h-2))B \otimes y\right)
\end{aligned}
$$

$$
\begin{aligned}
= \; &\left(\frac{1}{2}(\mu - h^2 + 2h)f_2(h-2) - 2f_3(h), \frac{1}{2}f_1(h-2) - 2f_4(h)\ ,\right. \\
&\frac{1}{2}(\mu h^2 + 2h)f_3(h-2) + f_5(h), \frac{1}{2}f_3(h-2) + f_6(h)\ , \\
&\left.\frac{1}{2}(\mu - h^2 + 2h)f_6(h-2), \frac{1}{2}f_5(h-2)\right).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&y.\,(\,f_1(h), f_2(h), f_3(h), f_4(h), f_5(h), f_6(h)\,) = \\
&\quad x.(f_1(h) \otimes x + f_2(h)B \otimes x + f_3(h) \otimes h + f_4(h)B \otimes h \\
&\quad + f_5(h) \otimes y + f_6(h)B \otimes y) \\
&= \frac{1}{2}(\mu - h^2 - 2h)f_2(h+2) \otimes x + \frac{1}{2}f_1(h+2)B \otimes x \\
&+ (\frac{1}{2}(\mu h^2 - 2h)f_3(h+2) - f_1(h)) \otimes h + (\frac{1}{2}f_3(h+2) - f_2(h))B \otimes h \\
&+ (\frac{1}{2}(\mu - h^2 - 2h)f_6(h+2) + 2f_3(h)) \otimes y \\
&+ (\frac{1}{2}f_5(h+2) + 2f_4(h))B \otimes y
\end{aligned}
$$

$$
\begin{aligned}
&= (\, \frac{1}{2}(\mu - h^2 - 2h) f_2(h+2), \frac{1}{2} f_1(h+2), \\
&\quad \frac{1}{2}(\mu h^2 - 2h) f_3(h+2) - f_1(h), \frac{1}{2} f_3(h+2) - f_2(h), \\
&\quad \frac{1}{2}(\mu - h^2 - 2h) f_6(h+2) + 2 f_3(h), \frac{1}{2} f_5(h+2) + 2 f_4(h) \,)\,.
\end{aligned} \tag{7.3}
$$

The action of yx is also needed, which help us determine the action of the Casimir element c. Using our previous computations, we have:

$$
\begin{aligned}
4yx.&(\, f_1(h), f_2(h), f_3(h), f_4(h), f_5(h), f_6(h) \,) \\
&= (\, (\mu - h^2 - 2h) f_1(h) - 4(\mu - h^2 - 2h) f_4(h+2), \\
&\quad (\mu - h^2 - 2h) f_2(h) - 4 f_3(h+2), \\
&\quad (\mu - h^2 - 2h + 8) f_3(h) + 2(\mu - h^2 - 2h) f_6(h+2) \\
&\quad - (\mu - h^2 + 2h) f_2(h-2), \\
&\quad (\mu - h^2 - 2h + 8) f_4(h) + 2 f_5(h+2) - 2 f_1(h-2), \\
&\quad (\mu - h^2 - 2h + 8) f_5(h) + 4(\mu - h^2 + 2h) f_4(h-2), \\
&\quad (\mu - h^2 - 2h + 8) f_6(h) + 4 f_3(h-2) \,)\,.
\end{aligned} \tag{7.4}
$$

Hence

$$
\begin{aligned}
c.&(\, f_1(h), f_2(h), f_3(h), f_4(h), f_5(h), f_6(h) \,) \qquad (7.5) \\
&= (\, (4h + \mu + 9) f_1(h) - 4(\mu - h^2 - 2h) f_4(h+2), \\
&\quad (4h + \mu + 9) f_2(h) - 4 f_3(h+2),
\end{aligned}
$$

$$
\begin{aligned}
&(\mu+9)f_3(h)+2(\mu-h^2-2h)f_6(h+2)\\
&-2(\mu-h^2+2h)f_2(h-2),\\
&(9+\mu)f_4(h)+2f_5(h+2)-2f_1(h-2),\\
&(-4h+\mu+9)f_5(h)+4(\mu-h^2+2h)f_4(h-2),\\
&(-4h+\mu+9)f_6(h)+4f_3(h-2)\ ).
\end{aligned}
$$

Let L^α be a maximal submodule of $L_{(\lambda,2)}$ where c acts as a scalar α. Then for any $f_1(h),\ldots,f_6(h)\in\mathbb{C}[h]$ such that

$$(f_1(h),f_2(h),f_3(h),f_4(h),f_5(h),f_6(h)\ )\in L^\alpha$$

we must have

$$(c-\alpha).(\ f_1(h),f_2(h),f_3(h),f_4(h),f_5(h),f_6(h)\ )=0. \tag{7.6}$$

Using (7.5) and (7.6) we have the following two systems of equations,

$$(4h+\mu+9-\alpha)f_1(h)-4(\mu-h^2-2h)f_4(h+2)=0, \tag{7.7}$$

$$(\mu+9-\alpha)f_4(h)+2f_5(h+2)-2f_1(h-2)=0, \tag{7.8}$$

$$(-4h+\mu+9-\alpha)f_5(h)+4(\mu-h^2+2h)f_4(h-2)=0, \tag{7.9}$$

and

$$(4h+\mu+9-\alpha)f_2(h)-4f_3(h+2)=0, \tag{7.10}$$

$$(\mu+9-\alpha)f_3(h)+2(\mu-h^2-2h)f_6(h+2)-2(\mu-h^2+2h)f_2(h-2)=0, \tag{7.11}$$

$$(-4h+\mu+9-\alpha)f_6(h)+4f_3(h-2)=0. \tag{7.12}$$

By equation (7.7) we have

$$f_1(h-2) = \frac{4(\mu - h^2 + 2h) f_4(h)}{4h + \mu + 1 - \alpha}, \tag{7.13}$$

and by (7.9) we have

$$f_5(h+2) = \frac{-4(\mu - h^2 - 2h) f_4(h)}{-4h + \mu + 1 - \alpha}. \tag{7.14}$$

