Resolving dominating partitions in graphs

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Abstract

A partition $\Pi = \{S_1, \ldots, S_k\}$ of the vertex set of a connected graph G is called a *resolving partition* of G if for every pair of vertices u and v, $d(u, S_j) \neq d(v, S_j)$, for some part S_j. The *partition dimension* $\beta_p(G)$ is the minimum cardinality of a resolving partition of G. A resolving partition Π is called *resolving dominating* if for every vertex v of G, $d(v, S_j) = 1$, for some part S_j of Π. The *dominating partition dimension* $\eta_p(G)$ is the minimum cardinality of a resolving dominating partition of G.

In this paper we show, among other results, that $\beta_p(G) \leq \eta_p(G) \leq \beta_p(G) + 1$. We also characterize all connected graphs of order $n \geq 7$ satisfying any of the following conditions: $\eta_p(G) = n$, $\eta_p(G) = n - 1$, $\eta_p(G) = n - 2$ and $\beta_p(G) = n - 2$. Finally, we present some tight Nordhaus-Gaddum bounds for both the partition dimension $\beta_p(G)$ and the dominating partition dimension $\eta_p(G)$.

Keywords: resolving partition, resolving dominating partition, metric location, resolving domination, partition dimension, dominating partition dimension.

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1 Introduction

Domination and location in graphs are two important subjects that have received a high attention in the literature, usually separately, but sometimes also both together. These concepts are useful to distinguish the vertices of a graph in terms of distances to a given set of vertices or by considering their neighbors in this set. Resolving sets were introduced independently in the 1970s by Slater [26], as *locating sets*, and by Harary and Melter [14], whereas dominating sets were defined in the 1960s by Ore [21]. Both types of sets have many and varied applications in other areas. For example, resolving sets have applications in robot navigation [26], combinatorial optimization [25], game theory [10], pharmaceutical chemistry [5] and in other contexts [2, 4]. On the other hand, dominating sets are helpful to design and analyze communication networks [8, 24] and to model biological networks [16].

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Many variations of location in graphs have since been defined (see survey [23]). For example, in 2000, Chartrand, Salehi and Zhang study the resolvability of graphs in terms of partitions [6], as a generalization of resolving sets when the vertices are classified in different types. A few years later, resolving dominating sets were introduced by Brigham, Chartrand, Dutton and Zhang [1] and independently by Henning and Oellermann [17] as metric-locating-dominating sets, combining the usefulness of resolving sets and dominating sets. Resolving dominating sets have been further studied in [3, 11, 19]. In this paper, following the ideas of these works, we introduce the *resolving dominating partitions*, as a way for distinguishing the vertices of a graph by using on the one hand partitions, and on the other hand, both domination and location.

1.1 Basic terminology

All the graphs considered are undirected, simple, finite and (unless otherwise stated) connected. Let v be a vertex of a graph G. The *open neighborhood* of v is $N_G(v) = \{w \in V(G) : vw \in E\}$, and the *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$ (we will write $N(v)$ and $N[v]$ if the graph G is clear from the context). The *degree* of v is $\deg(v) = |N(v)|$. The minimum degree (resp. maximum degree) of G is $\delta(G) = \min\{\deg(u) : u \in V(G)\}$ (resp. $\Delta(G) = \max\{\deg(u) : u \in V(G)\}$). If $\deg(v) = 1$, then v is said to be a *leaf* of G.

The distance between vertices $v, w \in V(G)$ is denoted by $d_G(v, w)$, or $d(v, w)$ if the graph G is clear from the context. The diameter of G is $\operatorname{diam}(G) = \max\{d(v, w) : v, w \in V(G)\}$. The distance between a vertex $v \in V(G)$ and a set of vertices $S \subseteq V(G)$, denoted by $d(v, S)$, is the minimum of the distances between v and the vertices of S, that is, $d(v, S) = \min\{d(v, w) : w \in S\}$.

Let $u, v \in V(G)$ be a pair of vertices such that $d(u, w) = d(v, w)$ for all $w \in V(G) \setminus \{u, v\}$, i.e., such that either $N(u) = N(v)$ or $N[u] = N[v]$. In both cases, u and v are said to be *twins*. Let W be a set of vertices of G. If the vertices of W are pairwise twins, then W is called a *twin set* of G.

Let $W \subseteq V(G)$ be a subset of vertices of G. The *closed neighborhood* of W is $N[W] = \cup_{v \in W} N[v]$. The subgraph of G induced by W, denoted by $G[W]$, has W as vertex set and $E(G[W]) = \{vw \in E(G) : v \in W, w \in W\}$.

The *complement* of G, denoted by $\overline{G}$, is the graph on the same vertices as G such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. Let G_1, G_2 be two graphs having disjoint vertex sets. The (*disjoint*) *union* $G = G_1 + G_2$ is the graph such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. The *join* $G = G_1 \vee G_2$ is the graph such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$.

1.2 Metric dimension and partition dimension

A set of vertices $S \subseteq V(G)$ of a graph G is a *resolving set* of G if for every pair of distinct vertices $v, w \in V(G)$, $d(v, x) \neq d(w, x)$, for some vertex $x \in S$. The *metric dimension* $\beta(G)$ of G is the minimum cardinality of a resolving set.

Resolving sets were introduced independently in papers [14] and [26] (in this last work they were called *locating sets*), and since then they have been widely investigated (see [4, 18, 25] and their references).

Let $\Pi = \{S_1, \ldots, S_k\}$ be a partition of $V(G)$. We denote by $r(u|\Pi)$ the vector of distances between a vertex $u \in V(G)$ and the elements of Π, that is, $r(u|\Pi) = (d(u, S_1), \ldots, d(u, S_k))$. If $u, v \in V(G)$, we say that a part S_i of Π *resolves* u and v if $d(u, S_i) \neq d(v, S_i)$. If $V' \subseteq V(G)$, we say that a part S_i of Π *resolves* V' if S_i resolves every pair of vertices of V'.

A partition $\Pi = \{S_1, \ldots, S_k\}$ is called a *resolving partition* of G if for any pair of distinct vertices $u, v \in V(G)$, $r(u|\Pi) \neq r(v|\Pi)$, that is, if the set $\{u, v\}$ is resolved by some part S_i of Π.

The *partition dimension* $\beta_p(G)$ of G is the minimum cardinality of a resolving partition of G. Resolving partitions were introduced in [6], and further studied in [7, 9, 12, 13, 22, 28]. In some of these papers the partition dimension of G is denoted by $pd(G)$. Next, some known results concerning this parameter are shown.

Theorem 1 ([6]). *Let G be a graph of order $n \geq 2$. Then,*

(1) $\beta_p(G) \leq \beta(G) + 1$.

(2) $\beta_p(G) \leq n - \mathrm{diam}(G) + 1$. Moreover, this bound is sharp.

(3) $\beta_p(G) = 2$ if and only if G is isomorphic to the path P_n.

(4) $\beta_p(G) = n$ if and only if G is isomorphic to the complete graph K_n.

(5) If $n \geq 6$, then $\beta_p(G) = n - 1$ if and only if G is isomorphic to either the star $K_{1,n-1}$, or the complete split graph $K_{n-2} \vee \overline{K_2}$, or the graph $K_1 \vee (K_1 + K_{n-2})$.

Remark 2. *Notice that the restriction $n \geq 6$ of Theorem 1(5) is tight, since $\beta_p(C_4) = 3$ and $\beta_p(C_4 \vee K_1) = 4$. Thus, in [6], the condition $n \geq 3$ of Theorem 3.3 is incorrect.*

Proposition 3 ([7]). *Given a pair of integers a, b such that $3 \leq a \leq b + 1$, there exists a graph G with $\beta_p(G) = a$ and $\beta(G) = b$.*

The remaining part of this paper is organized as follows. In Section 2, we introduce the dominating partition number $\eta_p(G)$ and show some basic properties for this new parameter. Finally, Section 3 is devoted to the characterization of all graphs G satisfying any of the following conditions: $\eta_p(G) = n$, $\eta_p(G) = n - 1$, $\eta_p(G) = n - 2$ and $\beta_p(G) = n - 2$ and to show some tight Nordhaus-Gaddum bounds for both the partition dimension $\beta_p(G)$ and the dominating partition dimension $\eta_p(G)$.

2 Dominating partition dimension

A set D of vertices of a graph G is a *dominating set* if $d(v, D) = 1$, for every vertex $v \in V(G) \setminus D$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set.

A set $S \subseteq V(G)$ is a *resolving dominating set*, if it is both resolving and dominating. The *resolving domination number* $\eta(G)$ of G is the minimum cardinality of a resolving dominating set of G. Resolving dominating sets were introduced in [1], and also independently in [17] (in this last work they were called *metric-locating-dominating sets*), being further studied in [3, 11, 15, 19, 20, 27].

As a straightforward consequence of these definitions, it holds that (see [3]):

$$\max\{\gamma(G), \beta(G)\} \leq \eta(G) \leq \gamma(G) + \beta(G).$$

A partition $\Pi = \{S_1, \ldots, S_k\}$ of $V(G)$ is called *dominating* if for every $v \in V(G)$, $d(v, S_j) = 1$ for some $j \in \{1, \ldots, k\}$. The *partition domination number* $\gamma_p(G)$ equals the minimum cardinality of a dominating partition in G.

Proposition 4. *For any non-trivial graph G, $\gamma_p(G) = 2$.*

Proof. Let S be a dominating set of cardinality $\gamma(G)$. Observe that the partition $\Pi = \{S, V(G) \setminus S\}$ is a dominating partition of G. Hence, $\gamma_p(G) = 2$, since G is non-trivial. □

Let $\Pi = \{S_1, \ldots, S_k\}$ be a partition of the vertex set of a non-trivial graph G. The partition Π is called a *resolving dominating partition* of G, *RD-partition* for short, if it is both resolving and dominating. The *dominating partition dimension* $\eta_p(G)$ of G is the minimum cardinality of an RD-partition of G. An RD-partition of cardinality $\eta_p(G)$ is called an $\eta_p(G)$*-partition* of G.

