Disjoint Skolem-type sequences and applications

by

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Abstract

Let $D = \{i_1, i_2, \ldots, i_n\}$ be a set of n positive integers. A Skolem-type sequence of order n is a sequence of i such that every $i \in D$ appears exactly twice in the sequence at position a_i and b_i, and $|b_i - a_i| = i$. These sequences might contain empty positions, which are filled with 0 elements and called hooks. For example, $(2, 4, 2, 0, 3, 4, 0, 3)$ is a Skolem-type sequence of order $n = 3$, $D = \{2, 3, 4\}$ and two hooks. If $D = \{1, 2, 3, 4\}$ we have $(1, 1, 4, 2, 3, 2, 4, 3)$, which is a Skolem-type sequence of order 4 and zero hooks, or a Skolem sequence.

In this thesis we introduce additional disjoint Skolem-type sequences of order n such as disjoint (hooked) near-Skolem sequences and (hooked) Langford sequences. We present several tables of constructions that are disjoint with known constructions and prove that our constructions yield Skolem-type sequences. We also discuss the necessity and sufficiency for the existence of Skolem-type sequences of order n where n is positive integers.

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Note:

The main results in thesis is in Chapter 3. Although the result is new and publishable, it consists of 16 cases, 8 based on paper [2]. Some of them are incorrect and the other 8 are also not all checked for correctness.

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Chapter 1

Introduction

Combinatorial design theory originated in 1776, when Euler posed the question of constructing two orthogonal latin squares of order 6 [10]. This was known as Euler's 36 Officers Problem. Euler conjectured that no solution occurs for order six. Subsequently, in 1782 [12] Euler wrote a paper, in which he conjectured that there exist orthogonal latin squares of all orders n except for $n \equiv 2 (\text{mod } 4)$.

Over the years, various designs have been discovered by combinatorial researchers such as Room Squares, Balanced Incomplete Block Designs, and 1-factorizations as well as other designs.

Informally, a combinatorial design can be defined as a way of selecting subsets from a finite set such that some conditions are satisfied. For example, suppose we have

$\{a, b, c, d, e, f, g\}$ a set of seven elements and we have to select 3-sets from the seven elements, such that each element occurs in three of the 3-sets and every intersection of two 3-sets has precisely one number. One example is $\{abc, ade, afg, bdf, beg, cdg, cef\}$, which is also called a Steiner triple system of order 7 and is denoted by $STS(7)$. It is known that an $STS(v)$ exists if and only if $v \equiv 1, 3 (\text{mod } 6)$. An $STS(v)$ is cyclic if it has an automorphism consisting of a single cycle of length v. A cyclic Steiner triple system of order v, denoted by $CSTS(v)$, exists if and only if $v \equiv 1, 3 (\text{mod } 6)$ and $v \neq 9$.

A triple system of order v and index λ, denoted $TS(v, \lambda)$, is a set V of v elements, together with a collection B of 3-element subsets of V called triples such that each 2-subset of V is a subset in precisely λ triples of B. $TS(v, \lambda)$ is cyclic if its automorphism group contains a v-cycle. A cyclic $TS(v, \lambda)$ is denoted by $CTS(v, \lambda)$.

In 1847, Kirkman studied triple systems when he formulated a problem called Kirkman Triple System Schoolgirls [15]. He posed the problem as follows: Fifteen young ladies in a school walk out three abreast for seven days in succession. It is required to arrange them daily, so that no two will walk twice abreast. Without the requirement of arranging the triples in days, the configuration is a Steiner triple system of order 15, and hence was known to Kirkman. Kirkman [16], presented his solution to this problem.

In 1897, Heffter [14], studied cyclic triple systems and he introduced his first and second difference problems to construct cyclic Steiner triple systems of order $6n+1$ and $6n+3$. Heffter's first difference problem is as follows: Can a set $\{1, 2, \ldots, 3n\}$ be partitioned into n ordered triples (a_i, b_i, c_i) with $1 \leqslant i \leqslant n$ such that $a_i + b_i \equiv c_i$ or $a_i + b_i + c_i \equiv 0 (\text{mod } 6n+1)$? If this partition is possible then $\{\{0, a_i + n, b_i + n\} : 1 \leqslant i \leqslant n\}$ will be the base blocks of a cyclic Steiner triple system of order $6n+1$. Heffter's second difference problem is as follows:

Can a set $\{1, 2, \ldots, 3n+1\} \backslash \{2n+1\}$ be partitioned into n ordered triples (a_i, b_i, c_i) with $1 \leqslant i \leqslant n$ such that $a_i + b_i \equiv c_i$ or $a_i + b_i + c_i \equiv 0 (\text{mod } 6n+3)$? If this partition is possible then $\{\{0, a_i+n, b_i+n\} : 1 \leqslant i \leqslant n\}$ with the base block $\{0, 2n+1, 4n+2\}$ having a short orbit of length $3n+1$ will be the base blocks of a cyclic Steiner triple system of order $6n+3$.

In 1957, [37] Skolem studied Steiner triple systems and constructed $STS(v)$ for $v = 6n+1$. He introduced the idea of Skolem sequences by asking if it is possible to distribute the numbers of the set $\{1, 2, \ldots, 2n\}$ into n ordered pairs (a_i, b_i) such that $b_i - a_i = i$ for $i = 1, 2, \ldots, n$. For example, the sequence $(4, 2, 3, 2, 4, 3, 1, 1)$ is a Skolem sequence of order 4. In the literature, Skolem sequences are also referred to as pure Skolem sequences. Skolem [38] proved that such a distribution exists whenever $n \equiv 0, 1 (\text{mod } 4)$. He extended his idea to that of the hooked Skolem sequence, he considered distributing the set $\{1, 2, \ldots, 2n-1, 2n+1\}$ into n ordered pairs (a_i, b_i) such that $b_i - a_i = i$ for $i = 1, 2, \ldots, n$, and conjectured that such distribution exists whenever $n \equiv 2, 3(\text{mod } 4)$. In 1961, O'Keefe [24] proved this conjecture to be true, and the solution requires leaving a space or zero for the missing integer called a hook in the $(2n)^{th}$ position. For example, the sequence $(3, 1, 1, 3, 2, 0, 2)$ is a hooked Skolem sequence of order three. The existence of (hooked) Skolem sequences of order n give a complete solution to Heffter's first problem, which leads to the construction of cyclic $STS(6n+1)$.

In 1966, Rosa [27] introduced other types of sequences by inserting a hook or zero in the middle of (hooked) Skolem sequences and called such sequences Rosa and hooked Rosa sequences of order n. He proved that a Rosa sequence exists whenever $n \equiv 0, 3(\text{mod } 4)$ and a hooked Rosa sequence exists whenever $n \equiv 1, 2(\text{mod } 4)$. These two types give a complete solution to Heffter's second difference problem, which leads to the construction of cyclic $STS(6n+3)$.

In 1958, Langford [17] observed his son playing with colored blocks and organizing them in sequences similar to Skolem sequences. However, he noticed that every integer i of the set $\{1, 2, \ldots, 2n\}$ can be arranged into disjoint pairs $\{(a_i, b_i) : 1 \leqslant i \leqslant n\}$ such that $\{b_i - a_i : 1 \leqslant i \leqslant n\} = \{d, d+1, \ldots, d+n-1\}$ and $\{d, d+1, \ldots, d+n-1\}$ is a sequence of n positive integers, where each i in the set appears exactly twice and the two appearances of i are exactly i element apart. He presented the case of three colors $n = 3$ as $(3, 1, 2, 1, 3, 2)$ by adding one to each term of $(2, 0, 1, 0, 2, 1)$ to yield a Skolem-type sequence of order $n + 1$. In 1959, Davies [11] and Priday [25] completely solved the case when $d = 2$. The combined works of Bermond, Brouwer, and Germa [4] in 1978 and Simpson [36] in 1983 showed that for all $d \geqslant 2$, and all admissible n, the necessary conditions for the existence of a (hooked) Langford sequence are also sufficient.

In 1981, Stanton and Goulden [39] used a pairing concept and asked for a set of $n-1$ pairs $P(1, n) \setminus m$ with each of the integers of $\{1, \dots, 2n-2\}$ appears exactly once and each of the integers of $\{1, \dots, m-1, m+1, \dots, n\}$ occurs as a difference exactly once. By using this concept they introduced near-Skolem sequences of order n and defect m. For example, we have the pairs $(1, 3)$,$(2, 8)$,$(4, 9)$,$(5, 6)$,$(7, 10)$ form a $P(1, 6) \setminus 4$, and this is a 4-near-Skolem sequence of order 6.

In 1994, Shalaby [28] presented the necessary conditions for the existence of near-Skolem sequences, and proved their sufficiency for all admissible orders.

Two sequences $S = (s_1, s_2, \dots, s_{2n})$ and $S^{'} = (s^{'}_1, s^{'}_2, \dots, s^{'}_{2n})$ of order n are defined to be disjoint if the pairs (a_i, b_i) and $(a^{'}_i, b^{'}_i)$, for $i = 1, 2, \dots, n$, where $a_i = b_i = a^{'}_i = b^{'}_i = i$, do not appear in the same locations in both sequences S and $S^{'}$. Two Steiner triple systems, (V, B_1) and (V, B_2), are disjoint if $B_1 \cap B_2 = \emptyset$. If (V, B_1)

and (V, B_2) are cyclic, then they are disjoint if they have no orbits in common. In 1993, Baker and Shalaby [2], used disjoint Skolem sequences to the existence problems for disjoint cyclic Steiner triple systems. They found the existence of four mutually disjoint Skolem sequences, three mutually disjoint hooked Skolem sequences, and two mutually disjoint near-Skolem sequences.

A λ-fold triple system of order v, denoted by $TS(v, \lambda)$, is a pair (V, B) where V is a v-set of points and B is a set of 3-subsets (blocks) such that any 2-subset of V appears in precisely λ blocks. In 2012, Shalaby and Silvesan [32] determined the intersection spectrum of (hooked) Skolem sequences with i pairs in common, for all admissible orders, where $i \in \{1, 2, \ldots, n-3, n\}$. They discussed cyclic λ-fold triple systems with a prescribed number of base blocks in common, and provided results for $\lambda = 2$, 3 and 4.

Given a $TS(v, \lambda)$, the fine structure of a triple system of index λ is the vector $(c_1, \ldots, c_\lambda)$, where c_i is the number of triples repeated precisely i times in the system. In 2014, Shalaby and Silvesan [31] proved that the necessary conditions are sufficient for the existence of two hooked Skolem sequences of order n with $0, 1, 2, \ldots, n-3$ and n pairs in common, and applied these results to the fine structure of cyclic λ-fold triple systems for $\lambda = 3$ and 4.

A set of m pairwise disjoint (hooked) Skolem sequences of order n forms a (hooked)

Skolem rectangle of order n and strength m. For example, the two disjoint hooked Skolem sequences of order 4 $(1,1,2,3,2,0,3)$ and $(3,1,1,3,2,0,2)$ are disjoint, and it forms a hooked Skolem rectangle of strength 2. In 2014, Linek, Mor and Shalaby [19] constructed (hooked) Skolem and Rosa rectangles and introduced direct constructions for Skolem and Rosa rectangles for $n \geqslant 20$, and proved the existence of six mutually disjoint Skolem sequences of order n for $n \equiv 0, 1 (\text{mod } 4)$, five mutually disjoint hooked Skolem sequences of order n for $n \equiv 2, 3 (\text{mod } 4)$, and four mutually disjoint Rosa sequences of order n for $n \equiv 0, 3 (\text{mod } 4)$. They applied these results to generate simple cyclic triple systems and disjoint cyclic triple systems.

In this thesis, we discuss some special cases of disjoint Skolem-type sequences. We start the paper with Baker and Shalaby [2], where the authors constructed disjoint Skolem-type sequences and related disjoint structures. We provide new disjoint results for (hooked) near-Skolem sequences and we also provide new disjoint results for (hooked) Langford sequences when $d \geqslant 3$ with finite exceptions of n. We present some applications for disjoint Skolem-type sequences.

In Chapter 2, we show a summary of disjoint Skolem-type sequences. We demonstrate some of the constructions of (hooked) Skolem sequences and (hooked) Rosa sequences. We emphasize the necessity in these sequences by presenting the same techniques that are used in proving the existence of Skolem related sequences [13].

For sufficiency, we directly construct the required sequences and produce tables that yield (hooked) Skolem sequences. We discuss known results of disjoint Skolem-type sequences. (See references [2] and [19]).

In Chapter 3, we discuss two disjoint (hooked) near-Skolem sequences of order n and defect m. We present necessity and sufficiency for such sequences. For sufficiency, we provide several constructions for some of the small cases for two disjoint near-Skolem sequences given in [28]. We produce new constructions for hooked near-Skolem sequences and prove that the constructions are disjoint with known constructions of hooked near-Skolem sequences given in [28].

In Chapter 4, we survey all the known results given in [32], [31], [18] and [4] of two disjoint (hooked) Langford sequences. We introduce results with $d = 3$ and 4 that are disjoint with the known results and the results given in [36]. For disjoint Langford sequences, we adjoin a known Langford sequence of order n and defect d to a Langford sequence of order n and defect d. For disjoint hooked Langford sequences, we adjoin a hooked Langford sequence of order n and defect d to a known Langford sequence of order n and defect d. However, our results are not considered to be complete, because most of them only work for higher n when $n > 2d - 1$, and are not valid for cases of small n.

In Chapter 5, we introduce the work of Shalaby and Silvesan [32], [35], and [34] and the work of Meszka and Rosa [22].

In Chapter 6, we conclude the results that we found in this thesis and we present open questions about Chapters 3 and 4.

Chapter 2

Disjoint Skolem-type sequences

In this chapter, we give a summary of the known results of disjoint Skolem-type sequences. We also present the necessary conditions and the sufficiency for the existence of (hooked) Skolem sequences. For the sufficiency, we provide constructions for the existence of (hooked) Skolem sequences of order n.

In 1992, Shalaby [30] proved the existence of disjoint (hooked) Skolem, near-Skolem sequences and $(n, 2)$-Langford sequences. He found at least four mutually disjoint Skolem sequences of order n and three mutually disjoint hooked Skolem sequences of order n, and applied the obtained results to the problem of disjoint cyclic Steiner triple systems and Mendelsohn triple systems.

Baker and Shalaby [2] proved that the maximum number of mutually disjoint (hooked) Skolem sequences of order n cannot exceed n for the case of Skolem sequences, and $n-1$ for the case of hooked Skolem sequences.

In 1993, Baker and Shalaby [2] studied the concept of disjoint Skolem sequences further, and applied this concept to several unsolved problems in design theory such as disjoint cyclic $STS(v)$. They derived necessary conditions for the existence of the maximum number of mutually disjoint (hooked) Skolem sequences of order n.

Lemma 2.0.1 *[2] The maximum number of mutually disjoint (hooked) Skolem sequences of order n is at most n in the case of Skolem sequences and $n-1$ in the case of hooked Skolem sequences.*

Proof The largest two numbers in any two of mutually disjoint Skolem sequences must occur in distinct positions. Skolem sequences have $2n$ positions. Therefore, it is impossible to have more than n mutually disjoint Skolem sequences. For the case of hooked Skolem sequences, there are only $2n-1$ positions in which to place the largest two numbers. The hook occurs in the $(2n)^{th}$ position. Thus the $(n)^{th}$ position also must be empty. Therefore, it is impossible to have more than $n-1$ mutually disjoint hooked Skolem sequences.■

We present basic definitions, necessary conditions, and sufficiency. We also present the known constructions for Skolem-type sequences starting with two papers: These are Skolem and Rosa rectangles and related designs [19], and disjoint Skolem sequences and related disjoint structures [2].

2.1 Definitions and Examples

In this section, we provide the basic definitions that we need in this chapter. Many generalizations of Skolem sequences exist. Here, we provide some known definitions of the Skolem-type sequences. (see reference [29] for more information).

Definition 2.1.1 *Let $D = \{i_1, i_2, \ldots, i_n\}$ be a set of n positive integers. A Skolem-type sequence of order n and $(m-2n)$ hooks, is a sequence $(s_1, s_2, \ldots, s_m)$ of positive integers $i \in D$ such that for each $i \in D$ there is exactly one $j \in \{1, 2, \ldots, 2n+m\}$ such that $s_j = s_{j-i} = i$.*

The integers $i \in D$ are called elements, and s_j and s_{j-i} are called positions. If the sequence above has a position that is not occupied by integers $i \in D$ and contains null elements denoted by 0, the sequence is called a hooked Skolem sequence.

Definition 2.1.2 *A Skolem sequence of order n is a sequence $S = (s_1, s_2, \ldots, s_{2n})$ of $2n$ integers which satisfies the conditions:*

1. *For every $k \in \{1, 2, \ldots, n\}$ there are exactly two positions $s_i, s_j \in S$ such that $s_i = s_j = k$.*

2. If $s_i = s_j = k,\ i < j$*, then* $j - i = k$.

Example 2.1.1 $(1, 1, 3, 4, 2, 3, 2, 4)$ *is a Skolem sequence of order* 4*. The pairs are* $(1, 2)$*,* $(3, 6)$*,* $(4, 8)$*,* $(5, 7)$.

Definition 2.1.3 *A* k*-extended Skolem sequence of order* n *is a sequence* $k-ext\ S = (s_1, s_2, \ldots, s_{2n+1})$ *of* $2n + 1$ *integers that satisfies conditions (1), (2) from Definition* 2.1.2 *and*

3. There is exactly one $i \in \{1, 2, \ldots, 2n + 1\}$ *such that* $s_i = 0$.

Note that the $s_i = 0$ is the hook that occurs in the sequence.

Example 2.1.2 *Let* $D = \{1, 2, 3\}$*,* $n = 3$ *,* $m = 7$ *and* $k = 6$*. We have* $S = (3, 1, 1, 3, 2, 0, 2)$ *is a* 6*-extended Skolem-type sequence or a hooked Skolem sequence of order* 3.

A hooked Skolem sequence of order n is an extended sequence with a hook in the $(2n)^{th}$ position. A Rosa sequence of order n is an extended sequence such that the hook in the $(n + 1)^{th}$ position. For example, $(2, 4, 2, 3, 0, 4, 3, 1, 1)$ is a Rosa sequence of order 4. A hooked Rosa sequence of order n is an extended sequence such that there are two hooks: one is in the $(n + 1)^{th}$ position and the other is in the $(2n + 1)^{th}$ position. For example, $(3, 1, 1, 3, 5, 0, 2, 4, 2, 5, 0, 4)$ is a hooked Rosa sequence of order 5.

Skolem [37] derived necessary conditions and the sufficiency for the existence of a Skolem sequence. O'Keefe [24] solved the existence problem for a hooked Skolem sequence. Later in this study, we will present the necessary conditions and the sufficiency for the existence of some of the Skolem-type sequences.

Definition 2.1.4 *A near-Skolem sequence of order* n *and defect* m*, is a sequence* $S = (s_1, s_2, \dots, s_{2n-2})$ *of integers* $s_i \in \{1, 2, \dots, m-1, m+1, \dots, n\}$ *that satisfies the following conditions:*

1. *For every* $k \in \{1, 2, \dots, m-1, m+1, \dots, n\}$*, there are exactly two positions* $s_i, s_j \in S$ *such that* $s_i = s_j = k$*.*

2. *If* $s_i = s_j = k$*, then* $j - i = k$*.*

Definition 2.1.5 *A hooked near-Skolem sequence of order* n *and defect* m*, is a sequence* $hS = (s_1, s_2, \dots, s_{2n-1})$ *of integers* $s_i \in \{1, 2, \dots, m-1, m+1, \dots, n\}$ *that satisfies conditions (1), (2) from Definition* 2.1.4 *and*

3. $s_{2n-2} = 0$.

We also refer to a (hooked) near-Skolem sequence of order n and defect m as a (hooked) m-near-Skolem sequence.

Remark 2.1.1 *Note that the definitions of (hooked) Skolem sequences can be obtained from (hooked) near-Skolem sequences by adding the difference* m *as in the* *previous definitions.*

Definition 2.1.6 *Two (hooked) Skolem sequences S and S' of order n, are disjoint if $s_i = s_j = k = s_t' = s_u'$ such that $(i,j) \neq (t,u)$ for all $k = 1, 2, \ldots, n$.*

Note that $|(i,j) \cap (t,u)| = 1$ is possible in disjoint Skolem-type sequences and disjoint (hooked) Langford sequences. We present an example of two disjoint Skolem sequences that have one element occurs at the positions j and t.

Example 2.1.3 *$S = (2,3,2,4,3,1,1,4)$ and $S' = (4,2,3,2,4,3,1,1)$ are two disjoint Skolem sequences of order 4. We have $(1,3)$,$(6,7)$,$(4,8)$,$(2,5)$ pairs for S and we also have $(1,5)$,$(2,4)$,$(3,6)$,$(7,8)$ pairs for S'. We find that every pair of elements appears in different locations in the sequences S and S'. We also find that $|(i,j) \cap (t,u)| = 1$, for example, element 1 exists in the pairs $(6,7)$ for S and $(7,8)$ for S', we notice that $j = t = 7$.*

Lemma 2.1.1 *Given a Skolem sequence $S = (s_1, s_2, \ldots, s_{2n})$, the reverse $\overleftarrow{S}= (s_{2n}, \ldots, s_2, s_1)$ is also a Skolem sequence.*

The reverse of a Skolem sequence is a Skolem sequence, the reverse of a Rosa sequence is a Rosa sequence, the reverse of a near-Skolem sequence is a near-Skolem sequence and the revers of a Langford sequence is a Langford sequence. Therefore, a Skolem sequence, a Rosa sequence, a near-Skolem sequence and a Langford sequence will be considered to be equivalent to their reverses.

Example 2.1.4 *Let* $S = (3, 4, 2, 3, 2, 4, 1, 1)$ *be a Skolem sequence of order* 4 *and* $\overleftarrow{S} = (1, 1, 4, 2, 3, 2, 4, 3)$ *be the reverse of* S *and it is also a Skolem sequence of order* 4*.* S *and* $\overleftarrow{S}$ *are two disjoint Skolem sequences of order* 4*. So,* S *is a reverse-disjoint sequence. To check disjointness, we simply check the positions of the elements in both sequences. If the positions of the elements are different in both sequences, we obtain disjoint sequences. For example, element* 1 *in the sequence* S *appears in positions* 7 *and* 8*, and also appears in positions* 1 *and* 2 *in the sequence* $\overleftarrow{S}$*. We follow the same process to check the disjointness for the remaining elements. We found that all the elements occur in different positions in both sequences* S *and* $\overleftarrow{S}$*, so* S *and* $\overleftarrow{S}$ *are disjoint. Thus,* S *is a reverse-disjoint sequence of order* 4*.*

Similarly, near-Skolem sequences can be represented as a set of disjoint integer pairs. However, the partition of the near-Skolem sequences of the set $\{1, 2, \ldots, 2n-2\}$ is represented as $n-1$ pairs.

