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A Hurwitz theory avatar of open–closed strings

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Abstract We review and explain an infinite-dimensional counterpart of the Hurwitz theory realization (Alexeevski and Natanzon, Math. Russ. Izv. 72:3–24, 2008) of algebraic open–closed-string model à la Moore and Lazaroiu, where the closed and open sectors are represented by conjugation classes of permutations and the pairs of permutations, i.e. by the algebra of Young diagrams and bipartite graphs, respectively. An intriguing feature of this Hurwitz string model is the coexistence of two different multiplications, reflecting the deep interrelation between the theory of symmetric and linear groups, S_∞ and $GL(\infty)$.

1 Introduction

It is an old idea to formulate the open–closed-string theory in purely algebraic terms (see Sect. 2 for details). This allows one to consider much simpler examples of string phenomena and involve basic mathematical constructions into the string theory framework. The idea is to formulate a set of axioms which has to be satisfied by the algebra of operators creating states, both on the boundary and in the bulk (world-sheet), in any two-dimensional open–closed topological field theory. This set of axioms was formulated and described in [2–5].

Having this axiomatic description in hands, one can look for simple examples that satisfy these axioms and, hence, could serve as simpler models for string theory, where some of its phenomena can be studied in detail.

One of the simplest examples of an appropriate theory was proposed in [1, 6] where the authors considered the theory of closed and open Hurwitz numbers which is actually the representation theory of symmetric (permutation) groups S_n. In this theory the boundary states (open strings) are enumerated by the bipartite graphs, while the bulk states (closed strings) are enumerated by the Young diagrams. However, the theory turns out to be oversimplified, since it is considered at fixed order of the symmetric group so that the number of states is finite.

On the other hand, if one considers the case of only closed (classical) Hurwitz numbers, i.e. closed strings, it turns out to be possible to glue together the theories at different orders (we denote the corresponding algebras A_n) of the symmetric groups and to embed them into the theory A corresponding to the infinite symmetric group, S_∞ so that the number of states becomes infinite and, moreover, the operators creating the states can be realized as differential operators from $GL(\infty)$ [7, 8]. This makes the theory much richer.

In this paper we repeat the same procedure for the open–closed case. As a first step, we reformulate the results of [7, 8] in terms of a map from $A=\oplus A_n$ (as a vector space) into the semi-infinite sequences of elements from A_n's, which induces a new product in the algebra A. Its non-trivial property is that the product of finite sums of elements from A remains a finite sum, and this product can be realized by differential operators [7, 8]. In the case of open–closed theory we are able to repeat this procedure and to construct a product with finiteness property, the map to semi-infinite sequences being borrowed from the Hurwitz theory. However, we were yet unable to realize this product by differential operators.

The paper is organized in the following way. In Sect. 2, we formulate the necessary set of axioms following [2–5, 9]. In Sect. 3, we briefly review the results of [1, 6] for constructing the open–closed theory corresponding to the symmetric group of fixed order. Finally, in Sect. 4 we construct the infinite-dimensional case and propose a map leading to the product with the finiteness property. This is our main result in this paper.

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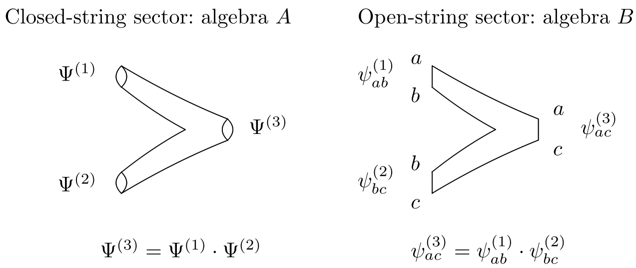
2 Open–closed duality in terms of Cardy–Frobenius algebras

In this section we follow [2–5, 9].

In string theory, the multiplication in the algebra of fields is associated with the sewing operation and with pant diagrams



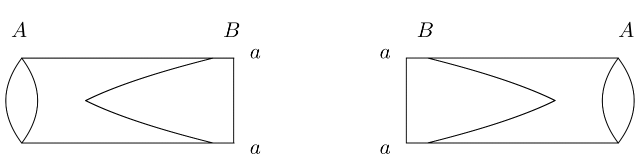
Here Ψ's are the fields in the closed sector and ψ_{ab} are those in the open one, we denote their algebras A and B correspondingly. The principal difference between the open and closed sectors is that in the former case the fields carry a pair of additional indices from the set of "boundary conditions" (or "D-branes"). In result $B = \oplus\mathcal{O}_{ab}$ splits into a combination of spaces corresponding to different boundary conditions. The sewing in the picture determines the algebra multiplication $\mathcal{O}_{ab} \otimes \mathcal{O}_{bc}$ which belongs to $\mathcal{O}_{ac}$ (no sum over b). Multiplications of all other elements are zero (e.g. $\mathcal{O}_{ab} \otimes \mathcal{O}_{cc} \to 0$). Diagonal subspaces $\mathcal{O}_{aa}$ are subalgebras of B, naturally associated with particular D-branes. They can be labeled both by a pair of indices aa or by single index a (very much like Cartan elements of the Lie algebras SL).

