$J-$HOLOMORPHIC CURVES IN A NEF CLASS

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ABSTRACT. Taubes established fundamental properties of $J-$holomorphic subvarieties in dimension 4 in [9]. In this paper, we further investigate properties of reducible $J-$holomorphic subvarieties. We offer an upper bound of the total genus of a subvariety when the class of the subvariety is $J-$nef. For a spherical class, it has particularly strong consequences. It is shown that, for any tamed J, each irreducible component is a smooth rational curve. It might be even new when J is integrable. We also completely classify configurations of maximal dimension. To prove these results we treat subvarieties as weighted graphs and introduce several combinatorial moves.

CONTENTS

1. INTRODUCTION

Let (M, J) be a closed, almost complex $4-$manifold. In this paper we study properties of reducible $J-$holomorphic subvarieties in M. Here J is not always assumed to be tamed.

Definition 1.1. *A closed set $C \subset M$ with finite, nonzero 2-dimensional Hausdorff measure is said to be an irreducible $J-$holomorphic subvariety if it has no isolated points, and if the complement of a finite set of points in C, called the singular points, is a connected smooth submanifold with $J-$invariant tangent space.*

A $J-$holomorphic subvariety Θ is a finite set of pairs $\{(C_i, m_i), 1 \leq i \leq n\}$, where each C_i is irreducible $J-$holomorphic subvariety and each m_i is a non-negative integer. The set of pairs is further constrained so that $C_i \neq C_j$ if $i \neq j$.

Pseudo-holomorphic subvarieties are closely related to, but clearly different from pseudo-holomorphic maps. They are the real analogues of one dimensional subvarieties in algebraic geometry. When J is understood, we will simply call a $J-$holomorphic subvariety a subvariety. An irreducible subvariety is said to be *smooth* if it has no singular points. A subvariety $\Theta = \{(C_i, m_i)\}$ is said to be connected if $\cup C_i$ is connected.

Taubes provides a systematic analysis of pseudo-holomorphic subvarieties in [9]. The knowledge of the structure of reducible $J-$holomorphic subvarieties is very important, in both the integrable case and the tamed case. Among others, two aspects are especially significant for applications. Firstly, under natural conditions, we need to know that the irreducible components are not too complicated. This point is used for example in the argument in [2] on the structure of rational curves. Secondly, we need to know the moduli space of the reducible subvarieties is not too large to ensure the existence of irreducible subvarieties. This is used in [6] for the study of Donaldons's tamed-to-compatible question and almost Kähler Nakai-Moishezon criterion. These aspects are the main focus of this paper.

Suppose C is an irreducible subvariety. Then it is the image of a $J-$holomorphic map $\phi : \Sigma \to M$ from a complex connected curve Σ, where ϕ is an embedding off a finite set. Σ is called the model curve and ϕ is called the tautological map. The map ϕ is uniquely determined up to automorphisms of Σ. This understood, the associated homology class e_C is defined to be the push forward of the fundamental class of Σ via ϕ. And for a subvariety Θ, the associated class e_Θ is defined to be $\sum m_i e_{C_i}$.

A special feature in dimension 4 is that, by the adjunction formula, the genus of a smooth subvariety C is given by $g_J(e_C)$ defined as follows. Given a class e in $H_2(M; \mathbb{Z})$, introduce the $J-$genus of e,

$$g_J(e) \quad = \quad \frac{1}{2}(e \cdot e + K_J \cdot e) + 1, \tag{1}$$

where K_J is the canonical class of J.

Moreover, when C is irreducible, $g_J(e_C)$ is non-negative. In fact, if Σ is the model curve of C, by the adjunction inequality in [7],

$$g_J(e_C) \geq g(\Sigma), \tag{2}$$

with equality if and only if C is smooth.

We investigate, under what conditions on the class e, $g_J(e)$ still bounds the total genus of any connected, reducible subvariety in e.

Definition 1.2. *The total genus $t(\Theta)$ of Θ is defined to be $\sum_i g_J(e_{C_i})$.*

Question 1.3. *Suppose e is a class with $g_J(e) \geq 0$ and $\Theta = \{(C_i, m_i)\}$ is a connected subvariety in the class e. Find general conditions such that*

$$g_J(e) \geq t(\Theta). \tag{3}$$

The study of Donaldson's "tamed to compatible" question and almost Kähler Nakai-Moishezon duality by Taubes' subvariety-current-form strategy [9, 6] led us to this problem in the case $g_J(e) = 0$ and J is tamed. This problem is very subtle when there are irreducible components with negative self-intersection and high multiplicity; incorrect assertions are easily made from geometric intuition (see e.g. Example 3.3).

In this paper, we settle it for $J-$nef classes. A class e is said to be *$J-$nef* if it pairs non-negatively with any $J-$holomorphic subvariety. Whenever there is a $J-$holomorphic subvariety representative in a $J-$nef class, we have $e \cdot e \geq 0$.

Theorem 1.4. *Suppose e is a $J-$nef class with $g_J(e) \geq 0$. Then* (3) *holds for any connected subvariety in the class e.*

To prove Theorem 1.4, we treat subvarieties as weighted graphs, and use *curve expansion* and *curve combination* to rearrange the multiply covered part. In fact, these techniques are also effective analyzing when the stronger bound

$$g_J(e) \geq \sum_i m_i g_J(e_{C_i}) \tag{4}$$

holds.

Notice that when $g_J(e) = 0$, we actually have equality in Theorem 1.4. This is because $g_J(e_{C_i}) \geq 0$ for all i since each C_i is irreducible. In turn this implies that $g_J(e_{C_i}) = 0$. Moreover, we have the following more precise result.

Theorem 1.5. *Suppose e is a $J-$nef class with $g_J(e) = 0$. Let Θ be a $J-$holomorphic subvariety in the class e.*

- *If Θ is connected, then each irreducible component of Θ is a smooth rational curve, and Θ is a tree configuration.*
- *If J is tamed, then Θ is connected.*

Here, for a tree configuration, we refer to Definition 4.2. In particular, distinct components in a tree configuration intersect at most once.

Recall that J is said to be tamed if there is a symplectic form ω such that the bilinear form $\omega(\cdot, J(\cdot))$ is positive definite. The tameness is necessary for the second bullet since otherwise there could be a null homologous $J-$holomorphic torus in Θ.

Thus, connected configurations in a $J-$nef spherical class match our geometric intuition: each component is a smooth rational curve. A particularly nice consequence is

Corollary 1.6. *Suppose J is a tamed almost complex structure and e is a class represented by a smooth J-holomorphic rational curve. Then for any $J-$holomorphic subvariety Θ in the class e, each irreducible component of Θ is a smooth rational curve.*

We will comment on various versions of this result in the literature ([2], [7], [8]) in 4.4.2.

For a $J-$nef spherical class, the irreducible part of the moduli space, when non-empty, is a smooth manifold of expected dimension. This is due to the "automatic regularity" of any smooth rational curve with non-negative self intersection. In Corollary 4.10 we further show that the reducible part always has smaller dimension. And if we assume that J is tame, by Proposition 4.5 in [6], this assumption would actually guarantee the irreducible part of the moduli space to be non-empty and hence the existence of a smooth rational curve in the given class. This is used as a crucial step in our study of Donaldson's tamed-to-compatible question and Nakai-Moishezon duality between the almost Kähler cone and the curve cone in [6]. Moreover, along with the techniques of [6], the second author is able to apply our main results to study when a symplectic surface is symplectically isotopic to an algebraic curve in a general ambient $4-$manifold.

We also investigate which stratum of the reducible part has codimension one. It is interesting that, in this case, the curve combination moves we applied to prove Theorem 1.4 have a nice interpretation as combinatorial blow-downs. This viewpoint makes it possible to classify the corresponding connected configurations in Theorem 4.23 when $b^+ = 1$. Precisely, these configurations are shown to be either successive blow-ups of a single smooth curve, or successive blow-ups of a comb configuration along the spike curve.

Finally, we would like to remark that, as pointed out by Gompf, the same arguments apply to closed holomorphic curves in a Stein manifold and all the results in this paper hold true as well.

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2. Pseudo-holomorphic subvarieties

We always assume M is a $4-$dimensional manifold with a fixed almost complex structure J.

2.1. **Properties of irreducible subvarieties.**

2.1.1. *Genus and adjunction number.* Let C be an irreducible subvariety. The geometric genus of C is defined to be the genus of its model curve C_0, and the arithmetic genus of C is $g_J(e_C)$. The adjunction inequality (2) says that the arithmetic genus is no less than the geometric genus.

The next result follows directly from the adjunction inequality.

Lemma 2.1. *If $g_J(e_C) = 0$, then C is a smooth rational curve.*

It is convenient to introduce the adjunction number.

Definition 2.2. *The adjunction number of e is given by*

$$adj(e) = e \cdot e + K_J \cdot e.$$

Notice that

$$2g_J(e) = adj(e) + 2.$$

By the adjunction inequality (2), $adj(e_C) \geq -2$.

2.2. **The moduli space.** In this subsection we fix a class e.

As in [9], we define the moduli space $\mathcal{M}_e$ of subvarieties in the class e: Any element Θ in $\mathcal{M}_e$ is a $J-$holomorphic subvariety with $e_\Theta = e$.

Definition 2.3. *A homology class $e \in H_2(M;\mathbb{Z})$ is said to be $J-$effective if $\mathcal{M}_e$ is nonempty.*

We use $\mathcal{M}_{irr,e}$ to denote the moduli space of irreducible subvarieties in class e. Let $\mathcal{M}_{red,e}$ denote $\mathcal{M}_e \setminus \mathcal{M}_{irr,e}$.

