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Abstract

We study the structure of 2-edge-connected simple graphs with many edges that have no spanning closed trail. X. T. Cai [2] conjectured that any 3-edge-connected simple graph G of order n has a spanning closed trail, if

$$|E(G)| \geq \binom{n-9}{2} + 16.$$

This bound is best-possible. We prove this conjecture, and we obtain a stronger conclusion.

1. INTRODUCTION

We follow the notation of Bondy and Murty [1], except that graphs have no loops, the graph of order 2 and size 2 is called a 2-cycle and denoted C_2, and K_1 is regarded as having infinite edge-connectivity. For a graph G, let $O(G)$ denote the set of vertices of odd degree in G. The set of natural numbers is denoted $\mathbf{N}$. Let $D_1(G)$ denote the set of vertices of degree 1 in G.

A graph G is called supereulerian if it has a spanning connected subgraph H whose vertices have even degree. A graph G is called collapsible if for every even set $X \subseteq V(G)$ there is a spanning connected subgraph H_X of G, such that $O(H_X) = X$. Thus, the trivial graph K_1 is both supereulerian and collapsible. Denote the family of supereulerian graphs by $\mathcal{SL}$, and denote the family of collapsible graphs by $\mathcal{CL}$. Obviously, $\mathcal{CL} \subset \mathcal{SL}$, and collapsible graphs are 2-edge-connected. Examples of graphs in $\mathcal{CL}$ include the cycles C_2, C_3, but not C_t if $t \geq 4$.

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Cai [2] conjectured that any 3-edge-connected simple graph G of order n with

$$|E(G)| \geq \binom{n-9}{2} + 16$$

is supereulerian. We shall show that any such graph is collapsible. The Petersen graph is one of infinitely many graphs that show that this inequality is best-possible.

2. THE REDUCTION METHOD

Let G be a graph, and let H be a connected subgraph of G. The contraction G/H is the graph obtained from G by contracting all edges of H, and by deleting any resulting loops. Even when G is simple, G/H may not be.

Theorem A (Catlin [3]) Let H be a subgraph of G. If $H \in \mathcal{CL}$ then

$$G \in \mathcal{SL} \iff G/H \in \mathcal{SL},$$

and

$$G \in \mathcal{CL} \iff G/H \in \mathcal{CL}. \square$$

In [3] it was shown that if H_1 and H_2 are both collapsible subgraphs of G with at least one common vertex, then $G[V(H_1) \cup V(H_2)] \in \mathcal{CL}$. Thus, any collapsible subgraph of G is contained in a unique maximal collapsible subgraph. For a graph G where H_1, H_2, ..., H_c are all the maximal collapsible subgraphs of G, define G' to be the graph obtained from G by contracting each H_i $(1 \leq i \leq c)$ to a distinct vertex. Since $V(G) = V(H_1) \cup .... \cup V(H_c)$, the graph G' has order c. We call the graph G' the reduction of G, and we call a graph reduced if it is the reduction of some graph. Any graph G has a unique reduction G' [3]. A graph is collapsible if and only if its reduction is K_1.

Let G be a graph. The arboricity of G, denoted $a(G)$, is the minimum number of forests whose union contains $E(G)$. Let $F(G)$ denote the minimum number of edges that must be added to G, to obtain a spanning supergraph containing two edge-disjoint spanning trees.

Theorem B Let G be a graph and let G' be the reduction of G. Then
(a) $G \in \mathcal{SL} \iff G' \in \mathcal{SL}$;
(b) G' is simple, G' has no 3-cycle, and $a(G') \leq 2$;
(c) $K_{3,3} - e$ ($K_{3,3}$ minus an edge) is collapsible;
(d) If $F(G) \leq 1$ then $G' \in \{K_1, K_2\}$;
(e) $G = G'$ if and only if G has no nontrivial collapsible subgraph;
(f) If $a(G) \leq 2$ then

$$|E(G)| + F(G) = 2|V(G)| - 2. \square$$

Parts (a), (b), (d) and (e) of Theorem B are proved in [3] and part (c) was proved in [4]. Part (f) is easy. A characterization of $F(G)$ appears in [6].

