SOME REMARKS ON THE SPECTRAL PROPERTIES OF TOEPLITZ OPERATORS

PIETRO AIENA AND SALVATORE TRIOLO

Abstract. In this paper we study some local spectral properties of Toeplitz operators T_ϕ defined on Hardy spaces, as the localized single valued extension property and the property of being hereditarily polaroid.

1. Introduction

This paper concerns some spectral properties of Toeplitz operators T_ϕ defined on the Hardy spaces $H^2(\mathbf{T})$, where $\mathbf{T}$ denotes the unit circle of $\mathbb{C}$, in particular, a local spectral property, the so-called single valued extension property (SVEP) and the property of being hereditarily polaroid. Example 4.8 provides un example of Toepliz operator with a continuous symbol ϕ on the unit circle $\mathbf{T}$ for which the single valued extension property fails for both T_ϕ and T'_ϕ, T'_ϕ the adjoint of T_ϕ. Consequently also the property of being hereditarily polaroid may fail, since this property entails the SVEP. The SVEP has some important consequences on the fine structure of the spectrum, so it has interest to determine conditions on ϕ which ensure SVEP for T_ϕ, or for T'_ϕ. We shall see that if $\phi \in H^\infty(\mathbf{T})$ then T_ϕ has SVEP and is hereditarily polaroid, while for continuous symbol the SVEP for T_ϕ (respectively, for T'_ϕ) holds if the orientation of the curve $\phi(\mathbf{T})$ traced out by ϕ is counterclockwise (respectively, the orientation of the curve $\phi(\mathbf{T})$ traced out by ϕ is clockwise). The SVEP is also ensured in some other special cases. Weyl's theorem for Toeplitz operators has been established first by Coburn [8]. In the last part we discuss Weyl type theorems for Toeplitz operators and the permanence of these theorems under functional calculus.

The results in this paper have not been published in journals, and part of these results have been included in the recent book by the first author [1].

2. Preliminaries

In what follows, by X we denote a complex infinite-dimensional Banach space, and by $L(X)$ the Banach algebra of all bounded linear operators defined on X. Let $T \in L(X)$, let $\alpha(T)$ and $\beta(T)$ denote the dimension of the kernel $\ker T$ and the codimension of the range $R(T) := T(X)$, respectively. Let

$$\Phi_+(X) := \{T \in L(X) : \alpha(T) < \infty \text{ and } T(X) \text{ is closed}\}$$

denote the class of all *upper semi-Fredholm* operators, and let

$$\Phi_-(X) := \{T \in L(X) : \beta(T) < \infty\}$$

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denote the class of all *lower semi-Fredholm* operators. If $T \in \Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$, the *index* of T is defined by ind $(T) := \alpha(T) - \beta(T)$. The semi-Fredholm spectrum (called also the *Wolf spectrum*) is the set

$$\sigma_{\rm sf}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_\pm(X)\},$$

while the *upper semi-Fredholm spectrum* is defined by

$$\sigma_{\rm usf}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_+(X)\}.$$

If $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$ denotes the set of all *Fredholm* operators, the set of *Weyl operators* is defined by

$$W(X) := \{T \in \Phi(X) : {\rm ind}\, T = 0\},$$

the class of *upper semi-Weyl operators* is defined by

$$W_+(X) := \{T \in \Phi_+(X) : {\rm ind}\, T \leq 0\}.$$

The classes of operators before defined give origin to the following spectra: the *Weyl spectrum*, defined by

$$\sigma_{\rm w}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W(X)\};$$

the *upper semi-Weyl spectrum*, defined by

$$\sigma_{\rm uw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_+(X)\}.$$

and the *lower semi-Weyl spectrum*, defined by

$$\sigma_{\rm lw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_-(X)\}.$$

A well known result of Fredholm theory establishes that for every $K \in K(X)$ we have

$$(1) \qquad \sigma_{\rm w}(T) = \sigma_{\rm w}(T+K), \quad \sigma_{\rm uw}(T) = \sigma_{\rm uw}(T+K), \quad \sigma_{\rm lw}(T) = \sigma_{\rm lw}(T+K)$$

(note that commutativity is not required). For an operator $T \in L(X)$ set

$$\rho_{\rm sf}^+(T) := \{\lambda \in \mathbb{C} : \lambda I - T \in \Phi_\pm(X),\ {\rm ind}\ (\lambda I - T) > 0\},$$

and

$$\rho_{\rm sf}^-(T) := \{\lambda \in \mathbb{C} : \lambda I - T \in \Phi_\pm(X),\ {\rm ind}\ (\lambda I - T) < 0\}.$$

Lemma 2.1. *If $T \in L(X)$ then*

(i) $\sigma_{\rm w}(T) = \sigma_{\rm sf}(T) \cup \rho_{\rm sf}^+(T) \cup \rho_{\rm sf}^-(T)$.

(ii) $\sigma_{\rm uw}(T) = \sigma_{\rm sf}(T) \cup \rho_{\rm sf}^+(T)$.

Proof. (i) The inclusion ($\subseteq$) is evident. Conversely, if $\lambda \notin \sigma_{\rm w}(T)$, then $\lambda I - T \in W(X)$, so $\lambda \notin \sigma_{\rm sf}(T) \cup \rho_{\rm sf}^+(T) \cup \rho_{\rm sf}^-(T)$.

(ii) The inclusion $\sigma_{\rm sf}(T) \cup \rho_{\rm sf}^+(T) \subseteq \sigma_{\rm uw}(T)$ is clear. Conversely, suppose that $\lambda \notin \sigma_{\rm sf}(T) \cup \rho_{\rm sf}^+(T)$. Then $\lambda I - T \in \Phi_\pm(X)$ and ind $(\lambda I - T) \leq 0$, so $\alpha(\lambda I - T) \leq \beta(\lambda I - T)$, which obviously implies that $\lambda I - T \in \Phi_+(X)$ and hence $\lambda \notin \sigma_{\rm uw}(T)$. Therefore, the equality (ii) holds. ∎

Lemma 2.2. *For every $T \in L(X)$ we have* ${\rm iso}\, \sigma_{\rm w}(T) \subseteq {\rm iso}\, \sigma_{\rm uw}(T) \subseteq {\rm iso}\, \sigma_{\rm sf}(T)$.

