

SEQUENTIAL GAMES UNDER POSITIONAL UNCERTAINTY

CHRISTOPHER D. GIBSON

Submitted in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy
in the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2019

©2019
Christopher D. Gibson
All rights reserved

ABSTRACT

SEQUENTIAL GAMES UNDER POSITIONAL UNCERTAINTY

CHRISTOPHER D. GIBSON

This dissertation focuses on sequential games of imperfect information. I study settings in which not only do agents face imperfect information in the traditional sense of not possessing all payoff-relevant information, but they also face uncertainty about their position of movement in the sequence. I have utilized this framework to study financial investment decisions by individuals, production decisions by firms, and implications on information aggregation in observational learning.

In order to study production decisions by firms I utilize a Stackelberg oligopoly model with a stochastic consumer demand. In this setting firms do not know their position of movement, and as a result of the stochastic demand they cannot infer from the prevailing price if another firm has yet entered the market. I find that as a result of uncertainty firms produce a higher quantity than they otherwise would have, resulting in a more competitive outcome. In fact, as the number of firms in the market increases, with positional uncertainty the equilibrium quantity actually exceeds the perfectly competitive quantity.

I then investigate the impact of positional uncertainty when agents must choose levels of investment in a financial asset. Investors receive a signal about the value of the asset but are not necessarily aware of their position in the sequence of investors. As a result, they are unsure to what extent the signal they receive represents profit-relevant information, or if the signal is “stale” in the sense that the information has been incorporated into the price by other investors. This results in more cautious levels of investment, and an asset price that does not represent the true underlying value.

To study the behavioral aspects of financial investment, I introduce in this model a notion of confidence. While much work in the area of behavioral finance has studied the role of

confidence over the accuracy of information, my interest is in confidence over the timing of information. I define an agent as overconfident if they believe they are more likely to have received the signal earlier than other agents, and are thus more likely to be early investors. The effect of overconfidence can overwhelm the cautious nature of positionally uncertain investors, even potentially leading to an overreaction to information. This effect can explain overvaluation of assets and volatility of prices in response to information.

In a model of observational learning, limited information about the history of actions slows the integration of information. However, I show that in the limit, even in the presence of limited histories complete learning occurs. In the environment of limited access to historical information I introduce uncertainty over position of action. This uncertainty even further dampens the process of learning from a welfare standpoint, but as the number of agents grows large complete learning still obtains in the limit for all levels of uncertainty.

The common finding in all these settings is that uncertainty about the order of action causes agents to be cautious about exploiting profitable opportunities. In the case of oligopoly this leads to more competitive outcomes, whereas in the cases of investment and social learning uncertainty leads to less effective information aggregation.

CONTENTS

LIST OF FIGURES	v
ACKNOWLEDGEMENTS	vi
DEDICATION	vii
1 SEQUENTIAL QUANTITY SETTING UNDER POSITIONAL UNCERTAINTY	1
1.1 INTRODUCTION	2
1.2 RELATED LITERATURE	3
1.3 THE MODEL	4
1.3.1 THE CASE OF NO UNCERTAINTY	6
1.4 INTRODUCING UNCERTAINTY: A UNIFORM INTERCEPT	6
1.4.1 IDENTICAL PRIORS	8
1.4.2 A COMMON PRIOR	11
1.4.3 THE CASE OF N FIRMS	15
1.5 A GENERALIZATION: A NORMAL INTERCEPT	18
1.6 CONCLUDING REMARKS	23
1.7 REFERENCES	24
1.8 APPENDIX FOR CHAPTER 1: PROOFS	25
2 THE ROLE OF CONFIDENCE OVER TIMING OF INVESTMENT INFORMATION	29
2.1 INTRODUCTION	30

CONTENTS CONT'D

2.2	RELATED LITERATURE	31
2.3	THE MODEL	33
2.3.1	THE CASE OF NO UNCERTAINTY	34
2.4	INTRODUCING UNCERTAINTY	37
2.4.1	TWO AGENTS	38
2.4.2	T IDENTICAL AGENTS	41
2.5	A NOTION OF CONFIDENCE	44
2.5.1	CONFIDENCE: THE MINDFUL INVESTOR	45
2.5.2	CONFIDENCE: THE MYOPIC INVESTOR	47
2.6	CONCLUDING REMARKS	51
2.7	REFERENCES	54
2.8	APPENDIX FOR CHAPTER 2: PROOFS	56
3	SOCIAL LEARNING WITH LIMITED HISTORIES	74
3.1	INTRODUCTION	75
3.2	RELATED LITERATURE	76
3.3	MODEL	78
3.3.1	PRIVATE SIGNALS	78
3.3.2	OBSERVABLE HISTORIES	79
3.3.3	EQUILIBRIUM	80
3.3.4	SOCIAL LEARNING	81
3.4	THE LINEAR CASE	81
3.4.1	FULLY OBSERVABLE HISTORY	82
3.4.2	LIMITED HISTORIES	84

CONTENTS CONT'D

- 3.4.3 POSITIONAL UNCERTAINTY 90
- 3.5 THE GENERAL CASE 95
- 3.6 CONCLUDING REMARKS 96
- 3.7 REFERENCES 98
- 3.8 APPENDIX FOR CHAPTER 3: PROOFS 99

LIST OF FIGURES

1.1	PLOT OF K WITH IDENTICAL PRIORS, $B=1$, AND $C=1/2$	9
1.2	TOTAL QUANTITY WITH IDENTICAL PRIORS, $B=1$, AND $C=1/2$	9
1.3	PLOT OF K WITH COMMON/IDENTICAL PRIORS, $B=1$, AND $C=1/2$	13
1.4	TOTAL QUANTITY WITH COMMON PRIORS, $B=1$, AND $C=1/2$	13
1.5	TOTAL QUANTITY FOR VARIOUS MARKET STRUCTURES, $B=1$ AND $C=1/2$	17
1.6	OPTIMAL QUANTITY AND PRICE UNDER NORMAL DEMAND INTERCEPT . .	21
1.7	POSTERIOR PROBABILITY OF FIRST AS A FUNCTION OF OBSERVED PRICE .	21
1.8	EFFECT OF NORMAL VARIANCE ON OPTIMAL QUANTITY AND PRICE	22
1.9	POSTERIOR BELIEF OF MOVING FIRST AS A FUNCTION OF PRICE	22
2.1	FINAL PRICE p_2 AS A FUNCTION OF PRIOR γ_1	40
2.2	FINAL MARKET PRICE FOR CERTAIN AND UNCERTAIN POSITION	43
2.3	FINAL MARKET PRICE FOR NEUTRAL AND MINDFUL OVERCONFIDENCE . .	47
2.4	FINAL MARKET PRICE FOR NEUTRAL AND MINDFUL UNDERCONFIDENCE .	47
2.5	FINAL MARKET PRICE FOR CERTAIN AND MINDFUL OVERCONFIDENCE . .	48
2.6	FINAL MARKET PRICE FOR CERTAIN AND MINDFUL UNDERCONFIDENCE . .	48
2.7	FINAL MARKET PRICE FOR NEUTRAL AND MYOPIC OVERCONFIDENCE . . .	49
2.8	FINAL MARKET PRICE FOR NEUTRAL AND MYOPIC UNDERCONFIDENCE . .	49
2.9	FINAL MARKET PRICE FOR CERTAIN AND MYOPIC OVERCONFIDENCE . . .	50
2.10	FINAL MARKET PRICE FOR CERTAIN AND MYOPIC OVERCONFIDENCE . . .	50
2.11	FINAL MARKET PRICE FOR ALL FORMS OF OVERCONFIDENCE	50

LIST OF FIGURES CONT'D

2.12	FINAL MARKET PRICE FOR ALL FORMS OF OVERCONFIDENCE	50
3.1	FINAL MARKET PRICE FOR CERTAIN AND UNCERTAIN POSITION	81
3.2	DECISION THRESHOLDS UNDER LIMITED HISTORIES OF OBSERVATION	87
3.3	MAXIMUM THRESHOLD IN STATE H	88
3.4	EXPECTED THRESHOLD IN STATE H	88
3.5	EXPECTED UTILITY IN FOR FULL AND LIMITED HISTORIES OF OBSERVATION	89
3.6	EXPECTED INFORMATION LOSS FROM LIMITED OBSERVATION	90
3.7	THRESHOLDS UNDER POSIONAL UNCERTAINTY FOR VARIOUS N	92
3.8	THRESHOLDS UNDER POSIONAL UNCERTAINTY FOR LARGE N AND VARIOUS γ	94
3.9	RELATIONSHIP BETWEEN EXPECTED UTILITY AND GAMMA IN STATE H	95

ACKNOWLEDGEMENTS

I would like to thank my primary advisors Pietro Ortoleva - who was extremely supportive from the start of my research career - and Qingmin Liu - whose support was invaluable in helping me complete my dissertation. I would also like to thank the other members of my committee - Mark Dean, Paolo Siconolfi, and Michael Woodford. I am a better researcher because of these incredible minds.

DEDICATION

To ZMB, without whom this may not have been possible.

CHAPTER 1

SEQUENTIAL QUANTITY SETTING UNDER POSITIONAL UNCERTAINTY

Abstract: In a Stackelberg oligopoly setting two firms set quantity without knowing whether they are the first or second in the market. I find that with a common prior positional uncertainty always leads to a more competitive level of quantity. This finding is exacerbated when firms do not share a common prior and the sum of their prior beliefs of moving first exceeds unity. Even in the presence of a common prior and many identical firms as the number of firms increases the equilibrium quantity in the presence of positional uncertainty can exceed that of perfect competition.

1.1 INTRODUCTION

Sequential models of firms deciding on whether to enter a market and the quantity to produce are as natural as the idea of competition itself. Under the assumption of free entry, firms look at the prevailing price and incumbent firms and enter if there are profitable opportunities. The sequential model has been extensively used to study the behavior of oligopolies, sequential quantity setting à la Stackelberg serving as the workhorse in this area. The standard result is that the leading firm anticipates the reaction of the following firm, enabling it to suppress downstream quantity and produce more than if they moved simultaneously.

The Stackelberg leader has a first-mover advantage because it can commit to a quantity before another firm enters the market. But of course this advantage depends on the leader *knowing* they are the leader. Likewise, the quantity decision of the following firm depends on their awareness that they are the follower. In practice this assumption may not withstand scrutiny, either in the case of duopoly or an arbitrary oligopoly setting. Since minimal effort would be required to determine whether there is an incumbent firm, the scrutiny would not target whether a firm knows if it is a follower. Rather, a firm may not know if it is a leader. That is, a firm deciding on quantity in a certain period may be unsure if a new entrant will subsequently infuse the market with supply, thereby introducing uncertainty to the profit-maximizing decision of the initial firm.

Notice that the strategic element of the Stackelberg model of oligopolistic competition begins and ends with the leading firm. The following firm merely takes the residual demand and sets quantity q to maximize profit subject to $p(q) = a - bq_1 - bq$. As far as the follower is concerned, they behave as a monopoly facing linear demand with intercept $a' = a - bq_1$. However, if the downstream firm believes there may be yet another follower, the problem becomes game theoretic with the follower responding to linear demand with intercept $a'' =$

$a' - bq_2$ after downstream firm 2 sets quantity q_2 . If there is any possibility of another firm entering a market, each firm essentially plays a Stackelberg competition game as a mix of a leader and a follower.

We will use a basic linear demand to model a Stackelberg competition setting. Two firms will be unsure of their position as leader or follower, but will face a prevailing market price p which is either the demand intercept $p = a$ (if they are the Stackelberg leader), or the residual price after the leader $p = a - bq_1$ (if they are the follower). In order that position cannot be perfectly inferred from the prevailing price, demand intercept a will be stochastic. The improper uniform distribution $a \sim U[0, \infty]$ will be the focus of analysis but the results apply to other distributions as well.

1.2 RELATED LITERATURE

To our knowledge no work has yet undertaken the study of sequential quantity setting in oligopoly markets with uncertainty over position. However, there has been much work on uncertainty in oligopoly markets with quantity setting firms, mostly focused on uncertainty over demand.

Gal-Or (1985) presents a model of linear demand with a normally distributed intercept, about which each firm receives a private noisy signal [4]. She shows that firms choosing quantity simultaneously after receiving informative signals have no incentive to share their private information about the demand intercept with other firms. Vives (1984) examines the case of heterogenous goods, confirming the result of Gal-Or if goods are complements but shows that information sharing is a dominant strategy if they are substitutes [6].

Other studies consider sequential quantity setting with stochastic demand. De Wolf and Smeers investigate a two period setting in which a Stackelberg leader chooses quantity without knowing the demand intercept, and a group of firms choose quantity simultaneously

in the second period after demand has resolved [1]. DeMiguel and Xu generalize this to multiple Stackelberg leaders choosing quantity simultaneously in the first period, and they identify conditions under which a unique equilibrium exists [2].

Ferreira and Ferreira (2009) study a two-period stochastic demand environment in which firms have a choice of which period to move. They identify conditions on the resolution of uncertainty in which a sequential decision is preferred to a simultaneous decision. If uncertainty is high and it is resolved in the second period, the first-mover advantage reverses, favoring the following firm that faces no uncertainty.

1.3 THE MODEL

We consider a multi-period market in which informed market participants (firms) receive signals and trade according to the information they infer from these signals. Each firm sets quantity in a market with linear demand $p(q) = p_0 - b \cdot q$, with p_0 determined stochastically from some distribution F over $[0, \infty]$ so that $\Pr(p_0 \leq p) = F(p)$ for all $p \in [0, \infty]$. Then given cost of production $c(q)$ and the order in which they move, firms set quantity to maximize profit. We will assume that cost of production takes the form $c(q) = c \cdot q^2$.

Due to the stochastic nature of demand, however, the order in which firms set quantity is unknown. When deciding on the quantity they wish to produce, firms only see the prevailing market price. This price could be the result of the stochastic draw p_0 (in the case that the firm moves first), or could be the residual price after quantity is set by another firm (in the case that the firm moves second).

While the stochastic demand intercept makes it impossible for either firm to perfectly infer their order of play, each has a prior belief $\Pr(First) = \mu$ that they are the first mover. Upon seeing the price p , firms use Bayesian updating to infer their posterior probability $\gamma(p) = \Pr(First|p)$ of being the first mover. In order to calculate this posterior then, they

must weight the probability that they are seeing the price $p = p_0$ as the first mover, or if they are seeing the residual price $p = p_0 - b \cdot q_1$ as the second mover.

In order to capture the role position plays in the strategic interaction it is useful to focus on the timing of the game.

t=0: The leading firm observes price $p_0 \sim F$ and decides on q_1 .

t=1: (i) The leading firm collects profits $q_1(p_0 - bq_1) - c \cdot q_1^2$.

(ii) The following firm observes price $p_1 = p_0 - bq_1$ and chooses q_2 .

t=2: Firm i collects profit $q_i(p_0 - b(q_1 + q_2)) - c \cdot q_i^2$ for $i = 1, 2$.

We will typically be looking at games of two firms, a leader and a follower, but our sequential quantity game in general takes the following form.

Definition 1.1. Let $\Lambda_N(F) = (N, F, \mu_i)_{i=1}^N$ denote an N -firm sequential entry oligopoly where firms face linear demand $p_0 - bQ$, $p_0 \in [0, \infty)$ given by distribution $F(\cdot)$, and firm i has prior belief μ_i^j of entering the market in position j .

In the case of $N = 2$, if $q^*(p - b \cdot q)$ is the best response for the follower to quantity q , expected profit is

$$\begin{aligned} \pi(p, q) &= \gamma(p) \{q(p - b \cdot q) - c \cdot q^2 + [q(p - b(q + q^*(p - b \cdot q))) - c \cdot q^2]\} \\ &\quad + (1 - \gamma(p))[q(p - b \cdot q) - c \cdot q^2] \end{aligned}$$

The posterior $\gamma(p)$ of setting quantity first can equivalently be viewed as the probability that another period will occur and the following firm will best respond to this quantity setting.

With this interpretation, the profit reduces to the intuitive form

$$\begin{aligned} \pi(p, q) &= q(p - b \cdot q) - c \cdot q^2 + \gamma(p) \cdot q[p - b(q + q^*(p - b \cdot q)) - c \cdot q^2] \\ &= (1 + \gamma(p))(q(p - b \cdot q) - c \cdot q^2) - \gamma(p) \cdot bq q^*(p - b \cdot q) \end{aligned}$$

Given the objective function each firm seeks to maximize, the standard notion of equilibrium follows naturally.

Definition 1.2. *An equilibrium of the game $\Lambda_2(F)$ is a function $q_i^*(\cdot), i = 1, 2$ such that for all p , $q_i^*(p)$ solves*

$$q_i = \arg \max_q \quad q(p - b \cdot q) - c \cdot q^2 + \gamma(p) \cdot q[p - b(q + q^*(p - b \cdot q)) - c \cdot q^2]$$

1.3.1 THE CASE OF NO UNCERTAINTY

To fix ideas we can look no further than the extreme cases where there is no uncertainty ($\gamma(p) = 0$ or $\gamma(p) = 1$). This is the reduced game form $\Lambda_2(\delta_p(p_0))$, where $\delta_p(p_0)$ is the Dirac measure that has mass only on $p = p_0$. In this extreme, each firm knows which is the quantity leader and which is the follower, so that the market reduces to the familiar Stackelberg oligopoly setting. Through backward induction, the first mover solves

$$\max_{q_1} \quad q(p - b \cdot q) - c \cdot q^2 + q[p - b(q + q^*(p - b \cdot q)) - c \cdot q^2]$$

where $q_2(p)$ maximizes $q_2((p_0 - b \cdot q_1) - q_2(b + c))$. Solving this system of equations the equilibrium in this case is

$$q_1 = p_0 \left(\frac{3b + 4c}{2(3b^2 + 8bc + 4c^2)} \right) \quad \text{and} \quad q_2 = p_0 \left(\frac{3b^2 + 12bc + 8c^2}{4(b + c)(3b^2 + 8bc + 4c^2)} \right)$$

1.4 INTRODUCING UNCERTAINTY: A UNIFORM INTERCEPT

Now return to the case of an uncertain linear demand, and therefore an uncertain order of quantity setting. The linear demand intercept p_0 is distributed over $[0, \infty)$ according to the distribution F . Now, however, suppose that every demand intercept $p_0 \in [0, \infty]$ is equally

likely, so that F is the improper uniform distribution.

Apart from the technical tractability the improper uniform distribution offers, we will see that there are compelling reasons to analyze this case. Not least of these reasons is that without additional information about the linear demand, each firm has no reason to believe any initial price p_0 to be more likely than any other. Moreover, while this particular distribution over p_0 sacrifices some generality, we will see later that the loss of generality is actually minimal. Not only is the uniform distribution the limiting case of many distributions for p_0 that may be of interest, but the cases short of the limits are locally well approximated by the normal distribution with little error as is demonstrated in section 4.

In order to determine the equilibrium in the uniform case it is necessary to first characterize the posterior $\gamma(p)$ of being the first mover under this distribution.

Lemma 1.1. *In the Stackelberg game $\Lambda_2(U[0, \infty))$, let $\gamma(p) = \Pr(\text{First}|p)$ be the posterior probability of being first upon observing price p and $\Pr(\text{First}) = \mu$ the prior probability. Then in a pure strategy equilibrium, $\gamma(p) \in \{0, \mu, 1\}$.*

This lemma shows that if any point is equally likely to be the initial price p_0 and if the observed p is a possible residual price from some initial p_0 , then it is equally likely that price p is observed by a first mover or a second mover, so the posterior collapses to the prior μ . The only cases in which the posterior will not be the prior is if either p cannot be the residual price from any initial p_0 , so price p is a “hole,” or if p is a “mass point” and as such is the residual price of a non-zero mass of initial p_0 .

In this case that p is a hole and there is no possible initial price such that $p_0 - b \cdot q^*(p_0) = p$, then the probability of being a follower is zero so a firm observing the price p knows they must be the first mover and $\gamma(p) = 1$. In the case that p is a mass point, there are uncountably infinitely many initial prices p_0 such that $p_0 - b \cdot q^*(p_0) = p$, and only one possible way for p to be the initial price. As such this mass overwhelms the probability of being the first mover, so that the posterior is $\gamma(p) = 0$.

These equilibrium pathologies involving holes or mass points in the support of $p - b \cdot q^*(p)$ both complicate equilibrium analysis and detract from its interest in describing reasonable market behavior. As such we will focus on an equilibrium devoid of such cases in order to focus on firms' reactions to market variables instead of abstract equilibrium considerations. In doing so we will highlight the interactions that result from a multi-period market with positional uncertainty, and how firms respond to beliefs of their own position as well as their beliefs over other firms' beliefs.

Given the absence of holes or mass points in the distribution, the posterior $\gamma(p) = \mu$ for all values of $p > 0$. This leads to a natural equilibrium result.

Proposition 1.1. *In the Stackelberg game $\Lambda_2(U[0, \infty))$ such that p_0 is distributed according to the improper uniform distribution, a linear equilibrium exists.*

Proposition 1 states that for a given firm i there is a constant k_i such that $q_i^*(p) = k_i p$. But to determine exactly what form this constant takes more needs to be said of the beliefs of each firm. In particular, we know each firm has some prior μ , but we have not yet considered how these priors may relate to one another. If both firms have identical structures and information it may be natural to assume they have identical priors as well. If firms' structures are not identical but their information over this asymmetry is shared, then they may not have identical priors but instead a common prior in that the sum of prior beliefs still sum to one. In fact the equilibrium composition and comparative statics will differ depending on how firms form these beliefs, differences that will be revealed in turn.

1.4.1 IDENTICAL PRIORS

In the case of identical priors if we denote $\Pr_i(\text{First}) = \mu_i$, then $\mu_i = \mu_j = \mu$ and the linear equilibrium takes simple form.

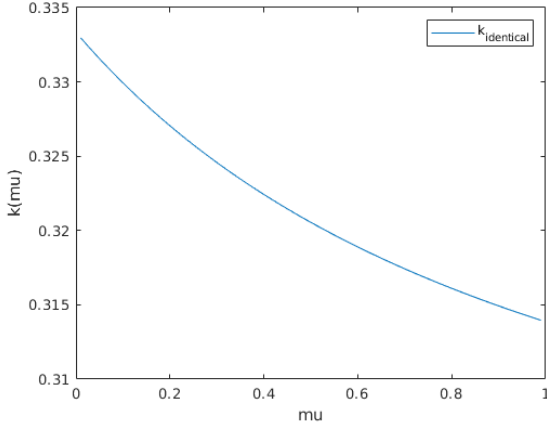


Figure 1.1: Plot of k with identical priors, $b=1$, and $c=1/2$

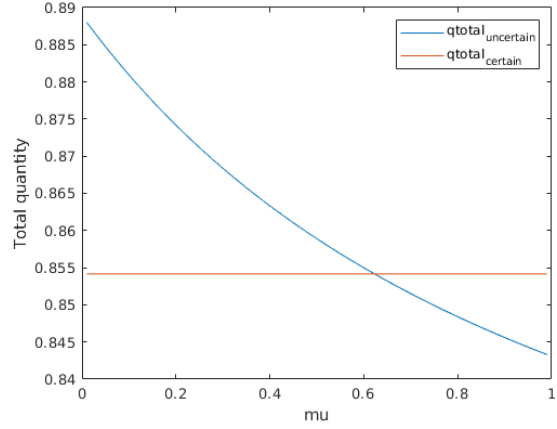


Figure 1.2: Total quantity with identical priors, $b=1$, and $c=1/2$

Proposition 1.2. *If priors are identical so that $\mu_i = \mu_j = \mu$, then a linear equilibrium of $\Lambda_2(U[0, \infty))$ exists and takes the form $q^*(p) = k \cdot p$ for all $p > 0$ with*

$$k = \frac{b(2 + 3\mu) + 2c(1 + \mu) - \sqrt{(b(\mu + 2) + 2c(\mu + 1))^2 - 8bc\mu(\mu + 1)}}{4b^2\mu}$$

It can be shown that for all values of parameters b and c , equilibrium parameter k is decreasing in prior belief μ . This is to be expected, as the prior μ is also the posterior $\gamma(p)$ of being the first mover. For a given price p , as the probability of being first increases, by definition the probability of a subsequent firm setting quantity in the market increases. Just as in the Stackelberg case the leading firm must reduce the production quantity in anticipation of the following firm's quantity reducing price even further, so too does the increase in the probability μ of a follower add weight to the trade-off between maximizing profit under current demand versus final demand if another firm were to enter.

Considering the parameters $b = 1$ and $c = 1/2$, the linear equilibrium constant takes the simplified form $k = \frac{3+4\mu-\sqrt{8\mu^2+16\mu+9}}{4\mu}$. Total quantity for initial price p_0 is then $kp_0 + k(p_0 - kp_0) = p_0(2k - k^2)$.

The plot on the left shows the relationship between prior μ and the linear parameter k . As described above k is a decreasing function of μ . The figure on the right plots total quantity as a proportion of initial price (q/p_0). As total quantity is a decreasing function of k it is also to be expected that it too would be a decreasing function of prior belief μ for the exact same reason. In fact, as the belief of each firm that they move first increases and each becomes more certain that their production will be followed with an additional infusion of quantity from the following firm, total quantity in the case of uncertainty actually drops below the Stackelberg equilibrium with no uncertainty.

Recalling from above that in the Stackelberg equilibrium $q_1 = p_0 \cdot \frac{b+2c}{6b^2+8bc+4c^2}$ and $q_2 = p_0 \cdot \frac{(3b^2+6bc+4c^2)}{4(b+c)(3b^2+4bc+2c^2)}$, with the parameters $b = 1$ and $c = 1/2$ total quantity $q_1+q_2 = 0.5238 \cdot p_0$. This total quantity is shown by the horizontal line in the right graph. As can be seen from this comparison, for low values of μ the total quantity in the uncertain case is higher than in the case of certainty but for high values of μ this relationship reverses. In fact this pattern holds for all values of b and c .

While we have no cause to question this pattern at the moment, we will see later there is indeed plenty of reason to expect that the introduction of uncertainty will lead to a strictly higher quantity. The Stackelberg case is that in which the order of quantity setting is commonly known: one agent has prior $\mu = 1$ of moving first and the other has prior $\mu = 0$. But if the quantity leader had even a little uncertainty of their position, the firm would have an incentive to increase quantity as the trade-off between maximizing current and final demand has shifted toward current. If at the same time the quantity follower had an equal amount of uncertainty in their position but in the reverse direction, this firm would have an incentive to decrease quantity as their quantity setting trade-off has shifted toward maximizing final demand. If the leading firm were able to anticipate the quantity reduction of the follower, the leading firm would be incentivized to even further increase quantity. And so forth.

While the end result of this iterative loop of backward induction is unclear in terms of how total quantity is effected, it is at least clear that being able to anticipate the rival firm's response to a new prior will mitigate the declining total quantity as μ increases. In the case of identical priors this anticipation fails because firms have – and expect the other firm to have – the exact same prior. Thus when their own prior changes each firm expects the prior of the other to change in exactly the same way. Barring the case where $\mu = 1/2$, the identical prior assumption comes with it the untenable shared belief that the total probability of moving first could exceed or fall short of unity, and moreover that firms are aware that they share this belief.

1.4.2 A COMMON PRIOR

The case of identical priors provided a simple solution characterizing the linear equilibrium that allowed for the analysis of firm behavior in the presence of uncertainty and how behavior changes with beliefs about their position of quantity setting. But valuable as a foothold into the problem at hand, the assumption that firms have exactly the same belief of moving first and are aware of this shared contradiction of probability theory seems an unlikely reality in which otherwise rational firms might operate.

In light of this incongruity a more fitting environment might be one in which firms correctly anticipate the rival firm's prior belief in relation to their own. The assumption that gives us this belief congruity is the common prior assumption. Defined in the usual way the common prior imposes the following structure on how the prior belief of each firm relate to one another.

Definition 1.3. *Firms i and j share a **common** prior if $\mu_i + \mu_j = 1$.*

The common prior assumption is useful because not only do priors beliefs accord with probability theory under this structure but also it introduces a consistency of beliefs that

would be expected of rational profit-maximizing firms. From a technical standpoint a market in which firms may have different beliefs μ introduces a layer of complication to firms' interaction, but we can still find a linear equilibrium, one with notably more desirable and realistic properties.

Proposition 1.3. *If firms have a common prior so that $\mu_i + \mu_j = 1$, then a linear equilibrium of $\Lambda_2(U[0, \infty))$ exists and takes the form $q^*(p) = k(\mu) \cdot p$ for all $p > 0$ with a linear parameter $k(\mu)$ of the form*

$$k(\mu) = \frac{b^2(3\mu^2 + \mu - 10) + 8bc(\mu + 1)(\mu - 2) + 4c^2(\mu + 1)(\mu - 2) + \sqrt{A(b, c, \mu)}}{4b^2(1 - \mu)(b(2 + \mu) + 2c(1 + \mu))}$$

where

$$\begin{aligned} A(b, c, \mu) = & b^4(\mu^2 - \mu - 6)^2 + 16b^3c(\mu^4 - 2\mu^3 - 7\mu^2 + 8\mu + 12) \\ & + 8b^2c^2(7\mu^4 - 14\mu^3 - 29\mu^2 + 36\mu + 44) + 16c^3(4b + c)(\mu^2 - \mu - 2)^2 \end{aligned}$$

In order to highlight the differences between the equilibrium under a common prior and that under an identical prior a sketch of the proof is useful. Each firm i solves for an equilibrium under the assumption that the other firm plays a linear strategy. However now the linear parameter depends on prior μ_i , as the prior is no longer the same for both firms. Firm i solves $\max_{k_i(\mu_i)} p k_i(\mu_i) [p - p k_i(\mu_i) p(b + c)] - b p k_i(\mu_i) \mu_i k_j(1 - \mu_i)$. This yields for each firm a first order condition for $k_i(\mu_i)$ and an inferred condition for $k_j(1 - \mu_i)$, from which the constants $k_i(\mu_i)$ and $k_j(\mu_j)$ can be solved.

As in the previous case – and for the same reason – it can be shown that k is decreasing in the prior belief μ . As the probability of being first increases, the trade-off between maximizing current and final demand shifts toward final and quantity is decreased. However

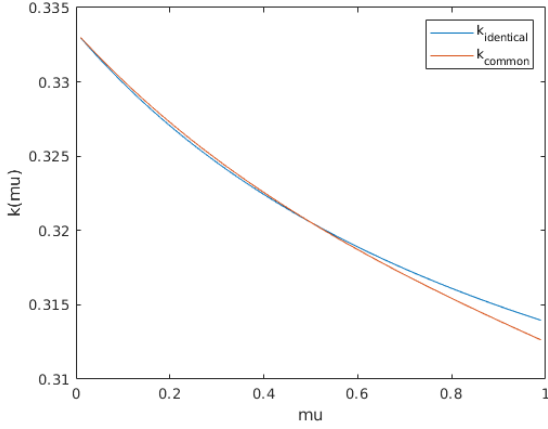


Figure 1.3: Plot of k with common/identical priors, $b=1$, and $c=1/2$

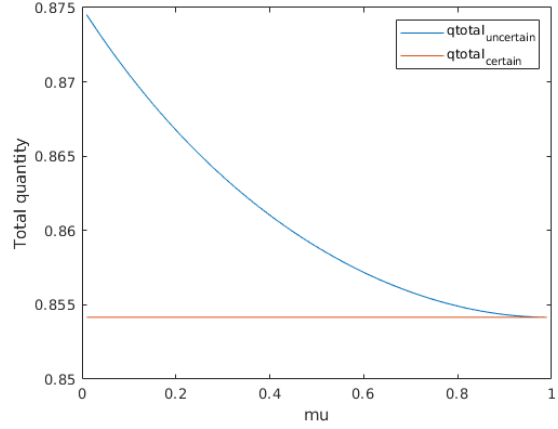


Figure 1.4: Total quantity with common priors, $b=1$, and $c=1/2$

unlike in the previous case, this decrease in quantity is amplified by the common prior realization that at the same time the prior belief of the other firm decreases, leading to an increase in quantity in the case of a following quantity setter.

For the parameters $b = 1$ and $c = 1/2$ the linear equilibrium parameter simplifies to $k(\mu) = \frac{20+4\mu-8\mu^2-\sqrt{32\mu^4-64\mu^3-152\mu^2+184\mu+256}}{4(2\mu+3)(1-\mu)}$. If initial price is p_0 and μ_1 is the prior belief of the leading firm, total quantity is $k(\mu_1)p_0 + k(1 - \mu_1)(p_0 - bk(\mu_1)p_0) = p_0[k(\mu_1) + k(1 - \mu_1) - bk(\mu_1)k(1 - \mu_1)]$

The plot on the left shows the inverse relationship of linear parameter k with μ . Moreover, this graph highlights the different behavior of the parameter k in the case of a common prior as compared to an identical prior. The two values of k meet at $\mu = 0$ and $\mu = 1/2$. when the prior is zero, in both cases the firm acts as a stand alone entity given the price, maximizing profit by equating marginal revenue and marginal cost, ruling out the possibility of a following quantity setter. When the prior $\mu = 1/2$ the priors are both identical and common so the cases overlap.

An increase in the prior will lead to a decrease in quantity as the firm becomes more confident that there is a follower. But now when $\mu \in (0, 1/2)$ the parameter k is higher than

in the case of an identical prior. In this region, while k still decreases with μ , this decrease is mitigated by the awareness given by the common prior assumption that the rival firm's prior is in the higher region ($1 - \mu > 1/2$), so that while the rival's declining belief of having a follower will lead to a quantity increase, this increase will be much lower than if their prior were less than $1/2$. As a result, under a common prior a firm with $\mu \in (0, 1/2)$ can afford less of a decrease in quantity in response to an increase in μ than if their rival shared the same prior $\mu \in (0, 1/2)$.

For $\mu > 1/2$ this logic reverses, and now any increase in prior μ is met with an equal and yet more formidable change in behavior from the rival firm. Equal in the sense that the rival's prior will decrease by the same magnitude with which μ increases, but more formidable in that the rival is in the more quantity-responsive region where $1 - \mu \in (0, 1/2)$. Thus increases to the prior $\mu > 1/2$ are exacerbated by the common prior assumption as compared to the case of identical priors.

The figure on the right shows the total quantity summed across the two periods ($2q_1 + q_2$) in proportion to initial price (q/p_0) in the case of a common prior compared with the Stackelberg case of no uncertainty. Unsurprisingly we see that these cases intersect when $\mu = 1$, when order is known. More interestingly, it is clear that in the case of uncertainty with a common prior, the total quantity in the market lies above the certain case for all values of μ .

As described, this is due to the inverse relationship between prior μ and linear parameter k , and the joint awareness of how this is influenced by the common prior. For prior $\mu < 1/2$, an increase in μ causes a decrease in k and the resultant linear quantity. But if at the same time the rival firm sees an decrease in its own μ , this and a higher residual price left by the leading firm causes an increase in quantity. These two factors, coupled with the more precipitous slope of the rival's parameter k in the high μ region more than compensate for the initial quantity drop, leading to a total quantity increase. The case of a decrease of

$\mu > 1/2$ is symmetric from the other firm's perspective, leading to a peak quantity at the neutral prior $\mu = 1/2$.

The common prior case characterizes a richer environment in which firms interact in the case of positional uncertainty. The mutual awareness that the change in a firm's own prior must be met with an equal change of the rival's prior restores a consistency to this interaction and a strengthening of the explanatory power of the model. The common prior solves the violations of probability theory suffered by the previously identical prior μ , and highlights a key result. **Introducing uncertainty in the position of quantity setting leads to a higher total level of production in equilibrium.**

This result is intuitive, as a movement away from certainty introduces to the following firm possibility being the leader, and to the leading firm the possibility of not having a follower. As we was the decrease in quantity of the former is more than compensated for by the increase in quantity from the latter, leading to a net increase in total quantity over the certain case.

For the remainder of the analysis we will focus on the case of the common prior. In many instances this will not matter as the cases intersect with a neutral prior, a natural assumption for otherwise identical firms. This assumption becomes even more important in the case of $n > 2$ identical firms, where both intuition and tractability call upon the neutral prior.

1.4.3 THE CASE OF N FIRMS

In a market of $N > 2$ firms, while tractability concerns impede an explicit solution for the linear equation parameter k the logic is very much the same. In such an environment we will assume the firms are identical and as such the trivial prior of $\mu = 1/N$ is assumed. To

illustrate, if there are three firms then profit for any given firm (say firm 1 without loss) is

$$\frac{1}{3} (q_1(p - bq_1) - cq_1^2) + \frac{1}{3} (q_1(p - b(q_1 + q_2)) - cq_1^2) + \frac{1}{3} (q_1(p - b(q_1 + q_2 + q_3)) - cq_1^2)$$

This can be simplified to $q_1(p - q_1(b + c)) - \frac{1}{3}bq_1(2q_2 + q_3)$. The technical complication arises from the observation that in a linear equilibrium, $q_3 = k(p - b(pk + k(p - bpk)))$, and iteratively when the market reaches N firms the parameter k must be solved from a polynomial equation of order N . While an explicit solution is no longer guaranteed the general case can still be solved implicitly for any N .

Proposition 1.4. *For the game $\Lambda_N(U[0, \infty))$ with $N \geq 2$ firms and a shared uniform prior $\mu_j^i = 1/N$ that firm i chooses quantity after $j - 1$ predecessors, a linear equilibrium $q = kp$ is defined implicitly by*

$$(1 - 2bk)(1 - (1 - bk)^N) = 2bcNk^2$$

We saw in the case of two firms that the introduction of position uncertainty resulted in a higher total quantity than if positions are certain. In fact, this is a result that generalizes to the case of N firms. Moreover since sequential quantity setting always results in a higher level of production than Cournot oligopoly, sequential quantity setting with positional uncertainty is too bounded below by Cournot.

The figure below shows the total output in the case of Cournot oligopoly, sequential quantity with positional uncertainty, and the perfectly competitive outcome (where firms make zero profit). As expected the case of sequential quantity setting with uncertainty lies above the simultaneous quantity setting of Cournot. But the surprising result is the relationship with the perfectly competitive quantity. It is known that Cournot converges to the perfectly competitive case as the number of firms goes to infinity, but it is striking how quickly the uncertain case converges. In fact, with the parameters $b = 1$ and $c = 1/2$, when the number of firms is more than 14 the quantity actually surpasses the perfectly competitive

outcome.

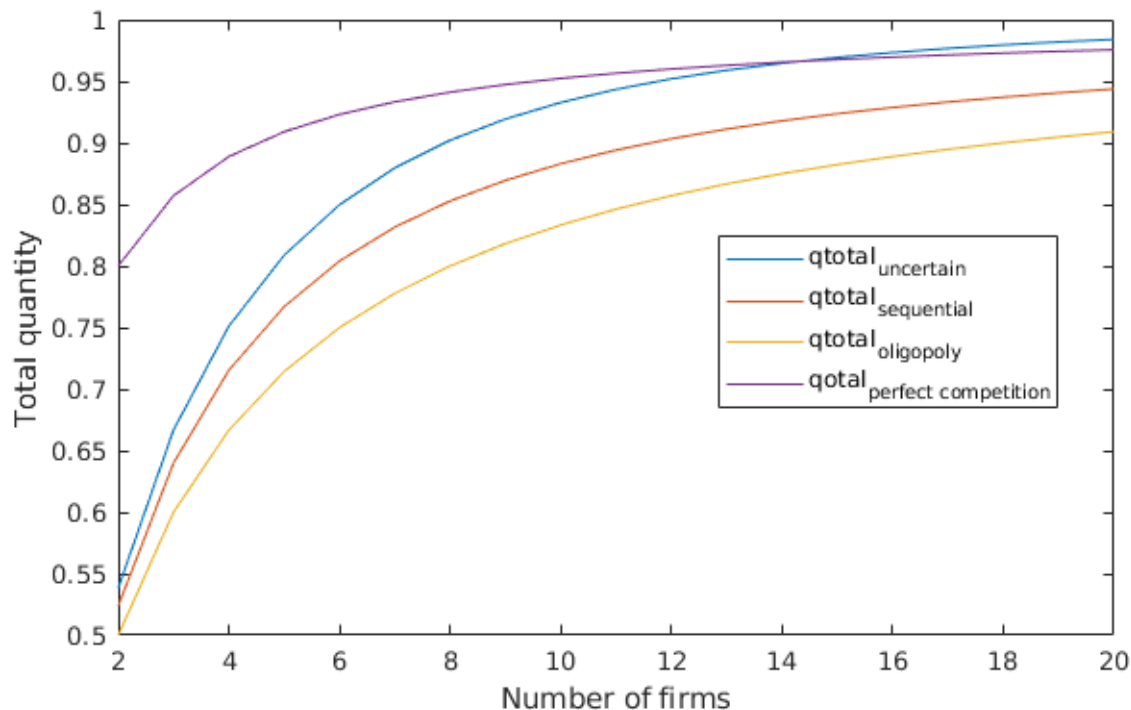


Figure 1.5: Total quantity for various market structures, $b=1$ and $c=1/2$

That the sequential quantity case with uncertainty exceeds the perfectly competitive outcome gives pause as it must imply that some firms make negative profit. While this is true, it is not true that all firms make losses, nor is it true that any firm expects losses ex-ante. It is only the leading firms who make losses, as they produce the most given the initial price p_0 , and as such are shouldered with consequences of this low probability event of being an early quantity setter.

