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Constructing a Concept of Number

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Numbers are concepts whose content, structure, and organization are influenced by the material forms used to represent and manipulate them. Indeed, as argued here, it is the inclusion of multiple forms (distributed objects, fingers, single- and two-dimensional forms like pebbles and abaci, and written notations) that is the mechanism of numerical elaboration. Further, variety in employed forms explains at least part of the synchronic and diachronic variability that exists between and within cultural number systems. Material forms also impart characteristics like linearity that may persist in the form of knowledge and behaviors, ultimately yielding numerical concepts that are irreducible to and functionally independent of any particular form. Material devices used to represent and manipulate numbers also interact with language in ways that reinforce or contrast different aspects of numerical cognition. Not only does this interaction potentially explain some of the unique aspects of numerical language, it suggests that the two are complementary but ultimately distinct means of accessing numerical intuitions and insights. The potential inclusion of materiality in contemporary research in numerical cognition is advocated, both for its explanatory power, as well as its influence on psychological, behavioral, and linguistic aspects of numerical cognition.

***Keywords*:** numerical cognition; numerical elaboration; extended cognition; Material Engagement Theory; materiality

Number is a mystery. Alternately deemed metaphysically real (Maddy, 1990) and created entirely by the powers of the human mind (Brouwer, 1981), numbers are considered "pure" concepts whose "abstract, non-empirical nature" describes but somehow lies beyond the physical world itself (Stewart, 2014, p. 8). Insight into the nature of number has been sought in the perceptual experience of quantity, something shared by so many species it would be surprising to find one that lacked, minimally, the ability to distinguish *more* from *less*. Bridging the gulf between the perceptual experience of quantity and elaborated forms of number has historically been challenging. More recently, number has been argued to be a cultural construct, informed by but distinct from shared quantity perception (Núñez, 2017).

Núñez's numerical–quantical distinction is a crucial starting point for understanding what numbers are: concepts realized and elaborated through the interaction of psychological processes, physiological characteristics, behaviors, and material forms and their attributes (e.g., quantity perception; pentadactyl limbs; pairing and handwriting; objects with quantity and attributes like persistence and manipulability). The first three span cultures, languages, and time. The perceptual experience of quantity appears universal even across the WEIRD/non-WEIRD divide (Henrich, Heine, & Norenzayan, 2010). Hands and feet with five digits are an unambiguous human physiological norm. Similarly distributed are behaviors like finger-counting (Domahs, Kaufmann, & Fischer, 2012) and the use of comparison and combinatorial strategies, phenomena whose universality is attested by the somatic basis of numbers, behavioral strategies

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like one-to-one correspondence, and the use of material forms to represent and manipulate numerical concepts (e.g., McCrink, Spelke, Dehaene, & Pica, 2013; Von den Steinen, 1894). Despite these commonalities, number systems demonstrate a wide variability in attributes like finite number (Greenberg, 1978) and organizing base (Comrie, 2011, 2013), as do their users in matters of how the fingers are used in counting (Bender & Beller, 2012) and the degree to which numbers are visualized as linear (Núñez, 2011).

Wide variability despite psychological, physiological, and behavioral commonalities suggests the material domain be examined for its influence on number concepts. Here it is argued that number concepts are constructed through material engagement, that their properties reflect and originate in different material forms incorporated into the cognitive system for numbers, and that numerical variability results primarily from different material forms and combinatorial choices used for representing and manipulating numbers. This materially informed variability creates conceptual differences between number systems and within number systems over time. Discussion is structured as follows: What numbers are as concepts and how they vary diachronically and synchronically is examined first, followed by a discussion of how material forms influence their content, structure, and organization. The role of particular material forms in elaborating numerical concepts is then outlined. Materiality and language are discussed as interacting and complementary but likely distinct means of accessing numerical intuitions and insights. Finally, a role for materiality in numerical cognition research is advocated.

The framework for this analysis is Material Engagement Theory (Malafouris, 2013), which considers cognition as *extended* (i.e., cognition is a system composed of brain, body, and materiality) and *enactive* (cognition *is* the interactivity between the components of the system). An extended/enactive framework for analyzing numerical cognition is appropriate, given that, for example, calculating with written numerical notations is an "amalgam of two [inseparable] activities, thinking (imagining actions) and scribbling (making ideal marks)" (Rotman, 2000, p. 39). Fingers and notations are also treated as physical forms with material qualities equivalent to those of physical devices like tallies and abaci, albeit possessing unique biological or symbolic qualities. This approach necessitates the conceptual boundaries of what is considered material be flexible, perhaps to an uncomfortable degree. However, it allows qualities like linearity, persistence, and manipulability to be examined across the full range of devices used to represent and manipulate numbers, and it provides insight into how fingers and notations fit within longer sequences of devices in numerical elaboration. The analysis focuses on material properties, how these interact with psychological and behavioral properties (e.g., quantity perception; comparison strategies), and how psychological–behavioral–material interactions inform the content, structure, and organization of numerical concepts.

1.0 Numbers as Concepts that Vary Diachronically and Synchronically

There is a tendency to think that what numbers are today is what numbers are everywhere and have always been. That is, a monolithic number is assumed for contemporary Western and non-Western peoples, as well as ancient societies and cultures (Rotman, 2000). However, today's Western numbers were elaborated over millennia. This elaboration has, among other things, turned one, zero, fractions, and negatives into numbers, and discarded any extensional conditions or restrictions on what numbers might apply to (Bender & Beller, 2011; Gouvêa, 2008; Ifrah, 1981, 2000; Klein, 1992; Rotman, 1987; Russell, 1920; Selin, 2000). Matters as seemingly self-evident and incontrovertible as $1 + 1 = 2$ have been questioned, and ways to prove them

formulated (Peano, 1889; Whitehead & Russell, 1927). Western numbers have become entities related to each other numerically in ways that are manipulable through operations and functions: The number 4 is itself; 2 + 2; 4 × 1; 10 – 6; 20 ÷ 5; √16; –1,365,908 ÷ –341,477; and any and simultaneously all of infinitely possible relations. The number 4 and its fellows are elements of a system in which the relations between them are as critical to what numbers are and what we can do with them as notes are to music and sounds are to language (Plato, 1892).

Yet not all numbers are characterized by infinite set of potential relations. The traditional numbers of the Oksapmin people of Papua New Guinea are an ordinal sequence counted on the body (Lean, 1992; Saxe, 2012). Their set of relations are those implicit to ordinal sequencing (e.g., *more than*, *less than*) and enable Oksapmin numbers to be incremented and decremented using positions on the body. Numbers related ordinally differ from numbers with infinite relations in ways that affect how they can be manipulated in performing arithmetical tasks. Asked to add and subtract with body-counting, Oksapmin respondents developed innovative strategies like double enumeration: Subtracting 7 from 16 involved creating "internal correspondences within the body system, using one series of body parts, in this case the thumb (1) through forearm (7) (the subtrahend), to keep track of the subtraction from 16 (the minuend)" (Saxe, 2012, p. 86). Quantity-wise, Oksapmin numbers are identical to Western numbers, but lack the ideas that 7, 9, and 16 are related beyond their ordinality and that operations like subtraction can be performed by manipulating those relations.[i] How such relations and operations are explicated materially and ultimately retained in forms like conceptual knowledge are discussed below.