Proof. Let $\lambda_0 \in \text{iso}\,\sigma_{\text{w}}(T)$. Then there exists an $\varepsilon > 0$ such that $\lambda I - T \in W(X)$ for all $0 < |\lambda| < \varepsilon$. This easily implies that $\lambda_0 \in \sigma_{\text{sf}}(T)$. In fact, if not, then $\lambda_0 I - T \in \Phi_+(X)$. By the continuity of the index function we then obtain $\text{ind}\,(\lambda_0 I - T) = 0$, i.e., $\lambda_0 \notin \sigma_{\text{w}}(T)$, a contradiction. Since, by Lemma 2.1, we have $\sigma_{\text{uw}}(T) = \sigma_{\text{sf}}(T) \cup \rho_{\text{sf}}^+(T)$, then $\mathbb{D}(\lambda_0, \varepsilon) \cap \sigma_{\text{uw}}(T) = \{\lambda_0\}$, so $\lambda_0 \in \text{iso}\,\sigma_{\text{uw}}(T)$.

Now, choose an arbitrary $\mu_0 \in \text{iso}\,\sigma_{\text{uw}}(T)$. To show the second inclusion it suffices to prove that $\mu_0 \in \sigma_{\text{sf}}(T)$. Assume that $\mu_0 \notin \sigma_{\text{sf}}(T)$. Then $\mu_0 I - T \in \Phi_\pm(X)$, and, since $\Phi_\pm(X)$ is an open subset of $L(X)$, there exists a $\delta > 0$ such that $\mu I - T \in \Phi_\pm(X)$ for all $\mu \in \mathbb{D}(\mu_0, \delta)$. Again by the continuity of the index function, there exists an $n \in \mathbb{Z} \cup \{-\infty, +\infty\}$ such that $\text{ind}\,(\mu I - T) = n$ for all $\mu \in \mathbb{D}(\mu_0, \delta)$. Note that $\sigma_{\text{uw}}(T) = \sigma_{\text{sf}}(T) \cup \rho_{\text{sf}}^+(T)$, by Lemma 2.1. If $n \leq 0$ then $\mu_o \notin \sigma_{\text{uw}}(T)$, a contradiction. If $n > 0$ then $\mathbb{D}(\mu_0, \delta) \subseteq \rho_{\text{uw}}(T) = \mathbb{C} \setminus \sigma_{\text{uw}}(T)$, and hence μ_0 is an interior point of $\sigma_{\text{uw}}(T)$, again a contradiction. Therefore, $\mu_0 \in \sigma_{\text{sf}}(T)$. ∎

Let $p(T) := p$ be the *ascent* of an operator T; i.e. the smallest non-negative integer p such that $\ker T^p = \ker T^{p+1}$. If such integer does not exist we put $p(T) = \infty$. Analogously, let $q(T) := q$ be the *descent* of T; i.e the smallest non-negative integer q such that $T^q(X) = T^{q+1}(X)$, and if such integer does not exist we put $q(T) = \infty$. It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$, see [**?**, Theorem 3.3]. Moreover, if $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$, then λ is a pole of the resolvent, see [14, Proposition 50.2], and in particular an isolated point of $\sigma(T)$.

The class of all *Browder operators* is defined

$$B(X) := \{T \in \Phi(X) : p(T), q(T) < \infty\};$$

while the class of all *upper semi-Browder operators* is defined

$$B_+(X) := \{T \in \Phi_+(X) : p(T) < \infty\}.$$

The *Browder spectrum* is defined by

$$\sigma_{\text{b}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \in B(X)\},$$

while the *upper semi- Browder spectrum* is defined by

$$\sigma_{\text{ub}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \in B_+(X)\}.$$

Obviously, $B(X) \subseteq W(X)$ and $B_+(X) \subseteq W_+(X)$, see [**?**, Theorem 3.4], so $\sigma_{\text{w}}(T) \subseteq \sigma_{\text{b}}(T)$ and $\sigma_{\text{uw}}(T) \subseteq \sigma_{\text{ub}}(T)$.
Recall that $R \in L(X)$ is said to be a Riesz operator if $\lambda I - T \in \Phi(X)$ for all $\lambda \neq 0$. The Browder spectra are invariant under Riesz commuting perturbations R, i.e.,

$$\sigma_{\text{b}}(T) = \sigma_{\text{b}}(T + R) \quad \text{and} \quad \sigma_{\text{b}}(T) = \sigma_{\text{b}}(T + R. \tag{2}$$

In the sequel we denote by $\sigma_{\text{ap}}(T)$ the *approximate point spectrum*, defined by

$$\sigma_{\text{ap}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\},$$

where an operator is said to be *bounded below* if it is injective and has closed range. All the spectra above defined are nonempty compact subsets of $\mathbb{C}$.

We now introduce a basic property of local spectral theory: an operator $T \in L(X)$ is said to have *the single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc U of λ_0, the only analytic function $f : U \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$ is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$. Evidently, an operator $T \in L(X)$ has SVEP at every point of the resolvent $\rho(T) := \mathbf{C} \setminus \sigma(T)$, and both T and T^* have SVEP at the points λ which belong to the boundary $\partial\sigma(T)$ of the spectrum, and in particular at all the isolated points of the spectrum. It is known that an operator T has SVEP at the points $\lambda \notin \sigma_{\rm ap}(T)$, and by duality T^* has SVEP at the points $\lambda \notin \sigma_{\rm s}(T)$. Hence if $\sigma_{\rm ap}(T)$ is contained in the boundary $\partial\sigma(T)$ then T has SVEP, and analogously, if $\sigma_{\rm s}(T)$ is contained in $\partial\sigma(T)$ then T^* has SVEP.

A bounded operator $T \in L(X)$ is said to be *polaroid* (respectively, a-polaroid) if every $\lambda \in \operatorname{iso}\sigma(T)$ (respectively, $\lambda \in \operatorname{iso}\sigma_{\rm ap}(T)$ is a pole of the resolvent. An operator $T \in L(X)$ is said to be *isoloid* (respectively, *a-isoloid*) if every $\lambda \in \operatorname{iso}\sigma(T)$ (respectively, $\lambda \in \operatorname{iso}\sigma_{\rm ap}(T)$ is an eigenvalue. Note that if $T \in L(X)$ is a-polaroid then T is polaroid and hence isoloid, and every a-polaroid is a-isoloid. Furthermore, every a-isoloid operator is isoloid.