Definition 2.1.7 *Two (hooked) near-Skolem sequences* $S = (s_1, s_2, \ldots, s_{2n-2})$ *and* $S' = (s_1', s_2', \ldots, s_{2n-2}')$ *of order* n *with defects* m_1*,* m_2 *(where* $1 \leqslant m_1, m_2 \leqslant n$*) are disjoint if* $s_i = s_j = k = s_t' = s_u'$ *such that* $(i, j) \neq (t, u)$ *for all* $k \in \{1, 2, \ldots, n\} \setminus \{m_1\}$

for the sequence S and $k \in \{1, 2, \dots, n\} \setminus \{m_2\}$ for the sequence $S^{'}$.

Example 2.1.5 *$S = (3, 4, 2, 3, 2, 4)$ and $S^{'} = (4, 2, 3, 2, 4, 3)$ are two disjoint near-Skolem sequences of order 4 and $m = 1$.*

Example 2.1.6 *$S = (5, 3, 1, 1, 3, 5, 2, 0, 2)$ and $S^{'} = (1, 1, 3, 5, 2, 3, 2, 0, 5)$ are two disjoint hooked near-Skolem sequences of order 5 and $m = 4$.*

Given a near-Skolem sequence of order n and defect m, $S = (s_1, s_2, \dots, s_{2n-2})$ and the reverse of S is $\overleftarrow{S} = (s_{2n-2}, s_{2n-3}, \dots, s_2, s_1)$. It is also a near-Skolem sequence of order n and defect m. If S and $\overleftarrow{S}$ are disjoint, S is called a reverse-disjoint sequence. We present the following examples for two disjoint (hooked) near-Skolem sequences of order n and defect m.

Example 2.1.7 *Let $S_1 = (3, 5, 6, 3, 1, 1, 5, 2, 6, 2)$ and $S_2 = (6, 2, 3, 2, 5, 3, 6, 1, 1, 5)$ be two disjoint near-Skolem sequences of order 6 and $m = 4$. We notice that S_2 is a reverse-disjoint sequence but S_1 is not because element 1 occurs at the same position in S_1 in addition to its reverse sequence.*

Example 2.1.8 *$S_1 = (5, 3, 1, 1, 3, 5, 2, 0, 2)$ and $S_2 = (1, 1, 3, 5, 2, 3, 2, 0, 5)$ are two disjoint hooked near-Skolem sequences of order 5 and $m = 3$.*

Definition 2.1.8 *A Langford sequence of order n and defect d denoted by L^n_d, is a sequence $L^n_d = (l_1, l_2, \dots, l_{2n})$ of $2n$ integers that satisfies:*

1. *For every $k \in \{d, d+1, \ldots, d+n-1\}$ there are exactly two positions $l_i, l_j \in L_d^n$ such that $l_i = l_j = k$.*

2. *If $l_i = l_j = k,\ i < j,$ then $j - i = k$.*

Example 2.1.9 $L_3^5 = (7,5,3,6,4,3,5,7,4,6)$ *is a Langford sequence of order* 5 *and defect* 3.

Definition 2.1.9 *A hooked Langford sequence of order n and defect d, denoted by hL_d^n, is a sequence $hL_d^n = (l_1, l_2, \ldots, l_{2n+1})$ of $2n+1$ integers that satisfies conditions (1), (2) from Definition* 2.1.9 *and*

3. $l_{2n} = 0$.

Example 2.1.10 $hL_2^6 = (7,5,3,6,4,3,5,7,4,6,2,0,2)$ *is a hooked Langford sequence of order* 6 *and defect* 2.

Definition 2.1.10 *Two (hooked) Langford sequences $L = (l_1, l_2, \ldots, l_{2n})$ and $L^{'} = (l_1^{'}, l_2^{'}, \ldots, l_{2n}^{'})$ of order n are disjoint if $l_i = l_j = k = l_t^{'} = l_u^{'}$ such that $(i,j) \neq (t,u)$ for all $k \in \{d, d+1, \ldots, d+n-1\}$.*

Given a Langford sequence of order n and defect d, $L = (l_1, l_2, \ldots, l_{2n})$ and the reverse of L is $\overleftarrow{L} = (l_{2n}, l_{2n-1}, \ldots, l_2, l_1)$. It is also a Langford sequence of order n and defect d. For example, we have a Langford sequence of order 5 and defect 3,

$L = (6, 4, 7, 5, 3, 4, 6, 3, 5, 7)$. It is clearly to see that L is a reverse-disjoint Langford sequence because when we reverse L we obtain $\overleftarrow{L} = (7, 5, 3, 6, 4, 3, 5, 7, 4, 6)$, which is a Langford sequence of order 5 and defect 3 that is disjoint with L.

Definition 2.1.11 *A Steiner triple system of order v, denoted by $STS(v)$, is a collection of 3-subsets, called triples or blocks, of a set V with v elements, such that every pair of elements occurs in exactly one block.*

2.2 Necessary Conditions for the Existence of the Four Types of Sequences

In this section, we provide the necessary conditions for the existence of (hooked) Skolem sequences and (hooked) Rosa sequences. However, we only present the proof for a Skolem sequence because the proofs for the other cases are similar.

2.2.1 Skolem sequences

The necessary conditions for the existence of a Skolem sequence of order n are $n \equiv 0, 1 \pmod 4$. The proof given here was discovered by Bang [3].

Lemma 2.2.1 *[3] A Skolem sequence of order n can only exist if $n \equiv 0, 1 \pmod 4$.*

Proof Consider the set of positions $\{(a_i, b_i) : i = 1, 2, \ldots, n\}$ such that $b_i - a_i = i$.

$\sum_{i=1}^{n} b_i - \sum_{i=1}^{n} a_i = \sum_{i=1}^{n}(b_i - a_i), = \sum_{i=1}^{n} i, = \frac{n(n+1)}{2}, \ldots (1)$

Note that these numbers a_i and b_i, $i = 1, 2, \ldots, n$ comprise the set $\{1, 2, \ldots, 2n\}$.

Therefore, $\sum_{i=1}^{n} b_i + \sum_{i=1}^{n} a_i = \sum_{i=1}^{2n} i$,

$= \frac{(2n)(2n+1)}{2}$,

$= n(2n + 1), \ldots (2)$

Adding (1) and (2) yields:

$2\sum_{i=1}^{n} b_i = \frac{n(n+1)}{2} + n(2n + 1) = \frac{n(5n+3)}{2}$.

This implies that $\sum_{i=1}^{n} b_i = \frac{n(5n+3)}{4}$.

Thus, $\frac{n(5n+3)}{4}$ must be an integer; and this happens only when $n \equiv 0, 1 (\text{mod } 4)$. Therefore, a Skolem sequence can exist only if $n \equiv 0, 1 (\text{mod } 4)$. ■

2.2.2 Hooked Skolem sequences

The necessary conditions for the existence of a hooked Skolem sequence of order n, which were derived by O'Keefe [24], are $n \equiv 2, 3 (\text{mod } 4)$.

Lemma 2.2.2 *[38] A hooked Skolem sequence of order n can only exist if $n \equiv 2, 3 (\text{mod } 4)$.*

Proof The proof is similar to the proof of necessary conditions for the existence of a Skolem sequence given in Lemma 2.2.1.■

2.2.3 Rosa sequences

The necessary conditions for the existence of (hooked) Rosa sequences were derived by Rosa [27]. He introduced the following results in Lemma 2.2.3 and Lemma 2.2.4

Lemma 2.2.3 *[27] A Rosa sequence of order n can only exist if $n \equiv 0, 3 \pmod 4$.*

Proof The idea of this proof is similar to the proof of necessary conditions for the existence of a Skolem sequence given in Lemma 2.2.1.■

2.2.4 Hooked Rosa sequences

Lemma 2.2.4 *[27] A hooked Rosa sequence of order n can only exist if $n \equiv 1, 2 \pmod 4$.*

Proof Again, the idea behind this proof is similar to the proof of necessary conditions for the existence of a Skolem sequence given in Lemma 2.2.1.■

2.3 Sufficiency for the Existence of (hooked) Skolem Sequences

In this section, we provide the sufficiency for the existence of (hooked) Skolem sequences by providing four tables, namely Table 2.1, Table 2.2, Table 2.3 and Table 2.4, of constructions that are not seen in the literature. These tables show evidence of the existence of (hooked) Skolem sequences of order n. We will only verify Table 2.1 as the verifications of the other tables are similar.

2.3.1 Skolem sequences

Skolem introduced a method for constructing Skolem sequences, and proved that the necessary conditions for the existence of Skolem sequences are also sufficient [37]. We prove this sufficiency by providing constructions that yield Skolem sequences. Our constructions include the element i and positions a_i and b_i, where i is an element in the sequence, and a_i and b_i are the positions of i. The relationship can be expressed as follows:

For $1 \leqslant i \leqslant n$, $a_i < b_i$, then $b_i - a_i = i$.

Theorem 2.3.1 *[37] A Skolem sequence of order n exists for all $n \equiv 0, 1 (\text{mod } 4)$.*

In the literature there are several constructions for the existence of Skolem sequences. We constructed Skolem sequences of order n and produced Table 2.1 and Table 2.2 of constructions.

Proof The constructions below yield Skolem sequences of order n for all $n \equiv 0, 1 (\text{mod } 4)$. There are two missing cases $n = 1$ and $n = 5$. These cases can be given by the sequences $(1, 1)$ and $(2, 4, 2, 3, 5, 4, 3, 1, 1, 5)$. Now, there are two cases for sufficiency: for $n \equiv 0 (\text{mod } 4)$, let $n = 4m$. The following construction yields

Row numbers	i	a_i	b_i	
(1)	1	$m+1$	$m+2$	
(2)	$4m$	1	$4m+1$	
(3)	$4m-2r-2$	$4m+2+r$	$8m-r$	$0 \leqslant r \leqslant 2m-2$
(4)	$4m-1$	$2m+2$	$6m+1$	
(5)	$2m-1$	$2m+1$	$4m$	
(6)	$4m-2r-1$	$r+1$	$4m-r$	$1 \leqslant r \leqslant m-1$
(7)	$2m-2r-3$	$m+3+r$	$3m-r$	$0 \leqslant r \leqslant m-3$

Table 2.1: A construction of a Skolem sequence of order n

Skolem sequences of order n for $m \geqslant 2$. We add the case of $n = 4$, which is satisfied by the sequence $(3, 4, 2, 3, 2, 4, 1, 1)$.

Case 1: $n \equiv 0 (\mathrm{mod}\ 4)$, let $n = 4m$. Omit row (7) when $m = 2$. In order to verify that this construction yields a Skolem sequence, it must be shown that each element of $\{1, 2, \ldots, 2n\}$ appears in a pair (a_i, b_i) exactly once, so that the differences $b_i - a_i = i$ are exactly the elements $i = 1, 2, \ldots, n$. Consider the pairs (a_i, b_i). It is clear that there are $n = 4m$ such pairs, and so there are exactly $2n = 8m$ elements, a_i and b_i. Thus, if every element of $\{1, 2, \ldots, 2n\} = \{1, 2, \ldots, 8m\}$ occurs in one of these pairs, then each of these elements must occur exactly once.

The elements $2, 3, \ldots, m$ occur in the pair $(r + 1, 4m - r)$ for $1 \leqslant r \leqslant m - 1$ from row (6). Both $m + 1$ and $m + 2$ are given by the pair $(m + 1, m + 2)$ in row (1). The elements $m+3, m+4, \ldots, 2m$ are given in the pairs $(m+3+r, 3m-r)$ for $0 \leqslant r \leqslant m-3$

in row (7). The elements $2m+1$ and $2m+2$ occur in the pairs $(2m+1, 4m)$ in row (5) and $(2m+2, 6m+1)$ in row (4), respectively. The pairs $(m+3+r, 3m-r)$ for $0 \leqslant r \leqslant m-3$ in row (7) give the elements $2m+3, 2m+4, \ldots, 3m$.

The elements $3m+1, 3m+2, \ldots, 4m-1$ occur in the pairs $(r+1, 4m-r)$ for $1 \leqslant r \leqslant m-1$ in row (6) while $4m$ appears in $(2m+1, 4m)$ in row (5). The elements $4m+2, 4m+3, \ldots, 6m-2$ are given by the pairs $(4m+2+r, 8m-r)$ for $0 \leqslant r \leqslant 2m-2$ in row (3). The element $6m+1$ appears in $(2m+2, 6m+1)$ in row (4). The remaining elements $6m+2, 6m+3, \ldots, 8m$ are presented in the pairs $(4m+2+r, 8m-r)$ for $0 \leqslant r \leqslant 2m-2$ in row (3). Finally, element 1 appears in the pair $(1, 4m+1)$ in row (2) while $4m+1$ appears in $(1, 4m+1)$ in row (2).

Thus, the proof is complete and the construction above yields a Skolem sequence. Therefore, all elements of $\{1, 2, \ldots, 8m\}$ occur in the pairs (a_i, b_i). Hence, each such element occurs exactly once as either a_i or b_i for some i.

Now, it must be verified that the differences $b_i - a_i$ give all values $1, 2, \ldots, 4m$ exactly once. There are $n = 4m$ differences, so it must only be shown that each element occurs at least once, which then implies that each occurs exactly once. First, $1 = (m+2) - (m+1)$ is the difference of $b_i - a_i$ from row (1). The differences $(3m-r) - (m+3+r) = 2m-2r-3$ for $0 \leqslant r \leqslant m-3$ in row (7) give the numbers $3, 5, \ldots, 2m-3$. The differences $(4m) - (2m+1)$ from row (5) give $2m-1$. The numbers $2m+1, 2m+3, \ldots, 4m-3$ are given by $(4m-r)-(r+1) = 4m-2r-1$ for $1 \leqslant r \leqslant m-1$ from row (6). $4m-1$ occurs as the difference $(6m+1)-(2m+2)$ from row (4). The elements $2, 4, \ldots, 4m$ are given by $(8m-r)-(4m+r+2) = 4m-2r-2$ for $0 \leqslant r \leqslant 2m-2$ from row (3). Finally, $4m$ occurs as the difference $(4m+1)-1$ from row (2).

Thus, the verification is complete. The sequences that are formed from the previous construction are Skolem sequences. We apply Table 2.1 and we obtain the following example.

Example 2.3.1 *For $n = 8$, we have* $(8, 5, 1, 1, 3, 7, 5, 3, 8, 6, 4, 2, 7, 2, 4, 6)$. *The pairs are* $(1, 9)$, $(2, 7)$, $(3, 4)$,$(5, 8)$, $(6, 13)$, $(11, 15)$, $(10, 16)$, $(12, 14)$.

Case 2: $n \equiv 1(\text{mod } 4)$, let $n = 4m+1$. The following construction gives Skolem sequences of order n for $m \geqslant 2$. Omit row (7) when $m = 2$. This completes the proof of Theorem 2.3.1.■

We apply Table 2.2 and we obtain the following example.

Example 2.3.2 *For $n = 9$, we have* $(9, 5, 8, 1, 1, 3, 5, 7, 3, 9, 8, 6, 4, 2, 7, 2, 4, 6)$. *The pairs are* $(1, 10)$, $(2, 7)$, $(3, 11)$, $(4, 5)$, $(6, 9)$, $(8, 15)$, $(12, 18)$, $(13, 17)$, $(14, 16)$.

Row numbers	i	a_i	b_i	
(1)	$4m-2r$	$4m+3+r$	$8m+3-r$	$1 \leqslant r \leqslant 2m-1$
(2)	$4m$	3	$4m+3$	
(3)	1	$m+2$	$m+3$	
(4)	$4m+1$	1	$4m+2$	
(5)	$2m+1$	2	$2m+3$	
(6)	$2m-1-2r$	$m+4+r$	$3m+3-r$	$0 \leqslant r \leqslant m-2$
(7)	$4m-3-2r$	$4+r$	$4m+1-r$	$0 \leqslant r \leqslant m-3$
(8)	$4m-1$	$2m+4$	$6m+3$	

Table 2.2: A construction of a Skolem sequence of order n

2.3.2 Hooked Skolem sequences

Theorem 2.3.2 *[24] A hooked Skolem sequence of order n exists for all $n \equiv 2, 3 \pmod 4$.*

In the literature there are several constructions for the existence of hooked Skolem sequences. We construct hooked Skolem sequences of order n and produced Table 2.3 and Table 2.4 of constructions.

Proof The constructions below yield hooked Skolem sequences of order n for all $n \equiv 2, 3 \pmod 4$.

Case 1: $n \equiv 2 \pmod 4$, let $n = 4m+2$. The required construction gives hooked Skolem sequences of order n for $m \geqslant 1$. Omit row (6) when $m = 1$.

Row numbers	i	a_i	b_i	
(1)	$2m+1$	$3m+1$	$5m+2$	
(2)	$2m+2r+3$	$m-r$	$3m+3+r$	$0 \leqslant r \leqslant m-1$
(3)	$2m-2r-1$	$m+1+r$	$3m-r$	$0 \leqslant r \leqslant m-1$
(4)	$2m+2$	$6m+3$	$8m+5$	
(5)	$4m+2$	$3m+2$	$7m+4$	
(6)	$4m-2r$	$4m+3+r$	$8m+3-r$	$0 \leqslant r \leqslant m-2$
(7)	$2m-2r$	$5m+3+r$	$7m+3-r$	$0 \leqslant r \leqslant m-1$

Table 2.3: A construction of a hooked Skolem sequence of order n

Example 2.3.3 *For $n = 6$, we have $(5,1,1,3,6,5,3,2,4,2,6,0,4)$. The pairs are $(1,6)$, $(2,3)$, $(4,7)$, $(5,11)$, $(8,10)$, $(9,13)$.*

Case 2: $n \equiv 3 (\text{mod } 4)$, let $n = 4m-1$. The following construction yields hooked Skolem sequences of order n for $m \geqslant 2$. Omit row (6) when $m = 2$. This completes the proof of Theorem 2.3.2.■

row numbers	i	a_i	b_i	
(1)	$2m-2r-1$	$m+r$	$3m-1-r$	$0 \leqslant r \leqslant m-1$
(2)	$4m-2r$	r	$4m-r$	$1 \leqslant r \leqslant m-1$
(3)	$2m+1$	$6m-2$	$8m-1$	
(4)	$4m-1$	$3m$	$7m-1$	
(5)	$2m-2r$	$5m-2+r$	$7m-2-r$	$0 \leqslant r \leqslant m-1$
(6)	$4m-2r-3$	$4m+r$	$8m-3-r$	$0 \leqslant r \leqslant m-3$

Table 2.4: A construction of a hooked Skolem sequence of order n

Example 2.3.4 *For $n = 7$, we have $(6, 3, 1, 1, 3, 7, 6, 4, 2, 5, 2, 4, 7, 0, 5)$. The pairs are $(1, 7)$, $(2, 5)$, $(3, 4)$, $(6, 13)$, $(8, 12)$, $(9, 11)$, $(10, 15)$.*

The following example demonstrates the existence of two disjoint (hooked) Skolem sequences of order n, implying the existence of four disjoint cyclic $STS(6n + 1)$.

Example 2.3.5 *Let $S = (1, 1, 4, 2, 3, 2, 4, 3)$ and $S' = (3, 4, 2, 3, 2, 4, 1, 1)$ be two disjoint Skolem sequences of order 4. We find the pairs (a_i, b_i) for $i = 1$, 2, 3 and 4. The pairs for the sequence S are $(1, 2)$, $(4, 6)$, $(5, 8)$, $(3, 7)$ and we have the difference systems $\{0, i, b_i + n\} \pmod{6n + 1}$ or $\{0, a_i + n, b_i + n\} \pmod{6n + 1}$ where $i = 1$, 2, 3 and 4. We obtain two solutions:*

1. $\{0, 1, 6\}$, $\{0, 2, 10\}$, $\{0, 3, 12\}$, *and* $\{0, 4, 11\} \pmod{25}$.

2. $\{0, 5, 6\}$, $\{0, 8, 10\}$, $\{0, 9, 12\}$, *and* $\{0, 7, 11\} \pmod{25}$.

We check the differences in $\mathbb{Z}_{25}$ and observe that all the non-zero elements appear *as differences twice in* $\mathbb{Z}_{25}$.

We follow the same process for the sequence S'. *The pairs* (a_i', b_i') *for* $i = 1,\ 2,\ 3$ *and* 4 *are* $(7,8)$, $(3,5)$, $(1,4)$, $(2,6)$ *and we have the difference systems* $\{0, i, b_i' + n\}$ $(\text{mod } 6n+1)$ *or* $\{0, a_i' + n, b_i' + n\}$ $(\text{mod } 6n+1)$ *where* $i = 1,\ 2,\ 3$ *and* 4. *We also obtain two solutions:*

1. $\{0,1,12\}$, $\{0,2,8\}$, $\{0,3,9\}$, *and* $\{0,4,10\}$ $(\text{mod } 25)$.
2. $\{0,11,12\}$, $\{0,7,9\}$, $\{0,5,8\}$, *and* $\{0,6,10\}$ $(\text{mod } 25)$.

We also observe that all the non-zero elements exist as differences twice in $\mathbb{Z}_{25}$. *Therefore, each sequence yields two disjoint cyclic* $STS(25)$.

2.3.3 Rosa sequences

Theorem 2.3.3 *[27] A Rosa sequence of order* n *exists for all* $n \equiv 0, 3 (\text{mod } 4)$.

Rosa [27], proved that the necessary conditions for the existence of a Rosa sequence of order n are sufficient by introducing constructions that yield Rosa sequences of order n for all $n \equiv 0, 3 (\text{mod } 4)$.

2.3.4 Hooked Rosa sequences

Theorem 2.3.4 *[27] A hooked Rosa sequence of order* n *exists for all* $n \equiv 1, 2 (\text{mod } 4)$.

Rosa [27] proved that the necessary conditions for the existence of a hooked Rosa sequence of order n are sufficient by introducing constructions that yield a hooked Rosa sequence of order n for all $n \equiv 1, 2 \pmod 4$.

We present an example showing that the existence of a Rosa sequence implies the existence of a cyclic $STS(6n+3)$.

Example 2.3.6 *Let* $R = (6,4,2,7,2,4,6,0,3,5,7,3,1,1,5)$ *be a Rosa sequence of order* 7. *We find the pairs of positions* (a_i, b_i) *for* $i = 1, 2, \ldots, 7$. *We have* $(13,14)$,$(3,5)$,$(9,12)$,$(2,6)$,$(10,15)$,$(1,7)$,$(4,11)$.

We take the base blocks of the form $\{0, i, b_i + n\} \pmod{45}$ *where* $i = 1, 2, \ldots, 7$. *We have* $\{0,1,7\}$,$\{0,2,12\}$,$\{0,3,19\}$,$\{0,4,13\}$,$\{0,5,22\}$,$\{0,6,14\}$, *and* $\{0,7,18\} \pmod{45}$.

Now, we observe that all the non-zero elements exist as differences twice in $\mathbb{Z}_{45}$. *We notice that* $\{0,15,30\}$ *is a missing base block, so its differences are also missing. It is clear to see that each base block will give a full orbit except for the missing base block, which will give a short orbit. So we will cyclically develop this base block* $\{0,15,30\} \pmod{45}$ *and the previous base blocks of the form* $\{0, i, b_i + n\}$ *to obtain blocks then add them all together. Thus, we obtain 330 blocks of cyclic* $STS(45)$.