Proposition 5. *If G is a non-trivial graph, then $\eta_p(G) = 2$ if and only if G is isomorphic to K_2.*

Proof. Certainly, $\eta_p(K_2) = 2$. Conversely, let G be a graph such that $\eta_p(G) = 2$. Take an $\eta_p(G)$-partition $\Pi = \{S_1, S_2\}$. Suppose that for some $i \in \{1, 2\}$, $|S_i| \geq 2$. Assume w.l.o.g. that $i = 1$ and take $u, v \in S_1$. As Π is a dominating partition, $r(u|\Pi) = (0, 1) = r(v|\Pi)$, contradicting that Π is a resolving partition. So, $|S_1| = |S_2| = 1$ and thus $G \cong K_2$. □

Proposition 6. *Let P_n and C_n denote the path and the cycle of order n, respectively. If $n \geq 3$, then $\eta_p(P_n) = \eta_p(C_n) = 3$.*

Proof. According to Proposition 5, it is sufficient to show, in both cases, the existence of an RD-partition of cardinality 3. Assume that $V = V(P_n) = V(C_n) = \{1, \ldots, n\}$; $E(P_n) = \{\{i, i+1\} : 1 \leq i < n\}$ and $E(C_n) = E(P_n) \cup \{\{1, n\}\}$. Consider the following sets of vertices:

$S_1 = \{1\}$, $S_1' = \{1, 2\}$, $S_2 = \{i \in V : i \text{ even}\}$, $S_2' = \{i \in V : i \neq 2, \text{ even}\}$, $S_3 = \{i \in V : i \neq 1, \text{ odd}\}$.

It is straightforward to check that $\Pi = \{S_1, S_2, S_3\}$ is an RD-partition of P_n, and also of C_n if n is odd, and that $\Pi' = \{S_1', S_2', S_3\}$ is an RD-partition of C_n, if n is even. □

Next, we show some results relating the dominating partition dimension η_p to other parameters such as the resolving domination number η, the partition dimension β_p, the order and the diameter.

Proposition 7. *For any graph G of order $n \geq 2$, $\eta_p(G) \leq \eta(G) + 1$.*

Proof. Suppose that $\eta(G) = k$. Notice that $k \leq n - 1$, since $n \geq 2$. Let $S = \{u_1, \ldots, u_k\}$ be a resolving dominating set of G. Then, $\Pi = \{\{u_1\}, \ldots, \{u_k\}, V(G) \setminus S\}$ is an RD-partition of G. □

Lemma 8. *Let G be a graph of order $n \geq 3$. Let $W \subsetneq V(G)$ be a twin set of cardinality $k \geq 2$.*

(1) *If W induces an empty graph, then $\eta_p(G) \geq \beta_p(G) \geq k$.*

(2) *If W induces a complete graph, then $\eta_p(G) \geq \beta_p(G) \geq k + 1$.*

(3) *If W is a set of leaves, then $\eta_p(G) \geq k + 1$.*

Proof. (1) Let W be a twin set of cardinality k. Since $d(w_1, v) = d(w_2, v)$ for every $w_1, w_2 \in W$ and for every $v \in V(G) \setminus \{w_1, w_2\}$, we have that different vertices of W must belong to different parts of any resolving partition. Hence, $\eta_p(G) \geq \beta_p(G) \geq k$.

Observe that, if $\beta_p(G) = k$, then every part of a resolving partition of cardinality k contains exactly one vertex of W.

(2) Suppose that W induces a complete graph and $W \subsetneq V(G)$. Since G is connected, there exists a vertex v adjacent to all the vertices of W. If $\beta_p(G) = k$ and Π is a resolving partition of cardinality k, then there is some vertex $w \in W$ such that v and w belong to the same part S of Π. Then, v and w are at distance 1 from any part of Π different from S, implying that $r(v|\Pi) = r(w|\Pi)$, a contradiction. Therefore, $\beta_p(G) \geq k + 1$.

(3) Assume that W is a twin set of leaves hanging from a vertex u. Suppose that $\eta_p(G) = k$ and Π is an RD-partition of cardinality k. Then, Π is also a resolving partition of cardinality k. Hence, there is some vertex $w \in W$ such that u and w belong to the same part S of Π. But in such a case, Π is not a dominating partition, because w is a leaf hanging from u. Therefore, $\eta_p(G) \geq k + 1$. □

Proposition 9. *Given a pair of integers a, b such that $3 \leq a \leq b + 1$, there exists a graph G with $\eta_p(G) = a$ and $\eta(G) = b$.*

Proof. Let $h = a - 2$ and $k = b - a + 2$. Take the caterpillar G of order $n = 2k + h$ displayed in Figure 1. The set $W = \{w_1, \ldots, w_h, u_1\}$ is a twin set of $h + 1$ leaves. Thus, by Lemma 8, we have $\eta_p(G) \geq h + 2$. Now, take the partition $\Pi = \{\{u_1, \ldots, u_k\}, \{v_1, \ldots, v_k\}, \{w_1\}, \ldots, \{w_h\}\}$. Clearly, Π is both a dominating and a resolving partition. Hence, $\eta_p(G) = h + 2 = a$.

To prove that $\eta(G) = b$, note first that every resolving dominating set S must contain all vertices from the twin set W except at most one. Observe also that for every $i \in \{1, \ldots, k\}$, either u_i or v_i must belong to S. Thus, $\eta(G) \geq h + k = b$. Now, take the set $S = \{w_1, \ldots, w_h, u_1, \ldots, u_k\}$. Clearly, S is both dominating and resolving. Hence, $\eta(G) = h + k = b$. □

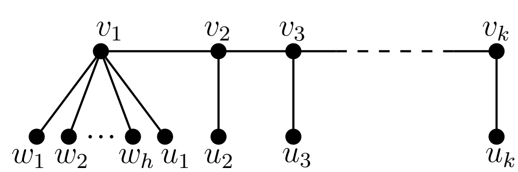


Figure 1: Caterpillar G of order $n = 2k + h$, $\eta_p(G) = h + 2$ and $\eta(G) = h + k$.

Next, a remarkable double inequality relating both the partition dimension and the dominating partition dimension is shown.

Theorem 10. *For any graph G of order $n \geq 3$, $\beta_p(G) \leq \eta_p(G) \leq \beta_p(G) + 1$.*

Proof. The first inequality follows directly from the definition of RD-partition. Let $\beta_p(G) = r$ and let $\Pi = \{S_1, \ldots, S_r\}$ be a resolving partition of G. If Π is a dominating partition, then $\eta_p(G) = \beta_p(G)$. Suppose that Π is not a dominating partition. Let $W = \{u \in V(G) : N[u] \subseteq S_i$ for some $i \in \{1, \ldots, r\}\}$. Note that $W \neq \emptyset$, since Π is not dominating, and that $S_i \backslash W \neq \emptyset$ for every $i \in \{1, \ldots, r\}$, since G is connected. In order to show that $\eta_p(G) \leq \beta_p(G) + 1$, we construct an RD-partition of cardinality $r + 1$.

Let $C_1, \ldots, C_s$ be the connected components of the subgraph $G[W]$ induced by W. Clearly, for every $i \in \{1, \ldots, s\}$, all vertices of C_i belong to the same part of Π. Next, we define a subset $W' \subseteq W$

as follows. If $|V(C_i)| = 1$, then add to W' the unique vertex of C_i. If $|V(C_i)| \geq 2$, then consider a 2-coloring of a spanning tree of C_i, choose one color and add to W' all vertices having this color. Note that, if $V(C_k) \subseteq S_{i_k}$ and a pair of vertices $x, y \in C_k$ are adjacent, then one endpoint of xy is in $W' \cup S_{i_k}$ and the other one belongs to $S_{i_k} \setminus W'$. Let $\Pi' = \{S'_1, \ldots, S'_r, W'\}$, where $S'_i = S_i \setminus W' \subseteq S_i$ for every $1 \leq i \leq r$. We claim that Π' is an RD-partition.

On the one hand, observe that the sets $S'_1, \ldots, S'_r, W'$ are nonempty by construction. On the other hand, notice that for every $u \in S_i$, $d(u, S_j) = d(u, w)$ for some vertex $w \in S_j \setminus W$ whenever $i \neq j$. Indeed, assume to the contrary that $d(u, S_j) = d(u, w)$ and $w \in S_j \cap W$. Since $w \in W$, we have $N[w] \subseteq S_j$. Thus, the vertex w' adjacent to w in a shortest (u, w)-path is also in S_j, implying that $d(u, S_j) \leq d(u, w') < d(u, w) = d(u, S_j)$, a contradiction. From this last observation, we conclude that $d(u, S_j) = d(u, S'_j)$ if $u \in S_i$ and $j \neq i$.

Next, we show that Π' is a dominating partition, i.e., that for any $u \in V(G)$, the vector $r(u|\Pi')$ has at least one component equal to 1. We distinguish two cases.

Case 1: $u \in W'$. Assume that $u \in S_i$, for some $i \in \{1, \ldots, r\}$. If u belongs to a trivial connected component of $G[W]$, then every neighbor of u is in S'_i. So, $d(u, S'_i) = 1$. If u belongs to a non-trivial connected component C_k of $G[W]$, then any neighbor of u with different color in the spanning tree of C_k considered in the construction of W' belongs to S'_i. So, $d(u, S'_i) = 1$.

Case 2: $u \in S'_i$, for some $i \in \{1, \ldots, r\}$. If $u \notin W$, as $u \in S'_i \setminus W = S_i \setminus W$, then u has a neighbor v in some S_j with $j \neq i$. Therefore, $d(u, S'_j) = 1$ if $v \in S'_j$, and $d(u, W') = 1$ if $v \in W'$. If $u \in W$, then u belongs to a non-trivial connected component of $G[W]$ and, by construction of W', u has a neighbor in W'. Thus, $d(u, W') = 1$.

Finally, we show that Π' is a resolving partition, i.e., that $r(u|\Pi') \neq r(v|\Pi')$ for every pair of distinct vertices $u, v \in V(G)$ belonging to the same part of Π'. We distinguish two cases.

Case 1: $u, v \in S'_i$ for some $i \in \{1, \ldots, r\}$. In such a case, $u, v \in S_i$. Since Π is a resolving partition, $d(u, S_j) \neq d(v, S_j)$ for some $j \neq i$. Using the observation above, we have that $d(u, S'_j) = d(u, S_j) \neq d(v, S_j) = d(v, S'_j)$ for some $j \neq i$. Therefore, $r(u|\Pi') \neq r(v|\Pi')$.

Case 2: $u, v \in W'$. If $u, v \in S_i$ for some $i \in \{1, \ldots, r\}$, then proceeding as in the previous case, we have $r(u|\Pi') \neq r(v|\Pi')$. Suppose thus that $u \in S_i$ and $v \in S_j$ with $i \neq j$. Notice that $d(u, S'_i) = 1$ and $N[v] \subseteq S_j$ because $v \in S_j$ and $v \in W' \subseteq W$. Thus, $d(v, S_i) \geq 2$, and so $d(v, S'_i) = d(v, S_i) \geq 2$. Finally, from $d(u, S'_i) \neq d(v, S'_i)$ we get that $r(u|\Pi') \neq r(v|\Pi')$. □

The following result is a direct consequence of Theorem 1(2) and Theorem 10.

