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EXPONENTIAL STABILITY IN THREE-DIMENSIONAL TYPE III THERMO-POROUS-ELASTICITY WITH MICROTEMPERATURES

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Abstract: We study the time decay of the solutions for the type III thermoelastic theory with microtemperatures and voids. We prove that, under suitable conditions for the constitutive tensors, the solutions decay exponentially. This fact is in somehow striking because it differs from the behaviour of the solutions in the classical model of thermoelasticity with microtemperatures and voids, where the exponential decay is not expected in the general case.

Keywords: type III thermoelasticity with voids, microtemperatures, exponential decay.

1. Introduction and basic equations

In the last fifty years new theories have been proposed to describe appropriately thermal phenomena. As it is known, the Fourier model when it is combined with the equation of heat

$$c\dot{\theta} = -\nabla \mathbf{q}$$

gives rise to some paradoxes which scientists prefer to avoid (in the above equation θ denotes the temperature, $\mathbf{q}$ the heat flux and c the thermal capacity). Among these new theories, let us cite the thermoelastic models proposed by Green and Lindsay [8], by Lord and Shulman [21] or the two temperatures theory proposed by Chen and Gurtin (see [3], among others). Green and Naghdi proposed three other thermoelastic theories that they named of type I, II and III, respectively [9, 10, 11]. The linear theory of type I coincides with the Fourier theory. In the types II and III a new independent variable, the thermal displacement, is introduced. The type III, that we consider here, is the most general because it contains the former two as particular or limit cases.

At the same time, a growing interest has been developed to introduce and analyse models where the microstructure of the material is taken into account [6, 12, 13, 14, 15, 16, 17, 22]. Microtemperatures have been also introduced (see, for instance, [2, 4, 5, 28, 29, 30]). These models are being studied, jointly with the Fourier heat conduction theory and also with the Green and Naghdi proposals. Iesan and Quintanilla [16] gave a model for the Green and Naghdi thermoelasticity of type II with microtemperatures and proved several qualitative properties for the solutions. Aouadi, Ciarletta and Passarella [1] extended some of these results to the type III model.

There are interesting physical applications of the thermoelasticity with voids, such as the study of solids with small distributed porous, let us mention, for instance, rocks, soils, woods, ceramics

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or even biological materials as bones. Microtemperatures have also been applied to physical situations (an amazing one is the heat conduction in blood).

From the mathematical point of view, all these theories drive to different systems of partial differential equations. Usually, the mechanical variables are determined by conservative equations, while the thermal variables, with the exception of the type II theory, determine dissipative processes. A natural problem to investigate is knowing when the dissipation mechanisms proposed by the thermal variables and the coupling terms are strong enough to make the thermoelastic perturbations to be damped in an exponential way. For the classical thermoelasticity with voids and microtemperatures, Casas and Quintanilla [2] proved that, in the one-dimensional case, the system of equations leads to perturbations that decay exponentially. A similar result holds for the type III thermoelasticity with microtemperatures for dimensions greater than one [23]. In this work we analyse the decay in time of the solutions to the system of equations that comes from the type III thermoelasticity with voids and microtemperatures and we prove that, again for dimensions greater than one, the solutions decay exponentially (an extension of [23]).

First of all, we describe the problem. We are going to work in a three dimensional bounded domain Ω with boundary smooth enough to allow the application of the divergence theorem. We will use the standard notation where "$,i$" means the partial derivative with respect to the variable x_i, a superposed dot means time derivative and summation on repeated indices is assumed.