Given the uncertainty in this environment, it is understandable that in the case of low probability events, an a priori optimal strategy yields losses a posteriori. The possibility of losses highlights another departure from certainty in quantity setting. Not only does the case of sequential quantity setting with positional uncertainty converge quickly (even surpassing) the perfectly competitive outcome, but this convergence comes at the expense of leading

firms' profits. While for a low number of firms there is a leading advantage in terms of profits, when the number of profits grows this advantage switches as the gains from being a quantity leader are outweighed by proximity to the final price. A leading firm can inject more quantity into the market than can a following firm, but if the price drops too much (if too many firms follow), the leader's high quantity turns out to be too high.

1.5 A GENERALIZATION: A NORMAL INTERCEPT

Previously the intercept p_0 that determined the linear demand curve was distributed over $[0, \infty)$ according to the improper uniform distribution so that any initial p_0 was equally likely. As a generalization, suppose now that demand intercept p_0 is distributed over $[0, \infty)$ according to the truncated normal distribution $N(\mu, \sigma^2)$. For ease of exposition suppose that $\mu = 0$ so that the truncated distribution is given by the density function $f(x, \sigma^2) = \sqrt{\frac{2}{\pi\sigma^2}} \exp\{-\frac{x^2}{2\sigma^2}\}$ with mean $\sqrt{\frac{2\sigma^2}{\pi}}$.

Notice that finding the equilibrium $q^*(p_0)$ which maps p_0 into residual price p according to $p = p_0 - bq^*(p_0)$ is equivalent to finding the inverse mapping of p to initial price p_0 according to $p_0^*(p) = p + bq^*(p_0^*(p))$. This latter mapping is solved as a fixed point problem but a unique solution will exist as long the mapping $p_0 \mapsto p$ is injective. Then upon seeing price p , the initial price, or demand intercept, if the firm is the first mover is p , while the initial price if second is $p_0^*(p)$. Then the probability of being first given the observed price p can be found through Bayesian updating as follows:

$$\gamma(p) = \frac{\Pr(F) \Pr(p|F)}{\Pr(F) \Pr(p|F) + \Pr(S) \Pr(p|S)} = \frac{\mu \exp\{-\frac{p^2}{2\sigma^2}\}}{\mu \exp\{-\frac{p^2}{2\sigma^2}\} + (1 - \mu) \exp\{-\frac{p_0^*(p)^2}{2\sigma^2}\}}$$

The assumption of a normal distribution is a generalization in the sense that as variance σ^2 increases, the distribution of p_0 increasingly resembles the improper uniform distribution,

converging to it in the limit. However, the updated probability $\gamma(p)$ of being the first hints at the complication the normal distribution introduces. This posterior can be reduced to $\mu/[\mu+(1-\mu)\exp\{-\frac{p_0^*(p)^2-p^2}{2\sigma^2}\}]$, which makes clear its dependency on the difference $p_0^*(p)^2-p^2$. However, as price increases the optimal quantity will increase, leading to an increase in the difference between initial and residual prices $p_0^*(p)-p$. This alone is not unique to the normal case, as we saw the same in the uniform case - constant in the case of a uniform intercept was not the difference between initial and residual prices but the ratio between them.

This problem becomes more complicated because now the rival firm's optimal response $q^*(p-bq)$ changes not just on the residual price $p-bq$, but also with the induced posterior $\gamma(p-bq)$. Moreover, since for any quantity q_1 the responding firm maximizes $q(p-bq_1)-q^2(b+c)-q\cdot b\gamma(p-bq_1)q_1$, varying q_1 will affect the responding firm's first order condition with respect to q both linearly and exponentially, so there is no closed form solution to the original first order condition with respect to q_1 . This points to a numeric solution.

The positional uncertainty and infinite state space of this problem join to present another complication: the problem is infinitely recursive. This is in itself is not new but the constant posterior of the uniform case allowed us to conjecture a linear equilibrium; the dynamic posterior here suggests a dynamic programming solution. However, the infinite recursiveness on both sides of the distribution - since for any price $p > 0$ it is always possible that there was a leader facing initial price $p_0^*(p)$ or will be a follower facing residual price $p-bq$ - leaves a dynamic programming problem with no initial point. Fortunately under the normal distribution we can find a "good enough" starting point in the following sense.

Lemma 1.2. *In the game $\Lambda_2(N)$ where demand intercept p_0 is distributed according to the truncated normal distribution on the interval $[0, \infty)$, then $\lim_{p \rightarrow 0} \gamma(p) = \mu$.*

The essence of this result is that since the posterior takes the form

$$\gamma(p) = \frac{\mu}{\mu + (1 - \mu) \exp \left\{ -\frac{p_0^*(p)^2 - p^2}{2\sigma^2} \right\}}$$

then if $p_0^*(p) \rightarrow 0$ as $p \rightarrow 0$ then $\gamma(p) \rightarrow \mu$. If this were not the case some $\varepsilon_2 > \varepsilon_1 > 0$ could be found such that $p < \varepsilon_1$ and $p_0 > \varepsilon_2$ for $p_0 - bq^*(p_0) = p$. But this induces a hole in the range $p \in (\varepsilon_1, \varepsilon_2)$ so that $\gamma(p) = 0$ which by assumption does not exist.

Lemma 1.3. *In the game $\Lambda_2(N)$ where demand intercept p_0 is distributed according to the truncated normal distribution on the interval $[0, \infty)$ and $q^*(\cdot)$ is an equilibrium, then $\lim_{p \rightarrow \infty} \gamma(p) = 1$.*

This result relies on the fact that

$$\gamma(p) = \frac{\mu \exp \left\{ \frac{p_0^*(p)^2 - p^2}{2\sigma^2} \right\}}{\mu \exp \left\{ \frac{p_0^*(p)^2 - p^2}{2\sigma^2} \right\} + (1 - \mu)}$$

and that as $p \rightarrow \infty$ the inducing $p_0^*(p)$ must increase in distance so $p_0^*(p)^2 - p^2$ diverges. If this were not true then

Then for a small enough initial price p_0 it can be assumed with little error that the a firm will behave as in the naïve case of a uniform prior, assuming the posterior of itself and any potential follower to be μ and setting the quantity $q(p) = k(\mu) \cdot p$ as above. From here, the best response $q^*(p)$ to any initial p can be determined recursively by choosing a suitably small starting point \underline{p} so $\gamma(\underline{p}) \approx \mu$ and iterating a finite number of steps.

Consider the case where each firm has prior $\Pr(F) = \mu$. Then for our canonical case of $b = 1$ and $c = 1/2$ let variance $\sigma^2 = 1$ to start and consider the lower bound for our numerical approximation of $\underline{p} = 1/100,000$.

For such a small lower bound \underline{p} we would expect that $\gamma(p) \approx \mu$ for p near \underline{p} , and that the optimal quantities for such prices would approximate $k(\mu) \cdot p$ as in the uniform case. Since

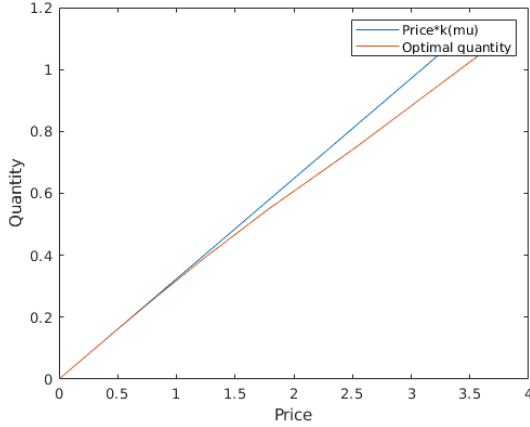


Figure 1.6: Optimal quantity and price under normal demand intercept

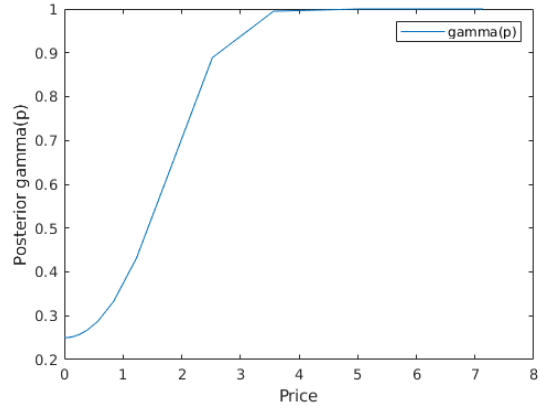


Figure 1.7: Posterior probability of first as a function of observed price

the lowest price player has posterior $\gamma(p) = \mu$ and assumes any follower will have the same, then by design this holds.

As figure 1.6 shows, the relationship between price and quantity is roughly linear but not quite. As described, the best response to a low price is approximately $p \cdot k(\mu)$, but as p increases the posterior moves away from $\gamma(p) \approx \mu$ and approaches $\gamma(p) = 1$.

However, the speed of this movement depends on the variance of the signal σ^2 . As the variance increases, high prices become less informative of position and the posterior does not update as much. This leads to the optimal q and $k^*(\mu) \cdot p$ coinciding for a larger number of prices. As figure 1.8 shows, while in the case of $\sigma^2 = 1$ the linear equilibrium with posterior $\gamma = \mu$ and the optimal quantity under the normal distribution diverged around $p = 1$, for $\sigma^2 = 100$ this difference is only perceptible near $p = 100$. This corresponds to the slowing of the posterior $\gamma(p)$ to update away from μ , as the following graph shows.

Beginning with a prior of $\mu = \frac{1}{4}$ the speed at which the posterior updates slows significantly. This is no surprise given that $\gamma(p) = \frac{\mu}{\mu + (1-\mu) \exp\left\{-\frac{p_0^*(p)^2 - p^2}{2\sigma^2}\right\}}$, so that $\lim_{\sigma^2 \rightarrow \infty} \gamma(p) = \mu$.

Given the stubbornness of $\gamma(p)$, the assumption of the previous section that price is distributed uniformly is even more appealing. A uniform price intercept is a good approximation

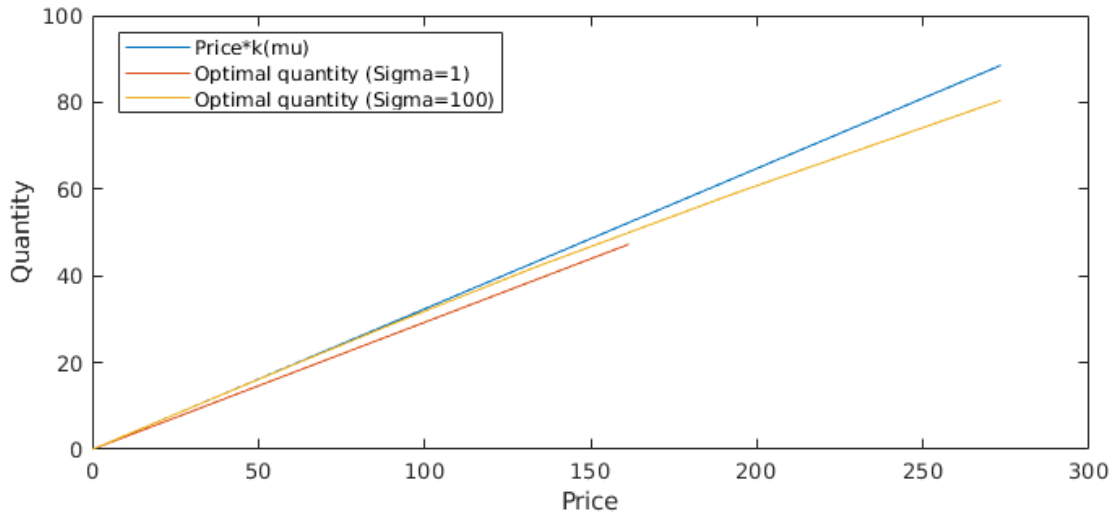


Figure 1.8: Effect of normal variance on optimal quantity and price

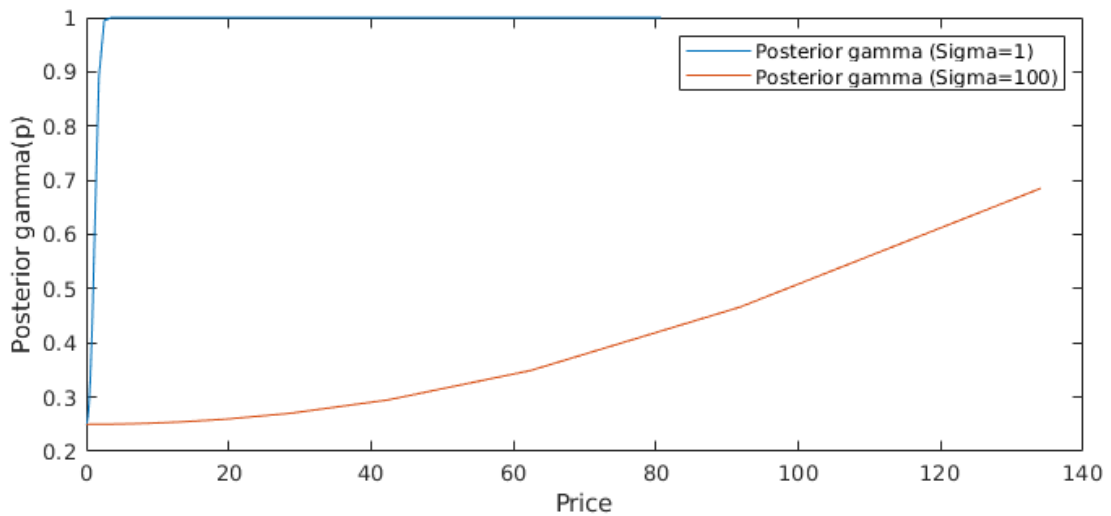


Figure 1.9: Posterior belief of moving first as a function of price

for small p and large variance σ^2 , offering credence to the improper uniform distribution as more than just a tractable choice.

1.6 CONCLUDING REMARKS

We have introduced a model of Stackelberg competition in which firms are unsure of their position as leader or follower. As a result, the probability of a competitor subsequently responding to quantity causes the downstream firm to reduce output, while the nonzero probability of being the follower causes the upstream firm to produce more than if position were perfectly known. Of these two opposing effects the incentive to increase production in response to the chance of being the downstream firm outweighs the incentive of the true follower to restrict quantity.

As a result of this interplay of incentives, uncertainty over position ultimately leads to a higher level of output in the market and a more competitive outcome for consumers. As the number of firms increases this difference widens, with the total quantity under positional uncertainty approaching – in some cases surpassing – quantity under perfect competition.

While the focus of this study was a model in which the stochastic demand intercept was uniformly distributed. The key feature of the uniform distribution that lends so much tractability is that since every price is equally likely, no price gives agents any more information about their position and the belief of being first remains as the prior. The results presented hold under other distributions, including the truncated normal. Moreover, as the variance of the intercept increases and the signal becomes less informative, the posterior belief remains close to the prior and the truncated normal case is well approximated by the uniform distribution.

BIBLIOGRAPHY

- [1] De Wolf, D. and Smeers, Y. (1997). “A stochastic version of a Stackelberg-Nash-Cournot equilibrium model.” *Management Science*, 43:2, pp. 190-197.
- [2] DeMiguel, V. and Xu, H. (2009). “A stochastic multiple-leader Stackelberg model: Analysis, computation, and application.” *Operations Research*, 57:5, pp. 1220-1235.
- [3] Ferreira, F. A. and Ferreira, F. (2009). “Quantity-setting competition under uncertain demand.” *Proceedings of the World Congress on Engineering*, 2009 Vol II, July 1-3, 2009.
- [4] Gal-Or, E. (1985). “Information sharing in oligopoly.” *Econometrica*, 53:2, pp. 329-343.
- [5] Raith, M. (1996). “A general model of information sharing in oligopoly.” *Journal of Economic Theory*, 71, pp. 260-288.
- [6] Vives, X. (1984). “Duopoly information equilibrium: Cournot and Bertrand.” *Journal of Economic Theory*, 34, pp. 71-94.

APPENDIX: PROOFS

Proof of Lemma 1. A firm observes p which induces a belief $\gamma(p)$. By Bayesian updating with prior $\Pr(\text{First}) = \mu$ and given $p_0 \sim U[0, a_0]$

$$\begin{aligned}\gamma(p) &= \frac{\Pr(p|F) \Pr(F)}{\Pr(p|F) \Pr(F) + \Pr(S) \Pr(p|S)} \\ &= \frac{\mu \Pr(p|F)}{\mu \Pr(p|F) + (1 - \mu) \Pr(p|S)} \\ &= \frac{\mu \lim_{\varepsilon \rightarrow 0} \int_{p-\varepsilon}^{p+\varepsilon} f(s) ds}{\mu \lim_{\varepsilon \rightarrow 0} \int_{p-\varepsilon}^{p+\varepsilon} f(s) ds + (1 - \mu) \lim_{\varepsilon \rightarrow 0} \int_{p-\varepsilon}^{p+\varepsilon} \int f(q = \frac{p_0-s}{b} | s) f(s) dp_0 \cdot ds}\end{aligned}$$

Where because initial price is distributed uniformly $f(s) = \frac{1}{a_0}$. The value of the posterior hinges on the integral $\int_{p-\varepsilon}^{p+\varepsilon} \int f(q = \frac{p_0-s}{b} | s) f(s) dp_0 \cdot ds$. There are three possibilities

- (i) There is no initial p_0 such that $q^*(p_0)$ satisfies $p = p_0 - b \cdot q^*(p_0)$ in which case we will say p is a hole in the support. If p is a hole then $f(q = \frac{p_0-p}{b} | p) = 0$ for all p_0 , so that $\lim_{\varepsilon \rightarrow 0} \int_{p-\varepsilon}^{p+\varepsilon} \int f(q = \frac{p_0-s}{b} | s) f(s) dp_0 \cdot ds = 0$. Thus if p is a hole in the distribution $\gamma(p) = 1$.
- (ii) p is a mass point so that there is a set P_0 with non-zero measure such that for all $p_0 \in P_0$, $q^*(p_0) = \frac{p_0-p}{b}$. Suppose the measure of $P_0 = \lambda > 0$. Also, since we are considering only full strategies, $f(q = \frac{p_0-p}{b} | p) = 1$ for all $p_0 \in P_0$. Then $\lim_{\varepsilon \rightarrow 0} \int_{p-\varepsilon}^{p+\varepsilon} \int f(q = \frac{p_0-s}{b} | s) f(s) dp_0 \cdot ds \geq \int_{P_0} f(s) ds = \lambda \cdot \frac{1}{a}$. And since the numerator of $\gamma(p)$ becomes arbitrarily small as $\varepsilon \rightarrow 0$, $\gamma(p) = 0$.
- (iii) If p is neither a mass point or a hole then for the set P_0 such that for all $p_0 \in P_0$, $q^*(p_0) = \frac{p_0-p}{b}$, then the measure of P_0 is zero since it can contain at most countably many points. Then $\int_{p-\varepsilon}^{p+\varepsilon} \int f(q = \frac{p_0-s}{b} | s) f(s) dp_0 \cdot ds = 2\varepsilon$ and $\gamma(p) = \mu$

Since this holds for all values of a_0 , it holds for the limiting case of the improper uniform

distribution as $a_0 \rightarrow \infty$. □

Proof of Proposition 2. Since we are looking for a pure strategy equilibrium with an infinite state space, according to Lemma 1 the posterior $\gamma(p) = \mu$. Moreover since firms have an identical prior $\Pr(F) = \mu$, each solves the problem

$$\max_q (1 + \mu)(q \cdot p - q^2(b + c)) - \mu q b q^*(p - b \cdot q)$$

Conjecture a linear equilibrium so $q^*(p) = k \cdot p$. Then the maximization problem becomes

$$\max_q (1 + \mu)(q \cdot p - q^2(b + c)) - \mu q b k (p - b \cdot q)$$

Solving for the optimal quantity and isolating the first order condition for q gives $q = p \left(\frac{1 + \mu(1 - bk)}{2b(1 + \mu(1 - bk)) + 2c(1 + \mu)} \right)$ so that $k = \left(\frac{1 + \mu(1 - bk)}{2b(1 + \mu(1 - bk)) + 2c(1 + \mu)} \right)$. Isolating k yields

$$k = \frac{b(2 + 3\mu) + 2c(1 + \mu) - \sqrt{(b(\mu + 2) + 2c(\mu + 1))^2 - 8bc\mu(\mu + 1)}}{4b^2\mu}$$

□

Proof of Proposition 3. As in the previous proposition firms have the posterior $\gamma_i(p) = \mu_i$, however now $\mu_i = 1 - \mu_j$, where the priors are not necessarily the same. Each firm solves

$$\max_q (1 + \mu)(q \cdot p - q^2(b + c)) - \mu_i q \cdot b k_j (p - b \cdot q)$$

Where k_j is the assumed constant of the rival firm as a function of their prior. As above this yields the equation

$$k_i = \frac{1 + \mu_i(1 - b k_j)}{2b(1 + \mu_i(1 - b k_j)) + 2c(1 + \mu_i)}$$

However, unlike the previous case k_j is not necessarily equal to k_i because they firms might have different priors. But due to the common prior assumption firm one solves for k_1 under the assumption that

$$k_j = \frac{1 + \mu_j(1 - bk_i)}{2b(1 + \mu_j(1 - bk_i)) + 2c(1 + \mu_j)}$$

Solving for $k(\mu)$ yields

$$k(\mu) = \frac{b^2(3\mu^2 + \mu - 10) + 8bc(\mu + 1)(\mu - 2) + 4c^2(\mu + 1)(\mu - 2) + \sqrt{A(b, c, \mu)}}{4b^2(1 - \mu)(b(2 + \mu) + 2c(1 + \mu))}$$

where

$$\begin{aligned} A(b, c, \mu) = & b^4(\mu^2 - \mu - 6)^2 + 16b^3c(\mu^4 - 2\mu^3 - 7\mu^2 + 8\mu + 12) \\ & + 8b^2c^2(7\mu^4 - 14\mu^3 - 29\mu^2 + 36\mu + 44) + 16c^3(4b + c)(\mu^2 - \mu - 2)^2 \end{aligned}$$

□

Proof of Proposition 4. Suppose all other firms play a linear strategy $q = k_1p$. Then firm i will choose quantity $q_i = kp_i$, where p_i is the residual price after $i - 1$ firms set quantities q_1, \dots, q_{i-1} . Then $q_{i+1} = k_1p_{i+1} = k_1(p_i - bk_1p_i) = k_1p_i(1 - bk_1)$, $q_{i+2} = k_1p_{i+2} = k_1(p_{i+1} - bk_1p_{i+1}) = k_1p_{i+1}(1 - bk_1) = k_1p_i(1 - bk_1)(1 - bk_1)$, and inductively, $q_{i+j} = p_i(1 - bk_1)k_1(1 - bk_1)^{j-1}$. If μ_i is the probability for firm i can be written as

$$\pi = q(p - q(b + c)) - qb \sum_{m=1}^n \mu_m \sum_{j=1}^{n-m} q_{i+j}$$

If we assume a uniform prior so that $\mu_m = \frac{1}{n}$ for all m

$$\begin{aligned}\pi &= q(p - q(b + c)) - qb \sum_{m=1}^n \frac{1}{n} \sum_{j=1}^{n-m} p(1 - bk)k_1(1 - bk_1)^{j-1} \\ &= q(p - q(b + c)) - qbp(1 - bk)k_1 \frac{1}{n} \sum_{m=1}^n (n - m)(1 - bk_1)^{m-1}\end{aligned}$$

and imposing that $q = kp$

$$= p^2 \left\{ k - k^2(b + c) - kb(1 - kb)k_1 \frac{1}{n} \sum_{m=1}^n (n - m)(1 - bk_1)^{m-1} \right\}$$

Solving for the optimal k gives first order condition

$$p^2 \left\{ 1 - 2k(b + c) - b(1 - 2kb)k_1 \frac{1}{n} \sum_{m=1}^n (n - m)(1 - bk_1)^{m-1} \right\} = 0$$

Using properties of geometric sums it can be shown that $\sum_{m=1}^n (n - m)r^{m-1} = \frac{n(1-r)-1+r^n}{(1-r)^2}$, so that replacing $r = 1 - bk_1$,

$$\sum_{m=1}^n (n - m)(1 - bk_1)^{m-1} = \frac{nbk_1 - 1 + (1 - bk_1)^n}{(bk_1)^2}$$

so that the first order condition becomes

$$p^2 \left\{ 1 - 2k(b + c) - b(1 - 2kb)k_1 \left(\frac{nbk_1 - 1 + (1 - bk_1)^n}{n(bk_1)^2} \right) \right\} = 0$$

Imposing symmetry in the equilibrium so that $k = k_1$ this reduces to

$$(1 - 2kb)(1 - (1 - kb)^n) = 2bcnk^2$$

□

CHAPTER 2

THE ROLE OF CONFIDENCE OVER TIMING OF INVESTMENT INFORMATION

Abstract: I present an investment environment wherein investors demand an asset based on perfectly informative signals, but face uncertainty about the timing of their information acquisition. I show that this reduces the demand and price for every period but that in the limit price as number of periods increases price converges to the true value of the asset. By introducing a concept of confidence over the time in which they receive a signal, I show that the impact of uncertainty can be exaggerated in either a negative or positive direction, with the limit price reflecting the true value of the asset depending on the type of confidence under consideration.

2.1 INTRODUCTION

Uncertainty is one of the most widely studied phenomena in all of economics. Without uncertainty, all decisions could be made through a combination of incorporating economically relevant variables and backward induction, yielding definitive answers and leaving economists (and people in general) to dedicate themselves to other pursuits. But uncertainty pervades. Outcomes of investment choices, information quality, and even the preferences of agents all suffer from the whims of uncertainty. As such, in order to accurately capture behavior the field of economics must accommodate and incorporate into models the reality of uncertainty in any form it may take.

One form of uncertainty that has garnered much attention in the realm of financial investment and firm profit maximizing decisions is over the quality of information. The final value of an uncertain decision can be found in the outcome into which uncertainty resolves itself, but when the decision must be made before such resolution the value lies solely in the quality of information over the possible outcomes. It is no wonder then that the quality of information is of such interest. But a metric over informational quality misses one of uncertainty's most important factors: timing. It is important not only to employ accurate information in making decisions in the face of uncertainty, but it is perhaps equally important to employ this information at the appropriate time.

In this paper I introduce a setting in which profit maximizing agents undertake decisions in the face of uncertainty. However, it is not the quality of information that is uncertain to agents, but rather the timing with which agents receive this information. To emphasize the effect of timing on information driven decisions, multiple agents will receive signals at different times, yet none will be aware of the order in which they receive this profit relevant information.

In order to isolate the role of positional uncertainty, investors will receive a perfectly

informative signal about the state of the world, in this case the value of an asset. While the asset's valuation is unambiguous, agents will must determine their investment strategies without knowing their position of movement. That is, they must face the uncertainty of other investors having already made their decisions, incorporating information into the asset price, thereby diminishing the value of the informative signal.

Upon a groundwork of behavior under positional uncertainty I build the notion of confidence. Agents who are equally likely to move in any particular period will be said to suffer from a confidence bias if they place any weight other than the uniform distribution on their beliefs of moving in any period. This notion of confidence encompasses both overconfidence, as is traditionally the focus in the behavioral literature, as well as underconfidence. Overconfidence will manifest in a type of front-loading of beliefs so that the agent believes it is more likely they will move earlier than later, expecting that greater gains to investment are possible than would be so with no such bias. Underconfidence will have the opposite quality, leading agents to place greater weight on the belief that they move in later periods.

The paper will proceed as follows. In section 2 I will discuss the most closely related literature; in section 3 I will introduce a basic model of investment; in section 4 I will introduce uncertainty; in section 5 I develop a notion of confidence that can change based on agents' beliefs in equilibrium; section 6 concludes. All proofs are relegated to the appendix unless they provide useful insight into the decision making process.

2.2 RELATED LITERATURE

Much work has been done on overconfidence in the trading of financial assets. Perhaps the most closely related work is that of Gervais, Odean (2001) [4]. In this model investors receive a perfect signal with a fixed probability or pure noise and must update their belief of receiving the informative signal. Through varying a confidence parameter they show that

belief of acquiring an informative can either converge to the case of perfect rationality for low levels of overconfidence, or diverge for high levels of confidence.

This work has many related elements including accounting for the confidence of agents and a multi-period investment setting. Among the many departures, however, is that here I investigate the role of confidence over position, not signal acquisition. Agents know they receive a perfect signal about the value of the asset but have imperfect information about the period in which they receive it. In addition in their setting agents receive signals and invest in each period. In order to isolate the role of confidence over positional uncertainty I restrict attention to one signal although the model generalizes to more frequent signals.

In other works Odean (2008b) shows that overconfidence in investors tends to lead to excessive trading and lower expected utility. Overconfident agents tend to overreact to salient information and underreact to trade relevant information, thereby preventing the information of rational agents from being fully reflected in market price [7]. Barber and Odean (2001) also find that men trade stocks 45% more than women, a finding hypothesized to come from overconfidence [1].

This excessive trading and overreaction to salient information is supported by an experiment comparing traders new to online trading to their previous gains (Barber and Odean 2002) [2]. It is found that while phone traders tended to beat the market, upon the switch traders tend to under-perform, a finding unexplained by the reduction in market frictions alone. It is hypothesized that overconfidence coupled with an increased trading speed cause online investors to increase their trading volume and reduce their performance.

Other studies show similar effects of confidence in other settings. Through FMRI scans Peterson (2005) shows that investor overconfidence may be related to reward system activation in the brain [8]. Handy and Underwood (2005) find that overconfidence increases price at which managers repurchase share prices [5], a finding backed up empirically by Shu et.al (2013) [9]. Other studies demonstrate how the salience of news stories can lead to

overconfidence and excessive trading (Barber, Odean 2008) [3] and that due to loss aversion traders tend to keep their assets when they suffer large losses disproportionately more often than when they enjoy small gains (Odean 1998) [6].

2.3 THE MODEL

I consider an environment in which agents receive information about the value of a financial asset. The previous value v_0 of the asset is unknown to investors but is assumed to have already been incorporated into the market price. Agents receive a signal η about how the value of the asset changes. Agents receive this signal privately and without distortion but share a prior belief with all market participants that it is drawn from the distribution $\eta \sim N(0, \sigma^2)$.

Agents wish to maximize the difference between the value of the asset and the price they pay. Upon receiving signal η they know the value of the asset is $v_t = \mathbb{E}[v_0] + \eta$, but they are unaware of the prior value v_0 . Upon viewing price p_t agents will choose their demand for the asset x in order to maximize $\mathbb{E}[x(v_t - p_t)]$. Importantly, there will be no short sale restrictions so that agents can demand a negative amount of the asset.

In addition to not knowing the initial value v_0 of the asset, agents are also unaware of their position of movement. If the agent moves at period t then $t - 1$ agents have already had the opportunity to move. In this setting position of movement refers to the time at which the signal is received, which is to say that if an agent moving at period t sees a price p_t , this price has already had information η incorporated into it by $t - 1$ other agents.

Notice that both elements of uncertainty are necessary to capture the idea of positional uncertainty. If the agent knew v_0 they could maximize $x(v_0 + \eta - p_t)$ without any information about their position of movement. Likewise if the agent were to know their position, through backward induction the agent could deduce how much information η was incorporated into

the price by the previous $t - 1$ agents.

In addition to the aforementioned informed traders there is a liquidity trader who demands an amount of the asset every period. This is necessary not only to capture the reality that investors participate in the market for reasons other than price (e.g. to raise capital or they are uninformed) but also to guarantee trade in a market with informed investors who present an information asymmetry for any price setting mechanism. Each period the liquidity trader will demand $z_t \sim N(0, \Omega)$ of the asset, an amount independent of process that yields η and independent of liquidity demands of other periods. All market participants share common knowledge of the i.i.d. z_t and its independence from η .

Finally there is a market maker that sets the price p_t each period. The market maker knows the prior distribution of η , z_t , and their independence from one another. Like the informed agents the market maker does not know the value v_0 of the asset at period 0, but in period 1 the dissemination of information η introduces the informational asymmetry. To combat this asymmetry the market maker sets a price each period in order to match the value v_t as closely as possible given current and historical demands for the asset. That is, $p_t = \mathbb{E}[v_t | \omega_t, h_t]$ where $\omega_t = x_t + z_t$, the sum of demand from the informed and liquidity traders, and $h_t = (w_i)_{i < t}$ is the historical series of market demand for each period.

2.3.1 THE CASE OF NO UNCERTAINTY

To gain a foothold into the decision making process faced by investors it is useful to start with the case of no uncertainty. Moreover, the case without uncertainty will provide a benchmark against which to compare decision making when agents do not know their position of movement.

Consider the investment setting as described with T periods and one agent moving in each period. Each agent knows their position $t \leq T$ and chooses demand to maximize the difference between the value of the asset and price per share. To describe how agents make

this decision, recall that they maximize $\mathbb{E}[x(v_t - p_t)]$. While v_t is perfectly known as agents know their position of movement, there remains uncertainty in the price.

As we will see the linear equilibrium takes the form $p_t = p_{t-1} + \lambda_t \omega_t$. Since demand ω_t includes liquidity traders that behave randomly, agents cannot perfectly predict price movements in period t and must take an expectation. The optimal demand then comes from maximizing $\mathbb{E}[x(v_t - (p_{t-1} + \lambda_t(x + z_t)))] = x(\mathbb{E}[v_0] + \eta - p_{t-1} - \lambda_t x)$, where it is assumed that price information in p_0 already contains v_0 ; in fact this assumption can (with some error induced by the liquidity trader) be verified by the agent through backward induction. The agent's optimal solution is then $x_t = \frac{\mathbb{E}[v_0] + \eta - p_{t-1}}{2\lambda_t}$.

The market maker sets a price attempting to match the asset's value, taking into account noise from the liquidity investor. Then in a linear equilibrium $p_t = \mathbb{E}[v_0 + \eta | \beta_t x_t + z_t, h_t]$. In equilibrium the value of β_t is known to the market maker so price setting becomes an exercise in signal extraction with noise $z_t \sim N(0, \Omega)$ induced by the liquidity trader and a prior belief $p_{t-1} - p_0$ of the value η . This yields an updated estimated value of the asset $p_t = p_{t-1} + \lambda_t \omega_t$. In this environment the equilibrium values of β_t and λ_t take a simple form.

Proposition 2.1. *For $T \in \mathbb{N} \cup \{\infty\}$ periods, if each agent knows their position $t \leq T$ then there exists a linear equilibrium of the form $p_t = p_{t-1} + \lambda_t \omega_t$, $x_1 = \beta_1 \eta$, and $x_t = \beta_t \eta + Z_t$ for $t > 1$ with*

$$\lambda_1 = \frac{1}{2} \sqrt{\frac{\sigma^2}{\Omega}}, \quad \beta_1 = \sqrt{\frac{\Omega}{\sigma^2}}, \quad \lambda_t = \frac{1}{\sqrt{8}}, \quad \beta_t = \frac{\sqrt{2}}{2^{t-1}} \text{ for } t > 1, \text{ and } Z_t \sim N(0, V_t)$$

Moreover $p_t = p_0 + \left[1 - \left(\frac{1}{2}\right)^t\right] \eta + Z'_t$ where $Z'_t \sim N(0, V'_t)$

As expected from the investor's first order condition, the equilibrium demand for the asset more or less halves each period in proportion to the value of the asset. In fact, for periods $t = 2$ and onward demand $x_t = \beta_t \eta$ exactly halves every period. The reason for this is that in equilibrium β is a ratio of the variance of liquidity trading Ω and of the market

maker's inferred variance from the procedure of noise signal updating. In the first period the market uses prior belief σ^2 of the asset's variance. But thereafter updated variance of the market maker is constant at $\frac{\Omega}{2}$. This result actually holds in a more general setting.

Lemma 2.1. *For any T period investment setting as above where agents demand $\beta_t \eta$ and $\beta_t = \frac{y}{\lambda_t}$ is a constant multiple of $\frac{1}{\lambda}$ and for any initial asset variance V_0 , variance is constant in all periods $t \geq 2$ and takes the form $V_{t-1} = y\Omega$*

A technical detail that explains the constancy of β_t and λ_t for periods $t \geq 2$ to be sure, the instant convergence of inferred variance is also interesting in its own right. Not only does this result apply to the present case where agents are aware of their position, but it also applies when agents face positional uncertainty. This can be seen from the fact that the term y above can be any function of priors over positions of movement, so as long as y is constant so too is the inferred variance v_{t-1} . Another surprising feature of the updated variance is that it is independent of the distribution of signal η , depending only the liquidity trading variance Ω .

In addition to being an expected consequence of the agent's first order condition the result that demand halves in each period also provides insight into the rationality of the market price updating. In equilibrium the change in price can be expressed as $p_t - p_{t-1} = \lambda_t(\beta_t \eta_t + z_t) = \frac{1}{2^t} \eta + \lambda_t z_t$. In each period the market receives half as much information as in the prior period so that the rate of information transmission slows. Price is thus a geometric series save for the error in each period resulting from the presence of liquidity demands. While liquidity traders introduce noise that prevents the market maker from perfectly inferring the value of η , thereby enabling an equilibrium in pure strategies, their presence also hinders the interpretation of price as the true value of the asset even at the limit. However, the fact that liquidity noise has mean zero allows us to at least comment on its expectation.

Corollary 2.1. $\mathbb{E}[p_t] = p_0 + \left[1 - \left(\frac{1}{2}\right)^t\right] \eta$ and $\lim_{t \rightarrow \infty} \mathbb{E}[p_t] = p_0 + \eta$.

The form of the error Z'_t is not important from the perspective of interpreting the price or its expectation. It will always introduce randomness that prevents the market price from perfectly reflecting the underlying value of the asset, but will always be present for reasons described above. This error does, however, take a convenient form.

Proposition 2.2. *In the above equilibrium for which $T \in \mathbb{N} \cup \{\infty\}$ periods and each agent knows their position $t \leq T$, the error term Z_t takes the form $Z_t = -\frac{1}{\lambda_t} \sum_{i=1}^{t-1} \left(\frac{1}{2}\right)^{t-i} \lambda_i z_i$ so that equilibrium demand for each period is $x_t = \beta_t \eta - \frac{1}{\lambda_t} \sum_{i=1}^{t-1} \left(\frac{1}{2}\right)^{t-i} \lambda_i z_i$ for $t > 1$. Moreover $p_t = p_0 + \left[1 - \left(\frac{1}{2}\right)^t\right] \eta + \sum_{i=1}^t \lambda_i z_i \left(\frac{1}{2}\right)^{t-i}$*

As the formulation of Z_t makes clear, each period noise from all previous periods becomes less relevant to price. But even though the Z_t follows a process in proportion to a geometric sum there is always $\lambda_t z_t$ incorporated in price p_t , preventing the price from converging to the true value of the asset. Fortunately a metric that is often referenced as an indication of an asset's value is the moving average, and with good reason.

Proposition 2.3. *In the above equilibrium for which $T \in \mathbb{N} \cup \{\infty\}$ periods and each agent knows their position $t \leq T$, $\text{plim}_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T p_t = p_0 + \eta$.*

As this proposition shows, the price may not converge to the true value of the asset but the moving average converges in probability. So in a probabilistic sense the market fully incorporates the value η .

2.4 INTRODUCING UNCERTAINTY

Now suppose agents face uncertainty over their position over movement, but suppose that agents assume with a common prior belief over the order. We begin with the case of two agents.

2.4.1 TWO AGENTS

As above suppose investors invest in an asset that evolves according to an unobservable process $v_{t+1} = v_t + \eta_{t+1}$ but in different periods each receives the same signal η about the process. Again there is a liquidity investor who demands $z_t \sim N(0, \Omega)$ independent of η_t .

In the case of two agents the common prior assumption provides that if agent 1 has prior belief $\Pr_1(F) = \gamma_1$ of moving first then it anticipates that agent 2 has prior belief $\Pr_2(F) = 1 - \gamma_1$. The agent receives a signal η about the how the value of the asset evolves but does not know the initial valuation and thus cannot infer if this valuation is already incorporated in price. With two agents, each can either be first or second and each observes a price p which may or may not incorporate the information η . Supposing p_t is the price before information enters the market,

- (i) If the agent is first then the observed price $p = p_t$
- (ii) If the agent is second then the observed price $p = p_t + \lambda_{t+1}y_{t+1}$, where y_{t+1} was demand from the first moving agent.

