Universal Intervals in the Homomorphism Order of Digraphs

Pablo Oviedo Timoneda

Advisor: Jan Hubička[1]

Co-advisor: Oriol Serra[2]

March, 2019

[1]Department of Applied Mathematics, Charles University
[2]Departament de Matemàtiques, Universitat Politècnica de Catalunya

Abstract

In this thesis we solve some open problems related to the homomorphism order of digraphs. We begin by introducing the basic concepts of graphs and homomorphisms and studying some properties of the homomorphism order of digraphs. Then we present the new results. First, we show that the class of digraphs containing cycles has the fractal property (strengthening the density property). Then we show a density theorem for the class of proper oriented trees. Here we say that a oriented tree is proper if it is not a path. Such result was claimed in 2005 but none proof have been published ever since. We also show that the class of proper oriented trees, in addition to be dense, has the fractal property. We end by considering the consequences of these results and the remaining open questions in this area.

Keywords: graph, digraph, homomorphism, partial order, homomorphism order, oriented tree, density, universality, fractal property.

Acknowledgements

First, I would like to express my deep gratitude to Jan Hubička, for introducing me such an amazing problems and encouraging me to try to solve them. He has always been patient by giving me advice and assistance every time I needed. This project would not have been possible without his guidance. I also want to thank Jaroslav Někesetřil for his wise advice and support, it is always an honour to have helpful discussions with him. My thanks are also extended to Oriol Serra for recommending me to work on this project and helping me whenever I needed. I wish to acknowledge the help provided by CFIS during the mobility and my years of learning, making always things easier. Finally, I would like to thank my family and friends, and specially Júlia, for their continuous support.

Contents

Preface

A homomorphism between two directed graphs (or digraphs) is a mapping between its vertex sets which preserves adjacency and direction of arcs. The behaviour and properties of homomorphisms have been extensively studied during the last two decades obtaining many good results. For this reason, the interest in this topic is increasing over the years. One concept which have been studied in particular is the homomorphism order. Given two digraphs G_1, G_2, we say that $G_1 \leq G_2$ if there exists a homomorphism from G_1 to G_2. The relation "$\leq$" is called the homomorphism order, and induces a quasiorder on the class of digraphs. This quasiorder can be extended into a partial order by choosing a representative for each equivalence class, in our case the so called core. A core of a digraph is its minimal homomorphic equivalent subgraph. Thus, the class of cores becomes a partially ordered set.

There are many results showing the richness of the homomorphism order on different classes of digraphs. In this thesis we contribute to this research by proving the following three new theorems. We say that a partially ordered set is universal if it contains every other countable partially ordered set as a suborder.

Theorem (3.3.2)**.** *Let G and H be two finite digraphs satisfying $G < H$. If the core of H is connected and contains a cycle, then the interval $[G, H]$ is universal.*

Theorem (4.3.3)**.** *Let T_1 and T_2 be two finite oriented trees satisfying $T_1 < T_2$. If the core of T_2 is different from a path, then there exists a oriented tree T such that $T_1 < T < T_2$.*

Theorem (4.4.1)**.** *Let T_1 and T_2 be two finite oriented trees satisfying $T_1 < T_2$. If the core of T_2 is different from a path, then the interval $[T_1, T_2]$ is universal.*

The first theorem strengthen the density property of the class of digraphs whose cores contain a cycle. The second result, which we consider to be a surprising result, was claimed in 2005 but the proof was never written. It shows the density property of the class of proper trees, which is the class of oriented trees whose cores are not paths. The third theorem strengthen the second one by showing that every interval of the class of proper trees, in addition to be dense, is universal.

The objective of this thesis is to introduce the homomorphism order and some concepts related to digraphs and homomorphisms in order to understand and prove the new results. The writing aims to be self-contained. However, we shall state some theorems without proof due to its difficulty and no so close relation to the main results. The text is structured as follows.

In Chapter 1 we first present the basic definitions and properties of homomorphisms and digraphs. Then we introduce some more advanced concepts as the product of digraphs, the exponential digraph and the cores, which will be used in the next chapters. The content of the chapter is based in the book "Graph and homomorphisms" by Hell and Nešetřil and we shall use its standard notation. To further information and a more general introduction on the topic we strongly recommend it [7].

In the first sections of Chapter 2 we introduce the homomorphism order. Then we characterise dense intervals and gaps in the homomorphism order of digraphs and the homomorphism order of undirected graphs. In the fourth section we prove Theorem 2.4.2 which is an original result and will be used to show Theorem 3.3.2.

In Chapter 3 we define the universal and fractal properties. Then we show that every interval in the homomorphism order of the class of graphs is universal, with only one trivial exception [3]. In the last section we prove Theorem 3.3.2.

In Chapter 4 we prove Theorems 4.3.3 and 4.4.1. Before that we first introduce some properties of the class of oriented trees and oriented paths. Every lemma of Chapters 3 and 4 used to show the new theorems is also a new result developed in this thesis.

Finally, in Chapter 5 we discuss the implications of the results and consider the remaining open questions in this area.

Chapter 1

Introduction to Digraphs and Homomorphisms

1.1 Basic Definitions

A *digraph* G is an ordered pair of sets (V, E) where $V = V(G)$ is a set of elements called *vertices* and $E = E(G)$ is a binary relation on V. The elements (u, v) of $E(G)$ are called *arcs* and we shall denote them as uv. An arc of the form (u, u) is called a *loop*. A digraph G is *finite* if $V(G)$ is finite. Note that in this case $E(G)$ would be also finite. We say that a digraph G is *symmetric*, or *irreflexive*, or etc., if $E(G)$ is symmetric, or irreflexive, or etc., respectively. Note that a digraph is irreflexive if and only if it doesn't contain any loop.

A *simple graph* or *graph* G is an ordered pair of sets (V, E) where $V = V(G)$ is a set of *vertices* and $E = E(G)$ is a set of *edges*, which are sets of vertices of size two. We shall denote an edge $\{u, v\}$ as uv or vu. Observe that we are using the same notation than arcs, so depending on the context uv will mean an arc (u, v) or an edge $\{u, v\}$. A graph G is *finite* if $V(G)$ is finite. Most commonly, in texts on graph theory, *graph* means "finite simple graph".

For this thesis we shall consider, for now on, every graph to be finite and simple, and every digraph to be finite and irreflexive.

Observe that one can also define graphs as symmetric digraphs by replacing each edge $\{u, v\}$ by the arcs (u, v) and (v, u). For this reason every definition or property on digraphs can be applied to graphs. In fact, we shall view the class of graphs as a subclass of the class of digraphs, via their corresponding symmetric digraphs.

An *orientation of* a graph G is a digraph obtained by replacing each edge $\{u, v\}$ by exactly one of the arcs (u, v) or (v, u). An *oriented graph* is a digraph obtained from an orientation of some graph. It can be observed that a digraph is an oriented graph if and only if it has not any pair of symmetric arcs. There are more natural transformations between graphs and digraphs but for the content of this thesis we are not interested in more of them.

We are giving below some basic definitions of graph theory. We shall give all of them in terms of digraphs, but they might also be applied to graphs considering a graph as a symmetric digraph.

Given a digraph G, if $uv \in E(G)$ we shall say that v is an *outneighbour* of u and u is an *inneighbour* of v. In the case of a graph we have that $uv = vu$. We shall say that u and v are adjacent as long as at least one of uv, vu is an arc of $E(G)$; in this case we shall also say that u and v are *neighbours*. The number of neighbours of a vertex u is called the *degree* of u.

Given two digraphs G and H, we say that G is a *subgraph* of H if $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$. In such case we shall write $G \subseteq H$. Given a digraph H and a subset $V(G) \subseteq V(H)$, the *digraph induced* by $V(G)$ is the digraph $G = (V(G), E(G))$ where $E(G) = \{uv \mid u, v \in V(G)\ uv \in E(H)\}$. In this case we say that G is an *induced subgraph* of H. Given two digraphs G, H such that $G \subseteq H$, then $H \backslash G$ is the digraph with $V(H) \backslash V(G)$ as set of vertices and $E(H) \backslash E(G)$ as set of arcs.

Finally, a digraph is *complete* if every pair of vertices are adjacent. We shall denote by K_n the complete graph with n vertices. We shall refer to an arbitrary orientation of K_n as $\vec{K}_n$. Note that $\vec{K}_1 = K_1$.

1.2 Definition of Homomorphism

Let G and H be two digraphs. A *homomorphism* from G to H is a mapping $f : V(G) \to V(H)$ such that if uv is an arc in G then $f(u)f(v)$ is an arc in H; in other words, $uv \in E(G)$ implies $f(u)f(v) \in E(H)$. A homomorphism from G to H is denoted by $f : G \to H$. If there exists a homomorphism from G to H we shall write $G \to H$ and we shall say that G is *homomorphic to* H. If there is no such homomorphism we shall write $G \nrightarrow H$. It is easy to check that the composition $f \circ g$ of homomorphisms $g : G \to H$ and $f : H \to X$ is a homomorphism from G to X. This implies that if G is homomorphic to H and H is homomorphic to X, then G is homomorphic to X.

Note that homomorphisms preserve not just adjacency, but also the direction of arcs. Thus, a homomorphism $f : G \to G$ induces a map $f : E(G) \to E(G)$ defined as $f(uv) = f(u)f(v)$.

Let G and H be two digraphs and $f : G \to H$ a homomorphism. The *image* of f, denoted $\mathrm{Im}(f)$ or $f(G)$, is the subgraph of H induced by the vertices $\{v \in V(H) \mid \exists u \in V(G) \text{ s.t. } f(u) = v\}$. The *preimage* of a vertex $v \in V(f(G))$, denoted $f^{-1}(v)$, is the set $\{u \in V(G) \mid f(u) = v\}$. Observe that every two vertices in $f^{-1}(v)$ are not adjacent since otherwise vv would be a loop in $E(H)$. An *independent set* is a digraph S such that every two vertices in $V(S)$ are not adjacent. This means that $f^{-1}(v)$ is an independent set.

An *isomorphism* from G to H is a bijective homomorphism $f : V(G) \to V(H)$ which also preserves non-adjacency. This means that a bijective mapping $f : V(G) \to V(H)$ is a isomorphism if $f(u)f(v) \in E(H)$ if and only if $uv \in E(G)$. From this fact we can observe that the mapping $f^{-1} : V(H) \to V(G)$ is also a homomorphism. We shall denote it as $f^{-1} : H \to G$. The composition $f \circ f^{-1} : G \to G$ is the identity on the digraph $G = (V(G), E(G))$. If there exists an isomorphism from G to H we shall say that G and H are *isomorphic*. Note that if $f : G \to H$ is a injective homomorphism then G is isomorphic to $f(G)$.

An *endomorphism* of a digraph G is a homomorphism from G to itself. The set of all endomorphisms of a digraph G is denoted by $\mathrm{End}(G)$. An *automorphism* is an isomorphism from a digraph G to itself. The set of all automorphisms of a digraph G, denoted $\mathrm{Aut}(G)$, is a group under composition. We have that $\mathrm{Aut}(G) \subseteq \mathrm{End}(G)$, but $\mathrm{End}(G)$ is not necessarily a group. It can be check that a bijective endomorphism is already an automorphism. For this reason endomorphisms differ from automorphisms in that their image could not be all G.

1.3 Properties of Homomorphisms

The fact that homomorphisms preserve adjacency and direction of arcs has interesting implications. One of the most direct implications is how paths and cycles behave under homomorphisms. So let's start this section with its definitions.

A *walk* in a digraph G is a sequence of vertices $v_0, v_1, \ldots, v_k \in V(G)$ together with a sequence of arcs $e_1, e_2, \ldots, e_k \in E(G)$ such that for each $i = 1, \ldots, k$, e_i is an arc joining v_{i-1} and v_i. The arcs of the form $v_{i-1}v_i$ are called *forward arcs* and the arcs of the form $v_i v_{i-1}$ are called *backwards arcs*. The integer k is called the *length* of the walk. The *net length* is the difference between the number of forwards arcs and the number of backward arcs. Note that in the case of a graph the net length is always equal to zero. Finally, a walk is *closed* if $v_0 = v_k$.

A *path* is a walk in which every vertex and arc in the sequences is different. Analogously, a *cycle* is a closed walk in which every vertex and arc in the sequences is different. Since a path and a cycle are walks, the definitions of length and net length are also applicable.

A walk, path, etc., in which all arcs are forward arcs are called *directed* walk, *directed* path, etc., respectively. It is often to talk about walks, paths, etc., when we are just in the case of a graph and refer to them as *oriented* walks, paths, etc., in the case of a digraph. However, as defined above, we shall consider them as digraphs, and not necessarily as graphs. We shall denote by P_k, C_k, $\vec{P}_k$ and $\vec{C}_k$ the path, cycle, directed path and directed cycle of length k respectively. Note that P_k has $k+1$ vertices and k arcs while C_k has k vertices and k arcs.

Proposition 1.3.1. *Let G and H be digraphs and $f : G \to H$ a homomorphism. If $v_0, \dots, v_k$ and $e_1, \dots, e_k$ is a walk in G then $f(v_0), \dots f(v_k)$ and $f(e_1), \dots, f(e_k)$ is a walk in H of the same length and net length.*

Proof. It is clear since homomorphism preserves adjacency and direction of arcs. □

The same argument can be applied for particular cases of walks.

Corollary 1.3.2. *Let G be a digraph.*

- *If $f_1 : P_k \to G$ is a homomorphism, then $f_1(P_k)$ is a walk in G.*
- *If $f_2 : C_k \to G$ is a homomorphism, then $f_2(C_k)$ is a closed walk in G.*
- *If $f_3 : \vec{P}_k \to G$ is a homomorphism, then $f_3(\vec{P}_k)$ is a directed walk in G.*
- *If $f_4 : \vec{C}_k \to G$ is a homomorphism, then $f_4(\vec{C}_k)$ is a directed closed walk in G.*

In all the cases the length and net length is preserved.

A digraph is *connected* if every two vertices are joined by a path. A *component* or *connected component* of a digraph G is the subgraph induced by a subset of vertices $S \subseteq V(G)$ such that there is not any vertex $v \in V(G) \backslash S$ adjacent to a vertex in S. In a digraph G the *distance* between two vertices in the same component $u, v \in G$ is the length of the shortest path joining them. It is denoted by $d_G(u, v)$.

Corollary 1.3.3. *Let $f : G \to H$ be a homomorphism. Then $d_H(f(u), f(v)) \leq d_G(u, v)$ for any $u, v \in G$.*

Proof. Let $u = u_0, \dots, u_k = v$ be the sequence of vertices of a path of length k in G. Since the image of a path of length k is a walk of the same length and every walk from $f(u)$ to $f(v)$ contains a path from $f(u)$ to $f(v)$, it follows that $d_H(f(u), f(v)) \leq k$. □

Colourings is one of the most studied concepts in graph theory in the last century. It is striking the amount of problems and applications related to it. Moreover, colourings are really related to homomorphisms. In fact, it is said that homomorphisms generalise colourings. Let's see why.

A graph G is *k-colorable* if there exists a partition of $V(G)$ into k independent sets. Such a partition is called a *k-colouring* of G. The *chromatic number* of a graph G, denoted $\chi(G)$, is the minimum k such that G is k-colorable.

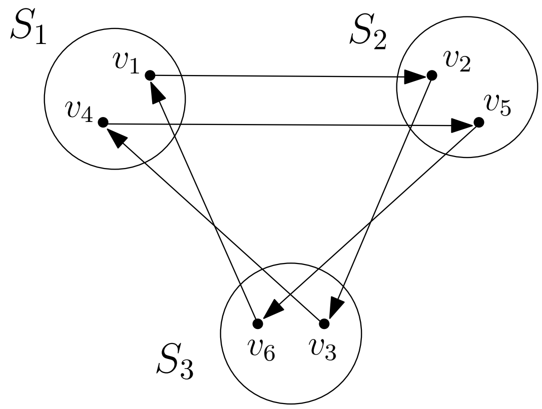

Figure 1.1: A $\vec{C}_3$-colouring of $\vec{C}_6$.

