A Mechanism Design Framework for Hiring Experts in E-Healthcare

Vikash Kumar Singh[a], Sajal Mukhopadhyay[a] and Fatos Xhafa[b]

[a] Dept. of Computer Science and Engineering, National Institute of Technology, Durgapur 713209, India; [b] Dept. of Computer Science, Universitat Politècnica de Catalunya, Barcelona 08034, Spain

ARTICLE HISTORY
Compiled October 30, 2019

ABSTRACT
We investigate the problem of hiring experts (motivated *socially* and *monetarily*) from outside of the hospital(s) in e-healthcare through the lens of mechanism design with and without money. This paper presents the mechanisms that handle the following scenarios: 1) Multiple patients and multiple experts with patients having *zero* budget, 2) Single or multiple patients and multiple experts with patient(s) having some *positive* budget. In this paper, for the first scenario, we have proposed algorithms based on the theory of *mechanism design without money* that satisfies several economic properties such as *truthfulness*, *pareto optimality*, and *core* allocation. Considering the second scenario, the *truthful* and *budget feasible mechanisms* are proposed. Through simulations, we evaluate the performance and validate our proposed mechanisms. The code of our proposed mechanisms is available online at: https://www.dropbox.com/sh/8lef3kzbwwwltj8/AAB_AsICQSvfdB0Wf1iY2L89a?dl=0

KEYWORDS
E-Healthcare; hiring experts; *core* allocation; pareto optimal; truthful; budget

1. Introduction

Over the past decades, most of the works in healthcare domain circumvent around the question: *how to effectively and efficiently schedule the healthcare resources such as nurses, physicians, and operation theatre(s) inside the hospitals?* Answering the above coined question, substantial works have been done to schedule resources (nurses (Berrada *et al.* 1996), physicians (V. Huele and Vanhoucke 2014), and operation theatre(s) (Guerriero and Guido 2011; Chan and Chen 2016; Cardoen *et al.* 2010a; Escobar-Rodriguez *et al.* 2014)) inside the hospitals in healthcare system. In our future references, hospital(s), medical unit(s), and organization(s) will be used interchangeably. However, how to hire (to schedule) resources (expert consultants (ECs) etc.) from outside the hospital(s) that is/are in-house for the patient(s) is mostly unaddressed (Starren *et al.* 2014; Singh *et al.* 2015, 2018a, 2017). It is observed that, with the prodigious growth of the communication media (say *video conferencing*, *Internet*, *smart phones* etc.), it may be an usual phenomena to have the consultancies by the experts (especially doctors) from outside the hospital(s). It is to be noted that the doctors can provide their consultancies by being present physically at the consultancy spot (where the patient is admitted) or virtually (using `video conferencing`, `Internet`, `smart phones` etc.). In our case, in order to add the pervasive flavour in

CONTACT F. Xhafa. Email: fatos@cs.upc.edu

the problem, we have considered that, doctors provide consultancy by their virtual presence also. In our future references ECs, doctors, and experts will be used interchangeably.

In this paper, the *experts hiring problem* is modelled through the robust concepts of mechanism design with and without money.[1] By *zero* budget, we mean that money is not involved in any sense in the hiring market. The idea behind studying the problem in *zero* budget case is to provide the expertise of the socially motivated doctors to the community (especially downtrodden) free of cost. On the other hand, the motivation behind investigating the problem in *positive* budget case is to provide experts to the patients who are monetarily well off, and in exchange of consultancy the patients will be charged by the experts.[2]

Previously, in (Singh *et al.* 2018a) the *experts hiring problem* is investigated with the constraint that the patients will be giving strict preference ordering over all the available experts or the subset of the available experts.[3] The *truthful* mechanisms were proposed for allocating the doctors to the patients[4]. However, a natural generalization of the *experts hiring problem* in *zero* budget case arises when each patient need not rank all the available or the subset of the available experts in *strict* ordering. Some of those might be indifferent among certain experts, so that preference lists may have *ties*. In this context, a *truthful* mechanism is proposed motivated by (Roughgarden 2016; Klaus *et al.* 2016) to cater the need of the more realistic version of the problem.

Considering the *positive* budget case, in (Singh *et al.* 2018b) the set-up with single patient and multiple experts with the constraint that patient have some *positive* budget, is considered. A *non-truthful* and *truthful* budget feasible mechanisms are proposed motivated by (Singer 2012, 2010; Khuller *et al.* 1999) for allocating experts to a patient. The more *realistic* version of the problem could be, to have multiple patients and multiple doctors, with the constraints that (1) each of the patient has some *positive* budget for the purpose of having *expertise* from outside of the admitted hospital, (2) the experts may have some preferred time slot (or availability time) for imparting the consultancy. In this paper, this generalized version of the problem is investigated and the *non-truthful* and *truthful* mechanisms are proposed motivated by (Singer 2012, 2010; Singh *et al.* 2018b; Khuller *et al.* 1999).

For further development of the paper, we denote the two cases with the parameter α; where $\alpha = 0$ denotes the *zero* budget case which will be referred as *case 1* henceforth and $\alpha > 0$ denotes the *positive* budget case which will be referred as *case 2* from now onwards. In this paper, first the experts hiring problem in zero budget case (*i.e.* $\alpha = 0$) is illustrated in detailed manner in section 3. Next, in section 4 the experts hiring problem in positive budget case (*i.e.* $\alpha > 0$) is studied.

[1] In this paper, the problems modelled using mechanism design with money and without money are studied under *positive* budget and *zero* budget cases respectively.

[2] The work done in this paper is the extension of the preliminary versions of the papers (Singh *et al.* 2018a,b) appeared in 3PGCIC 2017 and AINA 2018 respectively.

[3] The preference ordering can be provided by considering several factors but not limited to qualification of the experts, organization to which the experts belong, experts professional experience, and may be the feedback from the patients etc. The difference in preference ordering comes from the fact that patients varies in above mentioned factors.

[4] Technically, by truthful mechanism we mean that the mechanism in which the agents can't gain by misreporting their private value.

1.1. Flow Diagram of Proposed Healthcare System

In order to have a better understanding of our proposed model we have presented the above discussed cases with the help of a flow chart shown in Figure 1.

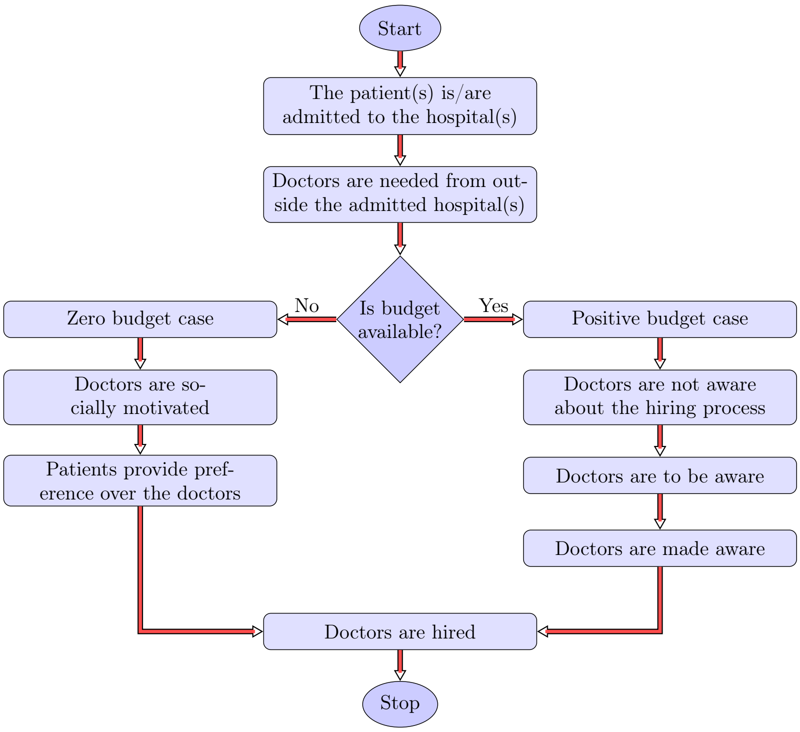


Figure 1. Flow diagram of our proposed system

The main contribution of this paper are:

- In this paper, the *experts hiring problem* has been investigated through the lens of mechanism design with and without money.
- For both the *zero* and *positive* budget case, the *non-truthful* and *truthful* mechanisms are proposed for allocating the doctor(s) to the patient(s).
- The simulations are performed for comparing our proposed mechanisms with a carefully crafted benchmark mechanisms.

The remainder of the paper is structured as follows. Section 2 elucidates the related works. Section 3 and 4 handles the *hiring experts* problem with *zero* and *positive* budget cases respectively. Experimental analysis is shown in section 5. Working of our model in real time is discussed in section 6. Finally, conclusions are drawn in Section 7.

2. Prior Related Works

This section contains a short description of the previous works and developments done in the domain of healthcare. The discussions will be mainly oriented in the following directions: (1) development and issues in the personal health information system (HIS), that contains the detailing of the patients medical records; (2) scheduling of healthcare resources (such as *operation theatres (OTs)*, *physicians*, *nurses* etc.)

inside the hospitals; and (3) scheduling of experts mainly doctors outside the in-house hospitals. The prior arts on scheduling the hospital resources such as *operation theatres (OTs)*, *physicians*, *nurses* (Berrada *et al.* 1996; Ko *et al.* 2017) etc. can be classified into two broad categories. One addressing the scenario of scheduling the hospital resources inside the in-house hospitals (Cardoen *et al.* 2010b; V. Huele and Vanhoucke 2014), with the other addressing, scheduling of experts outside the in-house hospitals (Starren *et al.* 2014; Singh *et al.* 2015; Pottayya *et al.* 2017). Our paper can be classified more in the second category. While, there are many fundamental questions that makes this healthcare research direction quite challenging. Our work finds relevance to some of them such as: a) *which experts are to be hired?* b) *how to have the expertise of the socially motivated experts for the patients (mainly downtrodden)?* c) *what is to be paid to the hired experts?*

- **Development and Issues in HIS:** In order to have an idea on the recent developments and challenges in the Health Information System (HIS) we recommend readers to go through (Stanimirovic 2015; Marcelo 2010; Paul *et al.* 2012). In (Stanimirovic 2015) the effort has been made for the development of HIS that partly based on three-level graph-based model (3LGM) and mainly based on a three-layer graph-based meta-model ($3LGM^2$). Further, they provide the guidelines for the cost and time effective implementation of HISs. In Atanasovski *et al.* (2018), a set of models that formalize the implementation of e-health system using Model Driven Architecture (MDA) as a framework, is presented. The study in (Ekblaw *et al.* 2016) developed a decentralized health record management system, named MedRec to handle Electronic Health Records (EHRs) using blockchain technology. The system provided patients with a comprehensive, immutable log and easy access to their medical information across different providers and various treatment sites. In Rahmadika and Rhee (2018) an architectural model for managing the Personal Health Information (PHI) data using block-chain technology is proposed. But there are still the issues of privacy, security, and integrity of the data in the proposed model. In order to have an overview of blockchain technology role and challenges in healthcare we recommend readers to go through Kumar *et al.* (2018). In Chiang *et al.* (2018), a personal health record system is established in the cloud to have an easy access of the patients' complete health record and enhance medical efficiency. In this, the proposed scheme ensures that any authorized medical personnel can obtain the decryption key only in the legal time interval and access the data.

- **In-house scheduling of healthcare resources:** In (Dexter and Macario 2004; Chan and Chen 2016; Guerriero and Guido 2011; Cardoen *et al.* 2010a,b; Samudra *et al.* 2016) the works have been done for allocating OTs on time to increase OTs efficiency. In (Bowers *et al.* 2016; Wang *et al.* 2007; Carter and Lapiere 2001; V. Huele and Vanhoucke 2014; Erhard *et al.* 2018; Tang *et al.* 2016) the different methods of scheduling the physicians in an emergency cases (may be critical operations) are discussed and presented. Several companies has developed physician scheduling software (ACEP 1998; ByteBloc 1995; MSI 1998) that will help in scheduling the physicians inside the hospital. In the series of physician scheduling literature, (Santos and Eriksson 2014) has investigated the physician scheduling problem and found that for timely and high quality care, the other healthcare resources such as patient, non-physician staff, room and equipment should also be scheduled and coordinated well along with physicians. Previously studied physician scheduling approaches (Carter and Lapiere 2001; V. Huele and Vanhoucke 2014) did not took this factor into consideration.

• **Scheduling experts outside the in-house hospitals:** As with the enhancement in the technologies, mainly communication media (say video conferencing, Internet, smart phones etc.), it may be an usual phenomena to think of providing the expertise of the medical staffs (*mainly doctors*) outside the in-house hospital by their virtual presence (Starren *et al.* 2014; Kakria *et al.* 2015) or by physical presence (Singh *et al.* 2015; Pottayya *et al.* 2017). Some literature works projected light, on the scarcity of healthcare facilities and non-availability of doctors in rural areas and proposed some feasible approaches in this direction (Mukherjee and Bhunia 2014; Mukherjee *et al.* 2016). In (Mukherjee and Bhunia 2014; Mukherjee *et al.* 2016), the efforts have been made to provide the basic healthcare services (mainly expertise by the doctors present in urban areas) to the patients residing in the rural areas. In order to establish such type of rural-urban consultancy arena, a remote healthcare framework has been proposed based on sensor-cloud (sensor technology and cloud computing are meld together) technologies. As this framework is capable of storing past and present data of the patients, so the doctors sitting remotely can access the data and provide the consultancy to the patients. With the rapid advancement in technologies, such as mobile network and cloud computing, a personalised and high-quality health monitoring is achieved. In Xu *et al.* (2017), a framework of an m-Health monitoring system based on a cloud computing platform (Cloud-MHMS) is designed to implement pervasive health monitoring. In (Starren *et al.* 2014) a doctor is providing the expertise through video conferencing to a patient admitted to other hospital with prior contact. In (Tekin *et al.* 2015) the context of the patient (such as *age*, *sex*, *medical report* etc.) is utilized to take the expertise of the doctors from outside the admitted hospitals in non-strategic setting. In (Singh *et al.* 2015, 2018b) the *strategic* case is considered and is solved using mechanism design with money and in (Singh *et al.* 2018a, 2017) mechanism design without money is utilized. *For the full version of (Singh et al. 2018b) and (Singh et al. 2018a) one can go through (Singh and Mukhopadhyay 2016b) and (Singh and Mukhopadhyay 2016a) respectively.*

3. Case 1: Hiring Experts Problem with $\alpha = 0$

In *zero* budget case, first the problem is studied under the consideration that, we have equal number of doctors and patients *say n*. Each of the patient initially allocated some in-house doctor and is providing the preference ordering (may or may not be strict) over all the available doctors for better expertise. But, one can think of the situation where there are n number of patients and m number of doctors such that $m \neq n$ ($m > n$ or $m < n$). Moreover, the constraints that each of the patient initially allocated an in-house expert is not feasible for $m \neq n$ case. Also, the patients providing the preference ordering over all the available doctors is not essential and can be relaxed for all the three different set-ups (*i.e.* $m = n$, $m < n$, and $m > n$). By relaxation, we mean that initially no in-house doctor will be assigned to patients for $m \neq n$ set-up and also the patients may give the preference ordering over the subset of the available doctors not necessarily in strict sense for all the three set-up.

In this section, we have developed three algorithms motivated by (Shapley and Scarf 1974; Roughgarden 2013, 2016; Klaus *et al.* 2016). Firstly, the RanPAM is given as a naive solution of our problem, that will help to understand better, the more robust **D***ominant* **s***trategy* **i***ncentive* **c***ompatible* (DSIC) mechanisms *i.e.* TOAM and TOAM-IComP. As an extension of TOAM the TOAM-IComP is proposed motivated by (Roughgarden 2016; Klaus *et al.* 2016) to cater the need of more realistic situations.[1] It is to be noted that, along with *truthfulness* the TOAM-IComP satisfies two other economic properties: *pareto optimality* and *the core*.

3.1. System Model and Problem Formulation

Our proposed model consists of n hospitals. In each hospitals several patients (or agents) from different income groups are admitted who need expert consultation from outside of the hospital. It is assumed that the participating patients cannot misreport their income group, it is taken care by hospital authorities. Based on the income group each hospitals provides one **b**elow **i**ncome **g**roup (BIG) patient for the category under consideration. The *third party* selects n doctors out of all available doctors based on the quality of the doctors. In this set-up each hospital needs exactly one expert consultant and each expert consultant can provide their service to one hospital at a time. In our model, expert consultation may be sought for several categories of diseases given as $x = \{x_1, x_2, \ldots, x_k\}$. The set of available expert consultants for a particular category $i \in \{1, \ldots, k\}$ is denoted by $\mathcal{S}_i = \{s_1^{x_i}, s_2^{x_i}, \ldots, s_n^{x_i}\}$. The set of available BIG patients of particular category $i \in \{1, \ldots, k\}$ is denoted by $\mathcal{P}_i = \{p_1^{x_i}, p_2^{x_i}, \ldots, p_n^{x_i}\}$. When i^{th} *category* is mentioned, the index will be assumed as $i \in \{1, \ldots, k\}$, otherwise $i \in \{1, \ldots, n\}$. Each agent $p_i^{x_i} \in \mathcal{P}_i$ has a preference ordering over all $s_i^{x_i} \in \mathcal{S}_i$.

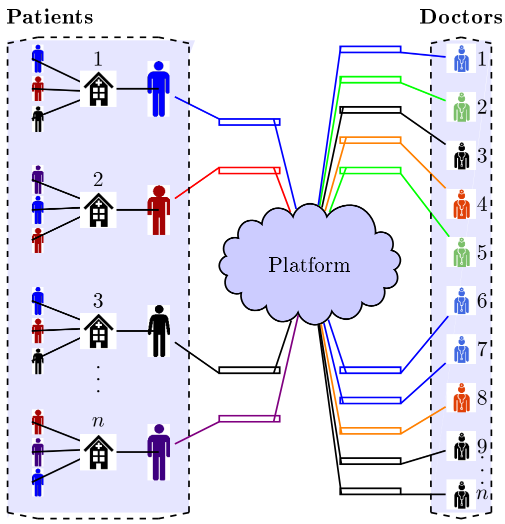


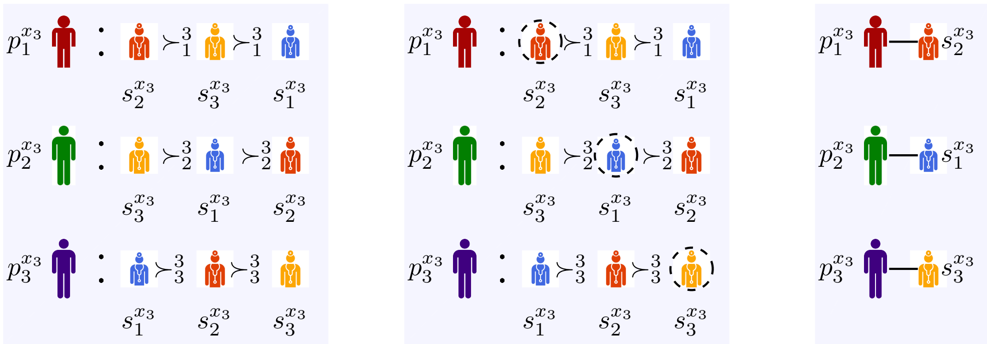
Figure 2. System model

The strict preference ordering of t^{th} agent $p_t^{x_i} \in \mathcal{P}_i$ in i^{th} category is denoted by $\succ_t^i$ over the set S_i, where $s_1^{x_i} \succ_t^i s_2^{x_i}$ means that in i^{th} category, t prefers $s_1^{x_i}$ to $s_2^{x_i}$. Whereas, the ties in the preference list of t^{th} agent $p_t^{x_i} \in \mathcal{P}_i$ in i^{th} category is denoted by $=_t^i$ over the set S_i, where $s_\ell^{x_i} =_t^i s_m^{x_i}$ means that in i^{th} category, t prefers equally $s_\ell^{x_i}$ and $s_m^{x_i}$. The set of preferences of all the agents in k different categories is denoted

[1]The TOAM mechanism is applicable for the scenarios with $m = n$ set-up in addition to the constraints that (1) each patient is providing the preference over all the experts not necessarily in strict fashion, and (2) each patient initially assigned an in-house expert. However, for the more realistic situations with $m \neq n$ set-up it can be seen that the initial assignments of the experts to the patients are not possible and also for both the set-ups ($m \neq n$, $m = n$) the patients may give the preference ordering over the subset of the available experts not necessarily in strict sense. So, for more realistic situations, implementation of TOAM may lead to some infinite loop. In these scenarios, TOAM-IComP is a much more viable option as initial assignments and preference over all the available experts are not required their.

by $\succ = \{\succ^1, \succ^2, \ldots, \succ^k\}$. Where, $\succ^i$ is the preference of all the agents in i^{th} category over the doctors in $\mathcal{S}_i$, represented as $\succ^i = \{\succ_1^i, \succ_2^i, =_3^i, \ldots, \succ_n^i\}$.[1] Given the preference of the agents, our proposed mechanisms allocates one doctor to one patient. Let us denote such an allocation vector by $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k\}$; where, each allocation vector $\mathcal{A}_i \in \mathcal{A}$ denotes the allocation vector of agents belongs to the i^{th} category denoted as $\mathcal{A}_i = \{a_1^i, a_2^i, \ldots, a_n^i\}$; where $a_t^i \in \mathcal{A}_i$ is a $(p_t^{x_i}, s_j^{x_i})$ pair.

Example 1: For our understanding purpose, let's say a category $x_3 \in x$ is selected from all the available categories.



(a) **(b)** **(c)**

Figure 3. **(a)** The patients belongs to category x_3 provides the strict preference ordering over the available doctor set $\mathcal{S}_3$. **(b)** By some arbitrary allocation rule, a doctor for each patient is selected from the preference list of the patients (shown by dashed circle). **(c)** The final patient-doctor allocation pair.