Corollary 11. *If G is a graph of order $n \geq 3$, then $\eta_p(G) \leq n - \operatorname{diam}(G) + 2$. Moreover, this bound is sharp, and is attained, among others, by P_n and $K_{1,n-1}$.*

Proposition 12. *If G is a graph of order $n \geq 3$ and diameter d such that $\eta_p(G) = k$, then $n \leq k\,(d^{k-1} - (d-1)^{k-1})$.*

Proof. Let $\Pi = \{S_1, \ldots, S_k\}$ be an RD-partition. If $u \in S_i$, then the i-th component of $r(u|\Pi)$ is 0, any other component is a value from $\{1, 2, \ldots, d\}$ and at least one component must be 1. Since there are $d^{k-1} - (d-1)^{k-1}$ such k-tuples, we have that $|S_i| \leq d^{k-1} - (d-1)^{k-1}$, and therefore, $n \leq \sum_{i=1}^{k} |S_i| \leq k(d^{k-1} - (d-1)^{k-1})$. □

3 Extremal graphs

In [6, 28], all graphs of order $n \geq 9$ satisfying $\beta_p(G) = n$, $\beta_p(G) = n - 1$ and $\beta_p(G) = n - 2$ were characterized. This section is devoted to approach the same problems for the dominating partition dimension $\eta_p(G)$. To this end, we prove a pair of technical lemmas.

Lemma 13. *Let $k \geq 2$ be an integer. Let G be a graph of order n containing a vertex u of degree d. If $n \geq 2k+1$ and $k \leq d \leq n-k-1$, then $\eta_p(G) \leq n-k$.*

Proof. Let $N(u) = \{x_1, \ldots, x_k, \ldots, x_d\}$. Let L be the set containing all leaves at distance 2 from u and let C be the set containing both all non-leaves at distance 2 and all vertices at distance at least 3 from u, i.e., $C = V(G) \setminus (N[u] \cup L)$. Assume that $|L| = l$ and $|C| = c$ and observe that $l + c = n - d - 1 \geq k$.

If $c \geq k$, then take the partition $\Pi = \{\{x_1, y_1\}, \ldots, \{x_k, y_k\}\} \cup \{\{z\} : z \notin \{x_1, \ldots, x_k, y_1, \ldots, y_k\}\}$, where $y_1, \ldots, y_k \in C$. Notice that Π is a resolving partition since, for every $i \in \{1, \ldots, k\}$, $\{u\}$ resolves the pair x_i, y_i, because $d(u, x_i) = 1 < 2 \leq d(u, y_i)$. Furthermore, for every $i \in \{1, \ldots, k\}$, vertex x_i is adjacent to u and vertex y_i is adjacent to a vertex different from x_i, because in the case y_i has degree 1, its neighbor does not belong to $N(u)$ by definition of C. So, Π is also a dominating partition and thus $\eta_p(G) \leq n - k$.

Now, assume $c < k$. Let $h = k - c$ and observe that $1 \leq h \leq l$ since $l + c \geq k$. First, we seek if it is possible to pair h vertices of L with h vertices of $N(u)$ satisfying that each pair is formed by non-adjacent vertices. Observe that this is equivalent to finding a matching M that saturates a subset L' of L of cardinality h in the bipartite graph H defined as follows: $N(u)$ and L are its partite sets, and if $x_i \in N(u)$ and $z \in L$, then $x_i z \in E(H)$ if and only if $x_i z \notin E(G)$. So, the degree in H of a vertex $z \in L$ is $\deg_H(z) = d - 1$. For every nonempty set $W \subseteq L$ with $|W| \leq k - 1$, we have $|W| \leq k - 1 \leq d - 1 \leq |N_H(W)|$, and for $W \subseteq L$ with $|W| = k$ we have $|W| \leq |N_H(W)|$ whenever $d \geq k + 1$ or $|N_H(W)| \geq k$. Therefore, according to Hall's Theorem, there exists a matching M saturating a subset L' of L of cardinality h, except for the case $h = k = d$, provided that $|N_H(W)| < k$ for every subset $W \subseteq L$ with $|W| = k$. Let M be such a matching, whenever it exists. We distinguish two cases.

Case 1: $h < k$. Consider the partition Π formed by the h pairs of the matching M, c pairs formed by pairing the vertices in C with c vertices in $N(u)$ not used in the matching M, and a part for each one of the remaining vertices formed only by the vertex itself. Part $\{u\}$ resolves each part of cardinality 2 and, by construction, Π is dominating. Thus, Π is an RD-partition, implying that $\eta_p(G) \leq n - k$.

Case 2: $h = k$. In such a case, $c = 0$ (i.e., $L = V(G) \setminus N[u]$). If $d > k$, then consider the partition Π formed by the k pairs of the matching M and a part for each one of the remaining vertices formed only by the vertex itself. As in the preceding case, it can be shown that Π is an RD-partition, and so $\eta_p(G) \leq n - k$.

If $d = h = k$ and there is a subset W of L of cardinality k with $|N_H(W)| \geq k$, then there exists a matching M between the vertices of W and the vertices of $N(u)$. Consider the partition Π formed by the k pairs of the matching M and a part for each one of the remaining vertices formed only by the vertex itself. As in the preceding case, it can be shown that Π is an RD-partition, and so $\eta_p(G) \leq n - k$.

Finally, if $d = h = k$ and there is no subset W of L of cardinality k with $|N_H(W)| \geq k$, then all vertices of L are leaves hanging from the same vertex of $N(u)$. We may assume without loss of generality that all vertices in L are adjacent to x_1. Let $y_1, \ldots, y_k \in L$ (they exist because $n \geq 2k+1$). Consider the partition $\Pi = \{\{u, y_1\}, \{x_2, y_2\}, \ldots, \{x_k, y_k\}\} \cup \{\{z\} : z \notin \{u, x_2, \ldots, x_k, y_1, \ldots, y_k\}\}$ (see Figure 2). Notice that Π is a resolving partition since, for every $i \in \{2, \ldots, k\}$, $P_1 = \{u, y_1\}$ resolves the pair x_i, y_i because $d(x_i, P_1) = d(x_i, u) = 1 < 2 = d(y_i, P_1)$; and $P_2 = \{x_2, y_2\}$ resolves the pair u, y_1, because $d(u, P_2) = d(u, x_2) = 1 < 2 = d(y_1, P_2)$. Besides, every vertex has a neighbor in another part by construction. Thus, Π is an RD-partition, implying that $\eta_p(G) \leq n - k$. □

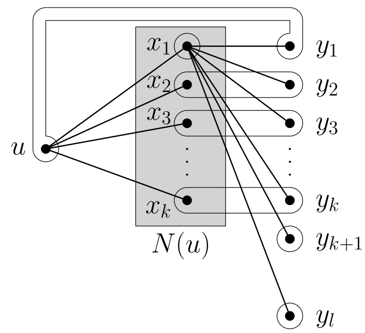


Figure 2: An RD-partition of cardinality $n - k$. There may be edges joining vertices of $N(u)$.

Lemma 14. *Let G be a graph order n.*

(1) *If $n \geq 5$ and $\text{diam}(G) \geq 3$, then $\eta_p(G) \leq n - 2$.*

(2) *If $n \geq 7$ and $\text{diam}(G) \geq 4$, then $\eta_p(G) \leq n - 3$.*

Proof. (1) Let $\text{diam}(G) = d$. If $d \geq 4$, then according to Corollary 11, $\eta_p(G) \leq n - d + 2 \leq n - 2$. Assume thus that $d = 3$ and take a vertex u of eccentricity $ecc(u) = 3$. If u is not a leaf, then $2 \leq \deg(u) \leq n - 3$ and, by Lemma 13, $\eta_p(G) \leq n - 2$. If u is a leaf, then consider the sets $D_i = \{v \mid d(u, v) = i\}$, $i \in \{1, 2, 3\}$. Take $x_i \in D_i$, $i \in \{1, 2, 3\}$ such that $\{ux_1, x_1x_2, x_2x_3\} \subseteq E(G)$. We distinguish cases depending on the cardinality of D_2.

Case 1: $|D_2| \geq 2$. Take a vertex $y_2 \in D_2 - x_2$. Note that $x_1y_2 \in E(G)$, since u is a leaf. Take the partition:

$$\Pi = \{\{x_1, x_2\}, \{x_3, y_2\} \cup \{\{z\} : z \neq x_1, x_2, x_3, y_2\}.$$

Clearly, Π is an RD-partition of G of cardinality $n - 2$. Thus, $\eta_p(G) \leq n - 2$.

Case 2: $|D_2| = 1$. Notice that $|D_3| \geq 2$ since $n \geq 5$. Take a vertex $y_3 \in D_3 - x_3$. Observe that $x_2y_3 \in E(G)$. Take the partition:

$$\Pi = \{\{x_1, x_2\}, \{u, y_3\} \cup \{\{z\} : z \neq u, x_1, x_2, y_3\}.$$

Clearly, Π is an RD-partition of G of cardinality $n - 2$. Thus, $\eta_p(G) \leq n - 2$.

(2) If $d \geq 5$, then according to Corollary 11, $\eta_p(G) \leq n - d + 2 \leq n - 3$. Assume thus that $d = 4$ and take a vertex u of eccentricity of $ecc(u) = 4$. Notice that $\deg(u) \leq n - 4$ and hence, according to Lemma 13 (case $k = 3$), $\eta_p(G) \leq n - 3$ whenever $\deg(u) \geq 3$. Suppose finally that $1 \leq \deg(u) \leq 2$ and consider the sets $D_i = \{v \mid d(u, v) = i\}$, $i \in \{1, 2, 3, 4\}$. Notice that $1 \leq |D_1| \leq 2$. Take $x_i \in D_i$, $i \in \{1, 2, 3, 4\}$ such that $\{ux_1, x_1x_2, x_2x_3, x_3x_4\} \subseteq E(G)$. We distinguish cases depending on the cardinality of D_1 and D_2.

Case 1: $|D_1| = 2$. Take a vertex $y_1 \in D_1 - x_1$. Take the partition:

$$\Pi = \{\{u, x_1\}, \{x_2, x_3\}, \{x_4, y_1\} \cup \{\{z\} : z \neq u, x_1, x_2, x_3, x_4, y_1\}.$$

Clearly, Π is an RD-partition of G of cardinality $n-3$. Thus, $\eta_p(G) \leq n-3$.

Case 2: $|D_1| = 1$ and $|D_2| \geq 2$. Take a vertex $y_2 \in D_2 - x_2$. Take the partition:

$$\Pi = \{\{u, x_4\}, \{x_1, x_2\}, \{x_3, y_2\} \cup \{\{z\} : z \neq u, x_1, x_2, x_3, x_4, y_2\}.$$

Clearly, Π is an RD-partition of G of cardinality $n-3$. Thus, $\eta_p(G) \leq n-3$.