2.4 Known results for disjoint Skolem-Type Sequences

In this section, we present several known results for (hooked) Skolem sequences and (hooked) Rosa sequences such as the results of Skolem, O'Keefe and Rosa. In [7], the results were given in the form of tables listing values of i, a_i, and b_i such that i is an element and a_i and b_i are the positions of that element in the sequence, and $b_i - a_i = i$ for all $1 \leqslant i \leqslant n$ where $a_i < b_i$. However, we do not present the required constructions here. Rather, we simply state the known results and refer the reader to reference [19] for the known constructions of disjoint (hooked) Skolem sequences and (hooked) Rosa sequences.

Many known results of (hooked) Skolem sequences and (hooked) Rosa sequences exist, and some of them are included in references [7], [11], [23], and some other work.

Now, we present the known results of disjoint Skolem-type sequences, starting from the paper of Baker and Shalaby [2] in which they proved the existence of disjoint (hooked) Skolem sequences and disjoint near-Skolem sequences.

In [2], Baker and Shalaby proved the existence of a reverse-disjoint near-Skolem sequence of order n. They reversed near-Skolem sequences of order n and defect m, and obtained two disjoint near-Skolem sequences for the majority of them. Shalaby [2] introduced the following theorems.

Theorem 2.4.1 *[2] For all $n \equiv 0, 1 \pmod 4$, $n \geqslant 4$, there exist at least four mutually disjoint Skolem sequences of order n.*

Proof [2] There are three cases to be considered in the proof, but we only display Case 1 and each case gives a reverse-disjoint Skolem sequence.

Case 1: $n \equiv 0 \pmod 8$, $n > 0$. Let $n = 8s$.

row numbers	i	a_i	b_i	
(1)	$8s-2r+1$	r	$8s-r+1$	$1 \leqslant r \leqslant 4s-1$
(2)	$8s-4r+2$	$8s+2r-1$	$16s-2r+1$	$1 \leqslant r \leqslant s-1$
(3)	$8s+4r$	$8s+2r+2$	$16s-2r+2$	$1 \leqslant r \leqslant s-1$
(4)	$4s-2r+2$	$10s+r+1$	$14s+3-r$	$1 \leqslant r \leqslant 2s-1$
(5)	$4s+2$	$4s$	$8s+2$	
(6)	$8s$	$4s+1$	$12s+1$	
(7)	1	$12s+2$	$12s+3$	
(8)	2	$10s-1$	$10s+1$	

Table 2.5: A construction of the existence of at least four mutually disjoint Skolem sequences of order n

For Case 2 and Case 3 see reference [2]. To check the disjointness we check all the values of (i, j), and whether their new positions in the reverse sequence $(2n+1-j, 2n+1-i)$ are distinct components.■

Example 2.4.1 *The following sequences are four mutually disjoint Skolem sequences of order* 4.

$(4, 1, 1, 3, 4, 2, 3, 2)$,

$(2, 3, 2, 4, 3, 1, 1, 4)$,

$(3, 4, 2, 3, 2, 4, 1, 1)$,

$(1, 1, 4, 2, 3, 2, 4, 3)$.

When we check the positions for the elements that occur in these sequences, we find that element 2*, which is in the first sequence, occurs in position* $(6, 8)$*. Its new* *position is* $(1,3)$ *in the second sequence,* $(3,5)$ *in the third sequence, and* $(4,6)$ *in the fourth sequence. We follow the same process for the other elements. We observe that the positions for the elements in the four sequences are distinct. Therefore, we conclude that these four sequences are mutually disjoint Skolem sequences of order* 4.

Theorem 2.4.2 *[2] For all* $n \equiv 2, 3 \pmod 4$*,* $n \geqslant 6$*, there exist at least* 3 *mutually disjoint hooked Skolem sequences of order* n.

Example 2.4.2 *The following sequences are three mutually disjoint hooked Skolem sequences of order* 6.

$(5, 6, 4, 1, 1, 5, 4, 6, 2, 3, 2, 0, 3)$,

$(4, 5, 3, 6, 4, 3, 5, 1, 1, 6, 2, 0, 2)$,

$(3, 1, 1, 3, 4, 5, 6, 2, 4, 2, 5, 0, 6)$.

As a result of Theorem 2.4.1 and Theorem 2.4.2, Baker and Shalaby [2] obtained Corollary 2.4.3 and Corollary 2.4.4 for the number of mutually disjoint, cyclic $STS(v)$, denoted by $n_c(v)$.

Corollary 2.4.3 *[2] For all* $v \geqslant 25$ *and* $v \equiv 1, 7 \pmod{24}$*,* $n_c(v) \geqslant 8$.

Corollary 2.4.4 *[2] For all* $v \geqslant 37$ *and* $v \equiv 13, 19 \pmod{24}$*,* $n_c(v) \geqslant 6$.

A Mendelsohn triple system of order v, $MTS(v)$, is a pair (V, B) where V is a set of v elements and B is a collection of cyclic triples of elements of V elements such that every ordered pair of distinct elements from V elements occur in exactly one triple.

Baker and Shalaby [2] observed that a cyclic Mendelsohn triple system can be obtained from a cyclic $STS(v)$ by replacing any base block $\{a,b,c\}$ of the cyclic $STS(v)$ by a set of two cyclic blocks $\{\langle a,b,c\rangle, \langle a,c,b\rangle\}$, where $\langle a,b,c\rangle$ is the triple containing the ordered pairs (a,b),(b,c), and (c,a). They improved on the known bounds for the numbers of disjoint cyclic Mendelsohn triple systems [9]. We show this by the following example.

Example 2.4.3 *Let* $S = (1,1,5,2,4,2,3,5,4,3)$ *be a Skolem sequence of order* 5*, and we find the pairs* $\{(a_i,b_i), 0 \leqslant i \leqslant n\}$*. The pairs are:* $(1,2)$*,* $(4,6)$*,* $(7,10)$*,* $(5,9)$*,* $(3,8)$*. To obtain a cyclic* $STS(6n+1)$ *we find the base blocks of the form* $\{(0,i,b_i+n) (\mathrm{mod}\ 6n+1) | i = 1,\ldots,n\}$*. The blocks are:*

$\{0,7,1\}$,$\{0,11,2\}$,$\{0,15,3\}$,$\{0,14,4\}$,$\{0,13,5\}$ $(\mathrm{mod}\ 31)$.

We notice that all the non-zero elements exist as differences twice in $\mathbb{Z}_{31}$*. We cyclically develop these base blocks* $(\mathrm{mod}\ 31)$ *and we obtain a cyclic* $STS(31)$*.*

We observe that if we replace each block $\{a,b,c\}$ *of a cyclic* $STS(6n+1,2)$ *in this example with* $\{\langle a,b,c\rangle, \langle a,c,b\rangle\}$*, the set of two cyclic blocks forms a cyclic Mendelsohn triple system of order* v*. For example, we replace the block* $\{1,8,2\}$ *with the set of two cyclic blocks* $\{\langle 1,8,2\rangle, \langle 1,2,8\rangle\}$*, and we follow same process for the rest of the* *blocks.*

Furthermore, Baker and Shalaby [2] produced the following additional improvement on the results obtained in [9]. They produced results for the number of mutually disjoint cyclic Mendelsohn triple systems of order v, denoted by $m_c(v)$.

Corollary 2.4.5 *[2] For all $v \geqslant 25$ and $v \equiv 1, 7 (\mathrm{mod}\ 24)$, $m_c(v) \geqslant 8$.*

Corollary 2.4.6 *[2] For all $v \geqslant 37$ and $v \equiv 13, 19 (\mathrm{mod}\ 24)$, $m_c(v) \geqslant 6$.*

In [32], Shalaby and Silvesan proved that there exist two (hooked) Skolem sequences of small orders that can have $0, 1, \ldots, n-3$, n pairs in common. In addition, they proved that this argument holds for larger orders. We show this by the following example.

Example 2.4.4 *The following two hooked Skolem sequences of order 7 have three pairs in common.*

$hS_1 = (3, 1, 1, 3, 6, 7, 2, 4, 2, 5, 6, 4, 7, 0, 5),$

$hS_2 = (6, 3, 1, 1, 3, 7, 6, 4, 2, 5, 2, 4, 7, 0, 5).$

The pairs are $(6, 13)$, $(8, 12)$, *(10,15).*

In particular, they constructed (hooked) Skolem sequences of order n by adjoining a (hooked) Skolem sequence of a smaller order with a (hooked) Langford sequence, which produced the following two theorems, provided that the necessary conditions for the existence of two (hooked) Skolem sequences of small orders with $0, 1, \ldots, n-3$, n pairs in common were sufficient.

Theorem 2.4.7 *[32] The necessary conditions are sufficient for two Skolem sequences of order n to have $0, 1, \ldots, n-3$ and n pairs in the same positions.*

Theorem 2.4.8 *[32] The necessary conditions are sufficient for two hooked Skolem sequences of order n to have $0, 1, \ldots, n-3$ and n pairs in the same positions.*

Shalaby and Silvesan used these results to the fine structure of a cyclic three-fold triple system and a cyclic four-fold triple system for $v \equiv 13, 19 \pmod{24}$. Finally, they extended these results to the fine structure of a cyclic Mendelsohn triple system. In addition to the results of Shalaby and Silvesan above, Silvesan [35] proved that there exist two cyclic Steiner triple systems of order $6n+1$ intersecting in $0, 1, 2, \ldots, n$ base blocks, and there exist two cyclic Steiner triple systems of order $6n+3$ intersecting in $1, 2, \ldots, n+1$ base blocks.

In [19], the authors presented several constructions for Skolem and Rosa rectangles that produced several results for disjoint (hooked) Skolem sequences, as well as disjoint (hooked) Rosa sequences. We state these results, and refer the reader to reference [19] for the required constructions.

Theorem 2.4.9 *[19] If $n \geqslant 1$ and $n \equiv 0, 1 \pmod{4}$, then there exist at least*

$\lfloor \log_3(2n+9) \rfloor - 1$ *disjoint Skolem sequences of order* n.

Theorem 2.4.10 *[19] If* $n \geqslant 13$ *and* $n \equiv 2, 3 (\mathrm{mod}\ 4)$, *then there exist at least* $\lfloor \log_3(2n+9) \rfloor - 2$ *disjoint hooked Skolem sequences of order* n.

As a result of Theorem 2.4.9 and Theorem 2.4.10, the authors of [19] produced two applications, namely disjoint and simple cyclic triple systems, and introduced Theorem 2.4.11 and Theorem 2.4.12.

Theorem 2.4.11 *[19] If* $n \geqslant 1$, $n \equiv 0, 1 (\mathrm{mod}\ 4)$ *and* $m = 2\lfloor \log_3(2n+9) \rfloor - 2$, *then there exist at least* m *disjoint* $STS(6n+1)$.

Theorem 2.4.12 *[19] If* $n \geqslant 13$, $n \equiv 2, 3 (\mathrm{mod}\ 4)$ *and* $m = 2\lfloor \log_3(2n-9) \rfloor - 4$, *then there exist at least* m *disjoint* $STS(6n+1)$.

The authors added the disjoint base blocks of the constructions that they found in [19], and obtained the following results:

Theorem 2.4.13 *[19] If* $n \geqslant 1$, *and* $m = 2\lfloor \log_3(2n+9) \rfloor - 2$, *then there exists a simple cyclic triple system* $CSTS_\lambda(6n+1)$ *for* $1 \leqslant \lambda \leqslant m$.

Theorem 2.4.14 *[19] If* $n \geqslant 1$, *and* $m = 2\lfloor \log_3(2n-9) \rfloor - 4$, *then there exists a simple cyclic triple system* $CSTS_\lambda(6n+1)$ *for* $1 \leqslant \lambda \leqslant m$.

We present the known results of direct constructions of Skolem and Rosa rectangles [19].

Theorem 2.4.15 *[19] If $n \geqslant 20$ and $n \equiv 0 \pmod 4$, then there exist six mutually disjoint Skolem sequences (i.e., a $6 \times n$ Skolem rectangle).*

It was shown in the proof of Theorem 2 given in [2] that there exists at least four mutually disjoint Skolem sequences of order $n \equiv 0 \pmod 4$. We present two reverse-disjoint Skolem sequences from the construction given in [19] that are disjoint with the four given in [2]. We follow the constructions of the proof of Theorem 2.4.16 given in [19]. We obtain S_1 and S_2.

Example 2.4.5 *Let $S_1 = (15, 13, 11, 9, 7, 5, 3, 19, 17, 3, 5, 7, 9, 11, 13, 15, 20, 18, 16, 12, 10, 8, 6, 4, 14, 17, 19, 4, 6, 8, 10, 12, 1, 1, 16, 18, 20, 2, 14, 2)$ be a Skolem sequence of order 20. We notice that S_1 is a reverse-disjoint sequence, so we obtain $S_2 = (2, 14, 2, 20, 18, 16, 1, 1, 12, 10, 8, 6, 4, 19, 17, 14, 4, 6, 8, 10, 12, 16, 18, 20, 15, 13, 11, 9, 7, 5, 3, 17, 19, 3, 5, 7, 9, 11, 13, 15)$, a Skolem sequence of order 20. Thus, S_1 and S_2 are disjoint Skolem sequences of order 20. Therefore, we have now two Skolem sequences S_1 and S_2 that are disjoint with the four given in [2].*

Theorem 2.4.16 *[19] If $n \geqslant 20$ and $n \equiv 1 \pmod 4$, then there exist six mutually disjoint Skolem sequences (i.e., a $6 \times n$ Skolem rectangle).*

Theorem 2.4.17 *[19] If $n \geqslant 20$ and $n \equiv 2 (\mathrm{mod}\ 4)$, then there exist five mutually disjoint hooked Skolem sequences (i.e., a $5 \times n$ hooked Skolem rectangle).*

It was shown in the proof of Theorem 4 given in [2] that there exists at least three mutually disjoint hooked Skolem sequences of order $n \equiv 2 (\mathrm{mod}\ 4)$. We present another three sequences from the construction given in [19] that are disjoint from the sequences given in [2]. In the following example, we follow the constructions of the proof of Theorem 2.4.18 given in [19] and obtain the sequences hS_1, hS_2 and hS_3.

Example 2.4.6 *The following are three hooked Skolem sequences of order 22.*

$hS_1 = (17, 15, 1, 1, 11, 9, 7, 5, 3, 21, 19, 3, 5, 7, 9, 11, 15, 17, 13, 22, 20, 18, 16, 14, 12,$ *10,8,6,4,19,21,13,4,6,8,10,12,14,16,18,20,22,2,0,2),*

$hS_2 = (2, 3, 2, 19, 3, 15, 13, 11, 9, 1, 1, 5, 22, 20, 7, 17, 5, 9, 11, 13, 15, 7, 19, 21, 18, 16,$ *14,12,10,8,6,4,17,20,22,4,6,8,10,12,14,16,18,0,21),*

and $hS_3 = (3, 4, 2, 3, 2, 4, 18, 16, 12, 10, 8, 6, 14, 22, 20, 21, 12, 6, 8, 10, 12, 1, 1, 16, 18,$ *19,14,15,13,11,9,7,5,17,20,22,21,5,7,9,11,13,15,0,19).*

Theorem 2.4.18 *[19] If $n \geqslant 20$ and $n \equiv 3 (\mathrm{mod}\ 4)$, then there exist five mutually disjoint hooked Skolem sequences (i.e., a $5 \times n$ hooked Skolem rectangle).*

As a result of the previous four theorems, the authors of [19] produced disjoint and simple cyclic triple systems for the smaller orders.

Theorem 2.4.19 *[19] For all* $v \equiv 1 (\mathrm{mod}\ 6)$ *and* $v \geqslant 37$*, there exists a simple cyclic triple system* $CSTS_\lambda(v)$*, for* $1 \leqslant \lambda \leqslant 10$*.*

Theorem 2.4.20 *[19] For all* $v \equiv 1, 7 (\mathrm{mod}\ 24)$ *and* $v \geqslant 49$*, there exists a simple cyclic triple system* $CSTS_\lambda(v)$*, for* $1 \leqslant \lambda \leqslant 12$*.*

In [19], the authors also produced several theorems of disjoint Rosa sequences, which we present as follows.

Theorem 2.4.21 *[19] If* $n \geqslant 44$ *and* $n \equiv 0, 3 (\mathrm{mod}\ 4)$*, then there exist at least* $\lfloor \log_3(2n + 9) \rfloor - 2$ *disjoint Rosa sequences of order* n*.*

Theorem 2.4.22 *[19] For* $n \equiv 0 (\mathrm{mod}\ 4)$*, there exist four mutually disjoint Rosa sequences (i.e., a* $4 \times n$ *Rosa rectangle).*

The four required mutually disjoint Rosa sequences are presented in tables given in [19]. We present two reverse-disjoint sequences obtained from the constructions given in [19].

Example 2.4.7 *We have* $R_1 = (18, 16, 14, 12, 10, 8, 6, 4, 2, 19, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 0, 17, 15, 13, 9, 7, 5, 3, 19, 11, 3, 5, 7, 9, 1, 1, 13, 15, 17, 20, 11)$ *and* $R_2 = (19, 17, 15, 13, 11, 9, 7, 5, 3, 20, 12, 3, 5, 7, 9, 11, 13, 15, 17, 19, 0, 18, 12, 14, 16, 10, 8, 6, 4, 20, 1, 1, 4, 6, 8, 10, 2, 14, 2, 18, 16)$*, which are two Rosa sequences of order* 20*. We* *notice that R_1 and R_2 are reverse-disjoint Rosa sequences of order* 20*. Thus, we have four mutually disjoint Rosa sequences of order* 20*.*

Theorem 2.4.23 *[19] For $n \equiv 3 \pmod 4$, and $n \geqslant 20$ there exists four mutually disjoint Rosa sequences (i.e., a $4 \times n$ Rosa rectangle).*

Finally, the authors in [19] produced results of near disjoint and near simple cyclic triple systems by using the constructions for Rosa rectangles.

Theorem 2.4.24 *[19] For all $v \equiv 3, 21 \pmod{24}$ and $v \geqslant 39$, there exist* 8 *mutually near disjoint cyclic $STS(v)$.*

Chapter 3

Disjoint (hooked) near-Skolem sequences

In this chapter, we prove that there exist disjoint hooked near-Skolem sequences of order n and defect m. We prove eight new cases of disjoint hooked near-Skolem sequences and we constructed tables for each case.

3.1 Introduction

Shalaby [28], [30] derived the necessary conditions for the existence of (hooked) near-Skolem sequences of order n and defect m, and he proved that the necessary conditions are sufficient. Baker and Shalaby [2] proved the existence of disjoint near-Skolem sequences of order n and defect m when they reversed m-near Skolem sequences and found that the sequences were reverse-disjoint except for the first case. We complete some of the missing small cases. For hooked near-Skolem sequences, we produce new constructions and prove that these constructions are disjoint with the known constructions given in [28]. We reverse Langford sequences for the small cases of (hooked) near-Skolem sequences. We refer the reader to Chapter 2 for the definitions of (hooked) near-Skolem sequences and (hooked) Langford sequences.

In 1981, Stanton and Goulden [39] introduced near-Skolem sequences to construct cyclic Steiner triple systems. They focused on a set of $n-1$ pairs $(P(1,n) \setminus m)$, where each integer of $\{1,2,\ldots,2n-2\}$ occurs exactly once and each integer of $\{1,2,\ldots,m-1,m+1,\ldots,n\}$ occurs as a difference exactly once. For example, $(5,6,1,1,4,5,3,6,4,3)$ is a 2-near-Skolem sequence of order 6, and this sequence corresponds to the pairs $(P(1,6) \setminus 2)$ that can be written as $(3,4)$, $(7,10)$, $(5,9)$, $(1,6)$ and $(2,8)$. In 1982, Billington [6] studied near-Skolem sequences to obtain several types of designs.

In 1987, Billington [5] conjectured that the necessary conditions are sufficient for the existence of extended Skolem sequences. Baker [1] proved that the necessary conditions are sufficient for the existence of extended Skolem sequences.

3.2 Necessity

In this section, we briefly present the necessary conditions for the existence of (hooked) near-Skolem sequences of order n and defect m.

Theorem 3.2.1 *[28] An m-near-Skolem sequence of order n, $m \leqslant n$, exists if and only if $n \equiv 0, 1 \pmod 4$ and m is odd, or $n \equiv 2, 3 \pmod 4$ and m is even.*

Proof [28]

$\sum_{r=1}^{n}(i+j) = 2n^2 - 3n + 1$, such that $r \neq m, r = s_i = s_j, \ldots (1)$
$\sum_{r=1}^{n}(j-i) = \frac{n(n+1)}{2} - m, \ldots (2)$

By subtracting (2) from (1) yields:

$2\sum_{r=1}^{n} i = \frac{3n^2-7n+2m+2}{2}$. So, $\sum_{r=1}^{n} i = \frac{n(3n-7)+2(m+1)}{4}$, therefore $\frac{n(3n-7)+2(m+1)}{4}$ must be an integer. If we solve for n and m we obtain the conditions of Theorem 3.2.1.

■

Theorem 3.2.2 *[28] A hooked m-near-Skolem sequence of order n, $m \leqslant n$ exists if and only if $n \equiv 0, 1 \pmod 4$ and m is even or $n \equiv 2, 3 \pmod 4$ and m is odd.*

Proof The proof for the case of a hooked near-Skolem sequence is similar to the proof above. We have:

$\sum_{r=1}^{n}(i+j) = 2n^2 - 3n + 2$, such that $r \neq m, r = s_i = s_j$, $\ldots(1)$
$\sum_{r=1}^{n}(j-i) = \frac{n(n+1)}{2} - m$, $\ldots(2)$

By subtracting (2) from (1) we obtain:

$2\sum_{r=1}^{n} i = \frac{3n^2-7n+4+2m}{2}$. This implies that $\frac{n(3n-7)+2(m+2)}{4}$ must be an integer. We obtain the conditions of Theorem 3.2.2 by solving for n and m.

■

3.3 Sufficiency

In this section, we prove that the previously stated necessary conditions are also sufficient for the existence of two disjoint hooked near-Skolem sequences of order n and defect m. We provide eight new constructions in the form of tables with the difference i for $i = 1, 2, \ldots, m-1, m+1, \ldots, n$ and the positions of a_i and b_i, that yield hooked near-Skolem sequences of order n and defect m. In addition, we observe that these eight new constructions are disjoint with the known constructions for hooked near-Skolem sequences of order n and defect m given in [28]. We refer the readers to see reference [28] for the known constructions of (hooked) near-Skolem sequences. We will prove the disjointness only for Case 1 of hooked near-Skolem sequences as the proofs of other cases are similar.

As we previously mentioned, Baker and Shalaby [2] proved that the reverse of near-Skolem sequences is disjoint, except for two cases, and they proved reverse-disjoint constructions for those cases. We provide the construction only for the first case with the correction for the small case when $m = n - 1$.