The set of patient in category x_3 is: $\mathcal{P}_3 = \{p_1^{x_3}, p_2^{x_3}, p_3^{x_3}\}$ and the set of doctors in category x_3 is: $\mathcal{S}_3 = \{s_1^{x_3}, s_2^{x_3}, s_3^{x_3}\}$. The set of preferences provided by patient set is given as: $\succ^3 = \{\succ_1^3, \succ_2^3, \succ_3^3\}$; where, $\succ_1^3 = (s_2^{x_3}, s_3^{x_3}, s_1^{x_3})$, $\succ_2^3 = (s_3^{x_3}, s_1^{x_3}, s_2^{x_3})$, $\succ_3^3 = (s_1^{x_3}, s_2^{x_3}, s_3^{x_3})$ as shown in Figure 3(a). Based on the preference ordering provided by the patients in $\mathcal{P}_3$, a doctor is allocated to a patient by some *arbitrary allocation rule* (detailed allocation rules are discussed later). The resultant allocation vector of an agents are given as: $\mathcal{A}_3 = \{a_1^3, a_2^3, a_3^3\}$, where $a_1^3 = (p_1^{x_3}, s_2^{x_3})$, $a_2^3 = (p_2^{x_3}, s_1^{x_3})$, $a_3^3 = (p_3^{x_3}, s_3^{x_3})$. So, $\mathcal{A}_3 = \{(p_1^{x_3}, s_2^{x_3}), (p_2^{x_3}, s_1^{x_3}), (p_3^{x_3}, s_3^{x_3})\}$ as shown in Figure 3(c).

Definition 1 (*Blocking coalition*)**.** For every $\mathcal{T}_i \subset \mathcal{P}_i$ let $\mathcal{A}_i(\mathcal{T}_i) = \{u \in \mathcal{A}_i : u_i^{p_i^{x_i}} \in \mathcal{T}_i,\ \forall i \in \mathcal{T}_i\}$ denote the set of allocations that can be achieved by the agents in $\mathcal{T}_i$ trading among themselves alone. Given an allocation $a \in \mathcal{A}_i$, a set $\mathcal{T}_i \subseteq \mathcal{P}_i$ of agents is called a *blocking coalition* (for a), if there exists a $u \in \mathcal{A}(\mathcal{T})$ such that $\forall i \in \mathcal{T}$ either $u_i \succ_i a_i$ or $u_i =_i a_i$ and at least one agent is better off *i.e.* for at least one $j \in \mathcal{T}_i$ we have $u_j \succ_j a_j$.

Definition 2 (*Core allocation*)**.** This property exhibits the fact that the allocation will be free of *blocking coalition*. In other words it says that if any subset of agents form a coalition and reallocates themselves via some internal reallocation, all of the members of the coalition can't be better off.

Definition 3 (*Truthfulness or DSIC*)**.** Let $\mathcal{A}_i = \mathcal{M}(\succ_i^i, \succ_{-i}^i)$ and $\hat{\mathcal{A}}_i = \mathcal{M}(\hat{\succ}_i^i, \succ_{-i}^i)$. TOAM is *truthful* if $a(i)^i \succeq_i^i \hat{a}(i)^i$, for all $p_i^{x_i} \in \mathcal{P}_i$.

[1] It is to be noted that patients are rational and strategic in nature. It means that, patients may mis-report their privately held preference list in order to gain.

Definition 4 (*Pareto optimality*)**.** An allocation $\mathcal{A}_i$ is *pareto optimal* if there exists no allocation $a_j^i \in \mathcal{A}_i$ such that any patient $p_i^{x_i} \in a_j^i$ can make themselves better off without making other patient(s) $p_k^{x_i} \in a_k^i$ worse off.

3.2. Proposed Mechanisms in Zero Budget Case

3.2.1. Random Pick-Assign Mechanism (RanPAM)

To better understand the model first we propose a *randomized algorithm* called RanPAM to assign doctors to the patients in the hospitals.[1]

3.2.1.1. Outline of RanPAM. Here, the central idea of the RanPAM is given.

RanPAM

For each category c_i:

(1) Randomly pick a patient from the available patients list.
(2) Next, randomly pick a doctor from the selected patient's preference list and allocate it.
(3) Remove the patient along with the allocated doctor from the consultancy arena.
(4) Repeat step 1-3 until all the patients does not allocated the doctors.

3.2.1.2. Detailed Random Pick-Assign Mechanism (RanPAM). The RanPAM consists of two stage allocation mechanism, namely; *Main* and *RanPAM allocation*. The idea behind the construction of the *Main* is to capture all the k categories present in the system. In each iteration of the *for* loop in line 2-5, a call to *RanPAM allocation* is made.

Algorithm 1: Main ($\mathcal{S}$, $\mathcal{P}$, x, $\succ$)

Output: $\mathcal{A} \leftarrow \phi$
1 **begin**
2 **for** *each* $x_i \in x$ **do**
3 $\quad$ $\mathcal{A}_i \leftarrow$ RanPAM allocation($\mathcal{S}_i$, $\mathcal{P}_i$, $\succ^i$)
4 $\quad$ $\mathcal{A} \leftarrow \mathcal{A} \cup \mathcal{A}_i$
5 **end**
6 return $\mathcal{A}$
7 **end**

In line 6, the final allocation $\mathcal{A}$ is returned. Considering the *RanPAM allocation*, from line 3 it is clear that the algorithm terminates once the list of patients in x_i category becomes empty. In line 4, the *rand()* function returns the index of the randomly selected patient and is stored in variable j. In line 5, the p^* data structure holds the patient present at the index returned in line 4. In line 6, a doctor is randomly selected from the patient j's preference list and is held in s^* data structure. Line 7 maintains the selected patient-doctor pairs of x_i category in $\mathcal{A}_i$. Line 8 and 9 removes the selected patients and selected doctors from the system. Line 10 removes the selected doctor from the preference lists of the remaining patients. In line 11, 12 the p^* and s^* are set to ϕ. The *RanPAM allocation* returns the final patient-doctor allocation pair set $\mathcal{A}_i$.

[1] It is to be noted that RanPAM is suffering from the blocking coalition. This leads to the violation of one of the economic properties in *zero budget* environment named as *core allocation*.

Algorithm 2: RanPAM allocation ($\mathcal{S}_i$, $\mathcal{P}_i$, $\succ^i$)

Output: $\mathcal{A}_i \leftarrow \phi$

begin

$j \leftarrow 0$, $p^* \leftarrow \phi$, $s^* \leftarrow \phi$

while $\mathcal{P}_i \neq \phi$ **do**

  $j \leftarrow rand(\mathcal{P}_i)$

  $p^* \leftarrow p_j^{x_i}$

  $s^* \leftarrow random(\succ_j^i)$

  $\mathcal{A}_i \leftarrow \mathcal{A}_i \cup (p^*, s^*)$

  $\mathcal{P}_i \leftarrow \mathcal{P}_i \setminus p^*$

  $\mathcal{S}_i \leftarrow \mathcal{S}_i \setminus s^*$

  $\succ_k^i \leftarrow \succ_k^i \setminus s^*, \forall k \in \mathcal{P}_i$

  $p^* \leftarrow \phi$

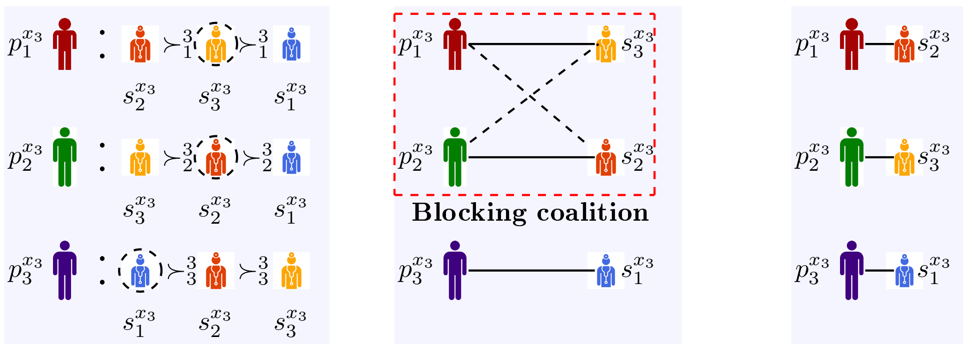
  $s^* \leftarrow \phi$

end

return $\mathcal{A}_i$

end

Example 2: Let's give a closer look with the help of illustrative example that RanPAM is suffering from *blocking coalition*. For an instance, let the category of all the patients and doctors be $x_3 \in x$. The set of patient is given as: $\mathcal{P}_3 = \{p_1^{x_3}, p_2^{x_3}, p_3^{x_3}\}$. The set of available doctors in the selected category *i.e.* x_3 is given as: $\mathcal{S}_3 = \{s_1^{x_3}, s_2^{x_3}, s_3^{x_3}\}$. The strict preference ordering revealed by the patient set $\mathcal{P}_3$ is shown in Figure 4(a). In the first iteration of the *while* loop, let $p_2^{x_3}$ be the patient selected randomly. Now, by the construction of mechanism, the mechanism randomly selects doctor $s_2^{x_3}$ from the available preference ordering of $p_2^{x_3}$.



(a) (b) (c)

Figure 4. **(a)** The patients belongs to category x_3 provides the strict preference ordering over the available doctor set $\mathcal{S}_3$. **(b)** By RanPAM, a doctor for each patient is selected from the preference list of the patients (shown by dashed circle). **(c)** The final patient-doctor allocation pair.

At the end of first iteration of while loop, the mechanism results in $(p_2^{x_3}, s_2^{x_3})$ pair. Similarly, in the next iteration of *while* loop, a patient $p_3^{x_3}$ is selected. The RanPAM selects $s_1^{x_3}$ from the $p_3^{x_3}$ preference list. Finally, $p_1^{x_3}$ allocated a doctor $s_3^{x_3}$ from preference list. The final allocation done by the RanPAM is shown in Figure 4(b). It can be seen from the preference list of $p_1^{x_3}$ and $p_2^{x_3}$, that both the agents do not get their best doctor from the available list of doctors. The most preferred doctor by $p_1^{x_3}$ *i.e.* $s_2^{x_3}$ is allocated to the agent $p_2^{x_3}$ and the most preferred doctor by $p_2^{x_3}$ *i.e.* $s_3^{x_3}$ is allocated to $p_1^{x_3}$. In this scenario, both the agent $p_1^{x_3}$ and $p_2^{x_3}$ can reallocate their current allocated doctor among themselves to make themselves better-off. The final allocation of $patient - doctor$ pair after reallocation among $p_1^{x_3}$ and $p_2^{x_3}$ (forming a blocking coalition) is shown in Figure 4(c).

Upper Bound Analysis: The time taken by the RanPAM is the sum of running times for each statement executed. Mathematically, the upper bound on the RanPAM for all the k categories is given as:

$$\left.\begin{aligned} T(n) &= \sum_{i=1}^{k}\left(1+\left(\sum_{i=1}^{n} i-1\right)\right) = \left(\sum_{i=1}^{k}1\right)+\left(\sum_{i=1}^{k}\sum_{i=1}^{n} i-1\right) \leq \left(\sum_{i=1}^{k}1\right)+\left(\sum_{i=1}^{k}\sum_{i=1}^{n} i\right) \\ &= k+\sum_{i=1}^{k}\frac{n(n+1)}{2} = k+\frac{kn(n+1)}{2} = \frac{kn^2+k(n+2)}{2} = O\left(kn^2\right) \end{aligned}\right\}$$

3.2.2. Truthful Optimal Allocation Mechanism (TOAM)

The proposed truthful mechanism needs to overcome several non-trivial challenges: firstly, the patients preferences are unknown and need to be reported in a truthful manner; secondly, the allocation of doctors made to the patients must satisfy the *core*. The previously discussed RanPAM mechanism failed to handle such challenges. To overcome these challenges, in this paper a *truthful* mechanism is proposed which is termed as TOAM. Along with, *truthfulness*, TOAM satisfies *pareto optimality*. The main idea of the TOAM is to develop a mechanism where the agents can't gain by manipulation. If there is no manipulation we can reach to the equilibrium of the system very quickly and the market become stable.

3.2.2.1. Outline of TOAM. In this section, the central idea of the TOAM is given.

<u>TOAM</u>

For each category c_i:

(1) First, randomly assign a doctor to each of the available patients.
(2) For each patient, point to the most preferred doctors from his/her preference ordering among the available one.
(3) Step 1 and 2 will result in a directed graph. Determine the directed cycle in the graph.
(4) Allocate the doctors to the patients following the directed cycle in the graph.
(5) Remove the patients along with the allocated doctors from the consultancy arena.
(6) Repeat step 2-5 until all the patients does not get the doctors.

3.2.2.2. Detailed Truthful Optimal Allocation Mechanism (TOAM). It is a four stage mechanism: *Main routine*, *Graph initialization*, *Graph creation* and *Optimal allocation*.

<u>Main routine:</u> The main idea behind the construction of *main routine* is to capture all the k categories present in the system. The input to the *main routine* are the set of sets of vertices representing all the available patients given as $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$; where $\mathcal{C}_i = \{c_1^{x_i}, c_2^{x_i}, \ldots, c_n^{x_i}\}$ is the set of vertices representing patients belonging to

x_i category, the set of sets of vertices representing all the available expert consultants (doctors) given as $\mathcal{Q} = \{\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_k\}$; where $\mathcal{Q}_i = \{q_1^{x_i}, q_2^{x_i}, \ldots, q_n^{x_i}\}$ is the set of vertices representing doctors belonging to x_i category, x represents the set of categories, and the preference lists of all the patients. The output is the allocation set $\mathcal{A}$.

Algorithm 3: Main routine ($\mathcal{C}$, $\mathcal{Q}$, x, $\succ$)

Output: $\mathcal{A} \leftarrow \phi$

1 **begin**
2 $\mathcal{Q}^* \leftarrow \phi$, $\mathcal{C}^* \leftarrow \phi$
3 **for** *each* $x_i \in x$ **do**
4 $\quad \mathcal{C}^* \leftarrow select(\mathcal{C})$
5 $\quad \mathcal{Q}^* \leftarrow select(\mathcal{Q})$
6 $\quad \mathcal{A}_i \leftarrow$ Graph initialization ($\mathcal{C}^*$, $\mathcal{Q}^*$, $\succ^i$)
7 $\quad \mathcal{A} \leftarrow \mathcal{A} \cup \mathcal{A}_i$
8 **end**
9 return $\mathcal{A}$
10 **end**

In Line 4, the $\mathcal{C}^*$ data structure temporarily holds the set of vertices returned by *select()* in x_i category. The $\mathcal{Q}^*$ data structure temporarily holds the set of vertices returned by *select()* in x_i category as depicted in line 5. In line 6 for each category x_i, a call to *graph initialization* is made to randomly allocate a doctor to each patient. The allocation set $\mathcal{A}$ maintains the allocation for each category in line 7. In line 9, the final allocation set $\mathcal{A}$ is returned.

Graph initialization: The input to the *graph initialization* phase are the set of vertices representing the patients in x_i category, the set of vertices representing the doctors in x_i category, and the preference profile of the patients in x_i category. The output of the *graph initialization* is the graph $\mathcal{G}$ in the form of adjacency matrix $\mathcal{F}$ representing the randomly allocated doctors to the patients. Line 2, initializes the adjacency matrix $\mathcal{F}$ of size $|\mathcal{V}_i| * |\mathcal{V}_i|$ to null matrix; where $\mathcal{V}_i = \mathcal{C}_i \cup \mathcal{Q}_i$.

Algorithm 4: Graph initialization ($\mathcal{C}_i$, $\mathcal{Q}_i$, $\succ^i$)

1 **begin**
2 $\mathcal{F} = \{0\}_{|\mathcal{V}_i| * |\mathcal{V}_i|}$
3 **for** *each vertex* $c_t^{x_i} \in \mathcal{C}_i$ **do**
4 $\quad q^* \leftarrow Select\_random(\mathcal{Q}_i)$
5 $\quad \mathcal{F}_{q^*, c_t^{x_i}} = 1$
6 $\quad \mathcal{Q}_i \leftarrow \mathcal{Q}_i \setminus q^*$
7 **end**
8 Graph creation ($\mathcal{C}_i$, $\mathcal{Q}_i$, $\mathcal{F}$, $\succ^i$)
9 **end**

The *for* loop in line 3 iterates over all the patients in the x_i category. In line 4, the *Select_random()* function takes the set of vertices $\mathcal{Q}_i$ (analogous to the doctors with x_i expertise area) as the input and returns the randomly selected vertex. The randomly selected vertex is held in q^* data structure. Line 5, places a directed edge from q^* to $c_t^{x_i}$. Line 6, removes the randomly allocated vertex held in q^* from $\mathcal{Q}_i$. In line 8, a call to *Graph creation* phase is done.

Graph creation: The input to the *graph creation* phase are the set of vertices representing the patients in x_i category, the set of vertices representing the doctors in x_i category, the adjacency matrix $\mathcal{F}$, and the preference profile of the patients in x_i category. The output of the *graph creation* is the adjacency matrix $\mathcal{F}$.

Algorithm 5: Graph creation ($\mathcal{C}_i$, $\mathcal{Q}_i$, $\mathcal{F}$, $\succ^i$)

1 **begin**
2 **for** *each vertex* $c_t^{x_i} \in \mathcal{C}_i$ **do**
3     $q^* \leftarrow Select\_best(\succ_t^i)$
4     $\mathcal{F}_{c_t^{x_i}, q^*} = 1$
5 **end**
6 Optimal allocation ($\mathcal{F}$, S)
7 **end**

In line 3, the *Select_ best()* function takes the strict preference ordering list of t^{th} agent as input and returns the best doctor from the available preference list. The q^* data structure holds the best selected doctor. Line 4 places a directed edge from $c_t^{x_i} \in \mathcal{C}_i$ to $q^* \in Q_i$. In line 6, a call to *Optimal allocation* phase is done.

Optimal allocation: Next target is to determine a finite cycle in a directed graph $\mathcal{G}$.

Algorithm 6: Optimal allocation ($\mathcal{F}$, S)

$\pi \leftarrow \phi$, $\hat{\mathcal{C}}^* \leftarrow \phi$, $\hat{\mathcal{Q}}^* \leftarrow \phi$
foreach $v_i \in \mathcal{V}$ **do**
    Mark $v_i \leftarrow$ unvisited
end
$\pi \leftarrow random(v_i \in \mathcal{V})$
Mark $\pi \leftarrow$ visited
$push(S, \pi)$
while S *is non-empty* **do**
    $\pi \leftarrow pop(S)$
    foreach π' *adjacent to* π **do**
        if π' *is unvisited* **then**
            Mark $\pi' \leftarrow$ visited
            $push(S, \pi')$
        end
        else if π' *is visited* **then**
            Exists a finite cycle.
        end
    end
end
Allocate each $v_i \in \mathcal{C}^*$ in cycle the doctors it points in $\mathcal{Q}^*$
forall $v_i \in \mathcal{C}^*$ *in the cycle* **do**
    $\hat{\mathcal{C}}^* \leftarrow \hat{\mathcal{C}}^* \cup v_i$
end
forall $v_i \in \mathcal{Q}^*$ *in the cycle* **do**
    $\hat{\mathcal{Q}}^* \leftarrow \hat{\mathcal{Q}}^* \cup v_i$
end
$\mathcal{C}^* \leftarrow \mathcal{C}^* \setminus \hat{\mathcal{C}}^*$; $\mathcal{Q}^* \leftarrow \mathcal{Q}^* \setminus \hat{\mathcal{Q}}^*$ `// Deletes the allocated patient and doctor nodes.`
$\mathcal{V} = \mathcal{Q}^* \cup \mathcal{C}^*$
if $\mathcal{C}^* \neq \phi$ *or* $\mathcal{Q}^* \neq \phi$ **then**
    Graph creation ($\mathcal{C}^*$, $\mathcal{Q}^*$, $\mathcal{F}$, $\succ$)
end

The input to the *Optimal allocation mechanism* is the adjacency matrix $\mathcal{F}$ returned from the previous stage and an empty stack S. Initially, all $v_i \in \mathcal{V}$ are marked unvisited. Random vertex $v_i \in \mathcal{V}$ is selected and after marking that v_i as visited is pushed into the stack. Line $10-18$, computes a finite directed cycle in the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ by following the outgoing arcs, until a vertex $v_i \in \mathcal{V}$ gets repeated. Line 20, reallocates as suggested by directed cycle. Each patient on a directed cycle gets the expert consultant better than the expert consultant it initially points to or the initially pointed expert consultant. Line $21-26$ keeps track of allocated patients and doctors. Line 27 deletes the patient nodes and the expert consultant nodes from the graph $\mathcal{G}$ that were reallocated in the previous step. A call is made to the *graph creation* phase to generate the updated graph from the available number of patients and the expert consultants until the patients set and doctor sets are not empty captured by line $29-31$.

Example 3: The detailed functioning of TOAM for category x_2 is illustrated in Figure 5.

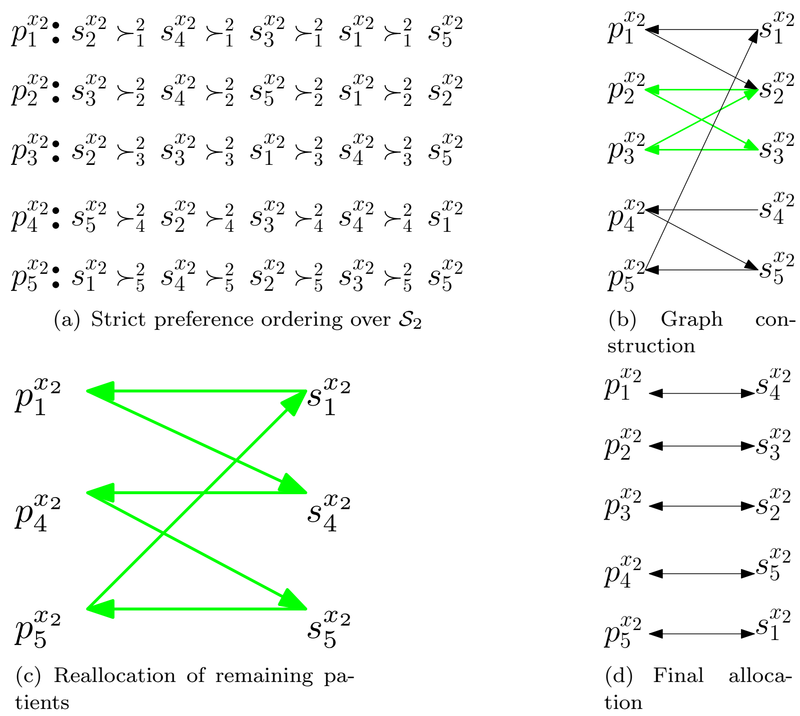


Figure 5. Detailed illustration of TOAM mechanism

The number of patients is $n = 5$ and the number of expert consultant (or doctors) is $n = 5$. The strict preference ordering given by the patient set $\mathcal{P}_2$ is shown in Figure 5(a). Following the *graph initialization* phase a directed edge is placed between the following pairs: $\{(s_1^{x_2}, p_1^{x_2}), (s_2^{x_2}, p_2^{x_2}), (s_3^{x_2}, p_3^{x_2}), (s_4^{x_2}, p_4^{x_2}), and(s_5^{x_2}, p_5^{x_2})\}$. Now, the *graph initialization* phase calls the *graph creation* phase. Following the *graph creation* phase, say, a patient $p_1^{x_2}$ is selected from $\mathcal{P}_2$. As, $s_2^{x_2}$ is the most preferred doctor in the preference list of $p_1^{x_2}$. So, a directed edge is placed from $p_1^{x_2}$ to $s_2^{x_2}$ as shown in Figure 5(b). The *for* loop of the *graph creation* phase places a directed edge between the remaining patients in $\mathcal{P}_2$ and doctors in $\mathcal{S}_2$, resulting in a graph shown in Figure 5(b). Now, running the optimal allocation phase on the directed graph shown in Figure 5(b), a cycle $(p_2^{x_2}, s_3^{x_2}, p_3^{x_2}, s_2^{x_2}, p_2^{x_2})$ is formed and allocation is done accordingly. Similarly, the remaining patients $\mathcal{P}_2 = \{p_1^{x_2}, p_4^{x_2}, p_5^{x_2}\}$ will be allocated a doctor as shown in Figure 5(c). The final allocation of *patient* $-$ *doctor* pair is shown in Figure 5(d).