Lemma 2.3. *If iso $\sigma_{\rm w}(T) = \emptyset$ then T is polaroid. If iso $\sigma_{\rm uw}(T) = \emptyset$ then T is a-isoloid.*

Proof. Let $\lambda \in \operatorname{iso}\sigma(T)$. Then either $\lambda \in \sigma_{\rm w}(T)$ or $\lambda \notin \sigma_{\rm w}(T)$. If $\lambda \in \sigma_{\rm w}(T)$ then $\lambda \in \operatorname{iso}\sigma_{\rm w}(T)$ and this is impossible. Therefore, $\lambda \notin \sigma_{\rm w}(T)$, so $\lambda I - T$ is Weyl and, since both T and T^* have SVEP at 0, see [**?**, Theorem 3.16 and Theorem 3.17], it then follows that $p(\lambda I - T) = q(\lambda I - T) < \infty$, i.e. λ is a pole of the resolvent.

Assume that $\operatorname{iso}\sigma_{\rm uw}(T) = \emptyset$ and $\lambda_0 \in \operatorname{iso}\sigma_{\rm ap}(T)$. Then there exists $\varepsilon > 0$ for which $\lambda I - T$ is bounded below for all $0 < |\lambda - \lambda_0| < \varepsilon$. We have either $\lambda_0 \in \sigma_{\rm uw}(T)$ or $\lambda_0 \notin \sigma_{\rm uw}(T)$. If were $\lambda_0 \in \sigma_{\rm uw}(T)$ then we would have $\lambda_0 \in \operatorname{iso}\sigma_{\rm uw}(T)$ and this is impossible. Hence, $\lambda_0 \notin \sigma_{\rm uw}(T)$, i.e, $\lambda_0 I - T \in W_+(X)$. Hence $\lambda_0 I - T$ has closed range, and since $\lambda_0 \in \sigma_{\rm ap}(T)$, it then follows that $\lambda_0 I - T$ is not injective, so T is a-isoloid. ∎

The set of all analytic functions defined on an open disc containing the spectrum $\sigma(T)$) will be denoted by $\mathcal{H}(\sigma(T))$. Then $f(T_\phi)$ is defined by the classical functional calculus. In the sequel we shall need the following simple result.

Lemma 2.4. *Let $T \in L(X)$ and $f \in \mathcal{H}(\sigma(T))$. Then iso $\sigma(f(T)) \subseteq f($iso $\sigma(T))$.*

Proof. Let $\lambda_0 \in \operatorname{iso}\sigma(f(T)) = \operatorname{iso} f(\sigma((T))$ and $\mu_0 \in \sigma(T)$ for which $\lambda_0 = f(\mu_0)$. Suppose that $\mu_0 \notin \operatorname{iso}\sigma(T)$. Then there exists a sequence μ_n which converges to μ_0. Evidently, fixed $c \in \mathbb{C}$, the set $\{\mu_j : f(\mu_j) = c\}$ is a finite set, since the function $g(\lambda) := c - f(\lambda)$ may have only a finite number of zeros in $\sigma(T)$). Hence $\{f(\mu_n) : n \in \mathbb{N}\}$ is an infinite set and $f(\mu_0) = \lambda_0 = \lim_{n\to\infty} f(\mu_n)$, so λ_0 is not an isolated point of $f(\sigma(T)$, a contradiction. Therefore, $\mu_0 \in \operatorname{iso}\sigma(T)$ and consequently $\lambda_0 \in f(\operatorname{iso}\sigma(T))$. ∎

Recall that $T \in L(X)$ is said to be *hereditarily polaroid* if any restriction $T|M$ of T on a invariant closed subspace M is polaroid. A proof of the following result may be found in [11].

Theorem 2.5. *Every hereditarily operator $T \in L(X)$ has SVEP.*

If $T \in L(X)$, the *quasi-nilpotent part* of T is defined by $H_0(T) := \{x \in X : ||Tx^n||^{1/n} \to 0\}$. Note that $\ker T^n \subseteq H_0(T)$ for all $n \in \mathbb{N}$, and that

$$H_0(\lambda I - T) \text{ closed } \Rightarrow T \text{ has SVEP } \lambda,$$

see [**?**].

A bounded operator $T \in L(X)$ is said to *belong to the class $H(p)$* if there exists a natural $p := p(\lambda)$ such that:

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \text{for all } \lambda \in \mathbb{C}. \tag{3}$$

Evidently, every $H(p)$ operator has SVEP. Since the restriction of a $H(p)$-operator to closed invariant subspace is still $H(p)$, see [17], it then follows that every $H(p)$-operator is hereditarily polaroid. It has been observed that every subscalar operator is $H(p)$ and in particular every subnormal operator is $H(p)$, see [1, Chapter 4]. Recall that a bounded operator $T \in L(H)$ on a Hilbert space H is said to be *hyponormal* if $T'T \geq TT'$. By an important result due to Putinar [18], every hyponormal operator is similar to a subscalar operator, see also [15, section 2.4]. Since the property of being $H(p)$ is preserved by quasi-affine transforms [17] then all hyponormal operators are $H(p)$.

3. Toeplitz operators

An important class of polaroid operators is provided by the Toeplitz operators on the classical Hardy spaces $H^2(\mathbf{T})$, where $\mathbf{T}$ denotes the unit circle of $\mathbb{C}$. To define the Hardy space $H^2(\mathbf{T})$, for $n \in \mathbb{Z}$, let χ_n be the function on $\mathbf{T}$ defined by

$$\chi_n(e^{it}) := e^{int} \quad \text{for all } n \in \mathbb{N}.$$

Let μ be the normalized Lebesque measure on $\mathbf{T}$, and $L^2(\mathbf{T})$ the classical Hilbert space defined with respect to μ. The set $\{\chi_n\}_{n\in\mathbb{Z}}$ is a orthogonal basis of $L^2(\mathbf{T})$. The *Hardy space* $H^2(\mathbf{T})$ is defined as the closed subspace of all $f \in L^2(\mathbf{T})$ for which

$$\frac{1}{2\pi}\int_0^{2\pi} f\chi_n \mathrm{dt} = 0 \quad \text{for } n = 1, 2, \dots.$$

The Hilbert space $H^2(\mathbf{T})$ is the closed linear span of the set $\{\chi_n\}_{n=0.1,\dots}$. Moreover, $H^2(\mathbf{T})$ is a closed subspace of $L^\infty(\mathbf{T})$. Let $H^\infty(\mathbf{T})$ denote the Banach space of all $\phi \in L^\infty(\mathbf{T})$ such that

$$\frac{1}{2\pi}\int_0^{2\pi} \phi\chi_n \mathrm{d}t = 0 \quad \text{for all } n = 1, 2, \dots.$$

$H^\infty(\mathbf{T})$ is a closed subalgebra of $L^\infty(\mathbf{T})$ and $H^\infty(\mathbf{T}) = \mathbf{L}^\infty(\mathbf{T}) \cap \mathbf{H^2}(\mathbf{T})$. If $\phi \in L^\infty(\mathbf{T})$ and $f \in L^2(\mathbf{T})$ then $\phi f \in L^2(\mathbf{T})$, so we may define an operator $M_\phi : L^2(\mathbf{T}) \to L^2(\mathbf{T})$ by

$$M_\phi f = \phi f \quad \text{for all } f \in L^2(\mathbf{T}),$$

where ϕf is the point-wise product.