Let G and H be two graphs. If there exists a homomorphism $f : G \to H$ it is often said that G is *H-colorable* or that f is a *H-colouring* of G. The reason to this is that if we have that $G \to H$ via a homomorphism f, then for each $v \in H$ if we take the independent set $f^{-1}(v)$ we obtain a partition of $V(G)$ into k independent sets, $k \leq |V(H)|$. This in fact, is the condition of a graph for being k-colorable. Then we can say that if $G \to H$ and $n = |V(H)|$ then G is n-colorable. From these facts we can deduce the following proposition.

Proposition 1.3.4. *A graph G is n-colorable if and only if $G \to K_n$. Moreover, the homomorphisms from G to K_n are precisely the n-colourings of G.*

Let H be a graph and let $n = |V(H)|$. We have that the condition of being homomorphic to H is stronger than being n-colorable. If a graph G is homomorphic to H we know that we can make a partition of $V(G)$ into n independent sets but, moreover, this partition might has some restrictions involving the non-adjacency of the vertices from two different independent sets. To think of homomorphism as a generalisation of colourings can be very useful to understand better the its behaviour and properties.

Corollary 1.3.5. *If $G \to H$ then $\chi(G) \leq \chi(H)$.*

Proof. Let $n = \chi(H)$. We know that $G \to H \to K_n$, then $G \to K_n$ which implies that $\chi(G) \leq n$. $\square$

There is a similar result concerning the odd girth. The *girth* of a digraph is the minimum length of a cycle in it. Similarly, the *odd girth* of a non bipartite graph is the minimum length of an odd cycle in it. It is known from Theorem 1.3.10 that the property of a graph of being non bipartite is equivalent to contain at least one odd cycle. The reason why we are interested in the odd girth and not in the girth will be seen in detail in Chapter 2. It is related to the fact that every bipartite graph is homomorphic to K_2.

Proposition 1.3.6. *Let G and H be two non bipartite graphs. If $G \to H$ then the odd girth of G is greater or equal to the odd girth of H.*

Proof. Let $v_0, \ldots, v_k = v_0$ be the sequence of vertices of an odd cycle in G of minimum length k. Let $f : G \to H$ be a homomorphism. Then $f(v_0), \ldots, f(v_k) = f(v_0)$ is a closed walk of length k in H. Since we can not obtain an odd number from the sum of even numbers, there exists at least one odd cycle in the sequence $f(v_0), \ldots, f(v_k) = f(v_0)$ of length less or equal to k. □

In Proposition 1.3.4 we have defined colourability in terms of homomorphisms. But this is not the only property of graph theory that can be expressed in such way. We say that a digraph G is *balanced* if every cycle in G has net length equal to zero. We denote by $\vec{T}_k$ the digraph with vertices $v_0, \ldots, v_k$ and arcs $v_i v_j$ for every $i < j$.

Proposition 1.3.7. *A digraph G with n vertices does not contain a directed cycle if and only if $G \to \vec{T}_{n-1}$*

Proof. It is easy to check that $\vec{T}_{n-1}$ does not contain a directed cycle. Suppose now that G contains a directed cycle $\vec{C} \subseteq G$. If $f : G \to \vec{T}_{n-1}$ is a homomorphism then we have that $f(\vec{C})$ is a directed cycle in $\vec{T}_{n-1}$ which is a contradiction.

Let G be a digraph of n vertices with no directed cycles. We shall now label each vertex v by the maximum number of arcs in a directed walk that ends in it. Since G is free of directed cycles, it is easy to check that this labelling is well defined and labels the vertices from 0 to n-1. Finally, this labelling induces a homomorphism from G to $\vec{T}_{n-1}$ by mapping each vertex with label i to the vertex $v_i \in \vec{T}_{n-1}$. □

Proposition 1.3.8. *A digraph G with n vertices is balanced if and only if $G \to \vec{P}_{n-1}$.*

Proof. As in the previous proof, it is easy to check that $\vec{P}_{n-1}$ is balanced. Suppose G is not balanced so there exists some cycle C in G with net length different from zero. If $f : G \to \vec{P}_{n-1}$ is a homomorphism then we have that $f(C)$ is a cycle in $\vec{P}_{n-1}$ with net length different from zero which is a contradiction since $\vec{P}_{n-1}$ is balanced.

Let G be a balanced graph with n vertices. We shall label its vertices by integers as follows. In each component of G pick one arbitrary vertex and label it to 0. Once a vertex has been labelled by the integer i, label all of its outneighbours by $i + 1$ and all of its inneighbours by $i - 1$. It is easy to check that these procedure will give every vertex a unique label since G is balanced. Once every vertex is labelled, we can shift the labels so that the smallest one starts with 0. Note that since G has n vertices the maximum label of a vertex will be at most $n-1$. This final labelling induces a homomorphism form G to $\vec{P}_{n-1}$ by mapping each vertex with label i to the vertex $v_i \in \vec{P}_{n-1}$. □

In the previous proof we have assigned a labelling to each vertex of a connected component of a balanced digraph. So given a connected balanced digraph G the previous labelling is unique and assigns each vertex an integer. We call the label of a vertex v the *level* of v, and we call the maximum level of a vertex in G the *height* of G.

Corollary 1.3.9. *If G and H are two balanced digraphs of the same height, then any homomorphism from G to H preserves the levels of vertices.*

Proof. As we have seen in the proof of Proposition 1.3.8, if G is a digraph of height k then there is a unique homomorphism from G to $\vec{P}_k$ which is the one that maps each vertex with level i to the vertex $v_i \in \vec{P}_k$. Suppose that there exists some homomorphism $f : G \to H$ which does not preserve the level of some vertex and let $g : H \to \vec{P}_k$ be a homomorphism. We know that g preserves the level of vertices. Then the composition $g \circ f : G \to H \to \vec{P}_k$ is a homomorphism from G to $\vec{P}_k$ which does not preserve the level of some vertex, which is a contradiction. □

There exists many cases in which the existence of some homomorphisms is equivalent to the non existence of some other homomorphisms. These cases are called *homomorphism dualities* and we shall focus on them in Section 2.5. There is one simple example of these dualities applied to graphs and it follows from the well known theorem of König, which states that a graph is *bipartite*, which means 2-colorable, if and only if it has not odd cycles. This theorem can be expressed in terms of a homomorphism duality.

Theorem 1.3.10 (König's theorem)**.** *A graph G satisfies $G \to K_2$ if and only if $C_k \nrightarrow G$ for every odd integer $k \geq 3$.*

There is also another simple example of a homomorphism duality, in this case applied to digraphs. The following proposition was shown in [15] by Nešetřil and Pultr.

Proposition 1.3.11. *A digraph G satisfies $G \nrightarrow \vec{T}_{k-1}$ if and only if $\vec{P}_k \to G$.*

Proof. The longest directed path in $\vec{T}_{k-1}$ has length $k-1$ while $\vec{P}_k$ is a directed path of length k, so $\vec{P}_k \nrightarrow \vec{T}_{k-1}$. It follows that if $\vec{P}_k \to G$ then $G \nrightarrow \vec{T}_{k-1}$.

Suppose that $\vec{P}_k \nrightarrow G$. Then the labelling of the proof of Proposition 1.3.7 is well defined, since G has not directed paths of length greater or equal to k and it labels the vertices of G from 0 to $k-1$. Thus the labelling induces a homomorphism $G \to \vec{T}_{k-1}$. □

This proposition implies the following well known fact that relates graphs and digraphs.

Corollary 1.3.12. *A graph G is k-colorable if and only if there exists an orientation of G which does not contain the directed path $\vec{P}_k$.*

Proof. If G is k-colorable we can make a partition of $V(G)$ into k independent sets $V_1, \dots, V_k$. We shall replace each edge in G by an arc as follows. Consider an edge uv in G. We know that $u \in V_i$ and $v \in V_j$ for some $0 \le i, j \le k$ such that $i \ne j$. We shall replace the edge uv by the arc uv if $i < j$ and by the arc vu otherwise. It is clear that this replacement give us an orientation of G homomorphic to $\vec{T}_{k-1}$. Then Proposition 1.3.11 implies that $\vec{P}_k \nrightarrow G$.

On the other hand, Let $\vec{G}$ be an orientation of G which does not contain the directed path $\vec{P}_k$. We know by Proposition 1.3.11 that there exists a homomorphism $f : \vec{G} \to \vec{T}_{k-1}$. Then the sets $f^{-1}(v_0), \dots, f^{-1}(v_{k-1}) \subseteq V(\vec{G}) = V(G)$, where $v_0, \dots, v_{k-1}$ are the vertices of $\vec{T}_{k-1}$, are a k-colouring of G. $\square$

1.4 Sum and Product

Given two digraphs G and H, the *disjoint union* or *sum* of G and H is the digraph $G+H$ which has the vertex set $V(G+H) = V(G) \sqcup V(H)$ and arcs $uv \in E(G+H)$ if $uv \in E(G)$ or $uv \in E(H)$. The same definition is applied to graphs. Note that the sum of two graphs is also a graph. As one could expect, the sum of digraphs has simple and interesting properties.

Proposition 1.4.1. *A digraph G is not connected if and only if G is equal to the sum of two digraphs.*

Proof. It is clear from the definition of sum that the sum of two digraphs is not connected. On the other hand, if G is not connected it has at least two components. Let G_1 be equal to one connected component and let G_2 be equal to $G \backslash G_1$. It is easy to check that $G = G_1 + G_2$. $\square$

More related to homomorphisms are the following properties.

Proposition 1.4.2. *Let G, H and X be digraphs.*

- $G \to G+H$ *and* $H \to G+H$.
- *If* $G \to X$ *and* $H \to X$ *then* $G+H \to X$.

Proof. Consider the two *inclusions* $i_G : G \to G+H$ and $i_H : H \to G+H$ defined as $i_G(u) = u$ for all $u \in G$ and $i_H(v) = v$ for all $v \in H$. It follows from the definition of $G+H$ that i_G and i_H are homomorphisms.

Moreover, if $f_G : G \to X$ and $f_H : H \to X$ are homomorphisms, then it is easy to check that the mapping $f : G+H \to X$ defined as $f(u) = f_G(u)$ for all $u \in G$ and $f(v) = f_H(v)$ for all $v \in H$ is also a homomorphism. □

Note that the homomorphism f defined in the previous proof satisfies $f_G = f \circ i_G$ and $f_H = f \circ i_H$, and it is the unique mapping which satisfies this property. In fact, this uniqueness property characterise the sum of digraphs and inclusions.

Theorem 1.4.3 (Characterisation of the Sum)**.** *For any digraphs G and H there exists a unique (up to isomorphism) digraph S and unique homomorphisms $s_G : G \to S$ and $s_H : H \to S$ such that for every digraph X to which G and H are homomorphic via $f_G : G \to X$ and $f_H : H \to X$, there exists a unique homomorphism $f : S \to X$ satisfying $f \circ s_G = f_G$ and $f \circ s_H = f_H$.*

Given two digraphs G and H, the *product* of G and H is the digraph $G\times H$ which has the vertex set $V(G \times H) = V(G) \times V(H)$ and arcs $(u,v)(u',v') \in E(G \times H)$ whenever uv and $u'v'$ are arcs in $E(G)$ and $E(H)$ respectively. See some examples in Figure 1.2. The same definition is applied to graphs. Note that the product of two graphs is also a graph. The product of digraphs has interesting properties and leads to important results on graph theory and, in particular, on homomorphisms. Some of its fundamental properties are the following ones.

Proposition 1.4.4. *Let G, H and X be digraphs.*

- $G \times H \to G$ *and* $G \times H \to H$.
- *If* $X \to G$ *and* $X \to H$ *then* $X \to G \times H$.

Proof. Consider the two *projections* $\pi_1 : G \times H \to G$ and $\pi_2 : G \times H \to H$ defined as $\pi_1(u,v) = u$ and $\pi_2(u,v) = v$ for all $(u,v) \in V(G \times H)$. It follows from the definition of $G \times H$ that π_1 and π_2 are homomorphisms.

Moreover, if $f_1 : X \to G$ and $f_2 : X \to H$ are homomorphisms, then it is easy to check that the mapping $f : X \to G \times H$ defined as $f(x) = (f_1(x), f_2(x))$ is also a homomorphism. □

Corollary 1.4.5. *For digraphs G and H, $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$.*

Proof. It easily follows from the fact that $G \times H \to G$ and $G \times H \to H$. □

We may ask ourselves in which cases the equality holds in the previous result. In fact, it has been conjectured that the equality always holds in the case of graphs. This is one of the most well known open problems in this area. For this reason we shall state the following conjecture.

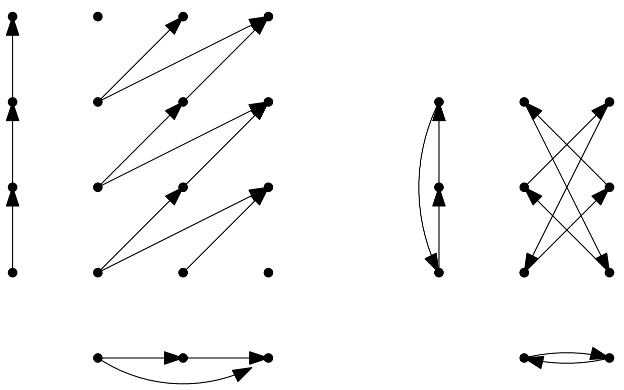

Figure 1.2: Product digraphs $\vec{P}_3 \times \vec{T}_2$ and $\vec{C}_3 \times K_2$.

Conjecture 1.4.6 (Hedetniemi's Conjecture)**.** *Let G and H be two graphs. Then $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$.*

Note that the homomorphism f defined in the proof of Proposition 1.4.4 satisfies $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$, and is the unique mapping which satisfies this property. In fact, as with the sum, this uniqueness characterise the product digraph and projections.

Theorem 1.4.7 (Characterisation of the Product)**.** *For any digraphs G and H, there exists a unique (up to isomorphism) digraph P and unique homomorphisms $p_1 : P \to G$ and $p_2 : P \to H$ such that for every digraph X homomorphic to G and H via $f_1 : X \to G$ and $f_2 : X \to H$, there exists a unique homomorphism $f : X \to P$ satisfying $p_1 \circ f = f_1$ and $p_2 \circ f = f_2$.*

Theorem 1.4.3 and Theorem 1.4.7 allows us to define both, the sum and the product, in a more general way. Both operations can be defined as the unique digraph which satisfies the properties of its characterisation theorem.

One might ask if both operations are commutative and associative. Moreover, if the product is distributive over the sum. Indeed, it is not difficult to see that the commutative, associative and distributive property holds for the sum and product of digraphs.

1.5 The Exponential Digraph

Let G and H be two digraphs. The *exponential digraph*, denoted H^G, is the digraph which vertices are the mappings $f : V(G) \to V(H)$ and which arcs are all ff' such that $f(u)f'(v) \in E(H)$ for every $uv \in E(G)$. Observe that the same definition can be applied to graphs.

Proposition 1.5.1. *Let G, H be two symmetric digraphs. Then H^G is also a symmetric digraph.*

Proof. Let $f, f' \in V(H^G)$ and suppose $ff' \in E(H^G)$. Let $uv \in E(G)$. Since G is symmetric, vu is also an arc of G. Then, $f(v)f'(u) \in E(H)$ since $ff' \in E(H^G)$, and $f'(u)f(v) \in E(H)$ since H is symmetric. So for every $uv \in E(G)$, $f'(u)f(v) \in E(H)$, which is the condition of $f'f$ to be an arc of H^G. Thus, H^G is a symmetric digraph. $\square$

Note that the exponential digraph is not an irreflexive digraph. Remember that a *loop* is an arc from a vertex to himself. We have considered all digraphs and graphs to be irreflexive, that is, without loops. However, the exponential digraph might have loops.