The system of equations that we want to study is the following (see [1], [15] or [16]):

(1.1)
$$\begin{cases} \rho\ddot{u}_i = (A_{ijkl}u_{k,l} - a_{ij}\theta + \zeta_{ij}\varphi + B_{ijkl}R_{k,l})_{,j} \\ J\ddot{\varphi} = (A_{ij}\varphi_{,j} - \alpha_{ij}\dot{R}_i + H_{ij}\tau_{,i})_{,j} - \zeta_{ij}u_{i,j} + \kappa\dot{\tau} - F_{ij}R_{i,j} - \xi\varphi \\ c\ddot{\tau} = -a_{ij}\dot{u}_{i,j} + (H_{ij}\varphi_{,i})_{,j} - (d_{ij}\dot{R}_i)_{,j} + (K_{ij}\tau_{,i} + K^*_{ij}\dot{\tau}_{,i})_{,j} - b_{ij}\dot{R}_{i,j} - \kappa\dot{\varphi} + (A^{(1)}_{ij}M_i)_{,j} \\ c_{ij}\ddot{R}_j = (B_{klij}u_{k,l} - b_{ij}\dot{\tau} + F_{ij}\varphi + C_{ijkl}R_{k,l})_{,j} - d_{ij}\dot{\tau}_{,j} + (C^*_{ijkl}\dot{R}_{k,l})_{,j} - \alpha_{ij}\dot{\varphi}_{,j} - A^{(2)}_{ij}\theta_{,j} - A^{(3)}_{ij}M_j \end{cases}$$

Here, u_i is the displacement vector, φ is the volume fraction, θ is the temperature and M_i are the microtemperatures. Moreover: τ is the *thermal displacement* introduced by Green and Naghdi and R_i are the *microthermal displacements*, defined respectively by:

$$\tau(\mathbf{x},t) = \tau_0(\mathbf{x}) + \int_0^t \theta(\mathbf{x},s)\,ds \text{ and } R_i(\mathbf{x},t) = R^0_i(\mathbf{x}) + \int_0^t M_i(\mathbf{x},s)\,ds.$$

As usual, ρ denotes the mass density, J the product of the mass density by the equilibrated inertia and c the thermal capacity. A_{ijkl} is the elasticity tensor, a_{ij}, ζ_{ij} and B_{ijkl} are, respectively, the coupling tensors between the displacement and the temperature, the displacement and the volume fraction, and the displacement and the microtemperatures. A_{ij}, $A^{(1)}_{ij}$, $A^{(2)}_{ij}$, $A^{(3)}_{ij}$, α_{ij}, H_{ij}, F_{ij}, d_{ij} and b_{ij} are other coupling tensors between the variables. K_{ij} is the tensor introduced by Green and Naghdi and it is usually called *rate conductivity*, K^*_{ij} is the thermal conductivity tensor, c_{ij} is a typical tensor of the theories with microtemperatures, and, finally, C_{ijkl} and C^*_{ijkl} are the specific type III tensors with microtemperatures.

We need to impose initial and boundary conditions. As initial conditions we assume

$$\begin{aligned} &u_i(\mathbf{x},0) = u^0_i(\mathbf{x}), \dot{u}_i(\mathbf{x},0) = v^0_i(\mathbf{x}), \varphi(\mathbf{x},0) = \varphi^0(\mathbf{x}), \dot{\varphi}(\mathbf{x},0) = \omega^0(\mathbf{x}) \\ &\tau(\mathbf{x},0) = \tau^0(\mathbf{x}), \dot{\tau}(\mathbf{x},0) = \theta^0(\mathbf{x}), R_i(\mathbf{x},0) = R^0_i(\mathbf{x}), \dot{R}_i(\mathbf{x},0) = M^0_i(\mathbf{x}) \text{ for } \mathbf{x} \in \Omega. \end{aligned} \tag{1.2}$$

We impose homogeneous Dirichlet boundary conditions:

$$u_i(\mathbf{x},t) = \varphi(\mathbf{x},t) = \tau(\mathbf{x},t) = R_i(\mathbf{x},t) = 0 \text{ for } \mathbf{x} \in \partial\Omega, t \geq 0. \tag{1.3}$$

The following symmetries are satified (see [1]):

(1.4)
$$A_{ijkl}=A_{klij}, A_{ij}=A_{ji}, K_{ij}=K_{ji}, K^*_{ij}=K^*_{ji}, C_{ijkl}=C_{klij}, C^*_{ijkl}=C^*_{klij}, c_{ij}=c_{ji}, A^{(3)}_{ij}=A^{(3)}_{ji}.$$