Let $f_4(h) \neq 0$. Applying (7.13) and (7.14) to (7.8) we have

$$\alpha^3 - (3\mu + 11)\alpha^2 + (3\mu^2 + 6\mu + 19)\alpha - (\mu^3 - 5\mu^2 + 3\mu + 9) = 0, \tag{7.15}$$

which implies that we have three solution of (7.15)

$$\alpha_1 = \mu + 1 = (\sqrt{\mu + 1} + 0)^2; \tag{7.16}$$

$$\alpha_2 = \mu + 5 + 4\sqrt{\mu + 1} = (\sqrt{\mu + 1} + 2)^2; \tag{7.17}$$

$$\alpha_3 = \mu + 5 - 4\sqrt{\mu + 1} = (\sqrt{\mu + 1} - 2)^2. \tag{7.18}$$

This is in agreement with the values in Theorem 7.1.1.

In this case, we have

$$f_1(h) = \frac{4(\mu - h^2 - 2h) f_4(h+2)}{4h + \mu + 9 - \alpha}, \tag{7.19}$$

and

$$f_5(h) = \frac{-4(\mu - h^2 + 2h) f_4(h-2)}{-4h + \mu + 9 - \alpha}. \tag{7.20}$$

Similarly, solving the second system of equations, we will get the same values of $\alpha_1, \alpha_2,$ and α_3. Moreover, we have

$$f_2(h) = \frac{4 f_3(h+2)}{4h + \mu + 9 - \alpha}, \tag{7.21}$$

and

$$f_6(h) = \frac{-4 f_3(h-2)}{-4h + \mu + 9 - \alpha}. \tag{7.22}$$

7.1.2 The submodule $\widetilde{L}_0$ where c acts as a scalar $\mu + 1$

Let $\mu \neq 0$, and $\alpha = \mu + 1$ (the Casimir constant related to the weight 0). In this case, we have

$$f_1(h) = \frac{(\mu - h^2 - 2h)f_4(h+2)}{h+2}, \tag{7.23}$$

$$f_5(h) = \frac{(\mu - h^2 + 2h)f_4(h-2)}{h-2}, \tag{7.24}$$

$$f_2(h) = \frac{f_3(h+2)}{h+2}, \tag{7.25}$$

$$f_6(h) = \frac{f_3(h-2)}{h-2}. \tag{7.26}$$

Since $\mu \neq 0$ and $f_1(h), f_2(h), f_5(h), f_6(h) \in \mathbb{C}[h]$, it follows that $f_3(h)$ and $f_4(h)$ must belong to $\mathbb{C}[h].h$.

Let $\widetilde{L}_0 = L^{\mu+1}$ be the submodule of $L_{(\lambda,2)}$ consisting of all eigenvectors of c with eigenvalue $\mu + 1$. Then any $v \in \widetilde{L}_0$ must be written as

$$\begin{aligned} v = (\ &(\mu - h^2 - 2h)g(h+2), f(h+2), hf(h), \\ &hg(h), (\mu - h^2 + 2h)g(h-2), f(h-2)\ ) \end{aligned} \tag{7.27}$$

for some $f(h), g(h) \in \mathbb{C}[h]$.

Proposition 7.1.2. *Let $\mu \neq 0$. Then $\widetilde{L}_0$ is isomorphic to M_λ^C.*

Proof. Define $\varphi : M_\lambda^C \longrightarrow \widetilde{L}_0$, where

$$\varphi(f(h)+g(h)B) = (\ (\mu - h^2 - 2h)g(h+2), f(h+2), hf(h), \\ hg(h), (\mu - h^2 + 2h)g(h-2), f(h-2)\ ). \tag{7.28}$$

It easy to show that φ is a linear bijective map, hence it is enough to show that φ is a module homomorphism. Let $f(h), g(h) \in \mathbb{C}[h]$, we have

$$\varphi(x.(f(h)+g(h)B)) = \varphi(\frac{1}{2}(\mu - h^2 + 2h)g(h-2) + \frac{1}{2}f(h-2)B) \\ = (\frac{1}{2}(\mu - h^2 - 2h)f(h), \frac{1}{2}(\mu - h^2 - 2h)g(h), \frac{1}{2}h(\mu - h^2 + 2h)g(h-2), \\ \frac{1}{2}hf(h-2), \frac{1}{2}(\mu - h^2 + 2h)f(h-4), \frac{1}{2}(\mu - h^2 + 6h - 8)g(h-4)\ ).$$