Proposition 1.5.2. *Let G and H be two digraphs. Then $G \to H$ if and only if H^G contains a loop.*

Proof. It follows from the definition of exponential digraph. In fact, a mapping $f : V(G) \to V(H)$ is a homomorphism if and only if $ff \in E(H^G)$ is a loop. $\square$

Proposition 1.5.3. *Let G, H and X be digraphs.*

- *X^{G+H} is isomorphic to $X^G \times X^H$.*
- *$X^{G\times H}$ is isomorphic to $(X^G)^H$.*

Proof. The digraph X^{G+H} has as its vertices the mappings $f : V(G + H) \to V(X)$. On the other hand, $X^G \times X^H$ has as its vertices all the pairs (f_G, f_H) of mappings $f_G : V(G) \to V(X)$ and $f_H : V(H) \to V(X)$. Let $\Phi : V(X^{G+H}) \to V(X^G \times X^H)$ be a mapping such that $\Phi(f) = (f|_G, f|_H)$, being $f|_G$ and $f|_H$ the restrictions of f to G and H respectively. Note that Φ is a bijective mapping. Let ff' be an arc of X^{G+H}. Then $f(u)f'(v) \in E(X)$ for every $uv \in G + H$. Note that all the arcs in $G + H$ are in G or in H. For this reason, ff' is an arc if and only if $f|_G(u)f'|_G(v) \in E(X)$ for every $uv \in G$ and $f|_H(u')f'|_H(v') \in E(X)$ for every $u'v' \in H$. This last condition implies that $f|_G f'|_G \in E(X^G)$ and $f|_H f'_H \in E(X^H)$, so $(f|_G, f|_H)(f'|_G, f'|_H) \in E(X^G \times X^H)$. In conclusion, $ff' \in E(X^{G+H})$ if and only if $\Phi(f)\Phi(f') \in E(X^G \times X^H)$. Thus, Φ is an isomorphism.

Let $\Theta : V((X^G)^H) \to V(X^{G\times H})$ be a mapping such that for each mapping $\varphi : H \to X^G$, $\Theta(\varphi)$ is a mapping from $G \times H$ to X that maps every vertex $(u, v) \in V(G \times H)$ to the vertex $(\varphi(v))(u)$. Note that Θ is a bijective mapping. We know by definition that $\varphi\varphi'$ is an arc of $(X^G)^H$ if and only if for every $vv' \in E(H)$, $\varphi(v)\varphi'(v') \in E(X^G)$; which means that $(\varphi(v))(u)(\varphi(v')(u')) \in E(X)$ for every $uu' \in E(G)$. Since $vv' \in E(H)$ and $uu' \in E(G)$ implies that $(u, v)(u', v') \in E(G \times H)$, it follows that $\Theta(\varphi)\Theta(\varphi') \in E(X^{G\times H})$. So in conclusion, $\varphi\varphi' \in E((X^G)^H)$ if and only if $\Theta(\varphi)\Theta(\varphi') \in E(X^{G\times H})$. Thus, Θ is an isomorphism. $\square$

Note that these properties are similar to the usual laws of exponentiation.

Let G be a non irreflexive digraph and let $vv \in E(G)$ be a loop. Observe that if $f : G \to H$ is a homomorphism then $f(v)f(v) \in E(H)$, which means that homomorphisms preserve loops.

Corollary 1.5.4. *Let G, H and X be digraphs. Then $G \times H \to X$ if and only if $H \to X^G$.*

Proof. From Proposition 1.5.3 we know that the digraph $X^{G\times H}$ is isomorphic to $(X^G)^H$. Then $X^{G\times H}$ has a loop if and only if $(X^G)^H$ has a loop. Finally, from Proposition 1.5.2, this is equivalent to $G \times H \to X$ if and only if $H \to X^G$. □

To finish the section, we make two more observations about the exponential digraph.

Proposition 1.5.5. *Let G and H be two digraphs.*

- *H is isomorphic to an induced subgraph of H^G.*
- *$G \times H^G \to H$.*

Proof. For each vertex $v \in V(H)$ consider the constant mapping $f_v : V(G) \to V(H)$ such that $f(u) = v$ for every $u \in V(G)$. Note that $f_v f_{v'} \in E(H^G)$ if and only if $vv' \in E(H)$. Then the mapping $\Phi : H \to \Phi(H) \subseteq H^G$ which maps the vertex $v \in V(H)$ to the vertex $f_v \in V(H^G)$ is an isomorphism.

Let $\varphi : V(G \times H^G) \to V(H)$ be a mapping such that $\varphi(v, f) = f(v)$ for every $v \in G$ and every mapping $f : G \to H$. Consider $(v, f)(v', f') \in E(G \times H^G)$. Then $vv' \in E(G)$ and $ff' \in E(H^G)$, so $f(v)f'(v') = \varphi(v, f)\varphi(v', f') \in E(H)$. Thus, φ is a homomorphism. □

1.6 Retracts and Cores

A *retraction* of a digraph G is a homomorphism $r : G \to H \subseteq G$ which satisfies $r(x) = x$ for all vertices $x \in V(H)$. If H admits a retraction from G we shall say that H is a *retract of* G. Retractions are at the heart of the problem of extending homomorphisms. However, we are interested in them since they allow us to define cores, which are one of the fundamental concepts of this thesis.

Proposition 1.6.1. *Let G be a digraph and let H be a subgraph of G. Then H is a retract of G if and only if any homomorphism $f : H \to X$ can be extended to a homomorphism $F : G \to X$.*

Proof. Suppose that H is a retract of G and let $f : H \to X$ be a homomorphism. We know there exists a retraction $r : G \to H$ such that $f(v) = v$ for every $v \in V(H)$. It follows, then, that $F = f \circ r : G \to X$ is a extension of f.

Consider the identity mapping $id : H \to H$. Suppose that id can be extended to a homomorphism $F : G \to H$. Then F is a retraction and thus H is a retract of G. □

We may observe that the composition of retractions is also a retraction. This implies that if a digraph K is a retract of H and H is a retract of G, then K is a retract of G. Note also that if G retracts to a proper subgraph H, then H must have strictly less number of vertices than G. So there must exists some subgraph of G which does not admit a retraction. For this reason we shall define the following concept.

A *core* is a digraph which does not retract to a proper subgraph. Cores are a fundamental concept to well define the homomorphism order as we shall see in Chapter 2.

Proposition 1.6.2. *Every digraph contains a core.*

Proof. It follows from the previous observations. □

The following proposition allows us to think about cores forgetting the concept of retraction.

Proposition 1.6.3. *A digraph G is a core if and only if G is not homomorphic to a proper subgraph.*

Proof. It is clear that if G retracts to a proper subgraph, then it is homomorphic to it. Conversely, if G is homomorphic to a proper subgraph, let H be a proper subgraph of G with the fewest number of vertices to which G is homomorphic. Then H is not homomorphic to a proper subgraph of itself. So any homomorphism $H \to H$ is an automorphism. Consider a homomorphism $f : G \to H$ and let $h = f|_H : H \to H$ be the restriction of f to H. Since h is an automorphism there exists an inverse automorphism h^{-1}. Observe that $h^{-1} \circ f$ is a retraction of G to H, and hence G is not a core. □

Observe that in the last proof we have shown that if H is a core, then every homomorphism $H \to H$ is an automorphism. This observation is really important since we shall use it several times during this thesis. For this reason let us state it as a Corollary.

Corollary 1.6.4. *Every homomorphism from a core to itself is an automorphism.*

Chapter 2

The Homomorphism Order

2.1 Homomorphic Equivalence

We say that two digraphs which are homomorphic to each other are *homomorphically equivalent*. It is easy to check that this relation is in fact an equivalence relation. An *equivalence relation* is a binary relation that is reflexive, symmetric and transitive. We could maybe think that two different digraphs which are homomorphically equivalent must have the same amount of vertices or arcs. But this is not true. However, there are a lot of properties that homomorphically equivalent digraphs will have in common. And they are all properties related to homomorphisms. One example is the chromatic number. It follows from Corollary 1.3.5 that two graphs which are homomorphically equivalent have the same chromatic number. And the same happens with the odd girth. But the property in which we are most interested is that homomorphically equivalent digraphs share the same core. This will allows us to split the set of all digraphs into equivalence classes via the homomorphic equivalence and choose for each class its correspondent core as its representative.

Proposition 2.1.1. *Every digraph is homomorphically equivalent to a unique (up to isomorphism) core.*

Proof. First of all, observe that every digraph is homomorphically equivalent to its core. Suppose now that H and H' are two different cores of a digraph G. From the transitive property of the equivalence relation H and H' are also homomorphically equivalent. Let $f : H \to H'$ and $g : H' \to H$ be homomorphisms. Since H and H' are cores, both $(f \circ g)$ and $(g \circ f)$ are automorphisms. Hence, H and H' are isomorphic. $\square$

Corollary 2.1.2. *Two homomorphically equivalent digraphs have the same (up to isomorphism) core.*

Proof. Let G and G' be two homomorphically equivalent digraphs and let H and H' be its core respectively. Observe that both G and G' are homomorphically equivalent to its respective cores. Since G and G' are also homomorphically equivalent, it follows from the transitive property that H and H' are homomorphically equivalent. Then, due to Proposition 2.1.1, H and H' are isomorphic. $\square$

With this results we know that every digraph in the same equivalence class has the same core. This is a very useful fact since we can generalise the results obtained for the core to all digraphs in its equivalence class. This is true since all digraphs in the same equivalence class have the same homomorphism properties. Some examples of cores are the complete graphs K_n. In fact, the set of all bipartite graphs is exactly the equivalence class which contains K_2 as its core. This is the reason why we are only interested in non bipartite graphs, since all bipartite graphs are homomorphically equivalent to K_2. Other examples in graphs are the odd cycles C_l. In digraphs, all directed paths $\vec{P}_k$ and directed cycles $\vec{C}_k$ are cores, as well as all digraphs $\vec{T}_n$. But there are plenty of more examples of cores. As matter of fact, asymptotically almost all digraphs are cores [7].

2.2 The Partial Order of Homomorphisms

A *partially ordered set* is a set $\mathcal{P}$ (not necessarily finite) together with a binary relation, usually denoted by $\leq$, satisfying the following properties:

- Reflexivity: $x \leq x$ for all $x \in \mathcal{P}$.
- Transitivity: $x \leq y$ and $y \leq z$ implies $x \leq z$ for all $x, y, z \in \mathcal{P}$.
- Antisymmetry: $x \leq y$ and $y \leq x$ implies $x = y$ for all $x, y \in \mathcal{P}$.

The relation $\leq$ is commonly referred to as a *partial order* on the set $\mathcal{P}$. For our purpose, since we are interested in the set of all digraphs, we shall only consider countable partially ordered sets.

Let $\vec{\mathscr{G}}$ be the set of all digraphs. Let us write $G \leq H$ for $G \to H$ (with $G, H \in \vec{\mathscr{G}}$). Observe that the relation $\leq$ ("being homomorphic to") is reflexive and transitive. We shall refer to this relation as the *homomorphism order*. However, it is not antisymmetric since homomorphically equivalent graphs might not be equal. A binary relation that is reflexive and transitive is called a *quasiorder*. Thus, the homomorphism order $\leq$ defines a quasiorder on $\vec{\mathscr{G}}$.

There are standard ways to transform a quasiorder into a partial order. One of them is by choosing a representative for each equivalence class. In our case we shall choose the cores to be the representative of each class as we have discussed in the previous section. Let $\vec{\mathscr{C}}$ be the set of all cores in $\vec{\mathscr{G}}$. Then the following theorem follows.

Theorem 2.2.1. $(\vec{\mathscr{C}}, \leq)$ *is a partially ordered set.*

In consequence, the homomorphism order is a partial order on $\vec{\mathscr{C}}$.

Let $\mathscr{G}$ be the set of all graphs and let $\mathscr{C}$ be the set of all cores in $\mathscr{G}$. Since we can view graphs as symmetric digraphs we have that $\mathscr{G} \subset \vec{\mathscr{G}}$. Note that the core of a symmetric digraph is also a symmetric digraph, then we also have that $\mathscr{C} \subset \vec{\mathscr{C}}$. It follows that $(\mathscr{C}, \leq)$ is a suborder of $(\vec{\mathscr{C}}, \leq)$. Since we shall also focus our interest in $(\mathscr{C}, \leq)$, let us state the following theorem.

Theorem 2.2.2. $(\mathscr{C}, \leq)$ *is a partially ordered set.*

The structure of the homomorphism order is rich in interesting properties that we shall discuss during this thesis, in particular in Chapter 3. Let's start with a simple one. A *lattice* is a partially ordered set in which every two elements have a least upper bound and a greatest lower bound.

Proposition 2.2.3. $(\vec{\mathscr{C}}, \leq)$ *is a lattice.*

Proof. Indeed, given two cores G and H, the least upper bound and greatest lower bound are the cores of the digraphs $G + H$ and $G \times H$ respectively. It follows from Proposition 1.4.2 that $G \leq G + H$, $H \leq G + H$, and if a core X satisfies $G \leq X$ and $H \leq X$ then $G + H \leq X$. Note that $G + H$ might not be a core but its core satisfies exactly the same inequalities in $(\vec{\mathscr{C}}, \leq)$. So the core of $G + H$ is the least upper bound of G and H. Analogously, it follows from Proposition 1.4.4 that the core of $G \times H$ is the greatest lower bound of G and H. $\square$

Recall that the sum and product of two graphs is also a graph. Then, applying the previous proof to graphs, it follows that $(\mathscr{C}, \leq)$ is a sublattice of $(\vec{\mathscr{C}}, \leq)$.

Another well known example of a lattice is the set of natural numbers ordered by divisibility. In this case the least upper bound is the least common multiple and the greatest lower bound is the greatest common divisor.

2.3 Incomparable Digraphs

A *total order* or *linear order* is a set with a transitive, antisymmetric and connex relation. A binary relation $\leq$ on a set $\mathcal{P}$ is *connex* if every pair of elements $a, b \in \mathcal{P}$ satisfies $a \leq b$ or $b \leq a$. Note that if a binary relation has the connex property then it is reflexive. It follows that a linear order is indeed a partial order, and that a partial order with the connex property is a linear order. In fact, it was shown in [13] that every partial order can be extend into a linear order. This statement is known as the order-extension principle. One of the most well known examples of a countably linear order is the set of rational numbers with the standard order $(\mathbb{Q}, \leq)$.

However, $(\vec{\mathscr{C}}, \leq)$ is not a linear order since there are pairs of digraphs G_1, G_2 such that $G_1 \nrightarrow G_2$ and $G_2 \nrightarrow G_1$, and hence, $G_1 \nleq G_2$ and $G_2 \nleq G_1$. In this case we shall say that G_1 and G_2 are *incomparable*. One way to obtain two incomparable graphs follows from the well known result of Erdös [1].

Theorem 2.3.1 ([1]). *For any positive integers k, l there exists a graph with chromatic number $\geq k$ and girth $\geq l$.*

Note that the previous theorem implies the same result but replacing girth by odd girth.

Corollary 2.3.2. *For any non bipartite graph G there exists a graph G' which is incomparable with it.*

Proof. Let G be a non bipartite graph with chromatic number $k \geq 3$ and odd girth $l \geq 3$. Consider a graph G' with chromatic number strictly greater than k and odd girth strictly greater than l. We know such graph exists from Erdös theorem. Then, it follows from Corollary 1.3.5 that $G' \nrightarrow G$ and from Proposition 1.3.6 that $G \nrightarrow G'$. Thus, G and G' are incomparable. □

Remember the definition of a graph of being k-colorable. Observe that the same definition can be applied to digraphs. A digraph is *k-colorable* if there exists a partition of $V(G)$ into k independent sets. And we can also define the *chromatic number* of digraphs analogously. In fact, if we consider the complete graph K_n as its respective symmetric digraph, Corollary 1.3.5 also holds for digraphs. With this in mind we can obtain a similar result to Corollary 2.3.2 in the case of digraphs.