Let P denote the projection of $L^2(\mathbf{T})$ onto $H^2(\mathbf{T})$.

Definition 3.1. *If $\phi \in L^\infty(\mathbf{T})$, the* Toeplitz operator with symbol ϕ T_ϕ *on* $H^2(\mathbf{T})$ *is defined by*

$$T_\phi f := P(\phi f) \quad \text{for } f \in H^2(\mathbf{T}).$$

The adjoint of the Hilbert space operator M_ϕ on $L^2(\mathbf{T})$ is $M_\phi' = M_{\overline{\phi}}$ and obviously, $M_\phi M_\phi' = M_\phi' M_\phi$, so M_ϕ is a normal operator. If $\phi \in H^\infty(\mathbf{T})$, the operator T_ϕ is the restriction of M_ϕ to the closed invariant subspace $H^2(\mathbf{T})$, so T_ϕ is subnormal.

Theorem 3.2. *If $\phi \in H^\infty(\mathbf{T})$, the Toeplitz operator T_ϕ is hyponormal. In particular, T_ϕ is hereditarily polaroid and hence has SVEP.*

Proof. T_ϕ, $\phi \in H^\infty(\mathbf{T})$, is subnormal and hence hyponormal, see Conway [9, Proposition 2.4.2]. The last statement is clear. Every hyponormal operator is hereditarily polaroid, and hence has SVEP, by Theorem 2.5. ∎

Denote by $\mathcal{H}_{nc}(\sigma(T))$ the set of all analytic functions defined in a neighborhood of $\sigma(T)$ such that f is nonconstant on each of the components of its domain. Recall that an operator K is said *algebraic* if there exists a polynomial h such that $h(T) = 0$. Example of algebraic operator are nilpotent operators and operators K for which a power K^n is finite-dimensional. The result of Theorem 3.2 may be improved as follows:

Theorem 3.3. *If $\phi \in H^\infty(\mathbf{T})$, then we have*

(i) *$f(T_\phi)$ is polaroid and has SVEP for every $f \in \mathcal{H}_{nc}(\sigma(T_\phi))$.*

(ii) *$f(T_\phi + K)$ is polaroid, and has SVEP for all algebraic operators K which commutes with T_ϕ and $f \in \mathcal{H}_{nc}(\sigma(T_\phi + K))$, while $f(T_\phi' + K')$ is a-polaroid:*

Proof. (i) T_ϕ is hereditarily polaroid, so, by [2, Theorem 2.4], $f(T_\phi)$ is polaroid for every $f \in \mathcal{H}_{nc}(\sigma(T_\phi))$. Furthermore, since T_ϕ has SVEP then $f(T_\phi)$ has SVEP for every $f \in \mathcal{H}_{nc}(\sigma(T_\phi))$, see [1, Theorem 2.86].

(ii) By [3, Theorem 2.15], $T_\phi + K$ is polaroid for every algebraic operator which commutes with T_ϕ, and hence $f(T_\phi + K)$ is polaroid for every $f \in \mathcal{H}_{nc}(\sigma(T_\phi + K))$, always by [2, Theorem 2.4]. By duality, then $f(T_\phi' + K')$ is polaroid. Since $T_\phi + K$ has SVEP, by [5, Theorem 2.3], then $f(T_\phi + K)$ has SVEP, again by [1, Theorem 2.86]. The SVEP for $f(T_\phi + K)$ entails that the approximate point spectrum $\sigma_{\mathrm{ap}}(f(T_\phi + K))$ coincides with $\sigma(f(T_\phi + K))$, hence $f(T_\phi' + K')$ is a-polaroid. ∎

The Toeplitz operators with continuous symbols are particulary amenable to study. Indeed, consider the case that the symbol ϕ is a continuous function on $\mathbf{T}$ and let $\mathbf{\Gamma} := \phi(\mathbf{T})$. Given $\lambda \notin \mathbf{\Gamma}$ denote by $wn(\phi, \lambda)$ the *winding number* of $\mathbf{\Gamma}$ with respect to λ (recall that the winding number $wn(\phi, \lambda)$ of a closed curve $\mathbf{\Gamma}$ in the plane around a given point λ is an integer representing the total number of times that curve travels counterclockwise around the point).

The following result, due to Coburn ([8]) plays a crucial role in characterizing the Toeplitz operators which are Fredholm:

Theorem 3.4. *Suppose that $\phi \in L^\infty(\mathbf{T})$ is not almost everywhere 0. Then either $\alpha(T_\phi) = 0$ or $\beta(T_\phi) = \alpha(T_\phi') = 0$.*

If the symbol $\phi \in L^\infty(\mathbf{T})$, Weyl operators T_ϕ may be characterized in the following way:

Corollary 3.5. *Suppose that $\phi \in L^\infty(\mathbf{T})$ is not almost everywhere 0. Then T_ϕ is Weyl if and only if T_ϕ is invertible. Consequently, $\sigma(T_\phi) = \sigma_{\rm w}(T_\phi)$.*

The following nice result is due to a number of authors, see for instance, Widom [19].

Theorem 3.6. *If $\phi \in C(\mathbf{T})$ then T_ϕ is a Fredholm operator if and only if ϕ does not vanish. In this case*

$$ind\; T_\phi = -wn(\phi, 0),$$

where $wn(\phi, 0)$ is the winding number of the curve traced by ϕ with respect to the origin. In particular, T_ϕ is Weyl, or equivalently, invertible, if and only if $wn(\phi, 0) = 0$.

Theorem 3.7. *If $\phi \in C(\mathbf{T})$ then*

$$\sigma(T_\phi) = \sigma_{\rm w}(T_\phi) = \sigma_{\rm b}(T_\phi) = \phi(\mathbf{T}) \cup \{\lambda \in \mathbb{C} : wn(\phi, \lambda) \neq 0\}.$$

and

$$\sigma_{\rm sf}(T) = \sigma_{\rm e}(T_\phi) = \phi(\mathbf{T}).$$

In particular, $\sigma(T_\phi) = \sigma_{\rm w}(T)$ is connected.