2.2.1. *Topology.* $\mathcal{M}_e$ has a natural topology. Let $|\Theta| = \cup_{(C,m)\in\Theta} C$ denote the support of Θ. Consider the symmetric, non-negative function, ϱ, on $\mathcal{M}_e \times \mathcal{M}_e$ that is defined by the following rule:

$$\varrho(\Theta, \Theta') = \sup_{z\in|\Theta|} \mathrm{dist}(z, |\Theta'|) + \sup_{z'\in|\Theta'|} \mathrm{dist}(z', |\Theta|). \tag{5}$$

The function ϱ is used to measure distances on $\mathcal{M}_e$, where the distance function dist is defined by an almost Hermitian metric on (M, J).

Given a smooth 2-form ν we introduce the pairing

$$(\nu, \Theta) = \sum_{(C,m)\in\Theta} m \int_C \nu.$$

The topology on $\mathcal{M}_e$ is defined in terms of convergent sequences:

A sequence $\{\Theta_k\}$ in $\mathcal{M}_e$ converges to a given element Θ if the following two conditions are met:

- $\lim_{k\to\infty} \varrho(\Theta, \Theta_k) = 0$.
- $\lim_{k\to\infty} (\nu, \Theta_k) = (\nu, \Theta)$ for any given smooth 2-form ν.

Definition 2.4. *Given a class e, introduce its $J-$dimension,*

$$\iota_e = \frac{1}{2}(e \cdot e - K_J \cdot e). \tag{6}$$

ι_e is the expected dimension of the moduli space $\mathcal{M}_e$.

2.2.2. *Smooth rational curves.* When e is a class represented by a smooth rational curve, we introduce

$$l_e = \max\{\iota_e, 0\}.$$

The following is an immediate consequence of the adjunction formula and the adjunction inequality (2).

Lemma 2.5. *If $g_J(e) = 0$, then*
- *$\iota_e = e \cdot e + 1$, where ι_e is defined in (6);*
- *every element in $\mathcal{M}_{irr,e}$ is a smooth rational curve.*

One special feature of the moduli space of smooth rational curves is the following automatic transversality ([4]), which is valid for an arbitrary almost complex structure.

Lemma 2.6. *Let e be a class represented by a smooth rational curve with $e \cdot e \geq -1$. Then $\mathcal{M}_{irr,e}$ is a smooth manifold of dimension $2l_e$.*

2.3. **$J-$nef class.**

Definition 2.7. *A homology class $e \in H_2(M;\mathbb{Z})$ is said to be $J-$nef if it pairs non-negatively with any $J-$effective class.*

The following lemma immediately follows from the positivity of intersections of distinct irreducible subvarieties.

Lemma 2.8. *If e is represented by an irreducible $J-$holomorphic subvariety and $e \cdot e \geq 0$, then e is a $J-$nef class.*

On the other hand, if e is $J-$nef and $J-$effective, $e \cdot e \geq 0$.

For a tamed almost complex structure J, the notion $J-$nef is a natural condition which guarantees the good behavior of the $J-$holomorphic subvarieties, as can be seen in many results in this paper and also Examples 3.3 and 4.27. Moreover, it is an important condition for constructing non-negative currents in [6] as we briefly explain below.

It is known that, when J is Kähler, in any big and nef cohomology class (*i.e.* a nef cohomology class with positive top power), there is a Kähler current. Here a current is a differential form with distribution coefficients. Hence it represents a real cohomology class when pairing with smooth closed forms in the weak sense. The Kähler currents play an intermediate role in [1] to construct Kähler forms. For the subvariety-current-form strategy, Taubes current is such an intermediate object, which is usually constructed through integrations over certain moduli space of subvarieties. Hence, our definition of $J-$nef mimics the original algebraic one, instead of the Kähler notion. Notice our definition does not require the existence of an almost Kähler form.

2.4. **When J is tamed.** Here is a well known fact that we will need in Section 4.

Lemma 2.9. *If J is tamed then the homology class e_C of any subvariety C is nontrivial.*

Here is a simple consequence of this fact and positivity of intersection.

Lemma 2.10. *Suppose $e\cdot e<0$ and $\mathcal{M}_{irr,e}$ is nonempty. Then $\mathcal{M}_e$ consists of a unique element.*

Proof. Let C be an irreducible variety in the class e. Suppose $\Theta=\{(C_i,m_i)\}$ is any subvariety in the class e. If each C_i is distinct from C, then $e\cdot e=e_C\cdot\sum m_ie_{C_i}$ is non-negative. This is impossible.

Suppose $C=C_i$ for some i, say $i=1$. Then by Lemma 2.9, the subvariety $\Theta'=(C_1,m_1-1)\cup\{(C_i,m_i),i\geq 2\}$ is empty, namely, $m_1=1$ and $\Theta=\{(C_1,1)\}$. □

Another basic fact is that $\mathcal{M}_e$ is compact by the Gromov compactness.

2.4.1. *K_J-spherical classes are $J-$effective.* Let S be the set of homology classes of M which are represented by smoothly embedded spheres.

The set of K_J-spherical classes is defined to be

$$S_{K_J}=\{e\in S|g_J(e)=0\}.$$

Proposition 2.11. *Let e be a class in S_{K_J}.*

- *Suppose $e\cdot e\geq -1$. Then for any symplectic form ω taming J, the Gromov-Taubes invariant of e is nonzero. In particular, $\mathcal{M}_e$ is nonempty, i.e. e is $J-$effective.*
- *If $e\cdot e\geq 0$, then M has to be rational or ruled, which has $b^+=1$.*

Proof. The first statement is a consequence of Taubes' symplectic Seiberg-Witten theory, see e.g. [5].

The second statement follows the first statement and [7]. □

3. Bounding the total genus

In this section, we prove Theorem 1.4.

3.1. **Two simple cases.** Suppose $\Theta=\{(C_1,m_1),\cdots,(C_n,m_n)\}$. Let $e_i=e_{C_i}$.

3.1.1. *Multiplicity one.* We first deal with the case where each m_i is equal to one.

Lemma 3.1. *Suppose $\Theta=\{(C_1,1),\cdots,(C_n,1)\}$, then both* (3) *and* (4) *hold.*

Proof. We compare the adjunction numbers:

$$adj(e)=\sum_i adj(e_i)+\sum_{i\neq j}e_i\cdot e_j.$$

By the adjunction inequality (2), $adj(e_i)\geq -2$. By the positivity of intersections, $e_i\cdot e_j\geq 0$ for any $i\neq j$.

If there are l components, then there are at least $l-1$ transversal intersection points. Thus

$$
\begin{aligned}
2g_J(e) = adj(e) + 2 \quad &= \textstyle\sum_i adj(e_i) + \sum_{i\neq j} e_i \cdot e_j + 2 \\
&\geq \textstyle\sum_{i=1}^{l} adj(e_i) + 2(l-1) + 2 \\
&= \textstyle\sum_{i=1}^{l} (adj(e_i) + 2) = 2\sum g_J(e_i).
\end{aligned} \tag{7}
$$

□

3.1.2. *One component.* Next we deal with the case that there is only one component.

Lemma 3.2. *Suppose $\Theta = \{(C_1, n)\}$ with $n > 1$. Both* (4) *and* (3) *hold if $e_1 \cdot e_1 > 0$. When $e_1 \cdot e_1 = 0$,* (3) *holds if $g_J(e_1) \geq 1$.*

Proof.

$$2[g_J(e) - n\, g_J(e_1)] = (n^2 - n)e_1 \cdot e_1 + (2 - 2n)$$

If $e_1 \cdot e_1 > 0$, then $g_J(e) - n\, g_J(e_1) \geq (n-1)(n-2) \geq 0$.

When $e_1 \cdot e_1 = 0$,

$$2[g_J(e) - g_J(e_1)] = (n-1)K_J \cdot e_1 = 2(n-1)(g_J(e_1) - 1).$$

□

On the other hand, if $e_1 \cdot e_1 < 0$, then (4) always fails and (3) could fail.

Example 3.3. *Suppose $M = \mathbb{CP}^2 \# 10\overline{\mathbb{CP}^2}$ and there is a smooth $J-$holomorphic genus one curve C in $-K_J$. Then the subvariety $\Theta = \{(C, 2)\}$ fails* (3) *since $g_J(-2K_J) = 0$ and $t(\Theta) = 1$.*

The multiplicity one case and the one component case are settled, even without the $J-$nef assumption.

We next introduce moves to reduce the general case to these two simple cases. To better describe these moves and their properties we view reducible curves as graph like objects, and introduce curve configurations.

3.2. **Nef, connected weighted graphs.**

Definition 3.4. *Here a weighted graph refers to a graph whose vertices are weighted by a pair of a $J-$effective class $\in H_2(M;\mathbb{Z})$ and a positive integer multiplicity.*

The edges are determined by the weighted vertices: there is an edge connecting two vertices whenever the intersection number of their classes is nonzero. Further, label each edge by the intersection number of the classes of its vertices.

The adjunction number and the self-intersection number of each vertex are those of its homology class.

Definition 3.5. *A curve configuration is a weighted graph satisfying the following two properties:*

- *the adjunction number of each vertex is at least -2.*

- *the label of each edge is positive.*

Specifically, to each reducible curve, we assign a weighted graph as follows: to each component C_i, assign the vertex, still denoted by C_i, weighted by the pair (e_i, m_i).

Notice that for each pair of intersecting components C_i, C_j, there is an edge connecting the corresponding vertices labeled by their intersecting number, and all edges arise this way. Clearly, the resulting weighted graph is a curve configuration due to the adjunction inequality (2) and the positivity of intersection. Moreover, the curve configuration is connected as a graph if and only if the reducible curve is connected.

Introduce the total class of a weighted graph in the obvious way. The adjunction number (resp. $J-$genus) of a weighted graph is then defined to be the adjunction number (resp. $J-$genus) of its total class.