Theorem 3.3.1 *[28] The necessary conditions for the existence of an m-near-Skolem sequence of order n are sufficient.*

Theorem 3.3.2 *[28] The necessary conditions for the existence of a hooked m-near-Skolem sequence of order n are sufficient.*

Lemma 3.3.3 *[28] The existence of a Skolem sequence of order t, $t \equiv 0, 1 \pmod 4$ implies the existence of a $(t+1)$-near-Skolem sequence of order q, where $q \geqslant 3t + 4$, and:*

(1) If $t \equiv 0 \pmod 4$ then $q \equiv 2, 3 \pmod 4$,

(2) If $t \equiv 1 \pmod 4$ then $q \equiv 0, 1 \pmod 4$.

Shalaby [28] proved eight cases yield m-near-Skolem sequences of order n and defect m and another eight cases yield hooked m-near-Skolem sequences of order n and defect m. He produced Lemma 3.3.3 to prove the small cases of (hooked) near-Skolem sequences. The idea of the this lemma is attaching a Skolem sequence of order n at the left of a hooked Langford sequence of order n and defect d to obtain a hooked near-Skolem sequences of order n and defect m. For example, let $S = (1, 1)$ be a Skolem sequence of order 1, and let $hL = (8, 3, 5, 7, 3, 4, 6, 5, 8, 4, 7, 0, 6)$ be a hooked Langford sequence of order $n = 6$ and defect $d = 2$. We attach the sequence S with the sequence hL to obtain $hS = (1, 1, 8, 3, 5, 7, 3, 4, 6, 5, 8, 4, 7, 0, 6)$, which is a hooked near-Skolem sequence of order 8 and defect 2.

Theorem 3.3.4 *[2] For all $n \geqslant 4$, m, n satisfying the conditions of Theorem* 3.2.1 *there exists reverse-disjoint m-near-Skolem sequences of order n.*

We produce a construction for Case 1 that gives a near-Skolem sequence of order n and defect m and we prove that this construction is disjoint with the known construction that we will show in Table 3.3.

Remark 3.3.1 *We will only verify Case 1 of a near-Skolem sequence and omit the verifications of the other cases as they are similar.*

Proof Necessity was proved in Theorem 3.2.1.

Case 1: $n \equiv 0 (\text{mod } 8)$. For this case we give two subcases, when $t \neq 4s - 1$ and when $t = 4s - 1$.

For $m = 1$, the reverse of the Langford sequence with $d = 2$ gives the 1-near-Skolem sequence. For $n \geq m > 1$, let $n = 8s$, $m = 2t + 1$. Table 3.1 illustrates when $t \neq 4s - 1$. Skip row (3) when $s = 1$, row (9) when $s = 1$ and row (8) when $s = 2$.

row numbers	i	a_i	b_i	
(1)	$8s-2r-1$	$r+1$	$8s-r$	$0 \leqslant r \leqslant 4s-t-2$
(2)	1	$4s+t$	$4s+t+1$	
(3)	$2t-2r-1$	$4s-t+r$	$4s+t-1-r$	$0 \leqslant r \leqslant t-2$
(4)	$8s-2r$	$4s-1+r$	$12s-1-r$	$0 \leqslant r \leqslant 1$
(5)	2	$14s-2$	$14s$	
(6)	$8s-4r$	$8s+2r-1$	$16s-1-2r$	$1 \leqslant r \leqslant 2s-1$
(7)	$4s-2$	$8s+2$	$12s$	
(8)	$4s-4r-2$	$10s+2r$	$14s-2-2r$	$1 \leqslant r \leqslant s-2$
(9)	$8s-4r-2$	$8s+2+2r$	$16s-2r$	$1 \leqslant r \leqslant s-1$

Table 3.1: A construction of disjoint near-Skolem sequences of order n and defect m

row numbers	i	a_i	b_i	
(1)	$8s-4r-2$	$7s+2r-1$	$15s-3-2r$	$1 \leqslant r \leqslant s-1$
(2)	1	$16s-3$	$16s-2$	
(3)	$4s-2$	$12s-3$	$16s-5$	
(4)	$8s-4r$	$8s-2+2r$	$16s-2-2r$	$1 \leqslant r \leqslant 2s-1$
(5)	2	$10s-3$	$10s-1$	

Table 3.2: A construction of disjoint near-Skolem sequences of order n and defect m

Table 3.2 illustrates when $t=4s-1$.

To verify that the construction of Table 3.1 provides near-Skolem sequences, it must be shown that each element of $\{1, 2, \ldots, 2n-2\}$ appears in a pair (a_i, b_i) exactly once where $i \neq m$ and m is the defect, and that the differences $b_i - a_i$ are exactly the elements $\{1, 2, \ldots, m-1, m+1, \ldots, n\}$. Considering the pairs (a_i, b_i), it is clear that there are $n = 8s$ such pairs, and so there are exactly $2n - 2 = 16s - 2$ positions a_i and b_i. Thus, if every element of $\{1, 2, \ldots, 2n-2\} = \{1, 2, \ldots, 16s-2\}$ occurs in one of these pairs, each of these elements must occur exactly once. When $t \geqslant 1$, we have the elements $1, 2, \ldots, 4s-t-1$ occur in the pairs $(r+1, 8s-r)$ for $0 \leq r \leq 4s-t-2$, from row (1). The elements $4s+t+2, 4s+t+3, \ldots, 8s$ occur in the pairs $(r+1, 8s-r)$ for $0 \leqslant r \leqslant 4s-t-2$, from row (1). The elements $4s-t, 4s-t+1, \ldots, 4s-2$ occur in the pairs $(4s-t+r, 4s+t-1-r)$ for $0 \leqslant r \leqslant t-2$, from row (3). Also, the elements $4s+1, 4s+2, \ldots, 4s+t-1$ occur in the pairs $(4s-t+r, 4s+t-1-r)$ for $0 \leqslant r \leqslant t-2$, from row (3). In row (4), the elements $4s-1, 4s$ occur in the pairs $(4s-1+r, 12s-1-r)$ for $0 \leqslant r \leqslant 1$ and the elements $12s-2, 12s-1$ occur in the pairs $(4s-1+r, 12s-1-r)$ for $0 \leqslant r \leqslant 1$. The elements $8s+1, 8s+3, \ldots, 12s-3$ occur in the pairs $(8s+2r-1, 16s-1-2r)$ for $1 \leqslant r \leqslant 2s-1$, from row (6). The elements $12s+1, 12s+3, \ldots, 16s-3$ occur in the pairs $(8s+2r-1, 16s-1-2r)$ for $1 \leqslant r \leqslant 2s-1$, from row (6). The elements $10s+2, 10s+4, \ldots, 12s-4$ occur in the pairs $(10s+2r, 14s-2-2r)$ for $1 \leqslant r \leqslant s-2$, from row (8). The elements

$12s+2, 12s+4, \ldots, 14s-4$ occur in the pairs $(10s+2r, 14s-2-2r)$ for $1 \leqslant r \leqslant s-2$, from row (8). The elements $8s+4, 8s+6, \ldots, 10s$ occur in the pairs $(8s+2+2r, 16s-2r)$ for $1 \leqslant r \leqslant s-1$, from row (9). The elements $14s+2, 14s+4, \ldots, 16s-2$ occur in the pairs $(8s+2+2r, 16s-2r)$ for $1 \leqslant r \leqslant s-1$, from row (9). Both $4s+t$ and $4s+t+1$ are given in the pair $(4s+t, 4s+t+1)$ in row (2). Both $8s+2$ and $12s$ are given in the pair $(8s+2, 12s)$ in row (7). Both $14s-2$ and $14s$ are given in the pair $(14s-2, 14s)$ in row (5).

Now, we verify that the differences $b_i - a_i = i$ where $i \neq m$ give the values $1, 2, \ldots, 8s$ exactly once. $1 = (4s+t+1) - (4s+t)$ is the difference of $b_i - a_i$, and occurs in row (2). The difference $(8s-r) - (r+1) = 8s - 2r - 1$ for $0 \leqslant r \leqslant 4s-t-2$ in row (1) gives the numbers $2t+3, 2t+5, \ldots, 8s-1$. The difference $(4s+t-1-r) - (4s-t+r) = 2t-2r-1$ for $0 \leqslant r \leqslant t-2$ in row (3) gives the numbers $3, 5, \ldots, 2t-1$. The difference $(12s-1-r) - (4s-1+r) = 8s-2r$ for $0 \leqslant r \leqslant 1$ in row (4) gives the numbers $8s-2, 8s$. The difference $(16s-1-2r) - (8s+2r-1) = 8s-4r$ for $1 \leqslant r \leqslant 2s-1$ in row (6) gives the numbers $4, 8, \ldots, 8s-4$. The difference $(14s-2-2r) - (10s+2r) = 4s-4r-2$ for $1 \leqslant r \leqslant s-2$ in row (8) gives the numbers $6, 10, \ldots, 4s-6$. The difference $(16s-2r) - (8s+2+2r) = 8s-4r-2$ for $1 \leqslant r \leqslant s-1$ in row (9) gives the numbers $4s+2, 4s+6, \ldots, 8s-6$. $2 = (14s) - (14s-2)$ is the difference of $b_i - a_i$, and occurs in row (5). Also, $4s-2 = (12s) - (8s+2)$ is the difference of $b_i - a_i$, and occurs in row (7). The verification is complete so that the construction above yields near-Skolem sequences.

The same argument holds for Table 3.2, which covers the case when $t = 4s - 1$. The elements $7s+1, 7s+3, \ldots, 9s-3$ occur in the pairs $(7s+2r-1, 15s-3-2r)$ for $1 \leqslant r \leqslant s-1$ in row (1). The elements $13s-1, 13s+1, \ldots, 15s-5$ occur in the pairs $(7s+2r-1, 15s-3-2r)$ for $1 \leqslant r \leqslant s-1$ in row (1). The elements $8s, 8s+2, \ldots, 12s-4$ occur in the pairs $(8s+2r-2, 16s-2-2r)$ for $1 \leqslant r \leqslant 2s-1$ in row (4). The elements $(12s, 12s+2, \ldots, 16s-4)$ occur in the pairs $(8s+2r-2, 16s-2-2r)$ for $1 \leqslant r \leqslant 2s-1$ in row (4). Both elements $16s-3$, $16s-2$ are given by the pair $(16s-3, 16s-2)$ in row (2). Both elements $12s-3$, $12s-5$ are given by the pair $(12s-3, 12s-5)$ in row (3). Both elements $10s-3$, $10s-1$ are given by the pair $(10s-3, 10s-1)$ in row (5). The difference $(16s-2)-(16s-3) = 1$ occurs in row (2). The difference $(16s-5)-(12s-3) = 4s-2$ occurs in row (3). The difference $(10s-1)-(10s-3) = 2$ occurs in row (5). The difference $(15s-2r-3)-(7s+2r-1) = 8s-4r-2$ for $1 \leqslant r \leqslant s-1$ gives the numbers $4s+6, 4s+10, \ldots, 8s-6$ in row (1). The difference $(16s-2r-2)-(8s+2r-2) = 8s-4r$ for $1 \leqslant r \leqslant 2s-1$ gives the numbers $4, 8, \ldots, 8s-4$ in row (4).

Finally, we check the reverse. The reverse is similar because we replace every difference $b_i - a_i$ by $-(a_i - b_i)$. We conclude that the differences and their reverses appear.

Now, we show the known construction that is given by Shalaby in [28] for the same case when $n \equiv 0 (\text{mod } 8)$.

For $m = 1$, the 1-near-Skolem sequence is a Langford sequence with $d = 2$ introduced in [11] and [25], so we skip all the subsequent cases when $m = 1$.

For $n \geqslant m > 1$, let $n = 8s$, $m = 2t + 1$. The required table is:

row numbers	i	a_i	b_i	
(1)	$8s-2r-1$	$8s+r-1$	$16s-r-2$	$0 \leqslant r \leqslant 4s-t-2$
(2)	1	$12s-t-2$	$12s-t-1$	
(3)	$2t-2r-1$	$12s-t+r$	$12s+t-r-1$	$0 \leqslant r \leqslant t-2$
(4)	$8s-2r$	$4s+r$	$12s-r$	$0 \leqslant r \leqslant 1$
(5)	2	$2s-1$	$2s+1$	
(6)	$8s-4r$	$2r$	$8s-2r$	$1 \leqslant r \leqslant 2s-1$
(7)	$4s-2$	$4s-1$	$8s-3$	
(8)	$4s-4r-2$	$2s+2r+1$	$6s-2r-1$	$1 \leqslant r \leqslant s-2$
(9)	$8s-4r-2$	$2r-1$	$8s-2r-3$	$1 \leqslant r \leqslant s-1$

Table 3.3: A construction of disjoint near-Skolem sequences of order n and defect m

The construction above yields near-Skolem sequences. We conclude that Table 3.1, Table 3.2 and Table 3.3 are disjoint. To check the disjointness, we observe that most of the even numbers occupy the positions $1, \ldots, n-2$ in Table 3.3. When we reverse them in Table 3.1 and Table 3.2, they occupy the positions $n+1, \ldots, 2n-2$, showing that they do not occupy the same positions. Most of the odd numbers occupy the positions $n-1, \ldots, 2n-2$ in Table 3.3. When we reverse these in Table 3.1 and Table 3.2, they occupy the positions $1, \ldots, n$, showing that they do not occupy the same positions.

We also observe that m_1, m_2 are not in conflict in the two sequences. For the case when $m = n-1$. If we reverse the sequence for $m = n-1$ in Table 3.3, it will not be disjoint because there will be one pair in common, namely the pair for element 1. In order to make the sequence disjoint, we reverse it and then reverse the elements in the positions $n-1,\ldots,2n-2$ again, but we keep the two largest even numbers in their positions. The pair for element 1 occupies the positions $2n-3$ and $2n-2$ and most of the even numbers will occupy the positions $n-1,\ldots,2n-4$. Thus, the two sequences are disjoint.

Case 2: $n \equiv 1 (\mathrm{mod}\ 8)$.

The constructions and solutions for the small cases for Case 2 of Theorem 3 given in [28] are reverse-disjoint.

Case 3: $n \equiv 2 (\mathrm{mod}\ 8)$.

The constructions and solutions for the small cases for Case 3 of Theorem 3 given in [28] are reverse-disjoint except for $n = 10$ and $10 \geqslant m \geqslant 4$, we provide the following disjoint sequences.

For $n = 10$ and $m = 4$, we have $(5, 10, 8, 6, 9, 5, 2, 7, 2, 6, 8, 10, 3, 9, 7, 3, 1, 1)$.

For $n = 10$ and $m = 6$, we have $(5, 10, 8, 4, 2, 5, 2, 4, 9, 7, 8, 10, 3, 1, 1, 3, 7, 9)$.

For $n = 10$ and $m = 8$, we have $(10, 6, 4, 2, 9, 2, 4, 6, 3, 7, 10, 3, 5, 9, 1, 1, 7, 5)$.

For $n = 10$ and $m = 10$, it is a Skolem sequence of order 9 which exists by Skolem [38].

Case 4: $n \equiv 3 (\mathrm{mod}\ 8)$.

Baker and Shalaby [2] proved constructions that are reverse-disjoint. The constructions for the small cases for Case 4 of Theorem 3 given in [28] are reverse-disjoint except for $n = 3$, $m = 2$, and for $n = 11$, $11 \geqslant m \geqslant 4$.

For $n = 3$ and $m = 2$ we provide disjoint sequence $(3, 1, 1, 3)$. For $n = 11$ and $11 \geqslant m \geqslant 4$ we also provide the following disjoint sequences.

For $n = 11$ and $m = 4$, we have $(10, 8, 6, 11, 9, 7, 1, 1, 6, 8, 10, 5, 7, 9, 11, 3, 5, 2, 3, 2)$.

For $n = 11$ and $m = 6$, we have $(10, 8, 4, 11, 9, 7, 4, 1, 1, 8, 10, 5, 7, 9, 11, 3, 5, 2, 3, 2)$.

For $n = 11$ and $m = 8$, we have $(10, 6, 4, 11, 9, 7, 4, 6, 1, 1, 10, 5, 7, 9, 11, 3, 5, 2, 3, 2)$.

For $n = 11$ and $m = 10$, we have $(8, 6, 4, 2, 11, 2, 4, 6, 8, 9, 7, 1, 1, 3, 5, 11, 3, 7, 9, 5)$.

Case 5: $n \equiv 4 (\text{mod } 8)$.

The constructions and solutions for the small cases for Case 5 of Theorem 3 in [28] are reverse-disjoint.

Case 6: $n \equiv 5 (\text{mod } 8)$.

The constructions and solutions for the small cases for Case 6 of Theorem 3 in [28] are reverse-disjoint.

Case 7: $n \equiv 6 (\text{mod } 8)$.

The constructions and solutions for the small cases for Case 7 of Theorem 3 in [28] are reverse-disjoint.

Case 8: $n \equiv 7 (\text{mod } 8)$. The constructions and solutions for the small cases for Case 8 of Theorem 3 in [28] are reverse-disjoint.

This completes the proof of Theorem 3.3.4 ■

Theorem 3.3.5 *For all $n \geqslant 4$, m,n satisfying the conditions of Theorem* 3.2.2, *there exist disjoint hooked m-near-Skolem sequences of order n.*

We produce cases for hooked m-near-Skolem sequences of order n and prove that these sequences are disjoint with hooked m-near-Skolem sequences given in [28]. We show the construction for Case 1 of a hooked near-Skolem sequence and verify that this construction yields hooked m-near-Skolem sequences of order n, and also verify that this construction is disjoint with the known construction for the same case given in [28]. We will omit the verifications of the remaining constructions since their verifications are similar to the one we will show and we only provide the constructions that we produced for disjoint hooked m-near-Skolem sequences of order n.

Proof .

Case 1: $n \equiv 0 (\mathrm{mod}\ 8)$

For hooked near-Skolem sequences with $m = 2$ and $m = 4$, we use the reverse of the Langford sequence in [36] when $d = 4s + 1$, $s \geqslant 1$, $t \geqslant 2s + 1$,(where t is odd numbers only). This gives the solutions for all the cases when $n \geqslant 16$. We add the sequence $(4, 1, 1, 3, 4, 0, 3)$ when $m = 2$ and the sequence $(3, 1, 1, 3, 2, 0, 2)$ when $m = 4$. We provide the cases of $n = 8$.

row numbers	i	a_i	b_i	
(1)	$8s-2r$	$8s-3+r$	$16s-3-r$	$0 \leqslant r \leqslant 4s-t-1$
(2)	$2t-2-2r$	$12s-t-1+r$	$12s+t-3-r$	$0 \leqslant r \leqslant t-4$
(3)	1	$12s-t-3$	$12s-t-2$	
(4)	3	$12s-4$	$12s-1$	
(5)	$8s+1-2r$	$4s-2+r$	$12s-1-r$	$1 \leqslant r \leqslant 2$
(6)	$4s-1$	$12s$	$16s-1$	
(7)	$4s-3-4r$	$2s+2r$	$6s-3-2r$	$0 \leqslant r \leqslant s-2$
(8)	$4s-5-4r$	$2s+3+2r$	$6s-2-2r$	$0 \leqslant r \leqslant s-3$
(9)	2	$2s-1$	$2s+1$	
(10)	4	$4s-2$	$4s+2$	
(11)	$8s-2r-5$	$r+1$	$8s-4-r$	$0 \leqslant r \leqslant 2s-3$

Table 3.4: A construction of disjoint hooked near-Skolem sequences of order n and defect m

For $n = 8$ and $m = 2$, we have $(8, 4, 1, 1, 6, 4, 5, 7, 8, 3, 6, 5, 3, 0, 7)$.

For $n = 8$ and $m = 4$, we have $(8, 1, 1, 2, 6, 2, 5, 7, 8, 3, 6, 5, 3, 0, 7)$.

For $n = 8$ and $m = 6$, we have $(3, 7, 4, 3, 8, 2, 4, 2, 7, 5, 1, 1, 8, 0, 5)$.

For $n = 8$ and $m = 8$, it is a hooked Skolem sequence of order 7 which exists by O'Keefe [24]. For $n \geqslant m > 4$ and $n > 8$, let $n = 8s$ and $m = 2t$, Table 3.4 is the required construction.

Skip row (8) when $s = 2$, skip row (2) when $t = 3$. To verify that the above construction provides hooked near-Skolem sequences of order n and defect m, it must be shown that each element of $\{1, 2, \dots, 2n-1\}$ appears in a pair (a_i, b_i) exactly once where $i \neq m$ and m is the defect, and that the differences $b_i - a_i$ are exactly the elements $1, 2, \dots, n-1$. Now consider the pairs (a_i, b_i) for $i = 1, 2, \dots, n$. It is easy to check that there are $n = 8s$ such pairs, and so there are exactly $2n - 1 = 16s - 1$ positions a_i and b_i. Thus, if every element of $\{1, 2, \dots, 2n - 1\} = \{1, 2, \dots, 16 - 1\}$ occurs in one of these pairs, each of these elements must occur exactly once. When $t \geqslant 1$, we have the elements $8s - 3, 8s - 2, \dots, 4s - t - 4$, which occur in the pairs $(8s - 3 + r, 16s - 3 - r)$ for $0 \leqslant r \leqslant 4s - t - 1$, from row (1). The elements $12s + t - 2, 12s + t - 1, \dots, 16s - 3$ occur in the pairs $(8s - 3 + r, 16s - 3 - r)$ for $0 \leqslant r \leqslant 4s - t - 1$, from row (1). The elements $12s - t - 1, 12s - t, \dots, 12s - 5$ occur in the pairs $(12s - t - 1 + r, 12s + t - 3 - r)$ for $0 \leqslant r \leqslant t - 4$, from row (2). Also, the elements $12s+1, 12s+2, \dots, 12s+t-3$ occur in the pairs $(12s-t-1+r, 12s+t-3-r)$ for $0 \leqslant r \leqslant t - 4$, from row (2). In row (5), while the elements $4s - 1, 4s$ occur in the pairs $(4s - 2 + r, 12s - 1 - r)$ for $1 \leqslant r \leqslant 2$, the elements $12s - 2, 12s - 3$ occur in the pairs $(4s - 2 + r, 12s - 1 - r)$ for $0 \leqslant r \leqslant 1$. The elements $2s, 2s + 2, \dots, 4s - 4$ occur in the pairs $(2s + 2r, 6s - 3 - 2r)$ for $0 \leqslant r \leqslant s - 2$, from row (7). The elements $4s+1, 4s+3, \dots, 6s-3$ occur in the pairs $(2s+2r, 6s-3-2r)$ for $0 \leqslant r \leqslant s-2$, from row (7). The elements $2s+3, 2s+5, \ldots, 4s-3$ occur in the pairs $(2s+3+2r, 6s-2-2r)$ for $0 \leqslant r \leqslant s-3$, from row (8). The elements $4s+4, 4s+6, \ldots, 6s-2$ occur in the pairs $(2s+3+2r, 6s-2-2r)$ for $0 \leqslant r \leqslant s-3$, from row (8). The elements $1, 2, \ldots, 2s-2$ occur in the pairs $(1+r, 8s-4-r)$ for $0 \leqslant r \leqslant 2s-3$, from row (11). The elements $6s-1, 6s, \ldots, 8s-4$ occur in the pairs $(1+r, 8s-4-r)$ for $0 \leqslant r \leqslant 2s-3$, from row (11). Both $12s-t-3$ and $12s-t-2$ are given in the pairs $(12s-t-3, 12s-t-2)$ in row (3). Both $8s+2$ and $12s$ are given in the pairs $(12s-4, 12s-1)$ in row (4). Both $12s$ and $16s-1$ are given in the pairs $(12s, 16s-1)$ in row (6). Both $2s-1$ and $2s+1$ are given in the pairs $(2s-1, 2s+1)$ in row (9). Both $4s-2$ and $4s+2$ are given in the pairs $(4s-2, 4s+2)$ in row (10).