A number that can be incremented and decremented ordinally involves the use of a contiguous material form to represent the sequence. Numbers that have not been collected onto a single material form (e.g., fingers, tallies) differ in the linearity of their organization. The Mundurukú use *fat*, *arms*, and *parents*—material forms with the requisite (approximate) cardinality but physically and conceptually unconnected—to represent *two*, *three*, and *four* (Rooryck, Saw, Tonda, & Pica, 2017).[ii] Their numerical mapping is nonlinear (Dehaene, Izard, Spelke, & Pica, 2008). These conditions are plausibly related: Numbers represented by disparate material forms might tend not to be organized in the same way influenced by contiguous forms like the fingers and tallies (e.g., linear with stable order), nor defined against one another to become more discrete. Numbers collected onto nonlinear material forms might also be less linearized. Yupno numbers are counted on the body in a sequence that crosses the hands, descends to cross the feet, rises to zigzag the head, and descends to zigzag the torso; altogether, direction reverses nine times horizontally and three vertically (Wassmann & Dasen, 1994). While portions of the sequence are linear (e.g., those crossing the hands), the complete pattern is not; this may be the reason Yupno number-users do not demonstrate a linearized mental number line (Núñez, Cooperrider, & Wassmann, 2012). As a cultural construct (Núñez, 2011), a mental number line is plausibly related to the linearity of the material form(s) used to represent and manipulate numbers. This is possibly a priming effect of material forms perceived visuospatially (Stoianov, Kramer, Umiltà, & Zorzi, 2008), with uncollected numbers showing a logarithmic distribution (Dehaene, 2003) and numbers collected onto a linear device being influenced toward linear structure. Increased linearization of the mental number line, in turn, likely facilitates the conceptualization of higher quantities in regularized and productive ways.

2.0 How Material Forms Inform Concepts Like Numbers

Material forms are integral to human cognition in ways that extend beyond their historically recognized role in storing mental content.[iii] This is particularly true of numbers, where material forms make the perceptual experience of quantity tangible (Malafouris, 2010) and comprensible (Frege, 1953). Material forms anchor and stabilize concepts, and their properties act as proxies for conceptual relationships (Hutchins, 2005). Given material form, quantity becomes discrete, persists, and can be represented and shared. It acquires content, structure, and organization from material properties like linearity and manipulability. These qualities increase the likelihood that someone interacting with an artifact will notice similarities and patterns in the quantities it instantiates, like each new notch on a tally being (one) more than the previous notch. Patterns and similarities, in turn, can be codified and will endure as artifacts, knowledge, behaviors, and language. Material forms support the visualization of numerical and mathematical concepts, and this pervades the full spectrum of mathematical elaboration (Dreyfus, 1991; Kaput, 1987), not only in the forms of gestures, objects, and diagrams but as notations and equations as well (Landy, 2010; Landy & Goldstone, 2007), forms whose semantic concision preserves numerical, spatial, topological, and geometric relations and facilitates pattern accessibility and informational manipulability in ways that paragraphs and discourse cannot (Larkin & Simon, 1987; Sfard & Linchevski, 1994).

Artifacts provide a medium for collaboration, enabling cultural systems like numbers to be realized and refined over generations of effort (given social investment in sustaining the requisite behaviors and specialization). Artifacts for counting place the realization of numbers within reach of societies comprised of average individuals. This is fortunate, for three reasons. Cumulatively, average capabilities and capacities have a regression-to-the-mean effect, so that the material forms used to represent and manipulate numbers (e.g., abaci, numerals) remain synchronized to average traits while becoming increasingly optimized for eliciting specific behavioral and psychological effects. The use of material forms as a common collaborative medium means that change in the cognitive system for numbers occurs on the level of the community of users, not just the individual. It also means that societies need not await a genius to invent such systems whole-cloth, assuming such massive invention were even possible.

Artifacts help mediate between what a society knows and an individual learns (Haas, 1996), interacting with psychological and physical abilities to pattern, habituate, and automate knowledge, behaviors, and skills. They enable problems to be decomposed into series of smaller tasks, which are more easily solved because they are smaller (Beer, 2003). This reduces a large, complex task to a sequence of relatively small and simple tasks (e.g., 98.713×1456.2 begins with 3×2) (Hodder, 2012). Decomposing problems also makes it easier for multiple individuals to collaborate in formulating solutions, increasing the potential for realizing novel outcomes (the two-heads-are-better-than-one effect writ large). Solutions can be encoded in artifacts, making the information available to other individuals and future generations. This accumulates social knowledge, distributing cognitive effort over space and time in a way that decreases the effort required by any particular generation (Hutchins, 1995). Societies need not reinvent numbers; rather, they use and extend the knowledge encoded in forms like abaci and numerals. This opens up further opportunities to refine or extend artifacts to new uses, affording additional possibilities for change (Damerow, 2010).

As more material forms are recruited into the cognitive system for numbers, qualities associated with older material forms can persist in the way newer forms are used, not in the

physical forms themselves but rather, through mental and behavioral mechanisms like knowledge, beliefs, and expectations; behavioral conditioning, patterning, and habituation; the predispositional effects of psychological, physiological, and behavioral capacities and capabilities; and enculturation effects (Tang et al., 2006). The persistence of material structure in mental/behavioral form tends to make numbers, relations, and operations independent of any particular material form used to represent and manipulate them; accordingly, numbers become irreducible to all the material forms they might take (Overmann, 2017a). That is, while conceptual content is related to the materiality that gives it form, there is more to the former than what the latter instantiates. Ancient Greek mathematicians used dot matrices to visualize and investigate numerical properties (the term *square* originated in matrices for numbers like 4, which consisted of 2×2 dots) (Klein, 1992). It was certainly possible for Neolithic Mesopotamians to have arranged four clay tokens in a two-by-two square. However, the Greeks conceived numbers as *entities*, while Neolithic Mesopotamians were more likely to have conceived four tokens as a *collection*. Four cones and four dots are nearly identical physically, but they represent distinct notions of the number 4. And this is exactly the difficulty: When we look at them, we superimpose our number 4 on their notations. The material form's representational intelligibility obscures differences in the conceptualization of number.

3.0 The Role of Particular Material Forms in Elaborating Numerical Concepts

The previous sections suggested that numerical content, structure, and organization are closely related but not reducible to the material form(s) used for representation and manipulation. Here the foundational idea is that the incorporation of material devices is the mechanism of numerical elaboration. The analysis is summarized in Table 1.

3.1 Initial Numbers: Perceptual Experience, Quantity Comparisons, Distributed Exemplars

The idea that numbers are cardinality shared by sets of objects (Russell, 1910, 1920), along with data on emerging number-words (e.g., Closs, 1993; Lean, 1992), suggest that numbers start as judgments of sameness and difference in pairs and single objects (e.g., two objects share quantity; a single object and a pair differ in quantity). The ability to abstract quantity, a categorical judgment of relation, while suppressing superfluous information (e.g., color) appears unique to humans (Christie & Gentner, 2007). The ethnographic literature documents the representation of such concepts through iconicity (e.g., recreating an exemplar's quantity with the matching quantity of fingers, objects, or syllables[iv]) and indexicality (pointing to an exemplar with gesture or words). As such exemplars can be distributed throughout an environment (as in the Mundurukú example) rather than being collected onto a contiguous material form, the associated number concepts tend to be limited to perceptible (i.e., mainly subitizable) quantities that are less discrete than they presumably become once collected onto a material form (where proximity and comparison act to define them against one another). Number-words are also collected into sequences, which tend not to be associated with linearized mental number lines (e.g., the Mundurukú and Yupno examples). Neither can number-words be manipulated into the kinds of (visual) patterns that stimulate numerical insights (e.g., adding or dividing words does not have the same potential for occasioning concepts of accumulation and divisibility as adding or dividing objects). Without further incorporation of material forms that can be used to scaffold concepts of higher quantity (e.g., the fingers on the hand and *five*), numbers will remain consistent with the capacity of processes influencing the subitizing range: object tracking (Carey, 2009), attention (Burr, Turi, & Anobile, 2010; Ester, Drew, Klee, Vogel, & Awh, 2012), and memory (Rooryck et al., 2017).