Proposition 2.3.3. *For any non balanced digraph G there exists a digraph G' which is incomparable with it.*

Proof. Let k be the chromatic number of G. Recall that if a digraph G is not balanced then it contains a cycle $C \subseteq G$ of length $l \geq 2$ and net length $d > 0$. Consider a graph of chromatic number strictly greater than k and girth strictly greater than l, and let G' be an orientation of such graph. It follows from Corollary 1.3.5 that $G' \nrightarrow G$. Suppose now that there exists a homomorphism $f : G \to G'$ and consider the cycle $C \subseteq G$. Then $f(C) \subseteq G'$ is a closed walk of length l and net length d, which implies that G' contains a cycle of length $l' \leq l$. This is a contradiction. Hence, G and G' are incomparable. □

2.4 Density

Given a partial ordered set $(\mathcal{P}, \leq)$ and two elements $a, b \in \mathcal{P}$, let us write $a < b$ to mean $a \leq b$ and $b \not\leq a$. A partially ordered set $(\mathcal{P}, \leq)$ is *dense* if for any pair of elements $a, b \in \mathcal{P}$ satisfying $a < b$ there exists an element $c \in \mathcal{P}$ such that $a < c < b$.

Observe that the partially ordered set $(\vec{\mathscr{C}}, \leq)$ is not dense; there is no digraph X satisfying $K_1 < X < \vec{K_2}$ since $\vec{K_2} \nrightarrow X$ implies that $X \to K_1$, which is equivalent to say that a digraph X does not contain any arc if it is an independent set. However, the interesting question is not if the homomorphism order is dense but which classes of digraphs are.

Let us start focusing on graphs. The partially ordered set $(\mathscr{C}, \leq)$ is also not dense since there is no graph X such that $K_1 < X < K_2$. But this is the only exception, and otherwise $(\mathscr{C}, \leq)$ is dense. This result was originally proved in [19] but the proof we give here is originally due to Perles and Nešetřil [14].

Theorem 2.4.1 ([14])**.** *Let graphs G, H be cores such that $G < H$ and $G \neq K_1$ or $H \neq K_2$. Then there exists a graph X such that $G < X < H$.*

Proof. First, observe that K_1 is homomorphic to every graph in $\mathscr{C}$. Secondly, K_2 is homomorphic to every graph in $\mathscr{C}$ except from K_1.

If $G = K_1$, then $H > K_2$ and we can take $X = K_2$. On the other hand, if $G \neq K_1$, then $G \geq K_2$ and $H > K_2$ so H is a non bipartite graph.

Let X' be a graph with chromatic number strictly greater than the chromatic number of the exponential graph G^H and odd girth strictly greater than the odd girth of H (which is ≥ 3 since H is non bipartite). We know such a graph exists from Theorem 2.3.1. Finally, let $X = G + (H \times X')$. Let's see that $G < X < H$.

It is clear that $G \to X$ via the inclusion. It is also clear that $X \to H$ since $G \to H$ and $H \times X' \to H$ via the projection (recall Proposition 1.4.2 and Proposition 1.4.4). Suppose now that $H \to X$. Since $H \nrightarrow G$, some component C of H satisfies $H' \nrightarrow G$. Then $C \to H \times X' \to X'$. So the odd girth of C must be greater or equal to the odd girth of X', which is a contradiction. Thus, $H \nrightarrow X$. Finally, suppose that $X \to G$. Then $H \times X' \to G$ which implies that $X' \to G^H$ by Corollary 1.5.4. But the chromatic number of X' exceeds the chromatic number of G^H which is a contradiction. Hence, $X \nrightarrow G$ and therefore $G < X < H$. □

The case of $(\vec{\mathscr{C}}, \leq)$ is not that simple as $(\mathscr{C}, \leq)$. However, we can also obtain the density property by requiring the upper digraph to contain a cycle. The following theorem is the first result developed in this thesis. It is a generalisation of a theorem which states the same but requiring the upper digraph to contain a directed cycle [7, Theorem 3.32].

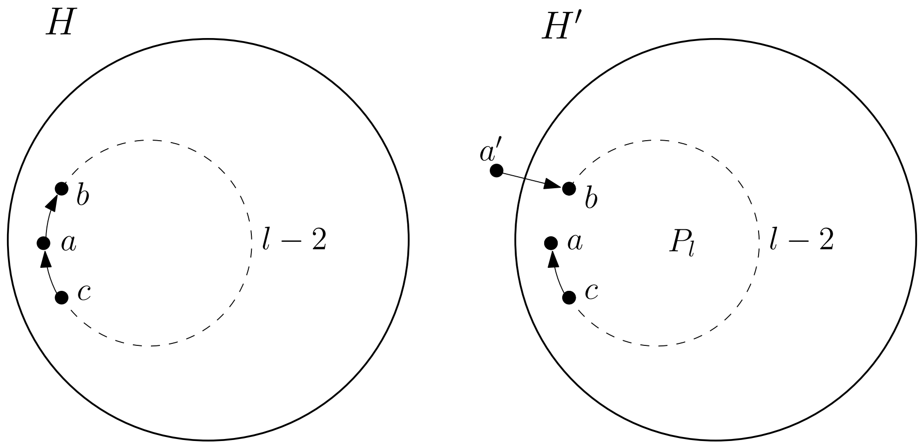

Figure 2.1: Digraph H and digraph H'. The arc joining the vertices a and c might be in the other direction.

Theorem 2.4.2. *Let digraphs G, H be cores satisfying $G < H$, where H is connected and contains a cycle. Then there exists a digraph X such that $G < X < H$.*

Proof. Let ab be an arc belonging to some cycle in H. Let c be the other vertex in the cycle adjacent to a, which means that ac or ca is an arc of the cycle. Note that if the considered cycle in H is isomorphic to $\vec{C_2}$, then b and c would be the same vertex and both ab and ba would be arcs of H. Otherwise $b \neq c$ and ba is not an arc of H. Let H' be a digraph obtained from H by adding a new vertex a', and replacing the arc ab by the arc $a'b$. Note that if we identify the vertex a' with the vertex a we obtain the digraph H. See Figure 2.1.

Let $n > |V(G)|$ and consider the complete graph K_n. Let X' be the digraph obtained from an arbitrary orientation of K_n by replacing each arc uv by a copy of H' identifying u with a' and v with a. Note that the vertices of K_n are in X'. We shall refer to them as *original vertices.*

Let $X = G + X'$. We claim that $G < X < H$.

It is clear that $G \to X$. Suppose now that $X' \to G$. Then, since $n > |V(G)|$ and X' contains n original vertices, at least two original vertices from X' will be mapped to the same vertex in G. That will induce a homomorphism $H \to G$ since every pair of original vertices in X' are joined by a copy of H'. Thus, $X \nrightarrow G$.

Consider $f : X' \to H$ that maps each original vertex to a and the rest of vertices to their corresponding vertex in H. It is easy to check that f is a homomorphism. On the other hand, suppose there exists an homomorphism $g : H \to X'$. Since H is a core, $f \circ g : H \to H$ is an automorphism. Then there exists $h : H \to H$ such that $h = (f \circ g)^{-1}$. Consider now $g \circ h : H \to X'$. Since $f \circ g \circ h = id_H$, $g \circ h$ maps the vertex $a \in V(H)$ to an original vertex $v_o \in V(X')$ and the vertex $b \in V(H)$ to the vertex b of some copy $H'_o \subset X'$ such that $v_o \in V(H'_o)$. Assume that $b \neq c$. It follows that the rest of vertices of the cycle will be mapped to their corresponding vertices of the same copy H'_o. Then $(g \circ h)(c)$ will not be adjacent to $v_o = (g \circ h)(a)$ but a and c are adjacent in H. This is a contradiction since $g \circ h$ is a homomorphism. In the case $b = c$, we shall have that $(g \circ h)(b)$ is not an inneighbour of $(g \circ h)(a)$ but ba is an arc in H, which is also a contradiction. So $H \nrightarrow X'$. Hence, $G < X < H$ as claimed. $\square$

Observe that the previous theorem is only valid when H is connected. However, we shall show the following strengthening.

Theorem 2.4.3. *Let digraphs G, H be cores satisfying $G < H$ and let $H_c \subseteq H$ be a connected component containing a cycle such that $H_c \nrightarrow G$. Then there exists a digraph X such that $G < X < H$.*

Proof. Observe that, analogously to the proof of Theorem 2.4.2, we can construct a digraph X'_1 such that $X'_1 < H_c$ and $X'_1 \nrightarrow G$. Then, $X = G + X'_1$ satisfies the desired result. $\square$

2.5 Homomorphism Dualities

There are cases in which the existence of some homomorphisms are equivalent to the non existence of certain other homomorphisms. We refer to this kind of characterization as *homomorphism duality*. Homomorphism dualities were first studied in [15] and were related to the homomorphism order in [16]. We have already seen some examples of homomorphism dualities in Section 1.3. But let us recall them and give a few more.

For every digraph X,

- $X \nrightarrow K_1$ if and only if $\vec{K}_2 \to X$.
- $X \nrightarrow \vec{T}_{k-1}$ if and only if $\vec{P}_k \to X$.
- $X \nrightarrow \vec{C}_k$ if and only if $C \to G$, for some oriented cycle C of net length not divisible by k.

For every graph G,

- $G \nrightarrow K_1$ if and only if $K_2 \to G$.
- $G \nrightarrow K_2$ if and only if $C_k \to G$, for some odd integer $k \geq 3$.

Observe that the simplest homomorphism duality statements involve just two digraphs. A *duality pair* is an ordered pair of digraphs (H, G) such that $X \nrightarrow G$ if and only if $H \to X$, for every digraph X. Observe that in particular, if we take $X = G$ or $X = H$, it implies that $H \nrightarrow G$. So if (H, G) is a duality pair then $H \nrightarrow G$. However, $G \to H$ might be possible. Is interesting to think about duality pairs as a partition of $\vec{\mathscr{G}}$ in two subsets S_1, S_2; S_1 consisting in all digraphs X_1 such that $X_1 \to G$ and S_2 consisting in all digraphs X_2 such that $H \to X_2$. With such a partition we shall have that $X_2 \nrightarrow X_1$ for any digraphs $X_1 \in S_1$ and $X_2 \in S$.

Proposition 2.5.1. *Given a duality pair (H', G) there is a unique (up to isomorphism) core H such that (H, G) is a duality pair.*

Proof. Suppose (H_1, G) and (H_2, G) are duality pairs. Then since $H_2 \nrightarrow G$, applying the definition of duality pair for (H_1, G), it follows that $H_1 \to H_2$. Analogously, $H_2 \to H_1$. So H_1 and H_2 are homomorphically equivalent, and hence, they are homomorphic to a unique (up to isomorphism) core. □

Proposition 2.5.2. *Given a duality pair (H, G') there is a unique (up to isomorphism) core G such that (H, G) is a duality pair.*

Proof. This proof is analogous to the previous one. □

The previous proposition motivates the following definition. If (H, G) is a duality pair we shall refer to the core of G as *the dual of H*.

Proposition 2.5.3. *Let (H, G) be a duality pair where H is a core. Then H is connected.*

Proof. Suppose H is not connected, so $H = H_1 + H_2$. Since H is a core, $H \nrightarrow H_1$ and $H \nrightarrow H_2$, which implies that $H_1 \to G$ and $H_2 \to G$ respectively. Then, $H \to G$ which is a contradiction. □

In order to state what we consider is the main result in the relation between homomorphism dualities and the homomorphism order let us define the concept of tree. An *oriented tree* is a connected digraph which does not contain any oriented cycle. An equivalent definition is that an *oriented tree* is a digraph in which every pair of vertices is joined exactly by a unique oriented path. We shall focus our interest on oriented trees in 4.

The following theorem, which we shall state without proof, characterise all duality pairs.

Theorem 2.5.4 ([16])**.** *If (H, G) is a duality pair, then H is homomorphically equivalent to an oriented tree. Conversely, if H is an oriented tree, then there exists a unique (up to isomorphism) core G such that (H, G) is a duality pair.*

More generally, we say that given two finite sets of digraphs $\mathcal{G}$ and $\mathcal{H}$, $(\mathcal{H}, \mathcal{G})$ is a *finite homomorphism duality* if for every digraph X, $X \nrightarrow G$ for all $G \in \mathcal{G}$ if and only if $H \to X$ for some $H \in \mathcal{H}$.

Theorem 2.5.5 ([4]). *If $(\mathcal{H}, \mathcal{G})$ is a finite homomorphism duality, then every digraph in $\mathcal{H}$ is homomorphically equivalent to an oriented tree. Conversely, if $\mathcal{H}$ is a finite set of trees, then there exists a unique (up to isomorphism) set of cores $\mathcal{G}$ such that $(\mathcal{H}, \mathcal{G})$ is a finite homomorphism duality.*

In fact, [16] and [4] give specific methods and constructions to obtain the dual of an oriented tree, or more generally, *the dual of* a finite set of trees.

2.6 Gaps and Duality Pairs

Let $(\mathcal{P}, \leq)$ be a partially ordered set and let $a, b \in \mathcal{P}$ satisfying $a \leq b$. The *closed interval* $[a, b]$ is the set of elements $x \in \mathcal{P}$ such that $a \leq x \leq b$. Note that $[a, b]$ contains at least the elements a and b. The *open interval* (a, b) is the set of elements $x \in \mathcal{P}$ such that $a < x < b$. An open interval might be empty. To avoid confusing between open intervals and duality pairs in $(\vec{\mathscr{C}}, \leq)$, we shall consider only closed intervals, and we shall refer to them just as *intervals*. An interval $[a, b]$ is a *gap* if there is no $x \in \mathcal{P}$ such that $a < x < b$, which is equivalent to say that the open interval (a, b) is empty.

Proposition 2.6.1. *A partially ordered set is not dense if and only if it contains at least one gap.*

Proof. It is clear from the definitions of density and gap. $\square$

Observe that we already know that there exists gaps in $(\vec{\mathscr{C}}, \leq)$ and $(\mathscr{C}, \leq)$ since they are not dense.

Corollary 2.6.2. *$[K_1, K_2]$ is the only gap in $(\mathscr{C}, \leq)$.*

Proof. It follows from Theorem 2.4.1. $\square$

However, $(\vec{\mathscr{C}}, \leq)$ is not that simple. We have seen in Theorem 2.4.2 that every interval $[F, H]$, where the core of H contains a cycle, is not a gap. Observe that the condition of not containing a cycle is the definition of being an oriented tree. So it appears that gaps in $(\vec{\mathscr{C}}, \leq)$, just as duality pairs, are closely related to trees. This is not a coincidence. Indeed, gaps and duality pairs are in one-to-one correspondence (see Figure 2.2).

Theorem 2.6.3 ([16]). *Let digraphs G, F, H be cores and let H be connected.*

- *If (H,G) is a duality pair then $[H \times G, H]$ is a gap.*
- *If $[F,H]$ is a gap then (H, F^H) is a duality pair.*

Moreover, if both cases happened, then G is homomorphically equivalent to F^H and F is homomorphically equivalent to $H \times G$.

Proof. Suppose (H,G) is a duality pair and suppose there exists a digraph X such that $H \times G < X < H$. Then $H \nrightarrow X$, which implies that $X \to G$, and since $X \to H$, we have by Proposition 1.4.4 that $X \to H \times G$, which is a contradiction.

On the other hand, suppose $[F,H]$ is a gap. We claim that $X \nrightarrow F^H$ if and only if $H \to X$ for any digraph X. Suppose $H \to X \to F^H$, then by Corollary 1.5.4 $H \times H \to F$, so $H \to F$, which is a contradiction. Suppose now that $X \nrightarrow F^H$ and $H \nrightarrow X$. From Corollary 1.5.4 we know that $H \times X \nrightarrow F$ since $X \nrightarrow F^H$. So $F < F + H \times X$. Due to $F \to H$, $H \times X \to H$, $H \nrightarrow F$ and $H \nrightarrow X$, it follows that $F + H \times X < H$. Thus, $F < F + H \times X < H$ contradicts that $[F,H]$ is a gap.