Multiplication operations satisfy a number of obvious relations [5]:

- *Closed-string sector (algebra A)*: associativity, commutativity
- *Open-string sector (algebra B)*: associativity

In the closed-string sector there are also an identity element $\mathbf{1}_A$ and a non-degenerate linear form $\langle \cdots \rangle_A$. Similarly, in the open sector in each space $\mathcal{O}_{aa}$ there are an identity element $\mathbf{1}_a$ and a non-degenerate linear form $\langle \cdots \rangle_a$, the latter providing at the same time the pairings of two elements $\psi_{ab} \in \mathcal{O}_{ab}$ and $\psi'_{ba} \in \mathcal{O}_{ba}$: $\langle \psi_{ab} \cdot \psi'_{ba} \rangle_a = \langle \psi'_{ba} \cdot \psi_{ab} \rangle_b$. Note that the identity element of the whole algebra B is given by the sum $\mathbf{1}_B = \sum_a \mathbf{1}_a$.

There is also the third crucial ingredient in the construction: *the open–closed duality* which comes from the possibility to interpret the annulus diagram in two dual ways. To this end, one needs to somehow relate the closed and open sectors. This is achieved by treating D-branes as states in the closed sector A via the diagram:



Algebraically, the requirement is that there are the homomorphisms

$$\phi_a : A \longrightarrow \mathcal{O}_{aa}, \tag{1}$$

one per each D-brane, and the dual maps

$$\phi^a : \mathcal{O}_{aa} \longrightarrow A \tag{2}$$

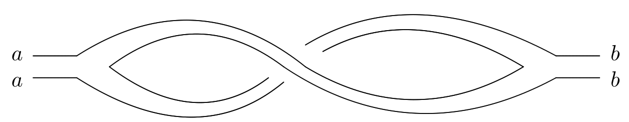
such that $\langle \phi^a(\psi_{aa})\Psi \rangle_A = \langle \psi_{aa}\phi_a(\Psi) \rangle_a$. The homomorphism ϕ_a preserves the identity: $\phi_a(\mathbf{1}_A) = \mathbf{1}_a$ and is central: $\phi_a(\Psi)\psi_{ab} = \psi_{ab}\phi_b(\Psi)$.

In terms of this homomorphisms one can write the open–closed duality in the form of the Cardy condition:

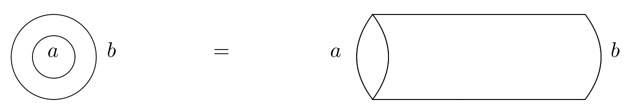
$$\sum_i \psi^i_{ba}\psi_{aa}\bar{\psi}^i_{ab} = \phi_b\big(\phi^a(\psi_{aa})\big) \tag{3}$$

where ψ^i_{ba} is a basis in $\mathcal{O}_{ba}$ and $\bar{\psi}^i_{ab}$ is its conjugated under the pairing.

The l.h.s. of this equation produces from the element ψ_{aa} an element of $\mathcal{O}_{bb}$ via the double twist diagram



which can be obtained in the closed-string channel (the r.h.s. of (3)) as



The pair of just described algebras A and B with a given homomorphism satisfying the Cardy condition is called Cardy–Frobenius (CF) algebra.

The Cardy condition can be also rewritten in the "converted form" (as an identity between combinations of correlation functions). To do this, first of all, we adjust our notation for the needs of Hurwitz theory and denote the elements of A and B through Δ and Γ. We also extend in the evident way the action of homomorphism to the whole diagonal part $B_d = \sum_a \mathcal{O}_{aa}$ of B: $\phi \equiv \sum_a \phi_a$ and similarly extend the linear form $\langle \psi_{ab} \rangle_B = \delta_{ab}\langle \psi_{ab} \rangle_a$ which immediately allows one to define the pairing for any two elements of B.

Then the Cardy relation can be rewritten as follows:

$$\sum_{\Gamma \in B} \langle \Gamma_{aa} \cdot \Gamma \cdot \Gamma_{bb} \cdot \bar{\Gamma} \rangle_B = \sum_{\Delta \in A} \big\langle \Gamma_{aa} \cdot \phi(\Delta)\big\rangle_B \big\langle \phi(\bar{\Delta}) \cdot \Gamma_{bb}\big\rangle_B. \tag{4}$$

The bars denote the duals: $\langle \Gamma \cdot \bar{\Gamma} \rangle_B = 1$ and $\langle \Delta \cdot \bar{\Delta} \rangle_A = 1$. Below we use the Cardy relation exactly in this form, only we omit the indices A and B in the linear forms.

3 Hurwitz theory

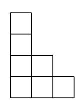
In Hurwitz theory the closed-string algebra is that of the Young diagrams (conjugation classes of permutations). This implies that the open-string fields will be labeled by pairs of Young diagrams with some additional data. Following [4] we identify them with bipartite graphs, conjugation classes of pairs of permutations.

A special feature of Hurwitz theory is additional decompositions of algebras $A = \oplus_n A_n$ and $B = \oplus_n B_n$. Homomorphisms $A_n \longrightarrow B_n$ and Cardy relations are straightforward only for particular values of n, while the entire algebra has a more sophisticated structure, which is only partly exposed in the present paper and deserves further investigation. In this section we describe the theory at fixed n following [1, 6].

3.1 Closed sector (algebra A_n)

Each permutation from the symmetric group S_n is a composition of cycles: for example, $6(34)(1527) \in S_7$ is the permutation

$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \downarrow & i & j & k & l & m & n & p \\ & m & p & l & k & j & n & i \end{array} \in [421] =$$



The lengths of cycles form an integer partition of n, and the ordered set of lengths is the Young diagram $\Delta = \{\delta_1 \geq \delta_2 \geq \cdots \geq \delta_{l(\Delta)} > 0\}$ of the size (number of boxes) $|\Delta| = \delta_1 + \delta_2 + \cdots + \delta_{l(\Delta)} = n$. The above-mentioned permutation is associated in this way with the Young diagram [421].