Now, we verify that the difference $b_i - a_i = i$ where $i \neq m$ gives the values $1, 2, \ldots, 8s$ exactly once. $1 = (12s-t-2)-(12s-t-3)$ is a difference of $b_i - a_i$, and occurs in row (3). $3 = (12s-1)-(12s-4)$ is a difference of $b_i - a_i$, and occurs in row (4). The difference $(16s-3-r)-(8s-3+r) = 8s-2r$ for $0 \leqslant r \leqslant 4s-t-1$ in row (1) gives the numbers $2t+2, 2t+4, \ldots, 8s$. The difference $(12s+t-3-r)-(12s-t-1+r) = 2t-2r-2$ for $0 \leqslant r \leqslant t-4$ in row (2) gives the numbers $6, 8, \ldots, 2t-2$. The difference $(12s-1-r)-(4s-2+r) = 8s+1-2r$ for $1 \leqslant r \leqslant 2$ in row (5) gives the numbers $8s-3, 8s-1$. The difference $(6s-3-2r)-(2s+2r) = 4s-3-4r$ for $0 \leqslant r \leqslant s-2$ in row (7) gives the numbers $5, 9, \ldots, 4s-3$. The difference

$(6s-2-2r)-(2s+3+2r) = 4s-4r-5$ for $0 \leqslant r \leqslant s-3$ in row (8) gives the numbers $7, 11, \ldots, 4s-5$. The difference $(8s-4-r)-(1+r) = 8s-2r-5$ for $0 \leqslant r \leqslant 2s-3$ in row (11) gives the numbers $4s-1, 4s-3, \ldots, 8s-5$. We find that $2 = (2s+1)-(2s-1)$ is a difference of $b_i - a_i$, and occurs in row (9). We have $(16s-1)-(12s) = 4s-1$ is a difference of $b_i - a_i$, and occurs in row (6). $4 = (4s+2)-(4s-2)$ is a difference of $b_i - a_i$, and occurs in row (10). The verification is complete. Therefore, the construction above yields hooked near-Skolem sequences.

Now we provide the known construction that is given by Shalaby in [28] for the same case when $n \equiv 0 (\text{mod } 8)$.

For $m = 2$, by Lemma $2(ii)(b)$ given in [28], the existence of the Skolem sequence $(1, 1)$ provides solutions for all the cases of $n \geqslant 8$.

For $n \geqslant m > 2$, let $n = 8s$, $m = 2t$, the required construction is:

row numbers	i	a_i	b_i	
(1)	$8s-2r$	$1+r$	$8s-r+1$	$0 \leqslant r \leqslant 4s-t-1$
(2)	$2t-2-2r$	$4s-t+r+3$	$4s+t-r+1$	$0 \leqslant r \leqslant t-3$
(3)	1	$4s-t+1$	$4s-t+2$	
(4)	$8s+1-2r$	$4s+r$	$12s+1-r$	$1 \leqslant r \leqslant 2$
(5)	$4s-1$	$4s+3$	$8s+2$	
(6)	$4s-2r-3$	$10s+1+r$	$14s-r-2$	$0 \leqslant r \leqslant 2s-3$
(7)	$8s-4r-5$	$8s+4+2r$	$16s-2r-1$	$0 \leqslant r \leqslant s-2$
(8)	2	$14s-1$	$14s+1$	
(9)	$8s-4r-7$	$8s+2r+3$	$16s-2r-4$	$0 \leqslant r \leqslant s-2$

Table 3.5: A construction of disjoint hooked near-Skolem sequences of order n and defect m

The construction above yields hooked near-Skolem sequences of order n and defect m. We conclude that Table 3.3.4 and Table 3.3.5 are disjoint. To check the disjointness, we observe that most of the even numbers occupy the positions $1, \ldots, n-10$ in Table 3.3.5. In Table 3.3.4 they occupy the positions $n-3, \ldots, n+1$, and therefore, they do not occupy the same positions in Table 3.3.5. Most of the odd numbers occupy the positions $2n-14, \ldots, 2n-1$ in Table 3.3.5, and the two largest odd numbers occupy the positions $n-7$ and $n-6$. Most of the odd numbers in Table 3.3.4 occupy the positions $1, \ldots, n-4$, so they do not occupy the same positions in Table 3.3.5. The two largest odd numbers in Table 3.3.4 occupy the positions $n-9$ and $n-8$. The pair for element 1 appears in the positions $n-9$ and $n-8$ in Table 3.3.5. In

Table 3.3.4, the pair of element 1 appears in the positions $2n-14$ and $2n-13$. Thus both of the sequences are disjoint.

Case 2: $n \equiv 1 (\text{mod } 8)$.

For hooked near-Skolem sequences with $m = 2$ and $m = 4$, we use the reverse of the Langford sequence in [36] when $d = 4s + 2$, $s \geqslant 1$, $t \geqslant 2s + 1$ (t is odd numbers only). This gives the solutions for all of the cases when $n \geqslant 17$. We add the sequence $(5, 1, 1, 3, 4, 5, 3, 0, 4)$ when $m = 2$ and the sequence $(5, 3, 1, 1, 3, 5, 2, 0, 2)$ when $m = 4$. We provide the cases of $n = 9$.

For $n = 9$ and $m = 2$, we have $(8, 4, 1, 1, 7, 4, 6, 9, 8, 5, 3, 7, 6, 3, 5, 0, 9)$.

For $n = 9$ and $m = 4$, we have $(8, 1, 1, 2, 7, 2, 6, 9, 8, 5, 3, 7, 6, 3, 5, 0, 9)$.

For $n = 9$ and $m = 6$, we have $(4, 2, 7, 2, 4, 5, 8, 9, 3, 7, 5, 3, 1, 1, 8, 0, 9)$.

For $n = 9$ and $m = 8$, we have $(6, 4, 2, 7, 2, 4, 6, 9, 3, 5, 7, 3, 1, 1, 5, 0, 9)$.

For all $n > m > 4$ and $n > 9$, let $n = 8s + 1$ and $m = 2t$. The required construction is.

row numbers	i	a_i	b_i	
(1)	$8s-2r$	$8s-1+r$	$16s-1-r$	$0 \leqslant r \leqslant 4s-t-1$
(2)	1	$12s-t-1$	$12s-t$	
(3)	$2t-2-2r$	$12s-t+1+r$	$12s+t-1-r$	$0 \leqslant r \leqslant t-4$
(4)	$4s+1-4r$	$2s-1+2r$	$6s-2r$	$0 \leqslant r \leqslant s-1$
(5)	$4s-1-4r$	$2s+2+2r$	$6s+1-2r$	$0 \leqslant r \leqslant s-2$
(6)	2	$2s-2$	$2s$	
(7)	4	$4s-1$	$4s+3$	
(8)	$8s+1-2r$	$4s+r$	$12s+1-r$	$0 \leqslant r \leqslant 1$
(9)	$8s-3-2r$	$r+1$	$8s-2-r$	$0 \leqslant r \leqslant 2s-4$
(10)	3	$12s-1$	$12s+2$	
(11)	$4s+3$	$12s-2$	$16s+1$	

Table 3.6: A construction of disjoint hooked near-Skolem sequences of order n and defect m

Case 3: $n \equiv 2 (\text{mod } 8)$.

For a hooked near-Skolem sequence with $m = 3$, we use the reverse of the Langford sequence in [36] when $d = 4s - 1$, $s \geqslant 2$, $t \geqslant 2s$ (t is odd numbers only). This gives the solutions for all of the cases when $n \geqslant 26$. We then add the sequence $(1, 1, 4, 5, 6, 2, 4, 2, 5, 0, 6)$. By using the same case of the reverse of the Langford sequence when $m = 1$, the solutions of all the cases of $n \geqslant 10$ are given but t in this case takes even numbers only. We add the sequence $(2, 0, 2)$ to the reverse of the Langford sequence. We provide all cases of $n = 10$ and $n = 18$.

For $n = 10$ and $m = 3$, we have $(6, 1, 1, 4, 8, 9, 6, 4, 10, 7, 5, 2, 8, 2, 9, 5, 7, 0, 10)$.

For $n = 10$ and $m = 5$, we have $(10, 8, 6, 1, 1, 9, 7, 4, 6, 8, 10, 4, 3, 7, 9, 3, 2, 0, 2)$.

For $n = 10$ and $m = 7$, we have $(4, 8, 6, 10, 4, 1, 1, 9, 6, 8, 5, 3, 6, 10, 3, 5, 9, 0, 6)$.

For $n = 10$ and $m = 9$, we have $(7, 5, 3, 10, 8, 3, 5, 7, 6, 1, 1, 4, 8, 10, 6, 4, 2, 0, 2)$.

For $n = 18$ and $m = 1$, we have,

$(3, 4, 8, 3, 10, 4, 6, 18, 16, 14, 8, 2, 6, 2, 10, 17, 15, 13, 11, 9, 7, 5, 12, 14, 16, 18, 5, 7, 9, 11,$
$13, 15, 17, 0, 12)$.

For $n = 18$ and $m = 3$, we have,

$(3, 12, 10, 3, 2, 6, 2, 18, 16, 14, 4, 6, 10, 12, 4, 17, 15, 13, 11, 9, 7, 1, 1, 14, 16, 18, 8, 7, 9, 11,$
$13, 15, 17, 0, 8)$.

For $n = 18$ and $m = 5$, we have,

$(18, 16, 14, 12, 10, 8, 6, 4, 17, 15, 13, 4, 6, 8, 10, 12, 14, 16, 18, 11, 1, 1, 9, 13, 15, 17, 2, 7, 2,$
$3, 11, 9, 3, 0, 7)$.

For $n = 18$ and $m = 7$, we have,

$(10, 1, 1, 3, 5, 8, 3, 18, 16, 5, 10, 12, 2, 8, 2, 17, 15, 13, 11, 9, 14, 6, 4, 12, 16, 18, 4, 6, 9, 11, 13,$
$15, 17, 0, 14)$.

For $n = 18$ and $m = 9$, we have,

$(7, 12, 10, 1, 1, 6, 4, 7, 18, 16, 4, 6, 10, 12, 8, 17, 15, 13, 11, 2, 14, 2, 8, 5, 3, 16, 18, 3, 5, 11, 13,$
$15, 17, 0, 14)$.

For $n = 18$ and $m = 11$, we have,

$(12, 2, 6, 2, 10, 1, 1, 18, 6, 4, 14, 8, 12, 4, 10, 17, 15, 13, 16, 8, 9, 7, 5, 3, 14, 18, 3, 5, 7, 9, 13, 15, 17, 0, 16)$.

For $n = 18$ and $m = 13$, we have,

$(1, 1, 12, 10, 2, 6, 2, 18, 4, 8, 14, 6, 4, 10, 12, 17, 15, 8, 16, 11, 9, 7, 5, 3, 14, 18, 3, 5, 7, 9, 11, 15, 17, 0, 16)$.

For $n = 18$ and $m = 15$, we have,

$(6, 10, 12, 2, 8, 2, 6, 1, 1, 16, 14, 10, 8, 4, 12, 17, 18, 4, 13, 11, 9, 7, 5, 3, 14, 16, 3, 5, 7, 9, 11, 13, 17, 0, 18)$.

For $n = 18$ and $m = 17$, we have,

$(2, 10, 2, 12, 1, 1, 8, 6, 4, 16, 14, 10, 4, 6, 8, 12, 18, 15, 13, 11, 9, 7, 5, 3, 14, 16, 3, 5, 7, 9, 11, 13, 15, 0, 18)$.

For all $n > m > 3$ and $n > 18$, let $n = 8s + 2$ and $m = 2t + 1$, the required construction is:

row numbers	i	a_i	b_i	
(1)	$8s-2r+1$	$8s+r$	$16s+1-r$	$0 \leqslant r \leqslant 4s-t-1$
(2)	1	$12s-t$	$12s-t+1$	
(3)	$2t-1-2r$	$12s-t+2+r$	$12s+t+1-r$	$0 \leqslant r \leqslant t-3$
(4)	$4s-2$	$2s+1$	$6s-1$	
(5)	3	$2s-1$	$2s+2$	
(6)	4	$2s$	$2s+4$	
(7)	2	$2s+3$	$2s+5$	
(8)	$8s-4-2r$	$1+r$	$8s-3-r$	$0 \leqslant r \leqslant 2s-3$
(9)	$4s-2-2r$	$4s+2+r$	$8s-r$	$1 \leqslant r \leqslant 2$
(10)	$4s-8-2r$	$2s+6+r$	$6s-2-r$	$0 \leqslant r \leqslant 2s-7$
(11)	$8s+4-2r$	$4s-1+r$	$12s+3-r$	$1 \leqslant r \leqslant 3$
(12)	$4s$	$12s+3$	$16s+3$	

Table 3.7: A construction of disjoint hooked near-Skolem sequences of order n and defect m

Case 4: $n \equiv 3 \pmod 8$. For hooked near-Skolem sequences with $m = 3$ and $m = 1$, we use the reverse of the Langford sequence in [36], $d = 4s$, $s \geqslant 1$, and $t \geqslant 2s$ (t is even numbers only). This gives the solutions for all the cases when $n \geqslant 11$. We add the sequence $(1, 1, 2, 0, 2)$ when $m = 3$ and the sequence $(2, 3, 2, 0, 3)$ when $m = 1$, to the reverse of the Langford sequence. We provide the remaining cases of $n = 11$.

For $n = 11$ and $m = 5$, we have $(6, 4, 2, 8, 2, 4, 6, 11, 9, 7, 10, 8, 3, 1, 1, 3, 7, 9, 11, 0, 10)$.

For $n = 11$ and $m = 7$, we have $(6, 9, 3, 5, 8, 3, 6, 10, 5, 11, 9, 4, 8, 1, 1, 4, 2, 10, 2, 0, 11)$.

For $n = 11$ and $m = 9$, we have $(6, 7, 2, 8, 2, 4, 6, 11, 7, 4, 10, 8, 5, 3, 1, 1, 3, 5, 11, 0, 10)$.

For $n = 11$ and $m = 11$, it is a hooked Skolem sequences of order 10 which is existed by O'Keefe [24].

For all $n \geqslant m > 3$, $n > 11$, let $n = 8s + 3$ and $m = 2t + 1$, the required construction is:

row numbers	i	a_i	b_i	
(1)	$8s-2r+3$	$2+r$	$8s+5-r$	$0 \leqslant r \leqslant 4s-t$
(2)	1	$4s-t+3$	$4s-t+4$	
(3)	$2t-1-2r$	$4s-t+5+r$	$4s+t+4-r$	$0 \leqslant r \leqslant t-3$
(4)	$4s+2$	1	$4s+3$	
(5)	$8s+4-2r$	$4s+3+r$	$12s+7-r$	$1 \leqslant r \leqslant 3$
(6)	2	$10s+2$	$10s+4$	
(7)	$8s-4-4r$	$8s+2r+7$	$16s+3-2r$	$0 \leqslant r \leqslant s-2$
(8)	$4s-2r$	$10s+5+r$	$14s+5-r$	$0 \leqslant r \leqslant 2s-2$
(9)	3	$16s+2$	$16s+5$	
(10)	$8s-4r-6$	$8s+6+2r$	$16s-2r$	$0 \leqslant r \leqslant s-3$

Table 3.8: A construction of disjoint hooked near-Skolem sequences of order n and defect m

Case 5: $n \equiv 4 (\text{mod } 8)$. For a hooked near-Skolem sequence with $m = 2$, we use the reverse of the Langford sequence in [36] when $d = 4s + 1$, $s \geqslant 1$, $t \geqslant 2s + 1$ (t is even numbers). This gives the solutions for all the cases when $n \geqslant 20$. We add the sequence $(4, 1, 1, 3, 4, 0, 3)$ to the reverse of the Langford sequence. We provide the remaining case of $n = 12$.

For $n = 12$ and $m = 2$, we have,

$(5, 1, 1, 3, 11, 5, 3, 7, 12, 10, 8, 6, 4, 9, 7, 11, 4, 6, 8, 10, 12, 0, 9)$.

For $n = 12$ and $m = 4$, we have,

$(2, 5, 2, 1, 1, 11, 5, 7, 12, 10, 8, 6, 3, 9, 7, 3, 11, 6, 8, 10, 12, 0, 9)$.

For $n = 12$ and $m = 6$, we have,

$(7, 3, 9, 11, 3, 1, 1, 7, 12, 10, 8, 9, 4, 2, 11, 2, 4, 5, 8, 10, 12, 0, 5)$.

For $n = 12$ and $m = 8$, we have,

$(6, 4, 2, 9, 2, 4, 6, 7, 12, 10, 5, 11, 9, 3, 7, 5, 3, 1, 1, 10, 12, 0, 11)$.

For $n = 12$ and $m = 10$, we have,

$(9, 5, 2, 4, 2, 7, 5, 4, 12, 9, 8, 11, 7, 6, 3, 1, 1, 3, 8, 6, 12, 0, 11)$.

For $n = 12$, $m = 12$, it is a hooked Skolem sequence of order 11 which is existed by O'Keefe [24].

For $n \geqslant m > 2$ and $n > 12$, let $n = 8s + 4$ and $m = 2t$, the required construction is:

row numbers	i	a_i	b_i	
(1)	$8s+4-2r$	$8s+r$	$16s+4-r$	$0 \leqslant r \leqslant 4s-t+1$
(2)	1	$12s-t+2$	$12s-t+3$	
(3)	$2t-2-2r$	$12s-t+4+r$	$12s+t+2-r$	$0 \leqslant r \leqslant t-3$
(4)	2	$16s+5$	$16s+7$	
(5)	$8s-2r+5$	$4s+r$	$12s+5-r$	$1 \leqslant r \leqslant 3$
(6)	3	$2s+1$	$2s+4$	
(7)	$8s-3-4r$	$1+2r$	$8s-2-2r$	$0 \leqslant r \leqslant s-1$
(8)	$8s-5-4r$	$4+2r$	$8s-1-2r$	$0 \leqslant r \leqslant s-1$
(9)	$4s-7-4r$	$2s+6+2r$	$6s-1-2r$	$0 \leqslant r \leqslant s-3$
(10)	$4s-5-4r$	$2s+3+2r$	$6s-2-2r$	$0 \leqslant r \leqslant s-3$
(11)	$4s-3$	2	$4s-1$	

Table 3.9: A construction of disjoint hooked near-Skolem sequences of order n and defect m

Case 6: $n \equiv 5 (\text{mod } 8)$. For a hooked near-Skolem sequence with $m = 2$, we use the reverse of the Langford sequence in [36] when $d \equiv 2 (\text{mod } 4)$, $d = 4s+2$, $s \geqslant 1$, $t \geqslant 2s+1$ (t takes even numbers only). This gives the solutions for all the cases of $n \geqslant 21$. We add the sequence $(5, 1, 1, 3, 4, 5, 3, 0, 4)$ to the reverse of the Langford sequence. We provide the remaining cases of $n = 5$ and $n = 13$.

For $n = 5$ and $m = 2$, we have $(5, 1, 1, 3, 4, 5, 3, 0, 4)$.

For $n = 5$ and $m = 4$, we have $(5, 3, 1, 1, 3, 5, 2, 0, 2)$.

For $n = 13$ and $m = 2$, we have

$(9, 5, 3, 13, 11, 3, 5, 1, 1, 9, 12, 10, 8, 6, 4, 11, 13, 7, 4, 6, 8, 10, 12, 0, 7)$. For all $n > m > 2$

row numbers	i	a_i	b_i	
(1)	$8s+4-2r$	$8s+r+2$	$16s+6-r$	$0 \leqslant r \leqslant 4s-t+1$
(2)	1	$12s+t+3$	$12s+t+4$	
(3)	$2t-2-2r$	$12s-t+4+r$	$12s+t+2-r$	$0 \leqslant r \leqslant t-3$
(4)	2	$16s+7$	$16s+9$	
(5)	$8s-2r+7$	$4s+r-2$	$12s+5-r$	$1 \leqslant r \leqslant 3$
(6)	3	$6s$	$6s+3$	
(7)	$8s-1-4r$	$1+2r$	$8s-2r$	$0 \leqslant r \leqslant s-1$
(8)	$8s-3-4r$	$4+2r$	$8s+1-2r$	$0 \leqslant r \leqslant s-2$
(9)	$4s-3-4r$	$2s+1+2r$	$6s-2-2r$	$0 \leqslant r \leqslant s-2$
(10)	$4s-1-4r$	$2s+2+2r$	$6s+1-2r$	$0 \leqslant r \leqslant s-2$
(11)	$4s+1$	2	$4s+3$	

Table 3.10: A construction of disjoint hooked near-Skolem sequences of order n and defect m

and $n > 5$, let $n = 8s+5$ and $m = 2t$, the required construction is Table 3.10:

Case 7: $n \equiv 6 (\text{mod } 8)$. For hooked near-Skolem sequences with $m = 3$ and $m = 1$, we use the reverse of the Langford sequence in [36] when $d = 4s-1$, $s \geqslant 2$, $t \geqslant 2s$ (t is even numbers only). This gives the solutions for all of the cases when $n \geqslant 22$. We add the sequence $(5,1,1,4,6,5,2,4,2,0,6)$ when $m = 3$ and the sequence $(2,5,2,4,6,3,5,4,3,0,6)$ when $m = 1$ to the reverse of the Langford sequence. We provide all of the cases when $n = 6$ and $n = 14$.

For $n = 6$ and $m = 3$, we have $(2,6,2,5,1,1,4,6,5,0,4)$.

For $n = 6$ and $m = 5$, we have $(3,6,4,3,1,1,4,6,2,0,2)$.

For $n = 14$ and $m = 3$, we have

$(4, 6, 1, 1, 4, 14, 12, 6, 2, 8, 2, 13, 11, 9, 7, 5, 10, 8, 12, 14, 5, 7, 9, 11, 13, 0, 10)$.

For $n = 14$ and $m = 5$, we have

$(6, 4, 1, 1, 14, 4, 6, 8, 13, 11, 9, 7, 14, 10, 3, 8, 12, 3, 7, 9, 11, 13, 2, 10, 2, 0, 14)$.

For $n = 14$ and $m = 7$, we have

$(3, 14, 5, 3, 6, 1, 1, 5, 10, 8, 6, 13, 11, 9, 12, 14, 4, 8, 10, 2, 4, 2, 9, 11, 13, 0, 12)$.

