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Zhi-Hong Chen
Butler University, chen@butler.edu

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Department of Mathematics
Wayne State University
Detroit Mi 48202

Abstract. Let G be a 3-edge-connected simple triangle-free graph of order n. Using a contraction method, we prove that if $\delta(G) \geq 4$ and if $d(u) + d(v) > n/10$ whenever $uv \in E(G)$ (or whenever $uv \notin E(G)$), then the graph G has a spanning eulerian subgraph. This implies that the line graph $L(G)$ is hamiltonian. We shall also characterize the extremal graphs.

Introduction.

We follow the notation of Bondy and Murty [3], except that graphs have no loops. The *line graph* $L(G)$ of graph G is a graph whose set of vertices is the set $E(G)$ of edges of G; two vertices e_1 and e_2 of $L(G)$ are adjacent if and only if e_1 and e_2 have a common vertex in G. For $v \in V(G)$, we define the *neighborhood* $N(v)$ of v in G to be the set of vertices adjacent to v in G. A *bond* is a minimal nonempty edge cut. We shall use P to denote the Petersen graph.

A graph is *eulerian* if it is connected and every vertex has even degree. An eulerian subgraph H is called a *dominating eulerian subgraph* of G if $E(G - V(H)) = \emptyset$. A graph G is called *supereulerian* if it has a spanning eulerian subgraph H. For a graph G, let $O(G)$ denote the set of vertices of odd degree in G. A graph G is called *collapsible* if for every even set $X \subseteq V(G)$ there is a spanning connected subgraph H_X of G, such that $O(H_X) = X$. Thus, the *trivial graph* K_1 is both supereulerian and collapsible. Denote the family of supereulerian graphs by $\mathcal{SL}$, and denote the family of collapsible graphs by $\mathcal{CL}$. Obviously, $\mathcal{CL} \subseteq \mathcal{SL}$, and collapsible graphs are 2-edge-connected. Examples of graphs in $\mathcal{CL}$ include the cycles C_2, C_3, but not C_t if $t \geq 4$.

Let G be a graph, and let H be a connected subgraph of G. The contraction G/H is the graph obtained from G by contracting all edges of H, and by deleting any resulting loops. Even when G is simple, G/H may not be.

In [5], Catlin showed that every graph G has a unique collection of maximal collapsible subgraphs $H_1, H_2, \cdots, H_c$. Define G_1 to be the graph obtained from G by contracting each H_i into a single vertex v_i', $(1 \leq i \leq c)$. Since $V(G) = V(H_1) \cup \cdots \cup V(H_c)$, the graph G_1 has order c and $V(G_1) = \{v_1', v_2', \cdots, v_c'\}$. We call the graph G_1 the *reduction* of G and call H_i the *preimage* of v_i' in G. In this paper we also say that G can be contracted to G_1 if G_1 is the reduction of G.

Any graph G has a unique reduction G_1 [5]. A graph is collapsible if and only if its reduction is K_1. We shall use $d(v)$ and $d_1(v)$ to mean the degree of a vertex v in G and G_1, respectively. A graph is *reduced* if it is the reduction of some other graph.

Theorem A (Catlin [5]). *Let G be a graph.*

(a) *G is reduced if and only if G has no nontrivial collapsible subgraphs.*

(b) *Let H be a collapsible subgraph of G. Then G is collapsible if and only if G/H is collapsible.*

(c) *Let H be a collapsible subgraph of G. Then G is supereulerian if and only if G/H is supereulerian.* ∎

In this note, we will discuss some best possible conditions for a triangle-free graph such that its line graph is hamiltonian.

There are some prior results on hamiltonian line graph of simple triangle-free graph.

Theorem B (Bauer [1]). *Let $G \subseteq K_{n,m}$ be bipartite, where $m \geq n \geq 2$. If $\delta(G) > m/2$, then $L(G)$ is hamiltonian.* ∎

Theorem C (Lai [10]). *Let G be a 2-edge-connected triangle-free simple graph on $n > 30$ vertices. If $\delta(G) > \frac{n}{10}$, then $L(G)$ is hamiltonian.* ∎

Remark: Several authors have studied the same kind of questions for simple graphs (see [2], [4], [5], [6], [7], [9] and [11]).

We shall use the following

Theorem D (Harary and Nash-Williams [8]). *The line graph $L(G)$ of a simple graph G with at least three edges contains a hamiltonian cycle if and only if G has a dominating eulerian subgraph.* ∎

Theorem E (Chen [6]). *Let G be a 3-edge-connected simple graph of order n. If every bond $E \subseteq E(G)$ with $|E| = 3$ satisfies the property that each component of $G - E$ has order at least $n/10$, then exactly one of the following holds:*

(i) *$G \in \mathcal{SL}$;*

(ii) *$n = 10s$ for some integer s, and G can be contracted to P (i.e. $G_1 = P$) such that the preimage of each vertex of P is a collapsible subgraph of G on exactly $s = n/10$ vertices.* ∎

Main Results.