The difference between the first price p and the second is that the second price already incorporates information about the asset's value from the first agent. Thus the remaining profit left to the second mover is less because the price relative the the value of the asset is higher. Given that the agent has no prior information about the value of the asset it must be assumed that $p_t = \mathbb{E}[v_t]$. Then agents solve

$$\begin{aligned}
 \max_x x \mathbb{E}[v_{t+1} - p_{t+1}] &= \max_x x \mathbb{E}[v_{t+1} - (p + \lambda_{t+1}\omega_{t+1})] \\
 &= \max_x x \cdot \gamma \mathbb{E}[v_t + \eta_{t+1} - (p_t + \lambda_{t+1}\omega_{t+1})] + \max_x x \cdot (1 - \gamma) \mathbb{E}[v_t + \eta_{t+1} - (p_t + \lambda_t y_t + \lambda_{t+1}\omega_{t+1})] \\
 &= \max_x x \cdot \gamma(\eta - \lambda_{t+1}x) - x \cdot (1 - \gamma)(\eta - \lambda_{t+1}x - \lambda_t y_t)
 \end{aligned}$$

where y_t is the expected quantity of the first mover in the event this agent is in fact choosing second. The profit maximization problem then becomes $\max_x x \cdot (\eta - \lambda_{t+1}x - (1 - \gamma)\lambda_t \mathbb{E}[y])$.

Notice that while $p = p_t$ or $p = p_t - \lambda_t y_t$, the maximization function does not contain the term p . This is because the agent does not know price p_t or value v_t , but on the expectation the best guess is that the market sets price $p_t = \mathbb{E}[v_t]$. Then these two terms cancel and the difference we are left with is that between the future valuation and current price.

Notice also that the linear price parameter λ_t is potentially different in every period. This is a result of the fact that the market maker's belief V_t of the informative signal η updates each period. However according to Lemma 1 the variance V_t is constant for all $t \geq 2$ as long as β_t is constant. Given that this is the sort of equilibrium of interest we will make the simplifying assumption that λ_t is constant for all t .

Assumption 2.1. *In any linear equilibrium price of the form $p_t = \alpha_t + \lambda_t \omega_t$ where ω_t is total market demand and α_t is a period specific constant, assume that $\lambda_t = \lambda$ for all t .*

With this additional assumption we are ready to characterize an equilibrium for two agents.

Proposition 2.4. *For $T = 2$ periods and agents do not know their position but have prior beliefs γ_1 and γ_2 and assume a common prior then there exists a linear equilibrium of the form $p_t = p_0(1 - \varphi) + \varphi p_{t-1} + \lambda \omega_t$ where p_0 is the price before information η entered the market, $x_1 = \beta_1(\gamma_1)\eta$ and $x_2 = \beta_2(\gamma_2)\eta$ with*

$$\beta_i(\gamma_i) = \frac{2 - \varphi(1 - \gamma_i)}{\lambda[4 - \varphi^2\gamma_i(1 - \gamma_i)]}, \quad \mathbb{E}_i[\beta_j(\gamma_i)] = \frac{2 - \varphi\gamma_i}{\lambda[4 - \varphi^2\gamma_i(1 - \gamma_i)]}$$

$$\lambda = \sqrt{\varphi}(1 - \varphi) \quad \text{and} \quad \varphi^2 + 3\varphi - 2 = 0 \quad (\varphi \approx 0.562)$$

From this result we can see the manner in which information about the asset's value translates into movements in the price. In equilibrium, $p_2 = p_0 + \lambda(\varphi x_1 + x_2) + \lambda(\varphi z_1 + z_2)$.

Then

$$\mathbb{E}[p_2] = \lambda(\varphi\beta_1\eta + \beta_2\eta) = \left\{ \frac{\varphi[2 - \varphi(1 - \gamma_1)]}{4 - \varphi^2\gamma_1(1 - \gamma_1)} + \frac{2 - \varphi(1 - \gamma_2)}{4 - \varphi^2\gamma_2(1 - \gamma_2)} \right\} \eta$$

and if agents share a common prior so that $\gamma_1 = 1 - \gamma_2$ this reduces to

$$= \left[\frac{2 + \varphi(2 - \gamma_1) - \varphi^2(1 - \gamma_1)}{4 - \varphi^2\gamma_1(1 - \gamma_1)} \right] \eta = \frac{\varphi[5 - \gamma_1(1 - \varphi)]\eta}{4 - \varphi^2\gamma_1(1 - \gamma_1)}$$

where the last equality comes from the fact that $\varphi^2 + 3\varphi = 2$.

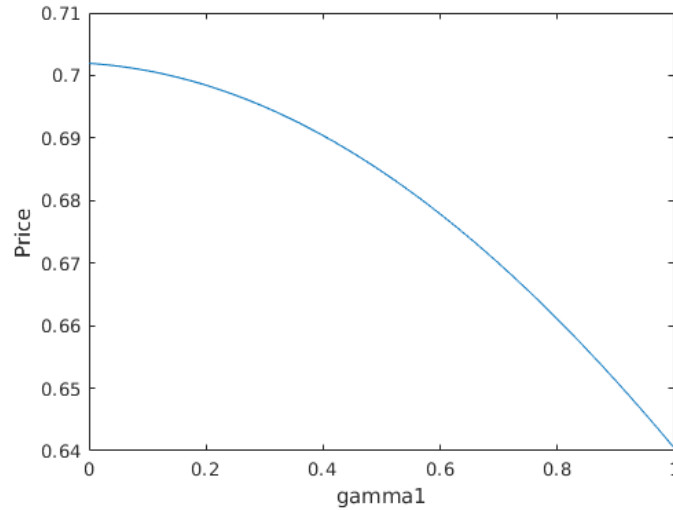


Figure 2.1: Final price p_2 as a function of prior γ_1

As figure 3.8 shows, as the common prior γ_1 increases, the degree to which the price reflects the informational content of demand diminishes. This is due to the weight φ on the demand of the first mover and the inability of the market maker to respond to changes in the common prior due to the information asymmetry.

2.4.2 T IDENTICAL AGENTS

We can generalize this case to one in which there are T agents, each receiving the signal η in a different period $t \leq T$ and sharing a common prior over their position of movement. A natural prior is uniform, where each agent believes that their probability of moving in period $t \leq T$ is $\Pr(t) = \frac{1}{T}$ for all periods. Furthermore, each agent believes that all other agents share this common prior.

As in the case of no uncertainty we will find a linear equilibrium in demand ω_t . Now, however, since agents do not know if the price they see is the original valuation or the price after $t-1$ periods of agents acting on information η , they will not assign a unit value to p_{t-1} . They weight the previous price based on their beliefs $\Pr(t)$ of moving in every t and their beliefs about other agents' actions. To compensate for this, in equilibrium price at period t will be a weight $\varphi < 1$ of the previous period price and current demand ω_t .

To see why this is, consider the pricing decision of the market maker. As before, each agent demands $x = \frac{\mathbb{E}[v_0] + \eta - p_{t-1}}{2\lambda}$ but now, with equal probability p_{t-1} could have information η incorporated in any number of periods $t \leq T-1$. Thus the agent will shade their demand down by the expected amount of information already incorporated into the price. Each period the market sets p_t in order to estimate $v_0 + \eta$. Then $p_t = \mathbb{E}[v_0 + \eta | \beta x_t + z_t] = p_0 + \frac{1}{\beta} \mathbb{E}[\beta \eta | \beta \eta + z_t] = p_0 + \frac{\Omega(p_{t-1} - p_0) + \beta V \omega_t}{\Omega + \beta^2 V}$ so that $p_t = p_0(1 - \varphi) + \varphi p_{t-1} + \lambda \omega_t$ where $\varphi = \frac{\Omega}{\Omega + \beta^2 V}$ and $\lambda = \frac{\beta V}{\Omega + \beta^2 V}$.

With this formulation price in each period is a φ discounted sum of previous demands plus initial price. If the agent moves in the second period price is $p_1 = p_0 + \lambda \omega_1$. If the agent moves in the third period then price is $p_2 = p_0 + \lambda \omega_2 + \varphi \lambda \omega_1$. Inductively if the agent moves in period t then $t-1$ agents move before and $p_{t-1} = p_0 + \sum_{i=1}^{t-1} \lambda \varphi^{(t-1)-i} \omega_i$. So to the agent, without knowledge of initial value v_0 , p_{t-1} is a combination of demand in previous periods, containing $p_0 = \mathbb{E}[v_0]$. This gives rise to a linear equilibrium of the following form.

Proposition 2.5. *For $T \in \mathbb{N} \cup \{\infty\}$ periods, if agents do not know their position but have a uniform and common prior belief over $t \leq T$ then there exists a linear equilibrium of the form $p_t = p_0(1 - \varphi) + \varphi p_{t-1} + \lambda \omega_t$ where p_0 is the price before information η enters the market and $x_t = \beta_t \eta$ with*

$$\beta_t = \frac{1}{\sqrt{\varphi}}, \quad \lambda = \sqrt{\varphi}(1 - \varphi), \quad \text{and} \quad \varphi = 1 - \frac{\varphi(1 - \varphi^T)}{T(1 - \varphi)}$$

This equilibrium can be solved down to the variable φ which itself cannot be solved for explicitly. Yet it still provides interesting insight. The most obvious result to note is that this equilibrium does not depend on liquidity noise Ω . This comes from the fact that the updated variance of η converges immediately as described above, so $\beta\lambda$ need not include this term. As the market maker gains information from demand each period, since the variance of η does not change, noise introduced by the liquidity traders offers no additional information.

Equilibrium behavior for the informed agents also accords closely to what we would expect. Since agents do not know which of the T positions they occupy when they choose their investment strategies, they tend to behave more cautiously than in the case with no uncertainty.

This figure compares the price for each number of time periods in the certain and uncertain cases, given that the true value of η is 1 and $p_0 = 0$. As we can see comparing the cases of certainty with uncertainty, as the number of periods T increases the information η is more quickly incorporated into the price of the asset in the certain case. Indeed in the certain case information is integrated at the geometric rate $1 - \frac{1}{2}^t$, while in the uncertain case the rate is not quite as fast.

While slower than in the case of certainty, we can say something about the rate of convergence to the true value η as the following proposition describes.

Proposition 2.6. *In the above equilibrium for which $T \in \mathbb{N} \cup \{\infty\}$ and agents do not know*

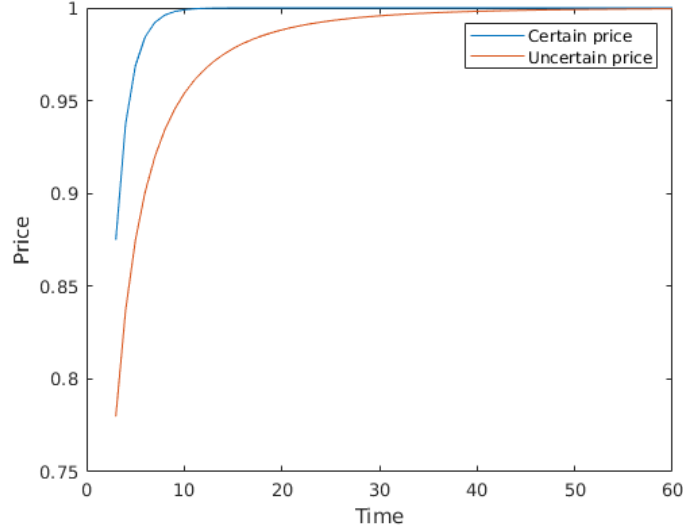


Figure 2.2: Final market price for certain and uncertain position

their position but have a uniform and common prior belief over $t \leq T$, $\mathbb{E}[p_t] = p_0 + \eta(1 - \varphi^t)$

In the case of positional uncertainty, for every number of possible time periods the price is lower than if position of movement were certain, but this price too converges at a (pseudo) geometric rate of $1 - \varphi^t$, with φ as defined above. The difference is that the φ is higher than the $\frac{1}{2}$ of the certain case for all t , and in fact $\lim_{t \rightarrow \infty} \varphi = 1$. However, since price depends on φ^t it is this term whose convergence determines the integration of signal η into the price as the number of periods T increases. As the figure makes clear this term indeed does converge to zero.

Proposition 2.7. *In the above equilibrium for which $T \in \mathbb{N} \cup \{\infty\}$ and agents do not know their position but have a uniform and common prior belief over $t \leq T$, $\lim_{T \rightarrow \infty} \mathbb{E}[p_t] = p_0 + \eta$ and $\text{plim}_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T p_t = p_0 + \eta$.*

As Proposition 7 describes we have an analogous limit result in the case of positional uncertainty - albeit with a slower rate of convergence. This slower convergence reflects the fact that symmetric agents are more cautious in acting on their signal as there may be up

to $T - 1$ periods of signal information already incorporated into the market price, making the gains uncertain. However, as the number of periods increases, the effect each agent has on equilibrium price by placing their optimal demand diminishes, so that demands in the certain and uncertain case merge and information η is fully incorporated.

2.5 A NOTION OF CONFIDENCE

Now that we have investigated the informed investing environment with certain and uncertain positions of movement, we can turn attention to how confidence plays a role in investment decisions. In particular, we saw in the environments with and without certainty that as the number of periods T increases price increased to the true value η of the asset. Furthermore we saw that this convergence was slower in the case of positional uncertainty but hardly by much; for $T \geq 40$ the prices were barely distinguishable.

Now we introduce the notion of confidence and attempt to answer the same questions. In particular, we would like to investigate in the presence of confidence over uncertain outcomes:

1. How does equilibrium price with confident agents compare to the case of no uncertainty?
2. How does equilibrium price with confident agents compare to the case of uncertainty with neutral agents possessing uniform priors over positions $t \leq T$?
3. As number of periods T grows large does price reflect the value η of the underlying asset?

In order to begin to answer these questions we will need to introduce a notion of confidence.

Definition 2.1. *In a T period investment setting, an agent is neutral in terms of confidence if their belief of moving in period t , $\Pr(t) = \mu_t$, is equal for all t so that $\mu_t = \frac{1}{T}$.*

Given this definition, in the uncertain case previously analyzed all agents were neutral. The concept of non-neutrality in terms of confidence takes the obvious definition.

Definition 2.2. *In a T period investment setting, an agent is non-neutral in terms of confidence if they are not confidence neutral. That is, if for some t_1, t_2 $\mu_{t_1} \neq \mu_{t_2}$.*

There are infinitely many ways in which an agent can stray from confidence neutrality. In order to narrow the scope of this definition, we will restrict attention to confidence over the first period. An agent will be said to be overconfident if she overweighs the probability of moving in the first period, and underconfident if she underweighs this probability.

Definition 2.3. *In a T period investment setting, an agent who has beliefs $\mu_1 = \frac{\gamma}{T}$, $\mu_t = \frac{T-\gamma}{T(T-1)}$ for $t \geq 2$ is overconfident if $\gamma > 1$ and underconfident if $\gamma < 1$.*

In the scope of this definition it is the belief of moving first that determines confidence. The probability of receiving the signal η in any other period is then spread uniformly across all other periods.

2.5.1 CONFIDENCE: THE MINDFUL INVESTOR

With these definitions regarding the confidence of investors over their uncertain position of movement we can define the equilibrium. Of course equilibrium behavior will depend on beliefs of other agents as well. In particular we begin with agents who are non-neutral ($\gamma \neq 1$) and take into account the non-neutrality of other agents. In this way we can think of these agents as “mindful” of their departure from neutrality and that other agents make the same departure. The market maker, unaware that investors behave anything other than fully rational, will set price exactly as before.

Proposition 2.8. *In a $T \in \mathbb{N} \cup \{\infty\}$ period investment setting, if informed agents do not know their position but hold a common belief $\mu_1 = \frac{\gamma}{T}$, uniform $\mu_t = \frac{T-\gamma}{T(T-1)}$ for $t > 2$ then*

there exists a linear equilibrium of the form $p_t = p_0(1 - \varphi) + \varphi p_{t-1} + \lambda \omega_t$ where p_0 is the price before information η enters the market and $x_t = \beta \eta$ with

$$\beta = \frac{(1 - \varphi)}{\frac{\lambda}{T-1} [(T - 1) - (2\varphi - 1)(\gamma - 1)]}, \quad \lambda = \sqrt{\varphi}(1 - \varphi), \quad \text{and} \quad \varphi = 1 - \frac{\varphi(1 - \varphi^T)}{T(1 - \varphi)}$$

As the parameter β makes clear γ has a predictable effect on demand for the asset. Agents tend to demand more (less) if $\gamma > 1$ ($\gamma < 1$) as is easily seen in the denominator into which γ enters negatively. When $\gamma = 1$ we return to the case of neutral uncertainty described above. Having no way to know or reason to suspect non-neutrality the market maker behaves as in the case of neutral agents. If the market maker were able to compensate for non-neutrality the price would more closely resemble that of the neutral case.

The figures below depict the movements of price as number of periods increases comparing the neutral case to the over/underconfident case when the true value of η is 1 and $p_0 = 1$. The figure on the left shows that in the case of overconfidence ($\gamma = 2$ here) the market price is always higher than in the confidence neutral case. Investors underweight the possibility that the price already contains information about the value η and thus demand more than they otherwise would. In fact as the graph shows, for early periods the price actually exceeds the value of η . With underconfidence we see even more caution than in the case of neutral uncertainty with the underconfident agent even further believing that the price already contains information about the value of the asset.

As the figures below demonstrate, comparing the results to initial market with no uncertainty paints an even more dramatic picture. In the overconfident case with just a few periods the price surpasses the geometric pricing schedule of no uncertainty. The underconfident case takes appreciably longer to integrate information about value into the price.

From these figures it does seem like eventually given enough periods the price does integrate the true value of η ; it appears that after 150 periods of investment the value is

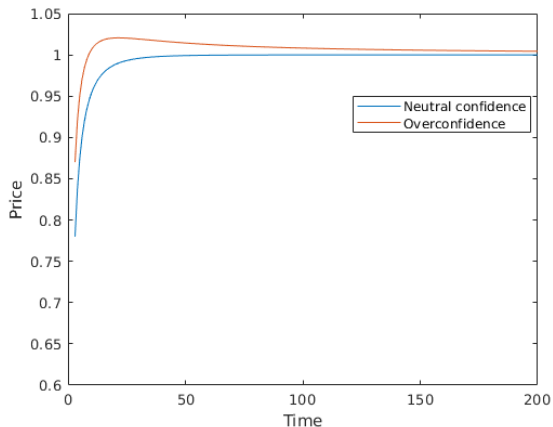


Figure 2.3: Final market price for neutral and mindful overconfidence

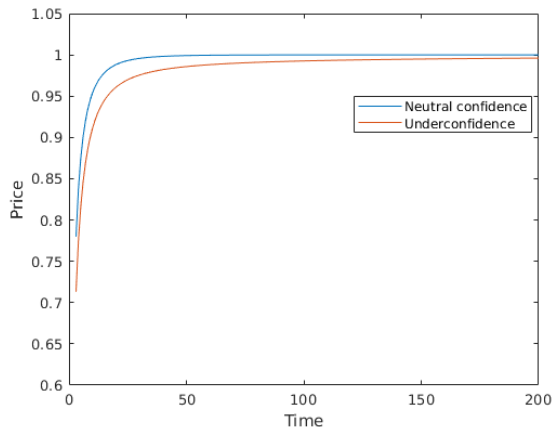


Figure 2.4: Final market price for neutral and mindful underconfidence

almost completely incorporated. In fact as with the cases of no uncertainty and neutral uncertainty we can say this unambiguously.

Proposition 2.9. *For a $T \in \mathbb{N} \cup \{\infty\}$ period investment setting, if informed agents hold a common belief $\Pr(t = 1) = \frac{\gamma}{T}$, uniform $\Pr(t) = \frac{T-\gamma}{T(T-1)}$ for $T > 2$, $\lim_{T \rightarrow \infty} \mathbb{E}[p_t] = p_0 + \eta$ and $\text{plim}_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T p_t = p_0 + \eta$.*

This proposition confirms that even if over(under)confident agents over(under)shoot the price for small T , for a large enough T all information about the value η is incorporated into the market price.

2.5.2 CONFIDENCE: THE MYOPIC INVESTOR

In the previous section we made the assumption that the non-neutral agent was “mindful” in the sense of being aware other agents share the same confidence bias. But it is at least as likely - if not more likely - that the agent is so confident that she believes she is the only agent with the informational advantage that increases (decreases) her likelihood of moving first. This would mean that in solving the maximization problem, it is assumed that other

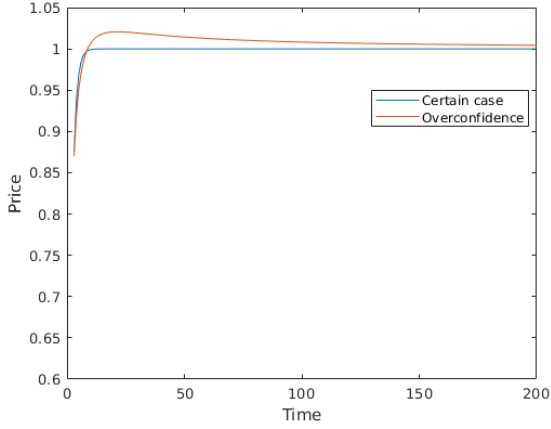


Figure 2.5: Final market price for certain and mindful overconfidence

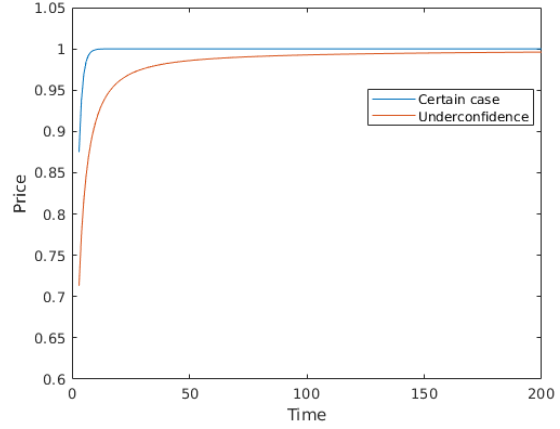


Figure 2.6: Final market price for certain and mindful underconfidence

agents behave as if they were neutral investors, and the confident investor would dismiss the possibility of others also biasing their belief of moving first.

Proposition 2.10. *In a $T \in \mathbb{N} \cup \{\infty\}$ period investment setting, if informed agents do not know their position but believe $\mu_1 = \frac{\gamma}{T}$, $\mu_t = \frac{T-\gamma}{T(T-1)}$ for $t > 2$ and believe other agents have a uniform prior $\Pr(t) = \frac{1}{T}$ for all $t \leq T$ then there exists a linear equilibrium of the form $p_t = p_0(1 - \varphi) + \varphi p_{t-1} + \lambda \omega_t$ where p_0 is the price before information η enters the market and $x_t = \beta_t \eta$ with*

$$\beta = \frac{1}{\lambda} \left[(1 - \varphi) + \frac{(\gamma - 1)(2\varphi - 1)}{2(T - 1)} \right], \quad \lambda = \sqrt{\varphi}(1 - \varphi), \quad \text{and} \quad \varphi = 1 - \frac{\varphi(1 - \varphi^T)}{T(1 - \varphi)}$$

Again the market maker sets price as in the neutral case having no information about the confidence bias of investors. As we have seen, even knowing the existence and magnitude of a bias is insufficient because of the many ways investors can operationalize their bias, mindfully and myopically among them.

In the following figures we see the comparison of naive confidence and the neutral and certain cases with the true value of $\eta = 1$ and $p_0 = 0$ as in all previous analyses. We

see again that demand is increasing in confidence γ which appears positively in both the β and δ terms. Clearly as $\gamma \rightarrow 1$ this approaches our previous equilibrium of confidence neutrality. The magnitude of this difference, however, is difficult to interpret from the first order conditions.

As figures 2.11 and 2.12 show, we have the same pattern of the overconfident investor

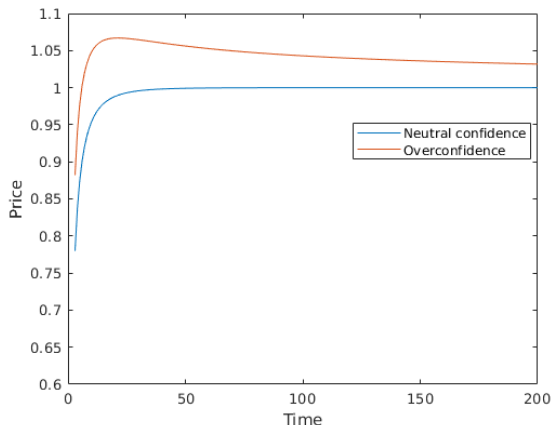


Figure 2.7: Final market price for neutral and myopic overconfidence

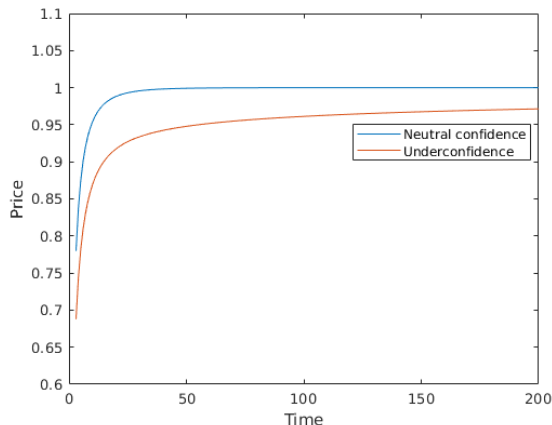


Figure 2.8: Final market price for neutral and myopic underconfidence

(left) investing so much more than in the neutral case that in very few periods price exceeds the true value of $\eta = 1$. Now, however, convergence of price to the true value of the asset seems questionable. Even after 200 periods the price of the over(under)confident investor over(under) estimates the value by about 3 percent; $p_{200} = 1.032$ ($p_{200} = 0.971$). Despite the persistence in price distortion the bias introduces, we can in fact establish a limit result.

Proposition 2.11. *In the above equilibrium for which $T \in \mathbb{N} \cup \{\infty\}$ and agents hold belief $\Pr(t = 1) = \frac{\gamma}{T}$, uniform $\Pr(t) = \frac{T-\gamma}{T(T-1)}$ for $T > 2$, and believe other agents hold a uniform prior $\Pr(t) = \frac{1}{T}$ for all $t \leq T$, $\lim_{T \rightarrow \infty} \mathbb{E}[p_t] = p_0 + \eta$ and $\text{plim}_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T p_t = p_0 + \eta$.*

While this limit result confirms that even in the case of myopic confidence we have that the asset price reflects its true value this convergence is extremely slow. This is of course

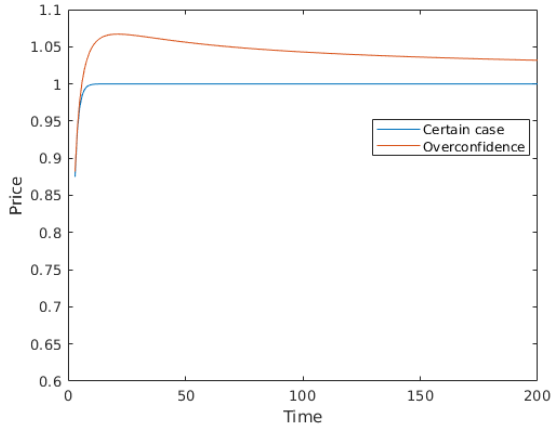


Figure 2.9: Final market price for certain and myopic overconfidence

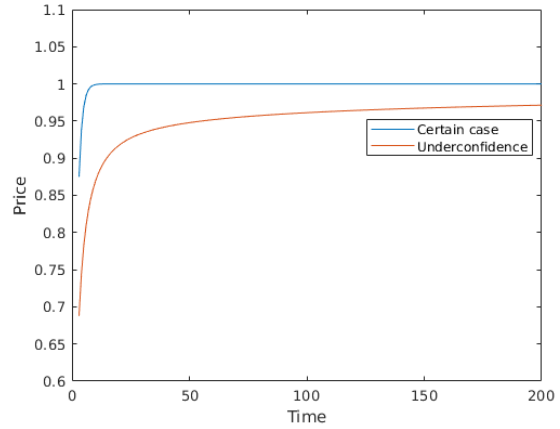


Figure 2.10: Final market price for certain and myopic underconfidence

due to the weighting of μ_1 that causes the agent to under/overestimate the probability that price already contains information about the value η from other agents. But even more than in the case of the mindfully confident investor, as more time periods/investors are added, the fact that each investor does not account for others' confidence γ prevents the bias from being spread over more and more periods as efficiently.

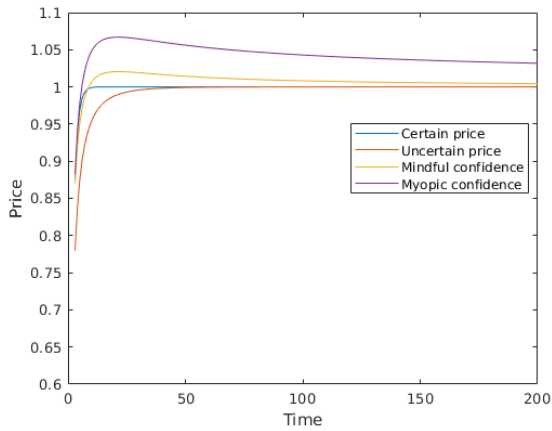


Figure 2.11: Final market price for all forms of overconfidence

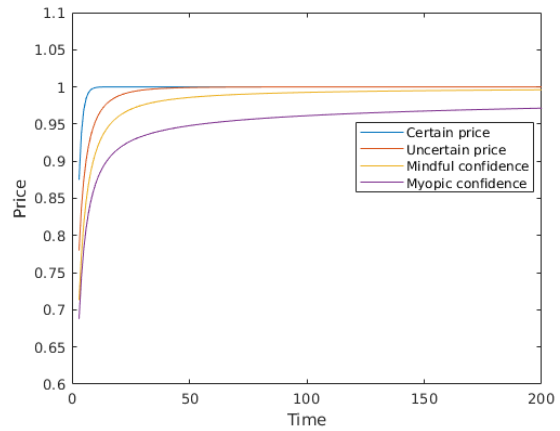


Figure 2.12: Final market price for all forms of overconfidence

The above figures show all of the cases together. As can be seen by the comparison,

although mindful confidence suffers some pathology for small T , after a relatively short time it converges to the certain and neutral cases. The cases of myopic confidence, however, seem to take their time. While they reach $\eta \pm 3\%$ in relatively short order, with increasing time periods T this difference does not seem to relent. This is due to the slow convergence of $\varphi \rightarrow 1$. While all other prices depended on the convergence of $\varphi^T \rightarrow 0$, the convergence of this series depends on the convergence of φ . This is, of course, a direct result of agents not considering the confidence biases of other agents.

2.6 CONCLUDING REMARKS

In a investment setting with informed investors, liquidity traders, and a market maker seeking to match the unknown value of an asset there are clear predictions in the case of certainty. Agents who face no uncertainty - either about the value of the asset or the number of investors who have acted before them - maximize profit in a linear equilibrium by halving the remaining value, leading to a rapid geometric convergence of the price to the asset's value. A generalization of this model wherein agents do not know the period in which they receive the informative signal, and as such do not know in which period they choose their demand, demonstrates a similar pattern that is slightly blunted by the uncertainty of how many investors had previously incorporated this profit relevant information into the price.

The introduction of confidence into this framework enriched the environment of uncertainty, allowing agents to differ in how they responded to not knowing the period when they receive the signal or how stale the information might be. Overconfident agents overweigh the probability of being first, leading to more demand than is profitable even in the case of certainty. This is reflected in a price that is higher than if the agents were neutral in terms of confidence, and possibly even higher than the value of the asset. Underconfident investors, conversely, tended to demand less of the asset than was profitable, leading to a price that

lagged every other case and took longer to converge to the true value.

One operationalization of confidence - “mindful” confidence - led to a higher/lower price than was otherwise profitable, and yet as the number of periods grew large the price converged to the value of the asset rather quickly. This result is appealing in that confidence biases of agents are not too disruptive to the information value of asset price given a suitably large number of periods. And yet, while the concept of mindful confidence allowed for agents’ beliefs to take into account that other agents share similar biases, the idea of being concurrently biased about one’s own beliefs and mindful of others is in a sense contradictory.

An agent may be overconfident that they are particularly shrewd observers of the financial news, picking up on value-relevant signals before others can catch on. But if they take into account that others act in the same way is it true that they are more adept at interpreting information? They may maintain an edge over some investors, but if they plan investment strategies based on others taking the same factors in mind and undertaking the same line of iterative induction, the belief that these investors are as naïve as all other seems to break down.

Out of this contradiction arose the notion of “myopic” confidence whereby investors are confident that they move first and discount the possibility that other investors share confidence biases. This concept conforms more to our idea of what it means to be too confident. In the setting of myopic confidence we found an even more exaggerated departure in demand behavior and price as a measure of value. Even though the price of the asset in this case converges to the true value in the limit it does so extremely slowly. In fact given a 200 period time horizon we saw the asset price still failed to converge.

In each of these investment environments the asset value was perfectly known (granted, by different investors at different times) and this value never changes. It may be of comfort to the informational value of price that in all but the most extreme case of myopic confidence price converges quickly to the true value. But of course in a more dynamic setting the value

is ever changing and signals are constantly being disseminated. If any of the above models were to be repeated every 5-10 periods the informational value at the limit would never have an opportunity to realize, leading to a potentially dramatic departure between the price of an asset and its value. Even if the effects of confidence did not accrue but canceled as a result of value fluctuation this still leaves the market with an undesirable level of volatility that reduces the appeal of investment and the ability of the market operate efficiently.

BIBLIOGRAPHY

- [1] Barber, B. and Odean, T. (2001), “Boys will be boys: Gender, overconfidence, and common stock investment.” *The Quarterly Journal of Economics*, 116, pp. 261-292.
- [2] Barber, B. and Odean, T. (2002), “Online investors: Do the slow die first?” *The Review of Financial Studies*, 15, pp. 455-487.
- [3] Barber, B. and Odean, T. (2008), “All that glitters: The effect of attention and news on the buying behavior of individual and institutional investors.” *The Review of Financial Studies*, 21, pp. 785-818.
- [4] Gervais, S. and Odean, T. (2001), “Learning to be overconfident.” *Review of Financial Studies*, 14, pp. 1-27.
- [5] Handy, J. F. and Underwood, S. (2015), “Overconfidence and the Timing of Share Repurchases” (June 23, 2015). Available at SSRN: <https://ssrn.com/abstract=2622097> or <http://dx.doi.org/10.2139/ssrn.2622097>
- [6] Odean, T. (1998a), “Are investors reluctant to realize their losses?” *The Journal of Finance*, 53:5, pp. 1775-1798.
- [7] Odean, T. (1998b), “Volume, volatility, price, and profit when all traders are above average.” *The Journal of Finance*, 53:6, pp. 1887-1934.

- [8] Peterson, R.L. (2005) “The neuroscience of investing: fMRI of the reward system.” *Brain Research Bulletin*, 67, pp. 391-397.
- [9] Shu, P.E., Yeh, Y.H., Chiang, T.L., and Hung, J.Y. (2013). “Managerial overconfidence and share repurchases.” *International Review of Finance*, 13:1, pp. 39-65.

APPENDIX: PROOFS

Proof of Proposition 1. Conjecture a linear price equilibrium of the form $p_t = p_{t-1} + \lambda_t \omega_t$ and consider the first agent's optimization problem. The anticipated market price is $p_1 = p_0 + \lambda_1 \omega_1$ so the agent solves

$$\max_{x_1} x_1 \mathbb{E}[v_0 + \eta - p_1 | \theta] = x_1 (\eta - \mathbb{E}[\lambda_1 \omega_1]) = x_1 (\eta - \lambda_1 x_1)$$

which yields the optimal quantity $x_1 = \frac{\eta}{2\lambda_1}$.

By induction, for a $t > 1$ conjecture that the optimal investment given λ_t is $x_t = \frac{1}{2^t \lambda_t} \eta + Z_t$ with $Z_t \sim N(0, V_t)$. Then the agent in period $t + 1$ solves

$$\max_{x_{t+1}} x_{t+1} \mathbb{E}[v_0 + \eta - p_{t+1} | \theta] = x_{t+1} \left(\eta - \left(\sum_{i=1}^t \lambda_i \omega_i + \lambda_{t+1} x_{t+1} \right) \right)$$

which yields equilibrium $x_{t+1} = \frac{\eta - \sum_{i=1}^t \lambda_i \omega_i}{2\lambda_{t+1}}$. By induction this holds for all preceding t so that $x_t = \frac{\eta - \sum_{i=1}^{t-1} \lambda_i \omega_i}{2\lambda_t}$ and $2\lambda_t x_t = \eta - \sum_{i=1}^{t-1} \lambda_i \omega_i$. Also notice that

$$\begin{aligned} x_{t+1} &= \frac{\eta - \sum_{i=1}^t \lambda_i \omega_i}{2\lambda_{t+1}} = \frac{\eta - \sum_{i=1}^{t-1} \lambda_i \omega_i - \lambda_t \omega_t}{2\lambda_{t+1}} = \frac{\eta - \sum_{i=1}^{t-1} \lambda_i \omega_i - \lambda_t x_t - \lambda_t z_t}{2\lambda_{t+1}} \\ &= \frac{2\lambda_t x_t - \lambda_t x_t - \lambda_t z_t}{2\lambda_{t+1}} = \frac{\lambda_t x_t - \lambda_t z_t}{2\lambda_{t+1}} \end{aligned}$$

By the induction assumption $\lambda_t x_t = \frac{1}{2^t} \eta + \lambda_t Z_t$ and

$$\begin{aligned} x_{t+1} &= \left(\frac{1}{2\lambda_{t+1}} \right) \left[\frac{1}{2^t} \eta + \lambda_t Z_t - \lambda_t z_t \right] \\ &= \left(\frac{1}{\lambda_{t+1}} \right) \left[\frac{1}{2^{t+1}} \eta + \frac{1}{2} \lambda_t Z_t - \frac{1}{2} \lambda_t z_t \right] \end{aligned}$$

so that $x_{t+1} = \frac{1}{2^{t+1} \lambda_{t+1}} \eta + Z_{t+1}$ where $Z_{t+1} = \frac{\lambda_t (Z_t - z_t)}{2\lambda_{t+1}} \sim N(0, \frac{\lambda_t^2 (V_t + \Omega)}{4\lambda_{t+1}^2})$. Letting $V_{t+1} =$

$\frac{\lambda_t^2(V_t+\Omega)}{4\lambda_{t+1}^2}$ gives $Z_{t+1} \sim N(0, V_{t+1})$. Then $x_{t+1} = \beta_{t+1}\eta + Z_{t+1}$ where $\beta_{t+1} = \frac{1}{2^{t+1}\lambda_{t+1}}$. Thus by induction this holds for all $t \leq T$.

Now consider the problem of the market maker. In each period the market maker sets the price in order to match the value of the asset. That is $p_t = \mathbb{E}[v_0 + \eta|\omega_t, h_t]$ where again $\omega_t = x_t + z_t$ is market demand, and h_t is the historical series of market demand. Then in period 1

$$\begin{aligned} p_1 &= \mathbb{E}[v_0 + \eta|\omega_1] = p_0 + \mathbb{E}[\eta|\beta_1\eta + z_1] = p_0 + \frac{1}{\beta_1}\mathbb{E}[\beta_1\eta|\beta_1\eta + z_1] \\ &= p_0 + \frac{\beta_1\sigma^2}{\beta_1^2\sigma^2 + \Omega}\omega_1 \end{aligned}$$

and so $\lambda_1 = \frac{\beta_1\sigma^2}{\beta_1^2\sigma^2 + \Omega}$. Moreover since $\beta_1 = \frac{1}{2\lambda_1}$ then

$$\begin{aligned} \lambda_1 &= \frac{\frac{1}{2\lambda_1}\sigma^2}{\left(\frac{1}{2\lambda_1}\right)^2\sigma^2 + \Omega} = \frac{\sigma^2}{\left(\frac{1}{2\lambda_1}\right)\sigma^2 + 2\Omega\lambda_1} \\ 2\sigma^2 &= \sigma^2 + 4\Omega\lambda_1^2 \\ \lambda_1 &= \sqrt{\frac{\sigma^2}{4\Omega}} \end{aligned}$$

so that

$$p_1 = \mathbb{E}[v_0 + \eta|\omega_1] = p_0 + \left(\sqrt{\frac{\sigma^2}{4\Omega}}\right)\omega_1$$

and the variance of the estimate of η is

$$V_1 = \frac{\beta_1^2\sigma^2\Omega}{\beta_1^2\sigma^2 + \Omega} = \frac{\left(\frac{1}{2\lambda_1}\right)^2\sigma^2\Omega}{\left(\frac{1}{2\lambda_1}\right)^2\sigma^2 + \Omega} = \frac{\sigma^2\Omega}{\sigma^2 + 4\lambda_1^2\Omega}$$

which reduces to $V_1 = \frac{\Omega}{2}$.