Also,

$$x.\varphi(f(h)+g(h)B) = x.(\ (\ (\mu - h^2 - 2h)g(h+2), f(h+2), hf(h), \\ hg(h), (\mu - h^2 + 2h)g(h-2), f(h-2)\ )\ )$$

$$= (\frac{1}{2}(\mu - h^2 - 2h)f(h), \frac{1}{2}(\mu - h^2 - 2h)g(h), \frac{1}{2}h(\mu - h^2 + 2h)g(h-2), \\ \frac{1}{2}hf(h-2), \frac{1}{2}(\mu - h^2 + 2h)f(h-4), \frac{1}{2}(\mu - h^2 + 6h - 8)g(h-4)\ )$$

$$= \varphi(x.(f(h)+g(h)B)).$$

Now

$$\varphi(y.(f(h)+g(h)B)) = \varphi(\frac{1}{2}(\mu - h^2 - 2h)g(h+2) + \frac{1}{2}f(h+2)B)$$

$$= (\frac{1}{2}(\mu - h^2 - 2h)f(h+4), \frac{1}{2}(\mu - h^2 - 6h - 8)g(h+4),$$
$$\frac{1}{2}h(\mu - h^2 - 2h)g(h+2), \frac{1}{2}hf(h+2),$$
$$\frac{1}{2}(\mu - h^2 + 2h)f(h), \frac{1}{2}(\mu - h^2 + 2h)g(h)\ ).$$

$$y.\varphi(f(h) + g(h)B) = y(\ (\ (\mu - h^2 - 2h)g(h+2), f(h+2), hf(h),$$
$$hg(h), (\mu - h^2 + 2h)g(h-2), f(h-2)\ )\ )$$

$$= (\frac{1}{2}(\mu - h^2 - 2h)f(h+4), \frac{1}{2}(\mu - h^2 - 6h - 8)g(h+4), \frac{1}{2}h(\mu - h^2 - 2h)g(h+2),$$
$$\frac{1}{2}hf(h+2), \frac{1}{2}(\mu - h^2 + 2h)f(h), \frac{1}{2}(\mu - h^2 + 2h)g(h)\ )$$
$$= \varphi(y.(f(h) + g(h)B)).$$

Again,

$$\varphi(h.(f(h) + g(h)B)) = \varphi(hf(h) + hg(h)B)$$

$$= (\ (\mu - h^2 - 2h)(h+2)g(h+2), (h+2)f(h+2), h^2 f(h),$$
$$h^2 g(h), (\mu - h^2 + 2h)(h-2)g(h-2), (h-2)f(h-2)\ )$$
$$= h.\varphi(f(h) + g(h)B)$$

Hence $M_\lambda^C \cong \widetilde{L}_0$. $\square$

7.1.3 The submodule $\widetilde{L}_2$ where c acts as a scalar $\mu+5+4\sqrt{\mu+1}$

Let $\alpha = \mu + 5 + 4\sqrt{\mu+1}$ (the Casimir constant related to the weight 2). Note that

$$(\mu - h^2 + 2h) = -(h - 1 - \sqrt{\mu+1})(h - 1 + \sqrt{\mu+1}),$$

and

$$(\mu - h^2 - 2h) = -(h + 1 - \sqrt{\mu + 1})(h + 1 + \sqrt{\mu + 1}).$$

Hence, in this case we have

$$f_1(h) = -\,(h + 1 + \sqrt{\mu + 1})f_4(h + 2), \tag{7.29}$$

$$f_5(h) = -\,(h - 1 - \sqrt{\mu + 1})f_4(h - 2), \tag{7.30}$$

$$f_2(h) = \frac{f_3(h + 2)}{h + 1 - \sqrt{\mu + 1}}, \tag{7.31}$$

$$f_6(h) = \frac{f_3(h - 2)}{h - 1 + \sqrt{\mu + 1}}. \tag{7.32}$$

Since $\mu \neq 0$ and $f_2(h), f_6(h) \in \mathbb{C}[h]$, it follows that $f_3(h)$ must belongs to $\mathbb{C}[h].(h - 1 - \sqrt{\mu + 1})(h + 1 + \sqrt{\mu + 1})$.

Let $\widetilde{L}_2 = L^{\mu+5+4\sqrt{\mu+1}}$ be the submodule of $L_{(\lambda,2)}$ consisting of all eigenvectors of c with eigenvalue $\mu + 5 + 4\sqrt{\mu + 1}$. Then for any $v \in \widetilde{L}_2$, v must be written as

$$\begin{aligned} v = (\ & -(h + 1 + \sqrt{\mu + 1})g(h + 2), (h + 3 + \sqrt{\mu + 1})f(h + 2), \\ & (h - 1 - \sqrt{\mu + 1})(h + 1 + \sqrt{\mu + 1})f(h), g(h), \\ & -(h - 1 - \sqrt{\mu + 1})g(h - 2), (h - 3 - \sqrt{\mu + 1})f(h - 2)\ ) \end{aligned} \tag{7.33}$$

for some $f(h), g(h) \in \mathbb{C}[h]$.

Proposition 7.1.3. *Let $\mu \neq 0$. Then $\widetilde{L}_2$ is isomorphic to $M^C_{\lambda+2}$.*

Proof. Note that in $M^C_{\lambda+2}$, the value

$$\begin{aligned}\mu' =& (\lambda+2)^2 + 2(\lambda+2)\\ =& \lambda^2 + 6\lambda + 8\\ =& \mu + 4\sqrt{\mu+1} + 4.\end{aligned}$$

Define $\varphi : M^C_{\lambda+2} \longrightarrow \widetilde{L}_2$, where

$$\begin{aligned}\varphi(f(h) + g(h)B) = \big(& -(h+1+\sqrt{\mu+1})f(h+2),\\ & -(h+3+\sqrt{\mu+1})g(h+2),\\ & -(h-1-\sqrt{\mu+1})(h+1+\sqrt{\mu+1})g(h), f(h),\\ & -(h-1-\sqrt{\mu+1})f(h-2),\\ & -(h-3-\sqrt{\mu+1})g(h-2)\ \big).\end{aligned}$$