Proof. Observe first that $\sigma_{\rm w}(T_\phi) \subseteq \sigma_{\rm b}(T_\phi) \subseteq \sigma(T_\phi)$, and hence, by Corollary 3.7, $\sigma_{\rm b}(T_\phi) = \sigma(T_\phi)$. Furthermore, $\sigma(T_\phi)$ is connected since it is formed from the union of Γ and certain components of the resolvent of T_ϕ. For the equality $\sigma_{\rm e}(T_\phi) = \phi(\mathbf{T})$, see Douglas [10, Chapter 7]. It remains only to show the equality $\sigma_{\rm sf}(T) = \phi(\mathbf{T}_\phi)$. Evidently, $\sigma_{\rm sf}(T_\phi) \subseteq \sigma_{\rm e}(T_\phi) = \phi(\mathbf{T}_\phi)$. To show the reverse inclusion assume that $\lambda \in \phi(\mathbf{T})$ and $\lambda \notin \sigma_{\rm sf}(T_\phi)$. Then $\lambda I - T_\phi$ is semi-Fredholm while $\lambda I - T_\phi$ is not Fredholm, so either $\alpha(\lambda I - T_\phi) = \infty$ or$\beta(\lambda I - T_\phi) = \infty$. By Corollary 3.4, if $\alpha(\lambda I - T_\phi) = \infty$ then $\beta(\lambda I - T_\phi) = 0$, so $\lambda I - T_\phi$ is surjective. But λ belongs to the boundary of the spectrum, so T_ϕ has SVEP at λ. This implies that $\lambda I - T_\phi$ is injective, see [1, Corollary 2.61], so we get a contradiction, since λ is a spectral point. Suppose the other case $\beta(\lambda I - T_\phi) = \infty$. Thus, again by Corollary 3.4, $\alpha(\lambda I - T_\phi) = 0$, so $\lambda I - T_\phi$ is injective. On the other hand, since λ belongs to the boundary of the spectrum, T'_ϕ has SVEP at λ and hence $q(\lambda I - T_\phi) < \infty$, see [1, Theorem 2.98]. This implies, by [1, Corollary 3.4], that $\beta(\lambda I - T_\phi) \leq \alpha(\lambda I - T_\phi) = 0$, again a contradiction. Therefore, $\lambda \in \sigma_{\rm sf}(T_\phi)$ and the proof is complete. ∎

The result of Corollary 3.7 may be improved. The spectra $\sigma(T_\phi)$ and the essential spectrum $\sigma_{\rm e}(T_\phi)$ are connected also if $\phi \in L^\infty(\mathbf{T})$, see [10, Corollary 7.47 and Theorem 7.45].

Theorem 3.8. *If $\phi \in C(\mathbf{T})$ then the following statements are equivalent:*

(i) *ϕ is nonconstant.*

(ii) *$iso\, \sigma_{\rm w}(T_\phi) = iso\, \sigma(T_\phi) = \emptyset$.*

Consequently, T_ϕ is polaroid.

Proof. If $\phi \in C(\mathbf{T})$ we have

$$\rho_{\rm w}(T_\phi) = \sigma(T_\phi) = \{\lambda \in \mathbb{C} : wn(\phi, \lambda) = 0\},$$

and

$$\rho_{\rm sf}^{+}(T_\phi) = \{\lambda \in \mathbb{C} : wn(\phi, \lambda) < 0\},$$

while

$$\rho_{\rm sf}^{-}(T_\phi) = \mathbb{C} \setminus \sigma_{\rm lw}(T_\phi) = \{\lambda \in \mathbb{C} : wn(\phi, \lambda) > 0\}.$$

From Lemma 2.1, we know that

$$\sigma_{\rm w}(T_\phi) = \sigma_{\rm sf}(T_\phi) \cup \rho_{\rm sf}^{+}(T_\phi) \cup \rho_{\rm sf}^{-}(T_\phi)$$

so $\sigma_{\rm w}(T_\phi)$ consists of $\mathbf{\Gamma} = \phi(\mathbf{T})$ and those holes with respect to which the winding number of ϕ is nonzero.

We see now that

$$\operatorname{iso} \sigma_{\rm w}(T_\phi) = \emptyset \Leftrightarrow \phi \text{ is non-constant.}$$

Indeed, if $\operatorname{iso} \sigma_{\rm w}(T_\phi) \neq \emptyset$, then, by Lemma 2.2, we have $\operatorname{iso} \sigma_{\rm sf}(T_\phi) \neq \emptyset$. Because $\mathbf{\Gamma} = \sigma_{\rm e}(\mathbf{T}_\phi)$ is connected, it then follows that $\mathbf{\Gamma}$ is a singleton and ϕ is constant. On the other hand, if ϕ is constant, for instance $\phi \equiv \lambda$, then it is obvious that $\sigma(T_\phi) = \sigma_{\rm w}(T_\phi) = \{\lambda\}$. Thus, $\operatorname{iso} \sigma_{\rm w}(T_\phi) = \{\lambda\}$. From Lemma 2.3 we conclude that every Toeplitz operator with continuous symbol is polaroid. ∎

Lemma 3.9. *If $\phi \in C(\mathbf{T})$ and $f \in \mathcal{H}(\sigma(T_\phi)$, then $f \circ \phi \in C(\mathbf{T})$ and there exists a compact operator K on $H^2(\mathbf{T})$ such that $T_{f\circ\phi} = f(T_\phi) + K$. Moreover, $\sigma(T_{f\circ\phi}) \subseteq f(\sigma(T_\phi))$.*

Proof. The first assertion is evident. The proof of the second assertion may be found in [13, Lemma 2.1]. The inclusion $\sigma(T_{f\circ\phi}) \subseteq f(\sigma(T_\phi))$ has been proved in [13, Lemma 3.1]. ∎

4. Weyl-type theorems for Toeplitz operators

The results of the previous section allow to establish Weyl type theorems for perturbations of Toepltz operators. Set

$$\pi_{00}(T) := \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(\lambda I - T) < \infty\},$$

and, analogously,

$$\pi_{00}^{a}(T) := \{\lambda \in \operatorname{iso} \sigma_{\rm ap}(T) : 0 < \alpha(\lambda I - T) < \infty\}.$$

Define

$$\Delta(T) := \sigma(T) \setminus \sigma_{\rm w}(T) \quad \text{and} \quad \Delta_a(T) := \sigma_{\rm ap}(T) \setminus \sigma_{\rm uw}(T).$$

Definition 4.1. *Let $T \in L(X)$.*

- *T is said to satisfy* Weyl's theorem, *((W)), if $\Delta(T) = \pi_{00}(T)$.*
- *T is said to satisfy* a- Weyl's theorem, *((aW)), if $\Delta_a(T) = \pi_{00}^{a}(T)$.*
- *T is said to have* property (w), *((w)), if $\Delta_a(T) = \pi_{00}(T)$.*

Remark 4.2. a-Weyl's theorem or property (w) for T entails Weyl's theorem for T, a-Weyl's theorem and property (w) are independent, see [6].