Definition 3.6. *A weighted graph is said to be nef if its total class pairs non-negatively with the class of each vertex.*

Here is an example of a nef curve configuration with total class e which is not the graph corresponding to a $J-$holomorphic reducible curve in class e.

Example 3.7. *Suppose $M = S^2 \times S^2$ with spherical classes $a = [S^2 \times pt]$ and $b = [pt \times S^2]$. Let J be such that $a - 2b$ has a $J-$holomorphic representative. Then $e = a + 2b$ is a $J-$nef class. The graph with two vertices, one labeled by $(a - b, 1)$ and the other by $(b, 3)$ is a nef curve configuration, but there is no corresponding $J-$holomorphic reducible curve.*

Lemmas 3.1 and 3.2 now takes the following form,

Lemma 3.8. *Given a connected curve configuration, if the multiplicity of each vertex is 1, then the sum of $J-$genus of vertices is bounded from above by the $J-$genus of its total class.*

Given any nef curve configuration with only one vertex weighted by (e_1, n), let $e = ne_1$. Then $g_J(e) \geq g_J(e_1)$ if $g_J(e) \geq 0$, and $g_J(e) \geq ng_J(e_1)$ if $e_1 \cdot e_1 > 0$.

Proof. The first statement is exactly a rephrase of Lemma 3.1 in the weighted graph language.

For the second statement, it follows from Lemma 3.2 and the following observation: By Lemma 2.8, $e_1 \cdot e_1 \geq 0$ since e_1 is a $J-$effective class and the weighted graph is nef. □

And Theorem 1.4 follows from

Proposition 3.9. *Given a connected, nef curve configuration whose total class has non-negative $J-$genus, then the sum of $J-$genus of vertices is bounded from above by the $J-$genus of its total class.*

3.3. **Curve expansion and curve combination.**

3.3.1. *Curve expansion.* We start with moves on vertices with non-negative self-intersection.

Given a weighted graph, for each vertex C with weight (e_C, m) such that $e_C \cdot e_C \geq 0$ and $m > 1$, replace it by m vertices, $C(k), 1 \leq k \leq m$, weighted by $(e_C, 1)$. This operation is called curve expansion.

Lemma 3.10. *Given a connected curve configuration with at least two vertices, the expanded weighted graph is still a connected curve configuration. If the original configuration is nef, so is the new one.*

The sum $\sum_i g_J(e_i)$ is always non-decreasing. The sum $\sum_i m_i g_J(e_i)$ is non-decreasing if curve expansion is not applied to vertex C with weight (e_C, m) such that $e_C \cdot e_C = 0$, $m > 1$ and $g_J(e_C) > 0$.

Consequently, Proposition 3.9 is true if the multiplicity of each vertex with negative self-intersection is 1.

Proof. Consider the expanded curve configuration.

Notice that the new vertices $C(k)$ have the same first weight and then the same adjunction number as that of C.

There are two kinds of new edges. If there is an edge connecting C with another vertex D in the original curve configuration, then there is an edge joining D with each $C(k)$ by an edge with the same positive label. Therefore the resulting weighted graph is connected. If the self-intersection number of C is positive, then there is an edge joining each pair of $C(k)$. Since the labels of these edges are also positive, the resulting weighted graph is a connected curve configuration with the same total class and the same total multiplicity.

The genus estimates essentially follow from Lemma 3.2. The inequality $g_J(me_C) \geq g_J(e_C)$ always holds when $e_C \cdot e_C \geq 0$. Hence $\sum_i g_J(e_i)$ is non-decreasing. If we are not applying expansion for (e_C, m) with $e_C \cdot e_C = 0$, $m > 1$ and $g_J(e_C) > 0$, the strong inequality $g_J(me_C) \geq m g_J(e_C)$ holds, which implies $\sum_i m_i g_J(e_i)$ is non-decreasing. □

Thus we may assume all the vertices with non-negative self-intersection have multiplicity 1.

Next we deal with vertices with negative self-intersections, especially -1 vertices. Here a vertex is called a -1 vertex if its class has self-intersection -1.

3.3.2. *Curve combination.* Given a connected curve configuration with the property that any vertex with multiplicity greater than 1 has negative self-intersection.

- (i) Suppose there are two adjoined vertices V_1, V_2 weighted by (D_i, n_i) with $n_1 = n_2 = n$. Collapse them to a vertex V weighted by $(D_1 + D_2, n)$. We call this move $(\text{i})_n$.
- (ii) Suppose there are two adjoined vertices V_1, V_2 weighted by (D_i, n_i) with $n_1 > n_2$, and $D_1 \cdot D_2 \geq -D_1 \cdot D_1$. Replace them by two vertices V, V' weighted by $(D_1 + D_2, n_2)$ and $(D_1, n_1 - n_2)$ respectively.

(iii) Suppose there is a -1 vertex E with multiplicity n_0, and there are neighboring vertices weighted by $(D_i, n_i), 1 \leq i \leq t$, with $D_i \cdot D_i \leq -2, 1 \leq i \leq t$ and

$$n_1 D_1 \cdot E + \cdots + n_t D_t \cdot E = n_0.$$

Replace them by t vertices weighted by $(D_i + (D_i \cdot E)E, n_i), 1 \leq i \leq t$. Notice that here we allow $n_i = 1$.

To record the value of t, we sometimes call this move (iii)$_t$.

The following simple observation is crucial for us:

Lemma 3.11. *If we apply any of the three moves above to a connected, nef curve configuration, the new weighted graph is a connected, nef curve configuration with the same total class. Moreover, it has the following properties:*

- *The sum of the multiplicities of vertices gets smaller.*
- *The sum $\sum_i g_J(e_i)$ is non-decreasing for any curve combination move.*
- *$\sum_i m_i g_J(e_i)$ is also non-decreasing for any curve combination move.*

Proof. Firstly, we notice that the first weight of each vertex is still a $J-$effective class since it is a linear combination of that of old vertices with non-negative coefficients.

To show that the new configuration is a curve configuration, we first verify the adj condition:

$$adj(D_1 + D_2) \geq 2 + adj(D_1) + adj(D_2) \geq -2 \tag{8}$$

for moves (i) and (ii), and

$$adj(D_i + (D_i \cdot E)E) = adj(D_i) + (D_i \cdot E)^2 + (2g_J(E) - 1)D_i \cdot E \geq -2 \tag{9}$$

for move (iii).

Next we verify the label condition. Clear for moves (i) and (ii). For move (iii), the label of each new edge is

$$(D_i + (D_i \cdot E)E) \cdot (D_j + (D_j \cdot E)E) = D_i \cdot D_j + (D_i \cdot E) \cdot (D_j \cdot E) > 0.$$

Let us prove the curve configuration is connected. It is clear for move (i). For move (iii), consider the collection of new vertices. The sum of their classes is the sum of the classes of the replaced vertices, so at least one of new vertices is connected to the rest of the configuration. Moreover, any two new vertices are adjoined to each other since we have shown that $(D_i + (D_i \cdot E)E) \cdot (D_j + (D_j \cdot E)E) > 0$.

For move (ii), we need the nefness condition. If V_1 is connected to another vertex in the original configuration other than V_2, then the new vertex V' is adjoined to the same vertex. Hence, the new configuration is connected. Otherwise, only V_2 is connected to other vertices. The graph is assumed to be nef, thus

$$e \cdot D_1 = (n_1 - n_2)D_1 \cdot D_1 + n_2(D_1 \cdot D_1 + D_1 \cdot D_2) \geq 0,$$

which implies

$$D_1 \cdot (D_1 + D_2) \geq \frac{n_1 - n_2}{n_2}(-D_1 \cdot D_1) > 0.$$

This shows the new configuration is still connected since V is connected to the vertices that D_2 was connected to.

For the first bullet of the properties, the sum of multiplicities are reduced by n_2 for the first two moves, and n_0 for the third.

For the second and the third bullets, the conclusion for first two moves follows from (8). For the third move, (9) implies

$$adj(D_i + (D_i \cdot E)E) \geq adj(D_i) + 2(D_i \cdot E)g_J(E).$$

This shows

$$\begin{aligned}
&\textstyle\sum_{i=1}^t n_i g_J(D_i + (D_i \cdot E)E) \\
&\geq \textstyle\sum_{i=1}^t (n_i g_J(D_i) + n_i D_i \cdot E g_J(E)) \\
&= \textstyle\sum_{i=1}^t n_i g_J(D_i) + n_0 g_J(E).
\end{aligned}$$

Similarly, $\sum_{i=1}^t g_J(D_i + (D_i \cdot E)E) \geq \sum_{i=1}^t g_J(D_i) + g_J(E)$. □

Here is an example how to apply these moves.

Example 3.12. *Consider the curve configuration in $\mathbb{CP}^2\#5\overline{\mathbb{CP}^2}$ with 4 vertices weighted by*

$$(H - E_1 - E_2 - E_3, n), (H - E_1 - E_4 - E_5, n), (E_1, 2n), (2H - E_2 - E_3 - E_4, 1).$$

The total class is

$$(2n + 2)H - (n + 1)E_2 - (n + 1)E_3 - (n + 1)E_4 - nE_5,$$

which has $J-$genus 0 and is Cremona equivalent to $(n + 1)H - nE_1$. Here Cremona equivalence refers to the equivalence under the group of diffeomorphisms preserving the canonical class K_J.