Consider a general $t > 1$. The market maker again sets price to match the expected value of the asset so that

$$\begin{aligned}
p_t &= \mathbb{E}[v_0 + \eta | \omega_t, h_t] = \mathbb{E}[v_0 + \eta | x_t + z_t, h_t] = \mathbb{E} \left[v_0 + \eta \left| \frac{1}{2\lambda_t} \left(\eta - \sum_{i=1}^t \lambda_i \omega_i \right) + z_t, h_t \right. \right] \\
&= \mathbb{E} \left[v_0 + \eta \left| \frac{1}{2\lambda_t} (\eta - (p_{t-1} - p_0)) + z_t, h_t \right. \right] = \mathbb{E} \left[v_0 + \eta \left| \frac{1}{2\lambda_t} (v_0 + \eta - p_{t-1}) + z_t, h_t \right. \right] \\
&= 2\lambda_t \mathbb{E} \left[\frac{1}{2\lambda_t} (v_0 + \eta - p_{t-1}) \left| \frac{1}{2\lambda_t} (v_0 + \eta - p_{t-1}) + z_t, h_t \right. \right] + p_{t-1}
\end{aligned}$$

Since p_{t-1} was the previous expectation of $v_0 + \eta$, $\frac{1}{2\lambda_t}(v_0 + \eta - p_{t-1}) \sim N(0, \left(\frac{1}{2\lambda_t}\right)^2 V_{t-1})$ where V_{t-1} is the previous variance estimate of η . Suppose that $V_{t-1} = \frac{\Omega}{2}$. If $V_t = \frac{\Omega}{2}$ as well then by induction this is the variance of η for all $t > 1$. Then the above expectation becomes

$$p_t = p_{t-1} + 2\lambda_t \frac{\left(\frac{1}{2\lambda_t}\right)^2 \frac{\Omega}{2}}{\left(\frac{1}{2\lambda_t}\right)^2 \frac{\Omega}{2} + \Omega} \omega_t = p_{t-1} + \frac{\left(\frac{1}{2\lambda_t}\right)}{\left(\frac{1}{2\lambda_t}\right)^2 + 2} \omega_t = p_{t-1} + \frac{1}{\left(\frac{1}{2\lambda_t}\right) + 4\lambda_t} \omega_t$$

Then

$$\lambda_t = \frac{1}{\left(\frac{1}{2\lambda_t}\right) + 4\lambda_t} \Rightarrow 8\lambda_t^2 + 1 = 2$$

so that $\lambda_t = \frac{1}{\sqrt{8}}$. Also,

$$V_t = \frac{\left(\frac{1}{2\lambda_t}\right)^2 V_{t-1} \Omega}{\left(\frac{1}{2\lambda_t}\right)^2 V_{t-1} + \Omega} = \frac{\left(\frac{1}{2\lambda_t}\right)^2 \frac{\Omega}{2} \Omega}{\left(\frac{1}{2\lambda_t}\right)^2 \frac{\Omega}{2} + \Omega} = \frac{\left(\frac{8}{4}\right) \frac{1}{2} \Omega}{\left(\frac{8}{4}\right) \frac{1}{2} + 1} = \frac{\Omega}{2}$$

and by induction $V_t = \frac{\Omega}{2}$ for all $t > 1$.

Since it has been shown that $\beta_t = \frac{1}{2^t \lambda_t}$, then by the formulation of λ_t it holds that $\beta_1 = \sqrt{\frac{\Omega}{\sigma^2}}$ and $\beta_t = \frac{\sqrt{2}}{2^{t-1}}$ for $t > 1$ as desired.

Lastly, $p_t = p_{t-1} + \lambda_t \omega_t$ so inductively

$$\begin{aligned} p_t &= p_0 + \sum_{i=1}^t \lambda_i \omega_i = p_0 + \sum_{i=1}^t \lambda_i (x_i + z_i) = p_0 + \sum_{i=1}^t \lambda_i \left(\frac{1}{2^i \lambda_i} \eta + Z_i \right) + \lambda_i z_i \\ &= p_0 + \sum_{i=1}^t \left(\frac{1}{2^i} \right) \eta + \sum_{i=1}^t \lambda_i (Z_i + z_i) = p_0 + \left[1 - \left(\frac{1}{2} \right)^t \right] \eta + Z'_t \end{aligned}$$

where Z'_t is a linear combination of independently normally distributed random variables with mean zero so $Z'_t \sim N(0, V'_t)$ □

Proof of Lemma 1. Suppose the agent demands $x_t = \beta_t \eta$ where $\beta_t = \frac{y}{\lambda_t}$. The market maker sets price so that $p_t = \mathbb{E}[v_t | \beta_t \eta + z_t] = \frac{V_{t-1} \beta_t}{V_{t-1} \beta_t^2 + \Omega}$, where V_{t-1} is the market maker's prior belief of the informative signal's variance. Then

$$\beta_t \lambda_t = \frac{\beta_t^2 V_{t-1}}{\beta_t^2 V_{t-1} + \Omega} = y$$

This yields $\beta_t^2 V_{t-1} = \frac{y\Omega}{1-y}$. When the market maker updates variance of the agent's signal given that liquidity noise $z_t \sim N(0, \Omega)$,

$$V_t = \frac{\beta_t^2 V_{t-1} \Omega}{\beta_t^2 V_{t-1} + \Omega} = \frac{\frac{y\Omega}{1-y} \Omega}{\frac{y\Omega}{1-y} + \Omega} = \frac{y\Omega^2}{y\Omega + (1-y)\Omega} = y\Omega$$

Since this was independent of the value V_{t-1} , variance will be $V_t = y\Omega$ for every period with only the possible exception of V_0 before variance can be updated from the prior belief. □

Proof of Proposition 2. The previous proof shows that the optimal quantity for the agent in period t is $x_t = \frac{\lambda_{t-1} x_{t-1} - \lambda_t z_{t-1}}{2\lambda_t}$. Then for $t = 2$, $x_2 = \frac{\lambda_1 x_1 - \lambda_2 z_1}{2\lambda_2} = \frac{\eta}{4\lambda_2} - \frac{1}{\lambda_2} \left(\frac{1}{2} \right) \lambda_1 z_t$ so that $\beta_2 = \frac{1}{4\lambda_2}$ and $Z_2 = -\frac{1}{\lambda_2} \sum_{i=1}^{2-1} \left(\frac{1}{2} \right)^{t-i} \lambda_i z_i$ so the result holds for $t = 2$.

By induction suppose that $x_t = \frac{1}{2^t \lambda_t} \eta - \frac{1}{\lambda_t} \sum_{i=1}^{t-1} \left(\frac{1}{2} \right)^{t-i} \lambda_i z_i$ for $t \geq 2$. Then $\lambda_t x_t =$

$\frac{1}{2^t}\eta - \sum_{i=1}^{t-1} \left(\frac{1}{2}\right)^{t-i} \lambda_i z_i$ and given that $x_{t+1} = \frac{\lambda_t x_t - \lambda_t z_t}{2\lambda_{t+1}}$,

$$\begin{aligned} x_{t+1} &= \left(\frac{1}{2\lambda_{t+1}}\right) \left[\left(\frac{1}{2^t}\eta - \sum_{i=1}^{t-1} \left(\frac{1}{2}\right)^{t-i} \lambda_i z_i\right) - \lambda_t z_t \right] \\ &= \left(\frac{1}{\lambda_{t+1}}\right) \left[\frac{1}{2^{t+1}}\eta - \sum_{i=1}^{t-1} \left(\frac{1}{2}\right)^{t+1-i} \lambda_i z_i - \frac{1}{2}\lambda_t z_t \right] \\ &= \left(\frac{1}{\lambda_{t+1}}\right) \left[\frac{1}{2^{t+1}}\eta - \sum_{i=1}^t \left(\frac{1}{2}\right)^{t+1-i} \lambda_i z_i \right] \\ &= \left(\frac{1}{\lambda_{t+1}}\right) \left[\frac{1}{2^{t+1}}\eta - \sum_{i=1}^{(t+1)-1} \left(\frac{1}{2}\right)^{(t+1)-i} \lambda_i z_i \right] \end{aligned}$$

so that $x_{t+1} = \frac{1}{2^{t+1}\lambda_{t+1}}\eta - \frac{1}{\lambda_{t+1}} \sum_{i=1}^{(t+1)-1} \left(\frac{1}{2}\right)^{(t+1)-i} \lambda_i z_i$. Thus $Z_{t+1} = -\frac{1}{\lambda_{t+1}} \sum_{i=1}^{(t+1)-1} \left(\frac{1}{2}\right)^{(t+1)-i} \lambda_i z_i$ and by induction the result holds for all $t \leq T$.

As noted in proposition 1

$$p_t = p_0 + \left[1 - \left(\frac{1}{2}\right)^t \right] \eta + Z'_t$$

where $Z'_t = \sum_{i=2}^t \lambda_i Z_i + \sum_{i=1}^t \lambda_i z_i$. From the above $Z_i = -\frac{1}{\lambda_i} \sum_{j=1}^{i-1} \left(\frac{1}{2}\right)^{i-j} \lambda_j z_j$ so that

$$\begin{aligned} \sum_{i=1}^t \lambda_i Z_i &= -\sum_{i=2}^t \sum_{j=1}^{i-1} \left(\frac{1}{2}\right)^{i-j} \lambda_j z_j = -\sum_{j=1}^{t-1} \sum_{i=j+1}^t \left(\frac{1}{2}\right)^{i-j} \lambda_j z_j = -\sum_{j=1}^{t-1} \lambda_j z_j \sum_{i=1}^{t-j} \left(\frac{1}{2}\right)^i \\ &= -\sum_{j=1}^{t-1} \lambda_j z_j \left[1 - \left(\frac{1}{2}\right)^{t-j} \right] \end{aligned}$$

Together,

$$\begin{aligned} \sum_{i=2}^t \lambda_i Z_i + \sum_{i=1}^t \lambda_i z_i &= \sum_{i=1}^t \lambda_i z_i - \sum_{j=1}^{t-1} \lambda_j z_j \left[1 - \left(\frac{1}{2}\right)^{t-j} \right] = \sum_{i=1}^t \lambda_i z_i - \sum_{i=1}^{t-1} \lambda_i z_i \left[1 - \left(\frac{1}{2}\right)^{t-i} \right] \\ &= \lambda_t z_t + \sum_{i=1}^{t-1} \lambda_i z_i \left(\frac{1}{2}\right)^{t-i} = \sum_{i=1}^t \lambda_i z_i \left(\frac{1}{2}\right)^{t-i} \end{aligned}$$

Then $Z'_t = \sum_{i=1}^t \lambda_i z_i \left(\frac{1}{2}\right)^{t-i}$ and $p_t = p_0 + \left[1 - \left(\frac{1}{2}\right)^t\right] \eta + \sum_{i=1}^t \lambda_i z_i \left(\frac{1}{2}\right)^{t-i}$ □

Proof of Proposition 3. Define the partial series $X_T = \frac{1}{T} \sum_{t=1}^T p_t$.

$$\begin{aligned} X_T &= \frac{1}{T} \sum_{t=1}^T p_t = \frac{1}{T} \sum_{t=1}^T \left[p_0 + \left(1 - \left(\frac{1}{2}\right)^t\right) \eta + \sum_{i=1}^t \lambda_i z_i \left(\frac{1}{2}\right)^{t-i} \right] \\ &= p_0 + \eta - \frac{\eta}{T} \left(1 - \left(\frac{1}{2}\right)^T\right) + \frac{1}{T} \sum_{t=1}^T \sum_{i=1}^t \lambda_i z_i \left(\frac{1}{2}\right)^{t-i} \end{aligned}$$

For each T variance of X_T (given that the z_i are independent) is

$$\begin{aligned} \text{Var}(X_T) &= \mathbb{E} \left[\left(\frac{1}{T} \sum_{t=1}^T \sum_{i=1}^t \lambda_i z_i \left(\frac{1}{2}\right)^{t-i} \right)^2 \right] = \frac{1}{T^2} \frac{\Omega}{8} \sum_{t=1}^T \sum_{i=1}^t \left(\frac{1}{2}\right)^{2(t-i)} \\ &= \frac{\Omega}{8T^2} \sum_{t=1}^T \sum_{j=0}^{t-1} \left(\frac{1}{4}\right)^j = \frac{\Omega}{8T^2} \sum_{t=1}^T \left(\frac{1 - \left(\frac{1}{4}\right)^t}{1 - \frac{1}{4}} \right) \\ &= \frac{\Omega}{6T^2} \sum_{t=1}^T \left(1 - \left(\frac{1}{4}\right)^t\right) = \frac{\Omega}{6T} - \frac{\Omega}{6T^2} \left(\left(\frac{1}{4}\right) \frac{1 - \left(\frac{1}{4}\right)^T}{1 - \frac{1}{4}} \right) \\ &= \frac{\Omega}{6T} - \frac{\Omega}{18T^2} \left(1 - \left(\frac{1}{4}\right)^T\right) = \frac{\Omega}{6T} - \frac{\Omega}{18T^2} + \frac{\Omega}{18T^2} \left(\frac{1}{4}\right)^T \end{aligned}$$

Let $\varepsilon > 0$. By Markov's inequality,

$$\begin{aligned} \Pr(|X_t - (p_0 + \eta)| \geq \varepsilon) &\leq \frac{\mathbb{E} \left[\left(p_0 + \eta - \frac{\eta}{T} \left(1 - \left(\frac{1}{2}\right)^T\right) + \frac{1}{T} \sum_{t=1}^T \sum_{i=1}^t \lambda_i z_i \left(\frac{1}{2}\right)^{t-i} - (p_0 + \eta) \right)^2 \right]}{\varepsilon^2} \\ &= \frac{1}{\varepsilon^2} \left\{ \left[\frac{\eta}{T} \left(1 - \left(\frac{1}{2}\right)^T\right) \right]^2 + \mathbb{E} \left[\left(\frac{1}{T} \sum_{t=1}^T \sum_{i=1}^t \lambda_i z_i \left(\frac{1}{2}\right)^{t-i} \right)^2 \right] \right\} \\ &= \frac{\left(\frac{\eta}{T}\right)^2 \left(1 - \left(\frac{1}{2}\right)^T\right)^2 + \frac{\Omega}{6T} - \frac{\Omega}{18T^2} + \frac{\Omega}{18T^2} \left(\frac{1}{4}\right)^T}{\varepsilon^2} \end{aligned}$$

so that

$$\Pr(|X_t - (p_0 + \eta)| < \varepsilon) \leq 1 - \frac{\left(\frac{\eta}{T}\right)^2 \left(1 - \left(\frac{1}{2}\right)^T\right)^2 + \frac{\Omega}{6T} - \frac{\Omega}{18T^2} + \frac{\Omega}{18T^2} \left(\frac{1}{4}\right)^T}{\varepsilon^2} \rightarrow 1 \text{ as } T \rightarrow \infty$$

so that $\text{plim}_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T p_t = p_0 + \eta$.

Note that $\lambda_1^2 = \frac{\sigma^2}{4\Omega}$. This was ignored for expositional clarity because $\text{Var}\left(\frac{1}{T}\lambda_1 z_1\right) = \frac{\sigma^2}{4T^2} \rightarrow 0 \iff \frac{\Omega}{8T^2} \rightarrow 0$. \square

Proof of Proposition 4. Recall that the agent solves the problem $\max_x x \cdot (\eta - \lambda_{t+1}x - (1 - \gamma)\lambda_t \mathbb{E}[y])$. This yields the optimal response to y of

Conjecture a linear price equilibrium of the form $p_t = p_0(1 - \varphi) + \varphi p_{t-1} + \lambda \omega_t$ for $t > 1$ where p_0 is the price before information η entered the market. The agent seeks to maximize

$$\begin{aligned} x \mathbb{E}[v_t - p_t] &= x \mathbb{E}[v_0 + \eta - p_0(1 - \varphi) - \varphi p_{t-1} - \lambda \omega_t] \\ &= x (\mathbb{E}[v_0] + \eta - p_0(1 - \varphi) - \varphi \mathbb{E}[p_{t-1}] - \lambda x) \end{aligned}$$

Which is maximized for $x = \frac{\varphi p_0 + \eta - \varphi \mathbb{E}[p_{t-1}]}{2\lambda}$ since $p_0 = \mathbb{E}[v_0]$. If the agent moves in the first period price is p_0 while second period price is $p_1 = p_0 + \lambda \omega_1$. With prior $\Pr(F) = \gamma_i$ the price $p_{t-1} = \gamma_i p_0 + (1 - \gamma_i) p_1 = p_0 + (1 - \gamma_i) \lambda \omega_1$. For a second period mover, $\mathbb{E}[\omega_1] = \mathbb{E}[x_1]$, demand can be written as

$$\begin{aligned} x &= \frac{\eta + \varphi(p_0 - p_{t-1})}{2\lambda} = \frac{\eta - \varphi(1 - \gamma_i)\lambda \mathbb{E}[x_1]}{2\lambda} \\ &= \frac{\eta}{2\lambda} - \frac{\varphi(1 - \gamma_i)\mathbb{E}[x_1]}{2} \end{aligned}$$

and as a result of the common prior belief expected quantity $\mathbb{E}[x_1]$ is

$$\mathbb{E}[x_1] = \frac{\eta}{2\lambda} - \frac{\gamma_i \varphi}{2} x_t$$

Thus in equilibrium

$$x_i = \frac{2 - \varphi(1 - \gamma_i)}{\lambda[4 - \varphi^2 \gamma_i(1 - \gamma_i)]} \eta \quad \text{and} \quad \mathbb{E}[x_j] = \frac{2 - \varphi \gamma_i}{\lambda[4 - \varphi^2 \gamma_i(1 - \gamma_i)]} \eta$$

so that

$$\beta_i(\gamma_i) = \frac{2 - \varphi(1 - \gamma_i)}{\lambda[4 - \varphi^2 \gamma_i(1 - \gamma_i)]} \quad \text{and} \quad \mathbb{E}[\beta_j(\gamma_i)] = \frac{2 - \varphi \gamma_i}{\lambda[4 - \varphi^2 \gamma_i(1 - \gamma_i)]}$$

The market maker sets price so that $p_t = \mathbb{E}[v_t | \omega_t, h_t]$ so

$$p_t = \mathbb{E}[v_t | \omega_t, h_t] = \mathbb{E}[v_0 + \eta | \omega_t, h_t] = p_0 + \mathbb{E}[\eta | \omega_t, h_t] = p_0 + \frac{1}{\beta} \mathbb{E}[\beta \eta | \beta \eta + z_t, h_t]$$

Recall that $z_t \sim N(0, \Omega)$ and the prior belief on the value is $\eta \sim N(p_{t-1} - p_0, V)$. Then $\beta \eta \sim N(\beta(p_{t-1} - p_0), \beta^2 V)$ and

$$p_t = p_0 + \frac{\Omega \beta(p_{t-1} - p_0) + \beta^2 V \omega_t}{\beta(\Omega + \beta^2 V)} = p_0 + \frac{\Omega(p_{t-1} - p_0) + \beta V \omega_t}{\Omega + \beta^2 V}$$

so that

$$\lambda = \frac{\beta V}{\Omega + \beta^2 V} \quad \text{and} \quad \varphi = \frac{\Omega}{\Omega + \beta^2 V}$$

If the market maker cannot distinguish between agent types price must be taken assuming a weighted average $\beta_t = \mu \beta(\gamma_1) + (1 - \mu) \beta(\gamma_2)$. Moreover if the market is not able to update

agents' posterior probability it must rely on $\gamma_i = \mu$ and $\gamma_j = 1 - \mu$. Then

$$\beta = \frac{2(1 - \mu(1 - \mu)\varphi)}{\lambda(4 - \mu(1 - \mu)\varphi^2)}$$

Moreover, if the market maker believes both investors are equally likely to move first, then $\mu = \frac{1}{2}$ and this reduces to $\beta = \frac{2}{\lambda(4+\varphi)}$. Given that $\beta\lambda = 1 - \varphi = \frac{\beta^2 V}{\Omega + \beta^2 V}$, this solves to $\beta^2 = \frac{(1-\varphi)\Omega}{V\varphi}$. Notice also that for any V_{t-1} updated variance is

$$V_t = \frac{\beta^2 V_{t-1} \Omega}{\beta^2 V_{t-1} + \Omega} = \frac{\left(\frac{(1-\varphi)\Omega}{1-\varepsilon_T}\right) \Omega}{\frac{(1-\varphi)\Omega}{1-\varepsilon_T} + \Omega} = \frac{(1-\varphi)\Omega}{(1-\varphi) + (1-\varepsilon_T)} = (1-\varphi)\Omega$$

so that $\beta = \sqrt{\frac{1}{\varphi}}$ and $\lambda = \sqrt{\varphi}(1 - \varphi)$. From the fact that $\beta\lambda = \frac{2}{4+\varphi} = 1 - \varphi$, $\varphi^2 + 3\varphi - 2 = 0$ so that $\varphi = \frac{\sqrt{17}-3}{2} \approx 0.562$ and $\lambda \approx 0.329$

□

Proof of Proposition 5. Conjecture a linear price equilibrium of the form $p_t = p_0(1 - \varphi) + \varphi p_{t-1} + \lambda \omega_t$ where p_0 is the price before information η entered the market. The agent seeks to maximize

$$\begin{aligned} x\mathbb{E}[v_t - p_t] &= x\mathbb{E}[v_0 + \eta - p_0(1 - \varphi) - \varphi p_{t-1} - \lambda \omega_t] \\ &= x(\mathbb{E}[v_0] + \eta - p_0(1 - \varphi) - \varphi p_{t-1} - \lambda x) \end{aligned}$$

Which is maximized for $x = \frac{\varphi p_0 + \eta - \varphi p_{t-1}}{2\lambda}$ since $p_0 = \mathbb{E}[v_0]$. If the agent moves in the second period price is $p_1 = p_0 + \lambda \omega_1$. If the agent moves in the third period then price is $p_2 = p_0 + \lambda \omega_2 + \varphi \lambda \omega_1$. Inductively if the agent moves in period t then $t - 1$ agents move before and $p_{t-1} = p_0 + \sum_{i=1}^{t-1} \lambda \varphi^{(t-1)-i} \omega_i$.

If there are T periods and each agent has the belief μ_t that they are moving in period t and since $\omega_t = x_t + z_t$ and z_t are independently distributed with zero mean, $\omega_t = x_t$

is the expectation for each period. Moreover since the agent has no additional information than the market before the first period when the signal η was released then $\mathbb{E}[v_0] = p_0$. Then optimal demand becomes

$$x = \frac{1}{2\lambda}(\varphi p_0 + \eta - \varphi p_{t-1}) = \frac{1}{2\lambda} \left[\eta - \sum_{t=2}^T \mu_t \left(\sum_{i=1}^{t-1} \lambda \varphi^{t-i} x_i \right) \right]$$

where the outer summation starts from $t = 2$ because when $t = 1$ the agent moves in the first period and there is no previous demand. Imposing agents' symmetry and the uniform belief over their period of movement, $x_i = x$ for all i and $\mu_t = \frac{1}{T}$ for all t . Demand then reduces to

$$\begin{aligned} x &= \frac{1}{2\lambda} \left[\eta - \sum_{t=2}^T \frac{1}{T} \sum_{i=1}^{t-1} \lambda \varphi^{t-i} x \right] = \frac{1}{2\lambda} \left[\eta - \frac{\lambda x}{T} \sum_{t=2}^T \sum_{i=1}^{t-1} \varphi^{t-i} \right] \\ &= \frac{1}{2\lambda} \left[\eta - \left(\frac{\lambda x \varphi}{T} \right) \frac{T(1-\varphi) - (1-\varphi^T)}{(1-\varphi)^2} \right] = \frac{\eta}{2\lambda} - \frac{x\varphi}{2} \left(\frac{1}{1-\varphi} - \frac{(1-\varphi^T)}{T(1-\varphi)^2} \right) \end{aligned}$$

which simplifies to

$$x = \frac{(1-\varphi)\eta}{\lambda \left[(2-\varphi) - \frac{\varphi(1-\varphi^T)}{T(1-\varphi)} \right]}$$

The market maker sets price such that $p_t = \mathbb{E}[v_t|\omega_t]$. Then

$$p_t = \mathbb{E}[v_t|\omega_t] = \mathbb{E}[v_0 + \eta|x_t + z_t] = p_0 + \mathbb{E}[\eta|\beta\eta + z_t] = p_0 + \frac{1}{\beta} \mathbb{E}[\beta\eta|\beta\eta + z_t]$$

Recall that $z_t \sim N(0, \Omega)$ and the prior belief on the value is $\eta \sim N(p_{t-1} - p_0, V)$. Then $\beta\eta \sim N(\beta(p_{t-1} - p_0), \beta^2 V)$ and

$$p_t = p_0 + \frac{\Omega\beta(p_{t-1} - p_0) + \beta^2 V \omega_t}{\beta(\Omega + \beta^2 V)} = p_0 + \frac{\Omega(p_{t-1} - p_0) + \beta V \omega_t}{\Omega + \beta^2 V}$$

so that

$$\lambda = \frac{\beta V}{\Omega + \beta^2 V} \text{ and } \varphi = \frac{\Omega}{\Omega + \beta^2 V}$$

For notational convenience let $\varepsilon_T = \frac{\varphi(1-\varphi^T)}{T(1-\varphi)}$. Then $\beta = \frac{1-\varphi}{\lambda[(2-\varphi)-\varepsilon_T]}$ and

$$\begin{aligned} \beta\lambda &= \frac{1-\varphi}{(2-\varphi)-\varepsilon_T} = \frac{\beta^2 V}{\Omega + \beta^2 V} \\ \beta^2 V[(2-\varphi)-\varepsilon_T] &= (1-\varphi)(\Omega + \beta^2 V) \\ \beta^2 &= \frac{(1-\varphi)\Omega}{V(1-\varepsilon_T)} \end{aligned}$$

Given that $\varphi = \frac{\Omega}{\Omega + \beta^2 V}$ the above implies that

$$\begin{aligned} \varphi &= \frac{\Omega}{\Omega + \frac{(1-\varphi)\Omega}{1-\varepsilon_T}} = \frac{1-\varepsilon_T}{1-\varepsilon_T + (1-\varphi)} \\ \varphi(2-\varphi-\varepsilon_T) &= 1-\varepsilon_T \\ (1-\varphi)^2 &= \varepsilon_T(1-\varphi) \\ \varphi &= 1-\varepsilon_T \end{aligned}$$

Notice also that for any V_{t-1} updated variance is

$$V_t = \frac{\beta^2 V_{t-1} \Omega}{\beta^2 V_{t-1} + \Omega} = \frac{\left(\frac{(1-\varphi)\Omega}{1-\varepsilon_T}\right) \Omega}{\frac{(1-\varphi)\Omega}{1-\varepsilon_T} + \Omega} = \frac{(1-\varphi)\Omega}{(1-\varphi) + (1-\varepsilon_T)} = (1-\varphi)\Omega$$

Then from above

$$\beta^2 = \frac{(1-\varphi)\Omega}{V(1-\varepsilon_T)} = \beta^2 = \frac{(1-\varphi)\Omega}{(1-\varphi)\Omega(1-\varepsilon_T)} = \frac{1}{\varphi}$$

so that $\beta = \sqrt{\frac{1}{\varphi}}$. It can be seen given the updating of the market maker that $\beta\lambda = 1 - \varphi$ so

that $\lambda = \sqrt{\varphi}(1 - \varphi)$ □

Proof of Proposition 6. From the above the price can be expanded as

$$\begin{aligned}
p_t &= (1 - \varphi)p_0 + \lambda\omega_t + \varphi p_{t-1} = (1 - \varphi)p_0 + \varphi p_0 + \sum_{i=1}^t \varphi^{t-i} \lambda \omega_t = p_0 + \sum_{i=1}^t \varphi^{t-i} \lambda (x + z_t) \\
&= p_0 + \sum_{i=1}^t \varphi^{t-i} \lambda x + \sum_{i=1}^t \varphi^{t-i} \lambda z_t = p_0 + \lambda x \sum_{j=0}^{t-1} \varphi^j + \sum_{i=1}^t \varphi^{t-i} \lambda z_t \\
&= p_0 + \lambda x \frac{1 - \varphi^t}{1 - \varphi} + \sum_{i=1}^t \varphi^{t-i} \lambda z_t = p_0 + \lambda \beta \eta \frac{1 - \varphi^t}{1 - \varphi} + \sum_{i=1}^t \varphi^{t-i} \lambda z_t \\
&= p_0 + (1 - \varphi) \eta \frac{1 - \varphi^t}{1 - \varphi} + \sum_{i=1}^t \varphi^{t-i} \lambda z_t = p_0 + (1 - \varphi^T) \eta + \sum_{i=1}^t \varphi^{t-i} \lambda z_t
\end{aligned}$$

and so $\mathbb{E}[p_t] = p_0 + (1 - \varphi^T) \eta$ □

Lemma 2.2. *If $x, \varphi \in (0, 1)$ and $\lim_{t \rightarrow \infty} x^t = \lim_{t \rightarrow \infty} \varphi^t = p$ for some $p \in (0, 1)$, then $\lim_{t \rightarrow \infty} t(1 - \varphi)^2 = \lim_{t \rightarrow \infty} t(1 - x)^2$ if such a limit exists.*

Proof. Let $\varepsilon > 0$ small enough so $0 < p - \varepsilon < p + \varepsilon < 1$ and choose $T \in \mathbb{N}$ such that $t \geq T$ implies both $p - \varepsilon < x^t < p + \varepsilon$ and $p - \varepsilon < \varphi^t < p + \varepsilon$. Then $(p - \varepsilon)^{1/t} < x < (p + \varepsilon)^{1/t}$, $(p - \varepsilon)^{1/t} < \varphi < (p + \varepsilon)^{1/t}$, and moreover $|\varphi - x| < |(p + \varepsilon)^{1/t} - (p - \varepsilon)^{1/t}|$. Then

$$\begin{aligned}
|t(1 - \varphi)^2 - t(1 - x)^2| &= t|\varphi^2 - x^2 - 2(\varphi - x)| = t|(\varphi - x)((\varphi + x) - 2)| \\
&< 4t|\varphi - x| < 4t|(p + \varepsilon)^{1/t} - (p - \varepsilon)^{1/t}|
\end{aligned}$$

Since this applies for all $t \geq T$ it will also apply in the limit. Note that

$$\begin{aligned}
\frac{d}{dt} k^{c/t} &= \frac{d}{dt} \exp \left\{ \frac{c}{t} \ln(k) \right\} = \exp \left\{ \frac{c}{t} \ln(k) \right\} \frac{-c}{t^2} \ln(k) \\
&= -\frac{c}{t^2} \ln(k) k^{c/t}
\end{aligned}$$

Then using L'Hôpital's Rule

$$\begin{aligned}
|t(1 - \varphi)^2 - t(1 - x)^2| &< \lim_{t \rightarrow \infty} 4t|(p + \varepsilon)^{1/t} - (p - \varepsilon)^{1/t}| = \lim_{t \rightarrow \infty} \frac{4|(p + \varepsilon)^{1/t} - (p - \varepsilon)^{1/t}|}{\frac{1}{t}} \\
&= \lim_{t \rightarrow \infty} \frac{4| -\frac{1}{t^2} \ln(p + \varepsilon)(p + \varepsilon)^{1/t} + \frac{1}{t^2} \ln(p - \varepsilon)(p - \varepsilon)^{1/t}|}{\frac{-1}{t^2}} \\
&= \lim_{t \rightarrow \infty} 4|\ln(p + \varepsilon)(p + \varepsilon)^{1/t} - \ln(p - \varepsilon)(p - \varepsilon)^{1/t}| \\
&= 4 \ln \left(\frac{p + \varepsilon}{p - \varepsilon} \right)
\end{aligned}$$

since both $(p - \varepsilon)^{1/t}$ and $(p + \varepsilon)^{1/t}$ converge to 1. For any $\delta > 0$ letting $\varepsilon < \frac{p(\exp\{\delta/4\}-1)}{\exp\{\delta/4\}+1}$ yields the result that $|t(1 - \varphi)^2 - t(1 - x)^2| < \delta$ and thus $\lim_{t \rightarrow \infty} t(1 - \varphi)^2 = \lim_{t \rightarrow \infty} t(1 - x)^2$. \square

Lemma 2.3. *In equilibrium, φ implicitly defined by $1 - \varphi = \frac{\varphi(1 - \varphi^T)}{T(1 - \varphi)}$ must converge to 1.*

Proof. In equilibrium $\varphi = \frac{\Omega}{\Omega + \beta^2 V}$ so $\varphi \in (0, 1)$. If $\lim_{T \rightarrow \infty} \varphi = p \in [0, 1)$ then $\varphi^T \rightarrow 0$ and $1 - \varphi = \frac{\varphi(1 - \varphi^T)}{T(1 - \varphi)} \rightarrow 0$ so $\varphi \rightarrow 1$. Thus it must be that $\varphi \rightarrow 1$. \square

Corollary 2.2. *In equilibrium, $\varepsilon_T = \frac{\varphi(1 - \varphi)}{T(1 - \varphi)}$ must converge to 0.*

Lemma 2.4. *For the above where $1 - \varphi = \frac{\varphi(1 - \varphi^T)}{T(1 - \varphi)}$, $\lim_{T \rightarrow \infty} \varphi^T = 0$.*

Proof. Since $\varphi \in (0, 1)$, $\lim_{T \rightarrow \infty} \varphi^T \in [0, 1]$. Suppose the series converges to some number inside the interval so $\lim_{T \rightarrow \infty} \varphi^T = p \in (0, 1)$. By definition $(1 - \varphi) = \frac{\varphi(1 - \varphi^T)}{T(1 - \varphi)}$ so that

$$\lim_{T \rightarrow \infty} T(1 - \varphi)^2 = \lim_{T \rightarrow \infty} \varphi(1 - \varphi^T) = 1 - p$$

Consider $x = p^{1/T}$. Clearly x^T converges to p and since $p \in (0, 1)$ $\lim_{T \rightarrow \infty} x = 1$. Then by the above lemma since the limit exists $\lim_{T \rightarrow \infty} T(1 - x)^2 = 1 - p$. However,

$$\begin{aligned}
\lim_{T \rightarrow \infty} T(1 - x)^2 &= \lim_{T \rightarrow \infty} \frac{1 - 2p^{1/T} + p^{2/T}}{\frac{1}{T}} = \lim_{T \rightarrow \infty} \frac{2\frac{1}{T^2} \ln(p)p^{1/T} - \frac{2}{T^2} \ln(p)p^{2/T}}{\frac{-1}{T^2}} \\
&= \lim_{T \rightarrow \infty} 2 \ln(p)(p^{2/T} - p^{1/T}) = 0
\end{aligned}$$

since both $p^{1/T} \rightarrow 1$ and $p^{2/T} \rightarrow 1$. This contradiction shows that p cannot be interior so that $\lim_{T \rightarrow \infty} \varphi^T \in \{0, 1\}$.

Suppose then that $\lim_{T \rightarrow \infty} \varphi^T = 1$. Recall that given the definition of ε_T ,

$$\begin{aligned} \lim_{T \rightarrow \infty} (1 - \varphi) &= \lim_{T \rightarrow \infty} \frac{\varphi(1 - \varphi^T)}{T(1 - \varphi)} = \lim_{T \rightarrow \infty} \frac{\varphi'(T) - \varphi^{T+1}[\ln(\varphi) + (T+1)\varphi'(T)/\varphi]}{(1 - \varphi) - T\varphi'(T)} \\ &= \lim_{T \rightarrow \infty} \frac{\varphi'(T) - \varphi^{T+1} \ln(\varphi) - (T+1)\varphi'(T)\varphi^T}{(1 - \varphi) - T\varphi'(T)} \\ &= \frac{\lim_{T \rightarrow \infty} [\varphi'(T)(1 - \varphi^T - T\varphi^T) - \varphi^{T+1} \ln(\varphi)]}{\lim_{T \rightarrow \infty} [(1 - \varphi) - T\varphi'(T)]} \\ &= \lim_{T \rightarrow \infty} \frac{\varphi'(T)(1 - \varphi^T - T\varphi^T)}{-T\varphi'(T)} = 1 \end{aligned}$$

since $\varphi^T \rightarrow 1, \varphi \rightarrow 1$. Thus $1 - \varphi \rightarrow 1$ so $\varphi \rightarrow 0$. Then it must be that φ converges to something less than 1, but if this is so then $\frac{\varphi(1-\varphi^T)}{T(1-\varphi)} \rightarrow 0$ which contradicts that $\lim_{T \rightarrow \infty} \varphi < 1$. The only remaining possibility is that $\lim_{T \rightarrow \infty} \varphi^T = 0$. \square

Proof of Proposition 7. Combining Proposition 6 and lemmas 2-4

$$\lim_{t \rightarrow \infty} \mathbb{E}[p_t] = \lim_{t \rightarrow \infty} p_0 + (1 - \varphi^T)\eta = p_0 + \eta.$$

Define the partial series $X_T = \frac{1}{T} \sum_{t=1}^T p_t$. Then

$$\begin{aligned} X_T &= \frac{1}{T} \sum_{t=1}^T p_t = \frac{1}{T} \sum_{t=1}^T \left[p_0 + (1 - \varphi^t)\eta + \sum_{i=1}^t \varphi^{t-i} \lambda z_t \right] \\ &= p_0 + \eta - \frac{\varphi(1 - \varphi^T)}{T(1 - \varphi)} \eta + \frac{1}{T} \sum_{t=1}^T \sum_{i=1}^t \varphi^{t-i} \lambda z_t \end{aligned}$$

For each T variance of X_T (given that the z_i are independent) is

$$\begin{aligned}
\text{Var}(X_T) &= \mathbb{E} \left[\left(\frac{1}{T} \sum_{t=1}^T \sum_{i=1}^t \lambda_i z_i \varphi^{t-i} \right)^2 \right] = \frac{1}{T^2} \lambda^2 \Omega \sum_{t=1}^T \sum_{i=1}^t \varphi^{2(t-i)} \\
&= \frac{\varphi(1-\varphi)^2 \Omega}{T^2} \sum_{t=1}^T \sum_{j=0}^{t-1} (\varphi^2)^j = \frac{\varphi(1-\varphi)^2 \Omega}{T^2} \sum_{t=1}^T \left(\frac{1 - (\varphi^2)^t}{1 - \varphi^2} \right) \\
&= \frac{\varphi(1-\varphi)^2 \Omega}{T^2(1-\varphi^2)} \sum_{t=1}^T (1 - (\varphi^2)^t) = \frac{\varphi(1-\varphi)\Omega}{T(1+\varphi)} - \frac{\varphi(1-\varphi)\Omega}{T^2(1+\varphi)} \sum_{t=1}^T (\varphi^2)^t \\
&= \frac{\varphi(1-\varphi)\Omega}{T(1+\varphi)} - \frac{\varphi(1-\varphi)\Omega}{T^2(1+\varphi)} \left(\frac{\varphi^2(1-\varphi^{2T})}{1-\varphi^2} \right) = \frac{\varphi(1-\varphi)\Omega}{T(1+\varphi)} - \frac{\varphi^3(1-\varphi^{2T})\Omega}{T^2(1+\varphi)^2} \\
&= \frac{\varphi(1-\varphi)\Omega}{T(1+\varphi)} - \left(\frac{1-\varphi^T}{T(1+\varphi)} \right) \left(\frac{1+\varphi^T}{T(1+\varphi)} \right) \varphi^3 \Omega
\end{aligned}$$

Let $\varepsilon > 0$. By Markov's inequality,

$$\begin{aligned}
\Pr(|X_t - (p_0 + \eta)| \geq \varepsilon) &\leq \frac{\mathbb{E} \left[\left(p_0 + \eta - \frac{\varphi(1-\varphi^T)}{T(1-\varphi)} \eta + \frac{1}{T} \sum_{t=1}^T \sum_{i=1}^t \varphi^{t-i} \lambda z_t - (p_0 + \eta) \right)^2 \right]}{\varepsilon^2} \\
&= \frac{1}{\varepsilon^2} \left\{ \left[\frac{\varphi(1-\varphi^T)}{T(1-\varphi)} \eta \right]^2 + \mathbb{E} \left[\left(\frac{1}{T} \sum_{t=1}^T \sum_{i=1}^t \varphi^{t-i} \lambda z_t \right)^2 \right] \right\} \\
&= \frac{\left(\frac{\varphi(1-\varphi^T)}{T(1-\varphi)} \right)^2 \eta^2 + \frac{\varphi(1-\varphi)\Omega}{T(1+\varphi)} - \left(\frac{1-\varphi^T}{T(1+\varphi)} \right) \left(\frac{1+\varphi^T}{T(1+\varphi)} \right) \varphi^3 \Omega}{\varepsilon^2}
\end{aligned}$$

so that

$$\Pr(|X_t - (p_0 + \eta)| < \varepsilon) \leq 1 - \frac{\left(\frac{\varphi(1-\varphi^T)}{T(1-\varphi)} \right)^2 \eta^2 + \frac{\varphi(1-\varphi)\Omega}{T(1+\varphi)} - \left(\frac{1-\varphi^T}{T(1+\varphi)} \right) \left(\frac{1+\varphi^T}{T(1+\varphi)} \right) \varphi^3 \Omega}{\varepsilon^2} \rightarrow 1$$

as $T \rightarrow \infty$ since $\varphi \rightarrow 1$ by Lemma 3, $\varphi^T \rightarrow 0$ by Lemma 4, and $0 \leq \frac{\varphi(1-\varphi^T)\Omega}{T(1+\varphi)} \leq \frac{\varphi(1-\varphi^T)\Omega}{T(1-\varphi)} \rightarrow 0$ by Lemma 3 which implies $\frac{\varphi(1-\varphi^T)\Omega}{T(1+\varphi)} \rightarrow 0$. Therefore $\text{plim}_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T p_t = p_0 + \eta$. \square

Proof of Proposition 8. Since the market maker is not aware of the confidence bias it

still sets $\lambda = \sqrt{\varphi}(1 - \varphi)$ and $\varphi = 1 - \frac{\varphi(1-\varphi^T)}{T(1-\varphi)}$. From the proof of Proposition 5 we was that the agent's optimal demand for the asset is

$$x_t = \frac{1}{2\lambda} \left[\eta - \sum_{t=2}^T \mu_t \left(\sum_{i=1}^{t-1} \lambda \varphi^{t-i} x_i \right) \right]$$

Now with a weight γ put on being a first mover, $\mu_1 = \frac{\gamma}{T}$ and all other beliefs $\mu_t = \frac{T-\gamma}{T(T-1)}$, and supposing that all other x_i are symmetric,

$$\begin{aligned} x &= \frac{1}{2\lambda} \left[\eta - \sum_{t=2}^T \frac{T-\gamma}{T(T-1)} \sum_{i=1}^{t-1} \lambda \varphi^{t-i} x_i \right] = \frac{\eta}{2\lambda} - \frac{T-\gamma}{T(T-1)} \frac{x_i}{2} \sum_{t=2}^T \sum_{j=1}^{t-1} \varphi^j \\ &= \frac{\eta}{2\lambda} - \frac{T-\gamma}{T(T-1)} \frac{x_i}{2} \sum_{t=2}^T \frac{\varphi - \varphi^T}{1-\varphi} = \frac{\eta}{2\lambda} - \frac{T-\gamma}{T(T-1)} \frac{x_i \varphi}{2(1-\varphi)} \left(T - \frac{1-\varphi^T}{1-\varphi} \right) \end{aligned}$$

Imposing symmetry of $x = x_i$ demand becomes

$$2x(1-\varphi) = \frac{(1-\varphi)\eta}{\lambda} - x \frac{T-\gamma}{T-1} \left(\varphi - \frac{\varphi(1-\varphi^T)}{T(1-\varphi)} \right)$$

which simplifies to

$$x = \frac{(1-\varphi)\eta}{\frac{\lambda}{T-1} [(T-1) - (2\varphi-1)(\gamma-1)]}$$

□

Proof of Proposition 9. From the proof of Proposition 6 we determined that

$$p_t = p_0 + \lambda x \frac{1-\varphi^t}{1-\varphi} + \sum_{i=1}^t \varphi^{t-i} \lambda z_t$$

So with demand $x = \frac{(1-\varphi)\eta}{\frac{\lambda}{t-1}[(t-1)-(2\varphi-1)(\gamma-1)]}$ expected price in time t is

$$\mathbb{E}[p_t] = p_0 + \frac{(1-\varphi)\eta}{\frac{1}{t-1}[(t-1)-(2\varphi-1)(\gamma-1)]} \left(\frac{1-\varphi^t}{1-\varphi} \right)$$

Then

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathbb{E}[p_t] &= \lim_{t \rightarrow \infty} \left\{ p_0 + \frac{(1-\varphi^t)\eta}{\frac{1}{t-1}[(t-1)-(2\varphi-1)(\gamma-1)]} \right\} \\ &= p_0 + \frac{\lim_{t \rightarrow \infty} (1-\varphi^t)\eta}{\lim_{t \rightarrow \infty} \frac{1}{t-1}[(t-1)-(2\varphi-1)(\gamma-1)]} = p_0 + \eta \end{aligned}$$

since $\varphi \rightarrow 1$, and $\varphi^t \rightarrow 0$. Thus $\lim_{t \rightarrow \infty} \mathbb{E}[p_t] = p_0 + \eta$.