Table 1. Analysis of Device Types

Type	Concept and transition	Capacities and properties	New capabilities	New limitations
Distributed exemplars	– Equivalences – Emergence of number concept	– Perception of quantity – Abstraction of quantity similarity or dissimilarity	– Recreated iconically (fingers, objects, syllables) and indexically (gesture, words)	– Unstructured – Limited to subitizing – Ephemeral
Fingers	– Equivalences – Imposition of basic structure	– Neurologically integrated with the perception of quantity – Ready availability – Psychological–behavioral–material bridge	– Linearity and stable order	– Limited capacity – Ephemeral
One dimension (e.g., tally)	– Collections related to enumerated objects – Use of material culture	– Linearity and stable order – Accumulation	– More capacity (higher numbers) – Persistent	– Visually indiscriminable – Not manipulable
Two dimensions (e.g., abacus)	– Collections related to each other, as well as to enumerated objects – Emergence of knowledge-based numeration	– Linearity and stable order (imposed) – More operations – Greater capacity (higher numbers)	– Grouped (more discriminable; productive) – Manipulable (more explicit relations, new operations, more complex operations)	– Loose (need for containment) – Neither concise nor persistent (not suitable for recording)
Written notations	– Conceptualized as entities (see note) – Numbers defined by relations	– Linearity and stable order (imposed) – Many operations – Two-dimensional structure (imposed) – Even greater capacity	– Integrity of form – Concise and persistent (suitable for recording) – Handwritten (literacy effects) – Ability to record large volumes of data – Whole–part relations – Greater calculational complexity – Manipulation by conceptual relations between fixed signs	– Fixed (not manipulable)

Note: Entities differ from collections in the copula used: *two and two* are *four* (beads on an abacus), but *two plus two* is *four* (notations) (Gowers, 2008). Table adapted from Overmann (2017), Concepts and how they get that way, Phenomenology and the Cognitive Sciences, available online 31 October 2017 through Springer Science+Business Media B.V. (https://doi.org/10.1007/s11097-017-9545-8).

3.2 Basic Structure: Fingers Influence Linearity and Stable Order

The prevalence of anatomical bases (e.g., 10, 5, and 20) in documented number systems implies that numbers are often collected onto the hand,[v] a typical material structure for representing the first non-subitizable quantities (five and ten[vi]). There are several reasons why the hand might be used enough to pattern the world's number systems and make finger-counting a behavior that spans significant differences in language, culture, and numerical elaboration: First, the angular gyrus, the region of the brain implicated in finger gnosia, finger-counting, and the ability to calculate, associates fingers and numbers (Roux, Boetto, Sacko, Chollet, & Trémoulet, 2003).[vii] Finger gnosia predicts numerical performance: The better someone "knows" her fingers, the more likely she will perform well on mathematical tasks (Gracia-Bafalluy & Noël, 2008; Marinthe, Fayol, & Barrouillet, 2001; Penner-Wilger et al., 2007; Reeve & Humberstone, 2011). The mental abacus, where participants move their fingers to manipulate an imaginary device, also associates fingers and numbers; interference experiments suggest that calculation does not depend on actually moving the fingers but involves motor-movement planning (Brooks, Barner, Frank, & Goldin-Meadow, 2014; Frank & Barner, 2012). Second, the use of the hand in representing quantity evokes the general tendency to judge the size of body and world by comparing them to each other (Mattens, 2013). Finally, the hand, as both actor and instrument, bridges the psychological, behavioral, and material dimensions of numeracy (Gallagher, 2013; Malafouris, 2013): The hand's quantity, position, and movement are appreciated thorough vision, proprioception, and interoception; it touches and manipulates objects being counted, as well as devices representing and manipulating numerical information, and of course it can itself act as a device.

While it is supported by its neurological underpinnings, finger-counting is learned behavior (as is *finger-montring*, display involving non-sequential finger patterns that facilitate biomechanical production and improve visual distinguishability). People blind from birth do not count on their fingers (Crollen, Mahe, Collignon, & Seron, 2011), presumably because it is behavior learned by watching social others and supported by visual interaction with the material structure of the hand and objects being enumerated. While finger-counting patterns vary cross-culturally (Domahs, Moeller, Huber, Willmes, & Nuerk, 2010; Huylebrouck, 1997), all known variants involve choosing some feature to start counting (e.g., an outside finger or finger segment) and proceeding sequentially in some fashion to another feature that finishes counting (Overmann, 2014). In other words, social groups tend to do the same thing—count with their fingers—in different ways. This suggests that patterns involve but are not determined by topographic sensorimotor structure (Harvey, Klein, Petridou, & Dumoulin, 2013); they appear to be mediated by material features of the hand and cultural exposure. Patterns are repeated enough to become habitual; the reasons for this become apparent when alternatives are considered: Using the fingers randomly would be less consistent and reliable, while using the fingers non-sequentially would be biomechanically awkward and less distinguishable visually. By comparison, starting with the same point, proceeding in the same fashion, and ending at the same point reduce demands on memory and attention, facilitate biomechanical production, and improve informational consistency and reliability, matters improved further by behavioral automaticity.

The hand's role in production is related to its natural grouping into fives and tens (twenties with toes). Once fingers and perhaps toes have been used on one person, the next natural grouping is repeating the sequence on the same person or using the fingers/toes of

another person. Natural grouping tends to limit the capacity of a single hand to five. Fingers also do not specify what they count, information that must be maintained elsewhere (e.g., typically, in memory or context). The hand is also perishable as a representational device, as it is needed for other purposes in fairly short order. These characteristics not only inform the basic numerical structure and organization, they also motivate the incorporation of other material forms.

3.3 Transition to Material Culture: Devices with a Single Dimension

New material forms are selected because of capabilities and properties they share with previous forms, and because they address their limitations. Fingers accumulate and represent, but typically not to quantities that necessitate grouping; they impose linearity and stable order, and they lack capacity and persistence. Devices that follow fingers include tallies, knotted strings, stringed beads, torn leaves, marks on the ground, body-counting, and pebbles. Notably, these devices supplement (not supplant) finger-counting, and they do not necessarily entail the availability of lexical names for all the quantities they represent (i.e., as devices like rosaries need not involve lexical counting). As a group, these devices support accumulation, influence or reinforce linearity and stable order, and provide capacity and persistence. They share the potential for scaffolding higher quantities, and with the increased number of exemplars, more intra- and inter-exponential relations (Beller & Bender, 2011; also see Figure 1). However, they also inject new limitations (e.g., not being manipulable). Fixed forms like tally marks have less capacity for removal (subtraction) and grouping (multiplication and division) than do manipulable forms like pebbles. Given the potential of these devices for accumulation (all) and removal/grouping (some), it cannot be coincidental that addition is generally the first arithmetical operation to emerge across number systems.[viii] They also represent the involvement of material forms beyond the body, most frequently as physical devices used communally,[ix] facilitating the accumulation and distribution of numerical knowledge across individuals and generations (Hutchins, 1995).

These devices influence numbers toward linearity and stable order for reasons similar to those for fingers: Use of the same sequence reduces demands on memory and attention and improves informational consistency and reliability. Knowledge and habit also predispose people toward certain expectations and behaviors in using (older) material forms, limiting the range of how (newer) forms might be used. Numbers that have acquired linearity and stable order from finger-counting are thus more likely to be used in similar fashion, even on forms whose structure is not inherently linear or ordered (e.g., pebbles). These devices have greater capacity and persistence than fingers, qualities that vary with the size and durability of the physical substance. Larger surfaces accumulate more marks than smaller ones. Tallies made of bone last longer than ones made of wood; devices made of plant materials last longer than marks on the ground or body, which are themselves more permanent than finger-counting and gestures.