First apply move (iii) to the -1 vertex $(E_1, 2n)$ to obtain the curve configuration with 3 vertices weighted by

$$(H - E_2 - E_3, n), (H - E_4 - E_5, n), (2H - E_2 - E_3 - E_4, 1).$$

Then apply move (i) to the first two vertices to obtain the curve configuration with 2 vertices weighted by

$$(2H - E_2 - E_3 - E_4 - E_5, n), (2H - E_2 - E_3 - E_4, 1).$$

3.4. **Nef, connected curve configuration with at least two vertices.**

3.4.1. *Rearrangement.*

Lemma 3.13. *Suppose a connected, nef curve configuration has at least two vertices. After applying curve expansion and appropriate curve combination moves (i), (ii), (iii) to -1 vertices, we would end up with a connected, nef curve configuration such that*

- *All vertices with non-negative self-intersection have multiplicity* 1*;*
- *The -1 vertices are not adjoined to each other. Moreover, any -1 vertex is not adjoined to a vertex with non-negative self-intersection;*
- *If vertices weighted by $(D_i, n_i), 1 \leq i \leq t$ with $D_i \cdot D_i \leq -2$ are all adjoined to a -1 vertex (E, n_0), then*

$$n_1 D_1 \cdot E + \cdots + n_t D_t \cdot E > n_0.$$

Proof. We apply move (i) first to each -1 vertex. After this is done we could assume that, for any -1 vertex, its multiplicity m is different from the multiplicity of any adjoined vertex.

We now apply move (ii) to each -1 vertex whenever it is adjoined to a vertex with self-intersection at least -1.

After applying moves (i) and (ii) repeatedly, we could assume that the second bullet is valid.

Given a -1 vertex weighted by (E, n_0), suppose the vertices that are adjoined to it are weighted by $(D_i, n_i), 1 \leq i \leq t$. Observe that, by the second bullet, each D_i has self-intersection ≤ -2 and

$$e \cdot E = n_1 D_1 \cdot E + \cdots + n_t D_t \cdot E - n_0.$$

Since the total class e is $J-$nef, we have a priori that

$$n_0 \leq n_1 D_1 \cdot E + \cdots + n_t D_t \cdot E.$$

If $n_0 = n_1 D_1 \cdot E + \cdots + n_t D_t \cdot E$, we are then in the situation to apply move (iii). This move may actually produce new -1 vertices and even vertices with non-negative self-intersection. If so, we apply curve expansion and curve combination moves (ii) and (i) again to rearrange so that the first and the second bullets are valid.

We notice that such rearrangement would stop in finite steps. This is because: (a) the total multiplicity is preserved after curve expansion, and it is reduced after each curve combination by the first bullet of Lemma 3.11, so we could only apply finitely many curve combination moves, (b) between two curve combination moves, the number of curve expansions is bounded by the total multiplicity.

□

3.4.2. *After rearrangement.*

Lemma 3.14. *For a connected, nef curve configuration satisfying all the three bullets in Lemma 3.13, if there is a vertex with multiplicity greater* *than* 1*, then the* $J-$*genus satisfies*

$$g_J(e) \geq 1 + \sum_i m_i g_J(e_i).$$

This technical lemma will be proved in the next subsection.

Example 3.15. *Here is one example of a connected, nef curve configuration satisfying all the three bullets in Lemma 3.13, and having a vertex with multiplicity greater than* 1*:* $\Theta = \{(C_1, 2), (C_2, 1), (C_3, 1), (C_4, 1), (C_5, 1)\}$ *with*

$$e_{C_1} = H - E_1 - E_2 - E_3, \quad e_{C_2} = e_{C_3} = e_{C_4} = H, \quad e_{C_5} = E_1.$$

Here $e = 5H - E_1 - 2E_2 - 2E_3$*, and* $g_J(e) = 4$*.*

3.5. **Proof of Proposition 3.9 and Lemma 3.14.** We first prove Proposition 3.9, which assumes Lemma 3.14.

3.5.1. *Proposition 3.9.*

Proof of Proposition 3.9. One vertex curve configuration case follows from the second half of Lemma 3.8. Hence we assume there are at least two vertices.

Denote the curve configuration by $G = \{(e_i, m_i)\}$. We apply the moves to get a curve configuration $G' = \{(e'_j, m'_j)\}$ as in Lemma 3.13. By the second bullet of Lemma 3.11 and Lemma 3.10, the sum $\sum_i g_J(e_i)$ is non-decreasing for any curve expansion and curve combination move.

By Lemma 3.14, if there is a vertex with multiplicity greater than 1, then

$$\begin{aligned} g_J(e) &\geq \textstyle\sum_j m'_j g_J(e'_j) + 1 \\ &\geq \textstyle\sum_j g_J(e'_j) + 1 \\ &\geq \textstyle\sum_i g_J(e_i) + 1. \end{aligned}$$

If the multiplicity of each vertex is 1, apply the first statement of Lemma 3.8 instead of Lemma 3.14, we obtain similarly $g_J(e) \geq \sum_i g_J(e_i)$.

□

3.5.2. *A stronger bound.* In fact, we can establish the stronger estimate

$$g_J(e) \geq \sum_i m_i g_J(e_{C_i}),$$

if there is no vertex having class me' with $e' \cdot e' = 0$, $m \geq 2$ and $g_J(e') \geq 1$ in any intermediate step of the rearrangement.

First of all, during the arrangement, if we never need to apply curve expansion for vertex C with weight (e_C, m) such that $e_C \cdot e_C = 0$, $m > 1$ and $g_J(e_C) > 0$, the sum $\sum_i m_i g_J(e_i)$ is non-decreasing by the third bullet of Lemma 3.11 and Lemma 3.10.

After the arrangement, if there is a vertex with multiplicity greater than 1, then by Lemma 3.14,

$$\begin{aligned} g_J(e) &\geq \sum_j m'_j g_J(e'_j)+1 \\ &\geq \sum_i m_i g_J(e_i)+1. \end{aligned}$$

If the multiplicity of each vertex is 1, apply the first statement of Lemma 3.8 instead of Lemma 3.14, we obtain $g_J(e)\geq \sum_i m_i g_J(e_i)$.

3.5.3. *Lemma 3.14.* It remains to prove Lemma 3.14.

Proof of Lemma 3.14. For a configuration as in Lemma 3.13, we suppose there are l vertices weighted by (e_i, m_i). Moreover, we define s_1, s_2, s_3 by requiring that

- If $1\leq i\leq s_1$, then $m_i\geq 2$ and $e_i\cdot e_i\leq -2$;
- If $s_1+1\leq i\leq s_1+s_2$, then $m_i\geq 2$ and $e_i\cdot e_i=-1$;
- if $s_1+s_2+1\leq i\leq s_1+s_2+s_3$, then $m_i=1$ and $e_i\cdot e_i\leq -1$;
- If $s_1+s_2+s_3+1\leq i\leq l$, then $m_i=1$ and $e_i\cdot e_i\geq 0$.

We further let

$$s=s_1+s_2.$$

With this understood we set up to show that if $s>0$ then

$$adj(e)\geq \sum_i m_i(adj(e_i)+2).$$

If $1\leq i\leq s$, write $m_ie_i=\sum_{1\leq k\leq m_i} e_i(k)$, with each $e_i(k)=e_i$.

$$\begin{aligned} adj(e) &= \sum_{1\leq i\leq s}\sum_{1\leq k\leq m_i} adj(e_i(k)) \\ &+\sum_{1\leq i\leq s}\sum_{1\leq k\neq k'\leq m_i} e_i(k)\cdot e_i(k')+\sum_{1\leq i\leq s}(m_ie_i)(e-m_ie_i) \\ &+\sum_{j>s} adj(e_j)+\sum_{j>s} e_j\cdot \sum_{i\leq s} m_ie_i+\sum_{j,k>s,j\neq k} e_j\cdot e_k. \end{aligned}$$

The adjunction terms satisfy

$$\sum_{1\leq i\leq s}\sum_{1\leq k\leq m_i} adj(e_i(k))+\sum_{j>s} adj(e_j)=\sum_{i=1}^{l} m_i\cdot adj(e_i).$$

We claim that the cross terms satisfy

$$\sum_{j>s} e_j\cdot \sum_{i\leq s} m_ie_i+\sum_{j,k>s,j\neq k} e_j\cdot e_k\geq 2(l-s).$$

To justify the claim, introduce $\alpha=e_1+\cdots+e_s$, and rewrite as

$$\sum_{j,k>s,j\neq k} e_j\cdot e_k+2\sum_{j>s} e_j\cdot\alpha+\sum_{j>s} e_j\cdot\sum_{i\leq s}(m_i-2)e_i. \tag{10}$$

Since $m_i\geq 2$ for $i\leq s$, the last term of (10) is non-negative.

To estimate the first two terms of (10), view the portion of the configuration involving vertices with $i\leq s=s_1+s_2$ as one single vertex $(\alpha, 1)$.

Notice here we use the assumption $s > 0$. Along with the remaining $l-s$ vertices, we obtain a graph with $l-s+1$ vertices. Notice that this graph is still connected.

Twice of the total labeling of this new graph is exactly the sum of the first two terms. For this graph of $l-s+1$ vertices to be connected, we need at least $l-s$ edges. Since each label is positive, we obtain the desired estimate.

The remaining terms

$$\begin{aligned} & \textstyle\sum_{1\le i\le s}\sum_{1\le k\neq k'\le m_i} e_i(k)\cdot e_i(k') + \sum_{1\le i\le s}(m_ie_i)(e-m_ie_i) \\ = & \textstyle\sum_{1\le i\le s} m_i(m_i-1)e_i\cdot e_i + \sum_{1\le i\le s}(m_ie_i)\cdot(e-m_ie_i) \\ = & \textstyle\sum_{1\le i\le s} m_ie_i(e-e_i). \end{aligned} \tag{11}$$

Sum them up, and notice that $m_i = 1$ if $i \ge s$, we have

$$\begin{aligned} adj(e) \ & \textstyle\ge \sum_{i=1}^{l} m_i\cdot adj(e_i) \\ & \textstyle+\sum_{1\le i\le s} m_ie_i(e-e_i) \\ & +2(l-s) \\ & \textstyle= \sum_{i=1}^{l} m_i\cdot (adj(e_i)+2) \\ & \textstyle+\sum_{1\le i\le s} m_ie_i(e-e_i) - \sum_{1\le i\le s} 2m_i. \end{aligned}$$

So it suffices to show that

$$\sum_{1\le i\le s} m_ie_i(e-e_i) - \sum_{1\le i\le s} 2m_i \ge 0.$$

We separate the discussion into the two cases $i \le s_1$ and $s_1+1 \le i \le s$.