It easy to show that φ is a linear bijective map, hence it is enough to show that φ is a module homomorphism. Let $f(h), g(h) \in \mathbb{C}[h]$. Then

$$\varphi(x.(f(h)+g(h)B)) = \varphi(\frac{1}{2}(\mu' - h^2 + 2h)g(h-2) + \frac{1}{2}f(h-2)B)$$

$$\begin{aligned}= \big(& -\frac{1}{2}(h+1+\sqrt{\mu+1})(\mu' - h^2 - 2h)g(h),\\ & -\frac{1}{2}(h+3+\sqrt{\mu+1})f(h),\\ & -\frac{1}{2}(h-1-\sqrt{\mu+1})(h+1+\sqrt{\mu+1})f(h-2),\\ & \frac{1}{2}(\mu' - h^2 + 2h)g(h-2),\\ & -\frac{1}{2}(h-1-\sqrt{\mu+1})(\mu' - h^2 + 6h - 8)g(h-4),\end{aligned}$$

$$-\frac{1}{2}(h-3-\sqrt{\mu+1})f(h-4)\;)\,,$$

and

$$\begin{aligned}
x.\varphi(f(h)+g(h)B) = x.(\; & -(h+1+\sqrt{\mu+1})f(h+2), \\
& -(h+3+\sqrt{\mu+1})g(h+2), \\
& -(h-1-\sqrt{\mu+1})(h+1+\sqrt{\mu+1})g(h), \\
& f(h), -(h-1-\sqrt{\mu+1})f(h-2), \\
& -(h-3-\sqrt{\mu+1})g(h-2)\;)
\end{aligned}$$

$$\begin{aligned}
= (\; & -\frac{1}{2}(h+1+\sqrt{\mu+1})(\mu-h^2-2h+4+4\sqrt{\mu+1})g(h), \\
& -\frac{1}{2}(h+3+\sqrt{\mu+1})f(h), \\
& -\frac{1}{2}(h-1-\sqrt{\mu+1})(h+1+\sqrt{\mu+1})f(h-2), \\
& -\frac{1}{2}(h-3-\sqrt{\mu+1})(h+1+\sqrt{\mu+1})g(h-2), \\
& -\frac{1}{2}(h-5-\sqrt{\mu+1})(\mu-h^2+2h)g(h-4), \\
& -\frac{1}{2}(h-3-\sqrt{\mu+1})f(h-4)\;)
\end{aligned}$$

Note that, in the first components,

$$\mu'-h^2-2h=(\mu+4\sqrt{\mu+1}+4-h^2-2h).$$

Also,

$$\begin{aligned}
\mu'-h^2+2h &= -\,(h-1-\sqrt{\mu'+1})(h-1+\sqrt{\mu'+1}) \\
&= -\,(h-3-\sqrt{\mu+1})(h+1+\sqrt{\mu+1}).
\end{aligned}$$

Moreover,

$$\begin{aligned}(\mu' - h^2 + 6h - 8) =& (\mu + 4\sqrt{\mu+1} - h^2 + 6h - 4) \\ =& -(h - 1 + \sqrt{\mu+1})(h - 5 - \sqrt{\mu+1})\end{aligned}$$

which make the fifth components in both equations equal. Hence $\varphi(x.(f(h+g(h)B))) = x.\varphi(f(h) + g(h)B)$. Now

$$\begin{aligned}\varphi(y.(f(h)+g(h)B)) &= \varphi(\frac{1}{2}(\mu^{'} - h^2 - 2h)g(h+2) + \frac{1}{2}f(h+2)B) \\ = (-&\frac{1}{2}(h+1+\sqrt{\mu+1})(\mu' - h^2 - 6h - 8)g(h+4), \\ &-\frac{1}{2}(h+3+\sqrt{\mu+1})f(h+4), \\ &-\frac{1}{2}(h-1-\sqrt{\mu+1})(h+1+\sqrt{\mu+1})f(h+2), \\ &\frac{1}{2}(\mu' - h^2 - 2h)g(h+2), \\ &-\frac{1}{2}(h-1-\sqrt{\mu+1})(\mu' - h^2 + 2h)g(h), \\ &-\frac{1}{2}(h-3-\sqrt{\mu+1})f(h)\ ),\end{aligned}$$

$$\begin{aligned}y.\varphi(f(h)+g(h)B) = y\,(&-(h+1+\sqrt{\mu+1})f(h+2), \\ &-(h+3+\sqrt{\mu+1})g(h+2), \\ &-(h-1-\sqrt{\mu+1})(h+1+\sqrt{\mu+1})g(h), \\ &f(h), -(h-1-\sqrt{\mu+1})f(h-2), \\ &-(h-3-\sqrt{\mu+1})g(h-2)\ )\end{aligned}$$

$$
\begin{aligned}
= (\ & -\frac{1}{2}(h+5+\sqrt{\mu+1})(\mu-h^2-2h)g(h+4), \\
& -\frac{1}{2}(h+3+\sqrt{\mu+1})f(h+4), \\
& -\frac{1}{2}(h-1-\sqrt{\mu+1})(h+1+\sqrt{\mu+1})f(h+2), \\
& -\frac{1}{2}(h+3+\sqrt{\mu+1})(h-1\sqrt{\mu+1})g(h+2), \\
& -\frac{1}{2}(h-1-\sqrt{\mu+1})(\mu-h^2+2h+4+4\sqrt{\mu+1})g(h), \\
& -\frac{1}{2}(h-3-\sqrt{\mu+1})f(h) \ ).
\end{aligned}
$$