Case 3: $|D_1| = 1$, $|D_2| = 1$ and $|D_3| \geq 2$. Take a pair of vertices $y_3, w \in D_3 \cup D_4 \setminus \{x_3, x_4\}$ such that $y_3 \in D_3$. Take the partition:

$$\Pi = \{\{x_1, w\}, \{x_2, x_3\}, \{x_4, y_3\} \cup \{\{z\} : z \neq x_1, x_2, x_3, x_4, y_3, w\}.$$

Clearly, Π is an RD-partition of G of cardinality $n-3$. Thus, $\eta_p(G) \leq n-3$.

Case 4: $|D_1| = 1$, $|D_2| = 1$ and $|D_3| = 1$. Take a pair of vertices $y_4, w_4 \in D_4 - x_4$. Note that $\{x_3y_4, x_3w_4\} \subseteq E(G)$. Take the partition:

$$\Pi = \{\{u, y_4\}, \{x_1w_4\}, \{x_2, x_3\} \cup \{\{z\} : z \neq u, x_1, x_2, x_3, x_4, y_4, w_4\}.$$

Clearly, Π is an RD-partition of G of cardinality $n-3$. Thus, $\eta_p(G) \leq n-3$. □

In [6], all graphs of order n satisfying $n-1 \leq \beta_p \leq n$ were characterized (see Theorem 1). We display a similar result for the dominating partition dimension η_p.

Theorem 15. *If G is a graph of order $n \geq 6$, then*

(1) *$\eta_p(G) = n$ if and only if G is isomorphic to either the complete graph K_n or the star $K_{1,n-1}$.*

(2) *$\eta_p(G) = n-1$ if and only if G is isomorphic to either the complete split graph $K_{n-2} \vee \overline{K_2}$, or the graph $K_1 \vee (K_1 + K_{n-2})$.*

Proof. (1) According to Theorem 10, if $\eta_p(G) = n$ then $n-1 \leq \beta_p(G) \leq n$. By direct inspection on graphs with $\beta_p(G) = n$ and $\beta_p(G) = n-1$ (see Theorem 1) the stated result is derived.

(2) It is a routine exercise to check that $\eta_p(K_{n-2} \vee \overline{K_2}) = \eta_p(K_1 \vee (K_1 + K_{n-2})) = n-1$. Conversely, let G be a graph such that $\eta_p(G) = n-1$. By Lemma 14(1), $\text{diam}(G) = 2$, since $G \ncong K_n$. Take a pair of vertices u, v such that $d(u, v) = 2$. By Lemma 13 (case $k = 2$), $\deg(u), \deg(v) \in \{1, n-2\}$. We distinguish three cases.

Case 1: $\deg(u) = \deg(v) = 1$. Let w be the vertex such that $N(u) = N(v) = \{w\}$. By Lemma 13, the rest of vertices of G have degree 1, as they are not adjacent neither to u nor to v. Hence, all vertices of G other than vertex w are leaves hanging from w, i.e., $G \cong K_{1,n-1}$, a contradiction.

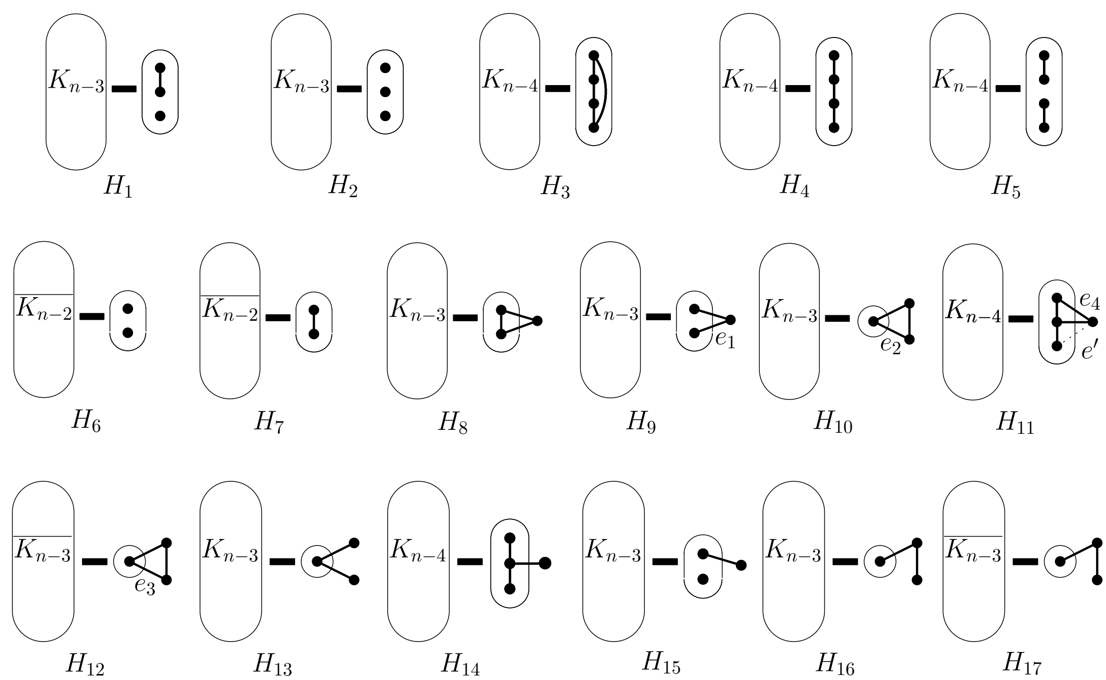
Case 2: $\deg(u) = \deg(v) = n-2$. In this case, $N(u) = N(v) = V(G) \setminus \{u, v\} = W$ and for all vertex $z \in W$, $\deg(z) \geq 2$. Then, by Lemma 13 (case $k = 2$), $\deg(z) \in \{n-2, n-1\}$.

If $\deg(z) = n-1$ for all $z \in W$, then G is isomorphic to the complete split graph $K_{n-2} \vee \overline{K_2}$.

If there is a vertex $t \in W$ such that $\deg(t) = n-2$, then let $s \in W$ be the vertex that is not adjacent to t. Observe that both t and s are adjacent to any other vertex of W. If $a, b \in W \setminus \{s,t\}$, then $\Pi = \{\{u,a\},\{s,b\}\} \cup \{\{z\} : z \neq a,b,u,s\}$ is an RD-partition, and thus $\eta_p(G) \leq n-2$.

Case 3: $\deg(u) = 1$ and $\deg(u) = n-2$. Let w be the vertex adjacent to u. Since the diameter is 2, every vertex $t \notin \{u,w,v\}$ is adjacent both to w and v. In particular, for all vertex $t \notin \{u,w,v\}$, $\deg(t) \geq 2$ and, by Lemma 13 (case $k=2$), $\deg(t) = n-2$ and then G is isomorphic to the graph $K_1 \vee (K_1 + K_{n-2})$. □

Next, we characterize those graphs with $\eta_p(G) = n-2$. Concretely, we prove that, for every integer $n \geq 7$, a graph of order n satisfies $\eta_p(G) = n-2$ if and only if it belongs to the family $\Lambda_n = \{H_1, \ldots, H_{17}\}$ (see Figure 3).



$H_1 \cong K_{n-3} \vee (K_2 + K_1)$	$H_2 \cong K_{n-3} \vee \overline{K_3}$	$H_3 \cong K_{n-4} \vee C_4$
$H_4 \cong K_{n-4} \vee P_4$	$H_5 \cong K_{n-4} \vee 2\,K_2$	$H_6 \cong K_{2,n-2}$
$H_7 \cong \overline{K_{n-2}} \vee K_2$	$H_8 \cong (K_{n-3} + K_1) \vee K_2$	$H_9 \cong (K_{n-3} + K_1) \vee \overline{K_2}$
$H_{10} \cong (K_{n-3} + K_2) \vee K_1$	$H_{11} \cong (K_{n-4} + K_1) \vee P_3 - e'$	$H_{12} \cong (\overline{K_{n-3}} + K_2) \vee K_1$
$H_{13} \cong (K_{n-3} + \overline{K_2}) \vee K_1$	$H_{14} \cong H_{11} - e_4$	$H_{15} \cong H_9 - e_1$
$H_{16} \cong H_{10} - e_2$	$H_{17} \cong H_{12} - e_3$	

Figure 3: The family Λ_n of all graphs of order $n \geq 7$ such that $\eta_p(G) = n-2$.

Proposition 16. *If $G \in \Lambda_n = \{H_1, \ldots, H_{17}\}$, then $\eta_p(G) = n-2$. Moreover, if $G \in \Lambda_n \setminus \{H_{12}, H_{17}\}$, then $\beta_p(G) = n-2$.*

Proof. According to Theorem 15, for every graph $H_i \in \Lambda_n$, $\beta_p(G) \leq \eta_p(G) \leq n-2$. Thus, it is enough to check that, for every graph $H_i \in \Lambda_n$, $\eta_p(H_i) \geq n-2$, and also that if $i \notin \{12, 17\}$, then $\beta_p(H_i) \geq n-2$.

Case 1: If $G \in \{H_6, H_7\}$, then it contains a twin set W of cardinality $n-2$ (see Figure 3) and thus, by Lemma 8, $\eta_p(G) \geq \beta_p(G) \geq n-2$.

Case 2: If $G \in \{H_1, H_2, H_8, H_9, H_{10}, H_{13}, H_{15}, H_{16}\}$, then there exists a set of vertices W of $n-3$ vertices of G such that W induces a complete graph (see Figure 3), and thus, according to Lemma 8, $\eta_p(G) \geq \beta_p(G) \geq (n-3)+1 = n-2$.

Case 3: If $G \in \{H_{12}, H_{17}\}$, then G is a graph with a twin set of $n-3$ leaves (see Figure 3) and, by Lemma 8, $\beta_p(G) \geq n-3$ and $\eta_p(G) \geq (n-3)+1 = n-2$.

Case 4: If $G \in \{H_3, H_4, H_5, H_{11}, H_{14}\}$, then there exists a twin set W of cardinality $n-4$ that W induces a complete graph (see Figure 3), and thus, by Lemma 8, $\eta_p(G) \geq \beta_p(G) \geq (n-4)+1 = n-3$. Suppose that there exists a resolving partition $\Pi = \{S_1, \ldots, S_{n-3}\}$ of cardinality $n-3$. Assume that $W = \{w_1, \ldots, w_{n-4}\}$ and $w_i \in S_i$, for every $i \in \{1, \ldots, n-4\}$, so that $S_{n-3} \cap W = \emptyset$. Notice also that all these graphs have diameter 2. We distinguish two cases.