From the second law of thermodynamics (see [15]) the following inequality must be satisfied:

$$K^*_{i,j}\xi_i\xi_j+(A^{(1)}_{ij}+A^{(2)}_{ij})\eta_i\xi_j+A^{(3)}_{ij}\eta_i\eta_j+C^*_{ijkl}\eta_{ij}\eta_{kl}\geq K_0(\xi_i\xi_i+\eta_i\eta_i+\eta_{ij}\eta_{ij}), \tag{1.5}$$

for a positive constant K_0 and for each pair of vectors ξ_i and η_i and for each tensor η_{ij}.

The type III theory proposes a frame where some new couplings appear [7, 20, 26, 27], couplings which were not present in the classical theory. Let us emphasize, for example, the tensors H_{ij} and B_{ijkl} of system (1.1). Those couplings are significant enough to make the qualitative results concerning the solutions of the new system of equations differ from the ones obtained for the classical theory or even from other known results for non-classical models. For example, Leseduarte, Magaña and Quintanilla [18] showed the exponential decay with respect to the time of the solutions for the one-dimensional type II thermoelasticity with voids when only the porous dissipation is present. Miranville and Quintanilla [24, 25] obtained the same result introducing viscosity effects in the model and even for the type II/III models. These are some nice examples of models where only one dissipation mechanism drives a system of three hyperbolic equations to the exponential stability. Magaña and Quintanilla [23] proved also the exponential decay of the solutions for the three-dimensional type III theory with microtemperatures, another relevant difference between the classical and the type three teories.

2. Existence and uniqueness of solutions

We use the semigroup arguments to give an existence theorem to the problem defined by (1.1)–(1.3). But first, we need to impose some assumptions over the constitutive coefficients. For each vector ξ_i, each pair of tensors ξ_{ij} and η_{ij} and each real number l, there exist positive constants $C_0, C_1, C_2, \rho_0, J_0$ and c_0 such that:

$$\begin{aligned}
&A_{ijkl}\xi_{ij}\xi_{kl}+2B_{ijkl}\xi_{ij}\eta_{kl}+C_{ijkl}\eta_{ij}\eta_{kl}+2\zeta_{ij}\xi_{ij}l+\xi\, l^2\geq C_0(\xi_{ij}\xi_{ij}+\eta_{ij}\eta_{ij}+l^2),\\
&A_{ij}\xi_i\xi_j+2H_{ij}\xi_i\eta_j+K_{ij}\eta_i\eta_j\geq C_1(\xi_i\xi_i+\eta_i\eta_j),\\
&c_{ij}\xi_i\xi_j\geq C_2\xi_i\xi_i,\ \rho\geq\rho_0>0,\ J\geq J_0>0,\ c\geq c_0>0.
\end{aligned} \tag{2.1}$$

These assumptions agree with the thermomechanical axioms and are compatible with the empirical observations. The first and second conditions in (2.1) can be interpreted by means of the elastic stability. The others are rather obvious.

The initial-boundary value problem (1.1)–(1.3) can be transformed into an abstract problem on a suitable Hilbert space. Let us introduce the following notation: $v_i=\dot{u}_i,\ \omega=\dot{\varphi},\ \theta=\dot{\tau},\ M_i=\dot{R}_i$. We consider the Hilbert space $\mathcal{H}$ defined by

$$\mathcal{H}=\{\mathcal{U}=(u_i,v_i,\varphi,\omega,\tau,\theta,R_i,M_i): u_i,\varphi,\tau,R_i\in W^{1,2}_0(\Omega), v_i,\omega,\theta,M_i\in L^2(\Omega)\},$$

where $W^{1,2}_0(\Omega)$ and $L^2(\Omega)$ are the usual Sobolev spaces. The inner product that we define in $\mathcal{H}$, which is equivalent to the usual one, is