Theorem 1. *Let G be a 3-edge-connected simple triangle-free graph of order n. If $\delta(G) \geq 4$ and if every $uv \in E(G)$ satisfies*

$$d(u) + d(v) \geq \frac{n}{10}, \tag{1}$$

then exactly one of the following holds:

(i) *$G \in \mathcal{SL}$;*

(ii) *$n = 10m$ for some integer $m \geq 8$, and G can be contracted to P such that the preimage of each vertex v_i ($1 \leq i \leq 10$) of P is either $K_{t,s}$ or $K_{t,s} - e$ for some e, where t and s are dependent on i, $t + s = m = n/10$ and $\min\{t, s\} \geq 4$.*

Proof: By (c) of Theorem A, and since P is not supereulerian, the conclusions (i) and (ii) are clearly mutually exclusive.

Let E be a bond of G with $|E| = 3$, and let H be a component of $G - E$. For any $e \in E(G)$, let n_e denote the number of edges of E adjacent in G to e. By $\delta(G) \geq 4$ and $|E| = 3$, we have $|V(H)| > 1$. Hence, H has an edge, say xy. By $\delta(G) \geq 4$ and $|E| = 3$, and since G is simple,

$$4 + 4 \leq d(x) + d(y) \leq 2(|V(H)| - 1) + n_{xy} \leq 2|V(H)| + 1,$$

and so $|V(H)| \geq 4 > 3 = |E|$. Then H has a vertex, say u, that is not incident with any edge of E. By $d(u) \geq \delta(G) \geq 4 > |E|$, u has a neighbor in H, say v, that is also not incident with any edge of E, and so $N(v) \subseteq V(H)$ and $N(u) \subseteq V(H)$. Since G is triangle-free, $N(u) \cap N(v) = \emptyset$. Hence, by (1),

$$|V(H)| \geq |N(u)| + |N(v)| = d(u) + d(v) \geq \frac{n}{10}.$$

By Theorem E, either $G \in \mathcal{SL}$, or $n = 10m$ for some $m \geq 8$ and G can be contracted to P such that all preimages $H_1, H_2, \cdots, H_{10}$ have order $m = n/10$.

Suppose G can be contracted to $G_1 = P$. Let $V(P) = \{v'_1, v'_2, \cdots, v'_{10}\}$. Thus $d_1(v'_i) = 3$ for $1 \leq i \leq 10$. The corresponding maximal collapsible subgraphs are $H_1, H_2, \cdots, H_{10}$. Each H_i ($1 \leq i \leq 10$) is joined to the remainder of G by a bond consisting of the $d_1(v'_i) = 3$ edges that are incident with v'_i in P. Then from above we can see that each H_i ($1 \leq i \leq 10$) has u_i and v_i in $V(H_i)$ such that

$$V(H_i) = N(v_i) \cup N(u_i) \text{ and } N(u_i) \cap N(v_i) = \emptyset.$$

Since only $d_1(v'_i) = 3$ edges of G have one end in H_i and by (1), it follows that H_i is $K_{t,s}$ or $K_{t,s} - e$ for some $e \in E(K_{t,s})$, where $t = |N(u_i)|$ and $s = |N(v_i)|$, and so $t + s = |V(H_i)| = n/10$ and $\min\{t, s\} \geq \delta(G) \geq 4$. ∎

Theorem 2. *Let G be a 3-edge-connected simple triangle-free graph of order n. If $\delta(G) \geq 4$ and if*

$$d(u) + d(v) \geq \frac{n}{10}, \tag{2}$$

whenever $uv \notin E(G)$, then exactly one of the following holds:

(i) $G \in \mathcal{SL}$;

(ii) $n = 20s$ *for some integer $s \geq 4$, and G can be contracted to P in such a way that the preimage of each vertex of P is either $K_{s,s}$ or $K_{s,s} - e$ for some edge e.*

Proof: Let E be a bond of G with $|E| = 3$, and let H be a component of $G - E$. From the proof of Theorem 1, we know that there is an edge, say uv, such that $N(v) \subseteq V(H)$ and $N(u) \subseteq V(H)$. Since G is triangle-free, $N(v) \cap N(u) = \emptyset$. Hence

$$|V(H)| \geq |N(u)| + |N(v)|. \tag{3}$$

Case 1 ($n \leq 80$). Since $\delta(G) \geq 4$, by (3),

$$|V(H)| \geq d(v) + d(u) \geq 2\delta(G) \geq 8 \geq \frac{n}{10}.$$

By Theorem E, it is easy to see that the theorem holds.