3. A GENERAL RESULT

Theorem 1. Let G be a 2-edge-connected simple graph of order n and let $p \in \mathbb{N} - \{1\}$. If

$$|E(G)| \geq \binom{n-p+1}{2} + 2p - 4, \tag{1}$$

then exactly one of these holds:

(a) The reduction of G has order less than p;

(b) Equality holds in (1), G has a complete subgraph H of order $n-p+1$, and the reduction of G is $G' = G/H$, a graph of order p and size $2p-4$;

(c) G is a reduced graph such that either

$$|E(G)| \in \{2n-4, 2n-5\} \text{ and } n \in \{p+1, p+2\}$$

or

$$|E(G)| = 2n-4 \text{ and } n = p+3.$$

Proof: The conclusions (a), (b), and (c) are clearly mutually exclusive.

Fix a reduced graph G_0, and suppose that G is a simple graph of order n with $G' = G_0$. Any 2-edge-connected graph G arises in this manner, for some value of G_0. Denote

$$V(G_0) = \{v_1, v_2, \ldots, v_c\},$$

and for each $1 \leq i \leq c$, let H_i denote the collapsible subgraph of G contracted to v_i by the reduction-contraction $G \longrightarrow G_0$. If $|E(G)|$ were maximum among all simple graphs G of order n with $G' = G_0$, then at most one H_i $(1 \leq i \leq c)$ is a nontrivial subgraph of G, and this H_i is a complete subgraph of order $n-c+1$. Therefore,

$$|E(G)| \leq |E(H_i)| + |E(G_0)| \leq \binom{n-c+1}{2} + |E(G_0)|, \tag{2}$$

with equality only if G has at most one nontrivial collapsible subgraph H_i and it is a complete subgraph of order $n-c+1$.

If $G_0 = K_1$, then (a) holds, since $p \geq 2$. Thus, we can suppose that $G_0 \neq K_1$. Since G is 2-edge-connected, so is its contraction G_0, and so $G_0 \neq K_2$. Hence by part (d) of Theorem B, $F(G_0) \geq 2$. By (b) of Theorem B, $a(G_0) \leq 2$, and so (f) of Theorem B gives

$$|E(G_0)| \leq 2c - 4. \tag{3}$$

By (2) and (3),

$$|E(G)| \leq \binom{n-c+1}{2} + 2c - 4,$$

with strict equality only if (2) or (3) holds strictly. This and the hypothesis of Theorem 1 give

$$\binom{n-p+1}{2} + 2p-4 \le |E(G)| \le \binom{n-c+1}{2} + 2c-4. \tag{4}$$

Simplification of (4) yields

$$2n(c-p) \le (c-p)(c+p+3) \tag{5}$$

Case 1 Suppose that $c = p$. Then equality holds throughout (4). This equality in (4) forces equality in (3) and in (2). Thus, (b) of Theorem 1 holds.

Case 2 Suppose that $c < p$. Then (a) of Theorem 1 holds.

Case 3 Suppose that $c > p$. This and (5) give

$$2n \le c+p+3. \tag{6}$$

By the definition of c, $c \le n$.

Subcase 3A Suppose that $c = n$. This and the hypothesis of Case 3 imply $p < n$, and so (6) and $c = n$ imply

$$p < n \le p+3. \tag{7}$$

Since $|V(G_0)| = c = n$, it follows that G is reduced, and so $G = G_0$. Hence by (3), $|E(G)| \le 2n-4$. To prove (c) of Theorem 1, it only remains to prove the appropriate lower bound on $|E(G)|$. If $n = p+1$, then (1) gives

$$|E(G)| \ge \binom{2}{2} + 2p-4 = 2p-3 = 2n-5.$$

If $n = p+2$, then (1) gives

$$|E(G)| \ge \binom{3}{2} + 2p-4 = 2p-1 = 2n-5.$$

If $n = p+3$, then

$$|E(G)| \ge \binom{4}{2} + 2p-4 = 2p+2 = 2n-4.$$

By (7), all cases have been considered.