Probability limit result achieved by applying Markov's law as in Proposition 7. □

Proof of Proposition 10. From the proof of Proposition 8 we saw that

$$x = \frac{\eta}{2\lambda} - \frac{T-\gamma}{T(T-1)} \frac{x_i \varphi}{2(1-\varphi)} \left(T - \frac{1-\varphi^T}{1-\varphi} \right)$$

If the agent believes all others act as though they have a uniform distribution over position

$t \leq T$, then $x_i = \frac{(1-\varphi)\eta}{\lambda}$ and

$$\begin{aligned} x &= \frac{\eta}{2\lambda} - \left(\frac{T-\gamma}{T-1} \right) \frac{(1-\varphi)\eta}{2(1-\varphi)\lambda} \left(\varphi - \frac{\varphi(1-\varphi^T)}{T(1-\varphi)} \right) \\ &= \frac{\eta}{2\lambda} \left[\frac{2(T-\gamma)(1-\varphi)}{T-1} + \frac{\gamma-1}{T-1} \right] \\ &= \frac{\eta}{\lambda} \left[(1-\varphi) + \frac{(\gamma-1)(2\varphi-1)}{2(T-1)} \right] \end{aligned}$$

□

Proof of Proposition 11. From the proof of Proposition 6 we determined that

$$p_t = p_0 + \lambda x \frac{1 - \varphi^t}{1 - \varphi} + \sum_{i=1}^t \varphi^{t-i} \lambda z_t$$

Given that demand is $x = \frac{(1-\varphi)\eta}{\lambda} + \frac{(\gamma-1)(2\varphi-1)\eta}{2\lambda(t-1)}$ from Proposition 10, the expected price in time t becomes

$$\mathbb{E}[p_t] = p_0 + (1 - \varphi^t)\eta + \left(\frac{(1 - \varphi^t)}{(t-1)(1-\varphi)} \right) \frac{(\gamma-1)(2\varphi-1)}{2}\eta$$

so that

$$\lim_{t \rightarrow \infty} \mathbb{E}[p_t] = p_0 + \eta + 0$$

since $\varphi^t \rightarrow 0$, and $\left(\frac{1-\varphi^t}{(t-1)(1-\varphi)} \right) \rightarrow 0$. Thus $\lim_{t \rightarrow \infty} \mathbb{E}[p_t] = p_0 + \eta$.

Probability limit result achieved by applying Markov's law as in Proposition 7. □

CHAPTER 3

SOCIAL LEARNING WITH LIMITED HISTORIES

Abstract: I adapt the standard observational learning environment and introduce a limited history of observation. When agents can only observe the action of the previous agent, complete learning still occurs but with a loss of welfare. When a limited history is coupled with uncertainty over position in the queue of actors, welfare further drops - increasing in uncertainty - but complete learning still occurs in the limit. These results are illustrated with a canonical linear model but learning holds in a more general setting satisfying the usual social learning assumptions.

3.1 INTRODUCTION

The social learning literature has highlighted the tension between the presence of sufficient information for full learning of the true state, and the rationality of agents ignoring their private information and joining a herd, leading to incomplete learning with positive probability. This literature was sparked by the work of Banerjee (1992) [2] and Bikhchandani, Hirshleifer, and Welch (1992) [3], who introduce a framework of identical agents receiving independent and identically distributed signals. The critical insight is that as a result of observing the full history of previous actions, agents may find it rational to ignore their private signal and infer the state of the world from the actions of previous decision makers. Since agents are identical this implies all future agents face the same decision, and an “information cascade” occurs whereby all agents ignore their private information.

There have been many extensions to this framework that allow for heterogeneous agent types, limited observable histories, or more generally the formation of networks of viewable histories, either exogenously formed or formed endogenously subject to a cost. However, little attention has been given to the social learning problem in which agents do not have full information about their position in the chain of decision makers.

The classic social learning example of deciding whether to eat at a restaurant or stay home suffices in demonstrating how strong are the assumptions of the social learning model. The story goes that a new restaurant opens in town and patrons must decide whether to visit the new eatery or stay home by using their private signal and observing the choices of others. As it goes, the agents later in line for dinner are able to infer the signals of earlier agents through their actions, with such inference either buttressing or altogether overriding their own private signal. But of course this depends on a full observation of the history of actions.

This assumption is actually a composite of two assumptions. The first is that every

previous action is observed. This assumption may be reasonable for early diners, but the idea of later diners spending all evening staking out the restaurant in order to make their decision is implausible. Even if such observation were possible indirectly, through word of mouth or consolidated review sources (i.e. Yelp or Google), observations are sure to get lost as the restaurant remains open over a longer period of time. And as time goes on, not only do observations get lost, but the number of choices get lost, highlighting the second assumption in social learning that agents know their position. It may be possible on opening night for a diner to know if they are among the first hundred patrons, but after the restaurant has been open a year diners might not even know if they are among the first hundred thousand!

It is the goal of this paper to investigate a social learning environment in which agents have only a limited history of observable actions. While unbounded signals ensure complete learning in the case of a fully observable history, a limited history leads to faster learning (in terms of convergence of decision thresholds), but a lower expected utility. Complete learning is also shown in the case of positional uncertainty but at the cost of a further decline in expected utility.

3.2 RELATED LITERATURE

The model presented here is most similar to the framework of Bikhchandani, Hirshleifer, and Welch (1992) [3] (henceforth BHW). BHW present a framework in which agents observe conditionally independent signals, as well as the actions of all previous agents. They show that rational agents enter into a herd, ignoring their private information when deciding on the choice of action, with positive probability.

The work of Banerjee (1992) [2] also helped spark the herding literature. This model differs in that agents face a continuum of choices, with only one (unknown) correct choice for the state of the world. In addition, only some agents receive an informative signal, and

it is only known to the agent whether or not she has a signal. As in BHW, Banerjee shows that agents rationally converge to a herd, even if they have an informative signal. This result is again driven by the fact that signals are not perfectly informative, so it may be more reasonable to discard private information by inferring the state of the world from the actions of others.

Smith and Sørensen (2000) [7] investigate social learning with heterogeneous agents. They discuss the concept of full learning, where the probability of taking the right action tends to 1 as the number of agents increases. They identify the importance of unbounded signals: if signals can be arbitrarily precise, there is always a probability of a strong signal overturning a herd. This differs from BHW in that signal precision is heterogeneous, so even in the presence of a strong herd a well-informed agent can change public opinion.

The notion of endogenous timing in social learning was explored by Gul and Lundholm (1995) [5]. They investigate a setting in which agents receive payoff-relevant signals, and attempt to guess the sum of these signals. Agents choose when to make their prediction, with an associated cost to waiting. They show that since agents with higher signals perceive a higher opportunity cost to waiting, they will act sooner. Given that higher signals convey more information, endogenizing the timing actually results in the efficient ordering of agents' actions.

Limited observable histories has been investigated through the idea of a “network” which describes the set of actions a given agent can observe. Acemoglu et al. (2011) [1] identify the unboundedness of networks as the condition that guarantees full learning. That is, as long as the size of a given agent's network is not bounded by some integer, convergence to the correct action occurs at the limit, so there is full learning. Song (2014) [8] arrives at a similar finding in a setting where networks are formed endogenously subject to a cost.

An early theoretical work addressing social learning with limited histories is Çelen, B. and Kariv, S. (2004) [4], wherein agents decide between actions sequentially with the aid

of a private signal and observation of the previous action. This model, however, featured a payoff as the sum of signals, a departure from the traditional framework of a correct action for each state. In fact in their model the state itself changes as the sum of signals oscillates between negative and positive. The present work attempts to apply the traditional social learning framework to a setting of limited observation, showing that complete learning still holds under the usual assumptions.

Perhaps the most closely related work is Monzón and Rapp (2014). In this model agents receive private signals but their observational history is limited to a random sampling of previous decision makers. They show that social learning persists under positional uncertainty, provided that action samples satisfy a stationarity assumption whereby they cannot be from the too distant past. This work also demonstrates the welfare loss of positional uncertainty. The focus of this work is a stationarity assumption, whereas at present we focus on the role of beliefs over position and how changes in these beliefs affects learning and welfare in equilibrium.

3.3 MODEL

Suppose N agents decide sequentially between one of two actions, $a \in \mathcal{A} = \{0, 1\}$ in an exogenously determined order. There are two states of the world, $\Omega = \{l, h\}$, and all agents agree that it is preferable to take action $a = 1$ in state h and $a = 0$ in state l . As such, agents share common risk-neutral vN-M utilities $u_n(1, h) = u_n(0, l) = 1$ and $u_n(0, h) = u_n(1, l) = 0$. In addition agents share a common flat prior $\Pr(h) = \Pr(l) = \frac{1}{2}$.

3.3.1 PRIVATE SIGNALS

Before deciding on an action each agent receives a private signal $\theta \in [-1, 1]$ about the state of the world. The signal θ is distributed according to $F = \{F_\omega(\theta)\}_{\omega \in \Omega}$, and conditional on

the state ω signals are drawn independently. We will assume the signal distributions are continuous and admit density functions.

Assumption 3.1 (C^1). *Signal distributions F_l and F_h are continuously differentiable. Denote their densities as f_l and f_h , respectively.*

We will require the usual (strict) monotone likelihood ratio property suggesting it is more likely to receive high values of signal θ in state h and low values in state l .

Assumption 3.2 (MLRP). *The distribution functions f_l and f_h satisfy the (strict) monotone likelihood ratio property in the sense that $\frac{f_h(\theta)}{f_l(\theta)}$ is strictly increasing in θ .*

We will assume that $F_h(\theta)$ and $F_l(\theta)$ are mutually absolutely continuous on the interval $[-1, 1]$. While this rules out any signal being perfectly informative of the state, we will assume that signals can come pretty close in the sense of an unbounded likelihood ratio.

Assumption 3.3 (Unbounded signal strength). *The informativeness of signal θ is unbounded in the sense that*

$$\lim_{\theta \rightarrow -1} \frac{f_h(\theta)}{f_l(\theta)} = 0 \quad \text{and} \quad \lim_{\theta \rightarrow 1} \frac{f_h(\theta)}{f_l(\theta)} = \infty$$

And finally, to avoid diverting the analysis from the implications of the learning environment through unnecessary complication, we will assume that the state dependent distributions are mutually symmetric about zero, though the results hold in the absence of this assumption.

Assumption 3.4 (Mutual Symmetry). *Signal distributions F_l and F_h are mutually symmetric in the sense that for all $\theta \in \text{supp}(F)$, $f_l(\theta) = f_h(-\theta)$.*

3.3.2 OBSERVABLE HISTORIES

In addition to receiving the conditionally independent private signal θ , agents observe a history $H_n \subseteq \{a_1, a_2, \dots, a_{n-1}\}$ of actions of preceding agents. Action profiles of the $n - 1$

agents who have moved by the start of period n take realizations $A_{n-1} \in \mathcal{A}^{n-1}$. Letting $H_n = A_{n-1}$ collapses the problem to the traditional sequential learning framework a la Smith and Sørensen. While our focus is social learning settings with limited observable histories, we will be interested in the traditional framework of fully observable histories as a baseline for comparison.

3.3.3 EQUILIBRIUM

The preliminaries above define a social learning game.

Definition 3.1. Let $\Gamma(H_n) = \{F, u_n, a_n, H_n\}_{n=1}^N$ denote a social learning game satisfying assumptions **(A1)**-**(A4)** with history $H_n \subset A_{n-1}$.

In equilibrium each agent chooses $a_n \in \mathcal{A}$ to maximize expected utility $\mathbb{E}[u(a_n, \omega) | \theta_n, H_n]$. Given the assumption of monotonicity on the likelihood ratio, a natural notion of equilibrium is that of a threshold $\hat{\theta}$ which if exceeded will induce an agent to take action $a_n = 1$.

Definition 3.2. Agent n follows a threshold strategy if

$$a_n = \begin{cases} 1 & \text{if } \theta_n \geq \hat{\theta}_n \\ 0 & \text{if } \theta_n < \hat{\theta}_n \end{cases}$$

for some $\hat{\theta}_n \in \text{supp}(F)$.

Since the probability distribution has no masses, the tie breaking rule for $\theta_n = \hat{\theta}_n$ will play no role in the analysis. Under the above assumptions, an equilibrium in which players utilize threshold strategies always exists.

Proposition 3.1. For social learning game $\Gamma(H_n)$ with $H_n \subset A_{n-1}$, a threshold strategy equilibrium exists.

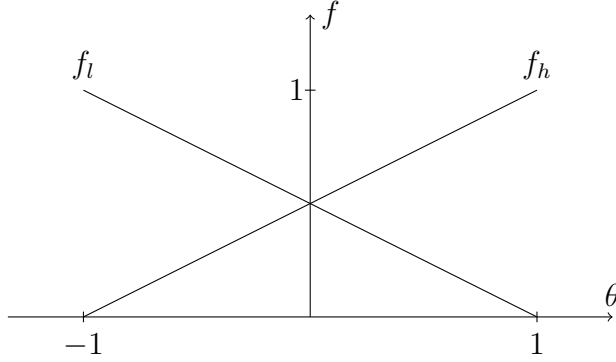


Figure 3.1: Final market price for certain and uncertain position

The existence of a threshold strategy equilibrium follows easily from the monotone likelihood ratio property. All proofs are relegated to the appendix.

3.3.4 SOCIAL LEARNING

Finally, we will examine the information aggregation properties of any equilibrium. In particular it will be of interest whether given a large enough game of social learning, agents tend to take the right action. For this we introduce a natural definition of learning.

Definition 3.3. *For a social learning game $\Gamma(H_n)$, we will say that complete learning occurs if $\lim_{n \rightarrow \infty} \Pr(a_n = 1|h) = 1$ and $\lim_{n \rightarrow \infty} \Pr(a_n = 0|l) = 1$.*

3.4 THE LINEAR CASE

To fix ideas consider the distribution functions $F_h(\theta) = \frac{1}{4}(1 + \theta)^2$ and $F_l(\theta) = 1 - \frac{1}{4}(1 - \theta)^2$ which admit linear densities $f_h(\theta) = \frac{1}{2}(1 + \theta)$ and $f_l(\theta) = \frac{1}{2}(1 - \theta)$.

The probability densities depicted above demonstrate the linear manner in which higher signals becoming more likely than low signals in state $\omega = h$. It can easily be verified that the densities f_l and f_h satisfy assumptions **(A1)**-**(A4)**.

3.4.1 FULLY OBSERVABLE HISTORY

Consider first the case where the complete history of actions taken by preceding agents is observable (e.g. $H_n = A_{n-1}$). The first agent has no predecessor and thus observes the history $A_0 = \emptyset$. Given that $a_1 = 1$ is preferred in state $\omega = h$ and $a_1 = 0$ is preferred in state $\omega = l$, with utilities u_n the agent will choose $a_1 = 1$ if and only if

$$\Pr(h|\theta_1) \geq \Pr(l|\theta_1) \iff \Pr(\theta_1|h) \Pr(h) \geq \Pr(\theta_1|l) \Pr(l) \iff f_h(\theta_1) \geq f_l(\theta_1)$$

and with flat prior $\Pr(h) = \frac{1}{2}$

$$\frac{1}{2}f_h(\theta) \geq \frac{1}{2}f_l(\theta) \iff \frac{1}{2}(1 + \theta) \geq \frac{1}{2}(1 - \theta)$$

which reduces to $\theta_1 \geq 0$ so that $\hat{\theta}_1 = 0$.

Having observed a_1 , the second agent will choose $a_2 = 1$ if and only if

$$\Pr(h|\theta_2, a_1) \geq \Pr(l|\theta_2, a_1) \iff \Pr(\theta_2, a_1|h) \geq \Pr(\theta_2, a_1|l)$$

The threshold $\hat{\theta}_2$ will depend on a_1 , with $\Pr(a_1 = 1|\omega) = \Pr(\theta \geq \hat{\theta}_1|\omega) = 1 - F_\omega(\hat{\theta}_1)$ and $\Pr(a_1 = 0|\omega) = F_\omega(\hat{\theta}_1)$. If $a_1 = 1$, then since $\hat{\theta}_1 = 0$ agent 2 will choose $a_2 = 1$ if

$$\begin{aligned} & \Pr(\theta_2, \theta_1 > 0|h) \Pr(h) \geq \Pr(\theta_2, \theta_1 > 0|l) \Pr(l) \\ & \iff f_h(\theta_2)(1 - F_h(0)) \Pr(h) \geq f_l(\theta_2)(1 - F_l(0)) \Pr(l) \\ & \iff \frac{1}{2}(1 + \theta_2)\left(1 - \frac{1}{4}(1 + 0)^2\right)\frac{1}{2} \geq \frac{1}{2}(1 - \theta_2)\left(\frac{1}{4}(1 - 0)^2\right)\frac{1}{2} \end{aligned}$$

where the second inequality comes from the conditional independence of the signal. The threshold then reduces to $\theta_2 \geq -\frac{1}{2}$. A similar calculation shows that agent 2 chooses the

high action after $a_1 = 0$ if $\theta \geq \frac{1}{2}$, yielding the conditional threshold

$$\hat{\theta}_2 = \begin{cases} -\frac{1}{2} & \text{if } a_1 = 1 \\ \frac{1}{2} & \text{if } a_1 = 0 \end{cases}$$

The equilibrium threshold for an arbitrary agent n is solved in much the same way, taking into account the entire history A_{n-1} leading up to the decision to act. As above, agent n will choose $a_n = 1$ if and only if

$$f_h(\theta_n) \Pr(A_{n-1}|h) \Pr(h) \geq f_l(\theta_n) \Pr(A_{n-1}|l) \Pr(l)$$

so that the threshold is defined by the likelihood ratio

$$\frac{1 + \hat{\theta}_n}{1 - \hat{\theta}_n} = \frac{\Pr(A_{n-1}|l) \Pr(l)}{\Pr(A_{n-1}|h) \Pr(h)}$$

While each threshold strategy $\hat{\theta}_n$ depends on the entire history of actions A_{n-1} , in comparing thresholds $\hat{\theta}_n$ and $\hat{\theta}_{n-1}$, the only informational asymmetry between agents n and $(n-1)$ is in the realization of θ_{n-1} , known only to $(n-1)$. Since the threshold $\hat{\theta}_{n-1}$ already contains information about the full history A_{n-2} up to the decision a_{n-1} , this suggests the possibility of a direct relationship between adjacent thresholds, enabling a recursive formulation of $\hat{\theta}_n$. Indeed this is the case.

Proposition 3.2. *For the social learning game with fully observable histories $\Gamma(A_{n-1})$ and canonical signal structure defined by $F_h(\theta) = \frac{1}{4}(1+\theta)^2$ and $F_l(\theta) = 1 - \frac{1}{4}(1-\theta)^2$, $\hat{\theta}_1 = 0$ and*

decision thresholds for $n \geq 2$ can be expressed recursively as

$$\hat{\theta}_n = \begin{cases} \frac{-1}{2+\hat{\theta}_{n-1}} & \text{if } a_{n-1} = 1 \\ \frac{1}{2-\hat{\theta}_{n-1}} & \text{if } a_{n-1} = 0 \end{cases}$$

Since the canonical case satisfies all of the traditional social learning assumptions that guarantee complete learning (e.g. MLRP, unbounded signals), it should be no surprise that the thresholds $\hat{\theta}_n$ converge and that complete learning is indeed achieved with a linear signal structure. Given the form of the decision thresholds, the conditional expectation is easily calculated as

$$\mathbb{E}[\hat{\theta}_n | \hat{\theta}_{n-1}, h] - \hat{\theta}_{n-1} = \frac{(\hat{\theta}_{n-1} + 1)^2(\hat{\theta}_{n-1} - 1)}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})}$$

and

$$\mathbb{E}[\hat{\theta}_n | \hat{\theta}_{n-1}, l] - \hat{\theta}_{n-1} = \frac{(\hat{\theta}_{n-1} + 1)(\hat{\theta}_{n-1} - 1)^2}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})}$$

enabling application of the Martingale Convergence Theorem to yield the result.

Proposition 3.3. *For the social learning game with fully observable histories $\Gamma(A_{n-1})$ and the canonical signal structure, state dependent thresholds $\hat{\theta}_n(\omega)$ converge with $\lim_{n \rightarrow \infty} \hat{\theta}_n(l) = 1$, $\lim_{n \rightarrow \infty} \hat{\theta}_n(h) = -1$, and complete learning occurs.*

3.4.2 LIMITED HISTORIES

Suppose now that instead of observing the entire history of preceding agents A_{n-1} , histories are limited in that each agent can only observe the predecessor's action. The viewable history is then $H_n = a_{n-1}$. The first two movers will behave the same way because they observe

the histories $A_0 = \emptyset$ and $A_1 = a_1$, respectively, exactly as before. Then $\hat{\theta}_1 = 0$; $\hat{\theta}_2 = -\frac{1}{2}$ if $a_1 = 1$ and $\hat{\theta}_2 = \frac{1}{2}$ if $a_1 = 0$. Now, however, $H_n \subsetneq A_{n-1}$ for $n \geq 2$ so agents will have less information with which to decide on an action a_n . With observable histories $H_n = a_{n-1}$, the thresholds are now defined by the likelihood ratio

$$\frac{1 + \hat{\theta}_n}{1 - \hat{\theta}_n} = \frac{\Pr(a_{n-1}|l) \Pr(l)}{\Pr(a_{n-1}|h) \Pr(h)}$$

Given that for each n there are only two possible histories H_n (with the exception of $n = 1$), we can reduce the decision of agent n to two thresholds

$$\hat{\theta}_n = \begin{cases} \bar{\theta}_n & \text{if } a_{n-1} = 1 \\ \underline{\theta}_n & \text{if } a_{n-1} = 0 \end{cases}$$

In the case of $n = 2$, $\bar{\theta}_2 = -\frac{1}{2}$ and $\underline{\theta}_2 = \frac{1}{2}$. Notice that $\bar{\theta}_n + \underline{\theta}_n = 0$ for $n = 2$. In fact this will be true for all n . Given the symmetry of the payoff function in states $\omega = \{l, h\}$ this result makes sense.

The departure from the case of fully observable histories begins with $n = 3$. Now agent n observes history $H_n = a_{n-1}$ but does not observe the action a_{n-2} . But the probability of a_{n-1} for a given state will depend on action a_{n-2} , which itself will depend on a_{n-3} and so on. The probability of observing an action then is

$$\Pr(a_{n-1}|\omega) = \sum_{A_{n-2} \in \mathcal{A}^{n-2}} \Pr(a_{n-1}|A_{n-2}, \omega) \Pr(A_{n-2}|\omega)$$

Given that the updated probability of a state depends on $n - 2$ unobservable previous actions the agent must account for 2^{n-2} possible history profiles, and the decision thresholds take

the cumbersome form

$$\frac{1 + \hat{\theta}_n}{1 - \hat{\theta}_n} = \frac{\sum_{A_{n-2} \in \mathcal{A}^{n-2}} \Pr(a_{n-1} | A_{n-2}, l) \Pr(A_{n-2} | l) \Pr(l)}{\sum_{A_{n-2} \in \mathcal{A}^{n-2}} \Pr(a_{n-1} | A_{n-2}, h) \Pr(A_{n-2} | h) \Pr(h)}$$

In the case of fully observable histories it was possible to solve for thresholds $\hat{\theta}_n$ recursively because both agents n and $(n-1)$ condition on A_{n-2} . But now agent $(n-1)$ conditions action a_{n-2} which is unobservable to agent n . Notice, however, that the thresholds $\bar{\theta}_{n-1}$ and $\underline{\theta}_{n-1}$ for agent $(n-1)$ depend on the action a_{n-2} according to

$$\frac{1 + \hat{\theta}_{n-1}}{1 - \hat{\theta}_{n-1}} = \frac{\sum_{A_{n-3} \in \mathcal{A}^{n-3}} \Pr(a_{n-2} | A_{n-3}, l) \Pr(A_{n-3} | l) \Pr(l)}{\sum_{A_{n-3} \in \mathcal{A}^{n-3}} \Pr(a_{n-2} | A_{n-3}, h) \Pr(A_{n-3} | h) \Pr(h)}$$

for $\hat{\theta}_{n-1} = \bar{\theta}_{n-1}$ or $\hat{\theta}_{n-1} = \underline{\theta}_{n-1}$ corresponding to $a_{n-2} = 1$ or $a_{n-2} = 0$, respectively. Since history A_{n-3} is unknown in both period n and $(n-1)$, this relationship enables player n to condition threshold $\hat{\theta}_n$ on only the two possible outcomes of a_{n-2} , greatly simplifying the problem and giving the following result.

Proposition 3.4. *For the social learning game with limited observable histories $\Gamma(a_{n-1})$ and the canonical signal structure, $\hat{\theta}_1 = 0$ and decision thresholds for $n \geq 2$ can be expressed recursively as*

$$\hat{\theta}_n = \begin{cases} \bar{\theta}_n = -\frac{1}{2}(1 + \bar{\theta}_{n-1}^2) & \text{if } a_{n-1} = 1 \\ \underline{\theta}_n = \frac{1}{2}(1 + \underline{\theta}_{n-1}^2) & \text{if } a_{n-1} = 0 \end{cases}$$

As alluded to above and as thresholds $\hat{\theta}_n$ clearly show, the symmetric signal structure implies that the thresholds are also symmetric about zero for every n .

Corollary 3.1. *For the social learning game with limited observable histories $\Gamma(a_{n-1})$ and the canonical signal structure, $\bar{\theta}_n + \underline{\theta}_n = 0$.*

The evolution of thresholds $\bar{\theta}_n$ and $\underline{\theta}_n$ is pictured below. As the figure shows, the bounds $\bar{\theta}_n$ and $\underline{\theta}_n$ diverge very quickly. This represents a higher standard of proof from signal θ_n in order deviate from previous action a_{n-1} .

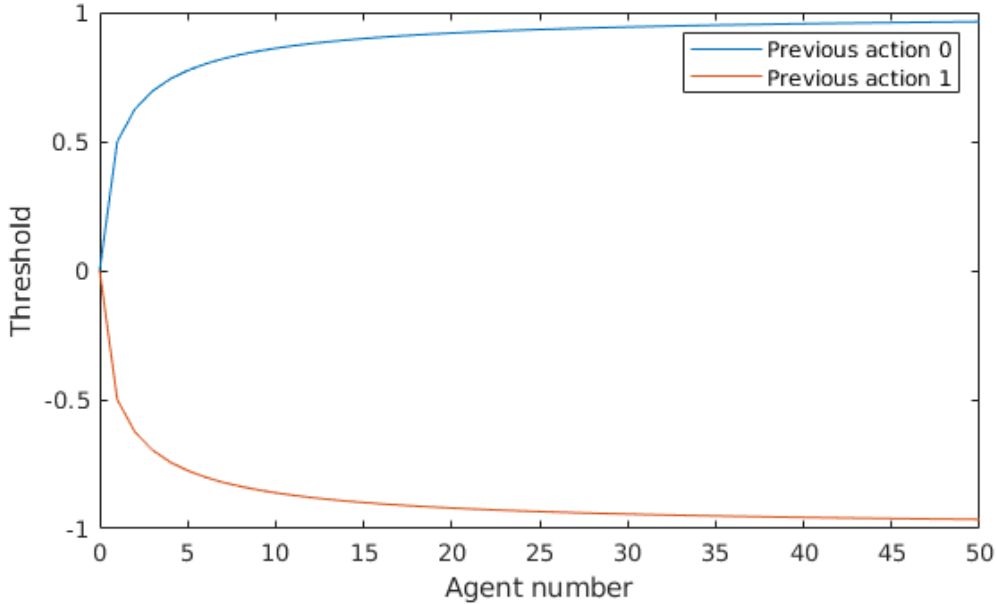


Figure 3.2: Decision thresholds under limited histories of observation

The thresholds partition the signal space into three regions. When $\theta_n > \underline{\theta}_n$, the agent will follow their signal and play $a_n = 1$ independent of previous action a_{n-1} , believing the state $\omega = h$ to be more likely. When $\theta_n < \bar{\theta}_n$ the agent will believe $\omega = l$ is more likely and play $a_n = 0$. When $\theta_n \in (\bar{\theta}_n, \underline{\theta}_n)$, the threshold for following the private signal is not surpassed and the agent will always follow the previous action a_{n-1} .

The figure depicting thresholds in the case of limited observable history suggests convergence to the limits of the distribution, so that as the periods advance the signal strength required to deviate from imitation of the predecessor increases. This would imply complete learning even in the case of limited histories, which the following result confirms.

Proposition 3.5. *For the social learning game with limited observable histories $\Gamma(a_{n-1})$ and the canonical signal structure, thresholds $\bar{\theta}_n$ if $a_{n-1} = 1$ and $\underline{\theta}_n$ if $a_{n-1} = 0$ converge with*

$\lim_{n \rightarrow \infty} \bar{\theta}_n = -1$, $\lim_{n \rightarrow \infty} \underline{\theta}_n = 1$, and complete learning occurs.

With fully observable histories, the threshold $\hat{\theta}_n$ was a recursive function of the previous agent's threshold, the function depending on the previous action a_{n-1} . Now, however, the threshold is solely determined by a_{n-1} , and as such sequences $\bar{\theta}_n$ and $\underline{\theta}_n$ take a predictable pattern. In fact as a result of this predictability of $\bar{\theta}_n$ and $\underline{\theta}_n$, it is possible that the martingale $\hat{\theta}_n$ derived from fully history game $\Gamma(A_{n-1})$ does not converge as quickly as the thresholds in limited history game $\Gamma(a_{n-1})$. In fact, as the following figure shows, on average this is the case.

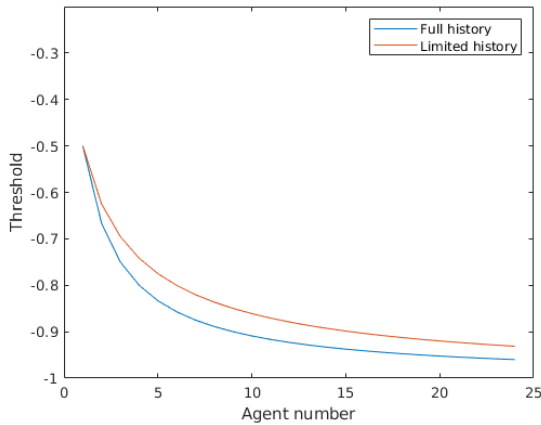


Figure 3.3: Maximum threshold in state h

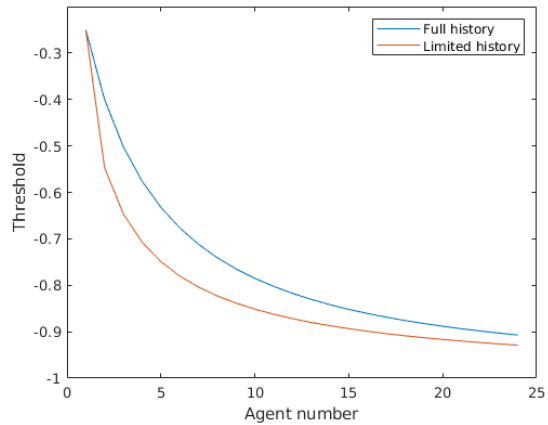


Figure 3.4: Expected threshold in state h

The panel on the left shows the maximal threshold values in the cases of full and limited history. In other words, these show the progression of the thresholds θ_n if $a_i = 1$ for all $i \leq n$. It is clear that with a history of only action $a_i = 1$ the threshold in the full information case converges more quickly than in the case of limited history. The right panel, however, shows that on average the threshold with limited history converges more quickly. In a sense, this reflects that with a limited history of observation, thresholds depend only on the previous action and are allowed to grow without respect to the full history. This fast growth is then reinforced by observing $a_{n-1} = 1$, given the strict threshold.

This interplay between history independence and a growing threshold suggests an increased possibility of error with limited observable histories. Comparing expected utilities highlights the welfare consequences of this error.

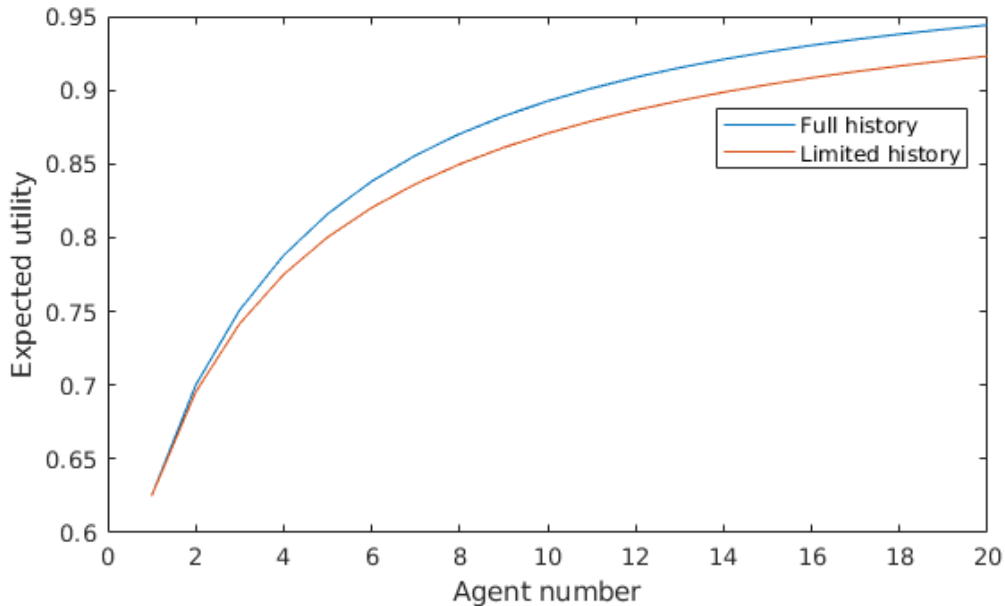


Figure 3.5: Expected utility in for full and limited histories of observation

Figure 3.5 shows $\mathbb{E}[u(\theta)|H_n = A_{n-1}]$ and $\mathbb{E}[u(\theta)|H_n = a_{n-1}]$, the expected utilities with full and limited histories. It shows, as we would expect, that on average utility is higher with full information than under a limited history of observation. Even though the thresholds converge faster on average with a limited history, suggesting faster learning, in fact this reflects the loss of information as a result of limited observations.

This is again a depiction of the progression of expected thresholds θ_n , but the shaded region of figure 3.6 shows where $\mathbb{E}[\theta_n|H_n = A_{n-1}] > \theta > \mathbb{E}[\theta_n|H_n = a_{n-1}]$. This is where the realization of signal θ falls between the thresholds in the full history case and the limited history case. For signals in this region, agent n would follow their signal with a full history, choosing $a_n = 1$ irrespective of the previous action, but would ignore the signal with a limited history, choosing $a_n = a_{n-1}$ even if $a_{n-1} = 0$. This increased possibility of error and

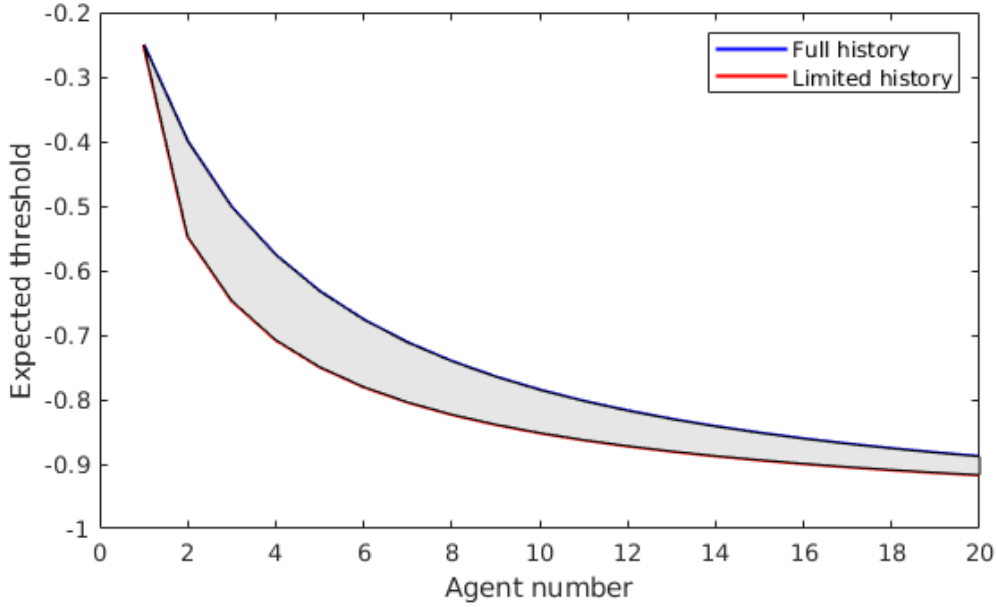


Figure 3.6: Expected information loss from limited observation

propensity to discard information drives down expected utility under limited observational history.