(2.2)
$$\langle\mathcal{U},\mathcal{U}^*\rangle=\frac{1}{2}\int_\Omega\Big(\rho v_i\overline{v}^*_i+J\omega\overline{\omega}^*+c\theta\overline{\theta}^*+c_{ij}M_i\overline{M}^*_j+2\mathcal{W}[(u_i,\varphi,\tau,R_i),(u^*_i,\varphi^*,\tau^*,R^*_i)]\Big)\,dV,$$

where a bar over a variable is used to denote its complex conjugate and

$$
\begin{aligned}
2\mathcal{W}[(u_i,\varphi,\tau,R_i),(u_i^*,\varphi^*,\tau^*,R_i^*)] &= A_{ijkl}u_{i,j}\overline{u}_{k,l} + B_{ijkl}(u_{i,j}\overline{R}^*_{k,l} + \overline{u}^*_{i,j}R_{k,l}) + C_{ijkl}R_{i,j}\overline{R}^*_{k,l} \\
&+ \zeta_{ij}(u_{i,j}\overline{\varphi}^* + \overline{u}^*_{i,j}\varphi) + \xi\varphi^2 + A_{ij}\varphi_{,i}\overline{\varphi}^*_{,j} + H_{ij}(\varphi_{,i}\overline{\tau}^*_{,j} + \overline{\varphi}^*_{,i}\tau_{,j}) + K_{ij}\tau_{,i}\overline{\tau}^*_{,j}.
\end{aligned}
$$

In order to obtain a synthetic expression for our problem we consider the operators defined below:

$$
\begin{array}{ll}
A_i(\mathbf{u}) = \frac{1}{\rho}(A_{ijkl}u_{k,l})_{,j} & B_i(\theta) = -\frac{1}{\rho}(a_{ji}\theta)_{,j} \\
C_i(\mathbf{R}) = \frac{1}{\rho}(B_{jikl}R_{l,k})_{,j} & D(\mathbf{v}) = -\frac{1}{c}a_{ij}v_{i,j} \\
E(\tau) = \frac{1}{c}(K_{ij}\tau_{,i})_{,j} & F_i(\varphi) = \frac{1}{\rho}(\zeta_{ij}\varphi)_{,j} \\
G(\theta) = \frac{1}{c}(K^*_{ij}\theta_{,i})_{,j} & H(\varphi) = \frac{1}{J}\big((A_{ij}\varphi_{,j})_j - \xi\varphi\big) \\
K(\mathbf{u}) = -\frac{1}{J}\zeta_{ij}u_{i,j} & N(\mathbf{M}) = -\frac{1}{J}(\alpha_{ij}M_i)_{,j} \\
Q(\tau) = \frac{1}{J}(H_{ij}\tau_{,i})_{,j} & S(\theta) = \frac{1}{J}\kappa\theta \\
T(\mathbf{R}) = -\frac{1}{J}(F_{ij}R_{i,j}) & V(\varphi) = \frac{1}{c}(H_{ij}\varphi_{,i})_{,j} \\
X(\mathbf{M}) = -\frac{1}{c}\big((d_{ij}M_i)_j + b_{ij}M_{i,j} - (A^{(1)}_{ij}M_i)_{,j}\big) & L_s(\mathbf{u}) = l_{si}\left(B_{klij}u_{k,l}\right)_{,j} \\
Z_s(\theta) = -l_{si}\left((b_{ji}\theta)_{,j} + d_{ij}\theta_{,j} + A^{(2)}_{ij}\theta_{,j}\right) & N_s(\mathbf{R}) = l_{si}\left(C_{jikl}R_{k,l}\right)_{,j} \\
P_s(\mathbf{M}) = l_{si}\left((C^*_{jikl}M_{k,l})_{,j} - A^{(3)}_{ij}M_j\right) & A^*_s(\varphi) = l_{si}(F_{ij}\varphi)_{,j} \\
B^*_s(\omega) = -l_{si}(\alpha_{ij}\omega_{,j}) & L^*(\omega) = -\frac{\kappa}{c}\omega
\end{array}
$$