Subcase 3B Suppose $c < n$. By the relations on c and by (6),

$$p < c < n < p+3. \tag{8}$$

Since each term of (8) is an integer,

$$c = p + 1; \quad n = p + 2. \tag{9}$$

But since G is a simple graph of order n, its reduction cannot have order $n-1$. By (9), $|V(G_0)| = c = n - 1$, and so the reduction of G cannot be G_0. This contradicts the definition of G_0 and G, and so Subcase 3B is impossible. □

4. THE REDUCTION OF 4-CYCLES

Suppose that a graph G contains a 4-cycle H. The subgraph H is not collapsible, and the equivalences of Theorem A do not apply in this case, if H is an induced subgraph. However, the theorem below provides an extension of the reduction method to subgraphs that are 4-cycles.

Let G be a graph containing an induced 4-cycle $xyzwx$, and define

$$E = \{xy, yz, zw, wx\}.$$

Define G/π to be the graph obtained from $G - E$ by identifying x and z to form a vertex v_1, by identifying w and y to form a vertex v_2, and by adding an edge v_1v_2.

Theorem C (Catlin [4, p. 241]) For the graphs G and G/π defined above, the following hold:
(a) If $G/\pi \in \mathcal{CL}$ then $G \in \mathcal{CL}$;
(b) $|V(G)| = |V(G/\pi)| + 2$;
(c) $|E(G)| = |E(G/\pi)| + 3$;
(d) If $G/\pi \in \mathcal{SL}$ then $G \in \mathcal{SL}$. □

5. SOME LEMMAS

Lemma 1. (Chen [7]) Let G be a simple 2-edge-connected graph of order at most 7. If G has at most two vertices of degree 2, then $G \in \mathcal{CL}$. □

Lemma 2. (Lai [8]) Let G be a simple connected graph of order at most 11. If $\delta(G) \geq 3$ then either G is the Petersen graph or the reduction of G is K_1 or K_2. □

Chen [7] had first proved Lemma 2 with the stronger hypothesis that $\kappa'(G) \geq 3$.

Lemma 3. Let G be a simple 2-edge-connected graph of order at most 8, and let $u \in V(G)$. If u is the only vertex of degree 2 in G, then $G \in \mathcal{CL}$.

Proof: Let G and u satisfy the hypothesis of Lemma 3. Then $G - u$ is connected. If $\kappa'(G - u) \geq 2$, then use Lemma 1 to see that $G - u \in \mathcal{CL}$. Then $G \in \mathcal{CL}$ follows. If $\kappa'(G - u) < 2$ then $G - u$ has a cut edge e such that some component, say H, of $G - u - e$ has no cut edge. Since u is the only vertex of degree 2 in G, H is nontrivial and H satisfies the hypothesis of Lemma 1 (with H in place of G of Lemma 1). Therefore, H is a nontrivial collapsible subgraph of G. Note that G/H also satisfies the hypothesis of Lemma 1 (with G/H in place of G of Lemma 1), and hence $G/H \in \mathcal{CL}$. By Theorem A, $G \in \mathcal{CL}$. □

Lemma 4 Any 3-edge-connected reduced graph of order 12 is 3-regular.

Proof: Let G be a 3-edge-connected reduced graph of order 12. By (e) of Theorem B,

(10) G has no nontrivial collapsible subgraph.

By way of contradiction, suppose that

(11) G is not 3-regular.

Then G has a vertex x with $d(x) \geq 4$. Since G is reduced, G is simple and has no 3-cycle, by (b) of Theorem B.

We claim

(12) x lies on a 4-cycle.

Suppose not. Since $d(x) \geq 4$ and $\delta(G) \geq 3$, at least 4 paths in G with origin x have length 1, and at least 8 paths with origin x have length 2. Since G has no 2-cycle and no 3-cycle, and since x is in no 4-cycle, no two of these 12 paths have the same terminus. Hence, $|V(G-x)| \geq 12$, a contradiction that proves (12).