Case 4.1: If $G \in \{H_3, H_4, H_5\}$, then $N[W] = V(G)$ and $|V(G) \setminus W| = 4$. Clearly, $|S_{n-3}| = 1$, since $r(z|\Pi) = (1, \ldots, 1, 0)$ for every $z \in S_{n-3}$. Notice also that $|S_i| \leq 2$ for $i \in \{1, \ldots, n-4\}$, since for every $x \in S_i$ we have $r(x|\Pi) = (1, \ldots, 1, \overset{i)}{0}, 1, \ldots, 1, h)$, with $h \in \{1, 2\}$. Hence, there must be exactly three sets of Π of cardinality 2 and we can suppose without loss of generality that $S_1 = \{w_1, x\}$, $S_2 = \{w_2, y\}$, $S_3 = \{w_3, z\}$ and $S_{n-3} = \{t\}$, where $\{x, y, z, t\} = V(G) \setminus W$. We know that $d(t, w_1) = d(t, w_2) = d(t, w_3) = 1$, hence $d(t, x) = d(t, y) = d(t, z) = 2$, a contradiction, because there is no vertex satisfying this condition in $V(G) \setminus W$.

Case 4.2: If $G \in \{H_{11}, H_{14}\}$, then $|N[W] \setminus W| = 3$. We may assume $N[W] \setminus W = \{a, b, c\}$ and $V(G) \setminus N[W] = \{z\}$ with $d(a, b) = d(b, c) = 1$, $d(b, z) = 1$ and $d(c, z) = 2$ in both graphs. Notice that S_{n-3} has as most one vertex from $\{a, b, c\}$, since $r(x|\Pi) = (1, \ldots, 1, 0)$ whenever $x \in \{a, b, c\} \cap S_{n-3}$. Moreover, $b \notin S_{n-3}$, because if $b \in S_{n-3}$, then $a \notin S_{n-3}$ so that $a \in S_i$, for some $i \in \{1, \ldots, n-4\}$, and then $r(a|\Pi) = r(w_i|\Pi) = (1, \ldots, 1, \overset{i)}{0}, 1, \ldots, 1, 1)$, a contradiction. So, we can assume without loss of generality that $\{w_1, b\} \subseteq S_1$. Thus, $S_{n-3} = \{z\}$, otherwise a or c should belong to S_{n-3}, so that $r(w_1|\Pi) = r(b|\Pi) = (0, 1, \ldots, 1, 1)$, a contradiction. Hence $c \in S_j$, for some $j \in \{1, \ldots, n-4\}$, but then $r(w_j|\Pi) = r(c|\Pi) = (1, \ldots, 1, \overset{j)}{0}, 1 \ldots, 1, 2)$, a contradiction. □

The remainder of this section is devoted to showing that these 17 graph families are the only ones satisfying $\eta_p(G) = n-2$.

First, note that as a direct consequence of Lemma 14(2) the following result is derived.

Corollary 17. *If G is a graph with $\eta_p(G) = n-2$, then $2 \leq \mathrm{diam}(G) \leq 3$.*

3.1 Case diameter 2

Let G be a graph such that $\eta_p(G) = n-2$ and $\mathrm{diam}(G) = 2$. We distinguish two cases depending whether $\delta(G) \geq n-3$ or $\delta(G) \leq n-4$. To approach the first case (notice that the restriction $\mathrm{diam}(G) = 2$ is redundant) we need the following technical lemma.

Lemma 18. *Let G be a graph of order $n \geq 7$ and minimum degree $\delta(G)$ at least $n-3$. If G contains at most $n-5$ vertices of degree $n-1$, then $\eta_p(G) \leq n-3$.*

Proof. Observe that the complement $\overline{G}$ of G is a (non-necessarily connected) graph with vertices of degree 0, 1 or 2. Thus, the components of $\overline{G}$ are either isolated vertices, or paths of order at least 2, or cycles of order at least 3. By hypothesis, G has at most $n-5$ vertices of degree $n-1$, therefore $\overline{G}$ has at least 5 vertices of degree 1 or 2. We distinguish three cases.

Case 1: $\overline{G}$ has only one non-trivial component. In such a case, $\overline{G}$ has al least a (non-necessarily induced) subgraph isomorphic to P_5. Let x_1, x_2, x_3, x_4 and x_5 be the vertices of this path, where $x_ix_{i+1} \in E(\overline{G})$ for $i = 1,2,3,4$. Let $z \notin \{x_1, x_2, x_3, x_4, x_5\}$. Consider the partition:

$$\Pi = \{\{x_1, x_3, x_5, z\}\} \cup \{\{v\} : v \notin \{x_1, x_3, x_5, z\}\}.$$

We claim that Π is an RD-partition of G (see Figure 4 (a)). Indeed, if $S_1 = \{x_2\}$ and $S_2 = \{x_4\}$, then $r(x_1|\Pi) = (2,1,\dots)$, $r(x_3|\Pi) = (2,2,\dots)$, $r(x_5|\Pi) = (1,2,\dots)$, $r(z|\Pi) = (1,1,\dots)$. Moreover, x_3 is adjacent in G to any vertex $w \notin \{x_1, x_2, x_3, x_4, x_5, z\}$, that exists because the order of G is at least 7. Therefore, Π is an RD-partition of G. Thus, $\eta_p(G) \leq n-3$.

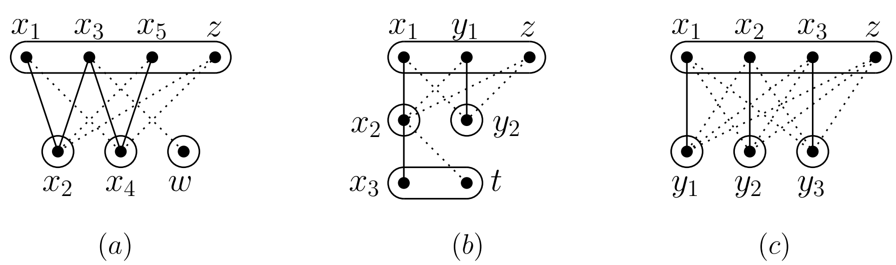


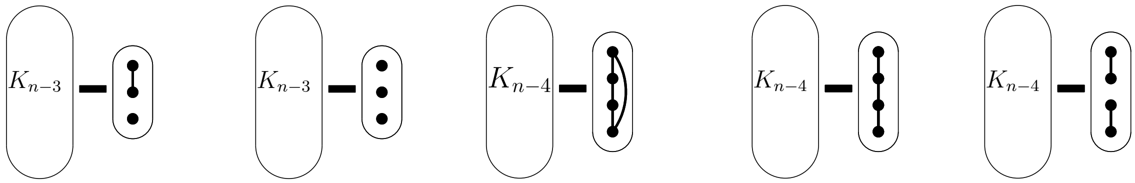
Figure 4: Solid (resp. dotted) lines mean adjacent (resp. non-adjacent) vertices in $\overline{G}$.

Case 2: $\overline{G}$ has at least two non-trivial components and one of them has order at least 3. If there is only one component of order ≥ 3, say C_1, then there is at least a component of order 2, say C_2. Otherwise, there are two components, say C_1 and C_2, of order at least 3. In both cases, we may assume that x_1, x_2, x_3 are vertices of C_1 and y_1, y_2 are vertices of C_2, such that $x_1x_2 \in E(\overline{G})$, $x_2x_3 \in E(\overline{G})$, $y_1y_2 \in E(\overline{G})$. Since $n \geq 7$, we may assume that there are two more vertices z and t such that at least one of them, say z, is not adjacent to y_2 in $\overline{G}$.

Consider the partition:

$$\Pi = \{\{x_1, y_1, z\}, \{x_3, t\}\} \cup \{\{v\} : v \notin \{x_1, x_3, y_1, t, z\}\}.$$

We claim that Π is an RD-partition of G (see Figure 4 (b)). Indeed, recall that two vertices are at distance 2 in G whenever they are adjacent in $\overline{G}$, and they are at distance 1 in G whenever



$H_1 \cong K_{n-3} \vee (K_2 + K_1)$ $H_2 \cong K_{n-3} \vee \overline{K_3}$ $H_3 \cong K_{n-4} \vee C_4$ $H_4 \cong K_{n-4} \vee P_4$ $H_5 \cong K_{n-4} \vee 2K_2$

Figure 5: Graphs of order $n \geq 7$, diameter $\text{diam}(G) = 2$ and minimum degree $\delta(G) \geq n - 3$ such that $\eta(G) = n - 2$.

they are not adjacent in $\overline{G}$. Hence, if $S_1 = \{x_2\}$ and $S_2 = \{y_2\}$, then $r(x_1|\Pi) = (2, 1, \dots)$, $r(y_1|\Pi) = (1, 2, \dots)$, $r(z|\Pi) = (1, 1, \dots)$, and $r(x_3|\Pi) = (2, \dots)$, $r(t|\Pi) = (1, \dots)$. Therefore, Π is an RD-partition of G and $\eta_p(G) \leq n - 3$.

Case 3: All non-trivial components of $\overline{G}$ have order 2. Then, $\overline{G}$ has at least 3 components that are copies of K_2. Let $\{x_i, y_i\}$, for $i = 1, 2, 3$, be the vertices of three of these copies, and let z be a vertex not belonging to them. Then,

$$\Pi = \{\{x_1, x_2, x_3, z\}\} \cup \{\{v\} : v \neq x_1, x_2, x_3, z\}$$

is an RD-partition of G (see Figure 4 (c)). Indeed, if $S_1 = \{y_1\}$, $S_2 = \{y_2\}$ and $S_3 = \{y_3\}$, then $r(x_1|\Pi) = (2, 1, 1, \dots)$, $r(x_2|\Pi) = (1, 2, 1, \dots)$, $r(x_3|\Pi) = (1, 1, 2, \dots)$ and $r(z|\Pi) = (1, 1, 1, \dots)$. Therefore, $\eta_p(G) \leq n - 3$. □

Proposition 19. *Let G be a graph of order $n \geq 7$, diameter 2 and minimum degree at least $n - 3$. If $\eta_p(G) = n - 2$, then $G \in \{H_1, H_2, H_3, H_4, H_5\}$ (see Figure 5).*

Proof. Let $\Omega \subseteq V(G)$ be the set of vertices of G of degree $n - 1$, which according to Lemma 18 contains at least $n - 4$ vertices. We distinguish cases depending on the cardinality of Ω.
Case 1: $|\Omega| \geq n - 2$. If $|\Omega| = n$, then $G \cong K_n$ and thus $\eta_p(G) = n$. Case $|\Omega| = n - 1$ is not possible. If $|\Omega| = n - 2$, then $G \cong K_{n-2} \vee \overline{K_2}$, and according to Theorem 15(2), $\eta_p(G) = n - 1$.