Conversely, given a Young diagram Δ, one can associate with it a direct sum of all permutations of the type Δ from the symmetric group $S_{|\Delta|}$, e.g.

$$[421] = \oplus i(jk)(lmnp)$$

where the sum goes over all $i, \dots, p = 1, \dots, 7$, which are all different, $i \neq \cdots \neq p$. In other words, the Young diagrams label the elements of *the center of the group algebra* of the symmetric group S_n. The multiplication (composition) of permutations induce a multiplication of Young diagrams of the same size, which we denote through $*$. For example,

$$A_1^* : [1] * [1] = [1],$$

A_2^*	[11]	[2]
[11]	[11]	[2]
[2]	[2]	[11]

(5)

A_3^*	[111]	[21]	[3]
[111]	[111]	[21]	[3]
[21]	[21]	$3 \cdot [111] + 3 \cdot [3]$	$2 \cdot [21]$
[3]	[3]	$2 \cdot [21]$	$2 \cdot [111] + [3]$

...

This multiplication is associative and commutative, and all the structure constants are positive integers, reflecting the combinatorial nature of this algebra A_n^*. It describes the closed sector of the Hurwitz model of string theory. Actually, at the next stage Δ plays the role of index a in the open sector.

One can also say that the Young diagrams label the conjugation classes of permutations: $\mu \sim g\mu g^{-1}$.

We also define the linear form on the algebra A_n:

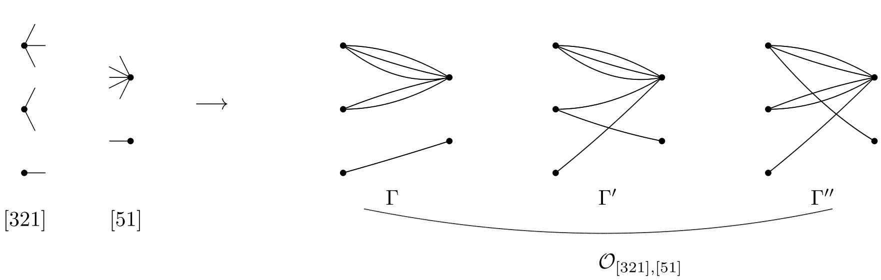
$$\langle \Delta \rangle_n = \begin{cases} 0 & \text{at } \Delta \neq [1^n], \\ \frac{1}{n!} & \text{at } \Delta = [1^n]. \end{cases} \tag{6}$$

3.2 Open sector (algebra B)

One can similarly consider the common conjugation classes of *pairs* of permutations of the same size:

$$[\mu, \nu] \sim \left[g\mu g^{-1}, g\nu g^{-1}\right], \quad \mu, \nu, g \in S_n.$$

Note that conjugation g is the same for μ and ν. Such classes are labeled by *the bipartite graphs*. For example, take two permutations from S_6, say, $i(jk)(lmn) \in [321]$ and $i(jklmn) \in [51]$. Represent the two Young diagrams by two columns of vertices, each vertex corresponds to a cycle and has a valence, equal to the length of the cycle:



After that a conjugation class gets associated with a graph obtained by connecting the vertices. Clearly, in our example there are three different bipartite graphs, i.e. three different conjugation classes: $\Gamma, \Gamma', \Gamma'' \in \mathcal{O}_{[321],[51]}$.

Note that the sizes of Young diagrams are equal to the numbers of edges in the graph: $|\Gamma| = \#(\text{edges in } \Gamma)$.

Bipartite graphs of the same size can be multiplied: the product $\Gamma_1 * \Gamma_2$ is non-vanishing, when the right Young diagram of Γ_1 coincides with the left Young diagram of Γ_2:

$$\Delta^r(\Gamma_1) = \Delta^l(\Gamma_2).$$

The product is then a sum of graphs with

$$\Delta^l(\Gamma_1 * \Gamma_2) = \Delta^l(\Gamma_1), \qquad \Delta^r(\Gamma_1 * \Gamma_2) = \Delta^r(\Gamma_2),$$

obtained by connecting the edges entering the same vertex in all possible ways. Formally,

$$[\mu, \nu] * [\mu'\nu'] = \sum_g [\mu, g\nu'g^{-1}] \cdot \delta(\nu, g\mu'g^{-1}). \tag{7}$$

This multiplication is still associative, but no longer commutative.

Technically one can label a bipartite graph by two cyclic representations with appropriately identified indices. For example, the three graphs from $\mathcal{O}_{[321],[51]}$ in the above example are

$$\Gamma = [i(jk)(lmn),\ i(jklmn)],$$
$$\Gamma' = [i(jk)(lmn),\ j(iklmn)],$$
$$\Gamma'' = [i(jk)(lmn),\ l(ijkmn)].$$

To multiply the so represented graphs one simply needs to appropriately rename the indices. For example, multiplying $\Gamma'' \in \mathcal{O}_{[321],[51]}$ with a graph from $\mathcal{O}_{[51],[2211]}$, one does the following:

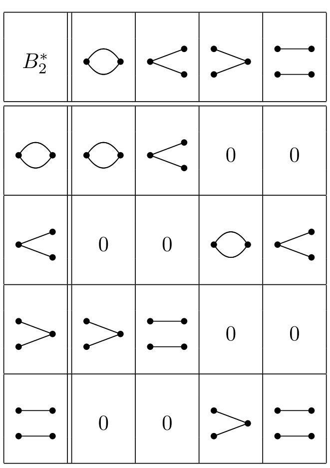
$$\begin{aligned}
&[i(jk)(lmn),\ l(ijkmn)] * [i(jklmn),\ ij(kl)(mn)] \\
&\quad = [i(jk)(lmn),\ l(ijkmn)] * [l(ijkmn),\ lj(ik)(mn)] \\
&\quad = [i(jk)(lmn),\ lj(ik)(mn)]
\end{aligned}$$

This algebra of bipartite graphs is the open-sector algebra B_n^* of the Hurwitz theory. The linear form on it is non-zero only on the *simple* graph, i.e. the graph with all connected components having two vertices:

$$\langle \Gamma \rangle_n = \begin{cases} 0 & \Gamma \text{ is not simple} \\ \frac{1}{|\mathbf{Aut}(\Gamma)|} & \Gamma \text{ is simple} \end{cases} \tag{8}$$

The simplest pieces of multiplication table are:

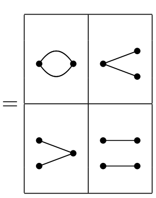
B_1^*: (9)



or

$$e_{ij} * e_{kl} = \delta_{jk} e_{il}$$

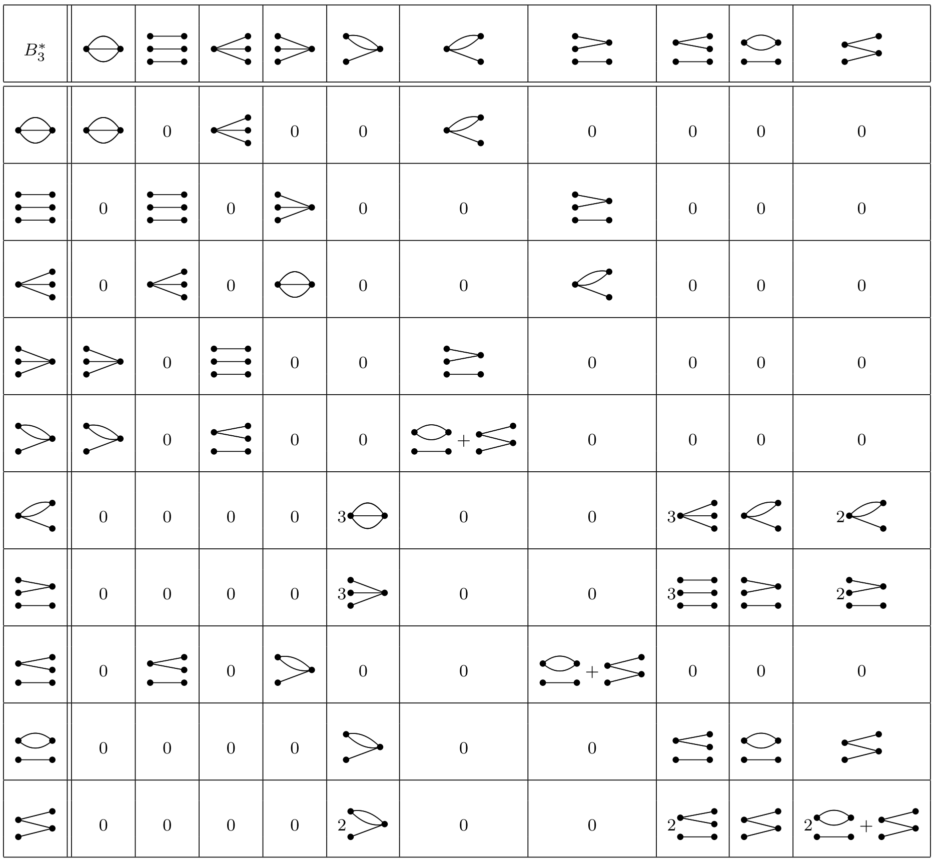
where

$$\{e_{ij}\} = \begin{array}{|c|c|} \hline e_{11} & e_{12} \\ \hline e_{21} & e_{22} \\ \hline \end{array} =$$ 

(10)

…

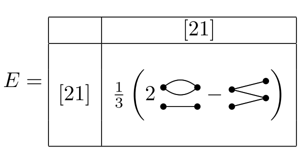
and, a little more complicated:

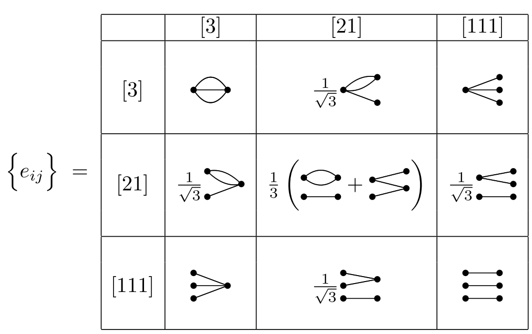


This table coincides with the combinatorial multiplication Table 1 from [4] (with misprint corrected in the right lowest corner). It can be also represented as the sum of the matrix algebras $M_3 \oplus M_1$:

$$
\begin{gathered}
e_{ij} * e_{kl} = \delta_{jk} e_{il}, \\
E * e_{ij} = e_{ij} * E = 0, \\
E * E = E
\end{gathered}
$$

where



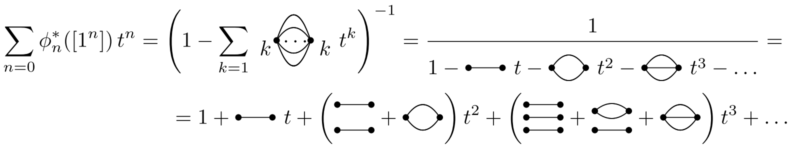


(11)