By (12), x lies on a 4-cycle, say $xyzwx$. Denote

$$E = \{xy, yz, zw, wx\}.$$

Define G/π to be the graph obtained from $G - E$ as described in Section 4 above. Thus, G and G/π satisfy Theorem C.

Since $\delta(G) \geq 3$ and $d(x) \geq 4$, we have

(13) $$d_{G/\pi}(v_1) \geq 4 \text{ and } \delta(G/\pi) \geq 3,$$

where v_1 is the vertex defined in Section 4. Let G_0 be the reduction of G/π. If $G = K_1$ then $G/\pi \in \mathcal{CL}$, and so (a) of Theorem C gives $G \in \mathcal{CL}$, contrary to the hypothesis of Lemma 4. Hence $G_0 \neq K_1$, and so by (b) of Theorem C,

(14) $$1 < |V(G_0)| \leq |V(G/\pi)| = |V(G)| - 2 = 10.$$

Case 1. Suppose that $\kappa'(G/\pi) < 2$. Then v_1v_2 is the only cut-edge of G/π, because G has no cut edge. Therefore, $G - E$ has two components, say G_1 and G_2, where $x, z \in V(G_1)$ and $y, w \in V(G_2)$.

Since the 4-cycle $xyzwx$ is an induced subgraph, $xz, wy \notin E(G)$. This, $\delta(G) \geq 3$, and the fact that G is simple imply that each G_i $(1 \leq i \leq 2)$ has a vertex of degree at least 3 that is not in $\{w, x, y, z\}$. Since G has order 12, since $\delta(G) \geq 3$, and since (10) precludes the presence of 3-cycles in G_i, this implies

$$5 \leq |V(G_i)| \leq 7, \qquad (1 \leq i \leq 2).$$

By $\delta(G) \geq 3$,

$$D_1(G_1) \cup D_1(G_2) \subseteq \{w, x, y, z\},$$

and these relations imply that each G_i, $1 \leq i \leq 2$, contains a nontrivial 2-edge-connected subgraph H_i, where H_i has at most two vertices of degree 2. Since $|V(H_i)| \leq 7$, Lemma 1 implies $H_i \in \mathcal{CL}$. Thus, H_i is a subgraph of G that contradicts (10).

Case 2 Suppose that $\kappa'(G/\pi) \geq 3$. Then $\kappa'(G_0) \geq \kappa'(G/\pi) \geq 3$. By this and (14), G_0 is nontrivial and satisfies the hypotheses of Lemma 2 and must therefore be the Petersen graph. This fact and (14) force $G_0 = G/\pi$, and so G/π is 3-regular, contrary to (13).

Case 3 Suppose that $\kappa'(G/\pi) = 2$. Since $\kappa'(G) \geq 3$, it follows that v_1v_2 is in every edge cut of size 2 in G/π. Denote $e_\pi = v_1v_2$. For the reduction G_0 of G/π, e_π lies in every edge cut of G_0 of size 2. By (b) of Theorem B,

(15) $$G_0 \text{ is simple.}$$

Subcase 3A Suppose that either $e_\pi \notin E(G_0)$ or $\kappa'(G_0) \geq 3$. In either case we must have $\kappa'(G_0) \geq 3$ and $1 < |V(G_0)| \leq 9$. This and (15) mean that G_0 is a counterexample to Lemma 2. Hence, Subcase 3A is impossible.

Subcase 3B Suppose that $e_\pi \in E(G_0)$ and $\kappa'(G_0) < 3$. Then

(16) $$\kappa'(G_0) = 2$$

and by a prior remark, e_π is in every edge cut of size 2 in G_0. If $\delta(G_0) \geq 3$, then by (14), (15), and Lemma 2, G_0 is the Petersen graph, contrary to (16). Hence,

(17) $$\delta(G_0) < 3.$$

Since e_π is in every edge cut of size 2 and by (16), (17) implies that G_0 has a unique vertex u (say) of degree 2, and u is incident with e_π. Denote $e_\pi = uv$ in $E(G_0)$.