Case 2: $|\Omega| = n - 3$. Let F be the subgraph of order 3 induced by $V(G) \setminus \Omega$, i.e., $F = G[V(G) \setminus \Omega]$. Notice that $|E(F)| \leq 1$. If $|E(F)| = 1$, then $G \cong H_1$. Otherwise, if $|E(F)| = 0$, then $G \cong H_2$.

Case 3: $|\Omega| = n - 4$. Consider the graph of order 4, $F = G[V(G) \setminus \Omega]$. Note that all vertices of F have degree either 1 or 2. There are thus three possibilities. If $F \cong C_4$, then $G \cong H_3$. If $F \cong P_4$, then $G \cong H_4$. If $F \cong 2K_2$, then $G \cong H_5$. □

Proposition 20. *Let G be a graph of order $n \geq 7$, diameter 2 and minimum degree at most $n - 4$. If $\eta_p(G) = n - 2$, then $G \in \{H_6, H_7, H_8, H_9, H_{10}, H_{11}, H_{12}, H_{13}, H_{14}\}$ (see Figure 6).*

Proof. By Lemma 13 for $k = 3$, we have that $\deg(w) \in \{1, 2, n - 3, n - 2, n - 1\}$, for every vertex $w \in V(G)$. Hence, $\delta(G) \leq 2$. We distinguish two cases.
Case 1: *There exists a vertex u of degree* 2. Consider the subsets $D_1 = N(u) = \{x_1, x_2\}$ and $D_2 = \{v \in V(G) : d(u, v) = 2\} = V(G) \setminus N[u]$, so that $|D_2| = n - 3$.
Case 1.1: *$G[D_2]$ is neither complete nor empty.* Then, there exist three different vertices $r, s, t \in D_2$ such that $rs \in E(G)$ and $rt \notin E(G)$. Let $y \in D_2 \setminus \{r, s, t\}$. We distinguish cases taking into account whether or not y and t are leaves.

- *Both y and t are leaves hanging from the same vertex.* Assume that they hang from x_1. Let $S_1 = \{u, y\}$ and $S_2 = \{x_2, s, t\}$. In such a case, S_2 resolves S_1, $\{r\}$ resolves the pair $\{s, t\}$ and S_1 resolves the pairs $\{x_2, s\}$ and $\{x_2, t\}$. Therefore, $\Pi = \{S_1, S_2\} \cup \{\{w\} : w \notin S_1 \cup S_2\}$ is a resolving partition. It can be easily checked that Π is also dominating. Hence, $\eta_p(G) \leq n - 3$, a contradiction.

- *Both y and t are leaves but not hanging from the same vertex, or neither y nor t are leaves.* If both y and t are leaves but not hanging from the same vertex, assume $x_1y \in E$ and $x_2t \in E$. Let $S_1 = \{x_2, y\}$ and $S_2 = \{x_1, s, t\}$. If neither y nor t are leaves and $N(t) \neq \{s, x_1\}$, let $S_1 = \{x_2, y\}$ and $S_2 = \{x_1, s, t\}$. If neither y nor t are leaves and $N(t) = \{s, x_1\}$, let $S_1 = \{x_1, y\}$ and $S_2 = \{x_2, s, t\}$. In all these cases, $\{u\}$ resolves S_1, $\{r\}$ resolves $\{s, t\}$, and $\{u\}$ resolves any other pair from S_2. Hence, $\Pi = \{S_1, S_2\} \cup \{\{w\} : w \notin S_1 \cup S_2\}$ is a resolving partition of G. It can be easily checked that Π is a dominating partition. Thus, $\eta_p(G) \leq n-3$, a contradiction.

- *Exactly one of the vertices y or t is a leaf.* We may assume that the leaf hangs from x_1. If t is a leaf, then take $S_1 = \{x_1, y\}$ and $S_2 = \{x_2, s, t\}$. If y is a leaf and $N(t) \neq \{x_1, s\}$ then take $S_1 = \{x_2, y\}$ and $S_2 = \{x_1, s, t\}$. In both cases, $\{r\}$ resolves $\{s, t\}$ and $\{u\}$ resolves any other pair in either S_1 or S_2. If y is a leaf and $N(t) = \{x_1, s\}$ then take $S_1 = \{u, y\}$ and $S_2 = \{x_2, s, t\}$. Then, $\{r\}$ resolves the pair $\{s, t\}$, S_1 resolves the other pairs from S_2; and S_2 resolves S_1. In all cases, $\Pi = \{S_1, S_2\} \cup \{\{w\} : w \notin S_1 \cup S_2\}$ is dominating partition. Thus, $\eta_p(G) \leq n - 3$, a contradiction.

Case 1.2: *$G[D_2]$ is either complete or empty.* Assume that $\deg(x_1) \leq \deg(x_2)$. Consider the subsets $N_1 = N(x_1) \cap D_2$ and $N_2 = N(x_2) \cap D_2$. Observe that $N_1 \cup N_2 = D_2$, and the sets $N_1 \setminus N_2$, $N_1 \cap N_2$ and $N_2 \setminus N_1$ are pairwise disjoint. Besides, $|N_2 \setminus N_1| \geq |N_1 \setminus N_2|$ because we have assumed $\deg(x_2) \geq \deg(x_1)$. Notice also that $\deg(x_2) \geq \deg(x_1) \geq 2$, as otherwise $\operatorname{diam}(G) \geq 3$. We distinguish two cases.

(1.2.1): $\deg(x_1) = 2$. Thus, $\deg(x_2) \geq (|D_2| - 1) + 1 \geq n - 3$.

- If $x_1x_2 \in E$, then $N_1 = \emptyset$ and $D_2 = N_2$. If $G[D_2] \cong K_{n-3}$, then $G \cong H_{10}$. If $G[D_2] \cong \overline{K_{n-3}}$, then $G \cong H_{12}$.

- If $x_1x_2 \notin E$, then $|N_1| = 1$ and $|N_2 \setminus N_1| = n - 4 \geq 3$. If $G[D_2] \cong \overline{K_{n-3}}$, then $\operatorname{diam}(G) \geq 3$. Hence, $G[D_2] \cong K_{n-3}$. Consider $y \in N_1$ and $z_1, z_2 \in N_2 \setminus N_1$. Let $S_1 = \{u, x_1\}$, $S_2 = \{x_2, z_2\}$ and $S_3 = \{y, z_1\}$ and consider the partition $\Pi = \{S_1, S_2, S_3\} \cup \{\{w\} : w \notin S_1 \cup S_2 \cup S_3\}$. Then, S_1 resolves both S_2 and S_3; and S_3 resolves S_1. Moreover, Π is a dominating partition of G. Thus, $\eta_p(G) \leq n - 3$, a contradiction.

(1.2.2): $\deg(x_1) \geq n - 3$. Hence, $\deg(x_2) \geq \deg(x_1) \geq n - 3$. In such a case, $|N_1| \geq n - 5$ and $|N_2| \geq n - 5$, and so $n - 7 \leq |N_1 \cap N_2| \leq n - 3$. We distinguish cases depending on the cardinality of $|N_1 \cap N_2|$.

- $|N_1 \cap N_2| = n-3$. Then, $N_1 = N_2 = V(G) \setminus N[u]$. If $x_1x_2 \in E$, then $G \cong H_8$ if $G[D_2] \cong K_{n-3}$, and $G \cong H_7$ if $G[D_2] \cong \overline{K_{n-3}}$. If $x_1x_2 \notin E$, then $G \cong H_9$ if $G[D_2] \cong K_{n-3}$, and $G \cong H_6$ if $G[D_2] \cong \overline{K_{n-3}}$.

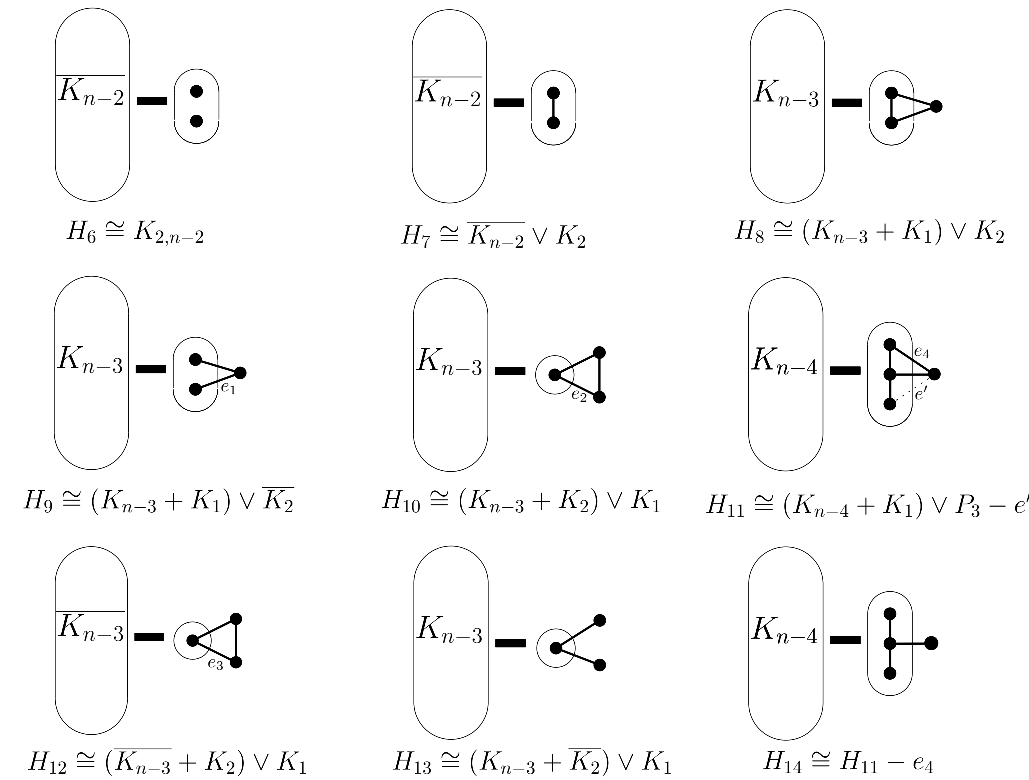


Figure 6: Graphs of order $n \geq 7$, diameter $\mathrm{diam}(G) = 2$ and minimum degree $1 \leq \delta(G) \leq 2$ such that $\eta(G) = n - 2$.