where l_{si} is the inverse of c_{ij}, that is, $l_{si}c_{ij} = \delta_{sj}$, being δ_{sj} the Kronecker delta. Using the above operators, problem (1.1)–(1.3) can be written as

$$
\frac{d}{dt}\mathcal{U}(t) = \mathcal{A}\mathcal{U}(t), \quad \mathcal{U}(0) = \mathcal{U}_0, \tag{2.3}
$$

where $\mathcal{U}_0 = \left(u_i{}^0, v_i{}^0, \varphi^0, \omega^0, \tau^0, \theta^0, R_i^0, M_i^0\right)$, and $\mathcal{A}$ is the following matrix operator

$$
\mathcal{A} = \begin{pmatrix}
0 & \mathbf{Id} & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{A} & 0 & \mathbf{F} & 0 & 0 & \mathbf{B} & \mathbf{C} & 0 \\
0 & 0 & 0 & Id & 0 & 0 & 0 & 0 \\
K & 0 & H & 0 & Q & S & T & N \\
0 & 0 & 0 & 0 & 0 & Id & 0 & 0 \\
0 & D & V & L^* & E & G & 0 & X \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{Id} \\
\mathbf{L} & 0 & \mathbf{A}^* & \mathbf{B}^* & 0 & \mathbf{Z} & \mathbf{N} & \mathbf{P}
\end{pmatrix}, \tag{2.4}
$$

where $\mathbf{A} = (A_i)$, $\mathbf{B} = (B_i)$, $\mathbf{C} = (C_i)$, $\mathbf{F} = (F_i)$, $\mathbf{L} = (L_s)$, $\mathbf{A}^* = (A^*_i)$, $\mathbf{B}^* = (B^*_i)$, $\mathbf{Z} = (Z_s)$, $\mathbf{N} = (N_s)$ and $\mathbf{P} = (P_s)$.

The domain of the operator $\mathcal{A}$ is $D(\mathcal{A}) = \{\mathcal{U} \in \mathcal{H} : \mathcal{A}\mathcal{U} \in \mathcal{H}\}$ which is dense in $\mathcal{H}$.

Lemma 2.1. *The operator $\mathcal{A}$ is dissipative. That is, for every $\mathcal{U} \in D(\mathcal{A})$, $\Re\langle \mathcal{A}\mathcal{U}, \mathcal{U}\rangle_{\mathcal{H}} \leq 0$.*

Proof. A direct calculation shows that

$$
\Re\langle \mathcal{A}\mathcal{U}, \mathcal{U}\rangle_{\mathcal{H}} = -\frac{1}{2}\int_\Omega \big(K^*_{ij}\theta_{,i}\overline{\theta}_{,j} + \frac{1}{2}(A^{(1)}_{ij} + A^{(2)}_{ij})(M_i\overline{\theta}_{,j} + \overline{M}_i\theta_{,j}) + A^{(3)}_{ij}M_i\overline{M}_j + C^*_{ijkl}M_{i,j}\overline{M}_{k,l}\big)dV.
$$

In view of assumptions (1.5), this quantity is negative and, therefore, the lemma is proved. □

Lemma 2.2. *The operator $\mathcal{A}$ satisfies that* 0 *belongs to the resolvent of $\mathcal{A}$ (in short, $0 \in \rho(\mathcal{A})$).*

Proof. This result can be proved straightaway by solving the corresponding system of equations. Usually, this can be done by using the Lax-Milgram lemma. □

Theorem 2.3. *The operator $\mathcal{A}$ generates a C_0-semigroup of contractions $S(t) = \{e^{\mathcal{A}t}\}_{t\geq 0}$ in $\mathcal{H}$ and, therefore, assuming that $\mathcal{U}_0 \in D(\mathcal{A})$, there exists a unique solution $\mathcal{U}(t) \in C^1([0,\infty),\mathcal{H}) \cap C^0([0,\infty), D(\mathcal{A}))$ to problem (2.3).*

Proof. The theorem is a direct consequence of the two previous lemmas and the use of the Lumer-Phillips corollary to the Hille–Yosida theorem. □