Theorem 4.3. [4] *Let $T \in L(X)$ be polaroid. Then we have*

(i) *If T' has SVEP then (W), (aW), and (w) hold T. Moreover, T' satisfies (W).*

(ii) *If T has SVEP then (W), (aW), (w), hold for T'. Moreover, T satisfies (W).*

For the proof of the following theorem see [4, Theorem 3.3].

Theorem 4.4. *Suppose that $T \in L(X)$ is polaroid and either T or T^* has SVEP. Then Weyl's theorem holds for both T and T^*.*

Evidently, in the case of Hilbert space operators in the statement of Theorem 4.4, the dual T^* may be replaced by the Hilbert adjoint T'.

In general, Theorem 4.4 cannot be applied to Toeplitz operators with continuous symbol, because, as we show in the next Example 4.8, the SVEP for both T_ϕ or T'_ϕ may fail. However, all Toeplitz operators with continuous symbol satisfy Weyl's theorem, since $\sigma(T_\phi) = \sigma_{\rm w}(T_\phi)$, and hence $\Delta(T_\phi) = \sigma(T_\phi) \setminus \sigma_{\rm w}(T_\phi) = \emptyset$. Moreover, if ϕ is non-constant, by Theorem 3.8 we have $\operatorname{iso}\sigma(T_\phi) = \operatorname{iso}\sigma_{\rm w}(T_\phi)$ and hence $\pi_{00}(T) = \emptyset$, so Weyl's theorem holds for T_ϕ.

The equivalence (i) $\Leftrightarrow$ (iii) of next result is due to Farenick and W. Y. Lee [13].

Theorem 4.5. *Let $\phi \in C(\mathbf{T})$ and $f \in \mathcal{H}(\sigma(T_\phi)$. Then the following are equivalent:*

(i) *$f(T_\phi)$ satisfies Weyl's theorem;*

(ii) *The spectral theorem holds for $\sigma_{\rm w}(T_\phi)$, i.e., $f(\sigma_{\rm w}(T_\phi)) = \sigma_{\rm w}(f(T_\phi))$;*

(iii) *$\sigma(T_{f\circ\phi}) = f(\sigma(T_\phi))$.*

Proof. (i) $\Leftrightarrow$ (ii) Every Toeplitz is operator is isoloid, since $\operatorname{iso}\sigma(T_\phi) = \emptyset$. Furthermore, by [1, Theorem 3.119], T_ϕ has stable sign index on $\rho_{\rm sf}(T_\phi)$ (i.e. $\operatorname{ind}(\lambda I - T_\phi)$ and $\operatorname{ind}(\mu I - T_\phi)$ are the same for all $\lambda, \mu \in \rho_{\rm sf}(T_\phi)$). Since T_ϕ satisfies Weyl's theorem, by Theorem [1, Theorem 6.52] then Weyl's theorem holds for $f(T_\phi)$ if and only if the spectral theorem holds for $\sigma_{\rm w}(T_\phi)$.
As observed before, the equivalence (i) $\Leftrightarrow$ (iii) has been proved in [13]. ∎

Weyl's theorem holds for $f(T_\phi)$ if we assume that f is injective.

Corollary 4.6. *If $\phi \in C(\mathbf{T})$ and $f \in \mathcal{H}(\sigma(T_\phi)$ is injective, then Weyl's theorem holds for $f(T_\phi)$ and $\sigma(T_{f\circ\phi}) = f(\sigma(T_\phi))$.*

Proof. The spectral mapping theorem holds for $\sigma_{\rm w}(T_\phi) = \sigma(T_\phi)$, see [1, Theorem 3.21] ∎

The spectral mapping theorem holds for $\sigma_{\rm w}(T_\phi) = \sigma(T_\phi)$ also if either T_ϕ or T'_ϕ have SVEP, see [1, Chaper 3]. In this case $f(T_\phi)$ satisfies Weyl's theorem for *all $f \in \mathcal{H}_{nc}(\sigma(T_\phi))$*:

Theorem 4.7. *Let $\phi \in C(\mathbf{T})$ and suppose that T_ϕ, or T'_ϕ, has SVEP. Then both $f(T_\phi)$ and $f(T'_\phi)$ satisfy Weyl's theorem for all $f \in \mathcal{H}_{nc}(\sigma(T_\phi))$. Moreover,*

$$\sigma(T_{f\circ\phi}) = f(\sigma(T_\phi)). \tag{4}$$

In particular, if T_ϕ has SVEP then both a-Weyl's theorem and property (w) hold for $f(T'_\phi)$, while if T'_ϕ has SVEP then both a-Weyl's theorem and property (w) hold for $f(T_\phi)$.

Proof. (i) The first assertion is a consequence of Theorem 4.5, since the spectral mapping theorem holds for $\sigma_{\rm w}(T_\phi) = \sigma(T_\phi)$ if T_ϕ, or T'_ϕ, has SVEP, see [1, Chaper 3]. Note that the first assertion is also a consequence of Theorem 4.9, since the SVEP for T_ϕ, or T'_ϕ entails the SVEP for $f(T_\phi)$, or $f(T'_\phi)$ for every $f \in \mathcal{H}_{nc}(\sigma(T_\phi))$, see [2, Chapter 2]. Since T_ϕ is polaroid then $f(T_\phi)$ is polaroid, by [2, Theorem 2.4], so both $f(T_\phi)$ and $f(T'_\phi)$ satisfy Weyl's theorem, by Theorem 4.4. The equality (4) has been observed before. If an operator $T \in L(X)$ has SVEP then Weyl's theorem, a-Weyl's theorem and property (w) are equivalent, see [6], so the latter statements are clear. ∎

The condition that f is injective in Corollary 4.6 or the condition that either T_ϕ, or T'_ϕ, has SVEP in Theorem 4.7, plays a crucial role. The next example shows that if $f \in \mathcal{H}(\sigma(T_\phi)$ is not injective, or if neither T_ϕ or T'_ϕ have SVEP, then Weyl's theorem holds for $f(T_\phi)$ may fail.