Case 2 ($n \geq 81$). Since $\delta(G) \geq 4$ and $|E| = 3$, either $N(u)$ or $N(v)$ has at least two vertices x and y which are not adjacent to any edges of E and then $N(x) \subseteq V(H)$ and $N(y) \subseteq V(H)$. We may assume that x and y are in $N(u)$. Since G is triangle-free, $xy \notin E(G)$. By (2),

$$2\max\{|N(x)|, |N(y)|\} \geq |N(x)| + |N(y)| = d(x) + d(y) \geq \frac{n}{10}.$$

We may assume

$$|N(x)| \geq \frac{n}{20}. \tag{4}$$

Since $n \geq 81$, $|N(x)| \geq 5$ and so we can find $w, z \in N(x)$ such that w and z are not adjacent to any edges of E and then $N(w) \subseteq V(H)$ and $N(z) \subseteq V(H)$. Since G is K_3-free, $wz \notin E(G)$. By (2),

$$2\max\{|N(w)|, |N(z)|\} \geq |N(w)| + |N(z)| = d(w) + d(z) \geq \frac{n}{10},$$

and so we may assume

$$|N(z)| \geq \frac{n}{20}. \tag{5}$$

Since $z \in N(x)$ and G is triangle-free, $N(x) \cap N(z) = \emptyset$. Since $N(x) \subseteq V(H)$, and $N(z) \subseteq V(H)$, by (4) and (5),

$$|V(H)| \geq |N(x)| + |N(z)| \geq \frac{n}{20} + \frac{n}{20} = \frac{n}{10}. \tag{6}$$

Therefore, by Theorem E, either $G \in \mathcal{SL}$, or G can be contracted to P such that the preimages $H_1, H_2, \cdots, H_{10}$ of vertices of P have order $\frac{n}{10}$.

Suppose that G can be contracted to P. Let $V(P) = \{v_1', v_2', \cdots, v_{10}'\}$. The corresponding maximal collapsible subgraphs are $H_1, H_2, \cdots, H_{10}$. From above and (6), and since $|V(H_i)| = n/10$, we can see that for each i ($1 \leq i \leq 10$), $V(H_i) = N(x_i) \cup N(z_i)$ for some $x_i, z_i \in V(H_i)$ with $N(x_i) \cap N(z_i) = \emptyset$ and $|N(x_i)| = n/20$, $|N(z_i)| = n/20$. Since only $d_1(v_i') = 3$ edges have exactly one end in H_i and by (2), H_i is either $K_{s,s}$ or $K_{s,s} - e$ for some $e \in E(K_{s,s})$, where s=n/20. ∎

Corollary 3. *Let G be a 3-edge-connected simple triangle-free graph on $n \geq 61$ vertices. If*

$$\delta(G) \geq \frac{n}{20},$$

then either $G \in \mathcal{SL}$ or $n = 20s$ for some integer $s \geq 4$ and G can be contracted to P in such a way that the preimage of each vertex of P is either $K_{s,s}$ or $K_{s,s} - e$ for some $e \in E(K_{s,s})$.

Proof: The inequalities $n \geq 61$ and $\delta(G) \geq n/20$ imply that $\delta(G) \geq 4$ and (2) holds in Theorem 2. Hence Corollary 3 follows. ∎

Remark: Let t and s be two integers with $t + s = 43$ and $\min\{t, s\} \geq 4$. Let G be the graph obtained by taking the union of bipartite graph $K_{t,s}$ and the Blanuša snark, and by identifying a pair of vertices, one from each component. Then G is a 3-edge-connected simple triangle-free graph of order $n = 60$ and

$$\delta(G) = 3 \geq \frac{n}{20},$$

and so for any two vertices u and v in G (no matter whether $uv \in E(G)$ or not),

$$d(u) + d(v) \geq 6 \geq \frac{n}{10}.$$

But the reduction of G is the Blanuša snark, which is a nonsupereulerian triangle-free cubic graph on 18 vertices, and so the graph G does not satisfy any conclusions of Theorem 1, 2 and Corollary 3. One can see that other reduced nonsupereulerian cubic graphs of order $n \leq 60$ can also be used to construct such graphs G. This shows that the condition $\delta(G) \geq 4$ in Theorem 1 and Theorem 2 is necessary and $n \geq 61$ in Corollary 3 is best possible in some sense.

By Theorem D and Theorem 1 or 2, we have the following

Corollary 4. *Let G be a 3-edge-connected simple triangle-free graph of order n. If $\delta(G) \geq 4$ and if*

$$d(u) + d(v) > \frac{n}{10},$$

whenever $uv \in E(G)$ (or whenever $uv \notin E(G)$), then $L(G)$ is hamiltonian. ∎

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