3. Exponential decay of solutions

We need to impose additional conditions over the constitutive tensors. We will assume that for every tensor ξ_{ij} and every vector ξ_i there exist two positive constants C and C^* such that

$$
\begin{aligned}
&B_{klij}\xi_{kl}\xi_{ij} \geq C\xi_{ij}\xi_{ij} \quad \text{or} \quad B_{klij}\xi_{kl}\xi_{ij} \leq -C\xi_{ij}\xi_{ij} \quad \text{and} \quad B_{ijkl} = B_{klij} \\
&H_{ij}\xi_i\xi_j \geq C^*\xi_i\xi_i \quad \text{or} \quad H_{ij}\xi_i\xi_j \leq -C^*\xi_i\xi_i \quad \text{and} \quad H_{ij} = H_{ji}.
\end{aligned} \tag{3.1}
$$

The above assumptions are quite natural in our context. However, two more technical conditions on some of the tensors are needed. Let us suppose that there exist two constants, m_1 and m_2 such that

$$
a_{ij} = m_1\zeta_{ij} \quad \text{and} \quad \alpha_{ij} = m_2\zeta_{ij}. \tag{3.2}
$$

Notice that for *isotropic and homogeneous materials* assumptions (3.2) are satisfied whenever $\zeta_{ij} = \zeta\delta_{ij}$ for a constant $\zeta \neq 0$. For transversally isotropic but non homogeneous materials we know that a_{ij}, α_{ij} and ζ_{ij} are 0 when $i \neq j$. In this case, conditions (3.2) will be satisfied whenever $a_{11}(\mathbf{x}) = m_1\zeta_{11}(\mathbf{x})$, $a_{22}(\mathbf{x}) = m_1\zeta_{22}(\mathbf{x})$, $a_{33}(\mathbf{x}) = m_1\zeta_{33}(\mathbf{x})$, $\alpha_{11}(\mathbf{x}) = m_2\zeta_{11}(\mathbf{x})$, $\alpha_{22}(\mathbf{x}) = m_2\zeta_{22}(\mathbf{x})$ and $\alpha_{33}(\mathbf{x}) = m_2\zeta_{33}(\mathbf{x})$.

We recall a characterization that ensures the exponential decay (see, for example, [19]).

Theorem 3.1. *Let $S(t) = \{e^{\mathcal{A}t}\}_{t\geq 0}$ be a C_0-semigroup of contractions on a Hilbert space. Then $S(t)$ is exponentially stable if and only if $i\mathbb{R} \subset \rho(\mathcal{A})$ and $\overline{\lim}_{|\lambda|\to\infty} \|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$.*

Lemma 3.2. *The operator $\mathcal{A}$ defined at (2.4) satisfies that $i\mathbb{R} \subset \rho(\mathcal{A})$.*

Proof. We follow the arguments given by Liu and Zheng ([19], page 25). Let us suppose that the intersection of the imaginary axis and the spectrum is non-empty. Therefore, there exist a sequence of real numbers λ_n with $\lambda_n \to \varpi$, $|\lambda_n| < |\varpi|$ and a sequence of vectors $\mathcal{U}_n = (\mathbf{u}_n, \mathbf{v}_n, \varphi_n, \omega_n, \tau_n, \theta_n, \mathbf{R}_n, \mathbf{M}_n)$ in $D(\mathcal{A})$ and with unit norm such that $\|(i\lambda_n\mathcal{I} - \mathcal{A})\mathcal{U}_n\| \to 0$.