3B(i). Suppose $|V(G_0)| \leq 8$. By (16) and by Lemma 3, $G_0 \in \mathcal{CL}$. Hence, $G/\pi \in \mathcal{CL}$ and by Theorem C, $G \in \mathcal{CL}$, contrary to the hypothesis of Lemma 5.

3B(ii). Suppose $|V(G_0)| \geq 9$. By (13), $\delta(G/\pi) \geq 3$, and so G/π has no vertex u of degree 2. Thus, G_0 is a proper contraction of G/π, and so by (14),

$$|V(G_0)| = 9, \qquad |V(G/\pi)| = 10.$$

Hence the contraction mapping $G/\pi \longrightarrow G_0$, being a reduction as well, identifies two vertices of $V(G/\pi)$ that are joined in G/π by multiple edges.

By the nature of the derivation of G/π from the simple graph G, any two vertices of G/π are joined by no more than two edges. Hence by the first part of (13), the contraction-mapping $G/\pi \longrightarrow G_0$ cannot involve an identification of v_1 with another vertex to form the vertex $u \in V(G_0)$, since u has degree 2. Instead, v_2 must be identified with a neighbor in G/π to form the vertex u in G_0, and so v_1 has degree at least 4 in G_0 as well as in G/π. Thus, $v = v_1$ in G_0. Let v' denote the other neighbor of u in G_0. Since e_π is in every edge-cut of size 2 in G_0, $\kappa'(G_0 - u) \geq 2$. By Lemma 3 (with $G_0 - u$ in place of G and with v' in place of u), $G_0 - u$ is collapsible of order 8. This contradicts the fact that G_0 is reduced. This contradiction concludes this subcase and it proves Lemma 4. □

Lemma 5 Let n be the smallest natural number such that there is a 2-edge-connected reduced graph G of order n and size $2n-4$, such that G is not $K_{2,n-2}$. Then $n \geq 14$ and G is 3-edge-connected.

Proof: Suppose that G is a smallest 2-edge-connected reduced graph with $|E(G)| = 2|V(G)|-4$, such that G is not $K_{2,n-2}$, where n denotes $|V(G)|$. Since G is reduced, $a(G) \leq 2$, by (b) of Theorem B. Hence, by (f) of Theorem B and by the definition of G,

(18) $$F(G) = 2.$$

If $\delta(G) = 2$ then G has a vertex u of degree 2. If $\kappa'(G-u) < 2$ then since G is 2-edge-connected, $G-u$ has a cut edge e, say, and if G_1 and G_2 denote the components of $G-u-e$, then it follows from (18) that $F(G_1)+F(G_2) = 1$. By (d) of Theorem B and since G is reduced, $\{G_1, G_2\} = \{K_1, K_2\}$. Since G is 2-edge-connected, this forces $G = C_4$. Since this contradicts the hypothesis of the lemma, we may conclude that $\kappa'(G-u) \geq 2$. Hence, by the minimality of G, $G - u = K_{2,n-3}$. Since G is reduced, (e) of Theorem B implies that u is not in a subgraph that is a 2-cycle, a 3-cycle, or $K_{3,3}$ minus an edge, for these three graphs are collapsible. Since $G \neq K_{2,n-2}$, it follows that

(19) $$\delta(G) \geq 3.$$

If $\kappa'(G) = 2$, then G has a cutset E of size 2, such that each component of $G - E$ is nontrivial, by (19). If $n < 14$ then the smallest component of $G - E$ satisfies the hypothesis of Lemma 1, and hence must be a nontrivial collapsible subgraph of G. This contradicts the hypothesis that G is reduced, and so $\kappa'(G) \neq 2$.