For $n = 14$ and $m = 9$, we have

$(1, 1, 10, 6, 3, 14, 12, 3, 8, 6, 13, 11, 10, 4, 7, 5, 8, 4, 12, 14, 5, 7, 11, 13, 2, 0, 2)$.

For $n = 14$ and $m = 11$, we have

$(1, 1, 2, 6, 2, 8, 4, 12, 10, 6, 4, 13, 14, 8, 9, 7, 5, 3, 10, 12, 3, 5, 7, 9, 13, 0, 14)$.

For $n = 14$ and $m = 13$, we have

$(6, 10, 8, 4, 14, 12, 6, 4, 1, 1, 8, 10, 11, 9, 7, 5, 3, 12, 14, 3, 5, 7, 9, 11, 2, 0, 2)$.

For all n, m where $n > m > 3$ and $n > 14$, when $n = 8s + 6$ and $m = 2t + 1$, the required construction is:

row numbers	i	a_i	b_i	
(1)	$8s+5-2r$	$r+2$	$8s+7-r$	$0 \leqslant r \leqslant 4s-t+1$
(2)	1	$4s-t+4$	$4s-t+5$	
(3)	$2t-1-2r$	$4s-t+6+r$	$4s+t+5-r$	$0 \leqslant r \leqslant t-3$
(4)	2	$10s+4$	$10s+6$	
(5)	$8s-2r+8$	$4s+r+3$	$12s+11-r$	$1 \leqslant r \leqslant 3$
(6)	3	$16s+8$	$16s+11$	
(7)	$8s-4r$	$8s+9+2r$	$16s+9-2r$	$0 \leqslant r \leqslant s-2$
(8)	$8s-2-4r$	$8s+8+2r$	$16s+6-2r$	$0 \leqslant r \leqslant s-3$
(9)	$4s+4-2r$	$10s+7+r$	$14s+11-r$	$0 \leqslant r \leqslant 2s$
(10)	$4s+6$	1	$4s+7$	

Table 3.11: A construction of disjoint hooked near-Skolem sequences of order n and defect m

Case 8: $n \equiv 7 (\text{mod } 8)$. For hooked near-Skolem sequences with $m = 3$ and $m = 1$, we use the reverse of the Langford sequence in [36] when $d \equiv 0 (\text{mod } 4)$, $d = 4s$, $s \geqslant 2$, $t \geqslant 2s$ (t is even numbers only). This gives the solutions for all of the cases when $n \geqslant 23$. We add the sequence $(6, 1, 1, 4, 5, 7, 6, 4, 2, 5, 2, 0, 7)$ when $m = 3$ and the sequence $(6, 4, 7, 5, 3, 4, 6, 3, 5, 7, 2, 0, 2)$ when $m = 1$ to the reverse of the Langford sequence. We provide all of the cases when $n = 7$ and $n = 15$.

For $n = 7$ and $m = 3$, we have $(6, 1, 1, 4, 5, 7, 6, 4, 2, 5, 2, 0, 7)$.

For $n = 7$ and $m = 5$, we have $(3, 6, 7, 3, 4, 1, 1, 6, 4, 7, 2, 0, 2)$.

For $n = 7$, $m = 7$, it is a hooked Skolem sequence of order 6 which is existed by

O'Keefe [24].

For $n = 15$ and $m = 3$, we have

$(4, 1, 1, 6, 4, 14, 12, 10, 2, 6, 2, 15, 13, 11, 9, 7, 5, 10, 12, 14, 8, 5, 7, 9, 11, 13, 15, 0, 8)$.

For $n = 15$ and $m = 5$, we have

$(10, 8, 6, 4, 1, 1, 14, 4, 6, 8, 10, 15, 13, 11, 9, 7, 12, 2, 3, 2, 14, 3, 7, 9, 11, 13, 15, 0, 12)$.

For $n = 15$ and $m = 7$, we have

$(3, 1, 1, 3, 4, 2, 14, 2, 4, 10, 8, 15, 13, 11, 9, 6, 12, 5, 8, 10, 14, 6, 5, 9, 11, 13, 15, 0, 12)$.

For $n = 15$ and $m = 9$, we have

$(11, 4, 2, 6, 2, 4, 8, 12, 10, 6, 15, 13, 11, 14, 8, 7, 5, 3, 10, 12, 3, 5, 7, 11, 13, 15, 0, 14)$.

For $n = 15$ and $m = 11$, we have

$(6, 8, 1, 1, 4, 9, 6, 14, 4, 8, 10, 15, 13, 2, 9, 2, 12, 7, 15, 3, 10, 14, 3, 5, 7, 13, 15, 0, 12)$.

For $n = 15$ and $m = 13$, we have

$(3, 6, 4, 3, 1, 1, 4, 6, 8, 12, 10, 15, 11, 9, 14, 2, 8, 2, 7, 5, 10, 12, 9, 11, 5, 7, 15, 0, 14)$.

For $n = 15$ and $m = 15$, it is a hooked Skolem sequence of order 14 which is existed by O'Keefe [24].

For all $n \geqslant m > 3$ and $n > 15$, let $n = 8s + 7$ and $m = 2t + 1$. For $n = 23$ only used lines $(*)$, and then added the following (i, j) pairs: $(24, 36)$, $(25, 29)$, $(26, 40)$, $(27, 43)$, $(28, 34)$, $(33, 41)$, $(35, 38)$, $(37, 39)$,$(22, 45)$, $(21, 42)$. The required construction is:

row numbers	i	a_i	b_i	
(1)	$^*8s-2r+3$	$1+r$	$8s-r+4$	$0 \leqslant r \leqslant 4s-t$
(2)	^{*1}	$4s-t+2$	$4s-t+3$	
(3)	$^*2t-1-2r$	$4s-t+4+r$	$4s+t+3-r$	$0 \leqslant r \leqslant t-3$
(4)	2	$14s+9$	$14s+11$	
(5)	$^*8s-2r+8$	$4s+r+1$	$12s+9-r$	$1 \leqslant r \leqslant 3$
(6)	3	$14s+7$	$14s+10$	
(7)	$^*4s+2$	$4s+5$	$8s+7$	
(8)	$4s-6-4r$	$10s+2r+11$	$14s+5-2r$	$0 \leqslant r \leqslant s-3$
(9)	$8s-4r$	$8s+2r+11$	$16s+11-2r$	$0 \leqslant r \leqslant s-1$
(10)	$4s-4$	$8s+8$	$12s+4$	
(11)	$8s-2-4r$	$8s+2r+10$	$16s+8-2r$	$0 \leqslant r \leqslant s-2$
(12)	4	$14s+4$	$14s+8$	
(13)	$4s-4r-8$	$10s+2r+10$	$14s-2r+2$	$0 \leqslant r \leqslant s-4$
(14)	$4s-2$	$10s+8$	$14s+6$	
(15)	$4s$	$8s+9$	$12s+9$	
(16)	$^*8s+7-2r$	$8s+6-r$	$16s+13-3r$	$0 \leqslant r \leqslant 1$

Table 3.12: A construction of disjoint hooked near-Skolem sequences of order n and defect m

This completes the proof of Theorem 3.3.5. ■

Chapter 4

Disjoint Langford sequences

4.1 Introduction

The study of constructing Langford sequences was introduced in 1958, when Langford [17] observed that his son had arranged colored blocks into three colored pairs (red, blue, and yellow), with one block between the red pair, two blocks between the blue pair, and three blocks between the yellow pair. He wrote the result as $(3, 1, 2, 1, 3, 2)$ or $(4, 2, 3, 2, 4, 3)$, and these sequences are known as Langford sequences of order 3 and defect $d = 2$.

Brouwer and Germa in 1976 [4], partitioned the set $\{1, 2, \ldots, 2n\}$ into n pairs (a_i, b_i) such that n numbers $b_i - a_i$, $1 \leqslant i \leqslant n$, are all the integers in the set $\{d, d +$

$1, \ldots, d+n-1\}$.

In 1959, Priday [25] and Davies [11] completely solved the case when $d = 2$. They called such sequences Langford when they do not contain hooks. Furthermore, Davies solved the problem of partitioning the set $\{1, 2, \ldots, 2n-1, 2n+1\}$ so that the differences (a_i, b_i) exhaust the set $\{2, 3, \ldots, n+1\}$, and he called such sequences hooked Langford.

It is shown in [4] that the necessary conditions for $\{d, d+1, \ldots, d+n-1\}$ to be Langford sequences are:

(i) $n \geqslant 2d-1$; and

(ii) $n \equiv 0$ or $1 (\mathrm{mod}\ 4)$ for d odd, $n \equiv 0$ or $3 (\mathrm{mod}\ 4)$, for d even. These conditions are also shown to be sufficient for $d = 3$ and $d = 4$. The conditions are shown to be sufficient for $n \geqslant 2d-1 (\mathrm{mod}\ 4)$ when n is odd.

In 1981, Simpson [36] established sufficiency for all even values of n satisfying (i) and he produced two theorems for (hooked) Langford sequences. He provided four tables of construction for Langford sequences of order n with defect d and five tables of construction for hooked Langford sequences of order n with defect d (see reference [36]).

A k-extended Langford sequence of defect d and length n is a sequence $(l_1, l_2, \ldots, l_{2n+1})$ in which l_k is an empty position that occurs in anywhere in the sequence and filled by 0, and each other element of the sequence comes from the set $\{d, d+1, \ldots, d+n-1\}$. Each $j \in \{d, d+1, \ldots, d+n-1\}$ occurs exactly twice in the sequence, and the two occurrences are separated by exactly $j-1$ elements. A hooked k-extended Langford sequence of defect d and length n is a partition of $\{1, \ldots, 2n+2\} \setminus \{2, k\}$ (where 2 is the second position in the sequence includes the hook that is also filled by 0) into differences $\{d, d+1, \ldots, d+n-1\}$. For example, we have $(2, 5, 2, 4, 0, 3, 5, 4, 3)$ a 5-extended Langford sequence of defect 2 and length 4. It is clearly to see that the empty position 0 is the extension $k = 5$. We also have $(5, 0, 4, 0, 3, 5, 4, 3)$ a hooked 4-extended Langford sequence of defect 3 and length 3. The extension $k = 4$ and the hook occurs at the second.

In 1998, Linek and Jiang [20] produced several constructions for (hooked)-extended Langford sequences with small defects for $d = 2$ and 3.

In 2003, Linek and Mor [18] considered the problem of constructing (hooked)-extended Langford sequences with large defects. They derived the necessary conditions for the existence of extended Langford sequences for $d \geqslant 4$ and all possible k extensions, but with a finite number of lengths n and differences D_i where $i = 0, 1$, and 2.

Two sequences have n pairs in common if n distinct entries occur in the same positions in the sequences. In 2012, Shalaby and Silvesan [32] checked the reverse sequences of all the known constructions of Langford sequences in [36], [4], and [18], to determine whether they are reverse-disjoint sequences or have pairs in common. They produced a table that shows the intersection between Langford sequences and their reverse sequences. In the references [32] and [35], they used the known Langford sequences given in [36], [4], and [18] and (hooked) Skolem sequences to construct two new Skolem sequences of order n with $0, 1, \ldots, n-3$ and n pairs in common. Moreover, they used these results and introduced triple systems with blocks in common for $\lambda = 2, 3$ and 4.

Shalaby and Silvesan [35] used Skolem sequences and Langford sequences to produce the fine structure of a $CTS(v, 2)$ for $v \equiv 1, 3 (\text{mod } 6)$, $v \neq 9$, and a $CTS(v, \lambda)$ for $v \equiv 1, 7 (\text{mod } 24)$ and $\lambda = 3, 4$.

4.2 Known results for disjoint Langford sequences

In this section, we present the known theorems of disjoint Langford sequences, beginning with the work of Shalaby and Silvesan [32]. When Shalaby and Silvesan reversed all the known constructions of Langford sequences, they found that some sequences are disjoint and some have pairs in common, and they tabulated the results. Baker and Shalaby [2] proved the existence of reverse-disjoint Langford sequences of order

n with defect 2 and produced the following lemma. (See reference [2] for the proof).

Lemma 4.2.1 *[2] For* $n \equiv 0, 1 (\mathrm{mod}\ 4)$*, there exists a reverse-disjoint Langford sequence of order* n *with* $d = 2$.

Example 4.2.1 *Let* $L = (4, 2, 3, 2, 4, 3)$ *be a Langford sequence of order* 3 *with* $d = 2$*, and let* $\overleftarrow{L} = (3, 4, 2, 3, 2, 4)$ *be its reverse and it is also a Langford sequence of order* 3 *and defect* 2*. We notice that* L *and* $\overleftarrow{L}$ *are disjoint, so* L *is a reverse-disjoint Langford sequence.*

Shalaby and Silvesan [32] found that the reverses of the known constructions given in [36], [18] and [4] yield two disjoint Langford sequences with a finite number of exceptions. They arranged the results they found in Table 4.1 given in [32].

We represent Table 4.1 given in [32], the last column of the table includes the complete reference for the instructions (reference, theorem number or table number, row number).

	n	d	Pairs in common	Reference
1	$4t$	$4s$	1 if $s \equiv 1 \pmod 3$	[36], 1(11)
		$s \geq 1$	1 if $s \equiv 2 \pmod 3$	[36], 1(12)
		$t \geq 2s$	0 if $s \equiv 0 \pmod 3$	[36], 1
		$4s+1$	0 if $s \equiv 1, 2 \pmod 3$	[36], 1
		$s \geq 1$	2 if $s \equiv 0 \pmod 3$	[36], 1(11), (12)
		$t \geq 2s+1$		
		$4s+2$	1 if $s \equiv 0, 2 \pmod 3$	[36], 1(14)
		$s \geq 1$	3 if $s \equiv 1 \pmod 3$	[36], 1(11), (12), (14)
		$t \geq 2s+1$		
		$4s-1$	0 if $s = 1$ or 2	[36], 1
		$s \geq 1$	$s-2$ if $s \equiv 2 \pmod 3$	[36], 1(8)
		$t \geq 2s$	$s \geq 5$	
			$s-1$ if $s \equiv 0, 1 \pmod 3$	[36], 1(8), (12)
			$s \geq 3$	
2	$4t$	$2t-e$	2 if $e \equiv 1 \pmod 3$	[4], 3(3), (4)
		$t \geq 2e+1$	0 if $e \equiv 0, 2 \pmod 3$	
3	$2d-1$	$d \geq 2$	0	[4], 2
	(mod 4)	d even		
		$d \geq 3$	0 if $n \neq 2d-1$	[4], 2
		d odd	1 if $n = 2d-1$	[4], 2(4)
4	$2d$	0 (mod 6)	1	[18] $1d$, (4)
		2 (mod 6)	1	[18] $1d$, (3)
		4 (mod 6)	0	[18] $1d$
5	$2d-1$	0 (mod 3)	$0, 1$ if $d = 3$	[18], $0a$ and [4], 2
			$0, 1, 2$ if $d \geq 6$	[18], $0a$ and [4], 2
		1 (mod 3)	$0, 1, 2$ if $d \geq 4$	[18], $0a$ and [4], 2
		2 (mod 3)	0 if $d = 2$	[18], $0a$ and [4], 2
			$0, 1, 3$ if $d = 5$	[18], $0a$ and [4], 2
			$0, 1, 2, 3$ if $d \geq 8$	[18], $0a$ and [4], 2

Table 4.1: The number of pairs in common between Langford sequences and their reverses

In [34], Shalaby and Silvesan used Langford sequences adjoined with (hooked) Skolem sequences to produce cyclic triple systems of order $(6n + 1)$ denoted by $CTS(6n+1)$. For example, consider a Skolem sequence of order 4, and adjoin it with a Langford sequence of order 12 and defect 5. We take the same Skolem sequence of order 4, and adjoin it with the reverse-disjoint Langford sequence of order 12 and defect 5. This gives two Skolem sequences of order 16 with four pairs in common. From this example, we can construct a cyclic triple system with base blocks of the forms $\{\{0, i, b_i + n\}(\text{mod } 6n + 1)\}$ for $i = 1, 2, \ldots, n$, and $\{\{0, a_i + n, b_i + n\}(\text{mod } 6n + 1)\}$ for $i = 1, 2, \ldots, n$, for each Skolem sequence. We have two cyclic $TS(6n + 1)$ with four repeated base blocks, two cyclic $TS(6n + 1)$ with three repeated base blocks, or two cyclic $TS(6n + 1)$ with two repeated base blocks.

We demonstrate an example that shows two disjoint cyclic $STS(6n+1)$ by using a Skolem sequence of order n and a Langford sequences of order n with defect d.

Example 4.2.2 *Let* $A = (12, 10, 8, 6, 11, 9, 7, -, -, 6, 8, 10, 12, 7, 9, 11)$ *be a sequence of order* 12 *and defect* 7 *formed by even and odd numbers and some free spaces in the middle of the sequence. We fit in these two spaces, a Skolem sequence of order 1,* $S_1 = (1, 1)$*. We attach a Langford sequence of defect 2 and order 4,* $L = (5, 2, 4, 2, 3, 5, 4, 3)$ *to the right of the sequence* A *and* $S_1 = (1, 1)$*. Thus, we obtain a Skolem sequence of order 12 as well as pairs* (a_i, b_i) *for* $i = 1, 2, \ldots, 12$*.*

$S = (12, 10, 8, 6, 11, 9, 7, 1, 1, 6, 8, 10, 12, 7, 9, 11, 5, 2, 4, 2, 3, 5, 4, 3)$, *the pairs are* $(8, 9)$, $(18, 20)$, $(21, 24)$, $(19, 23)$, $(17, 22)$, $(4, 10)$, $(7, 14)$, $(3, 11)$, $(6, 15)$, $(2, 12)$, $(5, 16)$, *and* $(1, 13)$.

We consider the base blocks of the following forms:

1.$\{\{0, a_i + n, b_i + n\} (\text{mod } 6n + 1)\}$ *for* $i = 1, 2, \ldots, n$, *so we obtain:* $\{0, 20, 21\}$, $\{0, 30, 32\}$, $\{0, 33, 36\}$, $\{0, 31, 35\}$, $\{0, 29, 34\}$, $\{0, 16, 22\}$, $\{0, 19, 26\}$, $\{0, 15, 23\}$, $\{0, 18, 27\}$, $\{0, 14, 24\}$, $\{0, 17, 28\}$, $\{0, 13, 25\}$*; and*

2.$\{\{0, i, b_i + n\} (\text{mod } 6n + 1)\}$ *for* $i = 1, 2, \ldots, n$, *so we obtain:* $\{0, 1, 21\}$, $\{0, 2, 32\}$, $\{0, 3, 36\}$, $\{0, 4, 35\}$, $\{0, 5, 34\}$, $\{0, 6, 22\}$, $\{0, 7, 26\}$, $\{0, 8, 23\}$, $\{0, 9, 27\}$, $\{0, 10, 24\}$, $\{0, 11, 28\}$, $\{0, 12, 25\}$. *We observe that all the non-zero elements exist as differences twice in* $\mathbb{Z}_{6n+1}$. *We develop each base block we obtained* $(\text{mod } 6n + 1)$.

Thus, we obtain two disjoint cyclic $STS(6n + 1)$.

Shalaby and Silvesan [32] modified Table 0a given in [18] by moving the pair $n = 2d - 1$ from the end of the sequence to the beginning and called such sequences modified Langford sequences. They arrived at two different disjoint Langford sequences of order $2d-1$ using the Langford sequence from [20], Table 0a, and the modified Langford sequence. They found one pair in common if $d = 5$ or $d \equiv 1 (\text{mod } 3)$, where $d \neq 4$, and two pairs in common if $d = 4$ or $d \equiv 0, 2 (\text{mod } 3)$, where $d \neq 5$ using the modified Langford sequence, and the Langford sequence from [4].

Example 4.2.3 *Let* $L = (11, 13, 7, 9, 6, 12, 10, 8, 5, 7, 6, 11, 9, 5, 13, 8, 10, 12)$ *and* $L' = (10, 11, 12, 13, 5, 6, 7, 8, 9, 5, 10, 6, 11, 7, 12, 8, 13, 9)$ *be two Langford sequences of defect* 5 *and order* 9*. Both sequences are not disjoint because there is one pair in common. The pair is* $(8, 16)$*.*

Example 4.2.4 *Let* $L = (7, 9, 6, 10, 8, 4, 5, 7, 6, 4, 9, 5, 8, 10)$

and $L' = (7, 8, 9, 10, 4, 5, 6, 7, 4, 8, 5, 9, 6, 10)$ *be two Langford sequences of defect* 4 *and order* 7*. The sequences are not disjoint because there are two pairs in common. The pairs are* $(1, 8)$ *and* $(4, 14)$*.*

Similarly, there is one pair in common if $d = 3$ or $d = 4$. There are no pairs in common if $d \equiv 0, 2 (\text{mod } 3)$, $d \neq 3$ and there are two pairs in common if $d \equiv 1 (\text{mod } 3)$, $d \neq 4$. Shalaby and Silvesan found three pairs in common for $d \equiv 2 (\text{mod } 3)$, $d \geqslant 5$, by taking the modified Langford sequence and the reverse of the Langford sequence from [18].

Shalaby and Silvesan [33] produced two more tables that give pairs in common between Langford sequences and their reverses. They added the pair $(1, 1)$ to the beginning or at the end of a Langford sequence $L_d^{n \equiv 2d-1 (\text{mod } 4)}$ (see reference [4], Theorem 2). The Langford sequence with the pair $(1, 1)$ appearing at the end of the sequence is denoted by $L_d^n(1, 1)$, and the Langford sequence with the pair $(1, 1)$ appearing at the beginning of the sequence is denoted by $(1, 1)L_d^n$. For example, we attach the pair

$(1, 1)$ at the beginning or at the end of a Langford sequence $(4, 2, 3, 2, 4, 3)$ of order 3 and defect 2, and we obtain a Skolem sequence $(1, 1, 4, 2, 3, 2, 4, 3)$ of order 4 that is a reverse-disjoint sequence. We also attach the pair $(1, 1)$ at the beginning or at the end of a Langford sequence $(5, 2, 4, 2, 3, 5, 4, 3)$ of order 4 and defect 2, and we obtain a Skolem sequence $(1, 1, 5, 2, 4, 2, 3, 5, 4, 3)$ of order 5 that is not a reverse-disjoint sequence.

We will represent this in a table given in [33], showing the number of pairs in common between $L_d^n(1,1)$ and their reverses.

	n	d	Sequence	Pairs in common	Reference
1	1 (mod 4)	9	$L_d^n(1,1)$	0	[4], 2
2	$4t$	$4s+1$ $s \geq 1$ $t \geq 2s+1$	$(1,1)L_d^n$	2 if $s = 1$ 3 if $s = 2$ $s+2$ if $s \equiv 2 \pmod 3$ $s \geq 5$ s if $s \equiv 0, 1 \pmod 3$ $s \geq 3$	[36], 1, (8), (13) [36], 1, (8)(11) [36], 1 (8), (11), (12) [36], 1, (8)
			$L_d^n(1,1)$	2 if $s \equiv 1 \pmod 3$ $s \geq 4$ 0 otherwise	[36], 1 (11), (12) [36], 1
3	$2d+3$	2 (mod 4) $d \neq 2$	$L_d^n(1,1)$	0	[4], 2

Table 4.2: The number of pairs in common between L_d^n with $(1,1)$ appended and their reverses

Shalaby and Silvesan [33] presented similar arguments for the case of hooked Langford sequences. They attached the triple $(2,0,2)$ to the sequence, so 0 can take the position of the hook. They reversed the obtained sequence and checked whether the sequences are disjoint or have pairs in common. The hooked Langford sequence with the triple $(2,0,2)$ is denoted by $hL_d^n * (2,0,2)$. For example, we attach the triple $(2,0,2)$ to a hooked Langford sequence $(9,5,3,7,8,3,5,4,6,9,7,4,8,0,6)$ of order 7 and defect 3. We obtain a Langford sequence $(9,5,3,7,8,3,5,4,6,9,7,4,8,2,6,2)$ of order 8 and defect 2.