These devices are also characterized by intra-exponential relations (Figure 1). The potential for relations begins with two or more elements (this condition arguably obtains with distributed exemplars and fingers, but is intensified by the proximity, contiguity, and capacity that material forms provide). Devices that accumulate and represent have the potential to help explicate the relations implicit to accumulation (e.g., *more than*) and comparison (*same as*). Devices with manipulability (e.g., pebbles) may support additional relations (e.g., regrouping four pebbles as two groups may explicate relations between two and four). *Plus one* can emerge from behaviors like making notches or stringing beads, as the embodied act of making another

notch can give rise to a concept of *one more*. However, devices like tallies and knotted strings lack natural grouping like those of the fingers, so they tend not to influence productive grouping (perhaps why anatomic grouping tends to persist); as a result, their inter-exponential relations tend to be shallow. The combination of some intra-exponential and few inter-exponential relations makes the devices one-dimensional.

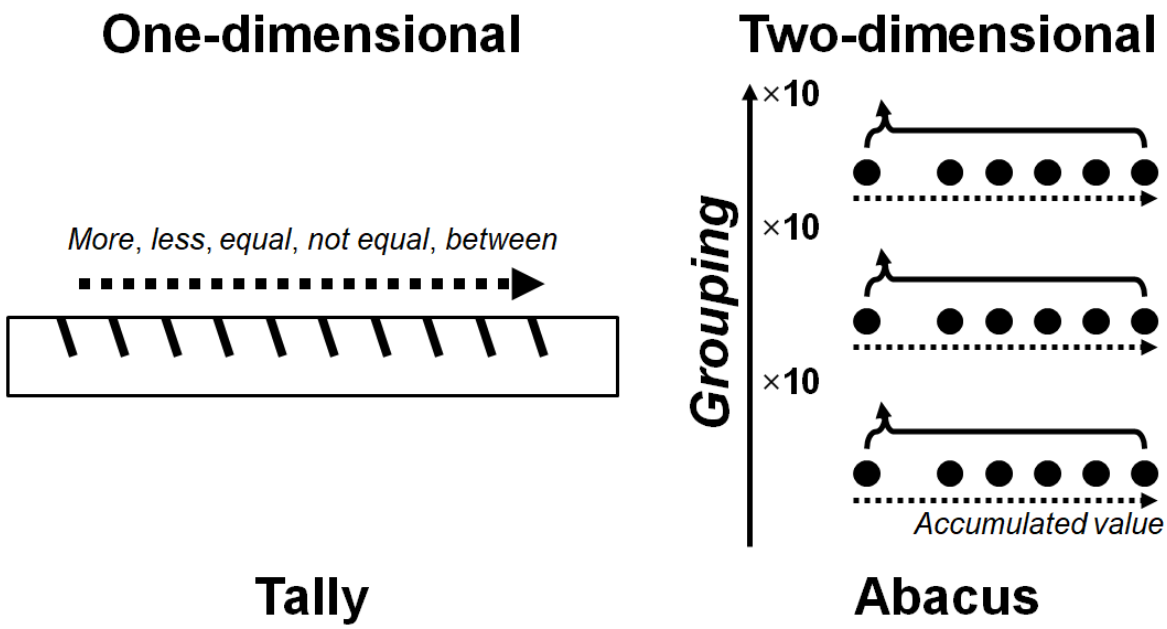

Figure 1. Exponential Dimensions of Counting Devices. Dimensionality is accumulation (a single, intra-exponential dimension) and grouping (a second, inter-exponential dimension). (Left) For material forms that accumulate but do not group, potential relations between numbers are those of an ordinal counting sequence (shown by the dashed line): *more than*, *less than*, *same as*, *not the same as*, *between*, and possibly *one more*. These intra-exponential relations are implicitly quantificational but not necessarily explicitly numerical (e.g., as a rosary can accumulate without number-words). Total value is achieved by accumulating along the single dimension. (Right) With material forms that also group, potential relations between numbers are those of the intra-exponential dimension (accumulation along the horizontal axis, shown by dashed lines) and those implicit to grouping (inter-exponential; accumulation along the vertical axis, shown by solid lines). Total value is achieved by accumulating along both dimensions. Western numbers have both dimensions (e.g., the numbers 0 through 9 and exponents 10^0, 10^1, 10^2, etc.); Oksapmin numbers have the first (but perhaps not the second, depending on how one views repetitions of the cycle); and Mundurukú numbers have neither. Figures based on Chrisomalis (2010).

One-dimensional devices also interact with quantity perception to impose a new limitation: Beyond the subitizing range, the quantity of elements is less visually discriminable. This necessitates that total value be obtained by recounting (which entails the availability of a lexical counting sequence) or comparison with an external standard (which entails the standard's availability), neither of which is particularly efficient. Visual indiscriminability can be overcome by grouping elements (e.g., on a fixed device like a tally, this can be accomplished by differences in spacing, length, or orientation during notch production). However, such conventions do not necessarily solve the need for manipulability (which affects devices like tallies more than knotted strings, as knots can be untied or cut off) or physical integrity (which affects loose

objects like pebbles). These considerations may, under social needs for increased enumeration, motivate the incorporation of new forms with greater manipulability and physical integrity.

3.4 Knowledge-Based Numeration: Devices with Intra- and Inter-Exponential Representation

Devices with grouping and manipulability potentialize the explication of inter-exponential relations, as well as operations beyond accumulation. They include the Mesopotamian clay tokens, the abacus, and notations (the last is treated in the next section). Clay tokens were used for accounting in Mesopotamia possibly as early as the tenth millennium BCE (Moore, 2000) and as late as the first millennium BC (MacGinnis, Monroe, Wicke, & Matney, 2014), but are particularly associated with the Neolithic (8300–4500 BCE) (Schmandt-Besserat, 1992). By the mid-to-late fourth millennium BCE, tokens were grouped (or "bundled"): One token of a higher unit was equivalent to between two and ten lower-unit tokens. Bundling relations were based on things like fingers (10 cycles) and metrology (as four quarts make a gallon today). They were encoded as conventions of token shape and size: In the system used to count most discrete objects, ten small cones were equivalent to one small sphere, six small spheres to one large cone (Nissen, Damerow, & Englund, 1993). Different combinations of shape–size conventions also designated the enumerated commodity (e.g., grain, fish), solving another problem, the need to represent what was being counted in the absence of writing.

Analysis of mathematical texts from later periods suggests that tokens may have been used with counting boards, making them an abacus-like device (Høyrup, 2000). Counting boards are not attested archaeologically (possibly because organic materials preserve poorly in the region; Coinman, 1996), nor are they depicted in reliefs or described textually. However, they are suggested by archaic and later cuneiform signs for words like *count*, *number*, and *chief administrator of the temple household* (i.e., someone who counted and used numbers), which vaguely resembled counting boards (Ifrah, 2000). Even without much formal organization of the surfaces on which they were placed, tokens were likely separated by quantity and ordered by magnitude.[x] Comingling tokens without regard to value would have degraded the intelligibility and accessibility of the represented information. Alternatively, gradation by increasing magnitude would have facilitated the location and exchange of higher and lower units in bundling/debundling. Further, calculations supported state-level bureaucratic management, suggesting social pressure to achieve reasonable levels of efficiency and effectiveness. Tokens likely acquired linearity and stable order from knowledge and habits acquired with earlier technologies (possible tallies are attested archaeologically [Coinman, 1996; Copeland & Hours, 1977; Davis, 1974; Reese, 2002; Tixier, 1974], finger-counting by lexical numbers and numerical organization [Edzard, 1980; Englund, 2004; Huehnergard & Woods, 2008]). In addition, when tokens began to be impressed in clay in the mid-to-late fourth millennium BCE, impressions were organized with linearity and stable order; this organization was not inherent in the physical form of either tokens or impressions, suggesting it was already in place when these forms were incorporated. Organization by linearity and stable order would ultimately facilitate the development of place value by the late third millennium (Robson, 2007).