Proposition 1. The Top Trading Cycle Algorithm (TTCA) is DSIC *(Gale and Shapley 1962; Shapley and Scarf 1974; Roughgarden 2013; Schummer and Vohra 2007).*

Proposition 2. The Top Trading Cycle Algorithm (TTCA) results in *unique core* allocation *(Gale and Shapley 1962; Shapley and Scarf 1974; Roughgarden 2013; Schummer and Vohra 2007).*

3.2.2.3. Several properties of TOAM. The proposed TOAM has several compelling properties. These properties are discussed next.

• **Computationally Efficient** The running time of TOAM will be the sum of the running time of *main routine*, *graph initialization*, *graph creation*, and *optimal allocation* phases. Line 2 of the *main routine* is bounded by $O(1)$. The *for* loop in line 3 executes for $k+1$ times. Line 4-7 are bounded by $O(1)$ for each iteration of the *for* loop. Line 6, takes the time equal to the time taken by *graph initialize* mechanism. For the time being, let the graph initialization mechanism is bounded by $O(N)$. Line 9 of *main routine* mechanism takes $O(1)$ time. So, the running time of *main routine* is bounded by: $O(1)+O(kN)+O(1) = O(kN)$. In *graph initialization*, in each execution of the *for* loop an edge is placed between the two vertices of the graph $\mathcal{G}$. Generating a partial directed graph $\mathcal{G}$ using line $3-7$ takes $O(n)$ time. Next, line 8 is bounded by the time taken by the *graph creation* mechanism. In the *graph creation* algorithm, the *for* loop contributes the major part of the running time *i.e.* $O(n)$. Line 6 of *graph creation* is bounded by the time taken by *optimal allocation*. For the time being let the time taken by by the *optimal allocation* be $O(M)$. So, the running time of graph creation algorithm is bounded by: $O(1)+O(n)+O(M)$. In, optimal allocation algorithm line 3 is bounded by $O(1)$. The total number of vertex is $n+n=2n$, so the outer *for* loop will take $O(n)$. Line $5-7$, is bounded by $O(1)$. the total number iterations of the innermost *while* loop of optimal allocation cannot exceed the number of edges in $\mathcal{G}$, and thus the size of S cannot exceed n. The *while* loop of optimal allocation algorithm is bounded by $O(n)$. Line $21-28$ of the mechanism can be executed in worst case $O(n^2)$. Line 30 in worst case is bounded by $O(n)$. The running time of *optimal allocation* is: $O(n)+O(1)+O(n)+O(1)+O(n^2)+O(n) = O(n^2)$. Total running time of TOAM: $O(n)+O(n^2) = O(n^2)$. Considering the k categories simultaneously we have $O(kn^2)$.

• **DSIC** The second property, that distinguishes the proposed TOAM from any direct revelation allocation mechanism is its DSIC property. In TOAM, the strict preference ordering revealed by the agents in any category $x_i \in x$ over the set of doctors $\mathcal{S}_i$ are unknown or private to the agents. As the strict preference ordering is private, any agent i belonging to category $x_j \in x$ can misreport their private information to make themselves better off. TOAM, an obvious direct revelation mechanism claims that agents in any category $i \in 1 \ldots k$ cannot make themselves better off by misreporting their private valuation, i.e. TOAM is DSIC.

Theorem 1. The TOAM is DSIC.

Proof. The truthfulness of the TOAM is based on the fact that each agent i gets the best possible choice from the reported strict preference, irrespective of the category $i \in 1 \ldots k$ of the agent i. It is to be noted that, the third party (or the platform) partition the available patients and doctors into different sets based on their category. The partitioning of doctors set $S = \{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_k\}$ is independent of the partitioning of the available patients into the set $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k\}$. So, if we select the patient set $\mathcal{P}_i \in \mathcal{P}$ and the doctor set $\mathcal{S}_i \in \mathcal{S}$ randomly from category $x_i \in x$ and show that for any agent $p_i^{x_i} \in \mathcal{P}_i$ misreporting the private information (in this case strict preference over $\mathcal{S}_i$) will not make the agent $p_i^{x_i}$ better-off, then its done. Our claim is that, if any agent belonging to x_i category, cannot be better off by misreporting their strict preference, then no agent from any category can be better off by misreporting the strict preference.

Fix category x_i. Let us assume that, if all the agents in x_i are reporting *truthfully*, then all the agents gets a doctor till the end of m^{th} iteration. From the construction of the mechanism in each iteration of the TOAM, at least a cycle $\Omega_i \in \Omega$ is selected. The set of cycles chosen by the TOAM in m iterations are: $\Omega = (\Omega_1, \Omega_2, \ldots, \Omega_m)$, where Ω_i is the cycle chosen by the TOAM in the i^{th} iteration, when agents reporting *truthfully.* Each agent in Ω_1, gets its first choice and hence no strategic agent can be benefited by misreporting. From the construction of the mechanism, no agent in Ω_i will ever be pointed by any agents in $\Omega_1, \ldots, \Omega_{i-1}$; if this is not the case, then agent i could have been belong to one of the previously selected cycle.
Once the doctor is allocated to the agent, the mechanism remove the agent along with the allocated doctor, and the strict preference list of the remaining agents are updated. Since, agent i gets its first choice outside of the doctors allocated in $\Omega_1, \ldots, \Omega_{i-1}$, it has no incentive to *misreport.* Thus, whatever agent i reports, agent i will not receive a doctor owned by an agent in $\Omega_1, \ldots, \Omega_{i-1}$. Since, the TOAM gives agent i its favourite doctor outside the selected cycle till now. Hence, agent i did not gain by misreporting the strict preference ordering. From our claim it must be true for any agents in any category $i \in 1 \ldots k$. Hence, TOAM is DSIC.

□

• **Core Allocation** The third property exhibited by the proposed TOAM is related to the *uniqueness* of the resultant allocation or in some sense *optimality.* The term used to determine the optimal allocation of TOAM is termed as *unique core allocation.* The claim is that, the allocation computed by the proposed TOAM is the *unique core allocation.*

Theorem 2. The allocation computed by TOAM is the *unique core allocation.*

Proof. The proof of *unique core allocation* for any category $x_i \in x$ can be thought of as divided into two parts. First, it is proved that the allocation vector $\mathcal{A}_i$ computed by TOAM for any category $x_i \in x$ is a *core allocation.* Once the allocation vector in x_i category is proved to be *The core*, the *uniqueness* of the *core* allocation for x_i is taken into consideration. Our claim is that, if the allocation $\mathcal{A}_i$ computed by TOAM for any arbitrary $x_i \in x$ is a *unique core allocation*, then the allocation computed by TOAM for all $x_i \in x$ will be a unique core allocation.

Fix category x_i. In order to prove the allocation computed by TOAM is a *core* allocation, consider an arbitrary sets of agents S^*, such that $S^* \subseteq P$. Let Ω_i is the cycle chosen by TOAM in the i^{th} iteration and $\delta(\Omega_i)$ is the set of agents allocated a doctor, when reporting *truthfully.* When TOAM will allocate the agents, at some cycle Ω_k, $i \in S^*$ will be included for the first time. In that case $\delta(\Omega_k) \cap S^* \neq \phi$. As any agent $i \in S^*$ is being included for the first time, it can be said that no other agent in S^* is included in the cycles $\Omega_1, \ldots, \Omega_{k-1}$. As the TOAM allocates the favourite doctor to any arbitrary agent $i \in \delta(\Omega_k)$ outside the doctors allocated to $\delta(\Omega_1), \ldots, \delta(\Omega_{k-1})$, it

can be concluded that $i \in \delta(\Omega_k)$ and $i \in S^*$ such that $\delta(\Omega_k) \cap S^* \neq \phi$ gets his favourite doctor at the k^{th} iteration. Hence no internal reallocation can provide a better doctor to any agent $i \in S^*$. Inductively, the same is true for any agent $j \in S^*$ that will satisfy $\delta(\Omega_k) \cap S^* \neq \phi$.
Now, we prove *uniqueness.* In TOAM, each agent in Ω_1 receives the best possible doctor from his preference list. Any core allocation must also do the same thing, otherwise the agents who didn't get the first choice could be better off with internal reallocation. So the core allocation agrees with the TOAM allocation for the agents in $\delta(\Omega_1)$. Now in TOAM, as all the agents in $\delta(\Omega_2)$ get their favourite doctors outside the set of doctors allocated to the agents $\delta(\Omega_1)$, any core allocation must be doing the same allocation, otherwise the agents in $\delta(\Omega_2)$ who didn't get their favourite choice can internally reallocate themselves in a better way. In this way we can inductively conclude that the core allocation must follow the TOAM allocation. This proves the uniqueness of TOAM.

Hence, it is proved that the allocation by TOAM for category x_i is a *unique core allocation.* From our claim it must be true for any agents in categories $i \ldots k$. Hence,the allocation computed by TOAM for any category $x_i \in x$ is the *unique core allocation.* $\square$

3.2.2.4. Correctness of the TOAM:. The correctness of the TOAM mechanism is proved with the *loop invariant* technique (Cormen *et al.* 2009; Gries 1987). The *loop invariant*: At the start of j^{th} iteration, the number of patient-doctor pairs to be explored are $n - \sum_{i=1}^{j-1} k_i$ in a category, where k_i is the number of patient-doctor pairs processed at the i^{th} iteration. Precisely, it is to be noted that $n - \sum_{i=1}^{j-1} k_i \leq n$. From definition of k_i, it is clear that the term k_i is non-negative. The number of patient-doctor pairs could be atleast 0. Hence, satisfying the inequality $n - \sum_{i=1}^{j-1} k_i \leq n$. We must show three things for this loop invariant to be true.
• **Initialization:** It is true prior to the first iteration of the loop. Just before the first iteration of the *while* loop, in optimal allocation mechanism $n - \sum_{i=1}^{j-1} k_i \leq n \Rightarrow n - 0 \leq n$ *i.e.* no patient-doctor pair is explored apriori in, say i^{th} category. This confirms that $\mathcal{A}_i$ contains no patient-doctor pair.
• **Maintenance:** For the loop invariant to be true, if it is true before each iteration of *while* loop, it remains true before the next iteration. The body of *while* loop allocates doctor(s) to the patient(s) with each doctor is allocated to one patient present in the detected cycle; *i.e.* each time $\mathcal{A}_i$ is incremented or each time n is decremented by k_i. Just before the j^{th} iteration the number of patient-doctor pairs allocated are $\sum_{i=1}^{j-1} k_i$, implies that the number of patient-doctor pairs left are: $n - \sum_{i=1}^{j-1} k_i \leq n$. After the j^{th} iteration, two cases may arise:
Case 1: If $k_j = n - \sum_{i=1}^{j-1} k_i$: In this case, all the k_j patient-doctor pairs will be exhausted in the j^{th} iteration and no patient-doctor pair is left for the next iteration. The inequality $n - (\sum_{i=1}^{j-1} k_i + k_j) = (n - \sum_{i=1}^{j-1} k_i) - k_j = (n - \sum_{i=1}^{j-1} k_i) - (n - \sum_{i=1}^{j-1} k_i)$ $= 0 \leq n$.
Case 2: If $k_j < n - \sum_{i=1}^{j-1} k_i$: In this case, j^{th} iteration allocates few patient-doctor pairs from the remaining patient-doctor pairs; leaving behind some of the pairs for further iterations. So, the inequality $n - (\sum_{i=1}^{j-1} k_i + k_j) \leq n = n - \sum_{i=1}^{j} k_i \leq n$ is satisfied.
From Case 1 and Case 2, at the end of j^{th} iteration the loop invariant is satisfied.
• **Termination:** In each iteration at least one patient-doctor pair is formed. This indicates that at some $(j+1)^{th}$ iteration the loop terminates and in line no. 8, S is exhausted, otherwise the loop would have continued. As the loop terminates and S is exhausted in $(j+1)^{th}$ iteration. We can say $n - \sum_{i=1}^{j} k_i = 0 \leq n$. This indicates that

all the agents are processed and each one has a doctor assigned when loop terminates.

3.2.3. Truthful Optimal Allocation Mechanism for InComplete Preference (TOAM-IComP)

3.2.3.1. Outline of TOAM-IComP. The central idea of the TOAM-IComP is given.

TOAM-IComP

For each category c_i:

(1) First, the n distinct random numbers are generated and assigned to patients.
(2) Next, based on the random number assigned, each time a patient is picked up from the patient list and a check is made, whether the preference list of the selected patient is empty or not;
- If not, then allocate the most preferred doctor to the patient from his/her preference ordering among the available one. Remove the patient along with the allocated doctor from the consultancy arena.
- Otherwise, remove the unallocated patient from the consultancy arena.

(3) Repeat step 2 until the patient list becomes empty.

3.2.3.2. Detailed TOAM-IComP. The input to the TOAM-IComP are: the set of n

Algorithm 7: TOAM-IComP ($\mathcal{S}_i$, $\mathcal{P}_i$, $\succ^i$)

begin
$\ell \leftarrow 0$, $\hat{p} \leftarrow \phi$, $\hat{s} \leftarrow \phi$, $\mathcal{B} \leftarrow \phi$
for i = *1 to n* **do**
  $\mathcal{B} \leftarrow \mathcal{B} \cup \{i\}$
end
for i = *1 to n* **do**
  swap $\mathcal{B}[i]$ with $\mathcal{B}[Random(i, n)]$
end
for *each* $p_j^{x_i} \in \mathcal{P}_i$ **do**
  $Assign(p_j^{x_i}, \mathcal{B}[\ell])$
  $\ell \leftarrow \ell + 1$
end
$\mathcal{P}_i \leftarrow$ Sort($\mathcal{P}_i$, $\mathcal{B}$)   // Sort $\mathcal{P}_i$ based on random number generated.
while $\mathcal{P}_i \neq \phi$ **do**
  $\hat{p} \leftarrow pick(\mathcal{P}_i)$ // Picks the patients based on the random number assigned.
  /* where, $j = 1, 2, \ldots, n$ */
  if $\succ_j^i \neq \phi$ **then**
    $\hat{s} \leftarrow Select\_best(\succ_j^i)$
    $\mathcal{F} \leftarrow \mathcal{F} \cup (\hat{p}, \hat{s})$
    $\mathcal{P}_i \leftarrow \mathcal{P}_i \setminus \hat{p}$
    $\mathcal{S}_i \leftarrow \mathcal{S}_i \setminus \hat{s}$
  end
  $\hat{p} \leftarrow \phi$, $\hat{s} \leftarrow \phi$
end
return $\mathcal{F}$
end

available patients in a particular category x_i, the set of m available doctors in a particular category x_i, and the set of preferences of all the patients for the available doctors in a x_i category. The output of the TOAM-IComP is the allocated patient-doctor pairs. In line 2, all the variables and data structures are initialized to 0 and ϕ respectively. In line 3-5 numbers 1 to n are captured in $\mathcal{B}$ data structure. Next, the generated list $\mathcal{B}$ is randomized using line 6-8. Line 9-12 assigns the distinct random numbers between 1 and n stored in $\mathcal{B}$ to the patients sequentially. In line 13, the patient list $\mathcal{P}_i$ is sorted based on the assigned random numbers. From line 14, it is clear that the mechanism terminates, once the patient list becomes empty. In line 15, using *pick()* function, patient is selected sequentially based on the number assigned. Line 16 checks the preference list of patient stored in $\hat{p}$. In line 17, the best available doctor is selected from the selected patient preference list by using $Select\_best()$ function. Line 18 maintains the selected patient-doctor pairs in $\mathcal{F}$ data structure. Line 19 and 20 removes the selected patients and selected doctors from their respective preference lists. Line 22 sets $\hat{p}$ and $\hat{s}$ to ϕ. The TOAM-IComP returns the final patient-doctor pair allocation set $\mathcal{F}$.

3.2.3.3. Upper Bound Analysis. The *random* number generator in line 3-12 is motivated by (Cormen *et al.* 2009) and is bounded above by n. When a *while* loop exits in the usual way (i.e., due to the inner loop header), the test is executed one time more than the body of the *while* loop. In line 14, the test is executed $(n+1)$ times, as their are n patients in $\mathcal{P}_i$. In line 13, the sorting is done that is bounded above by $n \lg n$. For each execution of *while* loop line $14-23$ will take constant amount of time.

$$\left.\begin{aligned}
T(n) &= \sum_{i=1}^{k}\left(\left(\sum_{i=1}^{n} O(1)\right) + \left(O(n \lg n)\right) + \left(\sum_{i=1}^{n} O(n)\right)\right)\\
&= \left(\sum_{i=1}^{k}\sum_{i=1}^{n} O(1)\right) + \left(\sum_{i=1}^{k} O(n \lg n)\right) + \left(\sum_{i=1}^{k}\sum_{i=1}^{n} O(n)\right)\\
&= O\left(\sum_{i=1}^{k}\sum_{i=1}^{n} 1\right) + O\left(\sum_{i=1}^{k} n \lg n\right) + O\left(\sum_{i=1}^{k}\sum_{i=1}^{n} n\right)\\
&= O\left(\sum_{i=1}^{k} n\right) + O\left(kn \lg n\right) + O\left(\sum_{i=1}^{k} n^2\right)\\
&= O\left(kn\right) + O\left(kn \lg n\right) + O\left(kn^2\right)\\
T(n) &= O\left(kn^2\right)
\end{aligned}\right\}$$

Proposition 3. *The Draw is DSIC* (Roughgarden 2016).

Proposition 4. *The outcome of The Draw is Pareto optimal* (Roughgarden 2016).

Theorem 3. The TOAM-IComP is DSIC.

Proof. Fix a category x_i. It is to be noted that, in TOAM-IComP the random numbers generated for the patients are independent of the preference list submitted by the patients. It means that, in which iteration of TOAM-IComP a patient is considered, is independent of his/her submitted preference list. From the construction of TOAM-

IComP, when a patient is considered he/she (henceforth he) will be getting his best doctor among the available doctors from his preference list. So, if any patient revealing the preference list other than the true preference list, then he will ends up getting the doctor worse than the doctor he will get while revealing the true preference list. From above argument it can be concluded that, it does not make any sense for the patients to mis-report their true preference list. From our claim it must be true for any agents in any category $i \in 1 \dots k$. Hence, TOAM-IComP is DSIC.
Similar argument can be given for the scenario when the patients have ties between the doctors in their respective preference list. □

Theorem 4. The outcome of TOAM-IComP is Pareto optimal.

Proof. Fix a category x_i. It can be seen from the construction of TOAM-IComP that at any k^{th} iteration, the patient under consideration gets the best doctor among the available doctors. As a thought experiment, let's run the TOAM-IComP and some "*unknown mechanism*" parallelly. The idea is to show that the allocation resulted by "*unknown mechanism*" is similar to that of TOAM-IComP, if not then in the allocation resulted by "*unknown mechanism*" some patients have been worsen off. So, this makes the TOAM-IComP Pareto optimal.
We will take the help of method of induction to prove the above claim.
Base case: Before the 1^{st} iteration *i.e.* in 0^{th} iteration the allocation by the two mechanisms are same *i.e. empty set.*
Inductive step: Let us say till ℓ^{th} iteration the two mechanisms results in same patient-doctor pairs. Thus, for the $(\ell+1)^{th}$ iteration the set of available doctors for the remaining patients will be similar for both the mechanisms. At this point, in TOAM-IComP any patient i considered will be getting the best possible doctor among the available doctors. But *say* with the same set of available doctors the "*unknown mechanism*" allocates the doctor other than allocated by the TOAM-IComP to patient i, then it is for sure that patient i is worsen off. If not, then both the mechanism would have resulted in same set of allocations, which is *optimal.* Hence, TOAM-IComP result in an outcome that is Pareto optimal.
From our claim it must be true for any patient in categories $i \dots k$. Hence, the outcome computed by TOAM-IComP for any category $x_i \in x$ is the *Pareto optimal.* Similar argument can be given for the scenario when the patients have ties between the doctors in their respective preference list. □

Theorem 5. The allocation computed by TOAM-IComP is the *unique core allocation.*

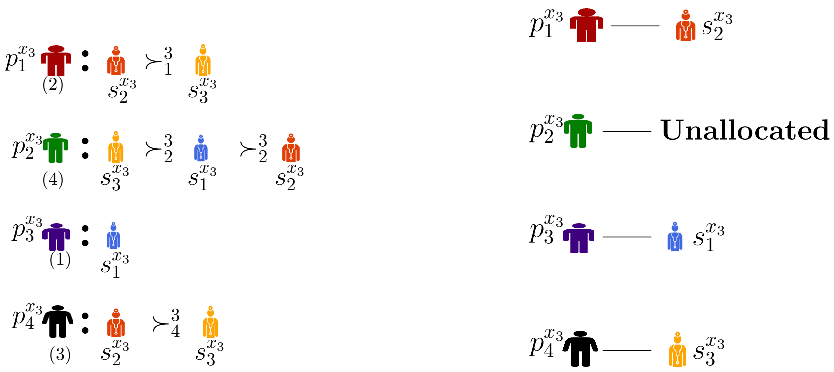
Proof. Fix a category x_i. It can be seen easily that TOAM-IComP results in a *unique allocation.* Now, talking about the resulted unique allocation to be "*The Core*". Let us say the two patients; patient i and j forms a coalition and reports their preference list by mutual collaboration (preference list other than their true preference list). As the random number assigned to the patients is independent of the preference lists reported by the patients. So, if at any particular time any of the patient i and j is considered, they will be allocated best doctor from the available doctors in their respective preference list. But, by mutual collaboration if they have manipulated their respective preference list then they will be getting either the allocation that they would have got if they haven't manipulated the list or the worse. It means that, by mutual interaction the patients can't gain. So, TOAM-IComP results in *core allocation.* From our claim it must be true for any agents in any category $i \in 1 \dots k$. Hence, TOAM-IComP results in *unique core allocation.*

Table 1. Running time and Economic properties of the proposed mechanisms

		Economic properties		
Proposed mechanisms	**Running time**	**The Core**	**Truthfulness**	**Pareto optimality**
RanPAM	$O(kn)$	✗	✗	✗
TOAM	$O(kn^2)$	✓	✓	✓
TOAM-IComP	$O(kn^2)$	✓	✓	✓

Similar argument can be given for the scenario when the patients have ties between the doctors in their respective preference list. □

Example 4: The detailed functioning of TOAM-IComP for category x_3 is illustrated in Figure 6. The number of patients is $n = 4$ and the number of expert consultants (or doctors) is $m = 3$. The strict preference ordering given by the patient set $\mathcal{P}_3$ is shown in Figure 6. Following line 3-12 of Algorithm 7, we generate the random numbers for the patients in $\mathcal{P}_3$. Now, based on the random number assigned as shown in Figure 6(a), first the patient $p_3^{x_3}$ is selected and assigned expert consultant $s_1^{x_3}$ from his preference list. Similarly, the remaining patients $p_1^{x_3}$, $p_4^{x_3}$, and $p_2^{x_3}$ are selected in the presented order and assigned the doctors $s_2^{x_3}$, $s_3^{x_3}$, and none respectively. The final allocation of *patient − doctor* pair is shown in Figure 6.