3.4.3 POSITIONAL UNCERTAINTY

Now suppose that in addition to observing only the action of the preceding agent, each agent does not know their position. Instead agents hold beliefs μ^n over their positions, where μ_i^n is the probability agent n places on moving in position i . Since the first mover easily deduces being first by the absence of any preceding action, we introduce an agent in position 0 that chooses as the first agent in the case of no positional uncertainty: $a_0 = 1$ if and only if $\theta_0 \geq 0$.

Suppose beliefs μ take the form $\mu_n^n = \gamma$ and $\mu_i^n = \frac{1-\gamma}{N-1}$ for $i \neq n$, so that the agent in position n has a belief $\gamma \in [0, 1]$ of their true position and spreads the additional probability $1 - \gamma$ uniformly across all other $N - 1$ positions. Then if agent n observes a_{n-1} the decision

threshold is determined as before

$$\begin{aligned}\frac{f_h(\hat{\theta}_n)}{f_l(\hat{\theta}_n)} &= \frac{\Pr(a_{n-1}|l) \Pr(l)}{\Pr(a_{n-1}|h) \Pr(h)} \\ &= \frac{\sum_{i=1}^N \mu_i^n \Pr(a_{i-1}|l) \Pr(l)}{\sum_{i=1}^N \mu_i^n \Pr(a_{i-1}|h) \Pr(h)}\end{aligned}$$

with the linear form of our signals and given that $\Pr(h) = \Pr(l)$

$$\begin{aligned}\frac{1 + \hat{\theta}_n}{1 - \hat{\theta}_n} &= \frac{\sum_{i=1}^N \mu_i^n \Pr(a_{i-1}|l) \Pr(l)}{\sum_{i=1}^N \mu_i^n \Pr(a_{i-1}|h) \Pr(h)} \\ \hat{\theta}_n &= \frac{\sum_{i=1}^N \mu_i^n (\Pr(a_{i-1}|l) - \Pr(a_{i-1}|h))}{\sum_{i=1}^N \mu_i^n (\Pr(a_{i-1}|l) + \Pr(a_{i-1}|h))}\end{aligned}$$

The assumed form of our probabilities μ^n yield the following result.

Proposition 3.6. *For the social learning game with limited observable histories $\Gamma(a_{t-1})$, positional uncertainty μ^n , and the canonical signal structure, a threshold equilibrium can be defined recursively as*

$$\begin{aligned}\bar{\theta}_1 &= \frac{N(\gamma - 2) + 1}{2(N - 1)} + 2 \left(\frac{1 - \gamma}{N - 1} \right) \sum_{i=0}^{N-1} \Pr(a_i = 1|l) \\ \underline{\theta}_1 &= \frac{N(3\gamma - 2) - 1}{2(N - 1)} + 2 \left(\frac{1 - \gamma}{N - 1} \right) \sum_{i=0}^{N-1} \Pr(a_i = 0|l)\end{aligned}$$

and for $n \geq 2$

$$\begin{aligned}\bar{\theta}_n &= 2 \left(\frac{N\gamma - 1}{N - 1} \right) \Pr(a_{n-1} = 1|l) + \bar{\theta}_1 - \frac{N\gamma - 1}{2(N - 1)} \\ \underline{\theta}_n &= 2 \left(\frac{N\gamma - 1}{N - 1} \right) \Pr(a_{n-1} = 0|l) + \underline{\theta}_1 - \frac{3(N\gamma - 1)}{2(N - 1)}\end{aligned}$$

As the thresholds make clear, the action dependent signals $\bar{\theta}_n$ and $\underline{\theta}_n$ exhibit the same

symmetry about zero as in the case without positional uncertainty.

Corollary 3.2. *For the social learning game with limited observable histories $\Gamma(a_{t-1})$, positional uncertainty μ^n , and the canonical signal structure, $\bar{\theta}_n + \underline{\theta}_n = 0$.*

The figure below shows the evolution of thresholds which display the downward trend that we have come to expect, suggestive of convergence to a limit.

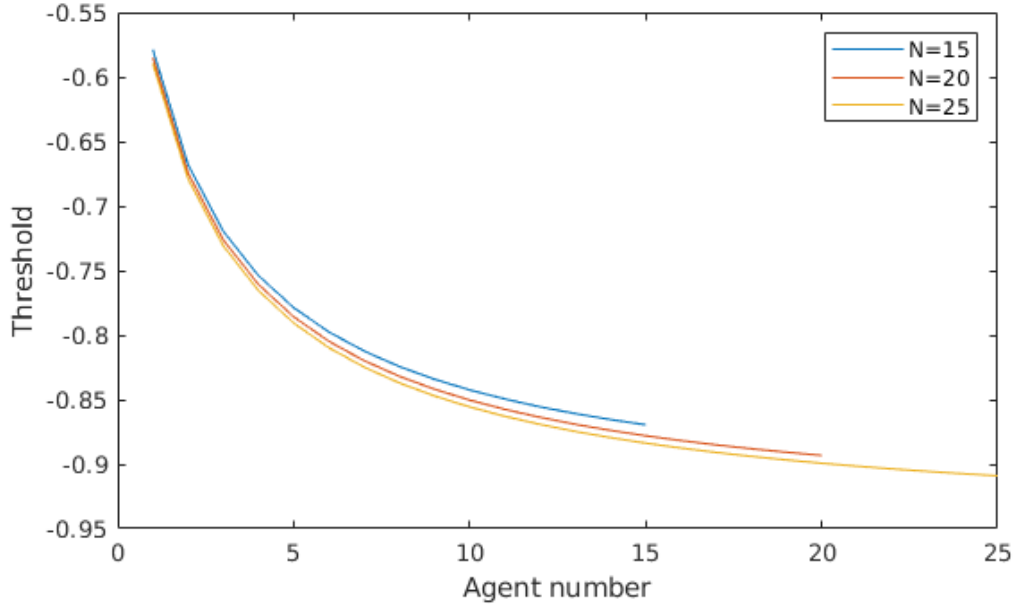


Figure 3.7: Thresholds under positional uncertainty for various N

Now, however, each agent n holds the belief $\gamma < 1$ that they act in the position n that they indeed do. This lack of certainty over position could translate into a lack of certainty over the true state of the world, leading to a limit $\bar{\theta} > -1$ or $\bar{\theta} < 1$. Fortunately, it turns out that if such a limit exists, this limit must be $\bar{\theta} = -1$ or $\bar{\theta} = 1$. While the speed of this convergence will depend on belief parameter γ , complete learning occurs in the limit for all beliefs.

Proposition 3.7. *For the social learning game with limited observable histories $\Gamma(a_{t-1})$, positional uncertainty μ^n , and the canonical signal structure, $\lim_{n \rightarrow \infty} \bar{\theta}_n = -1$ and $\lim_{n \rightarrow \infty} \underline{\theta}_n =$*

1 if such limits exist. Moreover, if these limits exist then complete learning occurs.

Notice also that the positional probability beliefs depend on N , and thus so do the thresholds. As the figure above shows, the larger is the number of agents N , the faster is the convergence of the threshold to its limit. As the number of agents increases, the belief of moving in any position other than n becomes diluted. This applies particularly to early positions where the predecessor faced a relatively low threshold, thereby inducing the successor to require a higher standard of proof. As this probability decreases, each agent relies more strongly on the true prior action a_{n-1} , thus leading to faster convergence of the threshold.

Since the object of interest will be the evolution of learning as the number of agents observing histories increases, we will focus on thresholds for large N , which take a convenient form.

Proposition 3.8. *For the social learning game with limited observable histories $\Gamma(a_{t-1})$, positional uncertainty μ^n , and the canonical signal structure*

$$\begin{aligned}\lim_{N \rightarrow \infty} \bar{\theta}_n &= -\frac{1}{2}(1 + \bar{\theta}_{n-1}^2) - \left(\frac{1-\gamma}{2}\right)(1 + \bar{\theta}_{n-1})^2 \\ \lim_{N \rightarrow \infty} \underline{\theta}_n &= \frac{1}{2}(1 + \bar{\theta}_{n-1}^2) + \left(\frac{1-\gamma}{2}\right)(1 + \bar{\theta}_{n-1})^2\end{aligned}$$

An obvious consequence of the form the thresholds take is that the introduction of term $\left(\frac{1-\gamma}{2}\right)(1 + \bar{\theta}_{n-1})^2$ leads to a lower (higher) value of threshold $\bar{\theta}_n(\underline{\theta}_n)$ for every n . This leads to a faster convergence to the limit, a rate which increases as belief γ decreases.

The above graph shows this relationship between γ and the rate of convergence, and in fact at the extreme of $\gamma \rightarrow 0$ the threshold converges immediately to $\bar{\theta}_n = -1$ for all n . Immediate convergence is the result of total positional uncertainty, whereby it makes more sense to each agent to follow the action of the previous agent because they have no concept

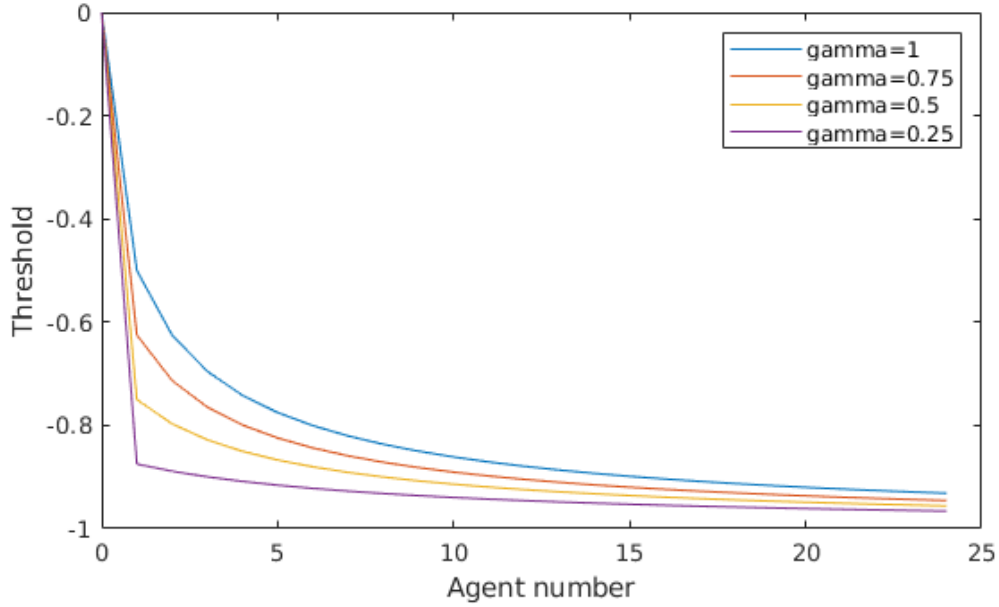


Figure 3.8: Thresholds under positional uncertainty for large N and various γ

of their own signal's informational value.

As in the case of limited history we can compare expected utility to get a more complete story of the welfare implications of this convergence. We would expect that the increased rate of convergence to lead to a further loss of welfare, increasing the region where agents ignore their information.

As expected, the figure above shows this exact result. The panel on the left shows a lower expected utility under positional uncertainty characterized by $\gamma = 0.75$, while the right panel shows an even further loss of utility for $\gamma = 0.5$. In fact expected utility in the case of limited history can be shown to take an explicit form.

Proposition 3.9. *Under positional uncertainty*

$$\mathbb{E}[u(\theta)|H_n = a_{n-1}, \mu^n] = \frac{1}{2\gamma}(1 + \bar{\theta}_n^2) + \frac{1-\gamma}{2\gamma}(\bar{\theta}_n^2 + 2\bar{\theta}_n - 1)$$

Comparative analysis on the parameter γ confirms the result that expected utility under

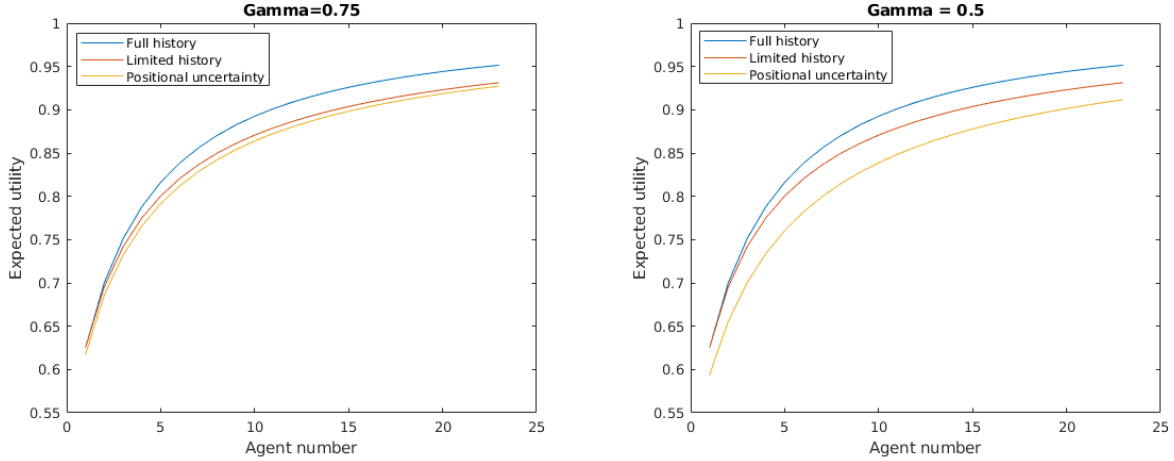


Figure 3.9: Relationship between expected utility and gamma in state h

positional uncertainty $\mathbb{E}[u(\theta)|H_n = a_{n-1}, \mu^n]$ indeed decreases as uncertainty γ increases.

3.5 THE GENERAL CASE

While much of the above used the canonical signal structure $F_h(\theta) = \frac{1}{4}(1 + \theta^2)$ and $F_l(\theta) = 1 - \frac{1}{4}(1 - \theta^2)$, many of the results hold for more general signal structures that satisfy the assumptions **(A1)**-**(A4)**. Of course, the result of complete learning should be no surprise, as it has been the focus of much theoretical work in the area of social learning.

Proposition 3.10. *For the social learning game with fully observable histories $\Gamma(A_{t-1})$ and a signal structure F satisfying **(A1)**-**(A4)**, state dependent thresholds $\hat{\theta}_n(\omega)$ converge with $\lim_{n \rightarrow \infty} \hat{\theta}_n(l) = 1$, $\lim_{n \rightarrow \infty} \hat{\theta}_n(h) = -1$, and complete learning occurs.*

The focus of this work, social learning in an environment with a limited history of observation, also features complete learning in the more general setting.

Proposition 3.11. *For the social learning game with limited observable histories $\Gamma(a_{t-1})$ and a signal structure F satisfying **(A1)**-**(A4)**, thresholds $\bar{\theta}_n$ if $a_{n-1} = 1$ and $\underline{\theta}_n$ if $a_{n-1} = 0$ converge with $\lim_{n \rightarrow \infty} \bar{\theta}_n = -1$, $\lim_{n \rightarrow \infty} \underline{\theta}_n = 1$, and complete learning occurs.*

To avoid placing a technical burden on the analysis by restricting the signal structure in ways that increase tractability but lack in obvious economic meaning, consider the following intuitive assumption.

Assumption 3.5. *Under limited histories with positional uncertainty $\bar{\theta}_n + \underline{\theta}_n = 0$ for all n .*

Under this assumption we have the general result of complete learning in an environment of positional uncertainty.

Proposition 3.12. *For the social learning game with limited observable histories $\Gamma(a_{t-1})$, positional uncertainty μ^n , and a signal structure F satisfying **(A1)**-**(A5)**, $\lim_{n \rightarrow \infty} \bar{\theta}_n = -1$ and $\lim_{n \rightarrow \infty} \underline{\theta}_n = 1$ if such limits exist. Moreover, if these limits exist then complete learning occurs.*

3.6 CONCLUDING REMARKS

The traditional model of social learning offers powerfully intuitive results on how the courtship of private information and observation leads to informed economic decision making. But this marriage is only as strong as the assumptions it stands upon. In particular, if the assumptions of fully observable histories and certainty about position in the sequence of actors come into question, there are behavioral and welfare consequences that alter the learning dynamic. By addressing this we gain a richer depiction of an environment in which agents learn from an appreciably less learned starting point.

The introduction of limited observation of preceding actions to the standard social learning model changes the integration of information, but the limit result of complete learning remains. With agents only conditioning on the previous action, the threshold equilibria take a predictable form, on average converging more quickly to the limit ensuring the correct action. Despite this, the increased possibility of discarding information increases the possibility for error, leading to a lower expected utility for each agent in finite time.

Complete learning in the limit continues to hold even when agents are uncertain of their position in the sequence. In fact, the threshold equilibria converge more quickly the higher is the uncertainty over position, exacerbating the reduction in expected utility of the limited history case. While the pace of learning in terms of welfare decreases with positional uncertainty, complete learning in the limit does not depend on its existence or magnitude.

BIBLIOGRAPHY

- [1] Acemoglu, D., M. A. Dahleh, I. Lobel, and A. Ozdaglar (2011): “Bayesian learning in social networks.” *Review of Economic Studies*, 78, pp. 1201-1236.
- [2] Banerjee, A.V. (1992). “A simple model of herd behavior.” *The Quarterly Journal of Economics*, 107:3, pp. 797-817.
- [3] Bikhchandani, S., Hirshleifer, D., and Welch, I. (1992). “A theory of fads, fashion, custom, and cultural change as informational cascades.” *Journal of Political Economy*, 100:5, pp. 992-1026.
- [4] Çelen, B. and Kariv, S. (2004). “Observational learning under imperfect information.” *Games and Economic Behavior*, 47, pp. 72-86.
- [5] Gul, F. and Lundholm, R. (1995). “Endogenous timing and the clustering of agents’ decisions.” *Journal of Political Economy*, 103:5, pp. 1039-66.
- [6] Monzón, I. and Rapp, M. (2014). “Observational learning with position uncertainty.” *Journal of Economic Theory*, 154, pp. 357-402.
- [7] Smith, L. and Sørensen, P. (2000). “Pathological outcomes of observational learning.” *Econometrica*, 68:2, pp. 371-398.
- [8] Song, Y. (2015): “Social Learning with Endogenous Network Formation,” unpublished. Available from <https://arxiv.org/abs/1504.05222v1>.

APPENDIX: PROOFS

Proof of Proposition 1. Suppose agent n observes history H_n and receives signal θ_n .

Then n will choose $a_n = 1$ if

$$\begin{aligned} \Pr(h|\theta_n, H_t) \geq \Pr(l|\theta_n, H_t) &\iff \frac{\Pr(\theta_n, H_t|h) \Pr(h)}{\Pr(\theta_n, H_t)} \geq \frac{\Pr(\theta_n, H_t|l) \Pr(l)}{\Pr(\theta_n, H_t)} \\ &\iff f_h(\theta_n) \Pr(H_t|h) \geq f_l(\theta_n) \Pr(H_t|l) \iff \frac{f_h(\theta_n)}{f_l(\theta_n)} \geq \frac{\Pr(H_t|l)}{\Pr(H_t|h)} \end{aligned}$$

Given that $\frac{f_h(\theta_n)}{f_l(\theta_n)}$ is strictly increasing in θ_n there must be some $\hat{\theta}_n$ for which $\Pr(h|\theta_n, H_t) \geq \Pr(l|\theta_n, H_t)$ if $\theta_n \geq \hat{\theta}_n$ and $\Pr(h|\theta_n, H_t) < \Pr(l|\theta_n, H_t)$ otherwise. Thus n follows a threshold strategy. \square

Lemma 3.1. For a general signal structure $\Pr(\theta_n \leq \theta|h) = F_h(\theta)$ and $\Pr(\theta_n \leq \theta|l) = F_l(\theta)$ that admit distribution functions f_h, f_l characterized by the monotone likelihood ratio property, cutoff strategies $\hat{\theta}_n$ are determined recursively by

$$\frac{f_h(\hat{\theta}_n)}{f_l(\hat{\theta}_n)} = \begin{cases} \frac{(1-F_l(\hat{\theta}_{n-1}))f_h(\hat{\theta}_{n-1})}{(1-F_h(\hat{\theta}_{n-1}))f_l(\hat{\theta}_{n-1})} & \text{if } a_{n-1} = 1 \\ \frac{F_l(\hat{\theta}_{n-1})f_h(\hat{\theta}_{n-1})}{F_h(\hat{\theta}_{n-1})f_l(\hat{\theta}_{n-1})} & \text{if } a_{n-1} = 0 \end{cases}$$

Proof. We determined that the threshold for each signal is implicitly defined by $\frac{f_h(\hat{\theta}_n)}{f_l(\hat{\theta}_n)} = \frac{\Pr(A_{n-1}|l) \Pr(l)}{\Pr(A_{n-1}|h) \Pr(h)}$. Since this is true for all n , $\frac{f_h(\hat{\theta}_{n-1})}{f_l(\hat{\theta}_{n-1})} = \frac{\Pr(A_{n-2}|l) \Pr(l)}{\Pr(A_{n-2}|h) \Pr(h)}$ so that $\Pr(A_{n-2}|l) \Pr(l) = \frac{f_h(\hat{\theta}_{n-1})}{f_l(\hat{\theta}_{n-1})} \Pr(A_{n-2}|h) \Pr(h)$.

In the case of $a_{n-1} = 1$, $\theta_{n-1} \geq \hat{\theta}_{n-1}$ and

$$\begin{aligned}
\frac{\Pr(A_{n-1}|l) \Pr(l)}{\Pr(A_{n-1}|h) \Pr(h)} &= \frac{\Pr(\theta_{n-1} \geq \hat{\theta}_{n-1}|l) \Pr(A_{n-2}|l) \Pr(l)}{\Pr(\theta_{n-1} \geq \hat{\theta}_{n-1}|h) \Pr(A_{n-2}|h) \Pr(h)} \\
&= \frac{(1 - F_l(\hat{\theta}_{n-1})) \Pr(A_{n-2}|l) \Pr(l)}{(1 - F_h(\hat{\theta}_{n-1})) \Pr(A_{n-2}|h) \Pr(h)} \\
&= \frac{(1 - F_l(\hat{\theta}_{n-1})) \frac{f_h(\hat{\theta}_{n-1})}{f_l(\hat{\theta}_{n-1})} \Pr(A_{n-2}|h) \Pr(h)}{(1 - F_h(\hat{\theta}_{n-1})) \Pr(A_{n-2}|h) \Pr(h)} \\
&= \frac{(1 - F_l(\hat{\theta}_{n-1})) f_h(\hat{\theta}_{n-1})}{(1 - F_h(\hat{\theta}_{n-1})) f_l(\hat{\theta}_{n-1})}
\end{aligned}$$

while if $a_{n-1} = 0$ then $\theta < \hat{\theta}_{n-1}$ and

$$\frac{\Pr(A_{n-1}|h) \Pr(h)}{\Pr(A_{n-1}|l) \Pr(l)} = \frac{F_l(\hat{\theta}_{n-1}) f_h(\hat{\theta}_{n-1})}{F_h(\hat{\theta}_{n-1}) f_l(\hat{\theta}_{n-1})}$$

Thus the cutoff strategy is defined recursively by

$$\frac{f_h(\hat{\theta}_n)}{f_l(\hat{\theta}_n)} = \begin{cases} \frac{(1-F_l(\hat{\theta}_{n-1}))f_h(\hat{\theta}_{n-1})}{(1-F_h(\hat{\theta}_{n-1}))f_l(\hat{\theta}_{n-1})} & \text{if } a_{n-1} = 1 \\ \frac{F_l(\hat{\theta}_{n-1})f_h(\hat{\theta}_{n-1})}{F_h(\hat{\theta}_{n-1})f_l(\hat{\theta}_{n-1})} & \text{if } a_{n-1} = 0 \end{cases}$$

□

Proof of Proposition 2. Applying the result of lemma 1 to our canonical signal structure

defined by $F_h(\theta) = \frac{1}{4}(1 + \theta)^2$ and $F_l(\theta) = 1 - \frac{1}{4}(1 - \theta)^2$, if $a_{n-1} = 1$,

$$\begin{aligned}
\frac{1 + \hat{\theta}_n}{1 - \hat{\theta}_n} &= \frac{\frac{1}{4}(1 - \hat{\theta}_{n-1})^2 \frac{1}{2}(1 + \hat{\theta}_{n-1})}{[1 - \frac{1}{4}(1 + \hat{\theta}_{n-1})^2] \frac{1}{2}(1 - \hat{\theta}_{n-1})} = \frac{(1 - \hat{\theta}_{n-1})(1 + \hat{\theta}_{n-1})}{3 - 2\hat{\theta}_{n-1} - \hat{\theta}_{n-1}^2} \\
&= \frac{(1 - \hat{\theta}_{n-1})(1 + \hat{\theta}_{n-1})}{(3 + \hat{\theta}_{n-1})(1 - \hat{\theta}_{n-1})} = \frac{1 + \hat{\theta}_{n-1}}{3 + \hat{\theta}_{n-1}} \\
\implies \hat{\theta}_n &= \frac{-1}{2 + \hat{\theta}_{n-1}}
\end{aligned}$$

and if $a_{n-1} = 0$

$$\begin{aligned}
\frac{1 + \hat{\theta}_n}{1 - \hat{\theta}_n} &= \frac{[1 - \frac{1}{4}(1 - \hat{\theta}_{n-1})^2]^{\frac{1}{2}}(1 + \hat{\theta}_{n-1})}{\frac{1}{4}(1 + \hat{\theta}_{n-1})^2 \frac{1}{2}(1 - \hat{\theta}_{n-1})} = \frac{3 + 2\hat{\theta}_{n-1} - \hat{\theta}_{n-1}^2}{(1 + \hat{\theta}_{n-1})(1 - \hat{\theta}_{n-1})} \\
&= \frac{(3 - \hat{\theta}_{n-1})(1 + \hat{\theta}_{n-1})}{(1 + \hat{\theta}_{n-1})(1 - \hat{\theta}_{n-1})} = \frac{3 - \hat{\theta}_{n-1}}{1 - \hat{\theta}_{n-1}} \\
\implies \hat{\theta}_n &= \frac{1}{2 - \hat{\theta}_{n-1}}
\end{aligned}$$

Thus we have the recursive result that

$$\hat{\theta}_n = \begin{cases} \frac{-1}{2 + \hat{\theta}_{n-1}} & \text{if } a_{n-1} = 1 \\ \frac{1}{2 - \hat{\theta}_{n-1}} & \text{if } a_{n-1} = 0 \end{cases}$$

□

Proof of Proposition 3. Given the recursive form of the threshold strategy determined for the canonical case,

$$\begin{aligned}
\mathbb{E}[\hat{\theta}_n | \hat{\theta}_{n-1}, h] &= \left(\frac{-1}{2 + \hat{\theta}_{n-1}} \right) \Pr(a_{n-1} = 1 | h) + \left(\frac{1}{2 - \hat{\theta}_{n-1}} \right) \Pr(a_{n-1} = 0 | h) \\
&= \left(\frac{-1}{2 + \hat{\theta}_{n-1}} \right) (1 - \Pr(a_{n-1} = 0 | h)) + \left(\frac{1}{2 - \hat{\theta}_{n-1}} \right) \Pr(a_{n-1} = 0 | h) \\
&= F_h(\hat{\theta}_{n-1}) \left(\left(\frac{1}{2 - \hat{\theta}_{n-1}} \right) - \left(\frac{-1}{2 + \hat{\theta}_{n-1}} \right) \right) + \left(\frac{-1}{2 + \hat{\theta}_{n-1}} \right) \\
&= \frac{1}{4} (1 + \hat{\theta}_{n-1})^2 \left(\frac{4}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})} \right) - \frac{1}{2 + \hat{\theta}_{n-1}} \\
&= \frac{1 + 2\hat{\theta}_{n-1} + \hat{\theta}_{n-1}^2}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})} - \frac{2 - \hat{\theta}_{n-1}}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})} \\
&= \frac{\hat{\theta}_{n-1}^2 + 3\hat{\theta}_{n-1} - 1}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})}
\end{aligned}$$

so that

$$\begin{aligned}
\mathbb{E}[\hat{\theta}_n | \hat{\theta}_{n-1}, h] - \hat{\theta}_{n-1} &= \frac{\hat{\theta}_{n-1}^2 + 3\hat{\theta}_{n-1} - 1}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})} - \hat{\theta}_{n-1} \\
&= \frac{\hat{\theta}_{n-1}^2 + 3\hat{\theta}_{n-1} - 1}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})} - \frac{4\hat{\theta}_{n-1} - \hat{\theta}_{n-1}^3}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})} \\
&= \frac{\hat{\theta}_{n-1}^3 + \hat{\theta}_{n-1}^2 - \hat{\theta}_{n-1} - 1}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})} \\
&= \frac{(\hat{\theta}_{n-1} + 1)^2(\hat{\theta}_{n-1} - 1)}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})}
\end{aligned}$$

Since $-1 \leq \hat{\theta}_{n-1} \leq 1$, all terms of $\mathbb{E}[\hat{\theta}_n | \hat{\theta}_{n-1}, h] - \hat{\theta}_{n-1}$ are positive except $(\hat{\theta}_{n-1} - 1) \leq 0$ so that $\mathbb{E}[\hat{\theta}_n | \hat{\theta}_{n-1}, h] \leq \hat{\theta}_{n-1}$. Then conditional on $\omega = h$, $\hat{\theta}_n$ is a supermartingale bounded below by -1 and thus must converge to a limit almost everywhere.

Similarly,

$$\begin{aligned}
\mathbb{E}[\hat{\theta}_n | \hat{\theta}_{n-1}, l] &= \left(\frac{-1}{2 + \hat{\theta}_{n-1}} \right) \Pr(a_{n-1} = 1|l) + \left(\frac{1}{2 - \hat{\theta}_{n-1}} \right) \Pr(a_{n-1} = 0|l) \\
&= \left(\frac{-1}{2 + \hat{\theta}_{n-1}} \right) \Pr(a_{n-1} = 1|l) + \left(\frac{1}{2 - \hat{\theta}_{n-1}} \right) (1 - \Pr(a_{n-1} = 1|l)) \\
&= (1 - F_l(\hat{\theta}_{n-1})) \left(\left(\frac{-1}{2 + \hat{\theta}_{n-1}} \right) - \left(\frac{1}{2 - \hat{\theta}_{n-1}} \right) \right) + \left(\frac{1}{2 - \hat{\theta}_{n-1}} \right) \\
&= \frac{1}{4}(1 - \hat{\theta}_{n-1})^2 \left(\frac{-4}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})} \right) + \frac{1}{2 - \hat{\theta}_{n-1}} \\
&= \frac{2 + \hat{\theta}_{n-1}}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})} - \frac{1 - 2\hat{\theta}_{n-1} + \hat{\theta}_{n-1}^2}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})} \\
&= \frac{-\hat{\theta}_{n-1}^2 + 3\hat{\theta}_{n-1} + 1}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})}
\end{aligned}$$

so that

$$\begin{aligned}
\mathbb{E}[\hat{\theta}_n | \hat{\theta}_{n-1}, l] - \hat{\theta}_{n-1} &= \frac{-\hat{\theta}_{n-1}^2 + 3\hat{\theta}_{n-1} + 1}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})} - \hat{\theta}_{n-1} \\
&= \frac{-\hat{\theta}_{n-1}^2 + 3\hat{\theta}_{n-1} + 1}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})} - \frac{4\hat{\theta}_{n-1} - \hat{\theta}_{n-1}^3}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})} \\
&= \frac{\hat{\theta}_{n-1}^3 - \hat{\theta}_{n-1}^2 - \hat{\theta}_{n-1} + 1}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})} \\
&= \frac{(\hat{\theta}_{n-1} + 1)(\hat{\theta}_{n-1} - 1)^2}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})}
\end{aligned}$$

Since $-1 \leq \hat{\theta}_{n-1} \leq 1$ all terms are nonnegative so that $\mathbb{E}[\hat{\theta}_n | \hat{\theta}_{n-1}, l] \geq \hat{\theta}_{n-1}$. Then conditional on $\omega = l$, $\hat{\theta}_n$ is a submartingale bounded above by 1 and thus must converge to a limit almost everywhere.

Let $\bar{\theta} = \lim_{n \rightarrow \infty} \hat{\theta}_n(h)$ be the limit of the supermartingale $\hat{\theta}_n$ conditional on the state $\omega = h$. Then

$$\begin{aligned}
\bar{\theta} &= \lim_{n \rightarrow \infty} \mathbb{E}[\hat{\theta}_n | \hat{\theta}_{n-1}, h] = \lim_{n \rightarrow \infty} \frac{\hat{\theta}_{n-1}^2 + 3\hat{\theta}_{n-1} - 1}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})} \\
\implies \bar{\theta} &= \frac{\bar{\theta}^2 + 3\bar{\theta} - 1}{(2 + \bar{\theta})(2 - \bar{\theta})}
\end{aligned}$$

which reduces to $(\bar{\theta} + 1)^2(\bar{\theta} - 1) = 0$. Then either $\bar{\theta} = 1$ or $\bar{\theta} = -1$. But as we saw above, $\hat{\theta}_{n-1} = 0$ if $n = 1$ so that $\mathbb{E}[\hat{\theta}_n | \hat{\theta}_{n-1}, h] \leq 0$ for all $n \geq 1$. This only leaves $\bar{\theta} = -1$.

Let $\underline{\theta} = \lim_{n \rightarrow \infty} \hat{\theta}_n(l)$ be the limit of the submartingale $\hat{\theta}_n$ conditional on the state $\omega = l$. Then

$$\begin{aligned}
\underline{\theta} &= \mathbb{E}[\hat{\theta}_n | \hat{\theta}_{n-1}, l] = \lim_{n \rightarrow \infty} \frac{-\hat{\theta}_{n-1}^2 + 3\hat{\theta}_{n-1} + 1}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})} \\
\implies \underline{\theta} &= \frac{-\underline{\theta}^2 + 3\underline{\theta} + 1}{(2 + \underline{\theta})(2 - \underline{\theta})}
\end{aligned}$$

which reduces to $(\underline{\theta} + 1)(\underline{\theta} - 1)^2 = 0$. Then either $\underline{\theta} = 1$ or $\underline{\theta} = -1$. But as we saw above, $\hat{\theta}_{n-1} = 0$ if $n = 1$ so that $\mathbb{E}[\hat{\theta}_n | \hat{\theta}_{n-1}, l] \geq 0$ for all $n \geq 1$. This only leaves $\underline{\theta} = 1$.