If we write the above expression in components, we obtain the following conditions:

$$
i\lambda_n\mathbf{u}_n - \mathbf{v}_n \to \mathbf{0}, \text{ in } \mathbf{W}^{1,2} \tag{3.3}
$$

$$
i\lambda_n\mathbf{v}_n - \mathbf{A}\mathbf{u}_n - \mathbf{F}\varphi_n - \mathbf{B}\theta_n - \mathbf{C}\mathbf{R}_n \to \mathbf{0}, \text{ in } \mathbf{L}^2 \tag{3.4}
$$

$$
i\lambda_n\varphi_n - \omega_n \to 0, \text{ in } W^{1,2} \tag{3.5}
$$

$$
i\lambda_n\omega_n - K\mathbf{u}_n - H\varphi_n - Q\tau_n - S\theta_n - T\mathbf{R}_n - N\mathbf{M}_n \to \mathbf{0}, \text{ in } \mathbf{L}^2 \tag{3.6}
$$

$$
i\lambda_n\tau_n - \theta_n \to 0, \text{ in } W^{1,2} \tag{3.7}
$$

$$i\lambda_n\theta_n - D\mathbf{v}_n - V\varphi_n - L^*\omega_n - E\tau_n - G\theta_n - X\mathbf{M}_n \to 0, \text{ in } L^2 \tag{3.8}$$

$$i\lambda_n\mathbf{R}_n - \mathbf{M}_n \to \mathbf{0}, \text{ in } \mathbf{W}^{1,2} \tag{3.9}$$

$$i\lambda_n\mathbf{M}_n - \mathbf{L}\mathbf{u}_n - \mathbf{A}^*\varphi_n - \mathbf{B}^*\omega_n - \mathbf{Z}\theta_n - \mathbf{N}\mathbf{R}_n - \mathbf{P}\mathbf{M}_n \to \mathbf{0}, \text{ in } \mathbf{L}^2. \tag{3.10}$$

In view of the dissipative term, we see that $\theta_n \to 0$ in $W^{1,2}$ and $\mathbf{M}_n \to \mathbf{0}$ in $\mathbf{W}^{1,2}$.

From (3.7) we also have that $\lambda_n\tau_n \to 0$ in $W^{1,2}$, and from (3.9), $\lambda_n\mathbf{R}_n \to 0$ in $\mathbf{W}^{1,2}$. If we multiply (3.8) by φ_n we obtain that

$$\langle i\lambda_n\theta_n, \varphi_n\rangle - \langle D\mathbf{v}_n, \varphi_n\rangle - \langle V\varphi_n, \varphi_n\rangle - \langle L^*\omega_n, \varphi_n\rangle - \langle E\tau_n, \varphi_n\rangle - \langle G\theta_n, \varphi_n\rangle - \langle X\mathbf{M}_n, \varphi_n\rangle \to 0. \tag{3.11}$$

In view of the definition of the operators and taking into account the convergence of $\theta_n, \mathbf{M}_n$ and the fact that φ_n is bounded in $W^{1,2}$ the above expression reduces to

$$\langle i\lambda_n\theta_n, \varphi_n\rangle - \langle D\mathbf{v}_n, \varphi_n\rangle - \langle V\varphi_n, \varphi_n\rangle - \langle L^*\omega_n, \varphi_n\rangle \to 0. \tag{3.12}$$

From (3.5), the last expression can be rewritten as

$$\langle i\lambda_n\theta_n, \varphi_n\rangle - \langle D\mathbf{v}_n, \varphi_n\rangle - \langle V\varphi_n, \varphi_n\rangle - i\lambda_n\langle L^*\varphi_n, \varphi_n\rangle \to 0. \tag{3.13}$$

Notice also that $\langle i\lambda_n\theta_n, \varphi_n\rangle = \langle \theta_n, -i\lambda_n\varphi_n\rangle$, and, hence, it tends to 0 because θ_n tends to 0 and $i\lambda_n\varphi_n$ is bounded (recall (3.5)). Therefore, we see that

$$\int_\Omega \left(H_{ij}(\varphi_n)_{,i}(\overline{\varphi_n})_{,j} + i\lambda_n a_{ij}(u_n)_{i,j}\overline{\varphi}_n + i\lambda_n\kappa|\varphi_n|^2\right) dV \to 0.$$

We want to prove that $(\varphi_n)_{,i} \to 0$. It will be sufficient to show that

$$\int_\Omega (a_{ij}(u_n)_{i,j}\overline{\varphi}_n)\, dV$$

tends to a real number. If we multiply (3.6) by φ_n we obtain

$$-\langle \omega_n, \omega_n\rangle + \langle H\varphi_n, \varphi_n\rangle + \langle K\mathbf{u}_n, \varphi_n\rangle \to 0. \tag{3.14}$$

From this last expression we see that $\langle K\mathbf{u}_n, \varphi_n\rangle$ tends to a real number. In view of the first condition in (3.2), it is clear now that $\varphi_n \to 0$ in $W^{1,2}$.