If $\kappa'(G) = 1$ then G has a cut edge e (say), and we denote by G_1 and G_2 the two components of $G - e$. By (18),

(20) $$F(G_1) + F(G_2) = 1.$$

By (19), G_1 and G_2 are nontrivial, and by (20), one of them, say G_1, has $F(G_1) = 0$. By (d) of Theorem B, G_1 is a nontrivial collapsible subgraph of G, contrary to (e) of Theorem B, since $G = G'$. Hence, $\kappa'(G) \neq 1$, and so we must have

$$\kappa'(G) \geq 3.$$

Hence, if $n \leq 11$ then by Lemma 2, $G \in \mathcal{CL}$ or G is the Petersen graph. Either case violates the definition of G. If $n = 12$ then by Lemma 4, G is 3-regular, and so $|E(G)| = 18$, contrary to the definition of G. Hence, $n \geq 13$. Finally, therefore, we suppose

$$n = 13,$$

and we shall derive a contradiction.

We claim that G has a 4-cycle. Suppose not, and let x be a vertex of degree $d(x) = \Delta(G)$ in G. Since G is reduced, x is in no cycle of length less than 5. Thus, each path of length at most 2 with origin x has a different terminus. There are $d(x)$ such paths of length 1 and at least $2d(x)$ of length 2, since $\delta(G) \geq 3$ by (19). Hence,

$$12 = |V(G - x)| \geq d(x) + 2d(x) = 3d(x), \tag{21}$$

with equality only if each neighbor of x has degree 3. By (19), $\Delta(G) \geq 3$, and since G has odd order, G is not 3-regular. This and (21) imply that

$$d(x) = 4, \tag{22}$$

and since equality holds in (21), each vertex adjacent to x has degree 3 in G. Since x is arbitrary, no two vertices of degree 4 in G are adjacent.

By $|E(G)| = 2n - 4 = 22$, by (19), and by $\Delta(G) = 4$, G has 5 vertices of degree 4 and 8 vertices of degree 3. Define

$$H = G - (\{x\} \cup N(x)).$$

By (22) and since the four vertices of $N(x)$ have degree 3 in G, $V(H)$ consists of 8 vertices, of which 4 have degree 4 and 4 have degree 3 in G. Since G has exactly 8 paths of length 2 with origin x and since each of these paths has a distinct terminus in $V(H)$, each vertex of $V(H)$ is adjacent in G to exactly one vertex not in $V(H)$. Hence, $V(H)$ consists of 4 vertices of degree 3 in H, and 4 vertices of degree 2 in H. In H there are 12 incidences of edges at the 4 vertices of degree 3, and there are only 8 incidences at the 4 vertices of degree 2. Therefore, two vertices of degree 3 in H are adjacent. These are adjacent vertices of degree 4 in G, a contradiction. This contradiction proves the claim that G has a 4-cycle.

Let $xyzwx$ be an induced 4-cycle in G. Define the graph G/π as in Section 4, so that Theorem C holds. Define

$$E = \{wx, xy, yz, zw\},$$

and denote the edge v_1v_2 of G/π by e_π.

Case 1 Suppose that e_π is a cut-edge of G/π. Then $G - E$ is disconnected. Define G_1 and G_2 to be the two components of $G - E$, where $2 \leq |V(G_1)| \leq |V(G_2)|$. Since $n = 13$, $2 \leq |V(G_1)| \leq 6$, and by (19), G_1 has at most 2 vertices of degree less than 3. Therefore, G_1 has a nontrivial 2-edge-connected simple subgraph H_1, say, with at most two vertices of degree 2. By Lemma 1, $H_1 \in \mathcal{CL}$, and so G has a nontrivial collapsible subgraph. Since G is reduced, this violates (e) of Theorem B.

Case 2 Suppose that e_π is not a cut edge of G/π. We claim

$$a(G/\pi) \leq 2. \tag{23}$$

Suppose not. By Nash-Williams' arboricity formula [9], G/π has a subgraph H (say) with

$$|E(H)| \geq 2|V(H)| - 1. \tag{24}$$

Now since G is reduced, $a(G) \leq 2$, and so H contains one or both vertices of $\{v_1, v_2\}$.