Finally, $\Pr(a_n = 1|h) = \Pr(\theta_n > \hat{\theta}_n|h) = 1 - F_h(\hat{\theta}_n) = 1 - \frac{1}{4}(1 + \hat{\theta}_n)^2$. Similarly $\Pr(a_n = 0|l) = \Pr(\theta_n \leq \hat{\theta}_n|l) = F_l(\hat{\theta}_n) = 1 - \frac{1}{4}(1 - \hat{\theta}_n)^2$. So then

$$\begin{aligned} \lim_{n \rightarrow \infty} \Pr(a_n = 1|h) &= \lim_{n \rightarrow \infty} 1 - \frac{1}{4} \left(1 + \hat{\theta}_n(h)\right)^2 = 1 - \frac{1}{4} \left(1 + \lim_{n \rightarrow \infty} \hat{\theta}_n(h)\right)^2 = 1 \\ \lim_{n \rightarrow \infty} \Pr(a_n = 0|l) &= \lim_{n \rightarrow \infty} 1 - \frac{1}{4} \left(1 - \hat{\theta}_n(l)\right)^2 = 1 - \frac{1}{4} \left(1 - \lim_{n \rightarrow \infty} \hat{\theta}_n(l)\right)^2 = 1 \end{aligned}$$

so that $\lim_{n \rightarrow \infty} \Pr(a_n = 1|h) = \lim_{n \rightarrow \infty} \Pr(a_n = 0|l) = 1$ and complete learning occurs. \square

Proof of Proposition 4. As shown above, the threshold for n is given by

$$\begin{aligned} \frac{f_h(\hat{\theta}_n)}{f_l(\hat{\theta}_n)} &= \frac{\Pr(a_{n-1}|l) \Pr(l)}{\Pr(a_{n-1}|h) \Pr(h)} \\ &= \frac{\sum_{A_{n-2} \in \mathcal{A}^{n-2}} \Pr(a_{n-1}|A_{n-2}, l) \Pr(A_{n-2}|l) \Pr(l)}{\sum_{A_{n-2} \in \mathcal{A}^{n-2}} \Pr(a_{n-1}|A_{n-2}, h) \Pr(A_{n-2}|h) \Pr(h)} \\ &= \frac{\sum_{a_{n-2} \in \{0,1\}} \sum_{A_{n-3} \in \mathcal{A}^{n-3}} \Pr(a_{n-1}|a_{n-2}, A_{n-3}, l) \Pr(a_{n-2}|A_{n-3}, l) \Pr(A_{n-3}|l)}{\sum_{a_{n-2} \in \{0,1\}} \sum_{A_{n-3} \in \mathcal{A}^{n-3}} \Pr(a_{n-1}|a_{n-2}, A_{n-3}, h) \Pr(a_{n-2}|A_{n-3}, h) \Pr(A_{n-3}|h)} \end{aligned}$$

Notice that since $H_{n-1} = a_{n-2}$ the action of $(n-1)$ is independent of history A_{n-3} so this reduces to

$$\begin{aligned} &= \frac{\sum_{a_{n-2} \in \{0,1\}} \sum_{A_{n-3} \in \mathcal{A}^{n-3}} \Pr(a_{n-1}|a_{n-2}, l) \Pr(a_{n-2}|A_{n-3}, l) \Pr(A_{n-3}|l)}{\sum_{a_{n-2} \in \{0,1\}} \sum_{A_{n-3} \in \mathcal{A}^{n-3}} \Pr(a_{n-1}|a_{n-2}, h) \Pr(a_{n-2}|A_{n-3}, h) \Pr(A_{n-3}|h)} \\ &= \frac{\sum_{a_{n-2} \in \{0,1\}} \Pr(a_{n-1}|a_{n-2}, l) \sum_{A_{n-3} \in \mathcal{A}^{n-3}} \Pr(a_{n-2}|A_{n-3}, l) \Pr(A_{n-3}|l)}{\sum_{a_{n-2} \in \{0,1\}} \Pr(a_{n-1}|a_{n-2}, h) \sum_{A_{n-3} \in \mathcal{A}^{n-3}} \Pr(a_{n-2}|A_{n-3}, h) \Pr(A_{n-3}|h)} \\ &= \frac{\sum_{a_{n-2} \in \{0,1\}} \Pr(a_{n-1}|a_{n-2}, l) \Pr(a_{n-2}|l)}{\sum_{a_{n-2} \in \{0,1\}} \Pr(a_{n-1}|a_{n-2}, h) \Pr(a_{n-2}|h)} \end{aligned}$$

The threshold for agent $(n - 1)$ is given by the solution

$$\frac{f_h(\hat{\theta}_{n-1})}{f_l(\hat{\theta}_{n-1})} = \frac{\Pr(a_{n-2}|l) \Pr(l)}{\Pr(a_{n-2}|h) \Pr(h)}$$

If thresholds $\bar{\theta}_{n-1}$ and $\underline{\theta}_{n-1}$ correspond to $a_{n-2} = 1$ and $a_{n-2} = 0$ respectively then

$$\frac{f_h(\bar{\theta}_{n-1})}{f_l(\bar{\theta}_{n-1})} = \frac{\Pr(a_{n-2} = 1|l) \Pr(l)}{\Pr(a_{n-2} = 1|h) \Pr(h)} \quad \text{and} \quad \frac{f_h(\underline{\theta}_{n-1})}{f_l(\underline{\theta}_{n-1})} = \frac{\Pr(a_{n-2} = 0|l) \Pr(l)}{\Pr(a_{n-2} = 0|h) \Pr(h)}$$

Then $\Pr(a_{n-2} = 1|h) \Pr(h) = \frac{f_l(\bar{\theta}_{n-1})}{f_h(\bar{\theta}_{n-1})} \Pr(a_{n-2} = 1|l) \Pr(l)$ and $\Pr(a_{n-2} = 0|l) \Pr(l) = \frac{f_h(\underline{\theta}_{n-1})}{f_l(\underline{\theta}_{n-1})} \Pr(a_{n-2} = 0|h) \Pr(h)$. Then the thresholds become

$$\begin{aligned} \frac{f_h(\hat{\theta}_n)}{f_l(\hat{\theta}_n)} &= \frac{\Pr(a_{n-1}|l) \Pr(l)}{\Pr(a_{n-1}|h) \Pr(h)} \\ &= \frac{\Pr(a_{n-1}|a_{n-2} = 1, l) \Pr(a_{n-2} = 1|l) \Pr(l) + \Pr(a_{n-1}|a_{n-2} = 0, l) \Pr(a_{n-2} = 0|l) \Pr(l)}{\Pr(a_{n-1}|a_{n-2} = 1, h) \Pr(a_{n-2} = 1|h) \Pr(h) + \Pr(a_{n-1}|a_{n-2} = 0, h) \Pr(a_{n-2} = 0|h) \Pr(h)} \\ &= \frac{\Pr(a_{n-1}|a_{n-2} = 1, l) \Pr(a_{n-2} = 1|l) \Pr(l) + \Pr(a_{n-1}|a_{n-2} = 0, l) \frac{f_h(\underline{\theta}_{n-1})}{f_l(\underline{\theta}_{n-1})} \Pr(a_{n-2} = 0|h) \Pr(h)}{\Pr(a_{n-1}|a_{n-2} = 1, h) \frac{f_l(\bar{\theta}_{n-1})}{f_h(\bar{\theta}_{n-1})} \Pr(a_{n-2} = 1|l) \Pr(l) + \Pr(a_{n-1}|a_{n-2} = 0, h) \Pr(a_{n-2} = 0|h) \Pr(h)} \end{aligned}$$

As noted, $\bar{\theta}_2 = -\frac{1}{2}$ and $\underline{\theta}_2 = \frac{1}{2}$ so that $\bar{\theta}_2 + \underline{\theta}_2 = 0$. Also, $\Pr(a_1 = 1|l) = (1 - F_l(0)) = \frac{1}{4}$ and $\Pr(a_1 = 0|h) = F_h(0) = \frac{1}{4}$ so that $\Pr(a_1 = 1|l) = \Pr(a_1 = 0|h)$. Conjecture that $\bar{\theta}_{n-1} + \underline{\theta}_{n-1} = 0$ and $\Pr(a_{n-2} = 1|l) = \Pr(a_{n-2} = 0|h)$ for $n \geq 4$. Then since $\Pr(h) = \Pr(l)$, this reduces to

$$\frac{f_h(\hat{\theta}_n)}{f_l(\hat{\theta}_n)} = \frac{f_h(\bar{\theta}_{n-1})[\Pr(a_{n-1}|a_{n-2} = 1, l) f_l(\underline{\theta}_{n-1}) + \Pr(a_{n-1}|a_{n-2} = 0, l) f_h(\underline{\theta}_{n-1})]}{f_l(\underline{\theta}_{n-1})[\Pr(a_{n-1}|a_{n-2} = 1, h) f_l(\bar{\theta}_{n-1}) + \Pr(a_{n-1}|a_{n-2} = 0, h) f_h(\bar{\theta}_{n-1})]}$$

If $a_{n-1} = 1$

$$\begin{aligned}
\frac{f_h(\hat{\theta}_n)}{f_l(\hat{\theta}_n)} &= \frac{f_h(\bar{\theta}_{n-1})[\Pr(\theta > \bar{\theta}_{n-1}|l)f_l(\underline{\theta}_{n-1}) + \Pr(\theta > \underline{\theta}_{n-1}|l)f_h(\underline{\theta}_{n-1})]}{f_l(\underline{\theta}_{n-1})[\Pr(\theta > \bar{\theta}_{n-1}|h)f_l(\bar{\theta}_{n-1}) + \Pr(\theta > \underline{\theta}_{n-1}|h)f_h(\bar{\theta}_{n-1})]} \\
&= \frac{f_h(\bar{\theta}_{n-1})[(1 - F_l(\bar{\theta}_{n-1}))f_l(\underline{\theta}_{n-1}) + (1 - F_l(\underline{\theta}_{n-1}))f_h(\underline{\theta}_{n-1})]}{f_l(\underline{\theta}_{n-1})[(1 - F_h(\bar{\theta}_{n-1}))f_l(\bar{\theta}_{n-1}) + (1 - F_h(\underline{\theta}_{n-1}))f_h(\bar{\theta}_{n-1})]} \\
&= \frac{f_h(\bar{\theta}_{n-1})[(1 - F_l(\bar{\theta}_{n-1}))f_l(-\bar{\theta}_{n-1}) + (1 - F_l(-\bar{\theta}_{n-1}))f_h(-\bar{\theta}_{n-1})]}{f_l(-\bar{\theta}_{n-1})[(1 - F_h(\bar{\theta}_{n-1}))f_l(\bar{\theta}_{n-1}) + (1 - F_h(-\bar{\theta}_{n-1}))f_h(\bar{\theta}_{n-1})]}
\end{aligned}$$

and by symmetry of the signal functions

$$= \frac{(1 - F_l(\bar{\theta}_{n-1}))f_h(\bar{\theta}_{n-1}) + F_h(\bar{\theta}_{n-1})f_l(\bar{\theta}_{n-1})}{(1 - F_h(\bar{\theta}_{n-1}))f_l(\bar{\theta}_{n-1}) + F_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1})}$$

and similarly if $a_{n-1} = 0$

$$\begin{aligned}
\frac{f_h(\hat{\theta}_n)}{f_l(\hat{\theta}_n)} &= \frac{f_h(\bar{\theta}_{n-1})[F_l(\bar{\theta}_{n-1})f_l(\underline{\theta}_{n-1}) + F_l(\underline{\theta}_{n-1})f_h(\underline{\theta}_{n-1})]}{f_l(\underline{\theta}_{n-1})[F_h(\bar{\theta}_{n-1})f_l(\bar{\theta}_{n-1}) + F_h(\underline{\theta}_{n-1})f_h(\bar{\theta}_{n-1})]} \\
&= \frac{f_h(\bar{\theta}_{n-1})[F_l(\bar{\theta}_{n-1})f_l(-\bar{\theta}_{n-1}) + F_l(-\bar{\theta}_{n-1})f_h(-\bar{\theta}_{n-1})]}{f_l(-\bar{\theta}_{n-1})[F_h(\bar{\theta}_{n-1})f_l(\bar{\theta}_{n-1}) + F_h(-\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1})]} \\
&= \frac{F_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1}) + (1 - F_h(\bar{\theta}_{n-1}))f_l(\bar{\theta}_{n-1})}{F_h(\bar{\theta}_{n-1})f_l(\bar{\theta}_{n-1}) + (1 - F_l(\bar{\theta}_{n-1}))f_h(\bar{\theta}_{n-1})}
\end{aligned}$$

For our canonical signal structure defined by $F_h(\theta) = \frac{1}{4}(1 + \theta)^2$ and $F_l(\theta) = 1 - \frac{1}{4}(1 - \theta)^2$, if

$$a_{n-1} = 1$$

$$\begin{aligned}
\frac{1 + \hat{\theta}_n}{1 - \hat{\theta}_n} &= \frac{\frac{1}{4}(1 - \bar{\theta}_{n-1})^2 \frac{1}{2}(1 + \bar{\theta}_{n-1}) + \frac{1}{4}(1 + \bar{\theta}_{n-1})^2 \frac{1}{2}(1 - \bar{\theta}_{n-1})}{[1 - \frac{1}{4}(1 + \bar{\theta}_{n-1})^2] \frac{1}{2}(1 - \bar{\theta}_{n-1}) + [1 - \frac{1}{4}(1 - \bar{\theta}_{n-1})^2] \frac{1}{2}(1 + \bar{\theta}_{n-1})} \\
&= \frac{(1 - \bar{\theta}_{n-1})^2(1 + \bar{\theta}_{n-1}) + (1 + \bar{\theta}_{n-1})^2(1 - \bar{\theta}_{n-1})}{(3 - 2\bar{\theta}_{n-1} - \bar{\theta}_{n-1}^2)(1 - \bar{\theta}_{n-1}) + (3 + 2\bar{\theta}_{n-1} - \bar{\theta}_{n-1}^2)(1 + \bar{\theta}_{n-1})} \\
&= \frac{(1 - \bar{\theta}_{n-1})(1 + \bar{\theta}_{n-1})[(1 - \bar{\theta}_{n-1}) + (1 + \bar{\theta}_{n-1})]}{6 - 2\bar{\theta}_{n-1}^2 + \bar{\theta}_{n-1}(4\bar{\theta}_{n-1})} \\
&= \frac{2(1 - \bar{\theta}_{n-1}^2)}{6 + 2\bar{\theta}_{n-1}^2} \\
\implies \hat{\theta}_n &= -\frac{1}{2}(1 + \bar{\theta}_{n-1}^2)
\end{aligned}$$

and if $a_{n-1} = 0$

$$\begin{aligned}
\frac{1 + \hat{\theta}_n}{1 - \hat{\theta}_n} &= \frac{[1 - \frac{1}{4}(1 - \bar{\theta}_{n-1})^2] \frac{1}{2}(1 + \bar{\theta}_{n-1}) + [1 - \frac{1}{4}(1 + \bar{\theta}_{n-1})^2] \frac{1}{2}(1 - \bar{\theta}_{n-1})}{\frac{1}{4}(1 + \bar{\theta}_{n-1})^2 \frac{1}{2}(1 - \bar{\theta}_{n-1}) + \frac{1}{4}(1 - \bar{\theta}_{n-1})^2 \frac{1}{2}(1 + \bar{\theta}_{n-1})} \\
&= \frac{(3 + 2\bar{\theta}_{n-1} - \bar{\theta}_{n-1}^2)(1 + \bar{\theta}_{n-1}) + (3 - 2\bar{\theta}_{n-1} - \bar{\theta}_{n-1}^2)(1 - \bar{\theta}_{n-1})}{(1 + \bar{\theta}_{n-1})^2(1 - \bar{\theta}_{n-1}) + (1 - \bar{\theta}_{n-1})^2(1 + \bar{\theta}_{n-1})} \\
&= \frac{6 - 2\bar{\theta}_{n-1}^2 + \bar{\theta}_{n-1}(4\bar{\theta}_{n-1})}{(1 + \bar{\theta}_{n-1})(1 - \bar{\theta}_{n-1})[(1 + \bar{\theta}_{n-1}) + (1 - \bar{\theta}_{n-1})]} \\
&= \frac{6 + 2\bar{\theta}_{n-1}^2}{2(1 - \bar{\theta}_{n-1}^2)} \\
\implies \hat{\theta}_n &= \frac{1}{2}(1 + \bar{\theta}_{n-1}^2)
\end{aligned}$$

Thus we have the recursive result that

$$\hat{\theta}_n = \begin{cases} -\frac{1}{2}(1 + \bar{\theta}_{n-1}^2) & \text{if } a_{n-1} = 1 \\ \frac{1}{2}(1 + \bar{\theta}_{n-1}^2) & \text{if } a_{n-1} = 0 \end{cases}$$

Finally, recall that we conjectured that $\bar{\theta}_{n-1} + \theta_{n-1} = 0$ and that $\Pr(a_{n-2} = 1|l) = \Pr(a_{n-2} =$

$0|h)$ and the resulting threshold $\hat{\theta}_n$ also satisfied $\bar{\theta}_n + \underline{\theta}_n = 0$. Moreover,

$$\begin{aligned} \Pr(a_{n-1} = 1|l) &= \Pr(a_{n-1} = 1|a_{n-2} = 1, l) \Pr(a_{n-2} = 1|l) \\ &\quad + \Pr(a_{n-1} = 1|a_{n-2} = 0, l) \Pr(a_{n-2} = 0|l) \\ &= (1 - F_l(\bar{\theta}_{n-1})) \Pr(a_{n-2} = 1|l) + (1 - F_l(\underline{\theta}_{n-1}))(1 - \Pr(a_{n-2} = 1|l)) \end{aligned}$$

and by symmetry of F_l and F_h and $\bar{\theta}_{n-1} + \underline{\theta}_{n-1} = 0$ this becomes

$$\begin{aligned} &= F_h(\underline{\theta}_{n-1}) \Pr(a_{n-2} = 1|l) + F_h(\bar{\theta}_{n-1})(1 - \Pr(a_{n-2} = 1|l)) \\ &= F_h(\bar{\theta}_{n-1}) + \Pr(a_{n-2} = 1|l)(F_h(\underline{\theta}_{n-1}) - F_h(\bar{\theta}_{n-1})) \end{aligned}$$

$$\begin{aligned} \Pr(a_{n-1} = 0|h) &= \Pr(a_{n-1} = 0|a_{n-2} = 1, h) \Pr(a_{n-2} = 1|h) \\ &\quad + \Pr(a_{n-1} = 0|a_{n-2} = 0, h) \Pr(a_{n-2} = 0|h) \\ &= F_h(\bar{\theta}_{n-1})(1 - \Pr(a_{n-2} = 0|h)) + F_h(\underline{\theta}_{n-1}) \Pr(a_{n-2} = 0|h) \\ &= F_h(\bar{\theta}_{n-1}) + \Pr(a_{n-2} = 0|h)(F_h(\underline{\theta}_{n-1}) - F_h(\bar{\theta}_{n-1})) \end{aligned}$$

So that

$$\Pr(a_{n-1} = 1|l) - \Pr(a_{n-1} = 0|h) = (\Pr(a_{n-2} = 1|l) - \Pr(a_{n-2} = 0|h))(F_h(\underline{\theta}_{n-1}) - F_h(\bar{\theta}_{n-1}))$$

and $\Pr(a_{n-1} = 1|l) = \Pr(a_{n-1} = 0|h)$ since $\Pr(a_{n-2} = 1|l) = \Pr(a_{n-2} = 0|h)$. So by induction, $\bar{\theta}_n + \underline{\theta}_n = 0$ and $\Pr(a_n = 1|l) = \Pr(a_n = 0|h)$ for all n . \square

Lemma 3.2. *In the case of limited histories in the sense that $A_n = a_n$, if $\Pr(a_{n-1} = 0|l) = \Pr(a_{n-1} = 1|h)$ then*

$$\Pr(a_{n-1} = 1|h) = \frac{f_l(\bar{\theta}_n)}{f_l(\bar{\theta}_n) + f_h(\bar{\theta}_n)}$$

Proof. As noted above, in the case of limited histories beliefs are recursively related according to

$$\Pr(a_{n-1} = 1|h) = \frac{f_l(\bar{\theta}_n)}{f_h(\bar{\theta}_n)} \Pr(a_{n-1} = 1|l) = \frac{f_l(\bar{\theta}_n)}{f_h(\bar{\theta}_n)} (1 - \Pr(a_{n-1} = 0|l))$$

and with $\Pr(a_{n-1} = 0|l) = \Pr(a_{n-1} = 1|h)$,

$$\Pr(a_{n-1} = 1|h) = \frac{f_l(\bar{\theta}_n)}{f_l(\bar{\theta}_n) + f_h(\bar{\theta}_n)}$$

□

Proof of Proposition 5. We found above that in equilibrium $\bar{\theta}_1 = 0$ and $\bar{\theta}_n = -\frac{1}{2}(1 + \bar{\theta}_{n-1}^2)$ for $n \geq 2$ so that

$$\bar{\theta}_{n+1} - \bar{\theta}_n = -\frac{1}{2}(1 + \bar{\theta}_n^2) - \bar{\theta}_n = -\frac{1}{2}(1 + \bar{\theta}_n)^2 < 0$$

Then $\bar{\theta}_n$ is a decreasing sequence bounded below by -1 and as such must converge. Moreover

$$\lim_{n \rightarrow \infty} |\bar{\theta}_{n+1} - \bar{\theta}_n| = \lim_{n \rightarrow \infty} \frac{1}{2}(1 + \bar{\theta}_n)^2 = 0 \iff \lim_{n \rightarrow \infty} \bar{\theta}_n = -1$$

And thus $\bar{\theta}_n \rightarrow -1$. Since we proved above that $\underline{\theta}_n = -\bar{\theta}_n$, $\underline{\theta}_n \rightarrow 1$.

We showed in the previous proposition that $\Pr(a_{n-1} = 1|l) = \Pr(a_{n-1} = 0|h)$. But since $\Pr(a_{n-1} = 1|l) = 1 - \Pr(a_{n-1} = 0|l)$ and $\Pr(a_{n-1} = 0|h) = 1 - \Pr(a_{n-1} = 1|h)$ it must also be that $\Pr(a_{n-1} = 0|l) = \Pr(a_{n-1} = 1|h)$.

By lemma 2,

$$\Pr(a_{n-1} = 1|h) = \frac{f_l(\bar{\theta}_n)}{f_l(\bar{\theta}_n) + f_h(\bar{\theta}_n)} = \frac{\frac{1}{2}(1 - \bar{\theta}_n)}{\frac{1}{2}(1 - \bar{\theta}_n) + \frac{1}{2}(1 + \bar{\theta}_n)} = \frac{1}{2}(1 - \bar{\theta}_n)$$

Then $\lim_{n \rightarrow \infty} \Pr(a_{n-1} = 1|h) = \lim_{n \rightarrow \infty} \frac{1}{2}(1 - \bar{\theta}_n) = 1$. Since $\Pr(a_{n-1} = 0|l) = \Pr(a_{n-1} = 1|h)$, $\lim_{n \rightarrow \infty} \Pr(a_{n-1} = 0|1) = 1$. Thus complete learning occurs. \square

Proof of Proposition 6. Assume $\Pr(a_{i-2}) = \frac{1}{2}$ given no a priori information possible about this action. As noted above,

$$\hat{\theta}_n = \frac{\sum_{i=1}^N \mu_i^n (\Pr(a_{i-1}|l) - \Pr(a_{i-1}|h))}{\sum_{i=1}^N \mu_i^n (\Pr(a_{i-1}|l) + \Pr(a_{i-1}|h))}$$

$$\begin{aligned} \Pr(a_{i-1}|l) + \Pr(a_{i-1}|h) &= \sum_{a_{i-2} \in \mathcal{A}} \Pr(a_{i-1}|a_{i-2}, l) \Pr(a_{i-2}|l) + \sum_{a_{i-2} \in \mathcal{A}} \Pr(a_{i-1}|a_{i-2}, h) \Pr(a_{i-2}|h) \\ &= F_l(\bar{\theta}_{n-1}) \Pr(a_{i-2} = 1|l) + F_l(\underline{\theta}_{n-1}) \Pr(a_{i-2} = 0|l) \\ &\quad + F_h(\bar{\theta}_{n-1}) \Pr(a_{i-2} = 1|h) + F_h(\underline{\theta}_{n-1}) \Pr(a_{i-2} = 0|h) \\ &= F_l(\bar{\theta}_{n-1}) \Pr(a_{i-2} = 1|l) + F_l(\underline{\theta}_{n-1})(1 - \Pr(a_{i-2} = 1|l)) \\ &\quad + F_h(\bar{\theta}_{n-1}) \Pr(a_{i-2} = 1|h) + F_h(\underline{\theta}_{n-1})(1 - \Pr(a_{i-2} = 1|h)) \end{aligned}$$

If $\Pr(a_{i-2}|l) \Pr(l) + \Pr(a_{i-2}|h) \Pr(h) = \frac{1}{2}$, $\Pr(a_{i-2}|l) + \Pr(a_{i-2}|h) = 1$ and

$$\begin{aligned} &= F_l(\bar{\theta}_{n-1}) \Pr(a_{i-2} = 1|l) + F_l(\underline{\theta}_{n-1})(1 - \Pr(a_{i-2} = 1|l)) \\ &\quad + F_h(\bar{\theta}_{n-1})(1 - \Pr(a_{i-2} = 1|l)) + F_h(\underline{\theta}_{n-1}) \Pr(a_{i-2} = 1|l) \\ &= \Pr(a_{i-2} = 1|l) [F_l(\bar{\theta}_{n-1}) - F_l(\underline{\theta}_{n-1}) - F_h(\bar{\theta}_{n-1}) + F_h(\underline{\theta}_{n-1})] + F_l(\underline{\theta}_{n-1}) + F_h(\bar{\theta}_{n-1}) \\ &= \Pr(a_{i-2} = 1|l) \left(\frac{1}{2}(\underline{\theta}_{n-1}^2 - \bar{\theta}_{n-1}^2) \right) + 1 + \frac{1}{4} (2(\underline{\theta}_{n-1} + \bar{\theta}_{n-1}) + \bar{\theta}_{n-1}^2 - \underline{\theta}_{n-1}^2) \\ &= 1 + \frac{1}{2}(\underline{\theta}_{n-1} + \bar{\theta}_{n-1}) + (\underline{\theta}_{n-1}^2 - \bar{\theta}_{n-1}^2) \left(\frac{1}{4} - \frac{1}{2} \Pr(a_{i-2} = 1|l) \right) \\ &= 1 + \frac{1}{2}(\underline{\theta}_{n-1} + \bar{\theta}_{n-1}) \left(1 + (\underline{\theta}_{n-1} - \bar{\theta}_{n-1}) \left(\frac{1}{2} - \Pr(a_{i-2} = 1|l) \right) \right) \end{aligned}$$

And the thresholds become

$$\bar{\theta}_n = \frac{\sum_{i=1}^N \mu_i^n (\Pr(a_{i-1} = 1|l) - \Pr(a_{i-1} = 1|h))}{\sum_{i=1}^N \mu_i^n (1 + \frac{1}{2}(\theta_{n-1} + \bar{\theta}_{n-1}) (1 + (\theta_{n-1} - \bar{\theta}_{n-1}) (\frac{1}{2} - \Pr(a_{i-2} = 1|l)))}$$

$$\underline{\theta}_n = \frac{\sum_{i=1}^N \mu_i^n (\Pr(a_{i-1} = 0|l) - \Pr(a_{i-1} = 0|h))}{\sum_{i=1}^N \mu_i^n (1 + \frac{1}{2}(\theta_{n-1} + \bar{\theta}_{n-1}) (1 + (\theta_{n-1} - \bar{\theta}_{n-1}) (\frac{1}{2} - \Pr(a_{i-2} = 1|l)))}$$

It is easy to see then that

$$\begin{aligned} \bar{\theta}_n + \underline{\theta}_n &= \frac{\sum_{i=1}^N \mu_i^n (\Pr(a_{i-1} = 1|l) - \Pr(a_{i-1} = 1|h)) + \sum_{i=1}^N \mu_i^n (\Pr(a_{i-1} = 0|l) - \Pr(a_{i-1} = 0|h))}{\sum_{i=1}^N \mu_i^n (1 + \frac{1}{2}(\theta_{n-1} + \bar{\theta}_{n-1}) (1 + (\theta_{n-1} - \bar{\theta}_{n-1}) (\frac{1}{2} - \Pr(a_{i-2} = 1|l)))} \\ &= \frac{\sum_{i=1}^N \mu_i^n ([\Pr(a_{i-1} = 1|l) + \Pr(a_{i-1} = 0|l)] - [\Pr(a_{i-1} = 1|h) + \Pr(a_{i-1} = 0|h)])}{\sum_{i=1}^N \mu_i^n (1 + \frac{1}{2}(\theta_{n-1} + \bar{\theta}_{n-1}) (1 + (\theta_{n-1} - \bar{\theta}_{n-1}) (\frac{1}{2} - \Pr(a_{i-2} = 1|l)))} = 0 \end{aligned}$$

Since $\bar{\theta}_n + \underline{\theta}_n = 0$, for each i

$$\begin{aligned} \Pr(a_{i-1}|l) + \Pr(a_{i-1}|h) &= 1 + \frac{1}{2}(\theta_{n-1} + \bar{\theta}_{n-1}) \left(1 + (\theta_{n-1} - \bar{\theta}_{n-1}) \left(\frac{1}{2} - \Pr(a_{i-2} = 1|l) \right) \right) \\ &= 1 \end{aligned}$$

and the threshold takes the form

$$\hat{\theta}_n = \sum_{i=1}^N \mu_i^n (\Pr(a_{i-1}|l) - \Pr(a_{i-1}|h))$$

Moreover, given that $\Pr(a_{i-1}|l) + \Pr(a_{i-1}|h) = 1$,

$$\hat{\theta}_n = 2 \sum_{i=1}^N \mu_i^n \Pr(a_{i-1}|l) - 1$$

If $\mu_n^n = \gamma$ and $\mu_i^n = \frac{1-\gamma}{N-1}$ for $i \neq n$ then

$$\begin{aligned}
\hat{\theta}_n &= 2\gamma \Pr(a_{n-1}|l) + 2 \sum_{i \neq n}^N \left(\frac{1-\gamma}{N-1} \right) \Pr(a_{i-1}|l) - 1 \\
&= 2\gamma \Pr(a_{n-1}|l) - 2 \left(\frac{1-\gamma}{N-1} \right) \Pr(a_{n-1}|l) + 2 \sum_{i=1}^N \left(\frac{1-\gamma}{N-1} \right) \Pr(a_{i-1}|l) - 1 \\
&= 2 \left(\frac{N\gamma-1}{N-1} \right) \Pr(a_{n-1}|l) - 1 + 2 \left(\frac{1-\gamma}{N-1} \right) \sum_{i=0}^{N-1} \Pr(a_i|l)
\end{aligned}$$

Consider $a_{n-1} = 1$. Since $\Pr(a_0 = 1|l) = \frac{1}{4}$ threshold $\bar{\theta}_1$ takes the form

$$\begin{aligned}
\bar{\theta}_1 &= 2 \left(\frac{N\gamma-1}{N-1} \right) \frac{1}{4} - 1 + 2 \left(\frac{1-\gamma}{N-1} \right) \sum_{i=0}^{N-1} \Pr(a_i = 1|l) \\
&= \frac{N(\gamma-2)+1}{2(N-1)} + 2 \left(\frac{1-\gamma}{N-1} \right) \sum_{i=0}^{N-1} \Pr(a_i = 1|l)
\end{aligned}$$

and for $n \geq 2$

$$\bar{\theta}_n = 2 \left(\frac{N\gamma-1}{N-1} \right) \Pr(a_{n-1} = 1|l) + \bar{\theta}_1 - \frac{N\gamma-1}{2(N-1)}$$

If $a_{n-1} = 0$, since $\Pr(a_0 = 0|l) = \frac{3}{4}$ threshold θ_1 takes the form

$$\begin{aligned}
\theta_1 &= 2 \left(\frac{N\gamma-1}{N-1} \right) \frac{3}{4} - 1 + 2 \left(\frac{1-\gamma}{N-1} \right) \sum_{i=0}^{N-1} \Pr(a_i = 0|l) \\
&= \frac{N(3\gamma-2)-1}{2(N-1)} + 2 \left(\frac{1-\gamma}{N-1} \right) \sum_{i=0}^{N-1} \Pr(a_i = 0|l)
\end{aligned}$$

and for $n \geq 2$

$$\theta_n = 2 \left(\frac{N\gamma-1}{N-1} \right) \Pr(a_{n-1} = 0|l) + \theta_1 - \frac{3(N\gamma-1)}{2(N-1)}$$

□

Lemma 3.3. For the canonical signal structure, if $\bar{\theta}_n + \underline{\theta}_n = 0$, then

$$\begin{aligned}\Pr(a_n = 1|l) &= \frac{1}{4}(1 + \bar{\theta}_n)^2 - \bar{\theta}_n \Pr(a_{n-1} = 1|l) && \text{and} \\ \Pr(a_n = 0|l) &= \frac{1}{4}(3 + \underline{\theta}_n)(1 - \underline{\theta}_n) + \underline{\theta}_n \Pr(a_{n-1} = 0|l)\end{aligned}$$

Proof.

$$\begin{aligned}\Pr(a_n = 1|l) &= \Pr(a_n = 1|a_{n-1} = 1, l) \Pr(a_{n-1} = 1|l) + \Pr(a_n = 1|a_{n-1} = 0, l) \Pr(a_{n-1} = 0|l) \\ &= (1 - F_l(\bar{\theta}_n)) \Pr(a_{n-1} = 1|l) + (1 - F_l(\underline{\theta}_n)) \Pr(a_{n-1} = 0|l) \\ &= (1 - F_l(\bar{\theta}_n)) \Pr(a_{n-1} = 1|l) + F_h(\bar{\theta}_n)(1 - \Pr(a_{n-1} = 1|l)) \\ &= \Pr(a_{n-1} = 1|l) \left[\frac{1}{4}(1 - \bar{\theta}_n)^2 - \frac{1}{4}(1 + \bar{\theta}_n)^2 \right] + \frac{1}{4}(1 + \bar{\theta}_n)^2 \\ &= \frac{1}{4}(1 + \bar{\theta}_n)^2 - \bar{\theta}_n \Pr(a_{n-1} = 1|l) \\ \Pr(a_n = 0|l) &= 1 - \Pr(a_n = 1|l) = \bar{\theta}_n \Pr(a_{n-1} = 1|l) - \frac{1}{4}(1 + \bar{\theta}_n)^2 \\ &= 1 - \frac{1}{4}(1 - \bar{\theta}_n)^2 - \bar{\theta}_n \Pr(a_{n-1} = 0|l) = \frac{1}{4}(3 + 2\bar{\theta}_n - \bar{\theta}_n^2) - \bar{\theta}_n \Pr(a_{n-1} = 0|l) \\ &= \frac{1}{4}(3 - \bar{\theta}_n)(1 + \bar{\theta}_n) - \bar{\theta}_n \Pr(a_{n-1} = 0|l) = \frac{1}{4}(3 + \underline{\theta}_n)(1 - \underline{\theta}_n) + \underline{\theta}_n \Pr(a_{n-1} = 0|l)\end{aligned}$$

□

Proof of Proposition 7. By definition $\bar{\theta}_n = 2 \left(\frac{N\gamma-1}{N-1} \right) \Pr(a_{n-1} = 1|l) + \bar{\theta}_1 - \frac{N\gamma-1}{2(N-1)}$ so that

$$\bar{\theta}_{n+1} - \bar{\theta}_n = 2 \left(\frac{N\gamma-1}{N-1} \right) (\Pr(a_n = 1|l) - \Pr(a_{n-1} = 1|l))$$

From lemma 3, $\Pr(a_n = 1|l) = \frac{1}{4}(1 + \bar{\theta}_n)^2 - \bar{\theta}_n \Pr(a_{n-1} = 1|l)$ and this becomes

$$\begin{aligned}\bar{\theta}_{n+1} - \bar{\theta}_n &= 2 \left(\frac{N\gamma - 1}{N - 1} \right) \left[\frac{1}{4}(1 + \bar{\theta}_n)^2 - \bar{\theta}_n \Pr(a_{n-1} = 1|l) - \Pr(a_{n-1} = 1|l) \right] \\ &= \left(\frac{N\gamma - 1}{2(N - 1)} \right) [(1 + \bar{\theta}_n)^2 - 4 \Pr(a_{n-1} = 1|l)(1 + \bar{\theta}_n)] \\ &= \left(\frac{N\gamma - 1}{2(N - 1)} \right) (1 + \bar{\theta}_n) [1 + \bar{\theta}_n - 4 \Pr(a_{n-1} = 1|l)]\end{aligned}$$

Suppose $\bar{\theta}_n$ converges to $\bar{\theta}$. Then

$$\left(\frac{N\gamma - 1}{2(N - 1)} \right) (1 + \bar{\theta}) \left[1 + \bar{\theta} - 4 \lim_{n \rightarrow \infty} \Pr(a_n = 1|l) \right] = 0$$

This is satisfied if $\bar{\theta} = -1$. If $\bar{\theta} > -1$ then

$$4 \lim_{n \rightarrow \infty} \Pr(a_n = 1|l) = 1 + \bar{\theta}$$

By definition $\underline{\theta}_n = 2 \left(\frac{N\gamma - 1}{N - 1} \right) \Pr(a_{n-1} = 0|l) + \underline{\theta}_1 - \frac{3(N\gamma - 1)}{2(N - 1)}$ so that

$$\underline{\theta}_{n+1} - \underline{\theta}_n = 2 \left(\frac{N\gamma - 1}{N - 1} \right) (\Pr(a_n = 0|l) - \Pr(a_{n-1} = 0|l))$$

From lemma 3, $\Pr(a_n = 0|l) = \frac{1}{4}(3 + \underline{\theta}_n)(1 - \underline{\theta}_n) + \underline{\theta}_n \Pr(a_{n-1} = 0|l)$ and this becomes

$$\begin{aligned}\underline{\theta}_{n+1} - \underline{\theta}_n &= 2 \left(\frac{N\gamma - 1}{N - 1} \right) \left(\frac{1}{4}(3 + \underline{\theta}_n)(1 - \underline{\theta}_n) + \underline{\theta}_n \Pr(a_{n-1} = 0|l) - \Pr(a_{n-1} = 0|l) \right) \\ &= 2 \left(\frac{N\gamma - 1}{N - 1} \right) \frac{1}{4} (1 - \underline{\theta}_n) ((3 + \underline{\theta}_n) - 4 \Pr(a_{n-1} = 0|l))\end{aligned}$$

Suppose $\underline{\theta}_n$ converges to $\underline{\theta}$. Then

$$2 \left(\frac{N\gamma - 1}{N - 1} \right) \frac{1}{4} (1 - \underline{\theta}) \left((3 + \underline{\theta}) - 4 \lim_{n \rightarrow \infty} \Pr(a_n = 0|l) \right)$$

This is satisfied if $\underline{\theta} = 1$. If $\underline{\theta} < 1$ then

$$4 \lim_{n \rightarrow \infty} \Pr(a_n = 0|l) = 3 + \underline{\theta}$$

Then we have two cases

(i)

$$\lim_{n \rightarrow \infty} \bar{\theta}_n = -1 \text{ and } \lim_{n \rightarrow \infty} \underline{\theta}_n = 1$$

(ii)

$$\lim_{n \rightarrow \infty} \Pr(a_n = 0|l) = \frac{1}{4}(3 + \underline{\theta}) \text{ and } \lim_{n \rightarrow \infty} \Pr(a_n = 1|l) = \frac{1}{4}(1 + \bar{\theta})$$

As shown above,

$$\bar{\theta}_n = 2 \left(\frac{N\gamma - 1}{N - 1} \right) \Pr(a_{n-1} = 1|l) - 1 + 2 \left(\frac{1 - \gamma}{N - 1} \right) \sum_{i=0}^{N-1} \Pr(a_i = 1|l)$$

If $\bar{\theta}_n \rightarrow \bar{\theta}$, then $\Pr(a_{n-1} = 1|l) \rightarrow \lim_{n \rightarrow \infty} \Pr(a_n = 1|l)$, $\sum_{i=0}^{N-1} \Pr(a_i = 1|l) \rightarrow N \lim_{n \rightarrow \infty} \Pr(a_n = 1|l)$, and

$$\bar{\theta} = 2 \lim_{n \rightarrow \infty} \Pr(a_n = 1|l) - 1$$

Similarly, if $\underline{\theta}_n \rightarrow \underline{\theta}$

$$\underline{\theta} = 2 \lim_{n \rightarrow \infty} \Pr(a_n = 0|l) - 1$$

In case (i) $\bar{\theta} = -1$ and $\underline{\theta} = 1$ so $\lim_{n \rightarrow \infty} \Pr(a_n = 1|l) = 0$ and $\lim_{n \rightarrow \infty} \Pr(a_n = 0|l) = 1$.

In case (ii)

$$\bar{\theta} = 2 \lim_{n \rightarrow \infty} \Pr(a_n = 1|l) - 1 = \frac{1}{2}(1 + \bar{\theta}) - 1 = \frac{1}{2}\bar{\theta} - \frac{1}{2}$$

so that $\bar{\theta} = -1$. Similarly, if $\underline{\theta}_n \rightarrow \underline{\theta}$ then

$$\underline{\theta} = 2 \lim_{n \rightarrow \infty} \Pr(a_n = 0|l) - 1 = \frac{1}{2}(3 + \underline{\theta}) - 1 = \frac{1}{2}\underline{\theta} + \frac{1}{2}$$

so that $\underline{\theta} = 1$. From above $\Pr(a_n = 1|h) = 1 - \Pr(a_n = 1|l)$, and by the definition of the limit, $\lim_{n \rightarrow \infty} \Pr(a_n = 1|h) = 1 - \lim_{n \rightarrow \infty} \Pr(a_n = 1|l) = 1 - \frac{1}{4}(1 + \bar{\theta}) = 1$ and $\lim_{n \rightarrow \infty} \Pr(a_n = 0|l) = \frac{1}{4}(3 + \underline{\theta}) = 1$

Thus in either case $\lim_{n \rightarrow \infty} \Pr(a_n = 1|h) = 1$ and $\lim_{n \rightarrow \infty} \Pr(a_n = 0|l) = 1$ so complete learning occurs. \square

Proof of Proposition 8. From proposition 6,

$$\begin{aligned} \bar{\theta}_1 &= \frac{N(\gamma - 2) + 1}{2(N - 1)} + 2 \left(\frac{1 - \gamma}{N - 1} \right) \sum_{i=0}^{N-1} \Pr(a_i = 1|l) \\ \bar{\theta}_n &= 2 \left(\frac{N\gamma - 1}{N - 1} \right) \Pr(a_{n-1} = 1|l) + \bar{\theta}_1 - \frac{N\gamma - 1}{2(N - 1)} \end{aligned}$$

From proposition 7, $\lim_{i \rightarrow \infty} \Pr(a_i = 1|l) = 0$ so that $\lim_{N \rightarrow \infty} 2 \left(\frac{1 - \gamma}{N - 1} \right) \sum_{i=0}^{N-1} \Pr(a_i = 1|l) = 0$. Thus $\lim_{N \rightarrow \infty} \bar{\theta}_1 = -\frac{2 - \gamma}{2}$ and $\lim_{N \rightarrow \infty} \bar{\theta}_n = 2\gamma \Pr(a_{n-1} = 1|l) - \frac{2 - \gamma}{2} - \frac{\gamma}{2} = 2\gamma \Pr(a_{n-1} = 1|l) - 1$. From lemma 3, $\Pr(a_{n-1} = 1|l) = \frac{1}{4}(1 + \bar{\theta}_{n-1})^2 - \bar{\theta}_{n-1} \Pr(a_{n-2} = 1|l)$. Moreover,

$\lim_{N \rightarrow \infty} \bar{\theta}_{n-1} = 2\gamma \Pr(a_{n-2} = 1|l) - 1$ which implies $\Pr(a_{n-2} = 1|l) = \frac{\bar{\theta}_{n-1} + 1}{2\gamma}$ and

$$\begin{aligned} \Pr(a_{n-1} = 1|l) &= \frac{1}{4}(1 + \bar{\theta}_{n-1})^2 - \bar{\theta}_{n-1} \frac{\bar{\theta}_{n-1} + 1}{2\gamma} \\ &= \frac{1}{4}(1 + 2\bar{\theta}_{n-1} + \bar{\theta}_{n-1}^2) - \frac{\bar{\theta}_{n-1}^2 + \bar{\theta}_{n-1}}{2\gamma} \\ &= \frac{1}{4\gamma}(\gamma + 2\bar{\theta}_{n-1}(\gamma - 1) + \bar{\theta}_{n-1}^2(\gamma - 2)) \end{aligned}$$

Then as $N \rightarrow \infty$,

$$\begin{aligned} \bar{\theta}_n &= 2\gamma \Pr(a_{n-1} = 1|l) - 1 = 2\gamma \left[\frac{1}{4\gamma}(\gamma + 2\bar{\theta}_{n-1}(\gamma - 1) + \bar{\theta}_{n-1}^2(\gamma - 2)) \right] - 1 \\ &= \frac{1}{2}(\gamma + \bar{\theta}_{n-1}^2(\gamma - 2)) + \bar{\theta}_{n-1}(\gamma - 1) - 1 \\ &= -\frac{2-\gamma}{2}(1 + \bar{\theta}_{n-1}^2) - (1-\gamma)\bar{\theta}_{n-1} \\ &= -\frac{1}{2}(1 + \bar{\theta}_{n-1}^2) - \frac{1-\gamma}{2}(1 + \bar{\theta}_{n-1})^2 \end{aligned}$$

Thus $\lim_{N \rightarrow \infty} \bar{\theta}_n = -\frac{1}{2}(1 + \bar{\theta}_{n-1}^2) - \frac{1-\gamma}{2}(1 + \bar{\theta}_{n-1})^2$ and since $\theta_n = -\bar{\theta}_n$ for all n , $\lim_{N \rightarrow \infty} \theta_n = \frac{1}{2}(1 + \bar{\theta}_{n-1}^2) + \frac{1-\gamma}{2}(1 + \bar{\theta}_{n-1})^2$. \square

Proof of proposition 9. By definition

$$\mathbb{E}[u(\theta)|H_n, h] = \Pr(a_n = 1|h) - \Pr(a_n = 1|l) = 1 - 2\Pr(a_n = 1|l)$$

From proposition 8, $\Pr(a_n = 1|l) = \frac{1}{4\gamma}(\gamma + 2\bar{\theta}_n(\gamma - 1) + \bar{\theta}_n^2(\gamma - 2))$, so that

$$\begin{aligned} \mathbb{E}[u(\theta)|H_n, h] &= 1 - 2 \left(\frac{1}{4\gamma} \right) (\gamma + 2\bar{\theta}_n(\gamma - 1) + \bar{\theta}_n^2(\gamma - 2)) \\ &= \left(\frac{1}{2\gamma} \right) (\gamma + 2\bar{\theta}_n(1 - \gamma) + \bar{\theta}_n^2(2 - \gamma)) \\ &= \frac{1}{2\gamma}(1 + \bar{\theta}_n^2) + \frac{1-\gamma}{2\gamma}(\bar{\theta}_n^2 + 2\bar{\theta}_n - 1) \end{aligned}$$

□

Lemma 3.4. *If $\frac{f_h(\theta)}{f_l(\theta)}$ exhibits the strict Monotone Likelihood Ratio Property in the sense that $\theta_1 > \theta_0$ implies $\frac{f_h(\theta_1)}{f_l(\theta_1)} > \frac{f_h(\theta_0)}{f_l(\theta_0)}$ then*

$$(i) \quad \frac{f_h(\theta)}{f_l(\theta)} > \frac{F_h(\theta)}{F_l(\theta)} \text{ for all } \theta \in \text{supp}(F)^\circ$$

$$(ii) \quad F_l \text{ strictly First Order Stochastically Dominates } F_h \text{ in that } F_l(\theta) > F_h(\theta) \text{ for all } \theta \in \text{supp}(F)^\circ$$

Proof. Let $\theta_0, \theta_1 \in \text{supp}(F)^\circ$ with $\theta_1 > \theta_0$. Then by the MLRP $f_h(\theta_1)f_l(\theta_0) > f_h(\theta_0)f_l(\theta_1)$ and integrating with respect to θ_0 ,

(i)

$$\begin{aligned} \int_{-\infty}^{\theta_1} f_h(\theta_1)f_l(\theta_0)d\theta_0 &> \int_{-\infty}^{\theta_1} f_h(\theta_0)f_l(\theta_1)d\theta_0 \\ \implies f_h(\theta_1)F_l(\theta_1) &> F_h(\theta_1)f_l(\theta_1) \\ \implies \frac{f_h(\theta_1)}{f_l(\theta_1)} &> \frac{F_h(\theta_1)}{F_l(\theta_1)} \end{aligned}$$

(ii) Integrating instead with respect to θ_1 ,

$$\begin{aligned} \int_{\theta_0}^{-\infty} f_h(\theta_1)f_l(\theta_0)d\theta_1 &> \int_{\theta_0}^{-\infty} f_h(\theta_0)f_l(\theta_1)d\theta_1 \\ \implies (1 - F_h(\theta_0))f_l(\theta_0) &> f_h(\theta_0)(1 - F_l(\theta_0)) \\ \implies \frac{(1 - F_h(\theta_0))}{(1 - F_l(\theta_0))} &> \frac{f_h(\theta_0)}{f_l(\theta_0)} \end{aligned}$$

Combining the above gives $\frac{(1-F_h(\theta_0))}{(1-F_l(\theta_0))} > \frac{F_h(\theta_1)}{F_l(\theta_1)}$, or $F_l(\theta_1) > F_h(\theta_1)$ for any interior θ_1 .