3.3 Relation between A_n and B_n

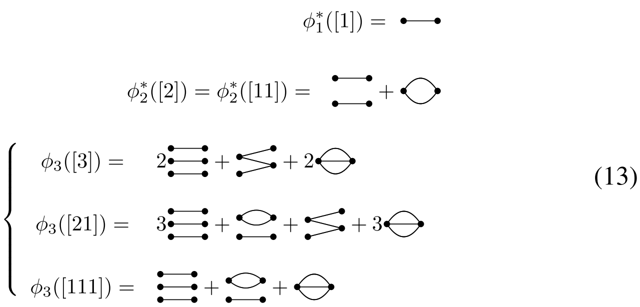
The $*$-homomorphism $\phi_n^* : A_n^* \longrightarrow B_n^*$ converts the Young diagrams from A_n^* into a certain linear combination of graphs from $\oplus_\Delta \mathcal{O}_{\Delta,\Delta}$ (but not Δ into $\mathcal{O}_{\Delta,\Delta}$ with the same Δ). The existence and manifest description of the homomorphism is done geometrically through association of graphs with the Hurwitz theory [1]. Here we just describe the answer.

The identity element of A_n^*, i.e. $[1^n] = [\underbrace{1, \ldots, 1}_{n}]$ is mapped into the identity element of B_n^* which is given by the formal series:



(12)

More generally:



(13)

$\dots$

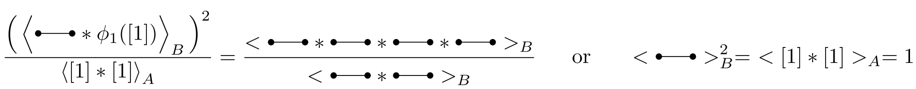
Remarkably, the homomorphism ϕ_n^* has a non-trivial kernel (coinciding with the non-trivial ideal in A_n^*). In particular,

$$\begin{aligned} &\ker\phi_1 = \emptyset,\\ &\ker\phi_2 = [2]-[11],\\ &\ker\phi_3 = [3]-[21]+[111],\\ &\dots \end{aligned} \tag{14}$$

For each n the Cardy relation (4) is satisfied [1], provided all the sums are over elements from A_n^* and B_n^* with the same n:

$$\begin{aligned} &\sum_{\Delta,\Delta'} \langle \Gamma_1 * \phi(\Delta)\rangle_B \left(\langle \Delta * \Delta'\rangle_A\right)^{-1} \langle \phi(\Delta') * \Gamma_2\rangle_B \\ &\quad = \sum_{\Gamma,\Gamma'} \langle \Gamma_1 * \Gamma * \Gamma_2 * \Gamma'\rangle_B \left(\langle \Gamma * \Gamma'\rangle_B\right)^{-1} \end{aligned} \tag{15}$$

For example:



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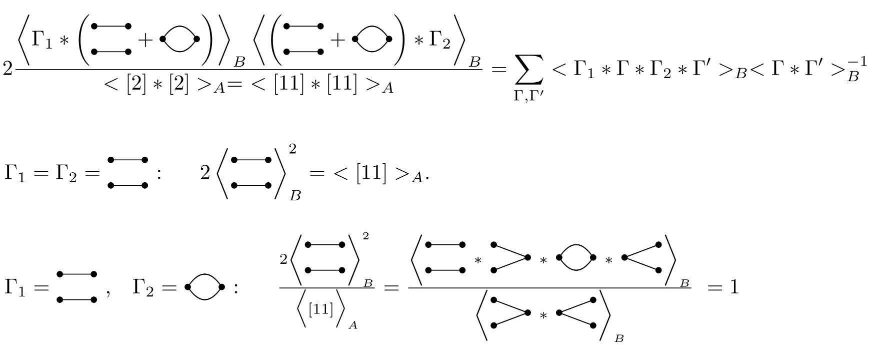


(17)

etc.

4 Unification of all A_n's and B_n's

4.1 $\circ$- versus $*$-multiplications and Universal CF Hurwitz algebra

For unification purpose one can consider the linear spaces $\mathcal{A}$ and $\mathcal{B}$ spanned by semi-infinite sequences of Young diagrams and bipartite graphs, respectively, containing exactly one element (perhaps, vanishing) of each size. The $*$-multiplication is then done termwise:

$$\begin{pmatrix} \Delta_1 \in A_1^* \\ \Delta_2 \in A_2^* \\ \Delta_3 \in A_3^* \\ \Delta_4 \in A_4^* \\ \dots \end{pmatrix} * \begin{pmatrix} \Delta_1' \\ \Delta_2' \\ \Delta_3' \\ \Delta_4' \\ \dots \end{pmatrix} = \begin{pmatrix} \Delta_1 * \Delta_1' \\ \Delta_2 * \Delta_2' \\ \Delta_3 * \Delta_3' \\ \Delta_4 * \Delta_4' \\ \dots \end{pmatrix} \quad \text{and}$$
$$\begin{pmatrix} \Gamma_1 \in B_1^* \\ \Gamma_2 \in B_2^* \\ \Gamma_3 \in B_3^* \\ \Gamma_4 \in B_4^* \\ \dots \end{pmatrix} * \begin{pmatrix} \Gamma_1' \\ \Gamma_2' \\ \Gamma_3' \\ \Gamma_4' \\ \dots \end{pmatrix} = \begin{pmatrix} \Gamma_1 * \Gamma_1' \\ \Gamma_2 * \Gamma_2' \\ \Gamma_3 * \Gamma_3' \\ \Gamma_4 * \Gamma_4' \\ \dots \end{pmatrix} \tag{18}$$

thus making the infinite algebras $\mathcal{A}^*$ and $\mathcal{B}^*$ of the linear spaces $\mathcal{A}$ and $\mathcal{B}$. The $*$-homomorphism $\phi^*: \mathcal{A}^* \longrightarrow \mathcal{B}^*$ is also defined termwise, and the Cardy relation also holds termwise, i.e. in the operator form (3) rather than in the converted one (4).