Case I: When $i \le s_1$, since the curve configuration is nef,

$$\sum_{1\le i\le s_1} m_ie_i(e-e_i) - \sum_{1\le i\le s_1} 2m_i \ge \sum_{1\le i\le s_1} -m_ie_i\cdot e_i - \sum_{1\le i\le s_1} 2m_i \ge 0.$$

The last inequality holds because $m_i \ge 2$ and $C_i^2 \le -2$ if $1 \le i \le s_1$.

Case II:

For $s_1+1 \le i \le s$, we need to be more careful.

$$e\cdot e_i = (\sum_{j\le s_1} m_je_j + \sum_{s+1\le j\le s+s_3} e_j + m_ie_i)\cdot e_i = \sum_{\{j\neq i: e_j\cdot e_i\neq 0\}} m_je_j\cdot e_i - m_i.$$

The equalities hold by the second bullet of Lemma 3.13.

Furthermore, by the third bullet of Lemma 3.13,

$$\sum_{\{j\neq i: e_j\cdot e_i\neq 0\}} m_je_j\cdot e_i - m_i \geq 1$$

for any $s_1+1\leq i\leq s$. Now, for $s_1+1\leq i\leq s$,

$$m_ie_i(e-e_i)=m_ie_i\cdot e-m_ie_i^2\geq m_i+m_i=2m_i.$$

Combining these two cases, we have proved the Lemma. □

4. Rational curves

When $g_J(e)=0$, we get more precise information.

4.1. **Tree of smooth components.**

Corollary 4.1. *Suppose e is a $J-$nef class with $g_J(e)=0$. If $\Theta=\{(C_i,m_i),1\leq i\leq l\}\in\mathcal{M}_{red,e}$ is connected, then each irreducible component is a smooth rational curve.*

Proof. Recall we have proved Theorem 1.4 in the last section, which says (3) holds. Since $g_J(e)=0$ and $g_J(e_i)\geq 0$, it follows from (3) that we must have $g_J(e_i)=0$ for each i. By the adjunction inequality (2), each C_i is a smooth rational curve. □

Now we show that Θ is a tree configuration.

Definition 4.2. *A connected weighted graph with l vertices is called a tree if the sum of the labels of the edges is $l-1$, which is the minimal number ensuring the graph to be connected.*

A tree graph is called a simple tree graph if further, each vertex has multiplicity 1.

Lemma 4.3. *Suppose $g_J(e)=0$ and $\Theta=\{(C_i,1),1\leq i\leq l\}$ is connected curve configuration with total class e, then $g_J(e_i)=0$ and the underlying graph is a simple tree.*

Proof. This follows from the argument in Lemma 3.1. More precisely, since $g_J(e)=0$, the estimate (7) has to be an equality. □

Lemma 4.4. *Suppose we apply any of the three curve combination moves to a connected, nef curve configuration. If each new vertex of the resulting curve configuration has $adj=-2$, then so does each replaced vertex of the initial curve configuration.*

Proof. We use the notation D_i as in 3.3.2.

For the first two moves, since $adj(D_i)\geq -2$, it follows from (8) that $-2=adj(D_1+D_2)$ if and only if

$$adj(D_1)=adj(D_2)=-2,\quad D_1\cdot D_2=1.$$

For the third move, since $adj(D_i) \geq -2$, $g_J(E) \geq 0$ and $D_i \cdot E > 0$, it follows from (9) that $-2 = adj(D_i + (D_i \cdot E)E)$ if and only if

$$adj(D_i) = -2, \quad g_J(E) = 0, \quad D_i \cdot E = 1.$$

□

Lemma 4.5. *Suppose $G = \{(e_i, m_i), 1 \leq i \leq l\}$ is a connected, nef curve configuration with each vertex having $adj = -2$. Let G' be the curve configuration obtained from G by a curve expansion or a curve combination. If G' is a tree, so is G.*

Proof. We only need to verify the change of the sum of labels is no smaller than the change of number of vertices. Let e be the total class of G.

For a curve expansion, a vertex weighted by (D, m) becomes m vertices weighted by $(D, 1)$. The number of vertices increases by $m-1$, and since G is connected, the sum of the labels increases by $(m-1)D \cdot (e - D) \geq m - 1$.

For curve combination move (i), the number of vertices decreases by 1. Suppose the two replaced vertices are C_1 and C_2. The sum of labels decreases by

$$(e_{C_1} \cdot \sum_{i \geq 3} e_{C_i} + e_{C_2} \cdot \sum_{i \geq 3} e_{C_i} + e_{C_1} \cdot e_{C_2}) - ((e_{C_1} + e_{C_2}) \cdot \sum_{i \geq 3} e_{C_i}) = e_{C_1} \cdot e_{C_2} = 1.$$

The last step is due to the $adj = -2$ assumption and Lemma 4.4.

For move (ii), the number of vertices is unchanged. Let the two replaced vertices be C_1 weighted by (D_1, n_1), and C_2 weighted by (D_2, n_2). Due to the $adj = -2$ assumption and Lemma 4.4, $D_1 \cdot D_2 = 1$. Hence $D_1 \cdot D_1 = -1$ because $D_1 \cdot D_2 \geq -D_1 \cdot D_1 > 0$. By nefness, $e \cdot D_1 \geq 0$. So C_1 should connect to vertices other than C_2. And the sum of labels would increase by

$$2(n_1 - n_2) - ((n_1 - n_2) + 1) \geq 0.$$

For move $(\mathrm{iii})_t$, the number of vertices would decrease by 1. The number of labels would increase at least by

$$\sum_{1 \leq i < j \leq t} (D_i \cdot D_j + 1) - (t + \sum_{1 \leq i < j \leq t} D_i \cdot D_j) \geq \frac{t(t-3)}{2} \geq -1. \tag{12}$$

□

Proposition 4.6. *Suppose e is a $J-$nef class with $g_J(e) = 0$. If G is connected curve configuration with class e and at least 2 vertices, then G is a tree graph.*

Proof. If $m_i = 1$ for each i, the assertion follows from Lemma 4.3.

Otherwise, we apply the curve expansion and combination moves to get a connected, nef curve configuration G' with class e and satisfying all the three bullets in Lemma 3.13. Notice that since $g_J(e) = 0$, Lemma 3.14 implies that each vertex of G' has multiplicity one. Therefore G' is a tree.

Then by Lemma 4.5, G is a tree as well. □

Corollary 4.7. *Suppose e is a $J-$nef class with $g_J(e) = 0$. If $\Theta = \{(C_i, m_i), 1 \leq i \leq l\} \in \mathcal{M}_{red,e}$ is connected, then the underlying weighted graph is a tree.*

4.2. **Dimension bound.** Suppose e is a $J-$nef class with $g_J(e) = 0$. If $\Theta = \{(C_i, m_i), 1 \leq i \leq n\} \in \mathcal{M}_{red,e}$ is connected, by Corollary 4.1 and Proposition 4.6, the underlying curve configuration of Θ is a tree with each vertex having genus 0.

4.2.1. l_G. In light of this, we introduce the following definition.

Definition 4.8. *The dimension of a tree graph G with vertices weighted by $\{(e_i, m_i)\}$ and having genus 0 is defined to be*

$$l_G = \sum_{i=1}^{n} l_{e_i}.$$

Recall that $l_{e_i} = \max\{\iota_{e_i}, 0\}$, and $\iota(e_i)$ is the $J-$dimension defined by (6). Since $g_J(e_i) = 0$, ι_{e_i} is equal to $e_i \cdot e_i + 1$ by Lemma 2.5.

We stratify $\mathcal{M}_{red,e}$ according to the underlying curve configuration. By Lemma 2.6, l_G is the complex dimension of the stratum corresponding to the curve configuration G.

Let $L = l_e$. By Lemma 2.6, L is the complex dimension of $\mathcal{M}_{irr,e}$, as long as $\mathcal{M}_{irr,e}$ is nonempty.

Example 4.9. *We illustrate Definition 4.8 with a graph having* 5 *vertices and dimension $L-1$. We will use notations as in Example 3.7. For the centered graph G with center vertex $(a-2b, 1)$ and four vertices $(b, 1)$, its dimension $l_G = 4$. In this case, $e = a + 2b$ and $L = l_e = 5$. So G is a codimension* 1 *graph.*

For a $J-$nef class, we have the following fact.

Lemma 4.10. *Suppose e is a $J-$nef class with $g_J(e) = 0$ and $L = l_e$. If G is a connected curve configuration with class e and $n \geq 2$ vertices, and vertices weighted by $\{(e_i, m_i), 1 \leq i \leq n\}$, then*

$$\sum_{i=1}^{n} m_i l_{e_i} \leq L - 1. \tag{13}$$

Lemma 4.10 is to be proved below. Before proving it, we first state an important corollary. Since $l_{e_i} \geq 0$ we have

Corollary 4.11. *Suppose e is a $J-$nef class with $g_J(e) = 0$ and $L = l_e$. If G is a connected curve configuration with class e and at least 2 vertices, then the dimension l_G of the stratum indexed by G satisfies the following bound,*

$$l_G \leq L - 1. \tag{14}$$

This is an analogue of Proposition 3.4 in [9], but valid for an arbitrary almost complex structure.