Subcase 2A Suppose $V(H) \cap \{v_1, v_2\} = \{v_1\}$. Then

$$|V(G[E(H)])| = |V(H)| + 1, \tag{25}$$

and we combine (25) with (24) to get

$$\begin{aligned} |E(G[E(H)])| &= |E(H)| \geq 2|V(H)| - 1 \\ &= 2|V(G[E(H)])| - 3. \end{aligned}$$

Since $a(G) \leq 2$, it follows that $G[E(H)]$ is one edge short of having two edge-disjoint spanning trees, i.e., $F(G[E(H)]) = 1$. Since G is reduced, (d) of Theorem B implies $G[E(H)] = K_2$. By (25), this gives

$$|V(H)| = |V(G[E(H)])| - 1 = 1.$$

This and (24) imply $|E(H)| \geq 2|V(H)| - 1 \geq 1$, and since H has no loop, we have a contradiction.

Subcase 2B Suppose $v_1, v_2 \in V(H)$. Then

$$|V(G[E(H) \cup E])| = |V(H)| + 2. \tag{26}$$

By (24) and (26),

$$\begin{aligned} |E(G[E(H) \cup E])| &= |E(H)| + 3 \geq 2|V(H)| + 2 \\ &= 2|V(G[E(H) \cup E])| - 2. \end{aligned} \tag{27}$$

Since $a(G) \leq 2$, (27) implies that the subgraph $G[E(H) \cup E]$ has two edge-disjoint spanning trees, i.e., $F(G[E(H) \cup E]) = 0$. Such a subgraph is collapsible (by (d) of Theorem B), contrary to the fact that G is reduced. This contradiction concludes Subcase 2B and proves the claim (23).

By (23), (f) of Theorem B gives

$$|E(G/\pi)| + F(G/\pi) = 2|V(G/\pi)| - 2.$$

By Theorem C, since $n = 13$, and since $|E(G)| = 2n - 4$,

$$|E(G/\pi)| = |E(G)| - 3 = 19$$

and

$$|V(G/\pi)| = |V(G)| - 2 = n - 2 = 11,$$

and combining these, we get $F(G/\pi) = 1$. Since G/π is 2-edge-connected in Case 2, (d) of Theorem B gives $G/\pi \in \mathcal{CL}$. By (a) of Theorem C, $G \in \mathcal{CL}$, a contradiction, since G is reduced and nontrivial. Hence, $n \geq 14$, and Lemma 5 is proved. □

Catlin [5] conjectured that no smallest number n exists that satisfies the hypothesis of Lemma 5.

6. PROOF OF CAI'S CONJECTURE

Theorem 2 Let G be a simple 3-edge-connected graph of order n. If

$$|E(G)| \geq \binom{n-9}{2} + 16, \tag{28}$$

then G is collapsible.

Proof: Let G satisfy the hypothesis of Theorem 2. If $G \in \mathcal{CL}$, then we are done. If not, then the reduction G' of G, has order at least 2 and is 3-edge-connected. By Lemma 2, either G' is the Petersen graph or G' has order $n \geq 12$.

But G also satisfies Theorem 1 with $p = 10$. By remarks of the prior paragraph, if conclusion (a) of Theorem 1 holds, then $G' = K_1$ and so $G \in \mathcal{CL}$. Conclusion (b) cannot hold, since the Petersen graph does not have size 16. If conclusion (c) holds, then G is a reduced graph of order $n \geq 12$, and either

$$|E(G)| \in \{19, 20\} \quad \text{and} \quad n = 12,$$

or

$$|E(G)| = 22 \quad \text{and} \quad n = 13.$$

By Lemma 4, if $n = 12$ then $|E(G)| = 18$, which is too small. By Lemma 5, if $n = 13$ and $|E(G)| = 22$ then $G = K_{2,11}$, contrary to the hypothesis that $\kappa'(G) \geq 3$. This exhausts the cases and proves Theorem 2. □

X. T. Cai [2] conjectured a weaker form of Theorem 2, in which "collapsible" is replaced by "supereulerian". It is easy to contruct graphs to show that (28) is best-possible, both in Theorem 2 and in Cai's conjecture. Let G be the simple graph obtained from a Petersen graph and the complete graph K_{n-9} by identifying one vertex from each graph. Then G has order $n = (n-9) + 10 - 1$, and if $n = 10$ or if $n \geq 13$ then $\kappa'(G) \geq 3$. Also,

$$|E(G)| = \binom{n-9}{2} + 15,$$

and since the reduction of G is the Petersen graph, G is not collapsible and (by (a) of Theorem B) G is not supereulerian. Hence, (28) is sharp.