- $|N_1 \cap N_2| = n - 4$. Then, $|N_2 \setminus N_1| + |N_1 \setminus N_2| = 1$. Thus, $|N_2 \setminus N_1| = 1$, $|N_1 \setminus N_2| = 0$ and $|N_1 \cap N_2| \geq 3$. If $G[D_2] \cong \overline{K_{n-3}}$, then $\mathrm{diam}(G) \geq 3$, a contradiction. If $G[D_2] \cong K_{n-3}$ and $x_1x_2 \in E$, then $G \cong H_{11}$. If $G[D_2] \cong K_{n-3}$ and $x_1x_2 \notin E$, then let $y_1, y_2, y_3 \in N_1 \cap N_2$ and let $z \in N_2 \setminus N_1$. Consider $S_1 = \{u, y_1\}$, $S_2 = \{x_2, y_2\}$, $S_3 = \{z, y_3\}$ and let $\Pi = \{S_1, S_2, S_3\} \cup \{\{w\} : w \notin S_1 \cup S_2 \cup S_3\}$. Then, $\{x_1\}$ resolves both S_2 and S_3, and S_3 resolves S_1. It is easy to check that it is a dominating partition. Therefore, $\eta_p(G) \leq n - 3$, a contradiction.

- $|N_1 \cap N_2| = n - 5$. Then, $|N_2 \setminus N_1| + |N_1 \setminus N_2| = 2$ and $|N_1 \cap N_2| \geq 2$. Let $y_1, y_2 \in (N_2 \setminus N_1) \cup (N_1 \setminus N_2)$ and $z_1, z_2 \in N_1 \cap N_2$, and let $S_1 = \{y_1, z_1\}$, $S_2 = \{y_2, z_2\}$ and $S_3 = \{u, x_1\}$. Then, $\Pi = \{S_1, S_2, S_3\} \cup \{\{w\} : w \notin S_1 \cup S_2 \cup S_3\}$ is an RD-partition of G. Indeed, S_1 resolves S_3 and, for $i \in \{1, 2\}$, S_i is resolved by S_1 if $y_i \in N_2 \setminus N_1$ and S_i is resolved by $\{x_2\}$ if $y_i \in N_1 \setminus N_2$. Besides, Π is dominating. Hence, $\eta_p(G) \leq n - 3$, a contradiction.

- $|N_1 \cap N_2| \in \{n - 6, n - 7\}$. In such a case, $|N_2 \setminus N_1| + |N_1 \setminus N_2| \in \{3, 4\}$. Since $|N_2 \setminus N_1| \geq |N_1 \setminus N_2|$, we have $|N_2 \setminus N_1| \geq 2$. Since $\deg(x_1) \geq n - 3$, we have $|N_1| \geq n - 5 \geq 2$. Let $y_1, y_2 \in N_1$ and $z_1, z_2 \in N_2 \setminus N_1$. If $S_1 = \{u, x_1\}$, $S_2 = \{y_1, z_1\}$ and $S_3 = \{y_2, z_2\}$, and $\Pi = \{S_1, S_2, S_3\} \cup \{\{w\} : w \notin S_1 \cup S_2 \cup S_3\}$, then S_1 resolves both S_2 and S_3, and S_2 resolves S_1. Moreover, Π is a dominating partition. Therefore, $\eta_p(G) \leq n - 3$, a contradiction.

Case 2: *There exists at least one vertex u of degree 1 and there is no vertex of degree 2.* Since diam$(G) = 2$, the neighbor v of u satisfies $\deg(v) = n-1$. Let Ω be the set of vertices different from v that are not leaves. Notice that there are at most two vertices of degree 1 in G, as otherwise all vertices in Ω would have degree between 3 and $n-4$, contradicting the assumption made at the beginning of the proof.

If there are exactly two vertices of degree 1, then $|\Omega| = n-3$. In such a case, as for every vertex $w \in \Omega$, $\deg(w) \geq n-3$, Ω induces a complete graph in G, and hence $G \cong H_{13}$.

Suppose next that u is the only vertex of degree 1, which means that Ω contains $n-2$ vertices, all of them of degree $n-3$ or $n-2$. Consider the (non-necessarily connected) graph $J = \overline{G[\Omega]}$. Certainly, J has $n-2$ vertices, all of them of degree either 0 or 1. Let L denote the set of vertices of degree 1 in J. Observe that the cardinality of L must be even. We distinguish three cases.

- If $|L| = 0$, then $G \cong K_1 \vee (K_1 + K_{n-2})$, and by Theorem 15 we have $\eta_p(G) = n-1$, a contradiction.
- If $|L| = 2$, then $G \cong H_{14}$.
- If $|L| \geq 4$, let $\{x_1, x_2, x_3, x_4\} \subseteq L$ such that x_1x_2 and x_3x_4 are edges of J, and let $y \in \Omega \setminus \{x_1, x_2, x_3, x_4\}$. Consider the partition $\Pi = \{S_1, S_2\} \cup \{\{w\} : w \notin S_1 \cup S_2\}$, where $S_1 = \{v, x_1\}$, $S_2 = \{u, x_3, y\}$. Observe that $\{x_2\}$ resolves S_1, $\{u, x_3\}$ and $\{u, y\}$, and $\{x_4\}$ resolves $\{x_3, y\}$. Besides, Π a is dominating partition. Therefore, $\eta_p(G) \leq n-3$, a contradiction. □

3.2 Case diameter 3

We consider the case $\eta_p(G) = n-2$ and diam$(G) = 3$.

Proposition 21. *Let G be a graph of order $n \geq 7$ and diameter 3. If $\eta_p(G) = n-2$, then $G \in \{H_{15}, H_{16}, H_{17}\}$ (see Figure 7).*

Proof. By Lemma 13 (case $k = 3$), every vertex has degree 1, 2, $n-3$, $n-2$ or $n-1$. Let u and v be two vertices such that $d(u, v) = 3$. In such a case, both u and v have degree at most $n-3$.

Notice that on the one hand, it is not possible to have neither $\{\deg(u), \deg(v)\} = \{2, n-3\}$ nor $\{\deg(u), \deg(v)\} = \{n-3\}$, as otherwise we would have more than n vertices because $N(u) \cap N(v) = \emptyset$, a contradiction.

On the other hand, if $\deg(u) = \deg(v) = 2$, then $\eta_p(G) \leq n-3$. Indeed, let ux_1x_2v be a (u, v)-path and let $D_i = \{z : d(u, z) = i\}$, for $i \in \{1, 2, 3\}$. Since $|D_1| = 2$, we may assume that $D_1 = \{x_1, y_1\}$. If $|D_2| \geq 2$, let $y_2 \in D_2 \setminus \{x_2\}$. If $x_1y_2 \in E$, let $S_1 = \{x_1, x_2\}$ and $S_2 = \{y_1, y_2, v\}$. If $x_1y_2 \notin E$, then $y_1y_2 \in E$, and consider $S_1 = \{y_1, x_2\}$ and $S_2 = \{x_1, y_2, v\}$. If $|D_2| = 1$, then v has a neighbor $z \in D_3$, so that z must be also adjacent to x_2. Let $S_1 = \{x_1, x_2, v\}$ and $S_2 = \{y_1, z\}$. In all cases, $\Pi = \{S_1, S_2\} \cup \{\{w\} : w \notin S_1 \cup S_2\}$ is an RD-partition, because it is dominating and $\{u\}$ resolves both S_1 and S_2. Hence, $\eta_p(G) \leq n-3$, a contradiction.

Therefore, we may assume that $\deg(u) = 1$ and that every vertex at distance 3 from u has degree 1, 2 or $n-3$. Let $D_i = \{x \in V(G) : d(u, x) = i\}$, for $i = 1, 2, 3$. Thus, $|D_1| = 1$. Let $D_1 = \{w\}$. We distinguish cases, depending on the cardinality of D_3.

Case 1: $|D_3| \geq 3$. Then, $\deg(w) \leq n-4$, and therefore, $\deg(w) = 2$, $|D_1| = |D_2| = 1$ and $|D_3| = n-3 \geq 4$. Let x be the only vertex in D_2. Notice that every vertex of D_3 is adjacent to x. We distinguish cases taking into account the degree of the vertices in D_3.

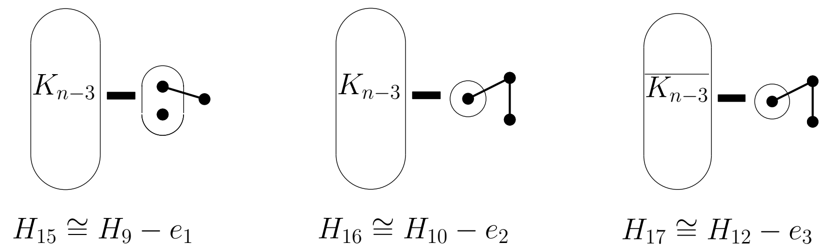


Figure 7: Graphs of order $n \geq 7$ and diameter 3 such that $\eta(G) = n - 2$.

- *There is a vertex of degree $n-3$ in D_3.* A vertex in D_3 of degree $n-3$ must be adjacent to all the other vertices of D_3. Therefore, there is exactly one vertex of degree $n-3$ in D_3 or every vertex in D_3 has degree $n-3$. In the last case, that is, if every vertex in D_3 has degree $n-3$, then D_3 is a clique and $G \cong H_{16}$. Otherwise, let y_1 be the only vertex in D_3 of degree $n-3$. Any other vertex in D_3 has degree 2, since it is adjacent to x and to y_1. Let $y_2, y_3, y_4 \in D_3 \setminus \{y_1\}$. Consider $S_1 = \{y_1, y_2\}$ and $S_2 = \{w, x, y_3\}$. Then, $\Pi = \{S_1, S_2\} \cup \{\{z\} : z \notin S_1 \cup S_2\}$ is an RD-partition of G. Indeed, it is dominating partition, $\{u\}$ resolves S_2 and $\{y_4\}$ resolves S_1 (see Figure 8(a)). Thus, $\eta_p(G) \leq n-3$, a contradiction.

- *Every vertex in D_3 has degree 1 or 2, and at least one of them has degree 2.* Then, $G[D_3]$ contains at least a copy of K_2. Let y_1 and y_2 be the vertices of such a copy of K_2, and take $y_3 \in D_3 \setminus \{y_1, y_2\}$. Consider $S_1 = \{w, y_1\}$, $S_2 = \{x, y_2\}$ and $S_3 = \{u, y_3\}$. It is straightforward to check that $\Pi = \{S_1, S_2, S_3\} \cup \{\{z\} : z \notin S_1 \cup S_2 \cup S_3\}$ is an RD-partition of G (see Figure 8(b)), and thus $\eta_p(G) \leq n-3$, a contradiction.

- *Every vertex in D_3 has degree 1.* Then, D_3 induces an empty graph and $G \cong H_{17}$.