The next step is to see that $\mathbf{u}_n \to 0$ in $\mathbf{W}^{1,2}$. To this end, we multiply (3.10) by $\mathbf{u}_n$:

$$\langle i\lambda_n\mathbf{M}_n, \mathbf{u}_n\rangle - \langle \mathbf{L}\mathbf{u}_n, \mathbf{u}_n\rangle - \langle \mathbf{B}^*\omega_n, \mathbf{u}_n\rangle - \langle \mathbf{N}\mathbf{R}_n, \mathbf{u}_n\rangle - \langle \mathbf{P}\mathbf{M}_n, \mathbf{u}_n\rangle \to 0.$$

From the definitions of $\mathbf{N}$ and $\mathbf{P}$ and taking into account that $\mathbf{u}_n$ is bounded we obtain

$$\langle \mathbf{M}_n, -i\lambda_n\mathbf{u}_n\rangle - \langle \mathbf{L}\mathbf{u}_n, \mathbf{u}_n\rangle - \langle \mathbf{B}^*\omega_n, \mathbf{u}_n\rangle \to 0.$$

Therefore, as $\lambda_n\mathbf{u}_n$ is bounded, we get

$$\int_\Omega (B_{klij}(\mathbf{u}_n)_{i,j}(\overline{\mathbf{u}}_n)_{l,k} + i\lambda_n\alpha_{ij}\varphi_n(\overline{\mathbf{u}}_n)_{i,j})\, dV \to 0.$$

Notice that our claim will be proved whether

$$\int_\Omega \alpha_{ij}\varphi_n(\overline{\mathbf{u}}_n)_{i,j} dV$$

tends to a real number. In order to prove this last statement we multiply (3.4) by $\mathbf{u}_n$:

$$-\langle \mathbf{v}_n, \mathbf{v}_n\rangle + \langle \mathbf{A}\mathbf{u}_n, \mathbf{u}_n\rangle + \langle \mathbf{F}\varphi_n, \mathbf{u}_n\rangle \to 0. \tag{3.15}$$

From the second condition in assumptions (3.2) we get

$$\int_\Omega B_{klij}(\mathbf{u}_n)_{i,j}(\overline{\mathbf{u}}_n)_{l,k}dV \to 0$$

and, therefore, $\mathbf{u}_n \to 0$ in $\mathbf{W}^{1,2}$. Finally, from (3.14) and (3.15), we obtain that ω_n and $\mathbf{v}_n$ tend to zero in L^2 and in $\mathbf{L}^2$, respectively. These facts lead to a contradiction and, hence, the lemma is proved. □

Lemma 3.3. *The operator $\mathcal{A}$ defined at (2.4) satisfies that* $\overline{\lim}_{|\lambda|\to\infty} \|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$.

Proof. The proof is similar to the one proposed for Lemma 3.2. □

Theorem 3.4. *The C_0-semigroup $S(t) = \{e^{\mathcal{A}t}\}_{t\geq 0}$ is exponentially stable. That is, there exist two positive constants M and α such that $\|S(t)\| \leq M\|S(0)\|e^{-\alpha t}$.*

Proof. The proof is a direct consequence of Lemma 3.2, Lemma 3.3 and Theorem 3.1. □

4. Conclusions

In this paper we have showed another difference between the classical and the type III thermoelasticity theories with respect to the time behaviour of the solutions. The perturbations for the type III thermoelasticity with microtemperatures and voids decay in an exponential way. This result is not true for the classical thermoelasticity. The exponential decay is a consequence of the new coupling tensors, which are not present in the classical theory.

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