Finally suppose that (H,G) is a duality pair and $[F,H]$ is a gap. It follows from Proposition 2.5.2 that G is homomorphically equivalent to F^H since both (H,G) and (H,F^H) are duality pairs. Corollary 1.5.4 implies that $H \times G \to F$ since $F^H \to G$. We know that $H \nrightarrow F$, so we must have that $F \to G$ from the duality condition. Then $F \to H \times G$ since we also have that $F \to H$. Hence, F is homomorphically equivalent to $H \times G$. $\square$

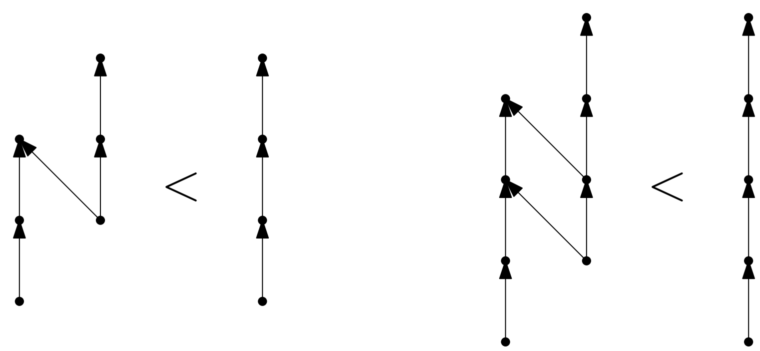

Figure 2.2: Gaps $[\vec{T}_2 \times \vec{P}_3, \vec{P}_3]$ and $[\vec{T}_3 \times \vec{P}_4, \vec{P}_4]$ corresponding to the duality pairs $(\vec{P}_3, \vec{T}_2)$ and $(\vec{P}_4, \vec{T}_3)$ respectively.

All in all, Theorem 2.5.4 and Theorem 2.6.3 characterise all duality pairs and gaps in $(\vec{\mathscr{C}}, \leq)$. We can conclude from both theorems that for every core T of an oriented tree, there exists unique digraphs G and F such that (T,G) is a duality pair and $[F,T]$ is a gap. Note that F must be a balanced digraph since it is homomorphic to an oriented tree. Moreover, G and F are related as shown in the last part of Theorem 2.6.3. And these are all duality pairs and gaps in $(\vec{\mathscr{C}}, \leq)$.

Chapter 3

Fractal Property of the Homomorphism Order

3.1 Universal Partially Ordered Sets

Given two partially ordered sets $(\mathcal{P}_1, \leq_1)$ and $(\mathcal{P}_2, \leq_2)$, an *embedding* from $(\mathcal{P}_1, \leq_1)$ to $(\mathcal{P}_2, \leq_2)$ is a mapping $\Phi : \mathcal{P}_1 \to \mathcal{P}_2$ such that for every $a, b \in \mathcal{P}_1$, $a \leq b$ if and only if $\Phi(a) \leq \Phi(b)$. It follows from its definition that embeddings are injective mappings. If such a mapping exists we shall say that $(\mathcal{P}_1, \leq_1)$ can be embedded into $(\mathcal{P}_2, \leq_2)$.

A partially ordered set $(\mathcal{P}, \leq)$ is *universal* if every partially ordered set can be embedded into it. Recall that from the purpose of this thesis, we are only considering countable partially ordered sets.

The existence of universal partially ordered sets have been proved several times [5, 12, 6]. Observe that embedding a universal partially ordered set into a partially ordered set $(\mathcal{P}, \leq)$, implies that $(\mathcal{P}, \leq)$ is also universal. For this reason, once you know there exists many different universal partially ordered sets, see [11] for more examples, a simple way to show that certain partially ordered set is universal is to embed one of them into it. So let's start showing that the homomorphism order of digraphs is universal by embedding a particular universal partially ordered set into it.

Let $\mathcal{P}'$ consists of all finite sets of the set of odd natural numbers, and let $\leq$ be a binary relation on $\mathcal{P}'$ such that for every $A, B \in \mathcal{P}'$, $A \leq B$ if and only if for every $a \in A$ there exists $b \in B$ such that b divides a. It is easy to check that $\leq$ is reflexive, transitive and antisymmetric, and hence, $(\mathcal{P}', \leq)$ is a partially ordered set.

Theorem 3.1.1 ([2]). *$(\mathcal{P}', \leq)$ is universal.*

Let us denote by $\vec{C}_n$ the directed cycle of length n. Observe that for every pair of natural numbers n and m, $\vec{C}_n$ is homomorphic to $\vec{C}_m$ if and only if m divides n.

Theorem 3.1.2. *The homomorphism order of digraphs* $(\vec{\mathscr{C}}, \leq)$ *is universal.*

Proof. We shall construct an embedding Φ from $(\mathcal{P}', \leq)$ into the homomorphism order.

Given a finite set of odd natural numbers A, let $\Phi(A)$ be the sum of directed cycles $\sum_{a \in A} \vec{C}_a$. It follows that for every $A, B \in \mathcal{P}'$, $\sum_{a \in A} \vec{C}_a \to \sum_{b \in B} \vec{C}_b$ if and only if $A \leq B$ with respect to the partial order in $(\mathcal{P}', \leq)$. Hence, Φ is an embedding. $\square$

We have actually shown that the class of disjoint union of directed cycles under the homomorphism order is universal.

To end the section, let us state the following theorem without proof.

Theorem 3.1.3 ([18])**.** *The homomorphism order of graphs* $(\mathscr{C}, \leq)$ *is universal.*

3.2 Fractal Property of $(\mathscr{C}, \leq)$

Let $(\mathcal{P}, \leq)$ be a partially ordered set. We shall say that $(\mathcal{P}, \leq)$ has the *fractal property* if every interval $[a, b] \subseteq (\mathcal{P}, \leq)$ is either universal or a gap. So if $[a, b]$ is a universal interval, then every partially ordered set can be embedded into it. In particular, $(\mathcal{P}, \leq)$ can be embedded into $[a, b]$, which is why this property is called "fractal".

This property was first introduced in [14]. Recently, it was shown that every interval in the homomorphism order of graphs $(\mathscr{C}, \leq)$ is universal, with the exception of $[K_1, K_2]$, which we know from Corollary 2.6.2 is the only gap in $(\mathscr{C}, \leq)$. Thus, it was shown that the homomorphism order of graphs has the fractal property.

Lemma 3.2.1. *Let graphs* G, H *be cores satisfying* $G < H$ *and* $G \neq K_1$ *or* $H \neq K_2$. *Then there exists incomparable graphs* X_1, X_2 *such that* $G < G + X_i < H$ *for* $i = 1, 2$.

Proof. Let X_1 be a graph given by Theorem 2.4.1 which satisfies $G < G + X_1 < H$. Since $X_1 \nrightarrow G$, some component C of X_1 satisfies $C \nrightarrow G$. Let X'' be a graph with chromatic number strictly greater than $\max\{|V(G^H)|, |V(X_1^H)|\}$ and odd girth strictly greater than the odd girth of C. Finally, let $X_2 = (H \times X'')$. Observe that it follows analogously to the proof of Theorem 2.4.1 that $G < G + X_2 < H$.

Let's see that X_1 and X_2 are incomparable. Suppose $X_1 \to X_2$. Then $C \to X_2$, so the odd girth of C must be greater or equal to the odd girth of X'', which is a contradiction. Suppose now that $X_2 \to X_1$. Then $H \times X'' \to X_1$, which implies by Corollary 1.5.4 that $X'' \to X_1^H$. Thus, the chromatic number of X'' must be less or equal to the chromatic number of X_1^H, which is a contradiction. □

Lemma 3.2.2. *For every connected non bipartite graph H there exists an integer l' such that for any two vertices $x, y \in V(H)$ and any $l > l'$ there exists a walk of length l starting with x and ending with y.*

Proof. Let $l' = 2|V(H)|$. Consider two vertices $x, y \in V(H)$ and an integer $l > l'$. If the parity of the length of the path joining x to y is equal to the parity of l, it is clear that there exists a walk of length l starting with x and ending with y since it can go consecutively forward and backward on the same edge. Otherwise, consider a vertex z that belongs to an odd cycle of H. Consider the walk starting with the path from x to z, continuing with the odd cycle, and ending with the path from z to x. Observe that the length of such walk, l_c, has odd parity. Then, the parity of the length remaining $l - l_c$ is the same as the parity of the path between x and y, so we can finish the walk as in the first case. □

Lemma 3.2.3. *Let graphs G, X, H be cores satisfying $G < X < H$, where H is connected. Then there exists a connected graph X', obtained by joining each component of X by long enough paths, such that $G < X' < H$.*

Proof. Assume that X is not connected, otherwise we are done. Let l' be the integer given by Lemma 3.2.2 for the graph H and let X' be the graph obtained by joining every pair of components of X by a path of length $l > \max\{l', |V(H)|\}$. It is clear that $G < X'$. Let's see that $X' \to H$. Let $f : X \to H$ be a homomorphism. Given a path P_l that joins two components, let x and y be its initial and terminal vertex, so $x, y \in V(X)$. Consider the homomorphism $g : P_l \to H$ that maps the path P to a walk of length l starting in $f(x)$ and ending in $f(y)$ given by Lemma 3.2.2. It follows that the mapping $h : X' \to H$ equal to f in the vertices of X and equal to the corresponding homomorphism g for each path P_l is a homomorphism. Suppose now that $f : H \to X'$ is a homomorphism. Then $f(H)$ must be homomorphic to one of the components of X since $l > |V(H)|$, which is a contradiction. Hence, $G < X' < H$. □

Theorem 3.2.4 ([3])**.** *Let graphs G, H be cores satisfying $G < H$ and $G \neq K_1$ or $H \neq K_2$. Then the interval $[G, H]$ is universal.*

Proof. Let H' be a connected component of H such that $H' \nrightarrow G$. Let X_1, X_2 be two incomparable graphs given by Lemma 3.2.1 such that $X_i < H'$ and $X_i \nrightarrow G$ for $i = 1, 2$. Assume that X_1 and X_2 are cores. By Lemma 3.2.3 we shall also assume that X_1, X_2 are connected. Let l' be the integer given by Lemma 3.2.2 for the graph H' and let $l > \max\{l', |V(X_1)|, |V(X_2)|\}$. We shall construct a graph Y consisting of graphs X_1, X_2 joined by two different paths of lengths $2l$ and $2l + 1$ as shown in Figure 3.1. We choose two distinguished vertices a, b to be the middle vertices of

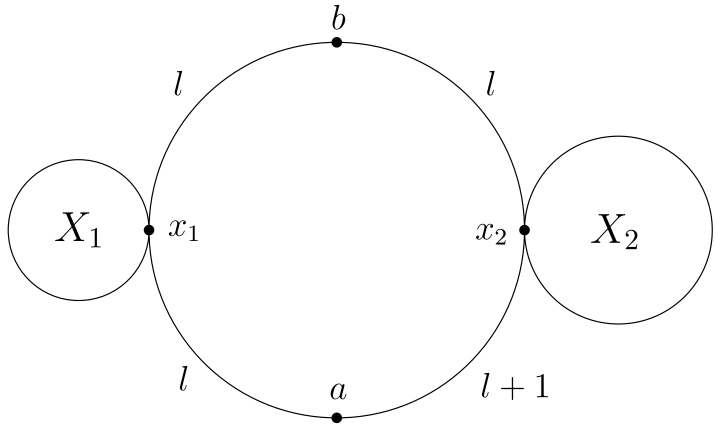

Figure 3.1: Graph Y.

these two paths. The vertices x_1, x_2 are picked arbitrary. Observe that by Lemma 3.2.2 there exists a homomorphism $f : Y \to H'$ such that $f(a) = f(b)$. Then, it is clear that $G < G + Y < H$.

We shall now construct an embedding from the homomorphism order of digraphs $(\vec{\mathscr{C}}, \leq)$ which we know is universal from Theorem 3.1.2, into the interval $[G, H]$.

Given a digraph F, let $\Phi(F)$ be the graph obtained by replacing each arc $uv \in E(F)$ by a copy of Y identifying vertices a with u and b with v. Observe that $G < G + \Phi(F) < H$. Consider a homomorphism $f : Y \to \Phi(F)$. Since X_1 and X_2 are incomparable connected cores, they must be mapped to a copy of itself in $\Phi(F)$ respectively. Moreover, since x_1 and x_2 are joined by two paths of length $2l$ and $2l + 1$, then x_1 and x_2 must be mapped to the vertices x_1, x_2 of the same copy of Y in $\Phi(F)$. We conclude that a homomorphism $f : Y \to \Phi(F)$ maps Y to some copy of it fixing all vertices in the paths joining X_1 and X_2. In particular, f fixes the vertices a, b.

We claim that for any pair of digraphs F_1, F_2, $F_1 \to F_2$ if and only if $\Phi(F_1) \to \Phi(F_2)$. Suppose $f : F_1 \to F_2$ is a homomorphism. Consider the mapping $g : \Phi(F_1) \to \Phi(F_2)$ equal to f on the vertices of F_1 and mapping each copy of Y corresponding to the arc uv to the copy of Y corresponding to the arc $f(u)f(v)$. It is easy to check that g is a homomorphism. Suppose now that $g : \Phi(F_1) \to \Phi(F_2)$ is a homomorphism. We have seen that every copy of Y in $\Phi(F_1)$ must be mapped to a copy of Y in $\Phi(F_2)$ fixing the vertices a and b. Consider the digraphs F_1, F_2 whose vertices are the vertices a, b of $\Phi(F_1)$ and $\Phi(F_2)$ and arcs uv if there exists a copy of Y such that $a = u$ and $b = v$ in $\Phi(F_1)$ and $\Phi(F_2)$ respectively. It follows that g induces a homomorphism $f : F_1 \to F_2$.

Finally, given a digraph $F \in \vec{\mathscr{C}}$, let $\Phi'(F) = G + \Phi(F)$. It follows that Φ' is an embedding from $(\vec{\mathscr{C}}, \leq)$ into the interval $[G, H]$. $\square$

3.3 Universal Intervals in $(\vec{\mathscr{C}}, \leq)$

In the previous section we have proved that the homomorphism order of graphs has the fractal property, which is that every interval in $(\mathscr{C}, \leq)$ is either universal or a gap. Moreover, every interval in $(\mathscr{C}, \leq)$ is universal with the only exception of the gap $[K_1, K_2]$. However, the case of digraphs is not that simple. In fact, it has an infinite number of gaps; one for each core of an oriented tree. We know from Theorem 2.4.2 that any interval $[G, H]$ where the core of H does not contain a cycle (or it is not an oriented tree) is dense. Furthermore, such intervals are universal.

Lemma 3.3.1. *Let digraphs G, H be cores satisfying $G < H$, where H is connected and contains a cycle. Then there exists incomparable graphs X_1, X_2 such that $G < G + X_i < H$ for $i = 1, 2$.*

Proof. Consider the shortest cycle in H and let H' be the digraph from Figure 3.2. The construction of H' is analogous to the proof of Theorem 2.4.2.

Let $n > \max\{|V(G)|, 3|V(H)|\}$ and consider the complete graph K_n. Let X_1 be the digraph obtained from an arbitrary orientation of K_n by replacing each arc uv by a copy of H' identifying u with a' and v with a. Note that the vertices of K_n are in X_1'. We shall refer to them as *original vertices*. It follows from the proof of Theorem 2.4.2 that $G < G + X_1 < H$.

Let S be a connected graph given by Theorem 2.3.1 with chromatic number and girth greater than $\max\{|V(G)|, |V(X_1)|\}$. Let $\vec{S}$ be an orientation of S containing a directed cycle $\vec{C} \subset \vec{S}$. Let X_2 be a digraph obtained from $\vec{S}$ by replacing each arc uv by a copy of H' identifying u with a' and v with a. Analogously, it follows that $G < G + X_2 < H$.