Again, when we return to the value of $\mu' = \mu + 4\sqrt{\mu+1} + 4$, we find that

$$\varphi(y.(f(h+g(h)B))) = y.\varphi(f(h)+g(h)B).$$

Finally,

$$
\begin{aligned}
\varphi(h.(f(h)+g(h)B)) =& \varphi(hf(h)+hg(h)B) \\
= (\ & -(h+1+\sqrt{\mu+1})(h+2)f(h+2), \\
& -(h+3+\sqrt{\mu+1})(h+2)g(h+2), \\
& -h(h-1-\sqrt{\mu+1})(h+1+\sqrt{\mu+1})g(h), \\
& \ hf(h), -(h-1-\sqrt{\mu+1})(h-2)f(h-2), \\
& -(h-3-\sqrt{\mu+1})(h-2)g(h-2) \ ) \\
=& h.\varphi(f(h)+g(h)B)
\end{aligned}
$$

Hence $M^C_{\lambda+2} \cong \widetilde{L}_2$. $\square$

7.1.4 The submodule $\widetilde{L}_{-2}$ where c acts as a scalar $\mu + 5 - 4\sqrt{\mu+1}$

Let $\mu \neq 0$ and let $\alpha = \mu + 5 - 4\sqrt{\mu+1}$ (the Casimir constant related the the weight -2). Since

$$(\mu - h^2 + 2h) = -(h - 1 - \sqrt{\mu+1})(h - 1 + \sqrt{\mu+1}),$$

and

$$(\mu - h^2 - 2h) = -(h + 1 - \sqrt{\mu+1})(h + 1 + \sqrt{\mu+1}),$$

it follows that,

$$f_1(h) = -(h + 1 - \sqrt{\mu+1})f_4(h+2), \tag{7.34}$$

$$f_5(h) = -(h - 1 + \sqrt{\mu+1})f_4(h-2), \tag{7.35}$$

$$f_2(h) = \frac{f_3(h+2)}{h + 1 + \sqrt{\mu+1}}, \tag{7.36}$$

$$f_6(h) = \frac{f_3(h-2)}{h - 1 - \sqrt{\mu+1}}. \tag{7.37}$$

Since $f_2(h), f_6(h) \in \mathbb{C}[h]$, it follows that

$$f_3(h) \in \mathbb{C}[h].(h - 1 + \sqrt{\mu+1})(h + 1 - \sqrt{\mu+1}).$$

Consider that $\widetilde{L}_{-2} = L^{\mu+5-4\sqrt{\mu+1}}$ be the submodule of $L_{(\lambda,2)}$ consisting of all eigenvectors of c with eigenvalue $\mu + 5 - 4\sqrt{\mu+1}$. Then for any $v \in \widetilde{L}_{-2}$, v must be written as

$$v = (\ -(h+1-\sqrt{\mu+1})g(h+2), (h+3-\sqrt{\mu+1})f(h+2), \qquad (7.38)$$
$$(h-1+\sqrt{\mu+1})(h+1-\sqrt{\mu+1})f(h), g(h),$$
$$-(h-1+\sqrt{\mu+1})g(h-2), (h-3+\sqrt{\mu+1})f(h-2)\ )$$

for some $f(h), g(h) \in \mathbb{C}[h]$.

Proposition 7.1.4. *$\widetilde{L}_{-2}$ is isomorphic to $M^C_{\lambda-2}$.*

Proof. Note that in $M^C_{\lambda-2}$, the value

$$\begin{aligned}\mu' =& (\lambda-2)^2 + 2(\lambda-2)\\ =& \lambda^2 - 2\lambda = \mu - 4\sqrt{\mu+1} + 4.\end{aligned}$$

Define $\varphi : M^C_{\lambda-2} \longrightarrow \widetilde{L}_{-2}$, where

$$\begin{aligned}\varphi(f(h)+g(h)B) = (& -(h+1-\sqrt{\mu+1})f(h+2),\\ & -(h+3-\sqrt{\mu+1})g(h+2),\\ & -(h-1+\sqrt{\mu+1})(h+1-\sqrt{\mu+1})g(h), f(h),\\ & -(h-1+\sqrt{\mu+1})f(h-2),\\ & -(h-3+\sqrt{\mu+1})g(h-2)\ ).\end{aligned}$$

It easy to show that φ is a linear bijective map, hence it is enough to show that φ is a module homomorphism. Let $f(h), g(h) \in \mathbb{C}[h]$, we have that

$$\begin{aligned}
\varphi(x.(f(h)+g(h)B)) =&\varphi(\frac{1}{2}(\mu'-h^2+2h)g(h-2)+\frac{1}{2}f(h-2)B)\\
=&\Big(-\frac{1}{2}(h+1-\sqrt{\mu+1})(\mu'-h^2-2h)g(h),\\
&-\frac{1}{2}(h+3-\sqrt{\mu+1})f(h),\\
&-\frac{1}{2}(h-1+\sqrt{\mu+1})(h+1-\sqrt{\mu+1})f(h-2),\\
&\frac{1}{2}(\mu'-h^2+2h)g(h-2),\\
&-\frac{1}{2}(h-1+\sqrt{\mu+1})(\mu'-h^2+6h-8)g(h-4),\\
&-\frac{1}{2}(h-3+\sqrt{\mu+1})f(h-4)\Big),
\end{aligned}$$