Whether or not Mesopotamian peoples can be credited with originating the abacus (Ifrah, 1981), that device spread throughout much of the ancient world; it was still used in Europe as late as the Middle Ages and remains in some use throughout the world (Donlan & Wu, 2017; Reynolds, 1993; Stone, 1972). Instead of representing inter-exponential relations with

conventions of shapes and sizes, as the Mesopotamian tokens had, abaci calculation elements generally have the same shape and size, with total (intra- and inter-exponential) value indicated by place within the full array. Despite this difference, abaci and tokens share the same limitations of physical form. First, their elements need containment. In Mesopotamia, this problem was addressed in the mid-to-late fourth millennium BCE by sealing tokens inside clay envelopes, an innovation that meant breaking the containers to regain access (this would be solved by impressing the outsides of envelopes with tokens before placing them inside, a development associated with the invention of writing). For abaci, elements in later designs were contained in some fashion (e.g., as the beads of the Roman abacus slid along grooves). Second, both were limited in the simultaneity and concision of the information they represented, making neither device suitable for storage. Represented information was relatively perishable and easily disarranged, and neither transportable nor easily recreated. These limitations would motivate the incorporation of a new material form: notations.

Before delving into notations, the impact of devices with grouping and manipulability on numbers and brains should be mentioned. As tokens were incorporated, numbers increased to an amazing extent: In Mesopotamia, tokens representing 600 have been dated to 8500–3500 BCE, and numerical signs for 3600 and 216,000 are attested in the third millennium BCE (Cuneiform Digital Library, 2015, Pennsylvania Sumerian Dictionary Online, 2015). The trend toward higher numbers is consistent with managing extensive agriculture, industrialized construction, and massive workforces. Grouping is productive, yielding higher numbers that increase potential intra- and inter-exponential relations and operations, concepts whose explication is facilitated by the use of manipulable forms. This meant that ancient number systems involved more facts than ever before: numbers, relations between numbers, operations for calculating, algorithms or sequences of calculations. Mesopotamian numeracy was transitioning to knowledge-based calculation, increasingly using mental knowledge rather than physical movements in calculation procedures. Arguably, this would involve brain regions like the angular gyrus implicated in recalling arithmetic facts (Grabner, Ansari, et al., 2009; Grabner, Ischebeck, et al., 2009). Simply, recalling arithmetic facts plausibly depends on such facts being available in the first place, and the new technologies were making them available to an unprecedented extent. This would be intensified even further by the notations that would follow.

3.5 Numbers as Entities: Handwritten Notations[xi]

Like their predecessors, written notations developed from capabilities and properties they shared with precursor technologies; responded to their limitations; and added new capabilities and limitations to the cognitive system for numbers. The properties shared by notations and material forms like tokens and tallies are underappreciated, perhaps because their differences seem more compelling. However, the importance of their similarities and differences is brought into focus when all these forms are compared for their material qualities, associated behaviors, and the influence of both on psychological processing.

Numerical notations and other material forms (abaci, tallies, fingers, etc.) all represent numerical information, and they do so in a manner distinct from that of written non-numerical language. That is, numerical notations are a non-glottographic (or semasiographic) notational system, like music (Gelb, 1980; Powell, 2009). The distinction between glottographic and non-glottographic writing (e.g., 7 vs. seven)[xii] is crucial: Both convey semantic information, but only glottographic writing identifies phonetic values (Hyman, 2006; Sampson, 1999). The lack of

phonetic specification means that non-glottographic writing "can be read with similar facility by speakers of different languages, or ... its reading has the character of paraphrase (i.e., two different 'readings' are likely to employ significant differences in word choice or syntactic construction)" (Hyman, 2006, p. 234). For example, 1 + 1 = 2 is *one plus one is two*, *one added to one make two*, *two is the sum of one plus one*, and other such variants in English, with analogous counterparts in other languages. This makes numerical notations translinguistic and decipherable in otherwise untranslatable or unknown languages and scripts (e.g., Linear A; proto-Elamite). The lack of phonetic specification implies that the phonetic value of number-words is not critical to understanding the material representation of numerical values, even if associated thoughts are experienced primarily in language.

Notations and other material forms are also similar in distributing numerical information over multiple elements. For example, the meaning of 745 is distributed over three numerals (7, 4, and 5), each of which assumes a place value understood mentally (as 7 × 100 plus 4 × 10 plus 5 × 1) rather than explicitly represented. There are numeral systems where the distribution is less compact: in Roman numerals, the number has twice as many elements: DCCXLV (one 500, two 100s, 10 before 50, and five). Ten Mesopotamian tokens would be required: one large cone marked with a small sphere (one 600), two large unmarked cones (two 60s), two small spheres (two 10s), and five small cones (five units). The fingers of 75 individuals would be needed. While intra- and inter-exponential representations differ among these forms, as does their physical substance, material qualities, and concision, they are similar in distributing numerical information over multiple semantically meaningful elements.

Numerical notations can also be distinguished from non-numerical writing by the fact that the former, like physical devices, are unambiguous in the information they represent (discussed below). In contrast, non-numerical writing requires phonetic specification to identify things like word choice, verbal tense, and noun case; without such specification, it is relatively ambiguous (Overmann, 2016a). Such qualities give numerical notations a contiguity with other physical forms that has no counterpart in non-numerical writing. For example, XXX (Roman numerals) and ●●● (Mesopotamian tokens) are semantically meaningful elements whose combinations represent the number 30; by comparison, *caput* (Latin) and a drawing of a head do not necessarily both specify the word *head*. In fact, in archaic Mesopotamian non-numerical writing, a head could mean *head*, *person*, or *capital* (a non-exhaustive list of potential meanings; it is precisely this ambiguity that motivates writing systems toward glottographic specificity).

These differences evoke the semiotic distinction between material and linguistic signs. Where material forms persist, spoken words are ephemeral (Malafouris, 2013). The most important difference is in *how* they mean. Material forms *instantiate* quantity: Three fingers, three cones, and III *are* three; they may be ambiguous regarding what they count, but they are unambiguous regarding their quantity. In contrast, language, spoken or written, *symbolizes* meaning: Associations between phonetic values and semantic meanings are conventional. Instantiation may explain why written numbers are, like other material representations of number, non-glottographic and unambiguous. It may explain why numerical notations are similar across significant linguistic and temporal differences: One, two, and three vertical or horizontal strokes are signs for subitizable numbers in many numerical notation systems (Chrisomalis, 2010). Instantiation may also inform conservation of form in numerical signs: Today's Hindu–Arabic 1, 2, and 3 are still essentially the same straight lines used since the beginning of writing (Branner, 2006; Chrisomalis, 2010; Ifrah, 2000; Martzloff, 1997; Nissen, 1986; Tompack, 1978),

despite their transmission through thousands of years and adoption by multiple cultures and languages. The same process yielded radical alteration in non-numerical writing, as scripts were adapted to different lexicons and phonemic inventories. This difference in semiotic function between material and linguistic signs means that numerals remain inherently material, despite being written, despite their written form becoming less depictive over time, and despite acquiring attributes that are mentally understood or behavioral rather than explicitly represented.

While these similarities are profound, if underappreciated, the differences are hardly trivial. First, whereas abaci and tokens are manipulable, notations are fixed. This means that notations are not a form that permits the explication of relations and operations. They are, however, a form that not only represents but whose concision facilitates the representation of large volumes of information (e.g., numerical relations). In Mesopotamia, numerical relations were explicated (most likely with tokens, whose parts could now also be compared to wholes represented notationally) and recorded in tables of relational data (e.g., tables of multiplication and reciprocals), which scribes reproduced and thus learned as part of their training (Proust, Donbaz, Dönmez, & Cavigneaux, 2007). This would have increased the proportion of knowledge in calculating (as compared to physical manipulations). Most importantly, notations are also handwritten, which entails a vastly different interaction between the psychological, physiological, behavioral, and material dimensions of numerical cognition than that involved in manipulating abaci beads or tokens. Simply, handwriting meant the fusiform gyrus becoming trained to recognize written objects and interact with Wernicke's, Broca's, and Exner's areas to associate number-signs with number-words and handwriting movements (Dehaene et al., 2010; Dehaene & Cohen, 2007, 2011; Nakamura et al., 2012; Overmann, 2016a).