4.2.2. *Lemma 4.10.*

Proof of Lemma 4.10. Notice that by the assumption, e is $J-$effective and $J-$nef, so $e \cdot e \geq 0$ by Lemma 2.8. Hence

$$L = l_e = \iota_e = e \cdot e + 1 \geq 1. \tag{15}$$

Let us first assume that $n = 1$. In this case, since G has least 2 vertices, $m_1 \geq 2$. By the adjunction formula, this is impossible if $e_1 \cdot e_1 = 0$. But if $e_1 \cdot e_1 > 0$, then $e \cdot e \geq m_1^2$, and $l_{e_1} = 1 + e_1 \cdot e_1$. Therefore

$$m_1 l_{e_1} = m_1 + m_1 e_1 \cdot e_1 = m_1 + \frac{1}{m_1} e \cdot e < 1 + e \cdot e = L.$$

Now we assume that $n \geq 2$. Then

$$L = \iota_e = \sum_{i=1}^{n} m_i e_i \cdot m_i e_i + \sum_{i=1}^{n} m_i e_i \cdot (e - m_i e_i) + 1. \tag{16}$$

Since G is connected and $n \geq 2$,

$$m_i e_i \cdot (e - m_i e_i) \geq m_i \tag{17}$$

for each i.

I. Let us start with the simple case where each e_i has $e_i \cdot e_i \geq 0$. Then $l_{e_i} = \iota_{e_i} = 1 + e_i \cdot e_i$ for each i, and $m_i^2 e_i \cdot e_i \geq m_i e_i \cdot e_i$. By (16) and (17),

$$L \geq 1 + \sum_{i=1}^{n} m_i l_{e_i}.$$

II. General case. Use $1, \cdots, k$ to label the vertices whose class has self-intersection at most -1. Notice that $l_{e_i} = 0$ for $i = 1, \cdots, k$.

Since G is connected, $e_j \cdot (e - m_j e_j) \geq 1$ for each $j \geq k+1$. Therefore L can be estimated as follows:

$$\begin{array}{rcl} L & = & 1 + e \cdot e \\ & = & 1 + \sum_{j=k+1}^{n} (m_j^2 e_j \cdot e_j + m_j e_j \cdot (e - m_j e_j)) + (\sum_{i=1}^{k} m_i e_i \cdot e) \\ & \geq & 1 + \sum_{j=k+1}^{n} m_j l_{e_j} + (\sum_{i=1}^{k} m_i e_i \cdot e) \\ & = & 1 + \sum_{j=1}^{n} m_j l_{e_j} + (\sum_{i=1}^{k} m_i e_i \cdot e). \end{array} \tag{18}$$

Finally, observe that, by the $J-$nefness of e, the last term $(\sum_{i=1}^{k} m_i e_i \cdot e)$ is non-negative. □

4.3. **Maximal dimension configurations.** We assume M has $b^+ = 1$. Let e continue to be a $J-$nef class with $g_J(e) = 0$.

If G is a connected curve configuration with class e and at least 2 vertices, we have shown in the two previous subsections that G is a tree graph (Proposition 4.6), and l_G is bounded above by $L - 1$ (Corollary 4.11).

In this subsection we classify all possible maximal dimension configurations with at least 2 vertices, i.e. configuration G with $L = 1 + l_G$.

Let G_- be the weighted subgraph containing each vertex whose class has self-intersection at most -1. Use $V_1, \cdots, V_k$ to label these vertices. Let G_+ be the weighted subgraph containing remaining vertices, use V_j with $j \geq k+1$ to label these vertices.

Lemma 4.12. *If $l_G = L - 1$ then $m_j = 1$ for $j \geq k+1$. Namely, the G_+ part is simple.*

Proof. Since e is $J-$nef and $l_{e_j} = 0$ for $j \leq k$, it follows from (18) that $L - 1 \geq \sum_{j=k+1}^{n} m_j l_{e_j}$. Since $l_{e_j} \geq 1$ for each $j \geq k+1$ and $l_G = \sum_{j=k+1}^{n} l_{e_j}$, we have the desired claim. □

From the proof above we actually have two more consequences of (18) under the assumption that $l_G = L - 1$. The first one is

$$e_i \cdot (e - e_i) = 1, \quad \text{if } k+1 \leq i \leq n, \tag{19}$$

and the second one is

$$e \cdot e_i = 0, \quad \text{if } i \leq k. \tag{20}$$

4.3.1. *When G_- is empty.*

Lemma 4.13. *If $l_G = L - 1$ and G_- is empty, then $n = 2$, and $m_1 = m_2 = 1$.*

Proof. In this case, by Lemma 4.12, $m_i = 1$ for each i. Moreover, $e_i \cdot (e - e_i) = 1$ for each i. Since G is connected, this is possible only if $n = 2$.

□

In the following we assume that G_- is not empty. We first show that G contains a centered subgraph.

4.3.2. *A centered subgraph.* A simple tree graph is called centered if there is a vertex, called the center vertex, which is adjoined to all the other vertices. Note that the graph in Example 4.9 is centered.

Lemma 4.14. *Assume G_- is non-empty and $l_G = L - 1$. Then*

- *The vertices in G_+ have the same weight with $m_i = 1$ and $e_i \cdot e_i = 0$.*
- *There is only one vertex in G_- which is adjoined to the vertices in G_+. Denote this vertex by V_1. V_1 has multiplicity one, and its class has self-intersection less than or equal to $k - n$.*
- *The weighted subgraph consisting of the vertex V_1 and vertices in G_+ is a centered graph with V_1 as the center.*
- *The weighted subgraph G_- is connected.*

Proof. For each i with $k+1 \leq i \leq n$, $l_{e_i} \geq 1$. Since $l_G = L-1$, it follows from Lemma 4.12 and (19) that, if there are $i \neq i' \geq k+1$ such that $e_i \cdot e_{i'} \neq 0$, then V_i and $V_{i'}$ are not adjoined to any other vertex. But this is impossible since G_- is non-empty and G is connected.

Hence, for $i \geq k+1$, the vertices V_i are not adjoined to each other. Since $b^+(M) = 1$, by light cone lemma, $e_i \cdot e_i = 0$ for each $i \geq k+1$, and for $i, i' \geq k+1$, $e_i = \alpha e_{i'}$. By the adjunction formula, $\alpha = 1$ for any pair. We have established the first bullet.

By the first bullet, the vertices in G_+ are disjoint. It follows from Lemma 4.12 and (19) that, each vertex in G_+ is adjoined to a unique vertex in G_-, and this vertex in G_- has to have multiplicity one. Since the classes of vertices in G_+ are the same, the vertices in G_+ are actually adjoined to the same vertex in G_-. Denote this vertex by V_1.

It follows from (20) that the class of V_1 has self-intersection less than or equal to $k - n$. We have now established both the second and the third bullets.

For the last bullet, it is a consequence of the second bullet since G is connected. $\square$

Next we show that G is a special kind of centered graph when G_- is not empty and there are no -1 vertices.

4.3.3. *When G_- is not empty and there are no -1 vertices.*

Lemma 4.15. *Suppose $l_G = L - 1$, G_- is not empty and there are no -1 vertices. Then G_- contains a unique vertex V. Furthermore, if $e_V \cdot e_V = -b$, then G is a centered graph with b teeth.*

Proof. We first show that all vertices of G have multiplicity 1. By the first bullet of Lemma 4.14, this is true for any vertex in G_+. Since there are no -1 vertices, no curve combination move is needed to achieve the configuration described in Lemma 3.13. Apply Lemma 3.14 to conclude that all vertices of self-intersection less than -1 also have multiplicity 1.

Now we show that every vertex in G_- is adjoined to at least two vertices of G. Since every vertex of G has multiplicity one, and each edge of G has label 1 by Proposition 4.6, we see that from (20), once a vertex in G_- is adjoined to only one other vertex of G, its self-intersection should be -1. But this is excluded by our assumption.

By the second bullet of Lemma 4.14 there is only one vertex V_1 in G_- which is adjoined to vertices in G_+. So any other vertex in G_- is adjoined to at least two vertices in G_-. By the last bullet of Lemma 4.14, G_- is connected. So if G_- has more than one vertex, V_1 is adjoined to at least one another vertex in G_-.

Thus, if $k \geq 2$, twice of the number of edges in G_- is at least

$$2(k-1) + 1 = 2k - 1 > 2(k-1).$$

This means that there must be a cycle in the weighted subgraph G_-. This implies that there is a cycle in G as well, which contradicts Proposition 4.7. Hence, there is only one vertex in G_- .

Finally, we conclude that G is a centered graph by the third bullet of Lemma 4.14. $\square$

The remaining case is that G_- contains -1 vertices. We start with the following observation.

4.3.4. *$\tilde{G}$ in Lemma 3.13.*

Lemma 4.16. *Suppose $\tilde{G}$ satisfies all the three bullets in Lemma 3.13 and $l_{\tilde{G}} = L-1$ or L. Then $\tilde{G}$ contains no -1 vertices.*

Proof. If $l_{\tilde{G}} = L$, then $\tilde{G}$ has only one vertex by Corollary 4.11. This vertex is not a -1 vertex since its class is just e, which is assumed to be $J-$nef.

Now let us assume that $l_{\tilde{G}} = L-1$. Notice that, if there is a -1 vertex E in $\tilde{G}$, then by the second and the third bullets of Lemma 3.13, we have $e_{\tilde{G}} \cdot E > 0$. But this contradicts to (20). □

In light of Lemma 4.16, we next analyze how l_G changes under curve moves.