and

$$\begin{aligned}
x.\varphi(f(h)+g(h)B) =& x.\Big(-(h+1-\sqrt{\mu+1})f(h+2),\\
&-(h+3-\sqrt{\mu+1})g(h+2),\\
&-(h-1+\sqrt{\mu+1})(h+1-\sqrt{\mu+1})g(h),\\
&f(h), -(h-1+\sqrt{\mu+1})f(h-2),\\
&-(h-3+\sqrt{\mu+1})g(h-2)\Big)\\
=&\Big(-\frac{1}{2}(h+1-\sqrt{\mu+1})(\mu-h^2-2h+4-4\sqrt{\mu+1})g(h),\\
&-\frac{1}{2}(h+3-\sqrt{\mu+1})f(h),\\
&-\frac{1}{2}(h-1+\sqrt{\mu+1})(h+1-\sqrt{\mu+1})f(h-2),\\
&-\frac{1}{2}(h-3+\sqrt{\mu+1})(h+1-\sqrt{\mu+1})g(h-2),\\
&-\frac{1}{2}(h-5+\sqrt{\mu+1})(\mu-h^2+2h)g(h-4),\\
&-\frac{1}{2}(h-3+\sqrt{\mu+1})f(h-4)\Big)
\end{aligned}$$

Note that, in the first components,

$$\mu' - h^2 - 2h = (\mu - 4\sqrt{\mu+1} + 4 - h^2 - 2h).$$

Also,

$$\begin{aligned}\mu' - h^2 + 2h =& -(h - 1 + \sqrt{\mu'+1})(h - 1 - \sqrt{\mu'+1})\\ =& -(h - 3 + \sqrt{\mu+1})(h + 1 - \sqrt{\mu+1}).\end{aligned}$$

Moreover,

$$\begin{aligned}(\mu' - h^2 + 6h - 8) =& (\mu - 4\sqrt{\mu+1} - h^2 + 6h - 4)\\ =& -(h - 1 - \sqrt{\mu+1})(h - 5 + \sqrt{\mu+1})\end{aligned}$$

which make the fifth components in both equations equal. Hence

$$\varphi(x.(f(h+g(h)B))) = x.\varphi(f(h) + g(h)B).$$

Now

$$\begin{aligned}\varphi(y.(f(h)+g(h)B)) &= \varphi(\frac{1}{2}(\mu - h^2 - 2h)g(h+2) + \frac{1}{2}f(h+2))\\ = (&-\frac{1}{2}(h+1-\sqrt{\mu+1})(\mu' - h^2 - 6h - 8)g(h+4),\\ &-\frac{1}{2}(h+3-\sqrt{\mu+1})f(h+4),\\ &-\frac{1}{2}(h-1+\sqrt{\mu+1})(h+1-\sqrt{\mu+1})f(h+2),\\ &\frac{1}{2}(\mu' - h^2 - 2h)g(h+2),\\ &-\frac{1}{2}(h-1+\sqrt{\mu+1})(\mu' - h^2 + 2h)g(h),\\ &-\frac{1}{2}(h-3+\sqrt{\mu+1})f(h)\ ),\end{aligned}$$

$$
\begin{aligned}
y.\varphi(f(h)+g(h)B) = y\,(& -(h+1-\sqrt{\mu+1})f(h+2), -(h+3-\sqrt{\mu+1})g(h+2),\\
& -(h-1+\sqrt{\mu+1})(h+1-\sqrt{\mu+1})g(h), f(h),\\
& -(h-1+\sqrt{\mu+1})f(h-2), -(h-3+\sqrt{\mu+1})g(h-2)\ )\\
= (& -\frac{1}{2}(h+5-\sqrt{\mu+1})(\mu-h^2-2h)g(h+4),\\
& -\frac{1}{2}(h+3-\sqrt{\mu+1})f(h+4),\\
& -\frac{1}{2}(h-1+\sqrt{\mu+1})(h+1-\sqrt{\mu+1})f(h+2),\\
& -\frac{1}{2}(h+3-\sqrt{\mu+1})(h-1+\sqrt{\mu+1})g(h+2),\\
& -\frac{1}{2}(h-1+\sqrt{\mu+1})(\mu-h^2+2h+4-4\sqrt{\mu+1})g(h),\\
& -\frac{1}{2}(h-3-\sqrt{\mu+1})f(h)\ ).
\end{aligned}
$$

Again, when we return to the value of $\mu' = \mu - 4\sqrt{\mu+1}+4$, we find that

$$\varphi(y.(f(h+g(h)B))) = y.\varphi(f(h)+g(h)B).$$

Finally,

$$
\begin{aligned}
\varphi(h.(f(h)+g(h)B)) =& \varphi(hf(h)+hg(h)B)\\
=& (\ -(h+1-\sqrt{\mu+1})(h+2)f(h+2),\\
& -(h+3-\sqrt{\mu+1})(h+2)g(h+2),\\
& -h(h-1+\sqrt{\mu+1})(h+1-\sqrt{\mu+1})g(h),\\
& \ hf(h), -(h-1+\sqrt{\mu+1})(h-2)f(h-2),\\
& -(h-3+\sqrt{\mu+1})(h-2)g(h-2)\ )\\
=& h.\varphi(f(h)+g(h)B)
\end{aligned}
$$

Hence $M^C_{\lambda-2} \cong \widetilde{L}_{-2}$. $\square$

Note another important result due to Kostant [Kos75, Corollary 5.5]):

Lemma 7.1.5. *Let c be the Casimir element of $U(\mathfrak{sl}_2(\mathbb{C}))$, and let M be an $\mathfrak{sl}_2(\mathbb{C})$-module where c acts as the scalar ρ. Assume that the values of the Casimir constant*

$$\left\{\alpha_i := (\sqrt{\rho} + 2i)^2 \mid i = -n, -n+1, \dots, n-1, n\right\}$$

are all distinct. Define

$$P_i = \{z \in M \otimes V(n) \mid \ c.z = \alpha_i z\},$$

such that if $0 \neq P_i$, P_i is the maximal submodule of $M \otimes V(n)$ in which c acts as the scalar α_i. Then

$$M \otimes V(n) = \bigoplus_{i=-n}^{n} P_i.$$

$\square$

Note that the values of the Casimir constant are not distinct in the cases when $\mu = -1, 0$.