The original spaces of Young diagrams and graphs, $A = \oplus_n A_n$ and $B = \oplus_n B_n$ can be embedded into $\mathcal{A}^*$ and $\mathcal{B}^*$ with the maps

$$\begin{aligned}&\rho : A = \oplus_n A_n \longrightarrow \mathcal{A},\\ &\sigma : B = \oplus_n B_n \longrightarrow \mathcal{B}.\end{aligned} \tag{19}$$

These embeddings have a triangular structure: ρ maps the element $\Delta \in A_n$ to the column with first $n-1$ entries zero, and similarly does σ. However, the embeddings are not $*$-homomorphisms. Still, because of the triangular form of the mappings, the images $\rho(A) \subset \mathcal{A}$ and $\sigma(B) \subset \mathcal{B}$ are $*$-subalgebras, i.e. $\rho(A) * \rho(A) \subset \rho(A)$ and $\sigma(B) * \sigma(B) \subset \sigma(B)$, so that one can define a new operation on A and B, which we call $\circ$-multiplication:

$$\begin{aligned}&\rho\big(\Delta \circ \Delta'\big) = \rho(\Delta) * \rho\big(\Delta'\big) \quad \text{and}\\ &\sigma\big(\Gamma \circ \Gamma'\big) = \sigma(\Gamma) * \sigma\big(\Gamma'\big).\end{aligned} \tag{20}$$

One can fix ρ and σ by giving their action on all the elements of A_n and B_n, respectively, and then continuing their action onto the whole A and B. If one admits infinite sums of elements to belong to A and B, respectively, σ and ρ can be continued to the isomorphisms $\mathcal{A}^* \cong A^\circ$, $\mathcal{B}^* \cong B^\circ$, i.e. every such a pair of embeddings determines a pair of algebras A° and B° with a homomorphism one to the other and with the Cardy relation satisfied (yet in the operator form (3)).

However, interesting embeddings are those giving rise to $\circ$-multiplication such that the products of finite sums of elements in A° and B° are also finite sums. We call such a pair of CF algebras A° and B° Universal CF Algebra (UCFA).[1]

In fact, one such pair can be manifestly constructed in the following way inherited from the open Hurwitz numbers (this is why we call this concrete UCFA Universal Hurwitz algebra).

The first embedding, ρ maps the element $\Delta \in A_n$ to the column with the $(n+k)$th entry of the form

$$\rho_{n+k}[\Delta] = \frac{(r_\Delta + k)!}{k!\, r_\Delta!}\, [\Delta, \underbrace{1, \dots, 1}_{k}] \tag{21}$$

where r_Δ is the number of lines of the unit length in Δ.

Similarly, the σ-embedding maps the element $\Gamma \in B_n$ to the column whose entries $\sigma_n(\Gamma)$ are

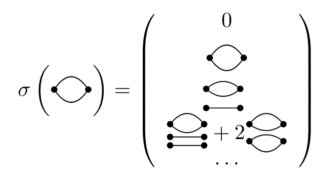
$$\sigma_n(\Gamma) = \begin{cases} \sum_{\Gamma_n \in \mathcal{E}_n(\Gamma)} \frac{|\mathbf{Aut}(\Gamma_n)|}{|\mathbf{Aut}(\Gamma_n \setminus \Gamma)||\mathbf{Aut}(\Gamma)|} \cdot \Gamma_n, & n \geq |\Gamma|,\\ 0, & n < |\Gamma|.\end{cases} \tag{22}$$

We call the graph with all connected components having two vertices as *simple graph*, and call *the standard extension of the graph* the graph obtained by adding simple connected components. Then, $\mathcal{E}_n(\Gamma)$ in (22) denotes the set of all degree n standard extensions of Γ.

4.2 Universal Hurwitz algebra: A°

In the simplest examples, these maps are

$$\rho([2]) = \begin{pmatrix} 0\\ [2]\\ [21]\\ [211]\\ \dots \end{pmatrix} \quad \text{and}$$