4.3.5. *l_G under curve moves.*

Lemma 4.17. *Let G' be obtained from G by a curve expansion. Then $l_G < l_{G'}$.*

Proof. This is clear since

$$D \cdot D + 1 < n(D \cdot D + 1)$$

if $D \cdot D \geq 0$ and $n > 1$. □

Lemma 4.18. *Let G' be obtained from G by a curve combination, which is not of type (i)$_1$ with $D_1 \cdot D_1 \neq -1$. Then $l_G \leq l_{G'}$. Furthermore, $l_G = l_{G'}$ if and only if the class of each new vertex of G' has negative self-intersection. In particular, under such a move, $l_G = l_{G'}$ if $l_G = L-1$ and G' has more than one vertices.*

Proof. For move (i)$_n$ with $n \geq 2$, the part modified has $l_{D_1} + l_{D_2} = 0$.

For move (iii), the part modified has $\sum_{i=1}^{t} l_{D_i} + l_E = 0$.

In these two cases, $l_G \leq l_{G'}$ since a new vertex V always has $l_V \geq 0$. The equality $l_G = l_{G'}$ holds if and only if V has negative self-intersection.

For move (ii), since $n_1 \geq 2$, we have $l_{D_1} = 0$. Meanwhile,

$$(D_1 + D_2) \cdot (D_1 + D_2) > D_2 \cdot D_2,$$

which implies $l_G \leq l_{G'}$. The equality $l_G = l_{G'}$ holds if and only if $(D_1 + D_2) \cdot (D_1 + D_2) < 0$.

For move (i)$_1$ with $D_1 \cdot D_1 = -1$, we have

$$D_2 \cdot D_2 + 1 < (D_1 + D_2) \cdot (D_1 + D_2) + 1.$$

Similarly, $l_G = l_{G'}$ if and only if $(D_1 + D_2) \cdot (D_1 + D_2) < 0$.

The last statement follows from Corollary 4.11. □

4.3.6. *$\tilde{G}$ in Lemma 3.13 revisited.* Given the two lemmas above, we have the following more precise description of $\tilde{G}$.

Lemma 4.19. *Suppose G contains a -1 vertex and $l_G = L-1$. We apply curve moves as in Lemma 3.13 to adjust G to a configuration $\tilde{G}$ satisfying all the three bullets there. Let $G_p, ..., G_1$ be the intermediate graphs. Then*

- *$l_{G_i} = L-1$,*
- *$l_{\tilde{G}} = L-1$ or L,*
- *There are no -1 vertices in $\tilde{G}$,*
- *If $\tilde{G}$ has at least 2 vertices, then it is either a graph with precisely 2 vertices as in Lemmas 4.13, or a centered graph as in 4.15,*
- *$\tilde{G}$ is a simple tree graph.*
- *$\tilde{G}_-$ contains at most one vertex.*

Proof. Notice that the curve combinations in Lemma 3.13 only involve -1 vertices, the first and second bullets follow from Lemmas 4.17, 4.18 and Corollary 4.11.

The third bullet follows from the second bullet and Lemma 4.16.

We now prove the fourth bullet. If $\tilde{G}$ has at least 2 vertices, by Corollary 4.11 and the second bullet, $l_{\tilde{G}} = L-1$. Since $\tilde{G}$ contains no -1 vertices by the third bullet, the statement follows from Lemmas 4.13, 4.15.

The last two bullets follows from the fourth bullet.

□

We will see that only the following restricted moves, which we call combinatorial blow-downs, are needed to obtain $\tilde{G}$ from G.

4.3.7. *Combinatorial blow-downs.*

Definition 4.20. *A simple combinatorial blow-down applied to a weighted graph G is the following process of removing a -1 vertex V of genus 0.*

1. *Either V is weighted by (v, t) and adjoined to only one vertex U weighted by (u, t) with $u \cdot v = 1$, then in the new graph these two vertices are removed and a new vertex U' weighted by $(u+v, t)$ is added.*
2. *Or V is weighted by (v, t_1+t_2) and adjoined to exactly two vertices U_1 weighted by (u_1, t_1) and U_2 weighted by (u_2, t_2) with edges labeled by one, i.e. $v \cdot u_1 = v \cdot u_2 = 1$, then these three vertices are replaced by two new vertices U_1' weighted by (u_1+v, t_1) and U_2' weighted by (u_2+v, t_2).*

The inverse process is called a simple combinatorial blow-up.

Geometrically, the first bullet corresponds to blowing up at a smooth point in the subvariety, the second bullet corresponds to blowing up at a transversal intersection point of two irreducible components.

4.3.8. *Each move is a combinatorial blow down.*

Lemma 4.21. *Suppose G contains a -1 vertex and $l_G = L-1$. Then after applying simple combinatorial blow-downs, G can be turned into a curve configuration $\tilde{G}$ with no -1 vertices. There are two cases:*

- *$\tilde{G}$ consists of only one vertex, whose class has non-negative self-intersection. In this case, except for the last blow-down, all the vertices involved in blow-downs have classes with negative self-intersection.*
- *$\tilde{G}$ is a centered graph. In this case, all the vertices involved in blow-downs have classes with negative self-intersection.*

Proof. We apply curve moves to adjust G to a configuration $\tilde{G}$ as in Lemma 4.19.

Let $G_p, ..., G_1$ be the intermediate graphs. It is convenient to sometimes write $G = G_{p+1}$ and $\tilde{G} = G_0$.

By Lemma 4.4 and Lemma 4.5, for each $i \geq 0$, each G_i is a tree graph of genus 0 vertices. Further, by Lemma 4.19, $G_0 = \tilde{G}$ is a *simple* tree graph and $\tilde{G}_-$ contains at most one vertex.

For $i \geq 1$, G_i contains at least 2 vertices, one of them is a -1 vertex. In fact, the move from G_i to G_{i-1} involves a -1 vertex of G_i. By Lemma 4.19, $\tilde{G}$ has no -1 vertices,

$$l_{G_i} = L-1 \text{ for } i > 0, \quad \text{and} \quad l_{\tilde{G}} = L-1 \text{ or } L. \tag{21}$$

No expansion moves: First we notice that curve expansion never occurs in the process. By Lemma 4.17 and (21), it could only possible occur in the last step, from G_1 to $\tilde{G}$ and when $l_{\tilde{G}} = L$. If this is the case, then $\tilde{G}$ has more than one vertex since expansion increases the number of vertices. However, this is impossible since in this case $\tilde{G}$ consists of a single vertex with multiplicity one due to the assumption $l_{\tilde{G}} = L$.

The move from G_q to G_{q-1} for $q \geq 2$: We know it is a *combination* move involving a -1 vertex. We will show that it is a simple combinatorial blow down.

We first make a general observation. Notice that for $q \geq 2$, $l_{G_q} = l_{G_{q-1}}$. Therefore, by Lemma 4.18, the classes of the new vertices in G_{q-1} have negative self-intersection.

I. Suppose for some $q \geq 2$ the move from G_q to G_{q-1} is a type (i) move. Then there are two adjoined vertices of the tree graph G_1, U_1 weighted by (u_1, t) and U_2 weighted by (u_2, t), one of them, say U_2, is a -1 vertex, and they are replaced by a vertex U of $\tilde{G}$ weighted by $(u_1 + u_2, t)$. Clearly, this move is just a type (1) simple combinatorial blow-down.

Notice that $(u_1 + u_2) \cdot (u_1 + u_2) = u_1 \cdot u_1 + 2 - 1 = u_1 \cdot u_1 + 1$ is negative. Therefore $u_1 \cdot u_1$ is negative as well.

II. Moves (ii) are not needed. Suppose for some $q \geq 2$ the move from G_q to G_{q-1} is a type (ii) move in the proof of Lemma 3.13. Since such a move is applied to a -1 vertex and another vertex whose class has self-intersection at least -1, the class of one new vertex of G_{q-1} has non-negative self-intersection. But this is impossible.

III. Suppose for some $q \geq 2$ the move from G_q to G_{q-1} is a type (iii)$_t$ move. Then there are t vertices U_i of G_1 weighted by (u_i, n_i), $1 \leq i \leq t$, and a -1 vertex V of G_1 weighted by (v, n_0), such that

$$u_i \cdot u_i \leq -2, \quad \text{and} \quad \sum_j n_i(u_i \cdot v) = n_0.$$

They are replaced by vertices W_i weighted by $(u_i + v, n_i)$.

Both G_q and G_{q-1} are tree graphs, and since the number of vertices of G_{q-1} is 1 less than that of vertices of G_q, the number of edges of G_{q-1} is also 1 less than that of labels of G_q. Apply the inequality (12), we find that it is only possible that $t = 1$ or 2.

When $t = 1$, the move is also a type (i) move, so it is a type (1) blow-down. As already shown, the classes of the involved vertices all have negative self-intersection.

Now assume that $t = 2$. Notice that $u_i \cdot v = 1$ since G_q is a tree graph. Hence this move is a type (2) simple combinatorial blow-down. We just need to verify the classes of the involved vertices all have negative self-intersection. This is true for U_1, U_2 and V by assumption. For W_i, this is also true since $(u_i + v) \cdot (u_i + v) \leq -2 + 2 - 1 = -1$.

The move from G_1 to $\tilde{G}$: The next step is to analyze the curve move from G_1 to $G_0 = \tilde{G}$.

I. Suppose this step is a type (i) move. We have already shown that it is a type (1) combinatorial blow-down. Here we have

Since $\tilde{G}$ is simple, t can only be equal to 1.

If $(u_1 + u_2) \cdot (u_1 + u_2) \geq 0$, then $l_{u_1+u_2} \geq l_{u_1} + l_{u_2} + 1$ and hence $l_{\tilde{G}} \geq l_{G_1} + 1 = L$. By Corollary 4.11, $\tilde{G}$ consists of a single vertex weighted by $(u_1 + u_2, 1)$. Thus this case corresponds to the first bullet of Lemma 4.21.