7. CONCLUDING REMARKS

Theorem D (Cai [2]) Let G be a 2-edge-connected simple graph of order n. If

$$|E(G)| \geq \binom{n-4}{2} + 6, \tag{29}$$

then exactly one of the following holds:

(i) $G \in \mathcal{SL}$;

(ii) Equality holds in (29) and G has a complete subgraph H of order $n-4$ such that $G/H = K_{2,3}$;

(iii) G is either $K_{2,5}$ or the cube minus a vertex.

Proof: Let G be a 2-edge-connected graph of order n satisfying (29), and let G' be the reduction of G. Then G satisfies the hypothesis of Theorem 1 with $p = 5$. If conclusion (a) of Theorem 1 holds, then G' is a 2-edge-connected reduced graph of order less than 5, and so $G' = K_1$. Hence, by (a) of Theorem B, $G \in \mathcal{SL}$. If (b) of Theorem 1 holds, then equality holds in (29) and G has a complete subgraph H of order $n-4$ such that G' is G/H, a graph of order 5 and size 6. By Lemma 5, $G/H = K_{2,3}$. If (c) holds, then G is a reduced graph such that either

$$|E(G)| \in \{2n-4, 2n-5\} \text{ and } n \in \{6,7\}$$

or

$$|E(G)| = 2n-4 \text{ and } n = 8.$$

By Lemma 5, if $|E(G)| = 2n-4$ for $n \in \{6,7,8\}$ then $G = K_{2,n-2}$, and so either $G \in \mathcal{SL}$ or $G = K_{2,5}$. If $|E(G)| = 2n-5$ and $n = 6$, then since $G = G'$ is 2-edge-connected and satisfies (b) of Theorem B, either G is a cube minus two adjacent vertices (hence in $\mathcal{SL}$) or G is contractible to $K_{2,3}$. If $|E(G)| = 2n-5$ and $n = 7$, then since $G = G'$ is 2-edge-connected and satisfies (b) of Theorem B, G is a cube minus a vertex. □

There are four contraction-minimal nonsupereulerian graphs of order at most 7, namely K_2, $K_{2,3}$, $K_{2,5}$ and $Q_3 - v$ (the cube minus a vertex). A consequence of this fact and Theorem 1 (with $p = 7$) is this:

Theorem 3 Let G be a connected simple graph of order $n \geq 10$. If

$$|E(G)| \geq \binom{n-6}{2} + 10, \tag{30}$$

then exactly one of the following holds:

(i) $G \in \mathcal{SL}$;

(ii) G is contractible to K_2 or $K_{2,3}$;

(iii) Equality holds in (30), G has a complete subgraph H of order $n-6$, and $G/H = K_{2,5}$. □

Conclusion (c) of Theorem 1 is precluded by the hypothesis $n \geq 10$ and because the only 2-edge-connected reduced graph of order $n = 10$ and size 16 is $K_{2,8}$ (by Lemma 5), which is supereulerian. There are several graphs of orders 8 and 9 that violate (30) and conclusions (i), (ii), and (iii). To see that (30) is best-possible, let G be a simple graph containing the complete subgraph $H = K_{n-6}$, $n \geq 10$, such that $G/H = Q_3 - v$. Then (30) barely fails and conclusions (i), (ii), and (iii) fail.

Veldman [10] uses lower bounds on $|E(G)|$ similar to those in this paper, in order to show that a given graph G has a cycle containing at least one end of each edge of G.

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