(a) Strict preference ordering (b) Final allocation

Figure 6. Detailed functioning of TOAM-IComP

4. Case 2: Hiring Experts Problem with $\alpha > 0$

In this section, the more realistic scenario of the doctors hiring problem is investigated, where unlike the *zero* budget case the patients are having some *positive* budget for the consultancy purpose. In *positive* budget case, first the more relaxed version of the problem is studied with a patient having some *positive* budget admitted to a hospital. The goal is to assign the set of experts to a patient so that the total payment made to the experts are within patient's budget. In the more general version, we have considered the case where there are multiple patients admitted to different hospitals and each patient is having some *positive* budget. On the other side of the market there are multiple doctors. The goal is to allocate the doctors to the patients so that the total payment made to the hired doctors are within patient's individual budget. It is to be noted that, the problem is studied under the consideration that doctors are not aware about the hiring concept.

In this section we have developed mechanisms: **No***n*-**t***ruthful* **b***udget* **c***onstraint (NoTBC) mechanism* motivated by (Khuller *et al.* 1999), and **T***ruthful* **b***udget* **c***onstraint (TBC) mechanism* motivated by (Singer 2012, 2010).

4.1. System Model and Problem Formulation

In this section, we formalize the doctors hiring problem where the multiple doctors are hired from outside of the hospital, for a patient having budget $\boldsymbol{\mathcal{B}'}$. The patient's budget $\boldsymbol{\mathcal{B}'}$ is a *public* information. The hospital to which a patient is admitted is having an accumulated, publicly known budget $\boldsymbol{\mathcal{B}}$, which will be utilized to inform about the hiring concept to the substantial number of ECs.

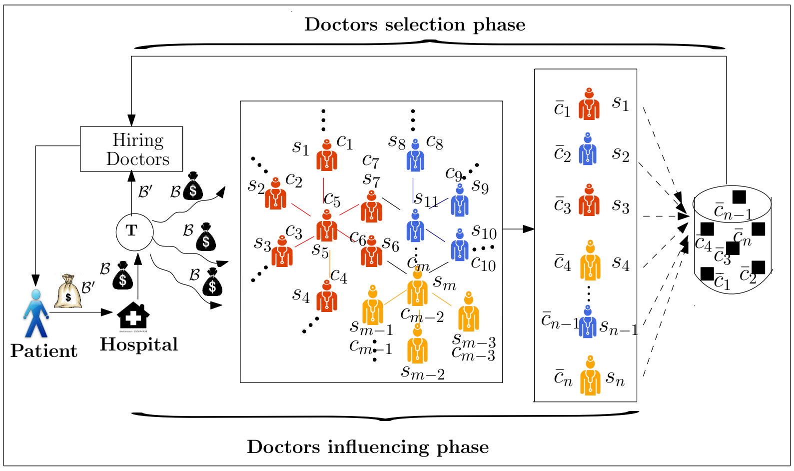


Figure 7. System model

The set of ECs is given as $\mathcal{S} = \{s_1, s_2, \ldots, s_m\}$; where each EC $s_i \in \mathcal{S}$ is assumed to be professionally connected with some $\chi_i \subseteq \mathcal{S} \backslash \{s_i\}$. The professional connections are given by a social graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of nodes representing ECs and $\mathcal{E}$ is the set of edges representing their professional connections in the social graph. Each s_i is associated with a hospital $\hbar_i \in \boldsymbol{\mathcal{H}}$.

Our model consists of two fold process. In the `first fold`, there is a social graph that is represented as $\mathcal{G}(\mathcal{V}, \mathcal{E})$ and publicly known *expert consultant activation function* given as $\boldsymbol{\mathcal{I}} : 2^{\mathcal{S}} \rightarrow \boldsymbol{\mathcal{R}}_{\geq 0}$. Given the subset $\varGamma \subseteq \mathcal{S}$ the value $\mathcal{I}(\varGamma)$ represents the expected number of doctors that are made aware about the hiring concept *i.e.* $I(\varGamma) = |\bigcup_{i \in \varGamma} \chi_i|$. Each node in the graph represents a doctor s_i that has a private cost (*aka* bid) c_i of being an initial adapter or the cost for spreading awareness about the hiring concept to other doctors. The cost vector of all the m doctors is given as: $\boldsymbol{\mathcal{C}} = \{c_1, c_2, \ldots, c_m\}$. It is to be noted that, the ECs are `rational` and `strategic` in nature. It means that, the ECs can gain by misreporting their private cost. As the ECs are `strategic`; so each $s_i \in \mathcal{S}$ may report their cost for being an initial adapter as c'_i instead of c_i in order to gain; where $c'_i \neq c_i$. The payment vector for the set $\varGamma$ is given as $\boldsymbol{\mathcal{P}}_{\varGamma} = \{\boldsymbol{\mathcal{P}}_{\varGamma_1}, \boldsymbol{\mathcal{P}}_{\varGamma_2}, \ldots, \boldsymbol{\mathcal{P}}_{\varGamma_k}\}$; where $\boldsymbol{\mathcal{P}}_{\varGamma_i}$ is the payment of $s_i \in \varGamma$. The objective of the first fold is to maximize the *expert consultant activation function* while the total payment is at most hospital's budget $\boldsymbol{\mathcal{B}}$.

In the `second fold`, we have a set of doctors consisting of doctors acted as leaders in the *first fold* and the aware doctors given as $\hat{\mathcal{S}} = \{s_1, s_2, \ldots, s_{i-1}, s_i, \ldots, s_n\}$ such that $n \leq m$. The quality vector of all the m ECs is given as $\boldsymbol{\mathcal{Q}} = \{\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_m\}$, where $\mathcal{Q}_i \in \boldsymbol{\mathcal{Q}}$ is the quality of i^{th} doctor. In general, the quality $\mathcal{Q}_i$ of a doctor s_i can be estimated using various parameters calculated later in the section. The publicly known *quality* function is given as $\boldsymbol{\mathcal{D}} : 2^{\hat{\mathcal{S}}} \rightarrow \boldsymbol{\mathcal{R}}_{\geq \mathbf{0}}$. Given a subset $\Upsilon \subseteq \hat{\mathcal{S}}$, the value $\boldsymbol{\mathcal{D}}(\Upsilon)$ denotes the sum of the qualities of all the doctors in Υ *i.e.* $\boldsymbol{\mathcal{D}}(\Upsilon) = \sum_{i \in \Upsilon} \mathcal{Q}_i$. For this fold, each doctor $s_i \in \hat{\mathcal{S}}$ will bid afresh their cost (*private*) for providing consultancy to the patient and is given as $\bar{c}_i$. The cost vector of all the n doctors is given as: $\bar{\boldsymbol{\mathcal{C}}} = \{\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_n\}$. The `strategic` behaviour of the doctors is continued in this fold also; so each $s_i \in \hat{\mathcal{S}}$ may report their cost of consultancy as $\bar{c}_i'$ instead of $\bar{c}_i$ in order to gain; where $\bar{c}_i' \neq \bar{c}_i$. Our objective is to determine the subset $\Upsilon \in \{\xi | \sum_{i \in \xi} \bar{c}_i \leq \boldsymbol{\mathcal{B}}'\}$ for which $\boldsymbol{\mathcal{D}}(\Upsilon)$ is maximized and the total payment should not exceed the patient's budget $\boldsymbol{\mathcal{B}}'$. The payment vector of the set Υ is given as $\hat{\boldsymbol{\mathcal{P}}} = \{\hat{\boldsymbol{\mathcal{P}}}_1, \hat{\boldsymbol{\mathcal{P}}}_2, \ldots, \hat{\boldsymbol{\mathcal{P}}}_r\}$.

4.1.1. Quality Determination

The parameters that determine the quality of each doctor s_i are: (1) `qualification` of s_i given as q_i (2) `success rate` of s_i given as sr_i (3) `experience` of s_i given as e_i (4) `hospital` to which s_i belong given as $\hbar_i$. So, the quality of doctor s_i is given as: $\mathcal{Q}_i = (w_1 \cdot q_i + w_2 \cdot sr_i + w_3 \cdot e_i + w_4 \cdot \hbar_i)$; where, $w_i \in [0, 1]$ such that $\sum_i w_i = 1$. The weighted sum of the some of the parameters considered in our case will result in the quality of the doctors.

4.1.2. Budget Distribution and Utilization

In our scenario, each fold is utilizing the budget from two independent sources. Firstly, talking about the hospital's budget it can be thought of as 1) the accumulated fund from the previously admitted patients say adding 5-6% of the total fees of each patients to the hospital fund. 2) Donation to the hospital by high profile persons or communities. Next, the source of the budget utilized in the second phase is the patient itself.

Definition 5 (*Marginal contribution* (Singer 2012))**.** The marginal contribution of an EC $s_i \in \mathcal{S}$ is the number of ECs informed about the hiring concept by the EC s_i given the set of $i-1$ ECs *i.e.* Γ_{i-1} already selected as the leaders. Mathematically, the marginal contribution of i^{th} EC given Γ_{i-1} is defined as: $\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_i}(\Gamma_{i-1}) = \boldsymbol{\mathcal{I}}(\Gamma_{i-1} \cup \{s_i\}) - \boldsymbol{\mathcal{I}}(\Gamma_{i-1})$

Definition 6 (*Quality contribution* (Singer 2010))**.** The quality contribution of an EC $s_i \in \mathcal{S}$ given a subset Υ_{i-1} of ECs already been selected is given as: $\boldsymbol{\mathcal{D}}_i(\Upsilon_{i-1}) = \boldsymbol{\mathcal{D}}(\Upsilon_{i-1} \cup \{s_i\}) - \boldsymbol{\mathcal{D}}(\Upsilon_{i-1})$ where $\boldsymbol{\mathcal{D}}(\Upsilon_i)$ denotes the sum of the qualities of all the doctors in Υ_i *i.e.* $\boldsymbol{\mathcal{D}}(\Upsilon_i) = \sum_i \mathcal{Q}_i$ given $\Upsilon_i = \{1, \ldots, i\}$ and $\boldsymbol{\mathcal{D}}(\Upsilon_0) = 0$ as $\Upsilon_0 = \phi$.

4.2. Proposed Mechanisms in Positive Budget Case

In this section, we present proposed mechanisms: **No***n*-**t***ruthful* **b***udget* **c***onstraint (NoTBC) mechanism* motivated by (Khuller *et al.* 1999) and **T***ruthful* **b***udget* **c***onstraint (TBC) mechanism* motivated by (Singer 2012, 2010).

4.2.1. NoTBC mechanism

It is a two pass mechanism consisting of **No***n*-**t***ruthful* **b***udget* **c***onstraint* **l***eader* **i***dentification (NoTBC-LI) mechanism* and **No***n*-**t***ruthful* **b***udget* **c***onstraint* **d***octor* **s***election (NoTBC-DS) mechanism.*

4.2.1.1. Outline of NoTBC mechanism. The central idea of the *NoTBC mechanism* is given.

NoTBC mechanism

Doctors Influencing Phase:

(1) Pick the doctor with the highest marginal contribution per cost value.
(2) Next, the check is made whether the cost for being an initial adapter of the doctor under consideration is less than or equal to the hospital's available budget or not. If yes, then he/she will be selected as the leader, otherwise not.
(3) Maintain the set of doctors aware about the hiring concept and reduce the hospital's available budget by the cost of the doctor selected as the leader.
(4) Remove the doctor selected as the leader from the doctor list.
(5) Follow step 1-4 until the doctor list become empty.
(6) Next, the payment of the selected doctors will be their revealed cost for being an initial adapter.

Doctors Selection Phase:

(1) Pick the doctor with highest quality contribution per cost value.
(2) Next, if the cost of consultancy of the doctor under consideration is less than or equal to the patient's available budget, then he/she will be selected for the consultancy purpose, otherwise not.
(3) Reduce the patient's available budget by the cost of consultancy of the selected doctor.
(4) Remove the doctor from the doctors list.
(5) Follow step 1-4 until the doctor set becomes empty.
(6) Next, the payment of the selected doctors will be their revealed cost of consultancy.

4.2.1.2. Detailed NoTBC mechanism. It is a two pass mechanism consisting of **No***n*-**t***ruthful* **b***udget* **c***onstraint* **l***eader* **i***dentification (NoTBC-LI) mechanism* and **No***n*-**t***ruthful* **b***udget* **c***onstraint* **d***octor* **s***election (NoTBC-DS) mechanism.*

NoTBC-LI mechanism In each iteration of *while* loop, a doctor with maximum marginal contribution per cost among the available doctors is considered and is selected only if its cost for being an initial adapter is less than the hospital's available budget. The payment of each doctors as a leader is their revealed cost.

NoTBC-DS Mechanism In each iteration of *while* loop, a doctor with maximum quality contribution per cost among the selected doctors by `NoTBC-LI mechanism` is considered and is hired only if its cost for the consultancy is less than the patient's available budget. The payment of each hired doctors is their revealed cost of consultancy.

Algorithm 8: NoTBC-LI mechanism ($\mathcal{G}$, $\mathcal{S}$, $\boldsymbol{\mathcal{B}}$, $\mathcal{C}$)

Output: $\hat{\mathcal{S}} \leftarrow \phi$, $\boldsymbol{\mathcal{P}}_{\Gamma} \leftarrow \phi$

1 $\bar{\mathcal{S}} \leftarrow \phi$ `// Set containing all the informed doctors.`

2 **while** $\mathcal{S} \neq \phi$ **do**

3 $\quad s_i \leftarrow argmax_{j \in \mathcal{S}} \left[\frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_j}(\Gamma_{j-1})}{c_j} \right]$

4 $\quad$ **if** $c_i \leq \boldsymbol{\mathcal{B}}$ **then**

5 $\quad\quad \Gamma \leftarrow \Gamma \cup \{s_i\}$; $\bar{\mathcal{S}} \leftarrow \bar{\mathcal{S}} \cup \{\chi_i\}$; $\boldsymbol{\mathcal{B}} \leftarrow \boldsymbol{\mathcal{B}} - c_i$

6 $\quad$ **end**

7 $\quad \mathcal{S} \leftarrow \mathcal{S} \setminus \{s_i\}$

8 **end**

9 $\hat{\mathcal{S}} = \Gamma \cup \bar{\mathcal{S}}$

10 **for** *each* $s_i \in \Gamma$ **do**

11 $\quad \boldsymbol{\mathcal{P}}_{\Gamma_i} \leftarrow c_i$; $\boldsymbol{\mathcal{P}}_{\Gamma} \leftarrow \boldsymbol{\mathcal{P}}_{\Gamma} \cup \{\boldsymbol{\mathcal{P}}_{\Gamma_i}\}$

12 **end**

13 return $\hat{\mathcal{S}}$, $\boldsymbol{\mathcal{P}}_{\Gamma}$

Algorithm 9: NoTBC-DS mechanism ($\hat{\mathcal{S}}$, $\boldsymbol{\mathcal{B}'}$, $\bar{\boldsymbol{\mathcal{C}}}$)

Output: $\Upsilon \leftarrow \phi$, $\hat{\boldsymbol{\mathcal{P}}} \leftarrow \phi$

1 **while** $\hat{\mathcal{S}} \neq \phi$ **do**

2 $\quad s_i \leftarrow argmax_{j \in \hat{\mathcal{S}}} \frac{\boldsymbol{\mathcal{D}}_j(\Upsilon_{j-1})}{\bar{c}_j}$

3 $\quad$ **if** $\bar{c}_i \leq \boldsymbol{\mathcal{B}'}$ **then**

4 $\quad\quad \Upsilon \leftarrow \Upsilon \cup \{s_i\}$; $\boldsymbol{\mathcal{B}'} \leftarrow \boldsymbol{\mathcal{B}'} - \bar{c}_i$

5 $\quad$ **end**

6 $\quad \hat{\mathcal{S}} \leftarrow \hat{\mathcal{S}} \setminus \{s_i\}$

7 **end**

8 **for** *each* $s_i \in \Upsilon$ **do**

9 $\quad \hat{\boldsymbol{\mathcal{P}}}_i \leftarrow \bar{c}_i$; $\hat{\boldsymbol{\mathcal{P}}} \leftarrow \hat{\boldsymbol{\mathcal{P}}} \cup \{\hat{\boldsymbol{\mathcal{P}}}_i\}$

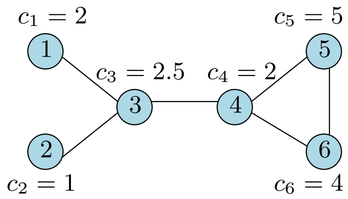
10 **end**

11 return Υ, $\hat{\boldsymbol{\mathcal{P}}}$

Example 5: Figure 8(a) show the initial configuration of the social graph along with cost distribution, and marginal contribution (m.c.). The quality vector of the nodes is given as: $\boldsymbol{\mathcal{Q}} = \{5,\ 1,\ 3,\ 5,\ 4,\ 5\}$. Higher the value, higher will be the quality. For understanding purpose we are taking the quality of the doctors as an integer value but in general it may not be the case. It is to be noted that the unit of cost and budget is taken as \$. We have considered hospital's budget to be 5. Using line 3 of the `Algorithm 8` the node 4 is considered. The condition $2 \leq 5$ for node 4 is satisfied. So, $\Gamma = \{4\}$ and $\bar{\mathcal{S}} = \{3,\ 5,\ 6\}$ as shown in Figure 8(b). Next, node 3 will be considered and $2.5 \leq 3$ for node 3 is satisfied. So, $\Gamma = \{4,\ 3\}$ and $\bar{\mathcal{S}} = \{3,\ 5,\ 6,\ 1,\ 2,\ 4\}$ as shown in Figure 8(c). So, we have $\hat{\mathcal{S}} = \{3,\ 5,\ 6,\ 1,\ 2,\ 4\}$. The payment of node 3 and node 4 are 2.5 and 2 respectively as shown in Figure 8(d). The total payment made to doctors $= 4.5 \leq 5$ (hospital's budget).

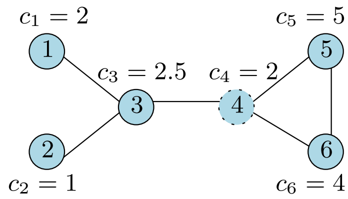
We have considered patient's budget to be 6. Using line 3 of `Algorithm 9` Node 1 is considered and selected as $2 \leq 6$ condition is satisfied. Next, Node 4 will be considered and $2 \leq 4$ for node 4 is satisfied. Next, Node 6 will be considered and $4 \leq 2$ for node 6 not satisfied. Next, node 3 will be considered and $2.5 \leq 2$ for node 3 is not satisfied. Next, node 2 will be considered and $1 \leq 2$ for node 2 is satisfied. Next, no node will be selected as cost of the remaining nodes is higher than remaining budget *i.e* 1. So,

$\Gamma = \{1,\ 4,\ 2\}$. The payment made to node 1, node 2, and node 4 is 2, 1, and 2. Total payment 5 is less than patient's budget 6.



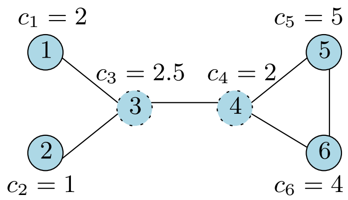
Node	1	2	3	4	5	6
m.c.	1	1	3	3	2	2
$\frac{m.c.}{c}$	0.5	1	1.2	1.5	0.4	0.5

(a) Initial configuration



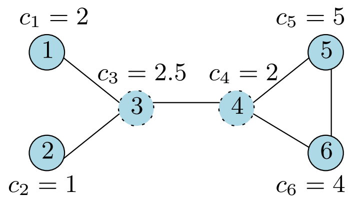
Node	1	2	3	4	5	6
m.c.	0	0	3	0	1	1
$\frac{m.c.}{c}$	0	0	1.2	0	0.2	.25

(b) Intermediate configuration



Node	1	2	3	4	5	6
m.c.	0	0	0	0	0	0
$\frac{m.c.}{c}$	0	0	0	0	0	0

(c) Intermediate configuration



Node	1	2	3	4	5	6
m.c.	0	0	0	0	0	0
price	0	0	2.5	2	0	0

(d) Payment determination

Figure 8. Working example of NoTBC mehanism

Lemma 1. The NoTBC mechanism is computationally efficient.

Proof. In `NoTBC-LI`, for m iteration of *while* loop we have $O(m^2)$. Thus, the running time of `NoTBC-LI` is $O(m^2)$. In `NoTBC-DS`, for m iteration (in worst case) of *while* loop we have $O(m^2)$. Thus, the running time of `NoTBC-DS` is $O(m^2)$. In both the cases, the payment determination will be linear in m. Thus, the computational complexity of `NoTBC` is given as $O(m^2)$. □

Lemma 2. The NoTBC mechanism is individually rational.