We will represent this in a table given in [33], showing the number of pairs in common between $hL_d^n * (2,0,2)$ and their reverses.

	n	d	r	Pairs in common	Reference
1	$2d+1+4r$	3	0	1	[36], 2(5)
			$r \geq 1$	0	[36], 2
		4	0	2	[36], 2, (5)(10$*$)
			$r \geq 1$	0	[36], 2
		d even, $d \geq 6$	$r = 0$	1	[36], 2, (5)
			$r = 1$	1	[36], 2, (10)
			$r \geq 2$	0	[36], 2
		d odd, $d \geq 5$	$r = 0$	1	[36], 2, (5)
			$r \geq 1$	0	[36], 2
2	$2d+1$	0 (mod 6)		1	[18], $2A$, (2)
		4 (mod 6)		1	[18], $2A$, (5)
		$d \neq 4$			
		2 (mod 6)		0	[18], $2A$
3	$4t+2$	$4s$		1 if $s = 2$	[36], 2, (11)
		$t - 2s = r$		2 if $s \equiv 2 \pmod 3$	[36], 2, (14), (16)
		$r \geq 0$		$s \geq 5$	
				0 if $s \equiv 0, 1 \pmod 3$	[36], 2
		$4s+2$		2 if $s \equiv 2 \pmod 3$	[36], 2, (14), (17)
		$t - 2s - 1 = r$		0 if $s \equiv 0, 1 \pmod 3$	[36], 2
		$r \geq 0$		$s \neq 1$	
				1 if $s = 1$	[36], 2, (11)
		$4s+3$		0 if $s \equiv 1 \pmod 3$	[36], 2
		$t - 2s - 1 = r$		1 if $s \equiv 0 \pmod 3$	[36], 2, (17)
		$r \geq 0$		$s \neq 0$	
				1 if $s \equiv 2 \pmod 3$	[36], 2, (14)
				$s \neq 0$	
				0 if $s = 0$	[36], 2
		$4s+1$		3 if $s = 2$	[36], 2, (9), (13), (16)
		$t - 2s \geq 0$		2 if $s = 3$	[36], 2, (9)
				$s - 1$ if $s \equiv 1 \pmod 3$	[36], 2, (9), (14)
				s if $s \equiv 0, 2 \pmod 3$	[36], 2, (9), (16)
				$s \geq 5$	

Table 4.3: The number of pairs in common between $hL_d^n * (2,0,2)$ and their reverses

We show examples for the case that includes one pair in common and the case that includes two pairs in common by attaching the pair $(2,0,2)$ to a hooked Langford sequence of order n and defect d.

Example 4.2.5 *We first form a hooked Langford sequence of order* 7 *and defect* 3

$hL = (8,5,7,9,3,6,5,3,8,7,4,6,9,0,4)$. *We attach the triple* $(2,0,2)$ *at the end of the sequence. We obtain* $L = (8,5,7,9,3,6,5,3,8,7,4,6,9,2,4,2)$, *a Langford sequence of order* 8 *and defect* 2. *When we reverse this sequence, we obtain* $\overleftarrow{L}=$ $(2,4,2,9,6,4,7,8,3,5,6,3,9,7,5,8)$ *a Langford sequence of order* 8 *and defect* 2. *It is clear to see that* L *and* $\overleftarrow{L}$ *sequences are not disjoint because there is one pair in common, the pair* $(4,13)$.

Example 4.2.6 *Let* $L = (5,8,4,9,7,5,4,3,6,8,3,7,9,2,6,2)$

and $\overleftarrow{L}= (2,6,2,9,7,3,8,6,3,4,5,7,9,4,8,5)$ *be two Langford sequences of defect* 2 *and order* 8. *These sequences are not disjoint because there are two pairs in common. The pairs are* $(4,13)$ *and* $(5,12)$.

4.3 Previous unpublished results

In this section, we prove four cases for two disjoint Langford sequences of order n and defect d, $(d > 2)$ and four cases for two disjoint hooked Langford sequences of order n and defect d, note that these cases are not seen in the literature. We show that two cases, of the four cases for two disjoint Langford sequences, provide two disjoint Langford sequences by modifying Simpson's approach (Theorem 1, Case 2 and Case 4 in [36]). We also show that the another two cases yield two disjoint Langford sequences with finite exceptions. Following these results, we produce four cases for two disjoint hooked Langford sequences of order n and defect d with finite exceptions by applying three steps for each case. Now, we present Theorems 1 and 2 given in [36].

Theorem 4.3.1 *[36] Necessary and sufficient conditions for the sequence* $\{d, d+1, \ldots, d+n-1\}$ *to be a Langford sequence are:*

1. $n \geqslant 2d-1$*; and*

2. $n \equiv 0$ *or* $1 \pmod 4$ *for* d *odd,* $n \equiv 0$ *or* $3 \pmod 4$*, for* d *even.*

Theorem 4.3.2 *[36] Necessary and sufficient conditions for the sequence* $\{d, d+1, \ldots, d+n-1\}$ *to be a hooked Langford sequence are:*

3. $n(n+1-2d)+2 \geqslant 0$*; and*

4. $n \equiv 2$ *or* $3 \pmod 4$ *for* d *odd,* $n \equiv 2$ *or* $1 \pmod 4$*, for* d *even.*

Now, we prove four cases of Langford sequences of order n and defect d.

Case 1: We use tables similar to those in [4]. For $d \equiv 2 \pmod 4$, let $n = 4t$ and suppose $d = 4s + 2$, $s \geqslant 1$, $t \geqslant 2s + 1$. The required construction yields disjoint Langford sequences with Langford sequences given in Theorem 1, Case 2 in [36].

Row numbers	a_i	b_i	$b_i - a_i$	$0 \leqslant j \leqslant$
(1)	$2t - 3s - j$	$2t + s + 3 + j$	$4s + 3 + 2j$	$t - 2s - 2$
(2)	$t - 2s + 1 - j$	$3t + s + 2 + j$	$2t + 3s + 1 + 2j$	$t - 2s - 1$
(3)	$6t - j$	$6t + 3s + 3 + j$	$3s + 3 + 2j$	$t - 2s - 2$
(4)	$5t - s - j$	$7t + 2s + 2 + j$	$2t + 3s + 2 + 2j$	$t - 2s - 2$
(5)	$5t - s + 1$	$7t + s + 2$	$2t + 2s + 1$	
(6)	$2t + 1 - j$	$4t + 2 + j$	$2t + 1 + 2j$	$s - 1$
(7)	$4t - s + 2 + j$	$t + s + 4 + 2j$	$2t + 2s + 2 + j$	$s - 2$
(8)	$3t + 2 - j$	$5t + 2 + j$	$2t + 2j$	s
(9)	$3t + s + 1 - j$	$7t + s + 3 + j$	$4t + 2 + 2j$	$s - 2$
(10)	$t - s + 1 - j$	$5t - s + 2 + j$	$4t + 1 + 2j$	$s - 1$
(11)	$2t - 3s + 1 + j$	$6t - s + 3 + 2j$	$4t + 2s + 2 + j$	$2s - 1$
(12)	$2t + 2 + j$	$6t - s + 2 + 2j$	$4t - s + j$	$s - 1$
(13)	$2t + s + 2$	$6t + 3s + 2$	$4t + 2s$	
(14)	$2t - s + 1$	$6t + s + 2$	$4t + 2s + 1$	
(15)	1	$4t + 1$	$4t$	

Table 4.4: A construction of disjoint Langford sequences of order n and defect d

Rows (7) and (9) are omitted for $s = 1$. Rows (1), (3), and (4) are omitted when $t = 2s + 1$. This construction yields Langford sequences of order n and defect d. We observe that this construction is disjoint with the construction of Theorem 1, Case 2 given in [36], because we take the last element that occurs at the end of the sequence and place it at the beginning. Thus, every position in the sequence will shift by one position to the right.

Case 2: Let $n = 4t$ and suppose $d = 4s + 1$, $s \geqslant 1$, $t \geqslant 2s + 1$. The required construction yields disjoint Langford sequences with Langford sequences given in Theorem 1, Case 4 in [36].

Row numbers	a_i	b_i	$b_i - a_i$	$0 \leqslant j \leqslant$
(1)	$2t - 3s - j$	$2t + s + 2 + j$	$4s + 2 + 2j$	$t - 2s - 2$
(2)	$t - 2s + 1 - j$	$3t + s + 2 + j$	$2t + 3s + 1 + 2j$	$t - 2s - 1$
(3)	$6t - s - j$	$6t + 3s + 1 + j$	$4s + 1 + 2j$	$t - 2s - 1$
(4)	$5t - s + 1 - j$	$7t + 2s + 1 + j$	$2t + 3s + 2j$	$t - 2s - 1$
(5)	$2t + 2 - j$	$4t + 2 + j$	$2t + 2j$	$s - 1$
(6)	$4t - s + 2 + j$	$6t + s + 3 + 2j$	$2t + 2s + 1 + j$	$s - 2$
(7)	$3t - j$	$5t + 1 + j$	$2t + 1 + 2j$	$s - 1$
(8)	$3t + s - j$	$7t + s + 1 + j$	$4t + 1 + 2j$	$s - 1$
(9)	$t - s - j$	$5t - s + 2 + j$	$4t + 2 + 2j$	$s - 2$
(10)	$2t - 3s + 1 + j$	$6t - s + 2 + 2j$	$4t + 2s + 1 + j$	$2s - 1$
(11)	$t - s + 1$	$3t + s + 1$	$2t + 2s$	
(12)	$2t + 3 + j$	$6t - s + 3 + 2j$	$4t - s + j$	$s - 2$
(13)	$2t - s + 2$	$6t - s + 1$	$4t - 1$	
(14)	$2t - s + 1$	$6t + s + 1$	$4t + 2s$	
(15)	1	$4t + 1$	$4t$	

Table 4.5: A construction of disjoint Langford sequences of order n and defect d

We omit rows (6), (9) and (12) when $s = 1$ and row (1) for $t = 2s + 1$. This construction yields Langford sequences of order n and defect d. We also observe that this construction is disjoint with the construction of Theorem 1, Case 4 given in [36], because we place the last element of the sequence at the beginning so that every position in the sequence will shift by one position to the right.

The sequences of Theorem 1, Case 1 and Case 3 given in [36] are not reverse-disjoint, and the existence of two disjoint Langford sequences for all admissible orders is still open for debate. We also demonstrate other new constructions by forming Langford sequences of order n and defect d and adjoining them with the Langford sequences Theorem 1, Case 1 and Case 3 given in [36]. This produces Langford sequences that are disjoint with those sequences in Theorem 1, Case 1 and Case 3 given in [36].

Case 3: Let $n = 4t + 3$ and suppose $d \equiv 0 (\mathrm{mod}\ 4)$, $d = 4s$, $s \geqslant 1$, $t \geqslant 8s - 1$. The following construction yields disjoint Langford sequences of order n and defect d.

Row numbers	a_i	b_i	$b_i - a_i$	$0 \leqslant j \leqslant$
(1)	$8s + t + 1 - j$	$13s + 3t + 1 + j$	$5s + 2t + 2j$	$t - 8s + 2$
(2)	$t + 11s - j$	$5t + 3s + 6 + j$	$4t - 8s + 6 + 2j$	$3s - 2$
(3)	$2t + 3s + 2 + j$	$6t + s + 6 + 2j$	$4t - 2s + 4 + j$	$6s - 2$
(4)	$2t + 9s + 1$	$6t + s + 5$	$4t - 8s + 4$	
(5)	$2t + 12s + 1 - j$	$4t + 8s + 3 + j$	$2t - 4s + 2 + 2j$	$3s - 1$
(6)	$2t + 12s + 2 + j$	$6t + s + 7 + 2j$	$4t - 11s + 5 + j$	$3s - 2$
(7)	$3t + 7s + 2$	$5t + 3s + 5$	$2t - 4s + 3$	
(8)	$3t + 10s - j$	$5t + 6s + 5 + j$	$2t - 4s + 5 + 2j$	$3s - 3$
(9)	$3t + 12s + 1 - j$	$7t + 5s + 5 + j$	$4t - 7s + 4 + 2j$	$3s - 1$
(10)	$4t + 5s + 4 + j$	$6t + 7s + 5 + 2j$	$2t + 2s + 1 + j$	$3s - 2$
(11)	$5t + 3s + 4 - j$	$7t + 8s + 5 + j$	$2t + 5s + 1 + 2j$	$t - 8s + 1$
(12)	$6t + s + 4 - j$	$6t + 13s + 3 + j$	$12s - 1 + 2j$	$t - 8s + 1$
(13)	$2t + 3s + 1 - j$	$2t + 15s + 1 + j$	$12s + 2j$	$t - 8s$
(14)	$1 + j$	$12s - 2 - j$	$12s - 3 - 2j$	$4s - 2$
(15)	$4s + j$	$16s - 2 - j$	$12s - 2 - 2j$	$4s - 1$

Table 4.6: A construction of disjoint Langford sequences of order n and defect d

Row (13) is omitted for $t = 8s - 1$. For the above construction, we simply form a Langford sequence of order n with $d = 4s$ and $n = 8s - 1$, $s \geqslant 1$. For example, we have $(9, 7, 5, 10, 8, 6, 4, 5, 7, 9, 4, 6, 8, 10)$ is a Langford sequence with $n = 7$, $d = 4$.

We use a known Langford sequence of order n given in [36] with $d = 4s - 1$, $s \geqslant 3$, $t \geqslant 2s$. We combine the sequence we formed with the known sequence given in [36], which results in a Langford sequence that is disjoint with the Langford sequence in Theorem 1, Case 1 given in [36].

Case 4: Let $n = 4t + 1$ and suppose $d = 4s - 1$, $s \geqslant 1$, $t \geqslant 8s - 3$. The following construction yields disjoint Langford sequences of order n and defect d.

Row numbers	a_i	b_i	$b_i - a_i$	$0 \leqslant j \leqslant$
(1)	$8s + t - 2 - j$	$13s + 3t - 3 + j$	$5s + 2t - 1 + 2j$	$t - 8s + 3$
(2)	$t + 11s - 3 - j$	$5t + 3s + 3 + j$	$4t - 8s + 6 + 2j$	$3s - 2$
(3)	$2t + 3s + 1 + j$	$6t + s + 4 + 2j$	$4t - 2s + 3 + j$	$6s - 4$
(4)	$2t + 12s - 4 - j$	$4t + 8s - 1 + j$	$2t - 4s + 3 + 2j$	$3s - 2$
(5)	$2t + 6s + 3 + j$	$6t + s + 3 + 2j$	$4t - s + j$	$3s - 2$
(6)	$2t + 15s - 4$	$6t + 13s - 3$	$4t - 2s + 1$	
(7)	$3t + 7s$	$5t + 3s + 2$	$2t - 7s + 2$	
(8)	$3t + 10s - 2 - j$	$5t + 6s + 2 + j$	$2t - 4s + 4 + 2j$	$3s - 3$
(9)	$3t + 13s - 4 - j$	$7t + 5s + 1 + j$	$4t - 8s + 5 + 2j$	$3s - 3$
(10)	$4t + 5s + 1 + j$	$6t + 7s + 1 + 2j$	$2t + 2s + j$	$3s - 3$
(11)	$5t + 3s + 1 - j$	$7t + 8s - 1 + j$	$2t + 5s - 2 + 2j$	$t - 8s + 3$
(12)	$2t + 3s - j$	$2t + 15s - 3 + j$	$12s - 3 + 2j$	$t - 8s + 2$
(13)	$6t + s + 2 - j$	$6t + 13s - 2 + j$	$12s - 4 + 2j$	$t - 8s + 2$
(14)	$1 + j$	$12s - 5 - j$	$12s - 6 - 2j$	$4s - 3$
(15)	$4s - 1 + j$	$16s - 6 - j$	$12s - 5 - 2j$	$4s - 2$

Table 4.7: A construction of disjoint Langford sequences of order n and defect d

Rows (12), and (13) are to be omitted when $t = 8s - 3$. We form a Langford sequence of order n with $d = 4s - 1$ and $n = 8s - 3$, $s \geqslant 1$. For example, we have $(6, 4, 7, 5, 3, 4, 6, 3, 5, 7)$, which is a Langford sequence with $n = 5$, $d = 3$.

We use a known Langford sequence of order n given in [36] with $d = 4s$, $s \geqslant 5$, $t \geqslant 2s$. We adjoin the sequence we formed with the known sequence given in [36], yielding a Langford sequence that is disjoint with a Langford sequence shown in Theorem 1, Case 3 given in [36].

Similarly, we produced four cases for hooked Langford sequences of order n and defect d ($d > 2$), by adjoining hooked Langford sequences of order n and defect d with the Langford sequences given in Theorem 1 in [36]. This produced hooked Langford sequences of order n and defect d that are disjoint with the known hooked Langford sequences given in Theorem 2 in [36].

We produced hooked Langford sequences in the case that $n \equiv 3 (\mathrm{mod}\ 4)$ for d odd, and $n \equiv 1 (\mathrm{mod}\ 4)$ for d even.

Case 1. We obtain a disjoint hooked Langford sequence of order $n = 4t - 1$, defect $d = 4s - 1$ and $t \geqslant 8s - 1$ by applying the following three steps.

Step 1: We form a hooked Langford sequence of order n and defect d, where $d = 4s - 1$, $s \geqslant 1$, and $n = 8s - 1$.

For example, if we form a hooked Langford sequence of order 7 and defect 3, we obtain $(9,7,5,3,8,6,3,5,7,9,4,6,8,0,4)$.

Step 2: We use a Langford sequence of order n and defect d given in [36], where $d = 4s+2$, $s = 3e-1$, $e \geqslant 1$, $t \geqslant 2s+1$.

One example is given by $(17, 23, 21, 26, 27, 28, 29, 25, 13, 11, 18, 19, 24, 14, 12, 10, 22, 17, 16, 20, 11, 13, 15, 21, 23, 10, 12, 14, 18, 26, 19, 27, 25, 28, 16, 29, 24, 15, 22, 20)$.

Step 3: We combine Step 1 and Step 2, and adjoin the hooked Langford sequence, which we formed with the one given in Step 2. We place the sequence in Step 1 at the end of the sequence in Step 2.

Case 2: We obtain a disjoint hooked Langford sequence of order $n = 4t+5$, defect $d = 4s$ and $t \geqslant 8s$ by applying the following three steps.

Step 1: We form a hooked Langford sequence of order n and defect d, where $d = 4s$, $s \geqslant 1$, and $n = 8s+1$.

For example, if we form a hooked Langford sequence of order $n = 9$ and defect $d = 4$, we obtain $(12, 10, 8, 6, 4, 11, 9, 7, 4, 6, 8, 10, 12, 5, 7, 9, 11, 0, 5)$.

Step 2: We use a Langford sequence of order n and defect d given in [36], where $d = 4s+1$, $s = 3e$, $e \geqslant 1$, and $t \geqslant 2s+1$. For example,

$(24, 32, 30, 20, 35, 36, 37, 38, 39, 40, 34, 27, 18, 16, 14, 25, 26, 19, 17, 15, 33, 31, 29, 20, 24, 21, 22, 28, 14, 16, 18, 23, 30, 32, 15, 17, 19, 13, 27, 35, 25, 36, 26, 37, 34, 38, 21, 39, 22,$

$40, 13, 29, 31, 33, 23, 28)$.

Step 3: We combine Step 1 and Step 2, and adjoin the hooked Langford sequence, which we formed with the one given in Step 2. We place the sequence in Step 1 at the end of the sequence in Step 2.

Case 3: We obtain a disjoint hooked Langford sequence of order $n = 4t + 7$, defect $d = 4s + 1$ and $t \geqslant 8s + 1$ by applying the following three steps.

Step 1: We form a hooked Langford sequence of order n and defect d, where $d = 4s + 1$, $s \geqslant 1$, and $n = 8s + 1$.

For example, if we form a hooked Langford sequence of order $n = 11$ and defect $d = 5$, we obtain $(15, 13, 11, 9, 7, 5, 14, 12, 10, 8, 5, 7, 9, 11, 13, 15, 6, 8, 10, 12, 14, 0, 6)$.

Step 2: We use a Langford sequence of order n and defect d given in [36], where $d = 4s$, $s = 3e + 1$, $e \geqslant 1$, and $t \geqslant 2s$. For example,

$(28, 40, 38, 36, 34, 41, 42, 43, 44, 45, 46, 47, 32, 21, 19, , 17, 29, 30, 31, 32, 39, 16, 22, 20, 18, 37, 35, 33, 28, 24, 25, 26, 17, 19, 21, 23, 27, 16, 34, 36, 38, 40, 18, 20, 22, 29, 41, 30, 42, 31, 43, 32, 44, 24, 45, 25, 46, 26, 47, 39, 33, 35, 37, 27)$.

Step 3: We combine Step 1 and Step 2, and adjoin the hooked Langford sequence, which we formed with the one given in Step 2. We place the sequence in Step 1 at the end of the sequence in Step 2.

Case 4: We obtain obtain a disjoint hooked Langford sequence of order $n = 4t + 13$, defect $d = 4s + 2$ and $t \geqslant 8s + 2$ by applying the following three steps.

Step 1: We form a hooked Langford sequence of order n and defect d, where $d = 4s + 2$, $s \geqslant 1$, and $n = 8s + 5$.

For example, if we form a hooked Langford sequence of order $n = 13$ and defect $d = 6$, we obtain

$(18, 16, 14, 12, 10, 8, 6, 17, 15, 13, 11, 9, 6, 8, 10, 12, 14, 16, 18, 7, 9, 11, 13, 15, 17, 0, 7)$.

Step 2: We use a Langford sequence of order n and defect d given in [36], where $d = 4s - 1$, $s = 3e + 1$, $e \geqslant 1$, and $t \geqslant 2s$. For example,

$(35, 33, 48, 46, 44, 42, 50, 51, 52, 53, 54, 55, 56, 57, 58, 40, 28, 26, 24, 22, 20, 36, 37, 38, 39, 21, 27, 25, 23, 49, 47, 45, 43, 41, 33, 35, 29, 30, 31, 32, 20, 22, 24, 26, 28, 34, 21, 42, 44, 46, 48, 23, 25, 27, 19, 40, 50, 36, 51, 37, 52, 38, 53, 39, 54, 29, 55, 30, 56, 31, 57, 32, 58, 19, 41, 43, 45, 47, 49, 34)$.