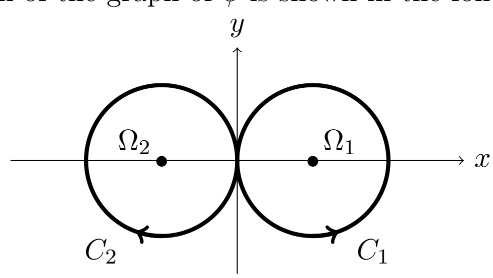
Example 4.8. Let ϕ be defined by

$$\phi(e^{i\theta}) := \begin{cases} -e^{2i\theta}+1 & \text{if } 0 \le \theta \le \pi, \\ e^{-2i\theta}-1 & \text{if } \pi \le \theta \le 2\pi. \end{cases}$$

In [13,] it has been observed that

$$\sigma(T_{\phi^2}) \neq [(\sigma(T_\phi))]^2,$$

so, by Theorem 4.5, Weyl's theorem fails for $f(T^2_\phi)$ in the case that $f(\lambda) = \lambda^2$. Now, the orientation of the graph of ϕ is shown in the following figure.



Let Ω_1 and Ω_2 be the interior of the circle C_1 and C_2, respectively. Since $wn(\phi,\lambda) = 1$ in Ω_1 and $wn(\phi,\lambda) = -1$ in Ω_2, we have

$$\text{ind}\,(\lambda I - T_\phi) < 0 \text{ for all } \lambda \in \Omega_1,$$

while

$$\text{ind}\,(\lambda I - T_\phi) > 0 \text{ for all } \lambda \in \Omega_2.$$

Observe that $\lambda I - T_\phi$ is Fredholm for every $\lambda \in \Omega_1 \cup \Omega_2$, since $\lambda \notin \phi(\mathbf{T}) = \sigma_e(\mathbf{T}_\phi)$, so, by [1, Corollary 2.106], the operator T_ϕ cannot have the SVEP, otherwise we would have $\text{ind}\,(\lambda I - T_\phi) \le 0$ for all $\lambda \in \Omega_2$, and, analogously, if T'_ϕ has the SVEP we would have $\text{ind}\,(\lambda I - T_\phi) \ge 0$ for all $\lambda \in \Omega_1$. A contradiction.

Note that Example 4.8 also provides an example of Toeplitz operator T_ϕ which is polaroid but not hereditarily polaroid, because T_ϕ does not have SVEP.

In general, for symbols $\phi \in L^\infty(\mathbf{T})$, the operators T_ϕ are not hyponormal, also if the symbol is continuous. For instance, the operator T_ϕ in the Example 4.8, cannot be hyponormal, since hyponormality entails SVEP. Toeplitz operators with continuous symbol which are hyponormal have been also studied by Farenick and W. Y. Lee [13].

Recall that given a compact set $\Omega \subseteq \mathbb{C}$, a *hole* of Ω is a bounded component of the complement $\mathbb{C} \setminus \Omega$. Since $\mathbb{C} \setminus \Omega$ always has an unbounded component, $\mathbb{C} \setminus \Omega$

is connected precisely when Ω has no holes.

In the next results we show that if orientation of the curve $\phi(\mathbf{T})$ does not change then either T_ϕ, or T'_ϕ, has SVEP.

Theorem 4.9. *Let $\phi \in C(T)$ and suppose that the orientation of the curve $\phi(\mathbf{T})$ with respect to each hole does not change. Then we have:*

(i) *If the orientation of the curve $\phi(\mathbf{T})$ traced out by ϕ with respect to each hole is counterclockwise then T_ϕ has SVEP.*

(ii) *If the orientation of the curve $\phi(\mathbf{T})$ traced out by ϕ with respect to each hole is clockwise then T'_ϕ has SVEP.*

Proof. (i) For simplicity we can suppose that $\sigma(T_\phi)$ has only a hole Ω_1 and that the orientation of $\phi(\mathbf{T})$ with respect to Ω_1 is counterclockwise. Then Ω_1 is a bounded component of $\mathbb{C} \setminus \phi(\mathbf{T})$. Denote by Ω_2 the unbounded component of $\mathbb{C} \setminus \phi(\mathbf{T})$. Then $wn(\phi, \lambda) > 0$ for every $\lambda \in \Omega_1$, while $wn(\phi, \lambda) = 0$ for every $\lambda \in \Omega_2$. Therefore, for every $\lambda \in \Omega_1$ we have

$$\text{ind}\,(\lambda I - T_\phi) = -wn(\phi, \lambda) < 0,$$

and consequently

$$\sigma(T_\phi) = \sigma_{\rm w}(T_\phi) = \Omega_1 \cup \phi(\mathbf{T}).$$

Now, if $\lambda \in \Omega_1$ the condition $\text{ind}\,(\lambda I - T_\phi) < 0$ entails that $\alpha(\lambda I - T_\phi) < \beta(\lambda I - T_\phi)$ and hence $\beta(\lambda I - T_\phi) > 0$. From Theorem 3.4 we have that $\alpha(\lambda I - T_\phi) = 0$, and $\lambda I - T_\phi$ has a closed range, since $\lambda \notin \sigma_{\rm e}(T_\phi) = \phi(\mathbf{T})$, and hence $\lambda I - T_\phi$ is Fredholm. Consequently, $\lambda \notin \sigma_{\rm ap}(T_\phi)$. Therefore, $\sigma_{\rm ap}(T_\phi) \subseteq \phi(\mathbf{T})$. Since $\phi(\mathbf{T})$ is the boundary of the spectrum $\sigma(T_\phi)$, then T_ϕ has the SVEP at every $\lambda \in \sigma_{\rm ap}(T_\phi)$. But every operator has SVEP at the points outside the approximate point spectrum, so T_ϕ has SVEP.