Let's see that X_1 and X_2 are incomparable. Suppose $X_2 \to X_1$. Since $\chi(S) > |V(X_1)|$, at least two original vertices of X_2 which are adjacent in S will be mapped to the same vertex in X_1. This induces a homomorphism $H \to X_1$, which is a contradiction. Suppose now that there exists a homomorphism $f : X_1 \to X_2$. Let $I \subset \vec{S}$ be a subgraph whose arcs are $\{xy \in E(\vec{S}) \mid \exists u \in V(X_1)$ such that $f(u) \in V(H'_{xy})\}$ where $H'_{xy} \subset X_2$ is the copy of H' corresponding to the arc xy. Since the girth of S is greater than $|V(X_1)|$, I must be an oriented tree. Recall that by Proposition 1.3.8 for any balanced digraph I, in particular any tree, there exists a homomorphism $I \to \vec{P}_k$ for some $k > 0$, where $\vec{P}_k$ is the directed path of length k. Observe that $\vec{P}_k \to \vec{C}$ for any $k > 0$. Then there exists a homomorphism $g : I \to \vec{P}_k \to \vec{C}$ for some $k > 0$. Let I', $\vec{P}'_k$ and $\vec{C}'$ be the digraphs obtained by replacing each arc of I, $\vec{P}_k$ and $\vec{C}$ by a copy of H' respectively. Then g induces a homomorphism $g' : I' \to \vec{P'}_k \to \vec{C}'$ identifying the arcs xy with their corresponding copies H'_{xy}. Observe that $\mathrm{Im}(f) \subset I'$ and $\vec{C}' \subset X_2$. Thus, $g' \circ f : X_1 \to X_2$ is a

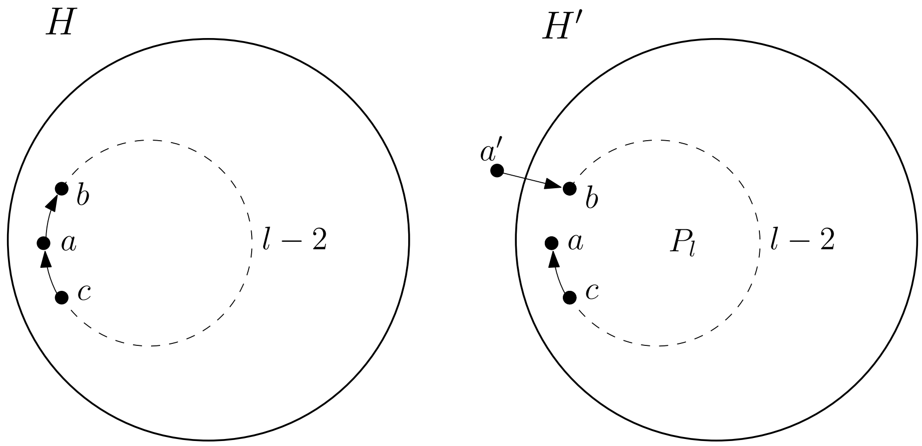

Figure 3.2: Digraph H, digraph H' and path P_l. The arc joining the vertices a and c might be in the other direction.

homomorphism. Let $v_o \in V(X_1)$ be an original vertex and let $H'_o \subset \vec{C}'$ be a copy of H' such that $(g' \circ f)(v_o) \in H'_o$. Since any other original vertex $v_i \in V(X'_1)$ is joined to v_o at least by a path of length l, $(g' \circ f)(v_i)$ must be mapped to H'_o or to one of the two copies of H' adjacent to H'_o. Hence, all original vertices of X_1 are mapped to at most three copies of H'. Considering that $\chi(K_n) > 3|V(H)|$, it follows that at least two original vertices from X_1 will be mapped to the same vertex in X_2. This induces a homomorphism $H \to X_2$ which is a contradiction. $\square$

Theorem 3.3.2. *Let digraphs G, H be cores satisfying $G < H$, where H is connected and contains a cycle. Then the interval $[G, H]$ is universal.*

Proof. Let X_1, X_2 be the digraphs of the proof of Lemma 3.3.1. Assume that X_1, X_2 are cores. Observe that X_1 and X_2 are connected since H is connected. Let X_1, X_2 be its respective cores. We know that $G < G + X_i < H$ for $i = 1, 2$. Let $m > \max\{|V(X_1)|, |V(X_2)|\}$. Consider the path P_l from Figure 3.2 and let x_1, x_2 be an original vertex of X_1 and $C' \subseteq X_2$ respectively (C' is a digraph from the proof of Lemma 3.3.1). Let Y be the digraph from Figure 3.3. Observe that x_1 is joined to x_2 by two different paths, one consisting in $2m$ consecutive paths P_l and the other consisting in $2m+1$ paths P_l in the opposite direction. Note that the vertices y_1, y'_1 are in a copy of H' so y_1 is identified with a' and y'_1 is identified with a. Observe that $Y \backslash X_2 \to X_1$ and $Y \backslash X_1 \to X_2$ due to the choice of x_1 and x_2. Finally, observe that $G < G + Y < H$.

We shall construct an embedding Φ from the homomorphism order of the class of all digraphs $(\vec{\mathscr{C}}, \leq)$ into the interval $[G, H]$.

Given a digraph $F \in \vec{\mathscr{C}}$, let $\Phi(F)$ be the digraph obtained by replacing each arc $uv \in E(F)$ by a copy of Y identifying u with y_1 and v with y_2. Observe that $G < G + \Phi(F) < H$. Consider a homomorphism $f : Y \to \Phi(F)$. Since X_1 and X_2

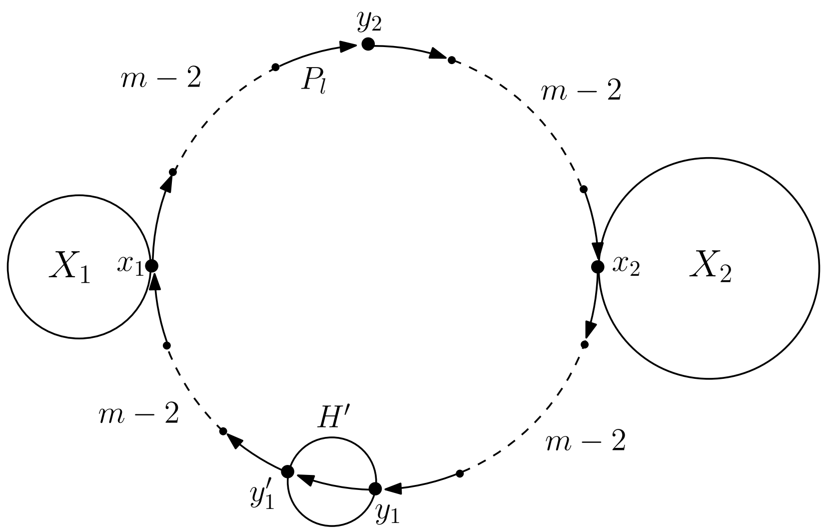

Figure 3.3: Digraph Y.

are connected incomparable cores, they must be mapped to a copy of itself in $\Phi(F)$ respectively. Suppose that the path P_l is not symmetric in respect to its middle point, so P_l has a direction. Then the only paths between two copies X_1 and X_2 in $\Phi(F)$ consisting on $2m$ consecutive forward paths P_l are those from x_1 to x_2 of the same copy of Y. Thus, $f(x_1) = x_1$, $f(x_2) = x_2$ and $f(y_2) = y_2$. And the same happens with the path consisting on $2m+1$ consecutive backward paths P_l, so $f(y_1) = y_1$. On the other hand, suppose that the path P_l is symmetric in respect its middle point. If there exists a homomorphism $H' \to P_l$, then the core of H, and hence H, must be a cycle (the one obtained by identifying the starting and the ending vertices of P_l). However, since P_l is symmetric, it implies that H can be collapsed into a path, which is a contradiction. So $H' \nrightarrow P_l$. Then the only pair of vertices, one of some copy of X_1 and the other of some copy of X_2 in $\Phi|(F)$, that are joined at the same time by a path consisting on $2m$ consecutive paths P_l , and by a path consisting on $2m+1$ consecutive paths P_l but containing a copy of H', are the vertices x_1 and x_2 of the same copy of Y. Thus, $f(x_1) = x_1$, $f(x_2) = x_2$, $f(y_2) = y_2$ and $f(y_1) = y_1$. We conclude that a homomorphism $f : Y \to \Phi(F)$ maps Y to some copy of it in $\Phi(F)$ fixing all vertices in both paths joining X_1 and X_2. In particular, f fixes the vertices y_1 and y_2.

We claim that for any pair of digraphs F_1, F_2, $F_1 \to F_2$ if and only if $\Phi(F_1) \to \Phi(F_2)$. Suppose $f : F_1 \to F_2$ is a homomorphism. Consider the mapping $g : \Phi(F_1) \to \Phi(F_2)$ equal to f on the vertices of F_1 and mapping each copy of Y corresponding to the arc uv to the copy of Y corresponding to the arc $f(u)f(v)$. It is easy to check that g is a homomorphism. Suppose now that $g : \Phi(F_1) \to \Phi(F_2)$ is a homomorphism. We have seen that every copy of Y in $\Phi(F_1)$ must be mapped to a copy of Y in $\Phi(F_2)$ fixing the vertices y_1 and y_2. Consider the digraphs F_1, F_2

whose vertices are the vertices y_1, y_2 of $\Phi(F_1)$ and $\Phi(F_2)$ and arcs uv if there exists a copy of Y such that $y_1 = u$ and $y_2 = v$ in $\Phi(F_1)$ and $\Phi(F_2)$ respectively. It follows that g induces a homomorphism $f : F_1 \to F_2$.

Finally, given a digraph $F \in \vec{\mathscr{C}}$, let $\Phi'(F) = G + \Phi(F)$. It follows that Φ' is an embedding from $(\vec{\mathscr{C}}, \leq)$ into the interval $[G, H]$. □

Observe that Lemma 3.3.1 and Theorem 3.3.2 are only valid when H is connected. However, we shall show the following strengthening.

Theorem 3.3.3. *Let digraphs G, H be cores satisfying $G < H$ and let $H_c \subseteq H$ be a connected component containing a cycle such that $H_c \nrightarrow G$. Then the interval $[G, H]$ is universal.*

Proof. Observe that, analogously to the proof of Lemma 3.3.1, we can construct two connected incomparable digraphs X_1, X_2 such that $X_i < H_c$ and $X_i \nrightarrow G$ for $i = 1, 2$. Then, analogously to the proof of Theorem 3.3.2, we can construct a digraph Y from X_1 and X_2, and a mapping Φ such that for any pair of digraphs F_1, F_2, $F_1 \to F_2$ if and only if $\Phi(F_1) \to \Phi(F_2)$. We end by considering the mapping $\Phi'(F) = G + \Phi(F)$, which is an embedding from $(\vec{\mathscr{C}}, \leq)$ into the interval $[G, H]$. □

We have proved that every interval in $(\vec{\mathscr{C}}, \leq)$ of the form $[G, H]$ where H contains a cycle is universal. The remaining cases are the intervals $[G, T]$ where T is an oriented tree. This cases are more complicated since there is no density theorem for them. In fact, every gap $[G, T]$ in the homomorphism order of digraphs satisfies that T is an oriented tree. We shall focus our interest on the class of oriented trees in the next chapter.

Chapter 4

Homomorphism Order of the Class of oriented trees

4.1 Homomorphisms and oriented trees

As we have already seen in the previous chapters, the class of oriented trees has unique properties and its study is necessary to understand completely the homomorphism order of digraphs. For that, let us start by recalling some basic definitions.

A *path* or *oriented path* is a digraph consisting in a sequence of different vertices $v_0, \ldots, v_k$ together with a sequence of different arcs $e_1, \ldots, e_k$ such that e_i is an arc joining v_{i-1} and v_i for each $i = 1, \ldots, k$. A *cycle* or *oriented cycle* its defined analogously to a path but with $v_0 = v_k$. A *tree* or *oriented tree* is a connected digraph containing no cycles. Note that, in particular, a path is a tree, so every definition and property which applies to trees is also valid for paths.

Proposition 4.1.1. *A digraph is a tree if and only if every pair of vertices is joined by a unique path.*

Proof. Let G be a digraph. Suppose G contains a cycle C. Let $v_0, v_1, \ldots, v_k = v_0$ be the sequence of vertices of C, then v_0, v_1 and $v_1, \ldots, v_k = v_0$ are two different paths between v_0 and v_1. Suppose G has a pair of different vertices a, b which are joined by two different paths $a = v_0, \ldots, v_k = b$ and $b = u_0, \ldots, u_l = a$, then $v_0, \ldots, v_k = u_0, \ldots, u_l = v_0$ is a cycle in G. □

The *height* of a tree is the maximum difference between forward and backward arcs of a subpath in it. Recall from Proposition 1.3.8 that, since every tree T is a balanced digraph, there exists a homomorphism $f : T \to \vec{P}_k$ for some integer $k > 0$. So given a tree T, consider the minimum $k > 0$ such that there exists a homomorphism $f : T \to \vec{P}_k$. Consider $\vec{P}_k$ as the path with vertices $0, 1, \ldots, k$ and arcs $01, 12, \ldots, (k-1)k$. The *level* of a vertex $v \in V(T)$ is the integer $f(v)$. Note that in this case the height of T is equal to k. Recall also that, by Corollary 1.3.9, every homomorphism between trees preserves the level of vertices. This implies the following proposition.

Proposition 4.1.2. *Let T_1 and T_2 be two trees. If $f : T_1 \to T_2$ is a homomorphism then the height of T_1 is less or equal to the height of T_2.*

A *leaf* is a vertex of a tree of degree one.

Proposition 4.1.3. *Let T be the core of a tree and let $v \in V(T)$ be a leaf. If vu is an arc of T then v is the only inneighbour of u. If uv is an arc of T then v is the only outneighbour of u.*

Proof. Suppose vu and wu are two different arcs of T. Then the mapping $f : V(T) \to V(T)$ defined as $f(v) = w$ and $f(x) = x$ for the rest of vertices is a homomorphism which is not injective. The other case is analogous. □

Observe that in particular, Proposition 4.1.3 implies that the core of a path starts and ends with two arcs in the same direction, with the only exception of the path $\vec{P}_1$ (or $\vec{K}_2$) which is the digraph consisting of one arc.

In order to prove the main theorems of this Chapter let us define a new term related to trees. Given a tree T, a vertex $u \in V(T)$ and a set of vertices $S \subseteq V(T)$, the *plank* from u to S, denoted $P(u, S)$, is the subgraph induced by the vertices of every path which starts with u and contains some vertex $v \in S$.

Lemma 4.1.4. *Let T be a tree and let $v, u \in V(T)$ be adjacent vertices. If $f : T \to T$ is an automorphism then $P(u, \{v\})$ is isomorphic to $P(f(u), \{f(v)\})$.*

Proof. Recall that if f is an automorphism then there exists an homomorphism $f^{-1} : T \to T$ such that $f \circ f^{-1}$ is the identity mapping.

First let's see that $f(P(u, \{v\}) \subseteq P(f(u), \{f(v)\})$. Suppose there exists a vertex $x \in P(u, \{v\})$ such that $f(x) \notin P(f(u), \{f(v)\})$. Then the path joining $f(x)$ to $f(u)$ does not contain the vertex $f(v)$. But applying f^{-1} to such path will imply that the path joining x to u neither contains the vertex v. This is a contradiction since $x \in P(u, \{v\})$.

Finally let's show that the number of vertices of $P(u, \{v\})$ is equal to the number of vertices of $P(f(u), \{f(v)\})$. This would imply that $f|_{P(u,\{v\})} : P(u, \{v\}) \to P(f(u), \{f(v)\})$ is in fact an isomorphism.

Suppose there exists a vertex $x \in V(T)$ such that $f(x) \in P(f(u)\{f(v)\})$ but $x \notin P(u, \{v\})$. Then the path joining $f(x)$ to $f(u)$ contains the vertex $f(v)$. But, as before, applying f^{-1} to such path will imply that the path joining x to u contains the vertex v, which is a contradiction since $x \notin P(u, \{v\})$. □

A digraph G is *rigid* if it is a core and the only automorphism $f : G \to G$ is the identity. The following Lemma will be used for proving the main theorems of this Chapter.

Lemma 4.1.5. *The core of a tree is rigid.*

Proof. Let T be the core of a tree. Let $f : T \to T$ be a homomorphism. Recall that f must be an automorphism since T is a core. Suppose f is different from the identity on T and let $u = \min_{v \in V(T)} \{d(v, f(v)) | v \neq f(v)\}$.

Let $u = v_0, v_1, \ldots, v_k = f(u)$ be the path that joins u with $f(u)$. Observe that $k \neq 1$, since otherwise u and $f(u)$ would be adjacent implying that u and $f(u)$ have different levels, but f is level preserving. Note also that $f(v_1)$ is adjacent to $f(u)$.