(23)

and the $\circ$-products are [11, 12]

$$\begin{aligned}\rho\big([1]\circ[1]\big) &= \rho\big([1]\big) * \rho\big([1]\big)\\ &= \begin{pmatrix} [1]\\ 2\,[11]\\ 3\,[111]\\ 4\,[1111]\\ \dots \end{pmatrix} * \begin{pmatrix} [1]\\ 2\,[11]\\ 3\,[111]\\ 4\,[1111]\\ \dots \end{pmatrix}\\ &= \begin{pmatrix} [1]*[1]\\ 4\,[11]*[11]\\ 9\,[111]*[111]\\ 16\,[1111]*[1111]\\ \dots \end{pmatrix} = \begin{pmatrix} [1]\\ 4\,[11]\\ 9\,[111]\\ 16\,[1111]\\ \dots \end{pmatrix}\\ &= \begin{pmatrix} [1]\\ 2\,[11]\\ 3\,[111]\\ 4\,[1111]\\ \dots \end{pmatrix} + 2 \begin{pmatrix} 0\\ [11]\\ 3\,[111]\\ 6\,[1111]\\ \dots \end{pmatrix}\\ &= \rho\big([1]\big) + 2\rho\big([11]\big)\end{aligned}$$

$$\begin{aligned}\rho\big([1]\circ[2]\big) &= \rho\big([1]\big) * \rho\big([2]\big)\\ &= \begin{pmatrix} [1]\\ 2\,[11]\\ 3\,[111]\\ 4\,[1111]\\ \dots \end{pmatrix} * \begin{pmatrix} 0\\ [2]\\ [21]\\ [211]\\ \dots \end{pmatrix}\\ &= \begin{pmatrix} 0\\ 2\,[11]*[2]\\ 3\,[111]*[21]\\ 4\,[1111]*[211]\\ \dots \end{pmatrix} = \begin{pmatrix} 0\\ 2\,[2]\\ 3\,[21]\\ 4\,[211]\\ \dots \end{pmatrix}\end{aligned}$$

[1] Note that originally the CF algebra was defined for finite-dimensional algebras. The subtlety of the infinite-dimensional case is discussed in [11, 12].

$$= 2\begin{pmatrix} 0 \\ [2] \\ [21] \\ [211] \\ \dots \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ [21] \\ 2\,[211] \\ \dots \end{pmatrix}$$
$$= 2\rho\big([2]\big) + \rho\big([21]\big)$$

The $\circ$-multiplication is evidently defined for any pair of Young diagrams or of bipartite graphs, without requiring them to have equal sizes (underlined are the diagrams of the same size, i.e. coming directly from the $*$-product):

$$\begin{array}{c|cccc} A^\circ & [1] & [11] & [2] & \dots \\ \hline [1] & \underline{[1]} + 2\cdot[11] & 2[11]+3[111] & 2[2]+[21] & \\ [11] & 2[11]+3[111] & \underline{[11]}+6\cdot[111]+6\cdot[1111] & \underline{[2]}+2\cdot[21]+[211] & \\ [2] & 2[2]+[21] & \underline{[2]}+2\cdot[21]+[211] & \underline{[11]}+3\cdot[3]+2\cdot[22] & \\ \dots & & & & \end{array} \qquad (24)$$

Note that even in UCFA the unit element is an infinite sum. For instance, in A° given by (23) it is

$$\sum_{k=1}^{\infty}(-1)^{k+1}\big[1^k\big]$$

and similarly in B°.

One more representation of the $\circ$-multiplication is in terms of the generating functions

$$J_\Delta(u) = \sum_{k\ge 0}\frac{(r_\Delta+k)!}{k!\,r_\Delta!}\,u^{|\Delta|+k}\,[\Delta, \underbrace{1,\dots,1}_{k}]. \qquad (25)$$

In these terms

$$J_{\Delta_1\circ\Delta_2}(v) = \oint J_{\Delta_1}(u) * J_{\Delta_2}\left(\frac{v}{u}\right)\frac{du}{u} = \sum_\Delta C^\Delta_{\Delta_1\Delta_2}J_\Delta(v) \qquad (26)$$

In [7, 8] the algebra A° was identified with the associative and commutative algebra of the cut-and-join operators,

$$\hat W(\Delta_1)\hat W(\Delta_2) = \sum_\Delta C^\Delta_{\Delta_1\Delta_2}\hat W_\Delta \qquad (27)$$

and for $\Delta = [\delta_1 \ge \delta_2 \ge \dots \ge \delta_{l(\Delta)} > 0] = [\dots, k+1, \underbrace{k,\dots,k}_{m_k}, k-1,\dots]$

$$\hat W_\Delta = \frac{1}{\prod_k m_k!\,k^{m_k}}\ :\prod_i \operatorname{Tr}\hat D^{\delta_i}: \qquad (28)$$

familiar also in the theory of matrix models. Here $\hat D_{\mu\nu}$ is the generator of the regular representation of $GL(\infty)$, which is manifestly realized as the differential operator on the space of functions of infinitely many matrix variables $X_{\mu\nu}$

$$D_{\mu\nu} = \sum_\rho X_{\mu\rho}\frac{\partial}{\partial X_{\nu\rho}} \qquad (29)$$

its degree is just the matrix multiplication (i.e. $(\hat D^2)_{\mu\nu} = \sum_\rho \hat D_{\mu\rho}\hat D_{\rho\nu}$ etc.) and the normal ordering implies that all $X_{\mu\nu}$ are placed to the left of all derivatives, see details in [7, 8]. This algebra is isomorphic also to the Ivanov–Kerov algebra [10].

4.3 Universal Hurwitz algebra: B°

An operator representation of the associative but non-commutative B° is an open question, to be discussed in the forthcoming paper [11, 12]. Here we just give a few examples of the $\circ$-product in this case.

Example 1 Let $\Gamma_{k,k}$ denote a graph with two vertices and k lines between them. Let $V_{R,R}$ be an element of B which is a collection of r_k copies of $\Gamma_{k,k}$ with various k, R is the corresponding Young diagram $R = \{k^{r_k}\}$.