Step 3: We combine Step 1 and Step 2, and adjoin the hooked Langford sequence, which we formed with the one in Step 2. We place the sequence in Step 1 at the end of the sequence in Step 2.

The following table presents the cases that we produced for disjoint (hooked) Langford sequences.

Row numbers	n	d	Pairs in common
(1)	$4t$	$4s+2, s \geqslant 1, t \geqslant 2s+1$	disjoint
(2)	$4t$	$4s+1, s \geqslant 1, t \geqslant 2s+1$	disjoint
(3)	$4t+3$	$4s, s \geqslant 1, t \geqslant 8s-1$	disjoint, $n \geqslant 8d-1$
(4)	$4t+1$	$4s-1, s \geqslant 1, t \geqslant 8s-3$	disjoint, $n \geqslant 7d-1$
(5)	$4t-1, t \geqslant 8s-1$	$d=4s-1, s \geqslant 1$	disjoint
(6)	$4t+5, t \geqslant 8s$	$d=4s, s \geqslant 1$	disjoint
(7)	$4t+7, t \geqslant 8s+1$	$d=4s+1, s \geqslant 1$	disjoint
(8)	$4t+13, t \geqslant 8s+2$	$d=4s+2, s \geqslant 1$	disjoint

Table 4.8: A construction for disjoint (hooked) Langford sequences

Chapter 5

Applications

5.1 Introduction

Let K be a set of positive integers and let λ be a positive integer. A pairwise balanced design (PBD(v, K, λ) or (K, λ)-PBD) of order v with block sizes from K is a pair (V, B'), where V is a finite set of cardinality v and B' is a family of subsets (blocks) of V that satisfies two properties:

1. If $B \in B'$, then $\mid B \mid \in K$;

2. Each pair of elements of V occurs together in exactly λ of the blocks B'. The integer λ is the index of the PBD. The notations PBD(v, K) and K-PBD of order v are often used when $\lambda = 1$.

For example, $\{1,2,4\}$,$\{2,3,5\}$,$\{3,4,6\}$,$\{5,6,1\}$,$\{4,5\}$,$\{2,6\}$,$\{1,3\}$ form a PBD $(6,\{2,3\},1)$. A PBD (v) is cyclic if its automorphism group contains a full cycle of length v. A PBD (v) is simple if it does not have repeated blocks.

A group divisible design is an ordered triple (P,G,B) where P is a finite set, G is a collection of sets called groups that partition P, and B is a set of subsets called blocks of P. A GDD of order v is cyclic if its automorphism group contains a full cycle of length v. A GDD is simple if it does not have repeated blocks.

$(P,G\cup B)$ is a PBD. The number of $|P|$ is the order of the group divisible design. So a group divisible design is a PBD with distinguished set of blocks, now called groups, which partition P. If a group divisible design has all groups of the same size, say g, and all blocks of the same size, say k, then we will refer to this design as a GDD(g,k). For example, let $V=\{1,2,3,4,5,6,7,8,9,10,11,12,13,14\}$ be a set of 14 elements. We obtain a GDD $(2,4)$ of order 14 that is a simple and a cyclic. The groups are $\{\{i,7+i\}|1\leqslant i\leqslant 7\}$, and the blocks are $\{\{i,1+i,4+i,6+i\}|1\leqslant i\leqslant 14\}$ (mod 14). We cyclically develop the base blocks for the groups and the blocks to obtain 14 blocks with no repetition. So, we have the following groups: $\{1,8\}$,$\{2,9\}$,$\{3,10\}$,$\{4,11\}$,$\{5,12\}$,$\{6,13\}$,$\{7,14\}$. We also have the blocks $\{1,2,5,7\}$,$\{2,3,6,8\}$,$\{3,4,7,9\}$,$\{4,5,8,10\}$,$\{5,6,9,11\}$,$\{6,7,10,12\}$,$\{7,8,11,13\}$,

$\{8,9,12,14\}$,$\{9,10,13,1\}$,$\{10,11,14,2\}$,$\{11,12,1,3\}$,$\{12,13,2,4\}$,$\{13,14,3,5\}$,

$\{14, 1, 4, 6\}$.

In the paper by Meszka and Rosa [22], near-Skolem sequences were used to determine cubic circulants as leaves, that we will define in next section, by finding the difference triples and the union of base blocks.

5.2 Definitions

In this section, we present examples and known definitions of a simple cyclic pairwise balanced design and a simple cyclic group divisible design. We also present several known definitions that we need in this chapter.

Definition 5.2.1 *A partial triple system of order v, denoted by $PTS\ (v)$, is a set V of v elements, and there is a collection B of 3-subsets of V, called triples or blocks such that every 2-subset of V is contained in, at most, one triple of B.*

Definition 5.2.2 *The leave of a partial triple system is a graph (V, E) where E is the set of unordered pairs not appearing in a triple of B.*

Example 5.2.1 *Let $V = (1, 2, 3, 4, 5, 6)$ be a partial triple system of order 6. We have $B = \{\{1, 3, 4\}, \{1, 5, 6\}, \{2, 3, 6\}, \{2, 4, 5\}\}$, which is a block of 3-subsets of V. The leave graph in this case is $\{1, 2\}$,$\{3, 5\}$,$\{4, 6\}$.*

Definition 5.2.3 *A cubic graph is a graph in which all vertices have a degree of three.*

A cubic graph is a 3-regular graph (regular graph means all the vertices have the same number of neighbors).

Example 5.2.2 *The Petersen graph is a well-known example of a cubic graph. It has ten vertices and each vertex must have three edges.*

Remark 5.2.1 *Let X be a graph, and x, y be the vertices of X. Therefore, $\overline{X}$ is then a complement of the graph X with the same vertex set as X. Where x and y are not adjacent in X, they are adjacent in $\overline{X}$.*

Definition 5.2.4 *A cubic circulant $(2n; s, n)$ is a cubic graph, the vertices of which are labelled $\mathbb{Z}_{2n}$, and its edges are $\{x, y\}$ if and only if $min(|x-y|, 2n-|x-y|) = s$ or n such that $s \in \{1, 2, \ldots, n-1\}$.*

5.3 Examples of the use of near-Skolem sequences and disjoint Langford sequences

Shalaby and Silvesan [34] proved that there are two cyclic Steiner triple systems of order $6n+1$ intersecting in $0, 1, 2, \ldots, n$ base blocks, denoted by $Int_c(6n+1)$, and there are two cyclic Steiner triple systems of order $6n+3$ intersecting in $1, 2, \ldots, n+1$ base blocks, which are in common, and denoted by $Int_c(6n+3)$.

Theorem 5.3.1 *[34]:* $Int_c(6n+1) = \{0, 1, 2, \ldots, n\}$.

Theorem 5.3.2 *[34]:* $Int_c(6n+3) = \{1, 2, \ldots, n+1\}$.

Example 5.3.1 *Let* $S = (3, 4, 2, 3, 2, 4, 1, 1, 12, 10, 8, 6, 13, 11, 9, 7, 5, 6, 8, 10, 12, 5, 7, 9, 11, 13)$ *be a Skolem sequence of order* 13*, where* S *consists of a Skolem sequence of order* 4*, that is attached with a Langford sequence of order* 9 *and defect* 5*. We can construct two cyclic Steiner triple systems of order* $6n+1$*, which have repeated base blocks in common.*

First, we obtain the pairs (a_i, b_i) *from* S*, where* $(b_i - a_i) = i$ *for all* $i = 1, 2, \ldots, n$*. We then take the base blocks of the forms:*

1.$\{\{0, a_i + n, b_i + n\} \pmod{6n+1}, i = 1, 2, \ldots, j\}$ *together with the base blocks* $\{\{0, i, b_i + n\} \pmod{6n+1}, i = j+1, \ldots, n\}$*;*

2. $\{\{0, a_i+n, b_i+n\} \pmod{6n+1}, i = 1, 2, \ldots, n\}$*. The pairs are* $(7, 8)$*,* $(3, 5)$*,* $(1, 4)$*,* $(2, 6)$*,* $(17, 22)$*,* $(12, 18)$*,* $(16, 23)$*,* $(11, 19)$*,* $(15, 24)$*,* $(10, 20)$*,* $(14, 25)$*,* $(9, 21)$*,* $(13, 26)$*. We take the following base blocks* $\pmod{79}$ *for* $i = 1, 2, 3, 4$ *and underline the repeated blocks:*

1.$\underline{\{0, 20, 21\}}$,$\underline{\{0, 16, 18\}}$,$\underline{\{0, 14, 17\}}$,$\underline{\{0, 15, 19\}}$ *combined with the base blocks*

$\{0,5,35\}$,$\{0,6,31\}$,$\{0,7,36\}$,$\{0,8,32\}$,$\{0,9,37\}$,$\{0,10,33\}$,$\{0,11,38\}$,$\{0,12,34\}$, $\{0,13,39\}$ (mod 79) *for* $j = 5, \ldots, 13$*;*

2.$\underline{\{0,20,21\}}$,$\underline{\{0,16,18\}}$,$\underline{\{0,14,17\}}$,$\underline{\{0,15,19\}}$,$\{0,30,35\}$,$\{0,25,31\}$,$\{0,29,36\}$, $\{0,24,32\}$,$\{0,28,37\}$,$\{0,23,33\}$,$\{0,27,38\}$,$\{0,22,34\}$,$\{0,26,39\}$ (mod 79) *for* $i = 1, \ldots, 13$.

From this example, we obtain a cyclic Steiner triple system of order 79 *with* 4 *repeated base blocks. We repeat this process when* $j = 1, \ldots, n$ *to obtain two cyclic Steiner triple systems of order* 79 *with* j *base blocks in common.*

Meszka and Rosa [22] considered cubic leaves on 10, 12, 16, 18 and 22 vertices of partial triple systems. They showed some examples of cubic graphs as leaves, and determined a cubic circulant graph as a leave by using some sequences such as extended Skolem sequences and near-Skolem sequences. (For more information, see reference [22]).

Theorem 5.3.3 *[22]*

Let $G = C(n; s, \frac{n}{2})$*, be a cubic circulant graph and let*

1. $n \equiv 4$ *or* $22 (\mathrm{mod}\ 24)$ *and* $s \equiv 1 (\mathrm{mod}\ 2)$*;*

2. $n \equiv 10$ *or* $16 (\mathrm{mod}\ 24)$ *and* $s \equiv 0 (\mathrm{mod}\ 2)$*;*

3. $n \equiv 6$ *or* $12 (\mathrm{mod}\ 24)$ *and* $s \equiv 1 (\mathrm{mod}\ 2)$*; and*

4. $n \equiv 0$ *or* $18 (\mathrm{mod}\ 24)$ *and* $s \equiv 0 (\mathrm{mod}\ 2)$, $s \neq \frac{n}{3}$.

Then G is a leave.

Proof [22] We will only demonstrate the case that uses near-Skolem sequences, as it is directly applicable to this proof. Let $n \equiv 22 (\mathrm{mod}\ 24)$, $n = 24t + 22$, $t \geqslant 0$. According to reference [29], a near Skolem sequence $S = (s_1, s_2, \ldots, s_{8t+6})$ of order $4t + 4$ and defect s exists for all odd s, $1 \leqslant s \leqslant 4t + 3$ where $s \in \{1, 3, \ldots, 4t + 3\}$. Each $k \in \{1, 2, \ldots, 4t + 4\} \setminus \{s\}$ forms the difference triple $(k, 4t + 4 + i, 4t + 4 + j)$ provided $s_i = s_j = k$. The union of orbits of triples $\{0, k, 4t + 4 + i + k\}$ (mod n), where $k \in \{1, 2, \ldots, 4t + 4\} \setminus \{s\}$, is a decomposition of $\overline{C}(n; s, \frac{n}{2})$ (where $\overline{C}(n; s, \frac{n}{2})$ is the complement of $C(n; s, \frac{n}{2})$) into triples for each odd $s \in \{1, 3, \ldots, 4t + 3\}$.■

Example 5.3.2 *Let* $S = (14, 12, 3, 4, 2, 3, 2, 4, 8, 6, 20, 18, 16, 12, 14, 6, 8, 10, 19, 17, 15, 13, 11, 9, 7, 1, 1, 10, 16, 18, 20, 7, 9, 11, 13, 15, 17, 19)$ *be a near-Skolem sequence of order* $4t + 4 = 20$ *where* $t = 4$ *and defects* $s \in \{1, 3, \ldots, 4t + 3\}$, *and in this example* $s = 5$. *Each* $k \in \{1, 2, \ldots, 20\} \setminus \{5\}$ *forms the difference triple* $(k, 4t + 4 + i, 4t + 4 + j)$ *provided* $s_i = s_j = k$. *We take the base blocks of the form* $\{0, k, 4t + 4 + i + k\}$ $(\mathrm{mod}\ 24t + 22 = 118)$, $k \in \{1, 2, \ldots, 20\} \setminus \{5\}$ *and we check the differences in* $\mathbb{Z}_{118}$. *The pairs* (a_i, b_i) *for* $i = 1, 2, \ldots, 20$ *are* $(1, 15)$,$(2, 14)$,$(3, 6)$,$(4, 8)$,$(5, 7)$,$(9, 17)$, $(10, 16)$,$(11, 31)$,$(12, 30)$,$(13, 29)$,$(18, 28)$,$(19, 38)$, $(20, 37)$,$(21, 36)$,$(22, 35)$, $(23, 34)$, $(24, 33)$,$(25, 32)$,$(26, 27)$.

The base blocks are $\{0,14,35\}$,$\{0,12,34\}$,$\{0,3,26\}$,$\{0,4,28\}$,$\{0,2,27\}$,$\{0,8,37\}$, $\{0,6,36\}$,$\{0,20,51\}$,$\{0,18,50\}$,$\{0,16,49\}$,$\{0,10,48\}$,$\{0,19,58\}$,$\{0,17,57\}$,$\{0,15,56\}$, $\{0,13,55\}$,$\{0,11,54\}$,$\{0,9,53\}$,$\{0,7,52\}$,$\{0,1,47\}$.

Now, we observe that all the non-zero elements exist in $\mathbb{Z}_{118}$ *as differences twice, except for the element* $\frac{n}{2} = 59$ *and the defect of* $s = 5$. *We conclude that the union of the short orbits of this triple* $\{0, k, 4t+4+i+k\} \pmod{24t+22}$, *where* $k \in \{1,2,\ldots,20\} \setminus \{5\}$, *is a decomposition of* $\overline{C}(118;5,59)$ *into triples for each* $s \in \{1,3,\ldots,19\}$.

Chapter 6

Conclusions and open questions

In this thesis, we have discussed the disjointness for Skolem-type sequences such as (hooked) Skolem sequences, (hooked) near-Skolem sequences, and (hooked) Langford sequences. We represented previous results from the literature, as well as new constructions for sufficiency for some of the Skolem-type sequences.

In Chapter 1, we introduced the topic and some known applications for disjointness for some of the Skolem-type sequences. In Chapter 2, we discussed several disjoint results of (hooked) Skolem sequences and we also discussed the known results of (hooked) Skolem sequences and (hooked) Rosa sequences. We plan to discuss some constructions, that are also not seen in the literature, of disjoint (hooked) Rosa sequences in a separate study. In Chapter 3, we discussed eight new cases for disjoint hooked near-Skolem sequences, and added small cases for disjoint near-Skolem sequences.

In Chapter 4, we provided four cases for two disjoint Langford sequences of order n only when $d = 4s + 2$ and $d = 4s + 1$. For $d = 4s$ and $d = 4s - 1$ we constructed two disjoint Langford sequences of order n and defect d by adjoining the known Langford sequences of order n and defect d to the Langford sequences of order n and defect d, which we formed. This solution satisfies condition two given in Theorem 4.3.1, but only satisfies condition one given in Theorem 4.3.1 when $n \geqslant 8d - 1$ for Case 3 and when $n \geqslant 7d - 1$ for Case 1. We provided four cases for two disjoint hooked Langford sequences of order n and defect d. We found that that condition three in Theorem 4.3.2 is satisfied with finite exceptions of n, and condition four given in Theorem 4.3.2 is completely satisfied. Therefore, we did not find cases yield two disjoint (hooked) Langford sequences for all admissible orders and all admissible defects, so this question remains open. Thus, our main objective for a future research is to find solutions for all of the remaining cases for two disjoint (hooked) Langford sequences of order n, as well as all admissible defects. We also provided the known constructions for two disjoint (hooked) Langford sequences.

Some questions that remain open regarding this topic are as follows:

1. Complete the solution for all remaining cases for disjoint (hooked) Langford

sequences of order n and all admissible defects.

2. Find applications for disjoint (hooked) Langford sequences of order n and defect m.

3. Find applications for disjoint (hooked) near-Rosa sequences of order n and defect m.

4. Find disjoint constructions for (hooked) near-Rosa sequences of order n and defect m for all admissible defects.

5. Find additional applications for disjoint (hooked) near-Skolem sequences of order n and defect m if possible.

6. Find additional disjoint constructions for (hooked) near-Skolem sequences of order n and defect m.

7. Construct hooked m-near Langford sequences of order n and defect d.

Bibliography

[1] C. A. Baker, Extended Skolem sequences. *J. Combin. Des.* **3(5)** (1995), 363-379.

[2] C. Baker and N. Shalaby, Disjoint Skolem sequences and related disjoint structures. *J. Combin. Des.*, **1(5)** (1993), 329-345.

[3] Th. Bang, On the sequence $[n\alpha]$ $n = 1, 2, \ldots$, supplementary note to the preceding paper by Th. Skolem. *Math. Scand.* **5** (1957), 69-76.

[4] J.-C. Bermond, A. E. Brouwer and A. Germa, Systémes de triplets et différences associées. *Colloq. CNRS, Problémes Combinatoires et théorie des Graphes*, Orsay 1976 (1978) 35-38.

[5] E. J. Billington, Cyclic balanced ternary designs with block size three and index two. *Ars Combin.* **23B**, (1987), 215-232.

[6] E. J. Billington, On Lambda coverings of pairs by triples repeated elements allowed in triples. *Util. Math.* **21C**, (1982), 187-203.

[7] A. Burgess, Construction of Skolem and related sequences, honours thesis submitted to Memorial University of Newfoundland, (1991).

[8] C. J. Colbourn, Disjoint cyclic Steiner triple systems. *Congressum Numerantium* **32** (1981), 205-212.

[9] C. J. Colbourn and M. J. Colbourn, Disjoint cyclic Mendelsohn triple systems. *Ars Combin* **11** (1981), 3-8.

[10] C. J. Colbourn and J. H. Dinitz, Mutually orthogonal Latin squares: a brief survey of constructions. Special issue on design combinatorics: in honor of S. S. Shrikhande **95** (2001), 9-48.

[11] R. O. Davies, On Langford's problem (II). *Math. Gaz.* **43** (1959), 253-255.

[12] L. Euler, Recherches sur une nouvelle espèce de quarrés magiques. *Veh. Zeeuw. Gen. Weten. Vlissengen* **9** (1782), 85-239.

[13] N. Francetić and E. Mendelsohn, A survey of Skolem-type sequences and Rosa's use of them. *Math. Slovaca* **59** (2009), 1, 39-76.

[14] L. Heffter, Uber tripelsysteme. *Math. Ann.* **49** (1897), 101-112.

[15] T. P. Kirkman, On a problem in combinations. *Cambridge and Dublin Math. J.* **2** (1847), 191-204.

[16] T.P. Kirkman, Note on an unanswered prize questions, *Cambridge and Dublin Math. J.* **5** (1850), 255-262.

[17] C. D. Langford, Problem. *Math. Gaz.* **42** (1958), 288.

[18] V. Linek and S. Mor, On partitions of $\{1, 2, \ldots, 2m + 1\} \setminus \{k\}$ into differences $d, \ldots, d + m - 1$: Extended Langford sequences of large defect. *J. Combin. Des.* **12** (2004), 421-442.

[19] V. Linek, S. Mor, and N. Shalaby, Skolem and Rosa rectangles and related designs. *Discrete Mathematics* **331** (2014), 53-73.

[20] V. Linek and Z. Jiang, Extended Langford sequences with small defect. *J. Combin. Theory Ser. A* **84** (1998), 38-54.

[21] V. Linek and N. Shalaby, The existence of (p, q)-extended Rosa sequences. *Discrete Math.* **308** (2008), 1583-1602.

[22] M. Meszka and R. Alexandra, Cubic leaves. *Australas. J. Combin.* **61** (2015), 114-129.

[23] R. S. Nickerson and D. C. B. Marsh, Problems and Solutions: Solutions of Elementary Problems: E 1845. *Amer. Math. Monthly* **74** (1967), 591-592.

[24] E. S. O'Keefe, Verification of a conjecture of Th. Skolem. *Math. Scand.* **9** (1961), 80-82.

[25] C. J. Priday, On Langford's problem (I). *Math. Gaz.* **43** (1959), 250-253.

[26] R. Rees and N. Shalaby, Simple and indecomposable twofold cyclic triple systems from Skolem sequences. *J. Combin. Des.* **8** (2000), 402-410.

[27] A. Rosa, Poznámka o cyklických Steinerových systémoch trojíc., *Math. Fyz. Čas* **16** (1966), 285-290.

[28] N. Shalaby, The existence of near-Skolem and hooked near-Skolem sequences, *Discrete Math.* **135** (1994), 303-319.

[29] N. Shalaby, Skolem and Langford sequences, The CRC Handbook of combinatorial designs, edited by C.J. Colbourn, J.H. Dinitz, CRC Press (2006), 612-616.

[30] N. Shalaby, Skolem sequences: Generalizations and Applications, PhD Thesis, McMaster University, Canada, (1992).

[31] N. Shalaby and D. Silvesan, The intersection spectrum of hooked Skolem sequences and applications. *Discrete App. Math.* **167** (2014), 239-260

[32] N. Shalaby and D. Silvesan, The intersection spectrum of Skolem sequences and its applications to λ-fold cyclic triple systems. *Discrete Math.* **312** (2012), 1985-1999.

[33] N. Shalaby and D. Silvesan, Supplement for the paper "The intersection spectrum of Skolem sequences and its applications to λ-fold cyclic triple systems", Unpuplish Trechnical Report, Available upon Request from the Authors.

[34] D. Silvesan and N. Shalaby, Cyclic block designs with block size 3 from Skolem-type sequences. *Des. Codes Cryptogr.* **63** (2012), 345-355.

[35] D. Silvesan, The spectrum of Skolem and hooked Skolem sequences with a prescribed number of pairs in common and applications, M.Sc Thesis, Memorial University of Newfoundland, (2007)

[36] J. E. Simpson, Langford sequences: perfect and hooked. *Discrete Math.* **44** (1983), 97-104.

[37] Th. Skolem, On certain distributions of integers in pairs with given differences. *Math. Scand.* **5** (1957), 57-68.

[38] Th. Skolem, Some remarks on the triple systems of Steiner. *Math. Scand.* **6** (1958), 273-280.

[39] R. G. Stanton and I. P. Goulden, Graph factorization, general triple systems, and cyclic triple systems. *Aequations Math.* **22** (1981), 1-28.