First we want to show that $f(v_1) = v_{k-1}$. Let's suppose that $f(v_1) \neq v_{k-1}$. Since f is an automorphism, $P(u, \{v_1\})$ is isomorphic to $P(f(u), \{f(v_1)\})$ by Lemma 4.1.4, and therefore $|V(P(u, \{v_1\}))| = |V(P(f(u), \{f(v_1)\}))|$. But $P(f(u), \{f(v_1)\}) \subset P(u, \{v_1\})$, which is a contradiction. It follows that $f(v_1) = v_{k-1}$. Now, let's consider two cases.

Suppose $k > 2$. Then $v_1 \neq v_{k-1}$. Observe that v_1 satisfies that $v_1 \neq f(v_1)$ and $d(v_1, f(v_1)) = k - 2$. But this is a contradiction, since k was the minimum such distance.

Suppose $k = 2$. Then $v_1 = v_{k-1} = f(v_1)$. By Lemma 4.1.4, $P(v_1, \{u\})$ is isomorphic to $P(v_1, \{f(u)\})$. Let $g : V(T) \to V(T)$ be a mapping equal to the identity on $T \backslash P(v_1, \{u\})$ and equal to f on $P(v_1, \{u)$. It is clear that $g : T \to T$ is a homomorphism since $f(v_1) = v_1$. So g must be an automorphism. However, observe that g is not injective since $g(u) = g(f(u))$. $\square$

4.2 The Class of Oriented Paths

Oriented paths is probably one of the simplest cases of digraphs. For this reason is important to understand its properties and in particular, its behaviour under homomorphisms. Homomorphisms between oriented paths have been studied in [8, 17, 7] and surprising results have been found. One example is the following theorem.

Theorem 4.2.1 ([8])**.** *Let G be a digraph and let P be a path. Then G is homomorphic to P if and only if any path homomorphic to G is also homomorphic to P.*

Recall that a *directed path* is a path which have all arcs in the same direction. We denote by $\vec{P}_k$ the directed path of length k. Note that $\vec{P}_k$ is a core for every $k > 0$.

Focusing our interest in the homomorphism order we have the following trivial result.

Proposition 4.2.2. *$\vec{P}_1$ and $\vec{P}_2$ are the only cores of a path of height one and two respectively. As a result, $[\vec{P}_1, \vec{P}_2]$ is a gap of the homomorphism order of paths.*

Let L_k be the path of height three given in Figure 4.1. We consider a and d to be the initial and ending vertex of L_k respectively. Note that $L_0 = \vec{P}_3$.

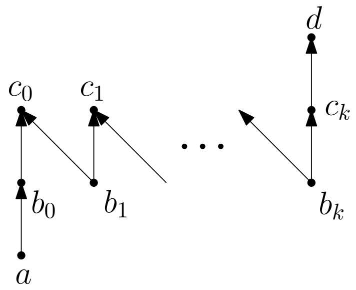

Figure 4.1: The path L_k (for $k \geq 0$).

Proposition 4.2.3. *$L_k \leq L_l$ if and only if $k \geq l$.*

Proof. It is clear since homomorphisms preserve adjacency and the level of vertices. □

Proposition 4.2.4. *The core of a path of height three is equal to L_k for some $k \geq 0$.*

Proof. Consider a path of height equal to three and let P be the path joining a leaf of level zero to a leaf of level three of minimum length. It can be seen that $P = L_k$ for some $k \geq 0$, and that the core of T is equal to P. □

Proposition 4.2.3 and Proposition 4.2.4 imply the following result.

Proposition 4.2.5. *$[L_{k+1}, L_k]$ is a gap of the homomorphism order of paths, for any $k \geq 0$.*

Let's denote $\mathcal{L}$ to the set of L_k for every $k \geq 0$. Observe that $(\mathcal{L}, \leq)$ is a linear order. In fact, $(\mathcal{L}, \leq)$ is isomorphic to the natural order of negative integers by associating each negative integer $(-k)$ with the path L_{k-1}.

As we have seen, the homomorphism order of the paths of height less or equal to three is really simple. First there is the gap $[\vec{P}_1, \vec{P}_2]$, and then there is the linear order $(\mathcal{L}, \leq)$ in the interval $[\vec{P}_2, \vec{P}_3]$. However, the homomorphism order of the paths of height greater or equal to four is more complex.

Theorem 4.2.6 ([17]). *Let paths P_1, P_2 satisfy $P_1 < P_2$ where P_2 is a path of height greater or equal to 4. Then there exists a path P such that $P_1 < P < P_2$.*

The proof of Theorem 4.2.6 is quite technical (making use of Theorem 4.2.1), and considers different cases. For this reason we shall skip it.

One might ask whether intervals of the form $[P_1, P_2]$ where P_2 is a path of height greater or equal to four are universal or not. This, in fact, is still an open question. Although it is not that strong, there is a result which relates the class of oriented paths with universality.

Theorem 4.2.7 ([9, 10]). *The class of oriented paths is universal.*

4.3 Density Theorem for Trees

In this section we shall show a density theorem for the class of oriented trees. This is one of the main results developed in this thesis. But first, let us show the following lemma which applies for all digraphs.

A *zig-zag* is a directed path which alternates forward and backward arcs. Observe that if a zig-zag has even length then the starting and ending vertex have the same level. On the other hand, if the length is odd the starting and ending vertex will have different level. The core of all zig-zags is the digraph $\vec{P_1}$ (or $\vec{K_2}$).

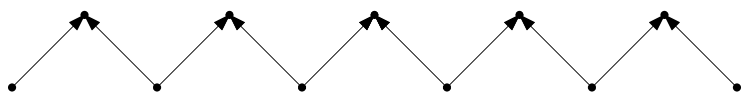

Figure 4.2: Zig-zag of length 10.

Given a digraph G and a zig-zag Z which starts in a vertex $v \in V(G)$, we say that the zig-zag is *proper* if there exists a homomorphism from Z to an arc of $G \backslash Z$.

Lemma 4.3.1. *Let digraphs G, X, H be cores satisfying $G < X < H$, where H is connected. Then there exists a connected digraph X', obtained from the joining of the components of X by proper and long enough zig-zags, such that $G < X' < H$.*

Proof. Assume that X is not connected, otherwise we are done. Consider a homomorphism $f : X \to H$. Let X_1, X_2 be two different components of X and let $x_1 \in V(X_1)$ and $x_2 \in V(X_2)$ be two vertices such that $d(f(x_1), f(x_2))$ is minimum.

Consider the digraph obtained from X by adding two new vertices x_1', x_2', joining x_1 to x_1' and x_2 to x_2' by a proper zig-zag of even length greater than $|V(H)|$, and joining x_1' to x_2' by the path in H from $f(x_1)$ to $f(x_2)$. Observe that f can be extended into a homomorphism from such digraph to H since the zig-zags can be mapped to an arc so $f(x_1') = f(x_1)$ and $f(x_2') = f(x_2)$. Now, let X' be the connected digraph obtained by joining each pair of components in X by the previous procedure. Analogously, f can be extended into a homomorphism $f' : X' \to H$. It is clear that $G < X'$ since $X \subset X'$. Finally, suppose there exists a homomorphism $g : H \to X'$. Since every zig-zag has length greater than $|V(H)|$, then H must be homomorphic either to one of the components of X or to some path $P \subset H$. The first can not be since $H \nrightarrow X$, and the second is a contradiction since H is a core. Hence, $G < X' < H$. □

In the previous section we have focused on the class of oriented paths. We know that paths are a particular case of trees. However, the class of trees which are different from paths has also interesting properties. For this reason, let us say that a tree is *proper* if its core is not a path. Observe that one particular characteristic of proper trees is that they always have a vertex of degree at least three.

In order to prove the density theorem we shall construct a tree $\mathcal{D}_n(T_2)$ from a given proper tree T_2 which will satisfy $T_1 < \mathcal{D}_n(T_2) < T_2$ for every tree $T_1 < T_2$.

Construction of $\mathcal{D}_n(T_2)$:

Let T_2 be the core of a proper tree. Then there exists a vertex $x \in V(T_2)$ such that x is adjacent to at least three different vertices, call them u, v, w. Without loss of generality we shall assume that ux and wx or xu and xu are arcs of T_2. In fact, we can assume that ux and wx are arcs; for the other case we would applied the same construction but changing the direction of all arcs appearing in it. Let $X' \subseteq V(T_2)$ be the set of vertices, different from u and w, which are adjacent to x. Note that X' is not empty since $v \in X'$. Let $X \subset T_2$ be the plank from x to X'. Let $U' = P(x, \{u\})$ and let $W' = P(x, \{w\})$. Let U and W be the tree obtained from U' and W' by removing the vertex x respectively. Note that $U \sqcup X \sqcup W \sqcup \{ux, wx\} = T_2$. See Figure 4.3.

Now, let $\mathcal{D}_1(T_2)$ be the tree from Figure 4.4, where W and W' are copies of the plank $W \subset T_2$, U is a copy of $U \subset T_2$, and X and X' are copies of $X \subset T_2$.

Finally, let $\mathcal{D}_n(T_2)$ be a tree consisting in n consecutive trees $\mathcal{D}_1(T_2)$ whose planks W' are identified with the planks W of the following trees, as shown in Figure 4.5. We shall refer to the vertices $w_i, a_i, u_i, x_i, b_i, x_i' \in \mathcal{D}_n(T_2)$ for $i = 1, \ldots, n$ as *labelled vertices*. Note that $\mathcal{D}_n(T_2)$ is a proper tree for any $n > 0$.

Lemma 4.3.2. *Let T_1 and T_2 be finite oriented trees such that T_2 is a proper tree and $T_2 \nrightarrow T_1$. If there exists a homomorphism $f : \mathcal{D}_n(T_2) \to T_1$, then every labelled vertex of $\mathcal{D}_n(T_2)$ is mapped to a different vertex of T_1.*

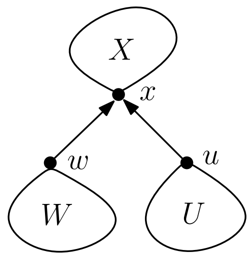

Figure 4.3: Tree T_2

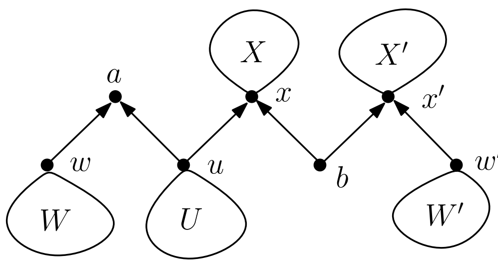

Figure 4.4: Tree $\mathcal{D}_1(T)$

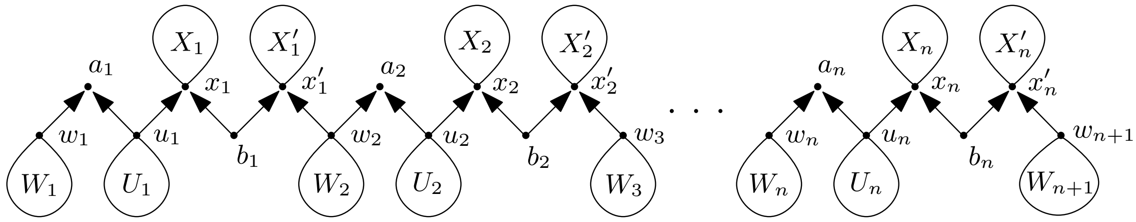

Figure 4.5: Tree $\mathcal{D}_n(T_2)$. Observe the enumeration of the vertices and planks of each tree $\mathcal{D}_1(T)$

Proof. Assume that T_2 is a core and consider a homomorphism $f : \mathcal{D}_n(T_2) \to T_1$. Observe that two consecutive labelled vertices can not be mapped via f to the same vertex since it would imply that T_1 contains a loop. Now, observe that if any pair of labelled vertices of distance two are mapped to the same vertex, it will induce a homomorphism $T_2 \to T_1$. This follows from the construction of $\mathcal{D}_n(T_2)$. See Figure 4.5. Finally, if two labelled vertices of distance greater or equal to three are mapped to the same vertex, it would imply that T_1 contains a cycle since every pair of labelled vertices of distance less than three are mapped to different vertices, but T_1 is a tree. We conclude that every labelled vertex has to be mapped to a different vertex of T_1. $\square$

Theorem 4.3.3. *Let T_1 and T_2 be two finite oriented trees satisfying $T_1 < T_2$. If T_2 is a proper tree, then there exists a tree T such that $T_1 < T < T_2$.*

Proof. Assume that T_2 is a core. Let $n > |V(T_1)|$ and consider the tree $\mathcal{D}_n(T_2)$ constructed from T_2. We claim that $T_1 < T_1 + \mathcal{D}_n(T_2) < T_2$.

Observe that there exists a homomorphism $h : \mathcal{D}_n(T_2) \to T_2$ which maps each vertex of $\mathcal{D}_n(T_2)$ to its corresponding vertex in T_2 (mapping the vertices a_i to x and the vertices b_i to either w or u for $i = 1, \ldots, n$). Suppose there exists a homomorphism $g : T_2 \to \mathcal{D}_n(T_2)$. Since the core of a tree is rigid, the plank U has to be mapped to some plank $U_i \subset \mathcal{D}_n(T_2)$ mapping all vertices of U to its corresponding vertices of U_i. Otherwise, $(h \circ g)(U)$ would be different from the identity on U, and hence, the composition $h \circ g : T_2 \to T_2$ would be a homomorphism different from the identity, contradicting Lemma 4.1.5. The same happens to the planks X and W;

W has to be mapped to some $W_j \subset \mathcal{D}_n(T_2)$ and X has to be mapped to either some X_k or $X'_k \subset \mathcal{D}_n(T_2)$. However, there are not three consecutive vertices u_i, x_k, w_j or u_i, x'_k, w_j in $\mathcal{D}_n(T_2)$. Then $\mathcal{D}_n(T_2) < T_2$.

Suppose there exists a homomorphism $f : \mathcal{D}_n(T_2) \to T_1$. By Lemma 4.3.2 we know that f must map every labelled vertex of $\mathcal{D}_n(T_2)$ to a different vertex of T_1. However, since $n > |V(T_1)|$, the number of labelled vertices of $\mathcal{D}_n(T_2)$ is greater than the number of vertices of T_1. Then $T_1 < T_1 + \mathcal{D}_n(T_2)$.

We end by considering the tree T consisting in the joining of T_1 and $\mathcal{D}_n(T_2)$ by a proper and long enough zig-zag. Then, by Lemma 4.3.1, $T_1 < T < T_2$. □

4.4 Fractal Property for Trees

In the previous section we have constructed a tree $\mathcal{D}_n(T_2)$ from a given proper tree T_2. Thanks to it we have proved a density theorem for the class of trees; every interval $[T_1, T_2]$ where T_2 is a proper tree is dense. Furthermore, we shall show that such intervals are in fact universal. The following theorem is one of the main results of this thesis.

Theorem 4.4.1. *Let T_1 and T_2 be two finite oriented trees satisfying $T_1 < T_2$. If T_2 is a proper tree, then the interval $[T_1, T_2]$ is universal.*

Proof. Assume that T_2 is a core. Let $n > |V(T_1)| + \max\{2|V(U)|, 2|V(X)|, 2|V(W)|\}$ and consider the tree $\mathcal{D}_n(T_2)$. We know from the proof of Theorem 4.3.3 that $T_1 < T_1 + \mathcal{D}_n(T_2) < T_2$.