Proof. From line 11 of Algorithm 8, we can see $\boldsymbol{\mathcal{P}}_{\Gamma_i} = c_i$ for each $s_i \in \Gamma$. Line 9 in Algorithm 9 shows that $\hat{\boldsymbol{\mathcal{P}}}_i = \bar{c}_i$. Therefore, we have payment for any winner is its cost. Hence, NoTBC mechanism is individually rational. □

Lemma 3. The NoTBC mechanism is budget feasible.

Proof. As it is clear that a doctor is included in the winning set only when the given condition in line 4 of Algorithm 8 and line 3 of Algorithm 9 is satisfied. As the payment in case of NoTBC is equal to the cost; the total payment will be at most the budget. Hence, NoTBC mechanism is budget feasible. □

4.2.2. *TBC Mechanism*

It is a two pass mechanism consists of **T***ruthful* **b***udget* **c***onstraint* **l***eader* **i***dentification* (TBC-LI) and **T***ruthful* **b***udget* **c***onstraint* **d***octor* **s***election* (TBC-DS) mechanisms.

4.2.2.1. Outline of TBC mechanism. The idea of the *TBC mechanism* is presented.

> **<u>TBC mechanism</u>**
>
> **Doctors Influencing Phase:**
>
> (1) Each time consider a doctor i with maximum marginal contribution per cost among the available doctors. But, the doctor i will be selected as the leader only when it satisfies the following condition: $c_i \leq \frac{\boldsymbol{\mathcal{B}}}{2}\left(\frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_i}(\Gamma_{i-1})}{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_i}(\Gamma_{i-1})+\boldsymbol{\mathcal{I}}(\Gamma_{i-1})}\right)$.
>
> (2) *Payment rule:* Drag the agent i outside the market and repeat step 1. Determine the largest index ℓ in the sorting of Γ' doctors (determined without s_i) that satisfies the condition in step 1.
>
> (3) Calculate $\boldsymbol{\mathcal{C}}_i^j = \frac{\boldsymbol{\mathcal{M}}^j_{\boldsymbol{\mathcal{C}}_i}(\Gamma'_{j-1})\cdot c_j}{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_j}(\Gamma'_{j-1})}$: *Maximal cost that doctor s_i can declare in order to be allocated instead of doctor in the j^{th} place in the sorting.*
>
> (4) Calculate $\boldsymbol{\Pi}_i^j = \frac{\boldsymbol{\mathcal{B}}\cdot\boldsymbol{\mathcal{M}}^j_{\boldsymbol{\mathcal{C}}_i}(\Gamma'_{j-1})}{\boldsymbol{\mathcal{I}}(\Gamma'_{j-1}\cup\{s_i\})}$: *Threshold payment.*
>
> (5) $\boldsymbol{\mathcal{P}}_{\Gamma_i} = max_{j\in[1..\ell+1]}\{min\{\boldsymbol{\mathcal{C}}_i^j, \boldsymbol{\Pi}_i^j\}\}$
>
> **Doctors Selection Phase:**
>
> (1) Doctors are sorted based on $\frac{\boldsymbol{\mathcal{D}}_i(\Upsilon_{i-1})}{\bar{c}_i}$ for all $s_i \in \hat{\mathcal{S}}$.
>
> (2) Greedily consider the doctor from the sorted list (in decreasing order), but the doctor will be hired only when it satisfies the following condition: $\bar{c}_i \leq \boldsymbol{\mathcal{B}}' \cdot \left(\frac{\boldsymbol{\mathcal{D}}_i(\Upsilon_{i-1})}{\boldsymbol{\mathcal{D}}_i(\Upsilon_{i-1})+\boldsymbol{\mathcal{D}}(\Upsilon_{i-1})}\right)$.
>
> (3) For each winning doctors the payment is: $\hat{\boldsymbol{\mathcal{P}}}_i = min\left\{\frac{\boldsymbol{\mathcal{D}}_i(\Upsilon_{i-1})\cdot\boldsymbol{\mathcal{B}}'}{\sum_{i\in\Upsilon}\boldsymbol{\mathcal{D}}(\Upsilon_i)}, \frac{\boldsymbol{\mathcal{D}}_i(\Upsilon_{i-1})\cdot\bar{c}_{\ell+1}}{\boldsymbol{\mathcal{D}}_{\ell+1}(\Upsilon_\ell)}\right\}$

4.2.2.2. Detailed TBC Mechanism. For first fold of hiring problem, we propose a TBC-LI mechanism motivated by (Singer 2012, 2010).

Allocation rule In this, a doctor with maximum marginal contribution per cost among the available doctors is considered. But the doctor is selected as the leader only when the ratio between their cost as the initial adapter and budget is less than or equal to half of the ratio between their marginal contribution and the value of the selected subset.

Algorithm 10: TBC-LI allocation mechanism ($\mathcal{G}$, $\mathcal{S}$, $\boldsymbol{\mathcal{B}}$, $\mathcal{C}$)

```
Output: Γ ← φ, Ŝ ← φ
S̄ ← φ; s_i ← argmax_{j∈S} [ M_{C_j}(Γ_{j-1}) / c_j ]
while c_i ≤ (B/2) ( M_{C_i}(Γ_{i-1}) / (M_{C_i}(Γ_{i-1}) + I(Γ_{i-1})) ) do
    Γ ← Γ ∪ {s_i}; S̄ ← S̄ ∪ {χ_i}
    s_i ← argmax_{j∈S\Γ} [ M_{C_j}(Γ_{j-1}) / c_j ]
end
Ŝ = Γ ∪ S̄
return Γ and Ŝ
```

Example 6: Figure 9(a) show the initial configuration of the social graph along with cost distribution, and marginal contribution (m.c.). The quality vector of the nodes is given as: $\boldsymbol{\mathcal{Q}} = \{5,\ 1,\ 3,\ 5,\ 4,\ 5\}$. Higher the value, higher will be the quality. For understanding purpose we are taking the quality of the doctors as an integer value but in general it may not be the case. It is to be noted that the unit of cost and budget is taken as \$. We have considered hospital's budget to be 10. Using line 1 of the `Algorithm 10` the node 4 is considered. The condition $2 \leq 5 \cdot (\frac{3}{3+0})$ for node 4 is satisfied. So, $\Gamma = \{4\}$ and $\bar{\mathcal{S}} = \{3,\ 5,\ 6\}$. Next, node 3 will be considered and $2.5 \leq 5 \cdot (\frac{3}{3+3})$ for node 3 is satisfied. So, $\Gamma = \{4,\ 3\}$ and $\bar{\mathcal{S}} = \{3,\ 5,\ 6,\ 1,\ 2,\ 4\}$. So, we have $\hat{\mathcal{S}} = \{3,\ 5,\ 6,\ 1,\ 2,\ 4\}$.

Node	1	2	3	4	5	6
m.c.	1	1	3	3	2	2

(a) Initial configuration

Node	1	2	3	4	5	6
m.c.	0	0	3	0	1	1

(b) Intermediate configuration

Figure 9. Detailed functioning of Algorithm 10

Payment rule The payment rule is motivated by (Singer 2010).

Algorithm 11: TBC-LI pricing mechanism (Γ, $\boldsymbol{\mathcal{B}}$, $\mathcal{C}$)

Output: $\boldsymbol{\mathcal{P}}_\Gamma \leftarrow \phi$

1 $\Gamma' \leftarrow \phi$, $\mathcal{S}' \leftarrow \phi$

2 **for** *each* $s_i \in \Gamma$ **do**

3 $\quad \mathcal{S}' \leftarrow \mathcal{S} \setminus \{s_i\}$

4 $\quad s_j \leftarrow argmax_{k \in \mathcal{S}'} \left[\frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{c}_k}(\Gamma'_{k-1})}{c_k}\right]$

5 $\quad$ **while** $c_j \leq \boldsymbol{\mathcal{B}}\left(\frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{c}_j}(\Gamma'_{j-1})}{\boldsymbol{\mathcal{M}}_{\boldsymbol{c}_j}(\Gamma'_{j-1}) + \boldsymbol{\mathcal{I}}(\Gamma'_{j-1})}\right)$ **do**

6 $\quad\quad \Gamma' \leftarrow \Gamma' \cup \{s_j\}$ `// ` Γ' ` is the set of leaders when ` s_i ` is not in the market.`

7 $\quad\quad \mathcal{S}' \leftarrow \mathcal{S}' \setminus \{s_j\}$

8 $\quad\quad s_j \leftarrow argmax_{k \in \mathcal{S}'} \left[\frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{c}_k}(\Gamma'_{k-1})}{c_k}\right]$

9 $\quad$ **end**

10 $\quad \Gamma' \leftarrow \Gamma' \cup \{s_j\}$ **for** *each* $s_j \in \Gamma'$ **do**

11 $\quad\quad$ Calculate $\boldsymbol{\mathcal{C}}_i^j = \frac{\boldsymbol{\mathcal{M}}^j_{\boldsymbol{c}_i}(\Gamma'_{j-1}) \cdot c_j}{\boldsymbol{\mathcal{M}}_{\boldsymbol{c}_j}(\Gamma'_{j-1})}$ and $\boldsymbol{\Pi}_i^j = \frac{\boldsymbol{\mathcal{B}} \cdot \boldsymbol{\mathcal{M}}^j_{\boldsymbol{c}_i}(\Gamma'_{j-1})}{\boldsymbol{\mathcal{I}}(\Gamma'_{j-1} \cup \{s_i\})}$

12 $\quad$ **end**

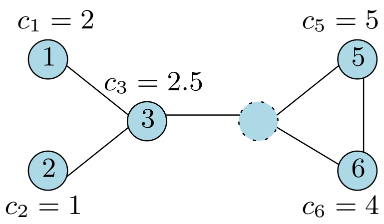
13 $\quad \boldsymbol{\mathcal{P}}_{\Gamma_i} \leftarrow max_{j \in [1..\ell+1]}\{min\{\boldsymbol{\mathcal{C}}_i^j, \boldsymbol{\Pi}_i^j\}\}$; $\boldsymbol{\mathcal{P}}_\Gamma \leftarrow \boldsymbol{\mathcal{P}}_\Gamma \cup \{\boldsymbol{\mathcal{P}}_{\Gamma_i}\}$

14 **end**

15 return $\boldsymbol{\mathcal{P}}_\Gamma$

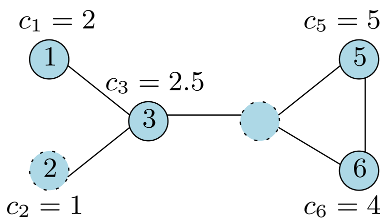
In this, for each doctor $s_i \in \Gamma$ consider running line $3 - 9$. Next, determine the largest index ℓ in the sorting of $|\Gamma'|$ doctors (determined without s_i) such that the ratio between their cost as the initial adapter and budget is less than or equal to the ratio between their marginal contribution and the value of the selected subset. Now for each point $j \in [1..\ell+1]$ find the maximal cost $\boldsymbol{\mathcal{C}}_i^j = \boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_i}^j(\Gamma'_{j-1}) \cdot \left(\frac{c_j}{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_j}(\Gamma'_{j-1})}\right)$ that doctor s_i can declare in order to be allocated instead of the doctor in the j^{th} place in the sorting; where $\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_i}^j(\Gamma'_{j-1})$ is the marginal contribution of doctor s_i when considered on j^{th} place is given as: $\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_i}^j(\Gamma'_{j-1}) = \boldsymbol{\mathcal{I}}(\Gamma'_{j-1} \cup \{s_i\}) - \boldsymbol{\mathcal{I}}(\Gamma'_{j-1})$. Now, if this cost does not exceed the threshold payment $\boldsymbol{\Pi}_i^j = \boldsymbol{\mathcal{B}} \cdot \frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_i}^j(\Gamma'_{j-1})}{\boldsymbol{\mathcal{I}}(\Gamma'_{j-1} \cup \{s_i\})}$ then the mechanism will declare s_i as the leader. Considering the maximum of the values at $j \in [1..\ell+1]$ results in the payment of s_i.

Example 7: Figure 10 shows the payment calculation of node 4. So, placing node 4 outside the market and utilizing line $3-9$ of Algorithm 11 on the configurations shown in Figure 10(a), Figure 10(b), and Figure 10(c) we find the critical point as $\ell = 2$ (index of node 3). Following Figure 10(d) at point 1 (index of node 2) the value $\mathcal{C}_4^1 = 3 \cdot (\frac{1}{1}) = 3$, and $\boldsymbol{\Pi}_4^1 = 10 \cdot (\frac{3}{3}) = 10$. So, $\min\{3, 10\} = 3$. Similarly, at point 2 (index of node 3) the value $\mathcal{C}_4^2 = 2 \cdot (\frac{2.5}{2}) = 2.5$, and $\boldsymbol{\Pi}_4^2 = 10 \cdot (\frac{2}{3}) = 6.66$. So, $\min\{2.5, 6.66\} = 2.5$. Considering point 3 (index of the first loser node $i.e$ node 6) we get $\mathcal{C}_4^3 = 2 \cdot (\frac{4}{1}) = 8$, and $\boldsymbol{\Pi}_4^3 = 10 \cdot (\frac{2}{5}) = 4$. So, $\min\{8,\ 4\} = 4$. The payment of node 4 is $\max\{3,\ 2.5,\ 4\} = 4$.



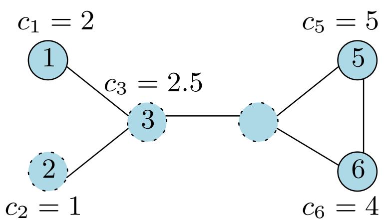
Node	1	2	3	5	6
m.c.	1	1	2	1	1

(a) Initial configuration



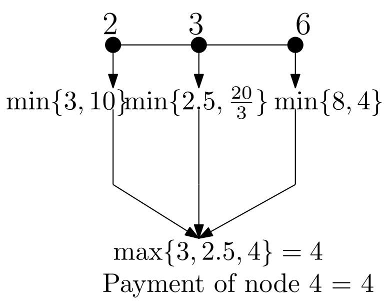
Node	1	2	3	5	6
m.c.	0	0	2	1	1

(b) Intermediate configuration



Node	1	2	3	5	6
m.c.	0	0	0	1	1

(c) Intermediate configuration



(d) Payment determination

Figure 10. Payment calculation of node 4

4.2.3. TBC-DS Mechanism

For the second fold of the doctors hiring problem, we propose a `TBC-DS mechanism` motivated by (Singer 2012, 2010).

Allocation rule In this, firstly the available doctors are sorted in decreasing order based on quality contribution by cost. Now, the doctors are greedily selected but will be hired only when the ratio of the selected doctor's cost of consultation and the patient's budget is less than or equal to the ratio between the quality contribution by the selected doctor and the value of the quality of the selected subset.

Algorithm 12: TBC-DS allocation mechanism ($\hat{\mathcal{S}}$, $\boldsymbol{\mathcal{B}}'$, $\bar{\boldsymbol{\mathcal{C}}}$)

Output: $\Upsilon \leftarrow \phi$
1 . $Sort(\hat{\mathcal{S}})$ `// Sorting based on` $\frac{\boldsymbol{\mathcal{D}}_i(\Upsilon_{i-1})}{\bar{c}_i}$ `for all` $s_i \in \hat{\mathcal{S}}$
2 **for** *each* $s_i \in \hat{\mathcal{S}}$ **do**
3   **if** $\frac{\bar{c}_i}{\boldsymbol{\mathcal{B}}'} \leq \left(\frac{\boldsymbol{\mathcal{D}}_i(\Upsilon_{i-1})}{\boldsymbol{\mathcal{D}}_i(\Upsilon_{i-1})+\boldsymbol{\mathcal{D}}(\Upsilon_{i-1})}\right)$ **then**
4         $\Upsilon \leftarrow \Upsilon \cup \{s_i\}$
5   **end**
6 **end**
7 return Υ

Example 8: Considering the set-up shown in Figure 9(a). We have utilized the same cost vector as given in Figure 6. The patient's budget is given as 8. The quality vector is given as $\boldsymbol{\mathcal{Q}} = \{5,\ 1,\ 3,\ 5,\ 4,\ 5\}$. The set of nodes informed by the leaders $\{4,\ 3\}$ is given as $\{3,\ 5,\ 6,\ 4,\ 1,\ 2\}$. So, the nodes 3, 5, 6, 4, 1, and 2 are sorted based on quality contribution per cost and is given as: $\{1, 4, 6, 3, 2, 5\}$. First node 1 is considered and the condition $2 \leq 8 \cdot (\frac{5}{5})$ for node 1 is satisfied. So, $\Upsilon = \{1\}$. Next, node 4 will be considered and the condition $2 \leq 8 \cdot (\frac{5}{10})$ for node 4 is satisfied. So, $\Upsilon = \{1,\ 4\}$.

Payment rule The `Payment rule` is motivated by (Singer 2010). For each s_i, it is defined as the minimum of the doctor's proportional share and the threshold payment. ℓ is the largest index that satisfies the condition in line 2 of Algorithm refalgo:10.

Algorithm 13: TBC-DS Pricing Mechanism (Υ, $\boldsymbol{\mathcal{B}}'$, $\bar{\boldsymbol{\mathcal{C}}}$)

Output: $\hat{\boldsymbol{\mathcal{P}}} \leftarrow \phi$
1 **for** *each* $s_i \in \Upsilon$ **do**
2     $\hat{\boldsymbol{\mathcal{P}}}_i \leftarrow min\left\{\frac{\boldsymbol{\mathcal{D}}_i(\Upsilon_{i-1})\cdot\boldsymbol{\mathcal{B}}'}{\sum_{i\in\Upsilon}\boldsymbol{\mathcal{D}}(\Upsilon_i)}, \frac{\boldsymbol{\mathcal{D}}_i(\Upsilon_{i-1})\cdot\bar{c}_{\ell+1}}{\boldsymbol{\mathcal{D}}_{\ell+1}(\Upsilon_\ell)}\right\}$; $\hat{\boldsymbol{\mathcal{P}}} \leftarrow \hat{\boldsymbol{\mathcal{P}}} \cup \hat{\boldsymbol{\mathcal{P}}}_i$
3 **end**
4 return $\hat{\boldsymbol{\mathcal{P}}}$

Example 9: The payment of doctors in $\Upsilon = \{1,\ 4\}$ is: $\hat{\boldsymbol{\mathcal{P}}}_1 = min\left\{\frac{5\times8}{10}, \frac{5\times4}{5}\right\} = 4$ and $\hat{\boldsymbol{\mathcal{P}}}_2 = min\left\{\frac{5\times8}{10}, \frac{5\times4}{5}\right\} = 4$.

Lemma 4. The TBC mechanism is computationally efficient.

Proof. In TBC-LI, line $1-5$ of Algorithm 10 is bounded above by m. In Algorithm 11, for each iteration of *for* loop line $3-14$ is bounded above by m. As we have m iterations in worst case, we have $O(m^2)$. Thus, the running time of TBC-LI mechanism is $O(m^2)$. In TBC-DS, line 1 of Algorithm 12 takes $O(m \lg m)$ time. Line $2-6$ takes $O(m)$ time. So, overall running time of Algorithm 12 is $O(m \lg m) + O(m) = O(m \lg m)$. Line $1-3$ of Algorithm 13 takes time $O(m)$. Thus, running time of TBC-DS mechanism is $O(m \lg m)$. The computational complexity of TBC mechanism is given as $O(m^2)+$

$O(m \lg m) = O(m^2)$. □

4.3. More General Setting

Till now, for simplicity purpose we have considered the set-up, where there is one patient having publicly known positive budget $\boldsymbol{\mathcal{B}'}$ admitted to a hospital and multiple doctors say m given as $\mathcal{S} = \{s_1, s_2, \ldots, s_m\}$. The goal is to allocate as many doctors as possible to a patient, so that the total payment made to the doctors is within patient's budget $\boldsymbol{\mathcal{B}'}$. But, one can think of the situation where there are multiple patients say k given as $\boldsymbol{\mathcal{P}} = \{p_1, p_2, \ldots, p_k\}$ may be admitted to k different hospitals and multiple doctors say m. Each of the patient p_i has a budget $\boldsymbol{\mathcal{B}'_i}$ which is *public* information. Also, each of the hospital report the consultancy time for the respective patient. Similar to the previous set-up, in this set-up it is assumed that the doctors are unaware about the hiring concept. In this model, the participating hospitals individual budget will be accumulated resulting in total budget say $\boldsymbol{\mathcal{B}}$ and will be utilized for the purpose of informing the substantial number of experts about the hiring concept. However, the source of each hospital's budget is similar to the previous scenario of case 2. Each of the doctor has the set of availability times for providing the consultancy. The goal is to allocate each patient $p_i \in \boldsymbol{\mathcal{P}}$ as many doctors as possible so that the total payment made to the hired doctors are within patients budget $\boldsymbol{\mathcal{B}'_i}$. Rest of the parameters for the general setting is similar to the previously discussed scenario in case 2.

For this setting, a mechanism is proposed for the `second fold` of the experts hiring problem named as **M***echanism for* **d***octors* **s***election in* **g***eneral* **s***etting* (MDS-GS). It is to be noted that, the mechanism for the `first fold` will be similar to the previous scenario of case 2.

4.3.1. Outline of MDS-GS

In this section, the underlying idea of the MDS-GS is presented.

MDS-GS

- Based on the reported availability time and consultancy time for consultancy, partition the agents set into two time slots *i.e.* `Morning slot (MS)` and `Evening slot (ES)`.
- In each time slot, assign a distinct random number to each of the available patients.
- Next, each of the patient is considered sequentially and the mechanisms proposed for the single patient setting, are executed for the two slots.
- Remove the set of experts allocated to a patient in current iteration from the consultancy arena.
- Iterate, until all the patients are considered.

4.3.2. Detailed MDS-GS

The input to the MDS-GS are: the set of n informed set of doctors *i.e.* $\hat{\mathcal{S}}$, the set of k available patients *i.e.* $\boldsymbol{\mathcal{P}}$. In line 1, the variable ℓ is initialized to 0. In line 2 the *slot* data structure holds the available slots *i.e.* MS and ES. In line 3 the agents are partitioned in morning slot (MS) and Evening slot (ES) based on their reported availability time and consultancy time. For each slot, using line $5-15$ the random numbers are generated and is assigned to each of the patients. In line 16, the patients present in each of the j^{th} slot is sorted based on the random number assigned. Line 17-22 executes the Algorithm 9, Algorithm 12, and Algorithm 13 for each of the patients in each time slot and each time deletes the doctors already allocated.