The following theorem summarizes all results we had so far in this section.

Theorem 7.1.6. *Let $\mu \in \mathbb{C} \setminus \{-1, 0\}$. Then*

$$L_{(\lambda,2)} = \widetilde{L}_2 \oplus \widetilde{L}_0 \oplus \widetilde{L}_{-2} \cong M^C_{\lambda+2} \oplus M^C_{\lambda} \oplus M^C_{\lambda-2}.$$

$\square$

7.1.5 Particular cases

If $\mu = 0$, then all previous calculations of $\widetilde{L}_0, \widetilde{L}_2$, and $\widetilde{L}_{-2}$ still work. The submodule $\widetilde{L}_2$ is still the maximal submodule of $L_{(\lambda,2)}$ on which c acts as the scalar $\mu + 5 +$

$4\sqrt{\mu+1} = 9$, but the difference is that $\widetilde{L}_0,$ and $\widetilde{L}_{-2}$ are not maximal with respect to the action of c. Our goal now is to find the maximal submodule on which c acts as the scalar 1. In this case we have

$$f_1(h) = -hf_4(h+2), \tag{7.39}$$

$$f_5(h) = -hf_4(h-2), \tag{7.40}$$

$$f_2(h) = \frac{f_3(h+2)}{h+2}, \tag{7.41}$$

$$f_6(h) = \frac{f_3(h-2)}{h-2}. \tag{7.42}$$

Since $f_5(h), f_6(h) \in \mathbb{C}[h]$, it follows that $f_3(h)$ must belong to $\mathbb{C}[h].h$. Let $\widehat{L^1}$ be the submodule of $L_{(\lambda,2)}$ consisting of all eigenvectors of c with eigenvalue 1. Then for any $v \in \widehat{L^1}$, v must be written as

$$\begin{aligned} v = (\, & -hg(h+2), f(h+2), hf(h), \\ & g(h), -hg(h-2), f(h-2)\,) \end{aligned} \tag{7.43}$$

for some $f(h), g(h) \in \mathbb{C}[h]$. It is clear that $\widetilde{L}_0$ and $\widetilde{L}_{-2}$ are submodules of the module $\widehat{L^1}$.

The second particular case is when $\mu = -1$ ($\lambda = -1$). All of the previous calculations for $\widetilde{L}_0$, $\widetilde{L}_2$, and $\widetilde{L}_{-2}$ work also in this case. The submodule $\widetilde{L}_0$ is still the maximal submodule of $L_{(\lambda,2)}$ on which c acts as the scalar $\mu+1=0$. The modules $\widetilde{L}_2$, and $\widetilde{L}_{-2}$ have the same Casimir constant equal to 4. Now we will find the maximal submodule on which c acts as the scalar 4. We have

$$f_1(h) = -(h+1)f_4(h+2), \tag{7.44}$$

$$f_5(h) = -(h-1)f_4(h-2), \tag{7.45}$$

$$f_2(h) = \frac{f_3(h+2)}{h+1}, \tag{7.46}$$

and

$$f_6(h) = \frac{f_3(h-2)}{h-1}. \tag{7.47}$$

Since $f_2(h), f_6(h) \in \mathbb{C}[h]$, also $f_3(h)$ must belong to $\mathbb{C}[h].(h-1)(h+1)$. Let $\widehat{L^4}$ be the subset of $L_{(\lambda,2)}$ of all elements such that c acts as the scalar 1. Then for any $v \in \widehat{L^4}$, v must be written as

$$\begin{aligned} v = (& -(h+1)g(h+2), (h+3+)f(h+2), \\ & (h-1-)(h+1+\sqrt{\mu+1})f(h), g(h), \\ & -(h-1)g(h-2), (h-3)f(h-2) \ ) \end{aligned} \tag{7.48}$$

for some $f(h), g(h) \in \mathbb{C}[h]$. which means that $\widehat{L^4} = \widetilde{L}_2 = \widetilde{L}_{-2}$.

Indeed, in the case when $\mu = -1$, it is easy to find an element $v \in L_{(\lambda,2)}$ such that $c(c-4).v \neq 0$ (for example let $v = 1 \otimes h$), which implies that $L_{(\lambda,2)}$ cannot be the direct sum of $\widetilde{L}_0$ and $\widetilde{L}_2$.