(ii) Analogously, suppose that $\sigma(T_\phi)$ has only a hole Ω_1, and that the orientation of $\phi(\mathbf{T})$ is clockwise. Then $wn(\phi, \lambda) < 0$ for every $\lambda \in \Omega_1$, so, if $\lambda \in \Omega_1$ then $\text{ind}\,(\lambda I - T_\phi) > 0$. Consequently,

$$\sigma(T_\phi) = \Omega_1 \cup \phi(\mathbf{T}),$$

If $\lambda \in \Omega_1$ the condition $\text{ind}\,(\lambda I - T_\phi) > 0$ entails that and $\alpha(\lambda I - T_\phi) > \beta(\lambda I - T_\phi)$ for all $\lambda \in \Omega_1$, so $\alpha(\lambda I - T_\phi) > 0$. From Theorem 3.4 we have that $\beta(\lambda I - T_\phi) = 0$, so $\lambda \notin \sigma_{\rm s}(T_\phi)$ and hence $\sigma_{\rm s}(T_\phi) \subseteq \phi(\mathbf{T})$. Hence T'_ϕ has SVEP at every $\lambda \in \sigma_{\rm s}(T_\phi)$. But for every operator, the adjoint has the SVEP outside the surjective spectrum, so T'_ϕ has the SVEP. ∎

The case (i) of Theorem 4.9 applies in particular to the case where ϕ is a trigonometric polynomial $\phi(e^{i\theta}) =: \Sigma_{k=n}^{-n} a_k e^{ik\theta}$, or also in the case that T_ϕ is hyponormal, since these operators have SVEP, and hence the index $\text{ind}\,(\lambda I - T_\phi)$ on a hole is less or equal to 0. Note that if ϕ is a trigonometric polynomial then T_ϕ may be not hyponormal, see [13].

Corollary 4.10. *If ϕ is a trigonometric polynomial on $\mathbf{T}$ then T_ϕ has SVEP. Consequently, $f(T_\phi)$ satisfies Weyl'theorem for every $f \in \mathcal{H}(\sigma_{T_\phi}$ and the spectral theorem holds for $\sigma_{\rm w}(T_\phi)$. The same holds if T_ϕ is hyponormal.*

Theorem 4.11. *If $\phi \in C(\mathbf{T})$ and $\sigma(T_\phi)$ has planar Lebesgue measure zero then both T_ϕ and T'_ϕ have the SVEP.*

Proof. The planar measure of $\sigma(T_\phi)$ is zero, because $\sigma(T_\phi) = \sigma_{\rm e}(T_\phi) = \phi(\mathbf{T})$ is a compact set consisting of $\phi(\mathbf{T})$ and some of its holes, so $\partial\sigma(T_\phi) = \phi(\mathbf{T}) = \sigma(\mathbf{T}_\phi)$, which is just a continuous curve. Therefore, both T_ϕ and T'_ϕ have the SVEP. ∎

If we assume that $\phi \in H^\infty(\mathbf{T})$, then, from Theorem 4.4 and Theorem 4.3, we obtain:

Corollary 4.12. *If $\phi \in H^\infty(\mathbf{T})$ then $f(T_\phi)$ satisfies Weyl's theorem for all $f \in \mathcal{H}_{nc}(\sigma(T_\phi))$. Moreover, for all algebraic operators K which commute with T_ϕ $f(T_\phi + K)$ satisfies Weyl'theorem, while $f(T'_\phi + K')$ satisfies a-Weyl and property (w).*

Proof. If $\phi \in H^\infty(\mathbf{T})$ then T_ϕ is hereditarily polaroid, by Theorem 3.2, so $T + K$ is polaroid, see [3, Theorem 2.15], and this implies that also $f(T_\phi + K)$ is polaroid for every $f \in \mathcal{H}_{nc}(\sigma(T_\phi + K))$, see [2, Theorem 2.4]. Since T_ϕ has SVEP, by Theorem 2.5, then $T_\phi + K$ has SVEP, see [5, Theorem 2.3] and hence $f(T_\phi + K)$ has SVEP, by [1, Theorem 2.86]. By Theorem 4.4 we then conclude that both $f(T_\phi + K)$ and $f(T'_\phi + K')$ satisfy Weyl's theorem. Since $f(T_\phi + K)$ has SVEP property (w) and a-Weyl's theorem for $f(T'_\phi + K')$ are equivalent, see [1, Theorem 6.96]. ∎

Corollary 4.13. *Let $\phi \in C(\mathbf{T})$. Then we have:*

(i) *If the orientation of the curve $\phi(\mathbf{T})$ traced out by ϕ is counterclockwise with respect to each hole, then $f(T_\phi)$ satisfies Weyl's theorem for every $f \in \mathcal{H}_{nc}(\sigma(T_\phi))$, while $f(T'_\phi)$ satisfies both a-Weyl's theorem and property (w).*

(ii) *If the orientation of the curve $\phi(\mathbf{T})$ traced out by ϕ is clockwise with respect to each hole, then $f(T'_\phi)$ satisfies Weyl's theorem for every $f \in \mathcal{H}_{nc}(\sigma(T_\phi))$, while $f(T_\phi)$ satisfies both a-Weyl's theorem and property (w).*

Proof. T_ϕ has SVEP, so, as observed above,$f(T_\phi)$ has SVEP, and $f(T_\phi)$ is polaroid. The SVEP for $f(T_\phi)$ entails, as observed before that property (w) and a-Weyl's theorem for $f(T_\phi)$ are equivalent. ∎

Corollary 4.14. *Let $\phi \in C(\mathbf{T})$ such that $\sigma(T_\phi)$ has planar Lebesgue measure zero. Then $f(T_\phi)$ satisfies a-Weyl's theorem and property (w) for all $f \in \mathcal{H}_{nc}(\sigma(T_\phi))$.*

Proof. T'_ϕ, and hence $f(T'_\phi)$, has SVEP, so property (w) and a-Weyl's theorem for $f(T_\phi)$ are equivalent. ∎

It should be noted that Toeplitz operator T_ϕ may satisfy Weyl's theorem, also if the symbol ϕ is not continuous, for an example see [13]. Theorem 4.4 does not apply, in general, to non commuting compact perturbations $T + K$ of a polaroid operator. The bilateral shift S on $\ell^2(\mathbb{Z})$ has Weyl spectrum $\sigma_{\rm w}(S) = \mathbf{T}$, so $\rho_{\rm w}(S) = \mathbb{C} \setminus \mathbf{T}$ is not connected, and hence, see [12], there exists a compact operator K for which the SVEP for $T + K$ fails. On the other hand also the property of being polaroid may be not preserved under compact perturbations, indeed there exists a compact perturbation of a polaroid operator T for which $T + K$ is not polaroid, see [16, Theorem 1.5].

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DEIM, Università di Palermo (Italia) Viale delle Scienze, I-90128 Palermo (Italy)
E-mail address: `pietro.aiena@unipa.it`

DEIM, Università di Palermo Viale delle Scienze, I-90128 Palermo (Italy)
E-mail address: `salvatore.triolo@unipa.it`