Algorithm 14: MDS-GS ($\hat{\mathcal{S}}$, $\boldsymbol{\mathcal{P}}$)

1 $\ell \leftarrow 0$
2 $Slot = \{MS,\ ES\}$
3 Based on the reported availability time and consultancy time for consultancy, partition the agents into MS and ES.
4 **for** *each* $j \in Slot$ **do**
5   **for** i = *1 to* $|j \cdot \boldsymbol{\mathcal{P}}|$ `// |j·P| is the number of patients in` j^{th} `slot.`
6   **do**
7 $\mathcal{R} \leftarrow \mathcal{R} \cup \{i\}$
8   **end**
9   **for** i = *1 to* $|j \cdot \boldsymbol{\mathcal{P}}|$ **do**
10   swap $\mathcal{R}[i]$ with $\mathcal{R}[Random(i, len(j))]$
11   **end**
12   **for** *each* $p_i \in j$ **do**
13 $Assign(p_i, \mathcal{R}[\ell])$
14 $\ell \leftarrow \ell + 1$
15   **end**
16 $\mathcal{P}^j \leftarrow$ Sort($\mathcal{P}^j$) `// Sort` $\mathcal{P}^j$ `based on random number assigned;` $\mathcal{P}^j$ `is the patient set in` j^{th} `slot.`
17   **for** *each* $p_i \in \mathcal{P}^j$ **do**
18   Execute Algorithm 9 `// For non-truthful mechanism.`
19   Remove allocated set of doctors by Algorithm 9 from slot j
20   Execute Algorithm 12 and Algorithm 13 `// For truthful mechanism.`
21   Remove the allocated set of doctors by Algorithm 12 from slot j
22   **end**
23 **end**

4.3.3. Upper Bound Analysis

The *random* number generator in line 5-15 is motivated by (Cormen *et al.* 2009) and is bounded above by k. The outer *for* loop in line $4-23$ will executed for 2 times and is bounded above by $O(1)$. In line 16, the sorting is done that is bounded above by $k \lg k$. Each iteration of *for* loop in line $17-22$ is bounded above by $O(m^2)$ (calculated earlier). Mathematically,

$$
\left.\begin{aligned}
T(n) &= \sum_{i=1}^{2}\left(\left(\sum_{i=1}^{k} O(1)\right) + \left(O(k \lg k)\right) + \left(\sum_{i=1}^{k} O(m^2)\right)\right) \\
&= \left(\sum_{i=1}^{2}\sum_{i=1}^{k} O(1)\right) + \left(\sum_{i=1}^{2} O(k \lg k)\right) + \left(\sum_{i=1}^{2}\sum_{i=1}^{k} O(m^2)\right) \\
&= O\left(\sum_{i=1}^{2}\sum_{i=1}^{k} 1\right) + O\left(\sum_{i=1}^{2} k \lg k\right) + O\left(\sum_{i=1}^{2}\sum_{i=1}^{k} m^2\right) \\
&= O\left(\sum_{i=1}^{2} k\right) + O\left(k \lg k\right) + O\left(\sum_{i=1}^{2} km^2\right) \\
&= O\left(k\right) + O\left(k \lg k\right) + O\left(km^2\right) \\
T(n) &= O\left(km^2\right)
\end{aligned}\right\}
$$

4.4. Analysis of Proposed Mechanisms

Lemma 5. In TBC-LI, the total payment made to the doctors are within hospital's budget $\boldsymbol{\mathcal{B}}$.

Proof. The proof is motivated by (Singer 2012). As the maximum payment that any winning EC i can be paid is $\mathbf{\Pi}_i^k = \frac{\boldsymbol{\mathcal{B}} \cdot \boldsymbol{\mathcal{M}}_{\mathcal{C}_k}(\Gamma_{i-1})}{\boldsymbol{\mathcal{I}}(\Gamma_{i-1} \cup \{i\})}$. The total payment of the ECs as leaders *i.e.* $\mathcal{T}_{\mathcal{I}}^{\mathcal{P}}$ is given as:

$$
\mathcal{T}_{\mathcal{I}}^{\mathcal{P}} = \sum_{i \in \hat{\mathcal{S}}} \mathcal{P}_i = \sum_{i=1}^{k} \boldsymbol{\mathcal{B}} \cdot \frac{\boldsymbol{\mathcal{M}}_{\mathcal{C}_i}(\Gamma_{i-1})}{\boldsymbol{\mathcal{I}}(\Gamma_{i-1} \cup \{s_i\})} = \sum_{i=1}^{k} \boldsymbol{\mathcal{B}} \cdot \frac{\boldsymbol{\mathcal{M}}_{\mathcal{C}_i}(\Gamma_{i-1})}{\boldsymbol{\mathcal{I}}(\Gamma_i)}
$$

$$
\leq \frac{\boldsymbol{\mathcal{B}}}{\boldsymbol{\mathcal{I}}(\Gamma_k)} \cdot \sum_{i=1}^{k} \boldsymbol{\mathcal{M}}_{\mathcal{C}_i}(\Gamma_{i-1}) = \frac{\boldsymbol{\mathcal{B}}}{\boldsymbol{\mathcal{I}}(\Gamma_k)} \cdot \sum_{i=1}^{k} \underbrace{\boldsymbol{\mathcal{I}}(\Gamma_{i-1} \cup \{s_i\})}_{\substack{\text{Informed ECs} \\ \text{by set } \Gamma_i}} - \underbrace{\boldsymbol{\mathcal{I}}(\Gamma_{i-1})}_{\substack{\text{Informed ECs} \\ \text{by set } \Gamma_{i-1}}}
$$

$$
= \frac{\boldsymbol{\mathcal{B}}}{\boldsymbol{\mathcal{I}}(\Gamma_k)} \cdot \sum_{i=1}^{k} \underbrace{\boldsymbol{\mathcal{I}}(\Gamma_i)}_{\substack{\text{Informed} \\ \text{ECs by} \\ \text{set } \Gamma_i}} - \underbrace{\boldsymbol{\mathcal{I}}(\Gamma_{i-1})}_{\substack{\text{Informed ECs} \\ \text{by set } \Gamma_{i-1}}} \Rightarrow \mathcal{T}_{\mathcal{I}}^{\mathcal{P}} \leq \boldsymbol{\mathcal{B}}
$$

Hence, it is proved that the incentive compatible total payment do not exceed the budget. This holds true for the general setting. □

Lemma 6. In TBC-LI mechanism, if any doctor s_i comes ahead of its current position say $i' < i$ by declaring a cost $c_{i'} < c_i$ then,

$$
\frac{\boldsymbol{\mathcal{B}}}{2}\left(\frac{\boldsymbol{\mathcal{M}}_{\mathcal{C}_{i'}}(\Gamma_{i'-1})}{\boldsymbol{\mathcal{M}}_{\mathcal{C}_{i'}}(\Gamma_{i'-1}) + \boldsymbol{\mathcal{I}}(\Gamma_{i'-1})}\right) > \frac{\boldsymbol{\mathcal{B}}}{2}\left(\frac{\boldsymbol{\mathcal{M}}_{\mathcal{C}_i}(\Gamma_{i-1})}{\boldsymbol{\mathcal{M}}_{\mathcal{C}_i}(\Gamma_{i-1}) + \mathcal{I}(\Gamma_{i-1})}\right)
$$

Proof. If the EC i by reporting $c_{i'}$ moves at position i' such that $i' < i$ as depicted in *Figure* 11 below:

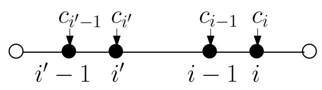


Figure 11. Pictorial representation

From the definition of $\mathcal{I}(\cdot)$ we can say: $\mathcal{I}(\Gamma_{i'-1}) < \mathcal{I}(\Gamma_{i-1})$. As the set $\Gamma_{i'-1}$ is smaller as compared to the set Γ_{i-1}, so from the definition of the monotone sub-modular marginal contribution property, it can be said:

$$\underbrace{\mathcal{M}_{\mathcal{C}_{i'}}(\Gamma_{i'-1})}_{\substack{\text{Marginal contribution}\\ \text{of } i' \text{ given } \Gamma_{i'-1}}} > \underbrace{\mathcal{M}_{\mathcal{C}_{i}}(\Gamma_{i-1})}_{\substack{\text{Marginal contribution}\\ \text{of } i \text{ given } \Gamma_{i-1}}} \tag{1}$$

The number of ECs leaders by the set $\Gamma_{i'}$ will be less than the number of ECs leaders by Γ_i. Mathematically,

$$\underbrace{\mathcal{M}_{\mathcal{C}_{i'}}(\Gamma_{i'-1})}_{\substack{\text{Marginal contribution}\\ \text{of } i' \text{ given } \Gamma_{i'-1}}} + \mathcal{I}(\Gamma_{i'-1}) < \underbrace{\mathcal{M}_{\mathcal{C}_{i}}(\Gamma_{i-1})}_{\substack{\text{Marginal contribution}\\ \text{of } i \text{ given } \Gamma_{i-1}}} + \mathcal{I}(\Gamma_{i-1})$$

$$\frac{1}{\mathcal{M}_{\mathcal{C}_{i'}}(\Gamma_{i'-1}) + \mathcal{I}(\Gamma_{i'-1})} > \frac{1}{\mathcal{M}_{\mathcal{C}_{i}}(\Gamma_{i-1}) + \mathcal{I}(\Gamma_{i-1})} \tag{2}$$

Combining equation 1, equation 2 and multiplying both side by $\frac{\mathcal{B}}{2}$, we get

$$\frac{\mathcal{B}}{2}\left(\frac{\mathcal{M}_{\mathcal{C}_{i'}}(\Gamma_{i'-1})}{\mathcal{M}_{\mathcal{C}_{i'}}(\Gamma_{i'-1}) + \mathcal{I}(\Gamma_{i'-1})}\right) > \frac{\mathcal{B}}{2}\left(\frac{\mathcal{M}_{\mathcal{C}_{i}}(\Gamma_{i-1})}{\mathcal{M}_{\mathcal{C}_{i}}(\Gamma_{i-1}) + \mathcal{I}(\Gamma_{i-1})}\right)$$

Hence, it is proved. □

Theorem 6. TBC-LI mechanism is monotone.

Proof. Fix i, c_{-i}, c_i, and $c_{i'}$. For mechanism `TBC-LI mechanism` to be *monotone*, we need to show that, any winning EC i with private cost c_i will still be considered in the winning set of ECs when declaring $c_{i'}$ such that $c_{i'} < c_i$ or any losing EC i with private cost c_i will still be considered in the losing set of ECs when declaring $c_{i'}$ such that $c_{i'} > c_i$. The proof is divided into two cases.

Case 1: In this case, the i^{th} winning EC deviates and reveals a cost of consultation $c_{i'} < c_i$. Again two cases can happen. If the EC i shows a small deviation in his/her (henceforth his) cost c_i *i.e.* $c_{i'}$ such that $c_{i'} < c_i$ and the current position of the EC i remains unchanged. In this situation, it can still be considered in the winning set. It is to be noted that, if the EC i reports a large deviation in his cost c_i *i.e.* $c_{i'}$ such that $c_{i'} < c_i$, then in this case by definition:

$$\frac{\mathcal{M}_{\mathcal{C}_1}(\Gamma_0)}{c_1} \geq \frac{\mathcal{M}_{\mathcal{C}_2}(\Gamma_1)}{c_2} \geq \frac{\mathcal{M}_{\mathcal{C}_3}(\Gamma_2)}{c_3} \geq \ldots \geq \frac{\mathcal{M}_{\mathcal{C}_n}(\Gamma_{n-1})}{c_n}$$

EC i will be placed some position ahead (say i') of its current position say i *i.e.* $i' < i$. This scenario is depicted in *Figure* 12 below.

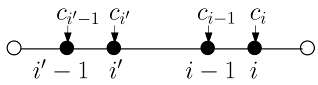


Figure 12. Pictorial representation

From Lemma 6 it can be said that if EC i is placed some position ahead by revealing a cost $c_{i'} < c_i$ then it must satisfy

$$\frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_{i'}}(\Gamma_{i'-1})}{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_{i'}}(\Gamma_{i'-1})+\boldsymbol{\mathcal{I}}(\Gamma_{i'-1})} > \frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_{i}}(\Gamma_{i-1})}{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_{i}}(\Gamma_{i-1})+\boldsymbol{\mathcal{I}}(\Gamma_{i-1})} \tag{3}$$

Let us suppose for the sake of contradiction that, when the EC $i \in \mathcal{S}$ comes ahead in ordering say at some position i' such that $c_{i'} < c_i$, then it is not considered in the winning set of the EC because it is not satisfying the given budget. If this is the case, then it means that:

$$c_{i'} > \frac{\boldsymbol{\mathcal{B}}}{2}\left(\frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_{i'}}(\Gamma_{i'-1})}{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_{i'}}(\Gamma_{i'-1})+\boldsymbol{\mathcal{I}}(\Gamma_{i'-1})}\right) \tag{4}$$

Combining our assumption $c_{i'} < c_i$ and equation 4 it can be concluded that:

$$\frac{\boldsymbol{\mathcal{B}}}{2}\left(\frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_{i'}}(\Gamma_{i'-1})}{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_{i'}}(\Gamma_{i'-1})+\mathcal{I}(\Gamma_{i'-1})}\right) < c_{i'} < c_i \tag{5}$$

Using condition in line 2 of *Algorithm* 10 and equation 5, we can say that:

$$\frac{\boldsymbol{\mathcal{B}}}{2}\left(\frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_{i'}}(\Gamma_{i'-1})}{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_{i'}}(\Gamma_{i'-1})+\mathcal{I}(\Gamma_{i'-1})}\right) < c_{i'} < c_i < \frac{\boldsymbol{\mathcal{B}}}{2}\left(\frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_{i}}(\Gamma_{i-1})}{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_{i}}(\Gamma_{i-1})+\mathcal{I}(\Gamma_{i-1})}\right)$$

$$\Rightarrow \frac{\boldsymbol{\mathcal{B}}}{2}\left(\frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_{i'}}(\Gamma_{i'-1})}{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_{i'}}(\Gamma_{i'-1})+\mathcal{I}(\Gamma_{i'-1})}\right) < \frac{\boldsymbol{\mathcal{B}}}{2}\left(\frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_{i}}(\Gamma_{i-1})}{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_{i}}(\Gamma_{i-1})+\mathcal{I}(\Gamma_{i-1})}\right)$$
$$\Rightarrow \frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_{i'}}(\Gamma_{i'-1})}{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_{i'}}(\Gamma_{i'-1})+\boldsymbol{\mathcal{I}}(\Gamma_{i'-1})} < \frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_{i}}(\Gamma_{i-1})}{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_{i}}(\Gamma_{i-1})+\boldsymbol{\mathcal{I}}(\Gamma_{i-1})}$$

So, it is a contradiction.

Case 2: In this case, the i^{th} losing EC deviates and reveals a cost of consultation $c_{i'}$ $> c_i$. Again two cases can happen. If the EC i shows a small deviation in his/her (henceforth his) cost c_i *i.e.* $c_{i'}$ such that $c_{i'} > c_i$ and the current position of the EC i remains unchanged. In this situation, it will be considered in the losing set. It is to be noted that, if the EC i reports a large deviation in his cost c_i *i.e.* $c_{i'}$ such that $c_{i'} > c_i$, then in this case by definition:

$$\frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_1}(\Gamma_0)}{c_1} \geq \frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_2}(\Gamma_1)}{c_2} \geq \frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_3}(\Gamma_2)}{c_3} \geq \ldots \geq \frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_n}(\Gamma_{n-1})}{c_n}$$

EC i will be placed some position ahead (say i') of its current position say i *i.e.* $i' > i$. This scenario is depicted in *Figure* 13 below.

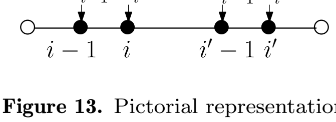


Figure 13. Pictorial representation

Analogous to the statement given in Lemma 6 it can be said that if EC i is placed some position ahead by revealing a cost $c_{i'} > c_i$ then it must satisfy

$$\frac{\boldsymbol{\mathcal{M}}_{\mathcal{C}_{i'}}(\Gamma_{i'-1})}{\boldsymbol{\mathcal{M}}_{\mathcal{C}_{i'}}(\Gamma_{i'-1}) + \boldsymbol{\mathcal{I}}(\Gamma_{i'-1})} < \frac{\boldsymbol{\mathcal{M}}_{\mathcal{C}_{i}}(\Gamma_{i-1})}{\boldsymbol{\mathcal{M}}_{\mathcal{C}_{i}}(\Gamma_{i-1}) + \boldsymbol{\mathcal{I}}(\Gamma_{i-1})} \tag{6}$$

Let us suppose for the sake of contradiction that, when the EC $i \in \mathcal{S}$ comes ahead in ordering say at some position i' such that $c_i' > c_i$, then it is not considered in the losing set of the EC because it is satisfying the given budget. If this is the case, then it means that:

$$c_{i'} < \frac{\boldsymbol{\mathcal{B}}}{2}\left(\frac{\boldsymbol{\mathcal{M}}_{\mathcal{C}_{i'}}(\Gamma_{i'-1})}{\boldsymbol{\mathcal{M}}_{\mathcal{C}_{i'}}(\Gamma_{i'-1}) + \mathcal{I}(\Gamma_{i'-1})}\right) \tag{7}$$

Combining our assumption $c_{i'} > c_i$ and equation 7 it can be concluded that:

$$\frac{\boldsymbol{\mathcal{B}}}{2}\left(\frac{\boldsymbol{\mathcal{M}}_{\mathcal{C}_{i'}}(\Gamma_{i'-1})}{\boldsymbol{\mathcal{M}}_{\mathcal{C}_{i'}}(\Gamma_{i'-1}) + \mathcal{I}(\Gamma_{i'-1})}\right) > c_{i'} > c_i \tag{8}$$

Using condition in line 2 of *Algorithm* 10 and equation 8, we can say that:

$$\frac{\boldsymbol{\mathcal{B}}}{2}\left(\frac{\boldsymbol{\mathcal{M}}_{\mathcal{C}_{i'}}(\Gamma_{i'-1})}{\boldsymbol{\mathcal{M}}_{\mathcal{C}_{i'}}(\Gamma_{i'-1}) + \mathcal{I}(\Gamma_{i'-1})}\right) > c_{i'} > c_i > \frac{\boldsymbol{\mathcal{B}}}{2}\left(\frac{\boldsymbol{\mathcal{M}}_{\mathcal{C}_{i}}(\Gamma_{i-1})}{\boldsymbol{\mathcal{M}}_{\mathcal{C}_{i}}(\Gamma_{i-1}) + \mathcal{I}(\Gamma_{i-1})}\right)$$

$$\Rightarrow \frac{\boldsymbol{\mathcal{B}}}{2}\left(\frac{\boldsymbol{\mathcal{M}}_{\mathcal{C}_{i'}}(\Gamma_{i'-1})}{\boldsymbol{\mathcal{M}}_{\mathcal{C}_{i'}}(\Gamma_{i'-1}) + \mathcal{I}(\Gamma_{i'-1})}\right) > \frac{\boldsymbol{\mathcal{B}}}{2}\left(\frac{\boldsymbol{\mathcal{M}}_{\mathcal{C}_{i}}(\Gamma_{i-1})}{\boldsymbol{\mathcal{M}}_{\mathcal{C}_{i}}(\Gamma_{i-1}) + \mathcal{I}(\Gamma_{i-1})}\right)$$

$$\Rightarrow \frac{\boldsymbol{\mathcal{M}}_{\mathcal{C}_{i'}}(\Gamma_{i'-1})}{\boldsymbol{\mathcal{M}}_{\mathcal{C}_{i'}}(\Gamma_{i'-1}) + \boldsymbol{\mathcal{I}}(\Gamma_{i'-1})} > \frac{\boldsymbol{\mathcal{M}}_{\mathcal{C}_{i}}(\Gamma_{i-1})}{\boldsymbol{\mathcal{M}}_{\mathcal{C}_{i}}(\Gamma_{i-1}) + \boldsymbol{\mathcal{I}}(\Gamma_{i-1})}$$

So, it is a contradiction.

Hence, the theorem is proved. □

Theorem 7. In TBC-DS, the function $\boldsymbol{\mathcal{D}} : 2^{\mathcal{S}} \Rightarrow \mathcal{R}_{\geq 0}$ is:

(a) **monotone:** If $\mathcal{S} \subseteq \mathcal{F}$ then $\boldsymbol{\mathcal{D}}(\mathcal{S}) \leq \boldsymbol{\mathcal{D}}(\mathcal{F})$, and
(b) **submodular:** If $\boldsymbol{\mathcal{D}}(\mathcal{S} \cup \{i\}) - \boldsymbol{\mathcal{D}}(\mathcal{S}) \geq \boldsymbol{\mathcal{D}}(\mathcal{F} \cup \{i\}) - \boldsymbol{\mathcal{D}}(\mathcal{F})$ $\forall \mathcal{S} \subseteq \mathcal{F}$.

Proof. To prove that the function is indeed monotone, let us suppose for the sake of contradiction that, if $\mathcal{S} \subseteq \mathcal{F}$ then $\boldsymbol{\mathcal{D}}(\mathcal{S}) > \boldsymbol{\mathcal{D}}(\mathcal{F})$. From the definition of *quality* function, we can say $\boldsymbol{\mathcal{D}}(\mathcal{S}) > \boldsymbol{\mathcal{D}}(\mathcal{F}) = \sum_{i\in\mathcal{S}} \mathcal{Q}_i > \sum_{i\in\mathcal{F}} \mathcal{Q}_i$. It is to be noted that under the given condition $\mathcal{S} \subseteq \mathcal{F}$, the sum of all the $\mathcal{Q}_i's$ over the set $\mathcal{F}$ will be greater than the sum of all the $\mathcal{Q}_i's$ over the set $\mathcal{S}$ *i.e.* $\sum_{i\in\mathcal{S}} \mathcal{Q}_i \leq \sum_{i\in\mathcal{F}} \mathcal{Q}_i$. So, the inequality $\boldsymbol{\mathcal{D}}(\mathcal{S}) > \boldsymbol{\mathcal{D}}(\mathcal{F}) = \sum_{i\in\mathcal{S}} \mathcal{Q}_i > \sum_{i\in\mathcal{F}} \mathcal{Q}_i$ cannot be true. Our assumption contradicts. Hence, the inequality if $\mathcal{S} \subseteq \mathcal{F}$ then $\boldsymbol{\mathcal{D}}(\mathcal{S}) \leq \boldsymbol{\mathcal{D}}(\mathcal{F})$ holds and the given function is *monotone*.
To prove that the function is *submodular*, as a thought experiment one can say that adding the same quantity *i.e.* in this case the quality value of any agent s_i to the sets

having relation $\mathcal{S} \subseteq \mathcal{F}$ will reflect the contribution by any agent s_i more in $\mathcal{S}$ than in $\mathcal{F}$. This completes the proof. $\square$

Theorem 8. TBC-LI is truthful.