The detailed structure of this module is as follows. Consider the submodule $U = c.L_{(\lambda,2)}$. Then U has the Casimir constant 4. Then $(c-4).U \neq 0$. Hence $W = c(c-4).L_{(\lambda,2)}$ is a non-zero submodule of U which also has the Casimir constant 4, that is, $(c-4).W = 0$. Indeed, the submodule W is just the submodule $\tilde{L}_2$(or $\tilde{L}_{-2}$), which is a simple submodule. Now consider the submodule $T = (c-4).L_{(\lambda,2)}$. Then

$$\begin{tikzcd}
 & & L_{(\lambda,2)} \arrow[dl, dash] \arrow[dr, dash] & \\
 & T \arrow[dl, dash] \arrow[dr, dash] & & U \arrow[dl, dash] \\
S \arrow[dr, dash] & & W \arrow[dl, dash] & \\
 & \{0\} & &
\end{tikzcd}$$

Figure 7.1: Submodules of $L_{(\lambda,2)}$ when $\mu = -1$

T admits two of the Casimir constants, 0 and 4. The submodules $S = (c-4)^2.L_{(\lambda,2)}$ and W are simple submodules of T, which are isomorphic to $\tilde{L}_0, \tilde{L}_2$ respectively. Moreover, $L_{(\lambda,2)} = U \oplus S$.

Remark 7.1.7. *This example shows that if we have Casimir constants which are not distinct (that is, the hypotheses of Kostant's Lemma 7.1.5 are not satisfied), then the module may decompose as a decomposition of root spaces but not as a decomposition of eigenspaces, see the enclosed picture.*

The lattice of submodules of $L_{(\lambda,2)}$ when $\mu = -1$ is shown in Figure 7.1.5.

7.2 Tensor products of M_λ^C and $V(2n)$

In this part, we will give a general result about the tensor products of the module M_λ^C and a simple finite dimensional module whose highest weight is even. First we quote a classical result (see, for example, the proof in [Maz09, Theorem 1.39]).

Theorem 7.2.1. *Consider simple finite dimensional modules $V(n)$ and $V(m)$, where $m \leq n$. Then*

$$V(n) \otimes V(m) \cong \bigoplus_{i=0}^{m} V(n+m-2i).$$

$\square$

Now let us define the module

$$L_{(\lambda,2n)} = M_{\lambda}^{C} \otimes V(2n), \tag{7.49}$$

Our next theorem is a generalization of Corollary 7.1.6 to the module $L_{(\lambda,n)}$. The proof of this theorem will follow the steps of [Maz09, Theorem 3.81] and [Nil15, Corollary 18].

Theorem 7.2.2. *Let $\mu \in \mathbb{C} \setminus \{-1, 0\}$. Then*

$$L_{(\lambda,2n)} \cong \bigoplus_{i=0}^{2n} M_{\lambda+2n-2i}^{C}.$$

Proof. Consider $L = L_{(\lambda,2n)}$ with $\lambda \in \mathbb{C} \setminus \{-2, -1, 0\}$. We will prove our theorem using induction on n. For $n = 1$ the result follows directly from Theorem 7.1.6.

For $n \geq 2$, using Theorem 7.2.1, we have

$$V(2) \otimes V(2n-2) \cong V(2n) \oplus V(2n-2) \oplus V(2n-4). \tag{7.50}$$

Now using (7.50), we have

$$M_{\lambda}^{C} \otimes V(2) \otimes V(2n-2) \cong M_{\lambda}^{C} \otimes (V(2n) \oplus V(2n-2) \oplus V(2n-4)) \tag{7.51}$$

$$
\begin{aligned}
&\cong M_\lambda^C \otimes V(2n) \oplus M_\lambda^C \otimes V(2n-2) \\
&\oplus M_\lambda^C \otimes V(2n-4) \\
\text{(by induction step)} \;\; &\cong M_\lambda^C \otimes V(2n) \oplus \bigoplus_{i=0}^{2n-2} M_{\lambda+2n-2-2i}^C \\
&\oplus \bigoplus_{i=0}^{2n-4} M_{\lambda+2n-4-2i}^C \\
&= M_\lambda^C \otimes V(2n) \oplus M_{\lambda+2n-2}^C \oplus \cdots \oplus M_{\lambda-2n+2}^C \\
&\oplus M_{\lambda+2n-4}^C \oplus M_{\lambda+2n-6}^C \oplus \cdots \\
&\oplus M_{\lambda-2n+6}^C \oplus M_{\lambda-2n+4}^C
\end{aligned}
$$

Now using Theorem 7.1.6, we have

$$
\begin{aligned}
M_\lambda^C \otimes V(2) \otimes V(2n-2) &\cong (M_{\lambda+2}^C \oplus M_\lambda^C \oplus M_{\lambda-2}^C) \otimes V(2n-2) \qquad (7.52) \\
&\cong M_{\lambda+2}^C \otimes V(2n-2) \oplus M_\lambda^C \otimes V(2n-2) \\
&\oplus M_{\lambda-2}^C \otimes V(2n-2) \\
\text{(by induction step)} \;\; &\cong \bigoplus_{i=0}^{2n-2} M_{\lambda+2n-2i}^C \oplus \bigoplus_{i=0}^{2n-2} M_{\lambda+2n-2-2i}^C \\
&\oplus \bigoplus_{i=0}^{2n-2} M_{\lambda+2n-4-2i}^C \\
&= M_{\lambda+2n-2}^C \oplus \cdots \oplus M_{\lambda-2n+6}^C \oplus M_{\lambda-2n+4}^C \\
&\oplus M_{\lambda+2n-2}^C \oplus \cdots \oplus M_{\lambda-2n+4}^C \oplus M_{\lambda-2n+2}^C \\
&\oplus M_{\lambda+2n-4}^C \oplus \cdots M_{\lambda-2n+2}^C \oplus M_{\lambda-2n}^C
\end{aligned}
$$

By Theorem 3.1.13, M_λ^C has a finite length (hence both Artinian and Noetherian).

Then, Krull–Schmidt theorem applies (see e.g. [Jac09]). To complete the proof, it remains to compare (7.51) and (7.52). $\square$

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