Let T be the core of $\mathcal{D}_n(T_2)$. First, we shall show that T must contain more than $|V(T_1)|$ labelled vertices. Consider a homomorphism $f : \mathcal{D}_n(T_2) \to T$. By Lemma 4.3.2, every labelled vertex of $\mathcal{D}_n(T_2)$ must be mapped to a different vertex of T, but $\mathcal{D}_n(T_2)$ has more labelled vertices than $|V(T_1)| + \max\{2|V(U)|, 2|V(X)|, 2|V(W)|\}$. It is easy to check that at least $|V(T_1)|$ labelled vertices of $\mathcal{D}_n(T_2)$ will be mapped to labelled vertices of T. Thus, T starts with some labelled vertex y and, after more than $|V(T_1)|$ consecutive labelled vertices, it ends with some labelled vertex z. See Figure 4.6. Let T' be the tree obtained from T by adding two new vertices y' and z' and joining y' to y and z' to z by a proper zig-zag of length 5 or 6 so y' and z' have the same level as shown in Figure 4.6. Finally, let T'' be the tree obtained by joining T_1 to T' by a proper and long enough zig-zag so $T_1 < T'' < T_2$ by Lemma 4.3.1.

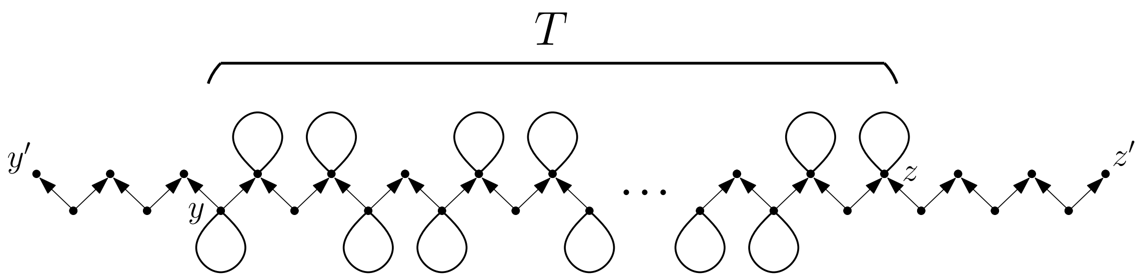

Figure 4.6: This is an example of how T' might look. The vertices y and z might be different from the ones in the figure but they must be labelled vertices of $\mathcal{D}_n(T_2)$.

Now, we shall construct an embedding Φ from the homomorphism order of the class of oriented paths, which we know is countably universal, into the interval $[T_1, T_2]$.

Given an oriented path P, let $\Phi(P)$ be the tree obtained by replacing each arc v_1v_2 in P by a copy of T'' identifying v_1 with y' and v_2 with z'. Observe that $T_1 < \Phi(P) < T_2$.

Let's see first that a homomorphism $f : T \to \Phi(P)$ can not map a labelled vertex of T to a vertex belonging to a zig-zag in $\Phi(P)$. Consider a vertex $x_i \in V(T)$. Let s be a vertex of $X_i \subset T$ adjacent to x_i and let S be the plank $P(x_i, \{s\}) \subset T$. Suppose $f(x_i)$ is a vertex belonging to a zig-zag in $\Phi(P)$. Note that $f(x_i)$ and x_i must have the same level. Observe that since $f(x_i)$ belongs to a zig-zag, $f(s)$ must have the same level of w and u. Let $h : \Phi(P) \to T_2$ be the homomorphism which maps each vertex of $\Phi(P)$ to its corresponding vertex in T_2 (mapping all the vertices of the zig-zags to x, u or w). Then $h \circ f|_S : S \to T_2$ is a homomorphism which maps the vertex s in S to either the vertex w or u in T_2. Note that $(h \circ f|_S)(x_i) = x$. Let $t : V(T_2) \to V(T_2)$ be a mapping equal to $h \circ f|_S$ for the vertices in $V(S)$ and equal to the identity mapping for the rest of vertices. It is easy to check that t is a homomorphism. Observe that t is a homomorphism different from the identity since $t(s)$ is equal to u or w. However this is a contradiction since T_2 is rigid by Lemma 4.1.5. Analogously, it can be checked that any labelled vertex of T can be neither mapped to the zig-zag between copies of T nor mapped to the zig-zag between a copy of T and a copy of T_1.

Let $f : T \to \Phi(P)$ be a homomorphism. It is clear that $f(y)$ can not belong to some copy of T_1 in $\Phi(P)$ since it would imply that the rest of labelled vertices of T would be mapped to the same copy of T_1, but $T \nrightarrow T_1$. Then $f(y)$ must belong to some copy of T in $\Phi(P_1)$. So let T_y be the copy of T such that $f(y) \in V(T_y)$. Observe that the rest of labelled vertices of T must be mapped into T_y, since non of them can be mapped to a vertex belonging to a zig-zag as we have seen above. We know from Lemma 4.1.5 that T is rigid, so $f : T \to T_y$ must be the identity mapping. This means that the vertex $y \in T$ must be mapped to the vertex $y \in T_y$ and the vertex $z \in T$ must be mapped to the vertex $z \in T_y$.

Finally, we claim that for any pair of paths P_1, P_2, $P_1 \to P_2$ if and only if $\Phi(P_1) \to \Phi(P_2)$.

Suppose $f : P_1 \to P_2$ is a homomorphism. Let's define $g : \Phi(P_1) \to \Phi(P_2)$ being equal to f on the vertices of P_1 and sending each copy of T'' corresponding to the arc v_1v_2 to the copy of T'' corresponding to the arc $f(v_1)f(v_2)$. Thus, g is a homomorphism.

Suppose now that $g : \Phi(P_1) \to \Phi(P_2)$ is a homomorphism. As we have seen above, each copy of T in $\Phi(P_1)$ must be mapped to a copy of T in $\Phi(P_2)$ identifying the vertices y and z of each copy respectively. Let S_1 and S_2 be two different copies of T in $\Phi(P_1)$ and let S'_1 and S'_2 be its respective copies of T'. Observe that if the intersection of S'_1 and S'_2 is not empty then the intersection of $g(S'_1)$ and $g(S'_2)$ is also not empty. In particular, suppose that the vertex z' in S'_1 is equal to the vertex y' in S'_2. In this case we have that the distance from the vertex z in S_1 to the vertex y in S_2 is at most 12. Then $g(S_1)$ can not be equal to $g(S_2)$ since vertices y and z in some copy of T are at distance many times greater than 12. So $g(S_1)$ and $g(S_2)$ must be mapped to two different copies of T in $\Phi(P_2)$ such that the vertex z' in $g(S'_1)$ is equal to the vertex y' in $g(S'_2)$. Finally, considering the paths P_1, P_2 whose vertices are the vertices y', z' of $\Phi(P_1)$ and $\Phi(P_2)$ and arcs uv if there exists a copy of T'' such that $y' = u$ and $z' = v$ in $\Phi(F_1)$ and $\Phi(F_2)$ respectively, it easily follows that g induces a homomorphism $f : P_1 \to P_2$. □

Theorem 4.4.1 is only valid for intervals $[T_1, T_2]$ where T_2 is a proper tree, so it is not a complete result for the class of oriented trees. However, it is a complete result if we only consider proper trees. Theorem 4.4.1 implies that every interval in the class of proper trees is universal.

Corollary 4.4.2. *The class of proper trees has the fractal property.*

A *forest* is a digraph containing no cycles. Observe that the difference between a forest and a tree is only its connectedness. So a forest is a digraph which is equal to a sum of trees. Then Theorem 4.4.1 can be generalised to the homomorphism order of the class of forests.

Theorem 4.4.3. *Let F_1 and F_2 be two forests satisfying $F_1 < F_2$ and let $T_2 \subseteq F_2$ be a connected component and a proper tree such that $T_2 \nrightarrow F_1$. Then the interval $[F_1, F_2]$ is universal.*

Proof. Assume that F_2 is a core. Note that T_2 is then a proper tree and a core. Let $n > |V(F_1)| + 2|V(T_2)|$ and consider the tree $\mathcal{D}_n(T_2)$. Analogously to the proof of Theorem 4.4.1 we can construct a mapping Φ from the class of paths to the class of proper trees such that for any path P, $\Phi(P) < T_2$ and $\Phi(P) \nrightarrow F_1$, and for any pair of paths P_1, P_2, $\Phi(P_1) \to \Phi(P_2)$ if and only if $P_1 \to P_2$. Finally, consider the mapping $\Phi'(P) = F_1 + \Phi(P)$. It is clear that Φ' is an embedding from the homomorphism order of paths into the interval $[F_1, F_2]$. $\square$

Let us say that a forest is *proper* if it is equal to a sum of proper trees.

Corollary 4.4.4. *The class of proper forests has the fractal property.*

Chapter 5

Concluding Remarks

5.1 Further Implications

We have seen in this thesis that the class of proper trees and the class of symmetric digraphs (or graphs) have the fractal property. These results are proved in Theorem 4.4.1 and Theorem 3.2.4 respectively. However, the class of finite digraphs, and even the class of oriented trees, is more complicated.

By Theorem 2.5.4 and Theorem 2.6.3 we have characterised all gaps in the homomorphism order of digraphs. In particular, we have shown that for every tree T there exists a balanced digraph G_T such that $[G_T, T]$ is a gap, and that all gaps have this form. Theorem 4.3.3 contributes to this result by implying that if $[G, T]$ is a gap and T is a proper tree, then G must contain a cycle.

The characterisation of universal intervals in the homomorphism order of digraphs seems to be complicated. Related to this issue, we have proved Theorem 3.3.2 and Theorem 4.4.1 (and generalised such results in Theorem 4.4.3 and Theorem 3.3.3 respectively). The first implies that every interval $[G, H]$ where the core of H contains a cycle (so it is not a tree) is universal. The second implies that every interval $[T, T_p]$ where T is a tree and T_p is a proper tree is also universal. Both theorems together have great implications. We know that given a proper tree T there exists a unique (up to homomorphic equivalence) digraph G_T such that $[G_T, T]$ is a gap. Consider now a interval $[G, T]$, where $G \neq G_T$, so there exists a digraph H such that $G < H < T$. If H is a tree then the interval $[H, T]$ is universal. Suppose H is not a tree, so it contains a cycle. Then the interval $[G, H]$ is universal and hence, $[G, T]$ is also universal since $[G, H] \subset [G, T]$. In conclusion, every interval $[G, T]$ where T is a proper tree is either universal or a gap.

Corollary 5.1.1. *The class of digraphs whose cores are not paths has the fractal property.*

The remaining cases are then the intervals of the form $[G, P]$ where the core of P is a path. We have already seen some properties of the class of oriented paths in Section 4.2. In fact, many of them holds for the homomorphism order of digraphs.

Observe that the definition of level and height is valid for balanced digraphs in general.

Proposition 5.1.2. *The core of a balanced digraph of height less or equal to three is a path.*

Proof. Let G be the core of a balanced digraph. If the height is equal to one or two it is clear that G must be a path. Assume that G has height equal to three. Let P be a path between a leaf of level 0 and a leaf of level 3 of minimum length of G. Note that P must be equal to the path L_l for some $l \geq 0$. Observe that $L_k \nrightarrow G$ for $0 \leq k < l$ since we have considered P to have minimum length. Let P' be a path. By Proposition 4.2.4 the core of P' is equal to $L_{l'}$ for some $l' \geq 0$. Then, by Proposition 4.2.3, $P' \to G$ if and only if $l' \geq l$. So $P' \to G$ if and only if $P' \to P$. Finally this implies by Theorem 4.2.1 that $G \to P$. Hence, the core of G is equal to P. □

This proposition implies that in the interval $[\vec{P}_2, \vec{P}_3]$ of the homomorphism order of digraphs there is only the linear order $(\mathcal{L}, \leq)$. It follows that every gap in the homomorphism order of the form $[G, T]$, where T is a tree of height less or equal to three, is either one of the two trivial gaps $[K_1, \vec{P}_1]$ or $[\vec{P}_1, \vec{P}_2]$, or one of the gaps in $(\mathcal{L}, \leq)$. Observe that the fact that $(\mathcal{L}, \leq)$ is in the homomorphism order already excludes the class of digraphs of having the fractal property as we have defined in this thesis. Intervals as $[L_2, \vec{P}_3]$, or even $[\vec{P}_2, \vec{P}_3]$, are neither a gap or universal.

5.2 Open Questions

We have characterised (in terms of universality and gaps) every interval $[G, H]$ where the core of H is not a path or it is a path of height less or equal to three. The remaining cases are intervals of the form $[G, P]$ where the core of P is a path of height greater or equal to four. For such intervals, we know that if $[G_P, P]$ is a gap then, by Theorem 4.2.6, the core of G_P must be different from a path. So G_P must be a proper tree or it must contain a cycle. However that is pretty much all the information we can obtain from the known results. It remains to be seen in which cases G_P contains a cycle or is a proper tree. We know that the first case is possible since we have seen an example in Figure 2.2 (the second gap). But we don't know if there exists gaps of the form $[G_P, P]$ where G_P is a proper tree and P is a path of height greater or equal to four. About universality, there is none result concerning universal intervals of the form $[G, P]$ where P is a path of height greater or equal to four.

It is also interesting to study the homomorphism order of the class of paths, which is a suborder of the homomorphism order of digraphs, so any result in it would be also valid for the second one. We have seen a density theorem for paths of height greater or equal to four. We might conjecture that such intervals are, in addition of dense, universal. Not so distinct case are intervals of the form $[T, P]$ where T is a proper tree and P is a path of height greater or equal to four. We believe that these intervals should be also dense, and moreover, universal. Then the homomorphism order of digraphs would have the fractal property with the linear order $(L, \leq)$ as its only exception. But it still remains to be seen.

References

[1] P. Erdős. Graph theory and probability. *Canad. J. Math.*, 11:34–38, 1959.

[2] J. Fiala, J. Hubička, and Y. Long. Universality of intervals of line graph order. *Europ. J. Combin.*, 41:221–231, 2014.

[3] J. Fiala, J. Hubička, Y. Long, and J. Nešetřil. Fractal property of the graph homomorphism order. *Europ. J. Combin.*, 66:101–109, 2017.

[4] J. Foniok, J. Nešetřil, and C. Tardif. Generalised dualities and maximal finite antichains in the homomorphism order of relational structures. *Europ. J. Combin.*, 29:881–899, 2008.

[5] R. Fraïssé. *Theory of Relations*, volume 118 of *Studies in Logic and the Foundation of Mathematics*. North-Holland, 1986.

[6] Z. Hedrlín. On universal partly ordered sets and classes. *J. Algebra*, 11:503–509, 1969.

[7] P. Hell and J. Nešetřil. *Graphs and Homomorphisms*, volume 28 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, 2004.

[8] P. Hell and X.Zhu. Homomorphisms to oriented paths. *Discrete Math.*, 132:107–114, 1994.

[9] J. Hubička and J. Nešetřil. Finite paths are universal. *Order*, 22:21–40, 2005.

[10] J. Hubička and J. Nešetřil. Universal partial order represented by means of oriented trees and other simple graphs. *Europ. J. Combin.*, 26:765–778, 2005.

[11] J. Hubička and J. Nešetřil. Some examples of universal and generic partial orders. In M. Grohe and J. A. Makowsky, editors, *Model Theoretic Methods in Finite Combinatorics*, pages 296–318. American Mathematical Society, 2011.

[12] J.B. Johnston. Universal infinite partially ordered sets. *Proc. Amer. Math. Soc.*, 7:507–514, 1956.

[13] E. Marczewsky. Sur l'extension de l'ordre partiel. *Fundamenta Mathematicae*, 16:386–389, 1930.

[14] J. Nešetřil. The homomorphism structure of classes of graphs. *Combin. Probab. Comput.*, 17:87–184, 1999.

[15] J. Nešetřil and A. Pultr. On classes of relations and graphs determined by subobjects and factorobjects. *Discrete Math*, 22:287–300, 1978.

[16] J. Nešetřil and C. Tardif. Duality theorems for finite structures (characterising gaps and good characterisations). *J. Combin. Theory Ser. B*, 80:80–97, 2000.

[17] J. Nešetřil and X. Zhu. Path homomorphisms. *Math. Proc. Camb. Phil. Soc.*, 120:207–220, 1996.

[18] A. Pultr and V. Trnková. *Combinatorial, Albebraic and Topological Representation of Groups, Semigroups and Categories*, volume 22 of *North-Holland Mathematical Library*. North-Holland, 1980.

[19] E. Welzl. Color-families are dens. *J. Theoret. Comput. Sci.*, 17:29–41, 1982.