□

Proof of Proposition 10. Recall from lemma 1 that the threshold strategy is defined recursively by

$$\frac{f_h(\hat{\theta}_n)}{f_l(\hat{\theta}_n)} = \begin{cases} \frac{(1-F_l(\hat{\theta}_{n-1}))f_h(\hat{\theta}_{n-1})}{(1-F_h(\hat{\theta}_{n-1}))f_l(\hat{\theta}_{n-1})} & \text{if } a_{n-1} = 1 \\ \frac{F_l(\hat{\theta}_{n-1})f_h(\hat{\theta}_{n-1})}{F_h(\hat{\theta}_{n-1})f_l(\hat{\theta}_{n-1})} & \text{if } a_{n-1} = 0 \end{cases}$$

Then

$$\begin{aligned} \mathbb{E} \left[\frac{f_h(\hat{\theta}_n)}{f_l(\hat{\theta}_n)} \middle| \theta_{n-1}, l \right] &= \Pr(a_{n-1} = 0|l) \frac{F_l(\hat{\theta}_{n-1})f_h(\hat{\theta}_{n-1})}{F_h(\hat{\theta}_{n-1})f_l(\hat{\theta}_{n-1})} + \Pr(a_{n-1} = 1|l) \frac{(1-F_l(\hat{\theta}_{n-1}))f_h(\hat{\theta}_{n-1})}{(1-F_h(\hat{\theta}_{n-1}))f_l(\hat{\theta}_{n-1})} \\ &= \frac{f_h(\hat{\theta}_{n-1})}{f_l(\hat{\theta}_{n-1})} \left[\Pr(a_{n-1} = 0|l) \frac{F_l(\hat{\theta}_{n-1})}{F_h(\hat{\theta}_{n-1})} + (1 - \Pr(a_{n-1} = 0|l)) \frac{(1-F_l(\hat{\theta}_{n-1}))}{(1-F_h(\hat{\theta}_{n-1}))} \right] \\ &= \frac{f_h(\hat{\theta}_{n-1})}{f_l(\hat{\theta}_{n-1})} \left[F_l \left(\frac{F_l}{F_h} - \frac{1-F_l}{1-F_h} \right) + \frac{1-F_l}{1-F_h} \right] \\ &= \frac{f_h(\hat{\theta}_{n-1})}{f_l(\hat{\theta}_{n-1})} \left[F_l \left(\frac{F_l(1-F_h) - F_h(1-F_l)}{F_h(1-F_h)} \right) + \frac{F_h(1-F_l)}{F_h(1-F_h)} \right] \\ &= \frac{f_h(\hat{\theta}_{n-1})}{f_l(\hat{\theta}_{n-1})} \frac{F_l^2 - 2F_lF_h + F_h}{F_h(1-F_h)} \\ &= \frac{f_h(\hat{\theta}_{n-1})}{f_l(\hat{\theta}_{n-1})} \frac{(F_l - F_h)^2}{F_h(1-F_h)} + \frac{f_h(\hat{\theta}_{n-1})}{f_l(\hat{\theta}_{n-1})} \end{aligned}$$

Thus $\mathbb{E} \left[\frac{f_h(\hat{\theta}_n)}{f_l(\hat{\theta}_n)} \middle| \theta_{n-1}, l \right] - \frac{f_h(\hat{\theta}_{n-1})}{f_l(\hat{\theta}_{n-1})} = \frac{f_h(\hat{\theta}_{n-1})}{f_l(\hat{\theta}_{n-1})} \frac{(F_l(\hat{\theta}_{n-1}) - F_h(\hat{\theta}_{n-1}))^2}{F_h(\hat{\theta}_{n-1})(1-F_h(\hat{\theta}_{n-1}))}$, and by strict First Order Stochastic Dominance this is strictly positive for interior signals θ . Then since it is a submartingale the likelihood ratio $\frac{f_h(\hat{\theta}_n)}{f_l(\hat{\theta}_n)}$ either converges to a limit or diverges, but since $F_l(\theta) - F_h(\theta) > 0$ for interior signals it cannot converge to a limit and hence must diverge. By the monotone

likelihood ratio property, since $\lim_{n \rightarrow \infty} \frac{f_h(\hat{\theta}_n)}{f_l(\hat{\theta}_n)} = \infty$ it must be that $\lim_{n \rightarrow \infty} \hat{\theta}_n = 1$. Similarly,

$$\begin{aligned}
\mathbb{E} \left[\frac{f_l(\hat{\theta}_n)}{f_h(\hat{\theta}_n)} \middle| \theta_{n-1}, h \right] &= \Pr(a_{n-1} = 0|h) \frac{F_h(\hat{\theta}_{n-1})f_l(\hat{\theta}_{n-1})}{F_l(\hat{\theta}_{n-1})f_h(\hat{\theta}_{n-1})} + \Pr(a_{n-1} = 1|h) \frac{(1 - F_h(\hat{\theta}_{n-1}))f_l(\hat{\theta}_{n-1})}{(1 - F_l(\hat{\theta}_{n-1}))f_h(\hat{\theta}_{n-1})} \\
&= \frac{f_l(\hat{\theta}_{n-1})}{f_h(\hat{\theta}_{n-1})} \left[\Pr(a_{n-1} = 0|h) \frac{F_h(\hat{\theta}_{n-1})}{F_l(\hat{\theta}_{n-1})} + (1 - \Pr(a_{n-1} = 0|h)) \frac{(1 - F_h(\hat{\theta}_{n-1}))}{(1 - F_l(\hat{\theta}_{n-1}))} \right] \\
&= \frac{f_l(\hat{\theta}_{n-1})}{f_h(\hat{\theta}_{n-1})} \left[F_h \left(\frac{F_h}{F_l} - \frac{1 - F_h}{1 - F_l} \right) + \frac{1 - F_h}{1 - F_l} \right] \\
&= \frac{f_l(\hat{\theta}_{n-1})}{f_h(\hat{\theta}_{n-1})} \left[F_h \left(\frac{F_h(1 - F_l) - F_l(1 - F_h)}{F_l(1 - F_l)} \right) + \frac{F_l(1 - F_h)}{F_l(1 - F_l)} \right] \\
&= \frac{f_l(\hat{\theta}_{n-1})}{f_h(\hat{\theta}_{n-1})} \frac{F_h^2 - 2F_hF_l + F_l}{F_l(1 - F_l)} \\
&= \frac{f_l(\hat{\theta}_{n-1})}{f_h(\hat{\theta}_{n-1})} \frac{(F_h - F_l)^2}{F_l(1 - F_l)} + \frac{f_l(\hat{\theta}_{n-1})}{f_h(\hat{\theta}_{n-1})}
\end{aligned}$$

Then $\mathbb{E} \left[\frac{f_l(\hat{\theta}_n)}{f_h(\hat{\theta}_n)} \middle| \theta_{n-1}, h \right] - \frac{f_l(\hat{\theta}_{n-1})}{f_h(\hat{\theta}_{n-1})} = \frac{f_l(\hat{\theta}_{n-1})}{f_h(\hat{\theta}_{n-1})} \frac{(F_h(\hat{\theta}_{n-1}) - F_l(\hat{\theta}_{n-1}))^2}{F_l(\hat{\theta}_{n-1})(1 - F_l(\hat{\theta}_{n-1}))}$ so the likelihood ratio $\frac{f_l(\hat{\theta}_n)}{f_h(\hat{\theta}_n)}$ is a submartingale conditional on $\omega = h$. Strict FOSD again implies that the likelihood ratio diverges so that $\hat{\theta}_n = -1$.

Together these results imply

$$\begin{aligned}
\lim_{n \rightarrow \infty} \Pr(a_n = 0|l) &= \lim_{n \rightarrow \infty} F_l(\hat{\theta}_n) = F(1) = 1 \\
\lim_{n \rightarrow \infty} \Pr(a_n = 1|h) &= \lim_{n \rightarrow \infty} (1 - F_h(\hat{\theta}_n)) = 1 - F(-1) = 1
\end{aligned}$$

so that complete learning occurs. □

Lemma 3.5. *Under the assumptions of social learning with limited history and general signals as in proposition 8, decision thresholds $\bar{\theta}_n$ if $a_{n-1} = 1$ and $\underline{\theta}_n$ if $a_{n-1} = 0$ satisfy $\bar{\theta}_1 < 0$ and $\bar{\theta}_n + \underline{\theta}_n = 0$. Moreover $\Pr(a_{n-2} = 1|l) = \Pr(a_{n-2} = 0|h)$ for all n .*

Proof. Consider the decision of the first agent. By the assumption of symmetry on F_l, F_h , agent 0 will play $a_0 = 1$ if and only if $\theta \geq 0$. As above, with prior $\Pr(h) = \frac{1}{2}$, agent 1 will

then set thresholds $\bar{\theta}_1$ if $a_0 = 1$ and $\underline{\theta}_1$ if $a_0 = 0$ such that

$$\frac{f_h(\bar{\theta}_1)}{f_l(\bar{\theta}_1)} = \frac{\frac{1}{2}(1 - F_l(0))}{\frac{1}{2}(1 - F_h(0))} \quad \text{and} \quad \frac{f_h(\underline{\theta}_1)}{f_l(\underline{\theta}_1)} = \frac{\frac{1}{2}(F_l(0))}{\frac{1}{2}(F_h(0))}$$

Then $\frac{f_h(\bar{\theta}_1)}{f_l(\bar{\theta}_1)} < 1$ since $F_l(0) > F_h(0)$ by strict FOSD. Thus $\bar{\theta}_1 < 0$. Also, since symmetry gives $F_l(0) = 1 - F_h(0)$,

$$\frac{f_l(-\bar{\theta}_1)}{f_h(-\bar{\theta}_1)} = \frac{f_h(\bar{\theta}_1)}{f_l(\bar{\theta}_1)} = \frac{1 - F_l(0)}{1 - F_h(0)} = \frac{F_h(0)}{F_l(0)}$$

so that $\frac{f_l(-\bar{\theta}_1)}{f_h(-\bar{\theta}_1)} = \frac{f_l(\underline{\theta}_1)}{f_h(\underline{\theta}_1)}$. Then by the strict monotonicity of the likelihood ratio, $\underline{\theta}_1 = -\bar{\theta}_1$.

In the base case of $n = 2$, $\bar{\theta}_{n-1} + \underline{\theta}_{n-1} = 0$ and $\Pr(a_{n-2} = 1|l) = 1 - F_l(0) = F_h(0) = \Pr(a_{n-2} = 0|h)$. Conjecture $\bar{\theta}_{n-1} + \underline{\theta}_{n-1} = 0$ and $\Pr(a_{n-2} = 1|l) = \Pr(a_{n-2} = 0|h)$ for general $n \geq 2$. As we in the proof of proposition 3 this implies that for general n the agent will set threshold $\bar{\theta}_n$ if $a_{n-1} = 1$ and $\underline{\theta}_n$ if $a_{n-1} = 0$ such that

$$\frac{f_h(\bar{\theta}_n)}{f_l(\bar{\theta}_n)} = \frac{(1 - F_l(\bar{\theta}_{n-1}))f_h(\bar{\theta}_{n-1}) + F_h(\bar{\theta}_{n-1})f_l(\bar{\theta}_{n-1})}{(1 - F_h(\bar{\theta}_{n-1}))f_l(\bar{\theta}_{n-1}) + F_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1})}$$

and similarly if $a_{n-1} = 0$

$$\frac{f_h(\underline{\theta}_n)}{f_l(\underline{\theta}_n)} = \frac{F_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1}) + (1 - F_h(\bar{\theta}_{n-1}))f_l(\bar{\theta}_{n-1})}{F_h(\bar{\theta}_{n-1})f_l(\bar{\theta}_{n-1}) + (1 - F_l(\bar{\theta}_{n-1}))f_h(\bar{\theta}_{n-1})}$$

By symmetry

$$\frac{f_l(-\bar{\theta}_n)}{f_h(-\bar{\theta}_n)} = \frac{f_h(\bar{\theta}_n)}{f_l(\bar{\theta}_n)} = \frac{(1 - F_l(\bar{\theta}_{n-1}))f_h(\bar{\theta}_{n-1}) + F_h(\bar{\theta}_{n-1})f_l(\bar{\theta}_{n-1})}{(1 - F_h(\bar{\theta}_{n-1}))f_l(\bar{\theta}_{n-1}) + F_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1})} = \frac{f_l(\underline{\theta}_n)}{f_h(\underline{\theta}_n)}$$

which implies $\frac{f_l(-\bar{\theta}_n)}{f_h(-\bar{\theta}_n)} = \frac{f_l(\underline{\theta}_n)}{f_h(\underline{\theta}_n)}$. By the strict monotonicity of the likelihood ratio $\underline{\theta}_n = -\bar{\theta}_n$.

Moreover,

$$\begin{aligned}
\Pr(a_{n-1} = 1|l) &= \Pr(a_{n-1} = 1|a_{n-2} = 1, l) \Pr(a_{n-2} = 1|l) \\
&\quad + \Pr(a_{n-1} = 1|a_{n-2} = 0, l) \Pr(a_{n-2} = 0|l) \\
&= (1 - F_l(\bar{\theta}_{n-1})) \Pr(a_{n-2} = 1|l) + (1 - F_l(\underline{\theta}_{n-1}))(1 - \Pr(a_{n-2} = 1|l))
\end{aligned}$$

and by symmetry of F_l and F_h and $\bar{\theta}_{n-1} + \underline{\theta}_{n-1} = 0$ this becomes

$$\begin{aligned}
&= F_h(\underline{\theta}_{n-1}) \Pr(a_{n-2} = 1|l) + F_h(\bar{\theta}_{n-1})(1 - \Pr(a_{n-2} = 1|l)) \\
&= F_h(\bar{\theta}_{n-1}) + \Pr(a_{n-2} = 1|l)(F_h(\underline{\theta}_{n-1}) - F_h(\bar{\theta}_{n-1}))
\end{aligned}$$

$$\begin{aligned}
\Pr(a_{n-1} = 0|h) &= \Pr(a_{n-1} = 0|a_{n-2} = 1, h) \Pr(a_{n-2} = 1|h) \\
&\quad + \Pr(a_{n-1} = 0|a_{n-2} = 0, h) \Pr(a_{n-2} = 0|h) \\
&= F_h(\bar{\theta}_{n-1})(1 - \Pr(a_{n-2} = 0|h)) + F_h(\underline{\theta}_{n-1}) \Pr(a_{n-2} = 0|h) \\
&= F_h(\bar{\theta}_{n-1}) + \Pr(a_{n-2} = 0|h)(F_h(\underline{\theta}_{n-1}) - F_h(\bar{\theta}_{n-1}))
\end{aligned}$$

So that

$$\Pr(a_{n-1} = 1|l) - \Pr(a_{n-1} = 0|h) = (\Pr(a_{n-2} = 1|l) - \Pr(a_{n-2} = 0|h))(F_h(\underline{\theta}_{n-1}) - F_h(\bar{\theta}_{n-1}))$$

and $\Pr(a_{n-1} = 1|l) = \Pr(a_{n-1} = 0|h)$ since $\Pr(a_{n-2} = 1|l) = \Pr(a_{n-2} = 0|h)$ by our induction conjecture. So by induction, $\bar{\theta}_n + \underline{\theta}_n = 0$ and $\Pr(a_n = 1|l) = \Pr(a_n = 0|h)$ for all n . \square

Proof of Proposition 11.

$$\begin{aligned}
\frac{f_h(\bar{\theta}_n)}{f_l(\bar{\theta}_n)} - \frac{f_h(\bar{\theta}_{n-1})}{f_l(\bar{\theta}_{n-1})} &= \frac{(1 - F_l(\bar{\theta}_{n-1}))f_h(\bar{\theta}_{n-1}) + F_h(\bar{\theta}_{n-1})f_l(\bar{\theta}_{n-1})}{(1 - F_h(\bar{\theta}_{n-1}))f_l(\bar{\theta}_{n-1}) + F_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1})} - \frac{f_h(\bar{\theta}_{n-1})}{f_l(\bar{\theta}_{n-1})} \\
&= \frac{f_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1}) - F_l(\bar{\theta}_{n-1})f_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1}) + F_h(\bar{\theta}_{n-1})f_l(\bar{\theta}_{n-1})^2}{f_l(\bar{\theta}_{n-1})[(1 - F_h(\bar{\theta}_{n-1}))f_l(\bar{\theta}_{n-1}) + F_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1})]} \\
&\quad - \frac{f_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1}) - F_h(\bar{\theta}_{n-1})f_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1}) + F_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1})^2}{f_l(\bar{\theta}_{n-1})[(1 - F_h(\bar{\theta}_{n-1}))f_l(\bar{\theta}_{n-1}) + F_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1})]} \\
&= \frac{(f_h(\bar{\theta}_{n-1}) - f_l(\bar{\theta}_{n-1}))(F_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1}) + F_h(\bar{\theta}_{n-1})f_l(\bar{\theta}_{n-1}))}{f_l(\bar{\theta}_{n-1})[(1 - F_h(\bar{\theta}_{n-1}))f_l(\bar{\theta}_{n-1}) + F_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1})]}
\end{aligned}$$

Then $\frac{f_h(\bar{\theta}_n)}{f_l(\bar{\theta}_n)} - \frac{f_h(\bar{\theta}_{n-1})}{f_l(\bar{\theta}_{n-1})} \leq 0 \iff f_h(\bar{\theta}_{n-1}) \leq f_l(\bar{\theta}_{n-1})$. Symmetry about 0 implies $\frac{f_h(0)}{f_l(0)} = 1$ and with the monotone likelihood ratio assumption $f_h(\bar{\theta}_{n-1}) \leq f_l(\bar{\theta}_{n-1})$ for $\bar{\theta}_{n-1} \leq 0$. Thus the likelihood ratio $f_h(\bar{\theta}_{n-1}) \leq f_l(\bar{\theta}_{n-1})$ is a decreasing sequence bounded below so it must converge. Moreover, strict FOSD implies that $f_h(\bar{\theta}_{n-1}) < f_l(\bar{\theta}_{n-1})$ if $\bar{\theta}_{n-1} < 0$. As we showed above $\bar{\theta}_1 < 0$ and by the monotonicity of the likelihood ratio $\theta_n < \bar{\theta}_{n-1}$ for all $n \geq 1$. Then the likelihood ratio strictly decreases in n until $F_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1}) + F_h(\bar{\theta}_{n-1})f_l(\bar{\theta}_{n-1}) = 0$, or when $F_l(\bar{\theta}_{n-1})$ and $F_h(\bar{\theta}_{n-1})$ converge to 0. Thus $\lim_{n \rightarrow \infty} \frac{f_h(\bar{\theta}_n)}{f_l(\bar{\theta}_n)} = 0$.

By lemma 2,

$$\Pr(a_n = 1|h) = \frac{f_l(\bar{\theta}_{n+1})}{f_l(\bar{\theta}_{n+1}) + f_h(\bar{\theta}_{n+1})} = \frac{1}{1 + \frac{f_h(\bar{\theta}_{n+1})}{f_l(\bar{\theta}_{n+1})}}$$

so that $\lim_{n \rightarrow \infty} \Pr(a_n = 1|h) = \frac{1}{1 + \lim_{n \rightarrow \infty} \frac{f_h(\bar{\theta}_{n+1})}{f_l(\bar{\theta}_{n+1})}} = 1$. By lemma 3 $\Pr(a_n = 1|l) = \Pr(a_n = 0|h)$ so $\lim_{n \rightarrow \infty} \Pr(a_n = 0|l) = 1$ and complete learning occurs. \square

Lemma 3.6. *Under mutual symmetry of F_l and F_h , if $\bar{\theta}_n + \theta_n = 0$ for all n*

$$\Pr(a_n = 1|l) - \Pr(a_{n-1} = 1|l) = F_h(\bar{\theta}_n) - \Pr(a_{n-1} = 1|l)[F_l(\bar{\theta}_n) + F_h(\bar{\theta}_n)]$$

$$\Pr(a_n = 1|h) - \Pr(a_{n-1} = 1|h) = F_l(\bar{\theta}_n) - \Pr(a_{n-1} = 1|h)[F_l(\bar{\theta}_n) + F_h(\bar{\theta}_n)]$$

and

$$\begin{aligned} \Pr(a_n = 1|l) \Pr(a_{n-1} = 1|h) - \Pr(a_{n-1} = 1|l) \Pr(a_n = 1|h) \\ = F_h(\bar{\theta}_n) \Pr(a_{n-1} = 1|h) - F_l(\bar{\theta}_n) \Pr(a_{n-1} = 1|l) \end{aligned}$$

Proof.

$$\begin{aligned} \Pr(a_n = 1|l) &= \Pr(a_n = 1|a_{n-1} = 1, l) \Pr(a_{n-1} = 1|l) + \Pr(a_n = 1|a_{n-1} = 0, l) \Pr(a_{n-1} = 0|l) \\ &= (1 - F_l(\bar{\theta}_n)) \Pr(a_{n-1} = 1|l) + (1 - F_l(\underline{\theta}_n)) \Pr(a_{n-1} = 0|l) \\ &= (1 - F_l(\bar{\theta}_n)) \Pr(a_{n-1} = 1|l) + F_h(\bar{\theta}_n)(1 - \Pr(a_{n-1} = 1|l)) \\ &= F_h(\bar{\theta}_n) + \Pr(a_{n-1} = 1|l)(1 - F_l(\bar{\theta}_n) - F_h(\bar{\theta}_n)) \end{aligned}$$

and

$$\begin{aligned} \Pr(a_n = 1|h) &= \Pr(a_n = 1|a_{n-1} = 1, h) \Pr(a_{n-1} = 1|h) + \Pr(a_n = 1|a_{n-1} = 0, h) \Pr(a_{n-1} = 0|h) \\ &= (1 - F_h(\bar{\theta}_n)) \Pr(a_{n-1} = 1|h) + (1 - F_h(\underline{\theta}_n)) \Pr(a_{n-1} = 0|h) \\ &= (1 - F_h(\bar{\theta}_n)) \Pr(a_{n-1} = 1|h) + F_l(\bar{\theta}_n)(1 - \Pr(a_{n-1} = 1|h)) \\ &= F_l(\bar{\theta}_n) + \Pr(a_{n-1} = 1|h)(1 - F_l(\bar{\theta}_n) - F_h(\bar{\theta}_n)) \end{aligned}$$

The first two desired equalities are easily obtained by rearranging the above equations while

the third is given by

$$\begin{aligned}
& \Pr(a_n = 1|l) \Pr(a_{n-1} = 1|h) - \Pr(a_{n-1} = 1|l) \Pr(a_n = 1|h) \\
&= [F_h(\bar{\theta}_n) + \Pr(a_{n-1} = 1|l)(1 - F_l(\bar{\theta}_n) - F_h(\bar{\theta}_n))] \Pr(a_{n-1} = 1|h) \\
&\quad - \Pr(a_{n-1} = 1|l)[F_l(\bar{\theta}_n) + \Pr(a_{n-1} = 1|h)(1 - F_l(\bar{\theta}_n) - F_h(\bar{\theta}_n))] \\
&= F_h(\bar{\theta}_n) \Pr(a_{n-1} = 1|h) - F_l(\bar{\theta}_n) \Pr(a_{n-1} = 1|l)
\end{aligned}$$

□

Lemma 3.7. *If $\hat{\theta}_n$ converges to a limit then $\Pr(a_n|l)$ and $\Pr(a_n|h)$ also converge.*

Proof. By lemma 7, $\Pr(a_n = 1|l) - \Pr(a_{n-1} = 1|l) = F_h(\bar{\theta}_n) - \Pr(a_{n-1} = 1|l)(F_l(\bar{\theta}_n) + F_h(\bar{\theta}_n))$.

If $\Pr(a_n|l)$ does not converge, let $\varepsilon > 0$ for which for any N there is always some $n \geq N$ with $|\Pr(a_n = 1|l) - \Pr(a_{n-1} = 1|l)| > \varepsilon$. Suppose $\Pr(a_n = 1|l) - \Pr(a_{n-1} = 1|l) > \varepsilon$. Then

$$\Pr(a_{n-1} = 1|l) < \frac{F_h(\bar{\theta}_n)}{(F_l(\bar{\theta}_n) + F_h(\bar{\theta}_n))} - \varepsilon$$

Then

$$\begin{aligned}
& \Pr(a_{n+1} = 1|l) - \Pr(a_n = 1|l) = F_h(\bar{\theta}_{n+1}) - \Pr(a_n = 1|l)(F_l(\bar{\theta}_{n+1}) + F_h(\bar{\theta}_{n+1})) \\
& = F_h(\bar{\theta}_{n+1}) - (F_h(\bar{\theta}_n) + \Pr(a_{n-1} = 1|l)(1 - F_l(\bar{\theta}_n) - F_h(\bar{\theta}_n))(F_l(\bar{\theta}_{n+1}) + F_h(\bar{\theta}_{n+1}))) \\
& = F_h(\bar{\theta}_{n+1}) - F_h(\bar{\theta}_n)(F_l(\bar{\theta}_{n+1}) + F_h(\bar{\theta}_{n+1})) - \Pr(a_{n-1} = 1|l)(1 - F_l(\bar{\theta}_n) \\
& \quad - F_h(\bar{\theta}_n))(F_l(\bar{\theta}_{n+1}) + F_h(\bar{\theta}_{n+1})) \\
& > F_h(\bar{\theta}_{n+1}) - F_h(\bar{\theta}_n)(F_l(\bar{\theta}_{n+1}) + F_h(\bar{\theta}_{n+1})) \\
& \quad - (1 - F_l(\bar{\theta}_n) - F_h(\bar{\theta}_n))(F_l(\bar{\theta}_{n+1}) + F_h(\bar{\theta}_{n+1})) \left(\frac{F_h(\bar{\theta}_n)}{F_l(\bar{\theta}_n) + F_h(\bar{\theta}_n)} - \varepsilon \right) \\
& = F_h(\bar{\theta}_{n+1}) - F_h(\bar{\theta}_n)(F_l(\bar{\theta}_{n+1}) + F_h(\bar{\theta}_{n+1})) + F_h(\bar{\theta}_n)(F_l(\bar{\theta}_{n+1}) + F_h(\bar{\theta}_{n+1})) \\
& \quad + \varepsilon(1 - F_l(\bar{\theta}_n) - F_h(\bar{\theta}_n))(F_l(\bar{\theta}_{n+1}) + F_h(\bar{\theta}_{n+1})) - \frac{F_h(\bar{\theta}_n)(F_l(\bar{\theta}_{n+1}) + F_h(\bar{\theta}_{n+1}))}{F_l(\bar{\theta}_n) + F_h(\bar{\theta}_n)} \\
& = \frac{F_h(\bar{\theta}_{n+1})F_l(\bar{\theta}_n) - F_h(\bar{\theta}_n)F_l(\bar{\theta}_{n+1})}{F_l(\bar{\theta}_n) + F_h(\bar{\theta}_n)} + \varepsilon(1 - F_l(\bar{\theta}_n) - F_h(\bar{\theta}_n))(F_l(\bar{\theta}_{n+1}) + F_h(\bar{\theta}_{n+1}))
\end{aligned}$$

Given the convergence of $\bar{\theta}_n$, n can be made large enough that

$$\begin{aligned}
& F_h(\bar{\theta}_{n+1})F_l(\bar{\theta}_n) - F_h(\bar{\theta}_n)F_l(\bar{\theta}_{n+1}) > \\
& \quad - \varepsilon(1 - F_l(\bar{\theta}_n) - F_h(\bar{\theta}_n))(F_l(\bar{\theta}_{n+1}) + F_h(\bar{\theta}_{n+1}))(F_l(\bar{\theta}_n) + F_h(\bar{\theta}_n))
\end{aligned}$$

so that $\Pr(a_{n+1}|l) > \Pr(a_n = 1|l)$ for $n \geq N$. Thus $\Pr(a_n = 1|l)$ is an increasing sequence, bounded above and must converge. If $\Pr(a_n = 1|l) - \Pr(a_{n-1} = 1|l) < -\varepsilon$ then $\Pr(a_n = 1|l)$ is a decreasing sequence bounded below and must converge. Since $\Pr(a_n = 0|l) = 1 - \Pr(a_n = 1|l)$ this must converge as well. Finally, an analogous proof shows the convergence of $\Pr(a_n|h)$. \square

Proof of Proposition 12. From above the likelihood ratio takes the form

$$\frac{f_h(\hat{\theta}_n)}{f_l(\hat{\theta}_n)} = \frac{\sum_{i=1}^N \mu_i^n \Pr(a_{i-1}|l) \Pr(l)}{\sum_{i=1}^N \mu_i^n \Pr(a_{i-1}|h) \Pr(h)}$$

With $\mu_n^n = \gamma$ and $\mu = \mu_i^n = \frac{1-\gamma}{N-1}$ for $i \neq n$ this becomes

$$\frac{f_h(\hat{\theta}_n)}{f_l(\hat{\theta}_n)} = \frac{(\gamma - \mu) \Pr(a_{n-1}|l) + \mu \sum_{i=0}^{N-1} \Pr(a_i|l)}{(\gamma - \mu) \Pr(a_{n-1}|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)}$$

If $\bar{\theta}_n$ is the threshold in response to $a_{n-1} = 1$ then

$$\begin{aligned} & \frac{f_h(\bar{\theta}_{n+1})}{f_l(\bar{\theta}_{n+1})} - \frac{f_h(\bar{\theta}_n)}{f_l(\bar{\theta}_n)} \\ &= \frac{(\gamma - \mu) \Pr(a_n|l) + \mu \sum_{i=0}^{N-1} \Pr(a_i|l)}{(\gamma - \mu) \Pr(a_n|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)} - \frac{(\gamma - \mu) \Pr(a_{n-1}|l) + \mu \sum_{i=0}^{N-1} \Pr(a_i|l)}{(\gamma - \mu) \Pr(a_{n-1}|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)} \\ &= \frac{[(\gamma - \mu) \Pr(a_n|l) + \mu \sum_{i=0}^{N-1} \Pr(a_i|l)][(\gamma - \mu) \Pr(a_{n-1}|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)]}{[(\gamma - \mu) \Pr(a_{n-1}|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)][(\gamma - \mu) \Pr(a_n|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)]} \\ &\quad - \frac{[(\gamma - \mu) \Pr(a_{n-1}|l) + \mu \sum_{i=0}^{N-1} \Pr(a_i|l)][(\gamma - \mu) \Pr(a_n|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)]}{[(\gamma - \mu) \Pr(a_{n-1}|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)][(\gamma - \mu) \Pr(a_n|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)]} \\ &= \frac{(\gamma - \mu)^2 [\Pr(a_n|l) \Pr(a_{n-1}|h) - \Pr(a_{n-1}|l) \Pr(a_n|h)]}{[(\gamma - \mu) \Pr(a_{n-1}|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)][(\gamma - \mu) \Pr(a_n|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)]} \\ &\quad + \frac{(\gamma - \mu) \mu \left[\sum_{i=0}^{N-1} \Pr(a_i|l) (\Pr(a_{n-1}|h) - \Pr(a_n|h)) + \sum_{i=0}^{N-1} \Pr(a_i|h) (\Pr(a_n|l) - \Pr(a_{n-1}|l)) \right]}{[(\gamma - \mu) \Pr(a_{n-1}|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)][(\gamma - \mu) \Pr(a_n|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)]} \end{aligned}$$

and with the results in lemma 7 this becomes

$$\begin{aligned} &= \frac{(\gamma - \mu)^2 [F_h(\bar{\theta}_n) \Pr(a_{n-1} = 1|h) - F_l(\bar{\theta}_n) \Pr(a_{n-1} = 1|l)]}{[(\gamma - \mu) \Pr(a_{n-1}|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)][(\gamma - \mu) \Pr(a_n|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)]} \\ &\quad + \frac{(\gamma - \mu) \mu \left[\sum_{i=0}^{N-1} \Pr(a_i|h) (F_h(\bar{\theta}_n) - \Pr(a_{n-1} = 1|l) [F_l(\bar{\theta}_n) + F_h(\bar{\theta}_n)]) \right]}{[(\gamma - \mu) \Pr(a_{n-1}|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)][(\gamma - \mu) \Pr(a_n|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)]} \\ &\quad - \frac{(\gamma - \mu) \mu \left[\sum_{i=0}^{N-1} \Pr(a_i|l) (F_l(\bar{\theta}_n) - \Pr(a_{n-1} = 1|h) [F_l(\bar{\theta}_n) + F_h(\bar{\theta}_n)]) \right]}{[(\gamma - \mu) \Pr(a_{n-1}|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)][(\gamma - \mu) \Pr(a_n|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)]} \end{aligned}$$

which converges to

$$\begin{aligned}
& \rightarrow \frac{\gamma^2 [F_h(\bar{\theta}) \Pr(a = 1|h) - F_l(\bar{\theta}) \Pr(a = 1|l)]}{[\gamma \Pr(a|h) + \Pr(a|h)][\gamma \Pr(a|h) + \Pr(a|h)]} \\
& + \frac{\gamma [\Pr(a|h)(F_h(\bar{\theta}) - \Pr(a = 1|l)[F_l(\bar{\theta}) + F_h(\bar{\theta})])]}{[\gamma \Pr(a|h) + \Pr(a|h)][\gamma \Pr(a|h) + \Pr(a|h)]} \\
& - \frac{\gamma [\Pr(a|l)(F_l(\bar{\theta}) - \Pr(a = 1|h)[F_l(\bar{\theta}) + F_h(\bar{\theta})])]}{[\gamma \Pr(a|h) + \Pr(a|h)][\gamma \Pr(a|h) + \Pr(a|h)]} \\
& = \frac{\gamma(1 + \gamma)[F_h(\bar{\theta}) \Pr(a = 1|h) - F_l(\bar{\theta}) \Pr(a = 1|l)]}{[\gamma \Pr(a|h) + \Pr(a|h)][\gamma \Pr(a|h) + \Pr(a|h)]}
\end{aligned}$$

Then $\lim_{n \rightarrow \infty} \frac{f_h(\bar{\theta}_{n+1})}{f_l(\bar{\theta}_{n+1})} - \frac{f_h(\bar{\theta}_n)}{f_l(\bar{\theta}_n)} = 0$ if $F_h(\bar{\theta}) \Pr(a = 1|h) = F_l(\bar{\theta}) \Pr(a = 1|l)$. And given that

$$\frac{f_h(\bar{\theta}_n)}{f_l(\bar{\theta}_n)} = \frac{\Pr(a_{n-1}=1|l)}{\Pr(a_{n-1}=1|h)},$$

$$\frac{f_h(\bar{\theta})}{f_l(\bar{\theta})} = \lim_{n \rightarrow \infty} \frac{f_h(\bar{\theta}_n)}{f_l(\bar{\theta}_n)} = \lim_{n \rightarrow \infty} \frac{\Pr(a_{n-1} = 1|l)}{\Pr(a_{n-1} = 1|h)} = \frac{\Pr(a = 1|l)}{\Pr(a = 1|h)} = \frac{F_h(\bar{\theta})}{F_l(\bar{\theta})}$$

so that $\frac{f_h(\bar{\theta})}{f_l(\bar{\theta})} = \frac{F_h(\bar{\theta})}{F_l(\bar{\theta})}$. But $\frac{F_h(\theta)}{F_l(\theta)} < 1$ for all interior θ and by symmetry and the MLRP $\frac{f_h(\theta)}{f_l(\theta)} > 1$ for $\theta > 0$, so $\bar{\theta} \in [-1, 0)$. By lemma 3 $\frac{f_h(\theta)}{f_l(\theta)} > \frac{F_h(\theta)}{F_l(\theta)}$ for all interior θ so the only remaining candidate is $\bar{\theta} = -1$. Indeed, by L'Hopital's rule, $\lim_{\theta \rightarrow -1} \frac{F_h(\theta)}{F_l(\theta)} = \lim_{\theta \rightarrow -1} \frac{f_h(\theta)}{f_l(\theta)} = \frac{f_h(-1)}{f_l(-1)}$ so that $\bar{\theta} = -1$.

From lemma 5, $\Pr(a_n = 1|h) = \Pr(a_n = 0|l) = \frac{F_l(\bar{\theta}_{n+1})}{F_l(\bar{\theta}_{n+1}) + F_h(\bar{\theta}_{n+1})}$

$$\lim_{n \rightarrow \infty} \frac{F_l(\bar{\theta}_{n+1})}{F_l(\bar{\theta}_{n+1}) + F_h(\bar{\theta}_{n+1})} = \lim_{\theta \rightarrow \bar{\theta}} \frac{1}{1 + \frac{F_h(\theta)}{F_l(\theta)}} = \lim_{\theta \rightarrow -1} \frac{1}{1 + \frac{f_h(\theta)}{f_l(\theta)}} = 1$$

so that $\lim_{n \rightarrow \infty} \Pr(a_n = 1|h) = \lim_{n \rightarrow \infty} \Pr(a_n = 0|l) = 1$ and complete learning occurs. \square