Then,

$$V_{R,R} * V_{R',R'} = \delta_{R,R'}V_{R,R} \qquad (30)$$

(with coefficient 1). The homomorphism acts on these elements as

$$\sigma_n(V_{R,R}) = \sum_{\Delta:\,|\Delta|=n}\prod_k\frac{(r_k+\delta_k)!}{r_k!\delta_k!}V_{R+\Delta,R+\Delta} \qquad (31)$$

(the sum of Young diagrams is simply $R+\Delta = \{k^{r_k+\delta_k}\}$).

Then

$$\sigma(V_{R,R}) * \sigma(V_{R,R}) = \sum_Y C^Y_{RR}\sigma(V_{R+Y,R+Y}) \qquad (32)$$

induces the $\circ$-product

$$V_{R,R}\circ V_{R',R'} = \sum_Y C^Y_{RR}V_{R+Y,R+Y} \qquad (33)$$

with the structure constants

$$C^{[n]}_{RR} = r_n(r_n+1),$$
$$C^{[1^n]}_{RR} = \frac{(r_1+n)!}{(n!)^2(r_1-n)!},$$
$$C^{[21]}_{RR} = r_1(r_1+1)r_2(r_2+1) = c[1]_{RR}c[2]_{RR},$$
$$\ldots$$

Important is appearance of factors r_1, r_1-1, r_1-2 etc.: they guarantee that the sum is finite.

Example 2 Similarly for $k>l$

$$\begin{aligned}\sigma(V_{1^k}) * \sigma(V_{1^l}) &= \frac{k!}{l!(k-l)!}\sigma(V_{1^k}) + \frac{(k+1)!}{(l-1)!(k+l-1)!}\sigma(V_{1^{k+1}}) \\ &\quad + \frac{(k+2)!}{2(l-2)!(k+2-l)!}\sigma(V_{1^{k+2}}) \\ &\quad + \frac{(k+3)!}{3(l-3)!(k+3-l)!}\sigma(V_{1^{k+2}}) + \cdots \end{aligned} \tag{34}$$

i.e.

$$V_{1^k} \circ V_{1^l} = \sum_{i=0}^{l} \frac{(k+i)!}{i!(l-i)!(k+i-l)!} V_{1^{k+i}}. \tag{35}$$

The sum is finite.

Example 3 Another extension is to arbitrary pair of $V_{R,R}$.

Take $V_{R+P,R+P}$ and $V_{R+Q,R+Q}$, i.e. the two diagrams have a common part R. Then for any k either p_k or q_k vanish, i.e. $p_kq_k=0$ and

$$\begin{aligned}\sigma(V_{R+P,R+P}) * \sigma(V_{R+Q,R+Q}) \\ = \sum_Y C^Y_{R+P.R+Q}\sigma(V_{R+P+Q+Y,R+P+Q+Y}).\end{aligned} \tag{36}$$

Then

$$\begin{aligned} C^{\emptyset}_{R+P,R+Q} &= \prod_k \frac{((r_k+p_k+q_k)!)^2}{(r_k+p_k)!(r_k+q_k)!p_k!q_k!}, \\ C^{[n]} &= C^{\emptyset}\frac{(r_n+p_n+q_n+1)(r_n-p_nq_n)}{(p_n+1)(q_n+1)} \\ &= C^{\emptyset}\frac{r_n(r_n+p_n+1)r_n}{p_n+1} \end{aligned} \tag{37}$$

where in the last formula we assumed that $q_n=0$, and p_n is arbitrary (though for other n the situation can be the opposite). Under the same assumption

$$\begin{aligned} C^{[n^2]} &= C^{\emptyset}\frac{(r_n-1)r_n(r_n+p_n+1)(r_n+p_n+2)}{2(p_n+1)(p_n+2)}, \\ C^{[m,n]} &= C^{\emptyset}\frac{r_mr_n(r_m+p_m+q_m+1)(r_n+p_n+q_n+1)}{(p_m+1)(q_m+1)(p_n+1)(q_n+1)}, \\ &\quad m\neq n, \\ &\ldots \end{aligned} \tag{38}$$

In the last formula one can have either $q_m=q_n=0$, or $q_m=p_n=0$, or $p_m=q_n=0$, or $p_m=p_n=0$.

In all these examples, one can see that the products of V_R are indeed finite sums, i.e. B° is a Universal algebra.

5 Summary

In this paper we formulated a procedure which allows one to glue together a set of finite-dimensional Cardy–Frobenius algebras (A_n, B_n) into an infinite-dimensional one. To this end, one fixes a map from the direct sum $(A=\oplus_n A_n, B=\oplus_n B_n)$ to the space of semi-infinite sequences $(\mathcal{A}=\{a_1,a_2,\ldots\}, \mathcal{B}=\{b_1,b_2,\ldots\})$, $a_i\in A_i$, $b_i\in B_i$, considered as the isomorphism of vector spaces. The product in the algebras A_n and B_n induces a componentwise product $*$ in $\mathcal{A}$ and $\mathcal{B}$ making a Cardy–Frobenius algebra of these vector spaces. Then, the fixed isomorphism induces a product $\circ$ in (A,B), in its turn making a Cardy–Frobenius algebra of it.

This procedure works equally well for any map, though non-trivial is the finiteness property of $\circ$-product which means that the product of finite sums of elements is a finite sum. We proposed a map with such a property inherited from the Hurwitz theory though the complete proof of the finiteness property would be achieved by constructing a realization of the constructed $\circ$-product in terms of differential operators (or infinite-dimensional matrices) from the universal enveloping algebra $U(GL(\infty))$, which is not available yet.

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