While a sign like 7 in cuneiform (𒌋𒌋𒌋) would still consist of seven elements, it would no longer be conceptualized as a collection, as seven small cones were, but would instead be conceived as an entity in its own right. Moreover, its concision yielded a historically unprecedented ability to record, learn, and apply large volumes of relational data. As a result, number-entities would become defined by their relations to one another. And their semantic meanings would not change if the signs symbolized rather than instantiated (i.e., as 7 symbolizes, 𒌋𒌋𒌋 instantiates; the former simplifies producing and apprehending multi-element signs), bundled (as the Roman V is 5 and cuneiform 𒌋 is 10), or valued by place (as 7 remains itself when multiplied by 10 or 100). This development would ultimately yield a concept of number fairly similar to today's Western notion (which is reasonable, given that Mesopotamian numbers are one of its deepest roots). Mesopotamian numbers still differed from Western ones, as their grouping was based on the 1:10 and 1:6 pattern that characterized the token-based system for counting most discrete objects (Rudman, 2007). Zero was at most a blank space that would, much later, inspire a sign for a blank space before ultimately becoming a number itself (Rotman, 1987). One was probably not a number, since even for the later Greeks it was the *unity*, a metaphysical notion that would not be revisited in the Western tradition until the Renaissance (Klein, 1992). And the operations that could be used with Mesopotamian numbers differed too (discussed below).

Writing had another crucial effect: It provided the ability to write non-numerical language, which was used to describe arithmetical operations. This need not have happened, as writing for numbers and writing for non-numerical language are dissociable (i.e., while many systems develop both, some systems develop only one or the other) (Chrisomalis, 2010). Interestingly, in a system of writing that used signs for entire words and phrases, Mesopotamian

arithmetical operations were conveyed not by symbols comparable to those of modern arithmetic (e.g., +, –) but by non-numerical descriptions of actions. Further, these narratives did not describe the same unified concepts that would ultimately emerge in the Western mathematical tradition. Mesopotamians had two types of addition and two subtractions (differentiated by whether or not the inputs could still be distinguished once the calculation was performed), several types of multiplication (one of which, multiplication by reciprocal, was analogous to division), and two bisections (differentiated by whether the half produced was "natural" or not) (Høyrup, 2002). The involvement of writing for non-numerical language and the lack of unified arithmetic concepts suggest two things. First, it cannot be coincidental that the ancient societies that invented writing—Mesopotamia, Egypt, China, and Mesoamerica—also developed mathematics. Not only was there similar pressure to manage social complexity through bureaucratic means, the ability to describe in writing supported the explication of arithmetical operations and development of more complex calculations (i.e., ones carried out to further places) and algorithms (longer sequences of operations). Second, it suggests that explicit operations, like initial realizations of accumulation, removing, and grouping, depend on the material forms used. This makes operations culturally constructed through interactions with material forms, just as numbers and relations are.

3.6 Beyond Numbers: Semasiographic Notations for Operations

The Western mathematical tradition eventually realized not just unified concepts for addition and subtraction but non-glottographic notations for them (e.g., + instead of *plus*, *add*, *you put together*, or *combine*) (Neal, 2002; Schulte, 2015). Non-glottographic notations for numbers and operations give mathematical equations a conceptual and physical manipulability (Landy, 2010; Landy & Goldstone, 2007), concision, and ability to represent relations of spatiality and transformational invariancy (Larkin & Simon, 1987; Sfard & Linchevski, 1994) with little counterpart in written non-numerical language (where the closest parallels may be things like crossword puzzles and anagrams). In Mesopotamia, non-glottographic signs for numbers meant that the phonetic values of Sumerian number-words were unrecorded for more than a thousand years after the availability of writing (and only then to help Semitic-speaking scribes learn the Sumerian number-words; Edzard, 1980; Pettinato, 1981). In the history of mathematics, the lack of non-glottographic signs and the concomitant necessity to use glottographic language "slowed" conceptual progress in algebra "for centuries," a circumstance that directly links conceptual development with representation that is more material and less linguistic (Sfard, 1991, p. 29). In contemporary mathematics, the non-glottographic sufficiency of both numbers and symbols has led some to question the role of language in mathematics altogether, inspiring movement to incorporate diagrams and pictures as a logical continuation of the historic transition from (glottographic) words to (non-glottographic) signs (Silver, 2017). (Admittedly, mathematicians have also proposed excluding *numbers*, on the premise that ontological determinations of what numbers are need not detain investigations of their structural relations and properties; Hellman, 1989.) Given the contiguity between numerical signs and other material representational forms, this suggests that the interaction between perceptual modalities (e.g., quantity, spatiality) and material forms may comprise a distinct pathway to numerical intuitions and insights, one that interacts with and complements but is separate from the access provided by language.

4.0 Materiality and Language: Interacting and Complementary but Likely Distinct

Both materiality and language are critical to numeracy, the ability to reason with numbers. In original contexts, materiality is the first, if not primary, means of access, since numbers start as the perceptual experience of quantity instantiated by material forms; made tangible through the use of material forms; and expressed through iconic and indexical use of material forms and language. Numbers depend on materiality to an extent that language does not: A society can have an oral tradition for millennia, but no society known has numbers without one or more material forms to represent them (broadly construed to include everything from distributed objects, fingers, devices, and notations). If language is a secondary, "slow and hesitant" route for numerical elaboration (Sfard & Linchevski, 1994, p. 198), it undoubtedly enables numbers to be communicated, developmentally acquired, learned, and applied in social contexts in ways far beyond those possible with only non-linguistic means (Barner, 2012; Ferrari, 2003; Rittle-Johnson, Siegler, & Alibali, 2003). If it is "almost universally accepted ... that at best language has only a secondary function with regards to the development of mathematical concepts, [and] mathematical and arithmetical thinking in particular is based on the active interaction with concrete [i.e., physical] objects... [nonetheless] there can be no doubt that the verbal communication of mathematical information becomes more and more important as the learning process progresses" (Damerow, 2010, pp. 150–151).

As means of accessing numerical cognition, materiality and language can be distinguished in terms of their cross-cultural prevalence, ease of acquisition, severability, and universal characteristics. While all human societies have language, not all societies have numbers (Everett, 2005), something that can reasonably be attributed to differences in social needs for numbers (Epps, Bowerin, Hansen, Hill, & Zentz, 2012), as well as behaviors with material forms. Acquisitional difficulty is the difference between learning Roman numerals and Latin, severability the fact that the latter is not a factor in the former, and vice versa. The universals of numerical language differs from those of non-numerical language, and spoken numbers differ from written ones (Chrisomalis, 2010; Comrie, 1989; Greenberg, 1978). Numeracy also involves specific neural reorganizations gained through enculturation and practice, similar to learning to read and write (Carreiras et al., 2009; de Cruz, 2012; Tang et al., 2006; Zamarian, Ischebeck, & Delazer, 2009).

As cognitive processes, numeracy and language demonstrate double-dissociation (Amalric & Dehaene, 2016; Ardila & Rosselli, 2002; Brannon, 2005; Carreiras, Monahan, Lizarazu, Duñabeitia, & Molinaro, 2015; Hannagan, Amedi, Cohen, Dehaene-Lambertz, & Dehaene, 2015; Monti, Parsons, & Osherson, 2012; Park, Chiang, Brannon, & Woldorff, 2014; Varley, Klessinger, Romanowski, & Siegal, 2005), establishing independence of form and function. The neurological basis for double-dissociation has been well established. Language preferentially involves Broca's and Wernicke's areas in the frontal and temporal lobes, while numeracy, numbers, and numerosity are associated with parietal activity (Amalric & Dehaene, 2016; Fias, Lammertyn, Caessens, & Orban, 2007; Orban et al., 2006), finger gnosia (Penner-Wilger et al., 2007; Reeve & Humberstone, 2011), and motor-movement planning (Brooks, Barner, Frank, & Goldin-Meadow, 2014; Frank & Barner, 2012). Non-numerical language also involves, as numerical words do not, the storage of about 10,000 specifiable words in long-term memory (the mental lexicon), and language and numbers link to different higher-order cognitive domains (Carreiras et al., 2015). Brains enculturated in different cultural systems for numbers

appear to process identical material stimuli differently, something that cannot be explained fully by language (Tang et al., 2006).