Proof. Fix EC j, c_{-j}, c_j, and $c_{j'}$. For `TBC-LI mechanism` to be *truthful*, we need to show that, it is not beneficial for any EC j to *underbid* or *overbid* say $c_{j'}$ such that $c_{j'} < c_j$ or $c_{j'} > c_j$ respectively.

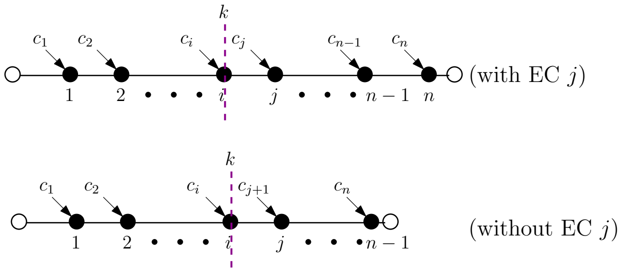


Figure 14. Pictorial representation

For each of the above possible scenarios, the proof is divided into two cases. Before going into the different cases let's consider the case where the EC j is reporting his true cost c_j. The pictorial representation of the possible set-up with n ECs are shown in Figure 14. The values from 1 to n represents the position (or index). Currently, our analysis lies around the index k and $k+1$; where k denote the index of the last EC ℓ that respects the allocation condition given in line 2 of *Algorithm 10*. As the ECs are sorted based on the marginal contribution per cost, so we can write

$$\frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_i}(\Gamma_{i-1})}{c_i} \geq \frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_j}(\Gamma_{i-1})}{c_j} \Rightarrow c_j \geq \frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_j}(\Gamma_{i-1}) \cdot c_i}{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_i}(\Gamma_{i-1})} \tag{9}$$

By using line 12 of *algorithm 11* and equation 9 it can be easily seen that, $c_j \geq \boldsymbol{\mathcal{C}}_j^i$. If this is the case then we can say $c_j \geq \boldsymbol{\Pi}_j^i$. In order to be allocated EC j must satisfy $\boldsymbol{\mathcal{C}}_j^i \leq \boldsymbol{\Pi}_j^i$ otherwise $\boldsymbol{\mathcal{C}}_j^i > \boldsymbol{\Pi}_j^i$ means not allocated. As we are taking the payment as:

- If $\boldsymbol{\mathcal{C}}_j^i = \boldsymbol{\Pi}_j^i \Rightarrow \boldsymbol{\mathcal{P}}_{\Gamma_j} = \min\{\boldsymbol{\mathcal{C}}_j^i, \boldsymbol{\Pi}_j^i\} = \boldsymbol{\Pi}_j^i = \boldsymbol{\mathcal{C}}_j^i \leq c_j$
- If $\boldsymbol{\mathcal{C}}_j^i > \boldsymbol{\Pi}_j^i \Rightarrow \boldsymbol{\mathcal{P}}_{\Gamma_j} = \min\{\boldsymbol{\mathcal{C}}_j^i, \boldsymbol{\Pi}_j^i\} = \boldsymbol{\Pi}_j^i = \boldsymbol{\Pi}_j^i \leq c_j$
- If $\boldsymbol{\mathcal{C}}_j^i < \boldsymbol{\Pi}_j^i \Rightarrow \boldsymbol{\mathcal{P}}_{\Gamma_j} = \min\{\boldsymbol{\mathcal{C}}_j^i, \boldsymbol{\Pi}_j^i\} = \boldsymbol{\mathcal{C}}_j^i \leq c_j$

If this is the case, then it can be concluded that $\boldsymbol{\mathcal{C}}_j^i \leq c_j$ or $\boldsymbol{\Pi}_j^i \leq c_j$. As the payment is less than the actual cost. Hence not allocated. Coming back to our *underbid* and *overbid* cases.

Scenario 1: Underbidding ($c_{j'} < c_j$) In this case, the j^{th} EC deviates and reveals a cost of consultation $c_{j'} < c_j$. This scenario give rise to two cases.

Case 1:When EC j is in losing set. If the EC j shows a small deviation in his/her (henceforth his) cost *i.e.* $c_{j'}$ such

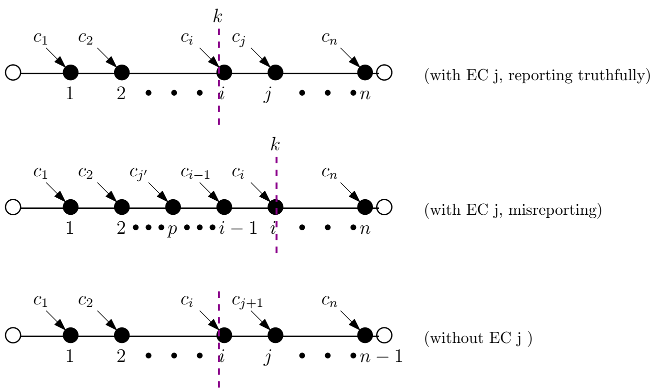


Figure 15. Pictorial representation

that $c_{j'} < c_j$ and the current position of the EC j remains unchanged. In this situation, it can still be considered in the losing set. It is to be noted that, if the EC j reports a large deviation in his cost c_j *i.e.* $c_{j'}$ in this case it will belong to winning set and will appear before EC i as shown in Figure 15. As the ECs are sorted based on the marginal contribution per cost, so we can write:

$$\frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_j}(\mathit{\Gamma}_{i-1})}{c_{j'}} \geq \frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_i}(\mathit{\Gamma}_{i-1})}{c_i} \Rightarrow c_{j'} \leq \frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_j}(\mathit{\Gamma}_{i-1}) \cdot c_i}{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_i}(\mathit{\Gamma}_{i-1})}$$

and $c_{j'} \leq \frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_j}(\mathit{\Gamma}_{i-1}) \cdot c_i}{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_i}(\mathit{\Gamma}_{i-1})} = \boldsymbol{\mathcal{C}}_j^i$ because from above we have got the relation $\boldsymbol{\mathcal{C}}_j^i \leq c_j$. This will lead to $c_{j'} \leq \boldsymbol{\mathcal{C}}_j^i \leq c_j$. The EC j is paid less than the actual cost.

Case 2: When EC j is in winning set. If the EC j shows a deviation in his cost such that $c_{j'} < c_j$ it will still belong to winning set and will appear before EC i as shown in Figure 16.

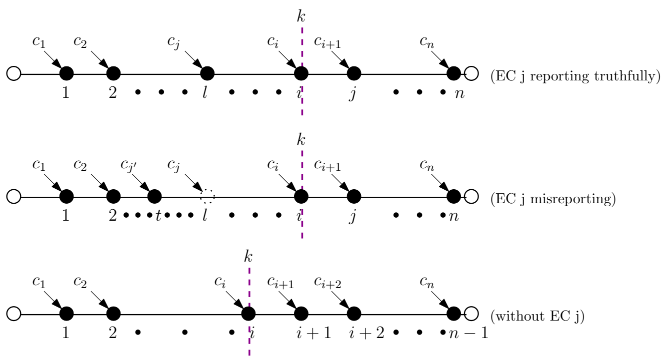


Figure 16. Pictorial representation

As the ECs are sorted based on the marginal contribution per cost, so we can write:

$$\frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_j}(\mathit{\Gamma}_{l-1})}{c_j} \geq \frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_i}(\mathit{\Gamma}_{i-1})}{c_i} \Rightarrow c_j \leq \frac{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_j}(\mathit{\Gamma}_{l-1}) \cdot c_i}{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{C}}_i}(\mathit{\Gamma}_{i-1})} = \boldsymbol{\mathcal{C}}_j^l$$

and for the case when the EC j deviates, then

$$\frac{\mathcal{M}_{\mathcal{C}_j}(\Gamma_{t-1})}{c_{j'}} \geq \frac{\mathcal{M}_{\mathcal{C}_i}(\Gamma_{i-1})}{c_i} \Rightarrow c_{j'} \leq \frac{\mathcal{M}_{\mathcal{C}_j}(\Gamma_{t-1}) \cdot c_i}{\mathcal{M}_{\mathcal{C}_i}(\Gamma_{i-1})} = \mathcal{C}_j^l$$

From above two equations it is clear that no matter what cost EC j is bidding, he will still be winning and be paid an amount $\mathcal{C}_j^l$. Hence, considering *Case 1* and *Case 2* it can be concluded that EC j does not gain by underbidding there true cost. In similar fashion, we can write the above mentioned equation for any position i before k and in the same way we can prove that $\mathcal{C}_j^i \leq c_j$.

Scenario 2: Overbidding ($c_{j'} > c_j$) In this case, the j^{th} EC deviates and reveals a cost of consultation $c_{j'} > c_j$. This scenario gives rise to two cases.

Case 1: When EC j is in losing set. If the EC j shows a small deviation in his/her (henceforth his) cost *i.e.* $c_{j'}$ such that $c_{j'} > c_j$ and the current position of the EC j remains unchanged. In this situation, it can still be considered in the losing set.

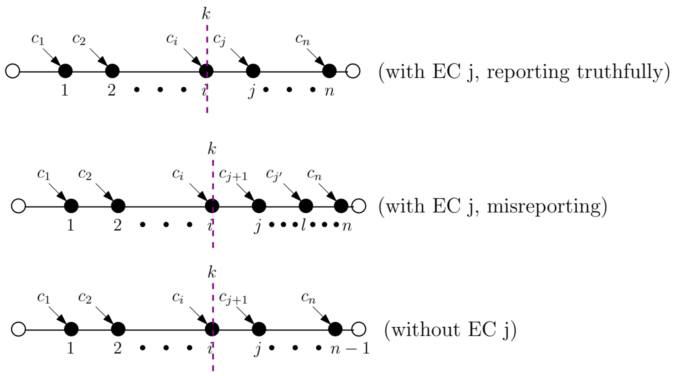


Figure 17. Pictorial representation

It is to be noted that, if the EC j reports a large deviation in his cost c_j *i.e.* $c_{j'}$ in this case it will belong to losing set and will still appear after EC i as shown in Figure 17. As the ECs are sorted based on the marginal contribution per cost, so we can write:

$$\frac{\mathcal{M}_{\mathcal{C}_i}(\Gamma_{i-1})}{c_i} \geq \frac{\mathcal{M}_{\mathcal{C}_j}(\Gamma_{i-1})}{c_i} \Rightarrow c_{j'} \leq \frac{\mathcal{M}_{\mathcal{C}_j}(\Gamma_{i-1}) \cdot c_i}{\mathcal{M}_{\mathcal{C}_i}(\Gamma_{i-1})}$$

and $c_{j'} \leq \frac{\mathcal{M}_{\mathcal{C}_j}(\Gamma_{i-1}) \cdot c_i}{\mathcal{M}_{\mathcal{C}_i}(\Gamma_{i-1})} = \mathcal{C}_j^i$. From above we have got the relation $\mathcal{C}_j^i \leq c_j$. This will lead to $c_{j'} \leq \mathcal{C}_j^i \leq c_j$. The EC j is paid less than the actual cost.

Case 2: When EC j is in winning set. If the EC j shows a deviation in his cost such that $c_{j'} > c_j$ and the current position of the EC j remains unchanged. In this situation, it can still be considered in the winning set. It is to be noted that, if the EC j reports a large deviation in his cost c_j *i.e.* $c_{j'}$ in this case it will belong to losing set and will appear after EC i as shown in Figure 18. Utilizing the definition of the marginal contribution per cost sorting and figure above:

$$\frac{\mathcal{M}_{\mathcal{C}_j}(\Gamma_{l-1})}{c_j} \geq \frac{\mathcal{M}_{\mathcal{C}_i}(\Gamma_{i-1})}{c_i} \Rightarrow c_j \leq \frac{\mathcal{M}_{\mathcal{C}_j}(\Gamma_{l-1}) \cdot c_i}{\mathcal{M}_{\mathcal{C}_i}(\Gamma_{i-1})} = \mathcal{C}_j^l$$

and for the case when the EC j deviates, then

$$\frac{\mathcal{M}_{\mathcal{C}_i}(\Gamma_{i-1})}{c_i} \geq \frac{\mathcal{M}_{\mathcal{C}_j}(\Gamma_{t-1})}{c_{j'}} \Rightarrow c_{j'} \geq \frac{\mathcal{M}_{\mathcal{C}_j}(\Gamma_{t-1}) \cdot c_i}{\mathcal{M}_{\mathcal{C}_i}(\Gamma_{i-1})} = \mathcal{C}_j^l$$

Now, if EC j deviates by large amount then it will belong to the losing set. From above two equations it is clear that no matter what cost EC j is bidding, he will be paid $\mathcal{C}_j^l$. Hence, considering *Case 1* and *Case 2* it can be concluded that EC j does not gain by overbidding there true cost. In similar fashion, we can write the above mentioned equation for any position i before k and in the same way we can prove that $\mathcal{C}_j^i \leq c_j$. Hence, max operator will still endure that if any agent j deviates and wins, then his payment will be less than the true cost and hence deviation may not give any gain.

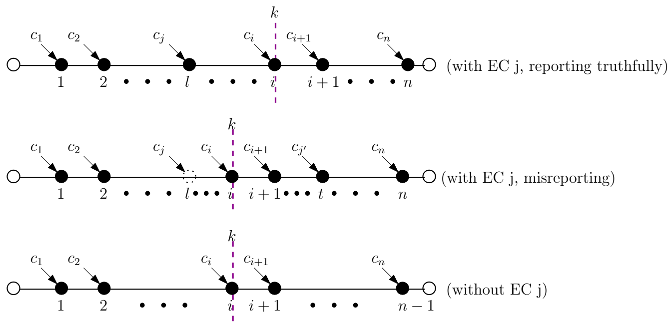


Figure 18. Pictorial representation

Hence, the theorem is proved. □

5. Experimental Findings

In this section, we compare the efficacy of the proposed mechanisms via simulations. For the *zero* budget case, the experiments are carried out in this section to provide a simulation based on the data (the strict preference ordering of the patients) generated randomly using the Random library in Python. Our proposed naive mechanism *i.e.* RanPAM is considered as a benchmark scheme and is compared with TOAM (in case of full preferences) and TOAM-IComP (in case of incomplete preferences).
For the *positive* budget case, we have compared our proposed mechanisms against the benchmark mechanism (`random mechanism`). In this, the doctors are selected randomly and are paid their declared cost. We have utilized the *coverage model* for the first fold of our hiring problem. The unit of cost and budget is \$. The experiments are performed for the single patient-multiple experts scenario. Similar graphical behaviour can be seen for multiple patient-multiple doctor scenario.

5.1. Simulation Set-Up

Case 1: For creating a real world healthcare scenario we have considered 10 different categories of patients and doctors for our simulation purpose. It is to be noted that, in each of the categories, some fixed number of patients and fixed number of doctors are present for taking consultancy and for providing consultancy respectively. One of the scenario that is taken into consideration for simulation purpose is, say there are equal number of patients and doctors present in each of the categories under consideration along with the assumption that each of the patients are providing strict preference ordering (generated randomly) over all the available doctors in the respective categories. This scenario is referred as *Scenario-1* in the rest of the paper. Next, the more general scenario with equal number of patients and doctors in each of the categories can be obtained by relaxing the constraint that all the available patients are providing the strict preference ordering over all the available doctors in categories under consideration. Here, it may be the case that, in each of the categories the patients are providing the strict preference ordering over the subset of the available doctors. This scenario is referred as *Scenario-2* in the rest of the paper.
In the series of different scenarios, next we have considered the utmost general set-up where there are n number of patients and m number of doctors such that $m \neq n$ ($m > n$ and $m < n$). In this, the patients are providing the strict preference ordering over the subset of the available doctors in each categories under consideration. The scenario with $m > n$ is referred as *scenario-3* and the scenario with $m < n$ is referred as *scenario-4* in the future references.

Case 2: For our simulation purpose, a social graph is generated randomly using `Networkx` package of python. It consists of `1000 nodes (doctors)` and approximately `28,250 edges`. The maximum and minimum degree a node can have is 10% and 1% of the total available nodes respectively. The cost of each node as initial adapter is uniformly distributed over $[30, 50]$, the cost of consultancy is uniformly distributed over $[35, 50]$, and quality is uniformly distributed over $[20, 50]$. The budget is considered in range $[100, 1000]$.

5.2. Performance Metrics

Case 1: The performance of the proposed mechanisms is measured under the banner of two important parameters:

• **Efficiency loss (EL).** It is the sum of the difference between the index of the doctor allocated from the agent preference list to the index of the most preferred doctor by the agent from his preference list. Mathematically, the EL is defined as: EL $= \sum_{i=1}^{n}(\overline{I}_{i_{\mathcal{A}}} - \overline{I}_{i_{\mathcal{MP}}})$ where, $\overline{I}_{i_{\mathcal{A}}}$ is the index of the doctor allocated from the initially provided preference list of the patient i, $\overline{I}_{i_{\mathcal{MP}}}$ is the index of the most preferred doctor in the initially provided preference list of patient i. Considering the overall available categories $x = \{x_1, x_2, \ldots, x_k\}$, the total efficiency loss (TEL) of the system is given as:

$$TEL = \sum_{x}\sum_{i=1}^{n}(\overline{I}_{i_{\mathcal{A}}} - \overline{I}_{i_{\mathcal{MP}}}) \tag{10}$$

• **Number of best allocation (NBA).** It measures the number of patients (say k) gets their best choice (most preferred doctor) from their provided preference list over the available number of doctors. It is the sum of the number of agents getting their most preferred doctor from their provided preference list.

Case 2: The performance metric includes the `Interested doctors set size`, and `Number of doctors hired`.

5.3. Simulation Directions

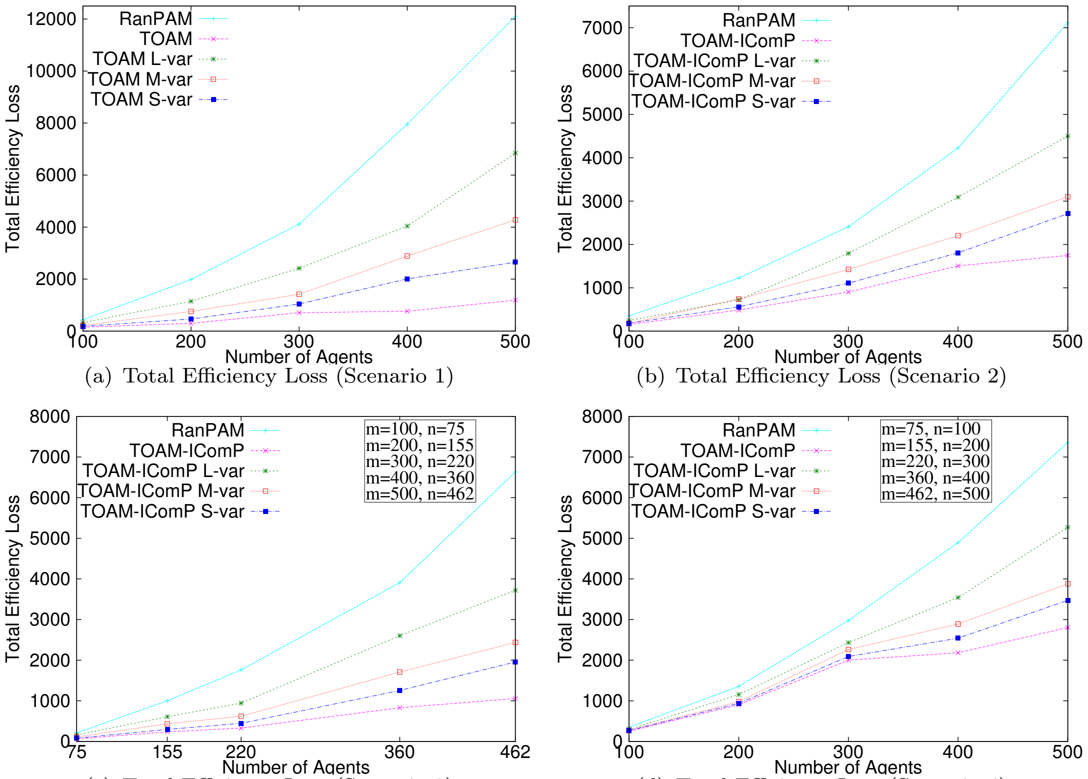
Considering case 1, as the benchmark scheme is vulnerable to manipulation, the two direction are seen for measuring the performance of RanPAM, TOAM, and TOAM-IComP. The two directions are:

- When all the agents (patients) are reporting their true preference list.
- When some of the agents (patients) are misreporting their true preference list.

5.4. Analysis of Results

Case 1: In this section, the result is simulated following the directions mentioned in Subsection 5.3. As the patients are varying their true preference list, the next question that comes is that, how many of the patients can vary their true preference list (*i.e.* what fraction of the total available patients can vary their true preference list?). To answer this question, the calculation is done using indicator random variable.
−**Expected amount of variation** The following analysis mathematically justifies the idea of choosing the parameters of variation. The analysis is motivated by (Cormen *et al.* 2009). Let $\mathcal{N}_i$ be the random variable associated with the event in which i^{th} patient varies his true preference ordering. Thus, $\mathcal{N}_i = \{i^{th}$ patient varies preference ordering$\}$. We have, from the definition of expectation that $E[\mathcal{N}_i] = \Pr\{i^{th}\ \textit{patient varies preference ordering}\}$. Let $\mathcal{N}$ be the random variable denoting the total number of patients vary their preference ordering. By using the properties of random variable, it can be written that $\mathcal{N} = \sum_{i=1}^{n} \mathcal{N}_i$. We wish to compute the expected number of variations, and so we take the expectation both sides and by linearity of expectation we can write $E[\mathcal{N}] = \sum_{i=1}^{n} E[\mathcal{N}_i] = \sum_{i=1}^{n}(\Pr\{i^{th}\ \textit{patient varies preference ordering}\})$ $= \sum_{i=1}^{n} 1/8 = n/8$. Here, $\Pr\{i^{th}\ \textit{patient varies preference ordering}\}$ is the probability that given a patient whether he will vary his true preference ordering. The probability of that is taken as $1/8$ (small variation). If the number of agents varies from $1/4$ and $1/2$, then the expected number of patient that may vary their preference ordering can be $n/4$ (medium variation), and $n/2$ (large variation) respectively.