If $(u_1 + u_2) \cdot (u_1 + u_2) < 0$, then $l_{\tilde{G}} = l_{G_1} = L - 1$ and $\tilde{G}_-$ is not empty. Hence $\tilde{G}$ is a centered graph as in Lemma 4.15. Moreover, notice also that $u_1 \cdot u_1 < -u_2 \cdot u_2 - 2u_1 \cdot u_2 = 1 - 2 = -1$. Thus this move is a combinatorial blow-down and the classes of all the vertices involved have negative self-intersection. This case corresponds to the second bullet of Lemma 4.21.

II. Suppose this step is a type (ii) move. Then there are two adjoined vertices of the tree graph G_1, U_1 weighted by (u_1, t_1) and U_2 weighted by (u_2, t_2) with $t_1 > t_2$ and

$$u_1 \cdot u_1 \geq -u_1 \cdot u_2 = -1. \tag{22}$$

One of them is a -1 vertex, and they are replaced by a vertex U weighted by $(u_1 + u_2, t_2)$ and a vertex V weighted by $(u_1, t_1 - t_2)$.

If U_1 is a -1 vertex. then the vertex V of $\tilde{G}$ is a -1 vertex. But $\tilde{G}$ doesn't contain any -1 vertex, so U_1 is not a -1 vertex, and from (22) we must have $u_1 \cdot u_1 \geq 0$. We then conclude that $t_1 = 1$ by Lemma 4.12. But then

$t_2 = 0$ since $t_2 < t_1$. This simply means that this step cannot be a type (ii) move.

III. Suppose this step is a type (iii)$_t$ move.

Since G_1 is a tree graph, $u_i \cdot v = 1$. Hence

$$(u_i + v) \cdot (u_i + v) \leq -2 + 2 - 1 = -1$$

for $1 \leq i \leq t$. Since $\tilde{G}_-$ contains at most one vertex, we have $t = 1$. Thus in this case $\tilde{G}$ is a centered graph.

Since $\tilde{G}$ is a simple graph, we have $n_1 = 1 = n_0$. In other words, this move is actually a (special case of) type (i)$_1$ move, and this case corresponds to the first bullet of Lemma 4.21.

We thus complete our proof. □

4.3.9. *Summary.* Thanks to Lemmas 4.13, 4.15, 4.21, we can completely describe those G with $l_G = L - 1$.

Proposition 4.22. *Suppose $b^+(M) = 1$, e is a $J-$nef class with $g_J(e) = 0$. Let G be a connected curve configuration with class e and $l_G = L - 1$.*

If G does not contain any -1 vertex, then G is a simple graph tree. Moreover, it is either a graph with precisely 2 vertices as in Lemmas 4.13, or a centered graph as in 4.15. If G contains a -1 vertex, then G is as described in Lemma 4.21.

4.3.10. *Maximal dimension strata of $\mathcal{M}_{red,e}$.* We translate Proposition 4.22 to the description of the maximal dimension strata of $\mathcal{M}_{red,e}$.

To state the result, for $\Theta \in \mathcal{M}_{red,e}$, write $\Theta = \Theta_+ \cup \Theta_-$, where Θ_- contains each pair (C, m) with $e_C \cdot e_C \leq -1$. Label the pairs in Θ_- by $(C_1, m_1), \cdots, (C_k, m_k)$. A pair $(C, 1)$ is called a -1 component if C is a smooth rational curve with $e_C \cdot e_C = -1$.

Theorem 4.23. *Suppose $b^+(M) = 1$, e is a $J-$nef class with $g_J(e) = 0$. Let $\Theta = \{(C_i, m_i), 1 \leq i \leq n\}$ be a subvariety in $\mathcal{M}_{red,e}$, and $L = l_e$. Then $L = 1 + \sum_{i=1}^n l_{e_{C_i}}$ only if each m_i is equal to 1.*

Moreover, Θ satisfies the following conditions:

- *If Θ_- is empty then $n = 2$, $e_{C_1} \cdot e_{C_2} = 1$, $e_{C_i} \cdot e_{C_i} \geq 0$.*
- *If Θ_- is not empty and there is no -1 component, then Θ_- consists of a unique component C_1 with $e_{C_1} \cdot e_{C_1} = 1 - n \leq -2$, and Θ_+ consists of at least $n - 1 \geq 2$ components $C_i, i \geq 2$, with $e_{C_i} = \cdots = e_{C_n}$ and $e_{C_i} \cdot e_{C_i} = 0$. Moreover, $e_{C_1} \cdot e_{C_2} = 1$. In short, Θ is a comb like configuration.*
- *Suppose Θ_- contains a -1 component. Then there are two cases.*
  1. *Θ is a successive blow-up of a smooth rational curve with non-negative self-intersection. And from the second blow-up, we only blow up at a point not lying in any component with non-negative self-intersection (there is at most one such component).*
  2. *Θ is a successive blow-up of a comb like configuration in the second bullet at points in C_1 and its proper transforms. Moreover, the blow-up points do not lie in C_i for each $i \geq 2$.*

Conversely, if Θ is as in any bullet above, then $L = 1 + \sum_{i=1}^{n} l_{e_{C_i}}$.

The next example illustrates the two cases containing a -1 component. Notations are as in Example 3.7.

Example 4.24. *For case (i) suppose we start off with a smooth rational curve in class b. Since $b \cdot b = 0$ we have $L = 1$. Then the proper transform of the b curve has negative self-intersection after one blow-up, so we can do any further blow-ups. However, if we start off with a smooth rational curve C in class $a + b$, then the blow-ups are restricted: only the first one can be at a point meeting the curve C (or its proper transform).*

For case (2) suppose we start off with the comb like configuration in Example 4.9 with C_1 being the smooth curve in class $a - 2b$. Then we can blow up at two different points in C_1 not lying in any of the four curves in class b.

Remark 4.25. *By Proposition 2.11, the condition $b^+(M) = 1$ is automatic if J is assumed to be tamed.*

4.4. **Tamed J.** In this subsection J is assumed to be a tamed almost complex structure on M.

Let e be a class in S_{K_J}. Recall that S_{K_J} is the set of K_J-spherical classes, defined to be $\{e \in S | g_J(e) = 0\}$. Here S is the set of homology classes which are represented by smoothly embedded spheres.

4.4.1. *Connectedness and $J-$nef class.*

Proposition 4.26. *Suppose $e \in S_{K_J}$ and $\Theta = \{(C_i, m_i)\} \in \mathcal{M}_e$. If $e \cdot e_{C_i} \geq 0$ for each i, then Θ is connected and each component C_i is a smooth rational curve.*

Proof. First observe that $e \cdot e = e \cdot \sum_i e_{C_i} \geq 0$. Hence $b^+(M) = 1$ by Proposition 2.11.

Suppose Θ is not connected. Since $b^+(M) = 1$, and each class e_{C_i} is nontrivial by Lemma 2.9, either Θ has a connected component D with negative self-intersection, or it consists of $p \geq 2$ homologous connected components, D_i, with self-intersection 0.

The first case is impossible since $e \cdot e_D = e_D \cdot e_D < 0$.

In the second case, denote $e' = e_{D_i}$. Then $-2 = K_J \cdot e = K_J \cdot pe'$. Thus $p = 2$. But $K_J \cdot e' = 1$ and $e' \cdot e' = 0$, which is impossible since K_J is characteristic.

Since Θ is a nef configuration, Proposition 3.9 implies each component C_i is a smooth rational curve. □

Example 4.27. *In $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$, if $E_1 - E_2$ is $J-$effective, then the class $3H - 2E_2$ in S_{K_J} is not $J-$nef, and there is a disconnected curve in this class with connected components in $3H - E_1 - E_2$ and $E_1 - E_2$.*

Proof of Theorem 1.5. The first bullet follows from Corollary 4.1 and Corollary 4.7.

The second bullet follows from Proposition 4.26. □

Proof of Corollary 1.6. When $e \cdot e < 0$, the conclusion follows from Lemma 2.10.

Suppose now that $e \cdot e \geq 0$. Observe first that $g_J(e) = 0$. Observe also that by Lemma 2.8, e is $J-$nef. Hence the conclusion follows from Proposition 4.26.

□

4.4.2. *Remarks on Theorem 1.5 and Corollary 1.6.* Examples 3.3 and 4.27 demonstrate that $J-$nefness is necessary for Theorem 1.5.

As mentioned in the introduction, Theorem 1.5 and Proposition 4.10 are crucial in [6] in applying Taubes's subvarieties-current-form's approach to Donaldson's tamed versus compatible question for an arbitrary tamed almost complex structure on rational manifolds.

Various versions of Corollary 1.6 have appeared in the literature. When J is integrable, it is used in the classification of rational surfaces in [2]. On page 521 in [2], a simple argument is given, but unfortunately, it is not correct. [1] Presumably there is a substitute for this argument, but we have not been able to find out if our result is new in this case.

When $M = \mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$ with $k \leq 9$, it is shown in [10] that for any tamed J, an irreducible curve C with $C \cdot C < 0$ must be a smooth rational curve. One can easily deduce Corollary 1.6 for such manifolds from this fact.

For a generic tamed J, McDuff [7] provided a more intricate argument and established several special cases, which are essential in characterizing rational symplectic $4-$manifolds in terms of embedded symplectic spheres with positive self-intersection. Recently, McDuff and Opshtein in [8] investigate the structure of generic pseudo-holomorphic curves in a relative setting. The reducible $J-$holomorphic curves considered there are limits of irreducible embedded $J'-$holomorphic curves for generic J' converging to J, and hence the Gromov compactness can be applied to bound the topological type of the reducible subvariety. A related general remark is that Corollary 1.6 applies to an arbitrary subvarieties in the moduli space. If the subvariety lies in a connected component of the moduli space which contains a smooth rational curve, one might be able to prove the result using the Gromov compactness.

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