In today's writing systems, there are significant overlaps between numerical and non-numerical signs: Both involve training effects in the brain (though subtle differences have also been documented; Grotheer, Ambrus, & Kovács, 2016), and written forms are to some extent substitutable (Carreiras, Duñabeitia, & Perea, 2007). The extent to which such effects are related to writing system qualities (e.g., significant elaboration, alphabetic sign–sound mapping, symbolic numerals) is unknown. If they have become more similar over time, in original writing systems, differences between the two are clear: Numerical representations instantiate quantity, are unambiguous regarding the quantity they mean, and remain semantically meaningful without phonetic specification, qualities that conserve written forms across time and transmission; non-numerical representations symbolize meaning, are relatively ambiguous in what word they intend, and reduce ambiguity by incorporating phonetic specification, qualities that radically change written forms and sign–sound mapping.

At higher-order levels of syntactic representation, numbers and language share fundamental similarities (e.g., both use fixed sets of rules that govern whether statements are well formed or transfer properties like truth between statements). These similarities have been used to argue that numbers represent a subset of language ("the human number faculty [is] essentially an 'abstraction' from human language, preserving the mechanism of discrete infinity and eliminating the other special features of language"; Chomsky, 1988, p. 169) or that both are informed by an underlying computational capacity responsible for properties like generativity (Chomsky, 2004). For language, it is claimed that discrete infinity, or at least its core computational capacity (merge or recursion), arose once, fully formed (Bolhuis, Tattersall, Chomsky, & Berwick, 2014; Hauser, Chomsky, & Fitch, 2002). This is difficult to reconcile with the observation that the particulates (i.e., finite, discrete units) involved in numerical generativity (e.g., numbers, relations, and operations) have emerged independently and become elaborated slowly through cultural acretion. Further, unlike those of language (Studdert-Kennedy, 2005), numerical particulates are meaningful in and of themselves. Cross-cultural tendencies for cognitive structure also manifest in domains other than numbers and language (e.g., reflexivity, symmetry, and transitivity characterize relations in kinship systems; Gilsdorf, 2012). This makes it unclear whether and to what extent cognitive structure is a single capability that fundamentally underlies and informs multiple domains (Chomsky, 2004), one or more domain-specific capabilities that influence other domains (Chomsky, 1988), properties that emerge through interaction between domains (Bybee, 2010), or some complex combination of all these possibilities.

For the purposes of the present discussion, it suffices to observe that language may be one of two highly interdependent means of accessing numerical cognition, and that it complements, explicates, and expresses insights gained through manuovisual interactions with materiality (Clark, 2006; Roepstorff, 2008). Interact with materiality enough, and it will occasion opportunities to name things and talk about them (i.e., lexical names for subitizable quantities); interact even more, and those names may become grammaticalized (i.e., grammatical number); interact further with higher (non-subitizable) quantities, and those names will become lexicalized (i.e., as rules for naming higher quantities). Simply, grammaticalization and lexicalization presuppose frequency of use (Bybee, 2010; Cacoullos & Walker, 2011; Heine, 2003) that in numbers reflects specific social conditions (e.g., requirements for numbers to manage affairs

within and between groups that increase with demographic factors like group size and contact). Grammaticalization and lexicalization not only imply something about the social need for numbers, they also show the influence of materiality on language (e.g., the association of lexical rules with the procedural system implies the involvement of motor movements [Ullman et al., 1997]; productive cycles of ten, five, and twenty suggest the use of the hands and feet as material forms for counting), making language a potential source of evidence for the material history of numbers.

Two cases will be used to illustrate how materiality and language interact in the cognitive system for numbers. The first involves Oksapmin numbers, where both material form (the body) and language are ordinal sequences (Saxe, 2012) that share qualities like sequential order and not being particularly manipulable (Overmann, 2016b). For the material form, body positions cannot be rearranged the way a group of pebbles can, while for language, words lack the cardinality and ordinality that materiality has. Words also lack the magnitude that yields numerical order (it is of course possible to count words or order them by increasing length, but this is not, in general, how they are most meaningful). Non-contrastive forms are reinforcing and predicted to yield a system characterized by stability over time, especially for small, relatively isolated groups (i.e., who are less likely to develop numbers in response to internal needs or encounter external number systems). The second case is that of Chinese and English numbers. In Chinese, both spoken and written numbers are perfectly regular and identical in their structure. English arguably uses the same numerals, which are also regular and decimally structured. However, English spoken numbers are atomic from one to ten, irregular from eleven to nineteen, and fairly regular above twenty. Chinese spoken numbers reinforce the linear structure of the written numerals, a quality thought to at least partly explain why Chinese speakers perform better at mathematical tasks than English speakers (Cantlon & Brannon, 2007). The latter example shows that contrasting material and linguistic forms is not invariably beneficial (Overmann, 2017b).

Language and materiality influence one another in other ways. Number-words may reinforce material linearity. Linguistic signs are temporally ordered as a function of producing the sounds of speech. Temporally sequential number-words might reinforce the linear ordering influenced by material forms like fingers and tallies. Numerical language, in turn, is influenced by the material forms used for counting. Lexical forms for small (subitizable) numbers emerge first across number systems, giving them the greatest potential for irregularity: When small numbers are named, the number system has little material or linguistic structuring; they are the most frequently used number-words (Davies & Gardner, 2013; Kilgarriff et al., 2014; Xiao & McEnery, 2004), subjecting them to memorization effects[xiii]; and they have the greatest longevity, increasing their exposure to processes of linguistic change. In comparison, higher numbers emerge through the use of material forms, increasing the likelihood their names will be influenced by material structure (e.g., as decimalization reflects finger-counting). The emergence of a numerical lexicon "merely reflects the development of more efficient, extralinguistic techniques" (Damerow, 2010, p. 212). Thus, as lexical rules for number-words incorporate material influence on production, their cross-linguistic variability reflects the use of different material forms and combinatorial choices across social groups.

5.0 Potential Implications for Numerical Research

In answering the question "What does it take to move from quantical cognition to numerical cognition, and how do these two forms relate to each other?" (Núñez, 2017, p. 421),

materiality should be considered. The material forms used to represent and manipulate numbers inform their content, organization, structure, and elaboration. This role may help explain, at least in part, the historical (and prehistorical) development of counting sequences and more elaborated forms such as arithmetic and mathematics from the perceptual experience of quantity. Material forms also have the potential to influence psychological, behavioral, and linguistic aspects of numerical cognition, a role that currently appears underappreciated and underexplored. Recognizing materiality as a part of the cognitive system for numbers has the potential to illuminate conceptual differences between cultural number constructs. In turn, cross-cultural numerical differences related to material form(s) may challenge aspects of current numerical research. This article concludes with the suggestion that materiality has a role in numerical cognition research, as illustrated by the questions that follow:

- When psychological reactions are compared in participants enculturated into differently elaborated number systems (e.g., systems with highly elaborated relations and operations distributed over multiple material forms compared to systems without relations, operations, or material distribution), does a monolithic notion of number suffice, or do elaborational differences have the potential to affect the constructs being measured? Are the effects of elaborational differences, if any, symmetrical? They might be predicted to affect participants enculturated into highly elaborated number systems differently than those enculturated into less elaborated number systems: The former may involve subsets of established concepts (i.e., more intuitive in having a conceptual basis), the latter concepts that are unknown or which significantly differ (i.e., having no or little conceptual basis).
- In numerical representation, some information is represented materially (explicit), while other knowledge is added mentally or behaviorally (implicit); the proportion of explicit representation and implicit knowledge varies between number systems (Zhang & Norman, 1995). How do proportionality differences affect matters like acquisitional ease and mathematical task performance? What changes when numerical representations are viewed, instead of being touched and physically manipulated?
- Beginning with the mathematicians of the ancient world, those who have taken mathematics to new heights have worked with number systems that have already become fairly elaborated. Today, even the simplest “mental” calculation draws upon embodied resources and the material prehistory of number: neural circuitry for motor-movement planning, concepts of numbers as entities with myriad relations, unified operational concepts, multiple representational forms. What happens to numerical prodigy and creative insight when a number system is relatively unelaborated? Are they underlying factors in number’s inception and early elaboration, new potentials that become actualized through exposure to more elaborated forms, or abilities that develop in conjunction with the co-evolution of cognition and culture?
- What are the implications of qualities shared between numerical notations (signs for numerals and operations) and other material forms used to represent quantity: non-glottographic, instantiation, unambiguous?
- The influence of (external) material forms on numerical language in particular and numerical cognition more generally may have implications for the structural relations between these domains. Simply, to what extent can numbers be viewed as concepts originating in language, (1) when materiality demonstrably influences their content, structure, and organization; whether and how they are realized, explicated, visualized,

manipulated, and elaborated; and aspects of number-words like productive grouping; and (2) when material representations of number can be understood without language (as documented in the ethnographic literature) or with a different language (as in Roman numerals with English number-words and concepts)? Is the relation between material forms and language consistent across the entire range of forms used to represent and manipulate numbers (i.e., distributed exemplars to notations)? How do materiality and language influence and change when they interact? What are the effects of structural similarities and dissimilarities between the material form(s) and numerical language across and within cultural numerical systems?

- What implications does non-glottographic representation have for the role of language in numerical cognition? Glottographic and non-glottographic forms may differentially involve phonetic recall. Is this related to the fact that most words for numbers are generated by means of lexical rules rather than stored in the mental lexicon (e.g., Ullman et al., 1997)? Does this differ for number-words stored in the mental lexicon? And to what degree is it feasible or desirable to remove language—in part or *in toto*—from numerical and mathematical representation and conceptualization?

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Notes

[i] If the need to add and subtract were indigenously developed (rather than posed through cultural contact), the cumbersomeness of strategies like double-enumeration would undoubtedly motivate change toward greater efficiency, informational accessibility, and reduced demand on working memory and attention. These qualities would likely be achieved by incorporating a new material form, one selected for what it shared with the previous system (e.g., the ability to increment and decrement ordinal sequences) and its ability to realize efficiency, etc.

[ii] The use of distributed material forms has been documented in other emerging number systems: The Abipones were described as using "the [four] fingers of an emu" and a "skin spotted with five different colours" to represent *four* and *five* (Dobrizhoffer, 1822, p. 168).

[iii] While many primates use tools, humans are unique in the degree to which they recruit, incorporate, and exploit material forms as a constitutive component of cognition. This ability is not identical to the term "symbolic reference," which can have the connotation of something the human brain does internally in response to external stimuli (as distinct from being an emergent property of their interaction).

[iv] The Mundurukú use words of one, two, three, and four syllables, respectively, for the corresponding numbers (Rooryck et al., 2017).

[v] Numbers appear to have a strong somatic basis, as attested by productive grouping (numerical bases) (Comrie, 2011; Hammarström, 2010; Lakoff & Núñez, 2000). Base 4 has been related to using the spaces between the fingers or omitting the thumb. Bases 8 and 32 might double base 4 (in the way base 10 uses both hands) or otherwise multiply it. Base 6 might involve the wrist in addition to the fingers, with base 12 formed by doubling. Alternatively, base 12 may be formed by using the segments of the fingers (Huylebrouck, 1997). Base 15 may use three hands instead of one (base 5) or two (base 10) hands, while base 20 may involve fingers and knuckles or both hands and feet (Comrie, 2011). Bases 24, 24, and 27, found in the body-counting systems of Papua New Guinea, use the hands and other parts of the body (Hammarström, 2010; Lean, 1992; Saxe, 2012). Base 60, used in Mesopotamia, appears to have been derived by combining a productive cycle of 10, which has an unambiguous anatomical basis, with a productive cycle of 6, which has a possible anatomical basis (Nissen et al., 1993; Rudman, 2007).

[vi] The somatic basis of numbers is also attested by embodied vocabulary: nouns like *digit* that mean both *finger* and *number* (found in unrelated languages like English and the languages of the Brazilian Aimoré and the Hudson Bay Inuit; Conant, 1896; Richardson, 1916), verbs like *count* that mean *finger* (found in the Siberian languages Chukchi and Koryak; Antropova & Kuznetsova, 1956), and words for *five*, *ten*, and *twenty* related to fingers, hands, fists, toes, and men (Swetz, 2009).

[vii] The angular gyrus has also been implicated in functions like higher-level cross-domain thinking (e.g., metaphorizing, often used to characterize numerical concepts), manipulation of numbers in verbal form, retrieval of arithmetic facts from memory, and mathematical competence (Dehaene, Piazza, Pinel, & Cohen, 2003; Grabner et al., 2007; Grabner, Ansari, et al., 2009; Grabner, Ischebeck, et al., 2009; Lakoff & Núñez, 2000; Ramachandran, 2004).

[viii] Despite their universality in counting, fingers are seemingly not a significant factor in developing arithmetical operations. Perhaps the embodied experience of quantity does not disappear when an extended finger is flexed in the same way it does when a pebble is removed from a pile, or the increased demand on working memory associated

with decrementing an ordinal sequence on a material form normally used to accumulate acts as disincentive (e.g., as in Oksapmin example).

[ix] Body-counting is an exception: Yupno body-counting is learned and used by adult men (Wassmann & Dasen, 1994). This restriction is plausibly related to the material form employed, as the body is cross-culturally associated with proscriptions that limit public touch and behavioral imitation.

[x] Tokens were like modern currency in consisting of a unit, multiples of the unit, and fractions of a unit. Assessing an amount of multiple tokens would have similarly involved separating them into like units arranged in denomination order. This would not require lines (like those on a counting board), though informal organization would limit operational complexity (Nagl, 1918).

[xi] While this discussion focuses on Mesopotamia (mainly as a matter of authorial familiarity), the mathematical traditions in Egypt, China, and Mesoamerica similarly involved fingers (e.g., decimal organization), one-dimensional devices (knotted cords and tallies), two-dimensional devices (abaci), notations, tables of relations, and writing for non-numerical language (Ascher & Ascher, 1981; Chrisomalis, 2010; Houston, 2008; Ifrah, 1981; Martzloff, 1997; Ritter, 2000). The resultant notational systems demonstrate similar characteristics of dimensionality and bases (Zhang & Norman, 1995), qualities developed from precursor technologies and retained in a mix of material and mental/behavioral forms.

[xii] A sign like 7 is semantically meaningful in a way that transcends language, at least in part because such signs inhabit organizational and structural patterns distinct from those of language. Though it designates the same quantity, *seven* adds phonetic information that ties it to English.

[xiii] Words used frequently tend to be memorized, giving them an increased potential for irregularity in comparison with infrequently used words, which tend to be rule-based and thus regular (Bybee, 2010). This principle applies to lexical numbers, where names for small numbers tend to be significantly irregular (e.g., the numbers *one* through *ten* in English); slightly higher numbers may be somewhat irregular (as *eleven* through *nineteen* in English have some regularity of form without conforming to the rules for naming higher quantities); and much higher numbers become regular (as *twenty* and higher are fairly regular in English).