In Figure 19(a) and Figure 19(b)-19(d), it can be seen that the total efficiency loss of the system in case of RanPAM is more than the total efficiency loss of the system in case of TOAM and TOAM-IComP respectively. This is due to the fact that, dissimilar to the RanPAM, TOAM and TOAM-IComP allocates the best possible doctors to the patients from their revealed preference list. Due to this reason, the value returned by equation 10 in case of TOAM and TOAM-IComP is very small as compared to RanPAM. In Figure 19(a), when the agents are varying (misreporting) their true preference ordering, then the TEL of the patients in case of TOAM with large variation (TOAM L-var) is more than the TEL in TOAM with medium variation (TOAM M-var) is more than the TEL in TOAM with small variation (TOAM S-var) is more than the TEL in TOAM without variation. As it is natural from the construction of the TOAM. Considering the case of incomplete preferences in Figure 19(b)-19(d), when the subset of agents are varying their true preference ordering, then the TEL of the patients in case of TOAM-IComP with large variation (TOAM-IComP L-var) is more than the TEL in TOAM-IComP with medium variation (TOAM-IComP M-var) is more than the TEL in TOAM-IComP with small variation (TOAM-IComP S-var) is more than the TEL in TOAM-IComP without variation. As this is evident from the construction of the TOAM-IComP.



(a) Total Efficiency Loss (Scenario 1)

(b) Total Efficiency Loss (Scenario 2)

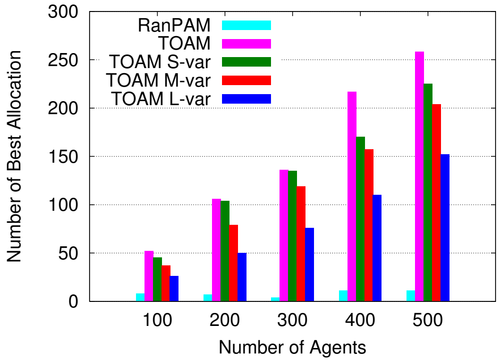
(c) Total Efficiency Loss (Scenario 3)

(d) Total Efficiency Loss (Scenario 4)

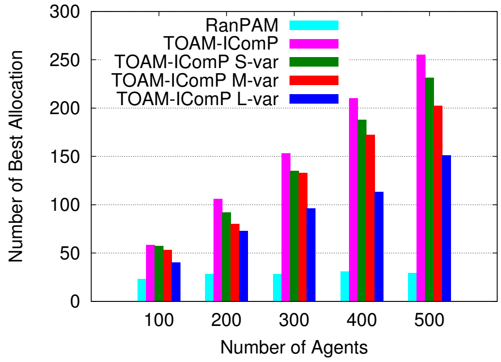
Figure 19. Total efficiency loss for different scenarios

Considering the case of our second parameter *i.e* NBA, in Figure 20(a) and Figure 20(b)-20(d), it can be seen that the NBA of the system in case of RanPAM is less than the NBA of the system in case of TOAM and TOAM-IComP respectively. This is due to the fact that, dissimilar to the RanPAM, TOAM and TOAM-IComP allocates the best possible doctors to the patients from their preference list. In Figure 20(a), when the agents are varying their true preference ordering, then the NBA of the patients in case of TOAM with large variation (TOAM L-var) is less than the NBA in TOAM with medium variation (TOAM M-var) is less than the NBA in TOAM with small variation (TOAM S-var) is less than the NBA in TOAM without variation. As it is natural from the construction of the TOAM.

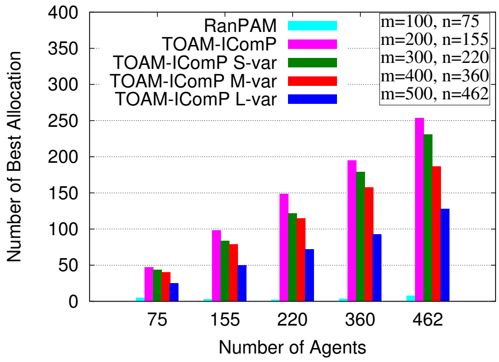
In Figure 20(b)-20(d), when the subset of agents are varying their true preference ordering, then the NBA of the patients in case of TOAM-IComP with large variation (TOAM-IComP L-var) is less than the NBA in TOAM-IComP with medium variation (TOAM-IComP M-var) is less than the NBA in TOAM-IComP with small variation (TOAM-IComP S-var) is less than the TEL in TOAM-IComP without variation. As this is evident from the construction of the TOAM-IComP.



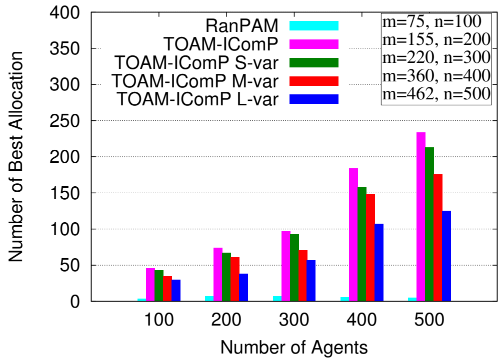
(a) Total Number of Best Allocation (Scenario 1)



(b) Total Number of Best Allocation (Scenario 2)



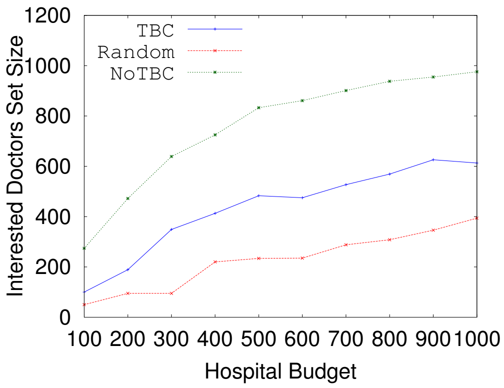
(c) Total Number of Best Allocation (Scenario 3)



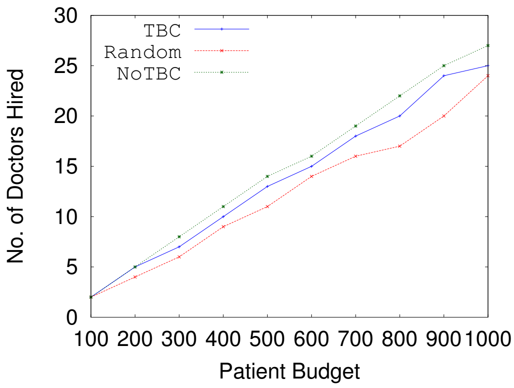
(d) Total Number of Best Allocation (Scenario 4)

Figure 20. Number of best allocation for different scenarios

Case 2: The simulation results shown in Figure 21(a) shows the comparison of the interested doctors set size *i.e.* the number of doctors acting as leaders and the number of doctors informed by the leaders about the hiring concept.



(a) Interested doctors set size with budget



(b) Hired doctors with budget

Figure 21. Simulation results for positive budget case

It is seen in Figure 21(a) that the interested doctors set size in case of NoTBC mechanism is higher than TBC mechanism and random mechanism. This nature of NoTBC mechanism is obvious due to the fact that the mechanisms (NoTBC-LI and

NoTBC-DS) are utilizing almost the complete quota of the available budgets whereas TBC mechanism is utilizing only a part of total budget. With the increase in budget, one can easily see the increasing gap between NoTBC mechanism and TBC mechanism. It can be seen evidently in Figure 21(b) that the number of doctors hired in case of NoTBC mechanism is higher than TBC mechanism and random mechanism. Similar reasoning can be given as above.

6. Realistic Implementation of Our Proposed Frameworks

In our country, several free medical camps are organized under the banners of *"Smile on Wheels"* (Smile 2017), *"UPPAHAR"* (Uppahar 2009), and several others on regular basis for health check ups, cataract surgeries, etc. In such type of existing system, there may exist a team of well known *ophthalmologist*, *physicians*, *dentists*, *dermatologists* and *general physicians* etc. In this, the expertise that are provided by the participating experts are mainly free of cost. Currently, the general practice in such existing system is that, irrespective of patients choice, doctors having speciality in certain domain are assigned randomly to the patient in an adhoc fashion. However, the existing system could be more structured if our developed framework is deployed in such scenarios. Also, deploying our framework to such existing system will lead to the allocation of favourable doctors among the available experts and not the random one, that will improve the quality of the healthcare service and also help in increasing the satisfaction level of patients in terms of healthcare services.

In our country, there are some hospitals that provide the comprehensive healthcare services to the unreached community free of cost. Among them, one of the hospitals is K G hospital, Coimbatore, India run by K. Govindaswamy Naidu Medical Trust. Talking about this hospital, it has 250 doctors, and 800 fully trained and experienced nurses and para-medical staff for providing the free medical services to the needy ones. At the time of writing this paper, they have already served around 85,000 free cataract surgeries, 200 free heart surgeries, 1,500 dialysis free of cost, screened over 300,000 people free of cost for blood pressure and saved 35,000 accident victims (Hospital 2018). Also, they have provided their services to the victims of several natural and man-made calamities such as *Kargil war*, *Gujrat earthquake*, and *Tsunami* (Hospital 2018).

As mentioned above, there motivation behind implementation of our proposed framework lies in the fact that other than in-house experts they will be empowered by some high profile experts present around the globe. This will lead to a two-fold gain a) improved healthcare facility and b) the projection of their identity at the global market.

7. Conclusions and Future Directions

In literature, several works have been done focusing on scheduling the healthcare resources (such as OTs, physicians, and nurses) inside the hospitals, in an efficient and effective manner. In addition to this, very few works have been done in the direction of scheduling the healthcare resources (especially doctors) outside the in-house hospitals, both in *strategic* and *non-strategic* setting. In this paper, we have investigated the *experts hiring problem* from outside the hospitals, in *strategic* setting. Due to the participation of *strategic agents* in our proposed healthcare system, the scenarios are modelled through the robust concepts of mechanism design with and without money. Here, we have investigated the hiring problem from outside the hospitals in *zero* budget and *positive* budget scenarios. The mechanisms are proposed that satisfies several economic properties such as *truthfulness*, *core allocation*, *pareto optimality*, and *budget feasibility*. Through theoretical and experimental analysis, it has been shown that the proposed mechanisms in case of zero budget *i.e.* TOAM and TOAM-ICoMP are *truthful*, *pareto optimal* and satisfies the *core* property. Also, in positive budget case it has been shown that TBC-LI mechanism is *truthful*, *monotone*, and *budget feasible*. Talking about the practical implication of our proposed system, currently, in any healthcare camp, the general practice is that, irrespective of the patients choice, doctors having speciality in certain domain are assigned randomly in an adhoc fashion. However, the existing system could be more structured if our developed framework is deployed in such scenarios. Also, deploying our framework to such existing system will lead to the allocation of favourable doctors among the available experts and not the random one, that will improve the quality of the healthcare service and also help in increasing the satisfaction level of patients in terms of healthcare services.

One of the future works in *zero budget* environment could be, the setting with n patients and m doctors ($m \neq n$ or $m == n$) where the members of the two participating communities, namely the patients and the doctors will be revealing the preference ordering *not necessarily in strict sense* over the members of the opposite community for a stipulated amount of time. In case of *positive budget* environment, one can think of investigating the *experts hiring problem* in combinatorial domain. In this set-up, we will have multiple patients and multiple experts, where each patient has a privilege to report the preferred set of experts they are interested in, among the available one along with their valuation.

Acknowledgement(s)

We would like to thanks the research students and faculty members of the Department of CSE, NIT Durgapur for their valuable suggestions during the course of this work. Further, we would like to thank Prof. Y. Narahari and members of the Game Theory Lab. at IISc Bangalore for their suggestions and directions. Finally, we thank the Government of India Ministry of Human Resource Development and Ministry of Electronics and Information Technology (MeitY) Government of India and Media Lab Asia (YFR fellowship grant under Vivsvesvaraya PhD scheme), for their funds.

References

ACEP, 1998. American college of emergency physicians, *in: Directory of Software in Emergency Medicine*, Dallas: (http://www.acep.org).

Atanasovski, B., Bogdanovic, M., Velinov, G., Stoimenov, L., Dimovski, A.S., Koteska, B., Jankovic, D., Skrceska, I., Kon-Popovska, M., and Jakimovski, B., 2018. On defining a model driven architecture for an enterprise e-health system, *Enterprise Information Systems*, 12 (8-9), 915–941.

Berrada, I., Ferland, J.A., and Michelon, P., 1996. A multi-objective approach to nurse scheduling with both hard and soft constraints, *Socio-Economic Planning Sciences*, 30 (3), 183–193.

Bowers, M.R., Noon, C.E., Wu, W., and Bass, J.K., 2016. Neonatal physician scheduling at the university of tennessee medical center, *Interfaces*, 46 (2), 168–182.

ByteBloc, 1995. Bytebloc software, epsked 3.0, *in: ByteBloc Software*, LongBeach: (http://www.bytebloc.com).

Cardoen, B., Demeulemeester, E., and Belien, J., 2010a. Operating room planning and scheduling: A literature review, *European Journal of Operational Research*, 201 (3), 921–932.

Cardoen, B., Demeulemeester, E., and Frank, E., 2010b. Operating room planning and scheduling problems: A classification scheme, *International Journal of Health Management and Information*, 1 (1), 71–83.

Carter, M.W. and Lapiere, S.D., 2001. Scheduling emergency room physicians, *Health Care Management Science*, 4 (4), 347–360.

Chan, H. and Chen, J., 2016. Provision-after-wait with common preferences, *in: Proceedings of the International Conference on Autonomous Agents and Multiagent Systems*, Richland, SC: IFAAMAS, AAMAS '16, 278–286.

Chiang, D.L., Huang, Y.T., Chen, T.S., and Lai, F.P., 2018. Applying time-constraint access control of personal health record in cloud computing, *Enterprise Information Systems*, 0 (0), 1–16.

Cormen, T.H., Leiserson, C.E., Rivest, R.L., and Stein, C., 2009. *Introduction to algorithms*, MIT press.

Dexter, F. and Macario, A., 2004. When to release allocated operating room time to increase operating room efficiency, *Anesthesia and Analgesia*, 98 (3), 758–762.

Ekblaw, A., Azaria, A., Halamka, J.D., and Lippman, A., 2016. A case study for blockchain in healthcare : "medrec" prototype for electronic health records and medical research data.

Erhard, M., Schoenfelder, J., Fuğener, A., and Brunner, J.O., 2018. State of the art in physician scheduling, *European Journal of Operational Research*, 265 (1), 1 – 18.

Escobar-Rodriguez, T., Escobar-Pérez, B., and Monge-Lozano, P., 2014. Technical and organisational aspects in enterprise resource planning systems implementation: Lessons from a spanish public hospital, *Enterprise Information Systems*, 8 (5), 533–562.

Gale, D. and Shapley, L., 1962. College admissions and the stability of marriage, *American Mathematical Monthly*, 69, 9–15.

Gries, D., 1987. *The Science of Programming*, Berlin, Heidelberg: Springer-Verlag, 1st ed.

Guerriero, F. and Guido, R., 2011. Operational research in the management of the operating theatre: A survey, *Journal of Health Care Management Science*, 14 (1), 89–114.

Hospital, K., 2018. https://en.wikipedia.org/wiki/g._bakthavathsalam, KG Hospital.

Kakria, P., Tripathi, N.K., and Kitipawang, P., 2015. A real-time health monitoring system for remote cardiac patients using smartphone and wearable sensors, *International Journal of Telemedicine and Applications*, 2015, 8:8–8:8.

Khuller, S., Moss, A., and Naor, J.S., 1999. The budgeted maximum coverage problem, *Information Processing Letters*, 70 (1), 39–45.

Klaus, B., Manlove, D.F., and Rossi, F., 2016. Matching under preferences, *in: Handbook of Computational Social Choice*, 333–355.

Ko, Y.W., Kim, D.H., Uhmn, S., and Kim, J., 2017. Nurse scheduling problem using backtracking, *Advanced Science Letters*, 23 (4), 3792–3795.

Kumar, T., Ramani, V., Ahmad, I., Braeken, A., Harjula, E., and Ylianttila, M., 2018. Blockchain utilization in healthcare: Key requirements and challenges, *in: IEEE* 20^{th} *International Conference on e-Health Networking, Applications and Services (Healthcom)*, Ostrava, Czech Republic, 1–7.

Marcelo, A.B., 2010. Health information systems: A survey of frameworks for developing countries., *Yearbook of Medical Informatics*, 25–9.

MSI, 1998. Msi software, physician scheduler 3.2, *in: MSI Software*, Fairfax: (http://www.msisoftware.com).

Mukherjee, N. and Bhunia, S.S., 2014. A sensor-cloud framework for provisioning remote health-care services, *in: Proceedings of Computing & Networking for Internet of Things (ComNet-IoT) workshop co-located with* 15^{th} *International Conference on Distributed Computing and Networking (ICDCN)*.

Mukherjee, N., Bhunia, S.S., and Bose, S., 2016. Virtual sensors in remote healthcare delivery: Some case studies, *in: Proceedings of the* 9^{th} *International Joint Conference on Biomedical Engineering Systems and Technologies (BIOSTEC 2016) - Volume 5: HEALTHINF, Rome,*

Italy, February 21-23, 2016., 484–489.
Paul, R.J., Ezz, I., and Kuljis, J., 2012. Healthcare information systems: A patient-user perspective, *Health Systems*, 1 (2), 85–95.
Pottayya, R.M., Lapayre, J.C., and Garcia, E., 2017. An adaptive videoconferencing framework for collaborative telemedicine, *in: Proceedings* 31^{st} *International Conference on Advanced Information Networking and Applications (AINA)*, Taipei, Taiwan, 197–204.
Rahmadika, S. and Rhee, K.H., 2018. Blockchain technology for providing an architecture model of decentralized personal health information, *International Journal of Engineering Business Management*, 10, 1847979018790589.
Roughgarden, T., 2013. CS364A: Algorithmic game theory (Stanford University), Lecture #9: Beyond quasi-linearity.
Roughgarden, T., 2016. CS269I: Incentives in computer science (Stanford University), Lecture #1: The draw and college admissions.
Samudra, M., Van Riet, C., Demeulemeester, E., Cardoen, B., Vansteenkiste, N., and Rademakers, F.E., 2016. Scheduling operating rooms: Achievements, challenges and pitfalls, *Journal of Scheduling*, 19 (5), 493–525.
Santos, M. and Eriksson, H., 2014. Insights into physician scheduling: A case study of public hospital departments in sweden, *International Journal of Health Care Quality Assurance*, 27 (2), 76–90.
Schummer, J. and Vohra, R.V., 2007. *Algorithmic Game Theory*, New York: Cambridge University Press, chap. Mechanism Design without Money, 209–242.
Shapley, L. and Scarf, H., 1974. On cores and indivisibility, *Journal of Mathematical Economics*, 1, 23–37.
Singer, Y., 2010. Budget feasible mechanisms, *in: Proceedings of* 51^{st} *Annual Symposium on Foundations of Computer Science (FOCS)*, 765–774.
Singer, Y., 2012. How to win friends and influence people, truthfully: Influence maximization mechanisms for social networks, *in: Proceedings of the* 5^{th} *International Conference on Web Search and Data Mining*, New York, NY, USA: ACM, WSDM '12, 733–742.
Singh, V.K. and Mukhopadhyay, S., 2016a. Hiring doctors in e-healthcare with zero budget, *CoRR*, abs/1606.08590.
Singh, V.K. and Mukhopadhyay, S., 2016b. Hiring expert consultants in e-healthcare with budget constraint, *CoRR*, abs/1610.04454.
Singh, V.K., Mukhopadhyay, S., and Das, R., 2018a. Hiring doctors in e-healthcare with zero budget, *in:* F. Xhafa, S. Caballé, and L. Barolli, eds., *Advances on P2P, Parallel, Grid, Cloud and Internet Computing: Proceedings of the* 12^{th} *International Conference on P2P, Parallel, Grid, Cloud and Internet Computing (3PGCIC-2017)*, Cham: Springer International Publishing, 379–390.
Singh, V.K., Mukhopadhyay, S., Debnath, N., and Chowdary, A.M., 2015. Auction aware selection of doctors in e-healthcare, *in: Proceedings of* 17^{th} *Annual International Conference on E-health Networking, Application and services (HealthCom)*, Boston, USA: IEEE, 363–368.
Singh, V.K., Mukhopadhyay, S., Sharma, A., and Roy, A., 2017. Hiring expert consultants in e-healthcare: A two sided matching approach, *CoRR*, abs/1703.08698.
Singh, V.K., Mukhopadhyay, S., Xhafa, F., and Sharma, A., 2018b. A budget feasible mechanism for hiring doctors in e-healthcare, *in: IEEE* 32^{nd} *International Conference on Advanced Information Networking and Applications (AINA)*, 785–792.
Smile, 2017. https://www.smilefoundationindia.org/smile_on_wheels.html#about, smile on Wheels programme.
Stanimirovic, D., 2015. Modelling the health information system in slovenia - operative, construction and implementation aspects, *International Journal of Engineering Business Management*, 7, 13.
Starren, J.B., Nesbitt, T.S., and Chiang, M.F., 2014. Telehealth, *Journal of biomedical informatics*, 541–560.
Tang, C., Freedman, H., and Sierminski, R., 2016. "Schedule change really disrupts a lot of

things!": Never-ending physician scheduling in a multi-function multi-setting practice, *in: IEEE* 29^{th} *International Symposium on Computer-Based Medical Systems (CBMS)*, 82–87.

Tekin, C., Atan, O., and Schaar, M.V.D., 2015. Discover the expert: Context-adaptive expert selection for medical diagnosis, *IEEE Transactions on Emerging Topics in Computing*, 3 (2), 220–234.

Uppahar, 2009. http://uppahar.in/default.aspx, uPPHAR.

V. Huele, C. and Vanhoucke, M., 2014. Analysis of the integration of the physician rostering problem and the surgery scheduling problem, *Journal of Medical Systems*, 38 (6), 1–16.

Wang, C.W., Sun, L.M., Jin, M.H., Fu, C.J., Liu, L., Chan, C.H., and Kao, C.Y., 2007. A genetic algorithm for resident physician scheduling problem, *in: Proceedings of the* 9^{th} *Annual Conference on Genetic and Evolutionary Computation*, New York, NY, USA: ACM, GECCO '07, 2203–2210.

Xu, B., Xu, L., Cai, H., Jiang, L., Luo, Y., and Gu, Y., 2017. The design of an m-health monitoring system based on a cloud computing platform, *Enterprise Information Systems*, 11 (1), 17–36.