Case 2: $|D_3| = 2$. Then, $|D_2| = n-4$. Let $D_3 = \{y_1, y_2\}$. Recall that both y_1 and y_2 have at least a neighbor in D_2. We distinguish cases taking into account the degree of the vertices in D_3.

- *There is a vertex of degree $n-3$ in D_3.* We may assume that this vertex is y_1, and it must be adjacent to y_2 and to all vertices in D_2. So, there is a vertex $x_1 \in D_2$ adjacent to both y_1 and y_2. Let $x_2 \in D_2 \setminus \{x_1\}$ and consider $S_1 = \{w, x_1, y_1\}$ and $S_2 = \{x_2, y_2\}$ Then, $\Pi = \{S_1, S_2\} \cup \{\{z\} : z \notin S_1 \cup S_2\}$ is a dominating partition, and $\{u\}$ resolves both S_1 and S_2 (see Figure 8(c)). Hence, $\eta_p(G) \leq n-3$, a contradiction.

- *Both vertices in D_3 have degree 1 or 2.* Let $x_1 \in D_2$ be a neighbor of y_1.

  If there exists a vertex $x_2 \in D_2 \setminus \{x_1\}$ not adjacent to y_2, let $x_3 \in D_2 \setminus \{x_1, x_2\}$. Consider $S_1 = \{w, x_1\}$, $S_2 = \{x_2, y_2\}$ and $S_3 = \{x_3, y_1\}$. Then, $\Pi = \{S_1, S_2, S_3\} \cup \{\{z\} : z \notin S_1 \cup S_2 \cup S_3\}$ is a dominating partition and $\{u\}$ resolves S_1, S_2 and S_3 (see Figure 8(d)). Therefore, $\eta_p(G) \leq n-3$, a contradiction.

  If all vertices in $D_2 \setminus \{x_1\}$ are adjacent to y_2, then $\deg(y_2) \geq n-5$, with means that $2 = \deg(y_2) = n-5$ and thus $n = 7$. Let $D_2 = \{x_1, x_2, x_3\}$ and consider $S_1 = \{w, x_1\}$, $S_2 = \{x_2, y_1\}$ and $S_3 = \{x_3, y_2\}$. Then, $\Pi = \{S_1, S_2, S_3\} \cup \{\{z\} : z \notin S_1 \cup S_2 \cup S_3\}$ is a dominating partition and $\{u\}$ resolves S_1, S_2 and S_3 (see Figure 8(e)). Therefore, $\eta_p(G) \leq n-3$, a contradiction.

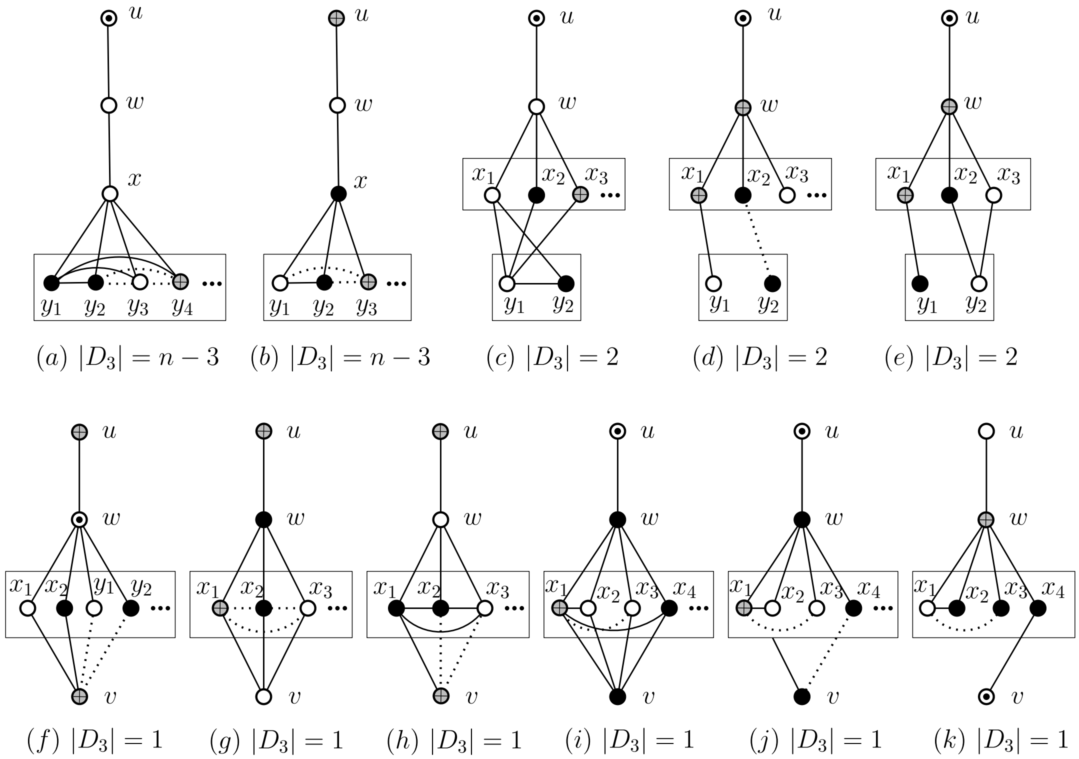


Figure 8: Solid (resp. dotted) lines mean adjacent (resp. non-adjacent) vertices. Vertices with the same "color" belong to the same part.

Case 3: $|D_3| = 1$. Then, $D_3 = \{v\}$ and $|D_2| = n-3$. We distinguish cases taking into account the degree of v and the subgraph induced by D_2.

- $\deg(v) = 2$. Let x_1 and x_2 be the two neighbors of v, and take $y_1, y_2 \in D_2 \setminus \{x_1, x_2\}$. Let $S_1 = \{u, v\}$, $S_2 = \{x_1, y_1\}$ and $S_3 = \{x_2, y_2\}$. Then, $\Pi = \{S_1, S_2, S_3\} \cup \{\{z\} : z \notin S_1 \cup S_2 \cup S_3\}$ is dominating partition such that $\{w\}$ resolves S_1, and S_1 resolves both S_2 and S_3 (see Figure 8(f)), implying that $\eta_p(G) \leq n-3$, a contradiction.

- $\deg(v) \in \{1, n-3\}$ *and* D_2 *induces an empty graph.*

  If $\deg(v) = n-3$, let $x_1, x_2, x_3 \in D_2$ and let $S_1 = \{u, x_1\}$, $S_2 = \{w, x_2\}$ and $S_3 = \{v, x_3\}$. Then, $\Pi = \{S_1, S_2, S_3\} \cup \{\{z\} : z \notin S_1 \cup S_2 \cup S_3\}$ is a dominating partition such that S_1 resolves both S_2 and S_3, and S_3 resolves S_1 (see Figure 8(g)), implying that $\eta_p(G) \leq n-3$, a contradiction.

  If $\deg(v) = 1$, then $G \cong H_{17}$.

- $\deg(v) \in \{1, n-3\}$ *and* D_2 *induces a complete graph.*

  If $\deg(v) = n-3$, then $G \cong H_{15}$.

  If $\deg(v) = 1$, let $x_1 \in D_2$ be the neighbor of v and $x_2, x_3 \in D_2 \setminus \{x_1\}$. Consider $S_1 = \{u, v\}$, $S_2 = \{w, x_3\}$ and $S_3 = \{x_1, x_2\}$. Then, $\Pi = \{S_1, S_2, S_3\} \cup \{\{z\} : z \notin S_1 \cup S_2 \cup S_3\}$ is a dominating partition such that S_1 resolves both S_2 and S_3, and S_3 resolves S_1 (see Figure 8(h)), implying that $\eta_p(G) \leq n-3$, a contradiction.

- $\deg(v) \in \{1, n-3\}$ *and* D_2 *induces neither a complete, nor an empty graph.*

  In that case, there exist vertices $x_1, x_2, x_3 \in D_2$ such that $x_1x_2 \in E(G)$ and $x_1x_3 \notin E(G)$.

  If $\deg(v) = n-3$, then $\deg(x_1) \geq 3$, and thus, $\deg(x_1) \geq n-3$. Hence, x_1 must be adjacent to any other vertex in D_2 different from x_3. Let $x_4 \in D_2 \setminus \{x_1, x_2, x_3\}$ and consider $S_1 = \{w, x_4, v\}$ and $S_2 = \{x_2, x_3\}$ Then, $\Pi = \{S_1, S_2\} \cup \{\{z\} : z \notin S_1 \cup S_2\}$ is a dominating partition such that $\{u\}$ resolves S_1 and $\{x_1\}$ resolves S_2 (see Figure 8(i)), implying that $\eta_p(G) \leq n-3$, a contradiction.

  Finally, suppose that $\deg(v) = 1$. If there is a leaf x in D_2, then $d(u, v) = d(x, v) = 3$. In such a case, interchanging the role of the vertices u and v, the preceding cases for $|D_3| \geq 2$ apply and we are done. So, we can assume that any vertex in D_2 has degree at least 2. Suppose that v is not adjacent to some vertex $x_4 \in D_2 \setminus \{x_1, x_2, x_3\}$. Notice that such a vertex exists whenever $n \geq 8$, because D_2 has at least 5 vertices. Let $S_1 = \{w, x_4, v\}$ and $S_2 = \{x_2, x_3\}$. Then, $\Pi = \{S_1, S_2\} \cup \{\{z\} : z \notin S_1 \cup S_2\}$ is a dominating partition such that $\{u\}$ resolves S_1 and $\{x_1\}$ resolves S_2. Therefore, Π is an RD-partition of G (see Figure 8(j)), and so $\eta_p(G) \leq n-3$, a contradiction.

  Finally, if $n = 7$ and the only vertex $x_4 \in D_2 \setminus \{x_1, x_2, x_3\}$ is adjacent to v, take $S_1 = \{x_2, x_3, x_4\}$ and $S_2 = \{u, x_1\}$. Then, $\Pi = \{S_1, S_2\} \cup \{\{z\} : z \notin S_1 \cup S_2\}$ is a dominating partition such that $\{v\}$ resolves both $\{x_2, x_4\}$ and $\{x_3, x_4\}$; S_2 resolves $\{x_2, x_3\}$; and S_1 resolves S_2. Therefore, Π is an RD-partition of G (see Figure 8(k)), and so $\eta_p(G) \leq n-3$, a contradiction. □

As a straightforward consequence of Propositions 16, 19, 20 and 21, the following result is obtained.

Theorem 22. *If G is a graph of order $n \geq 7$, then $\eta_p(G) = n-2$ if and only if $G \in \Lambda_n$ (see Figure 3).*

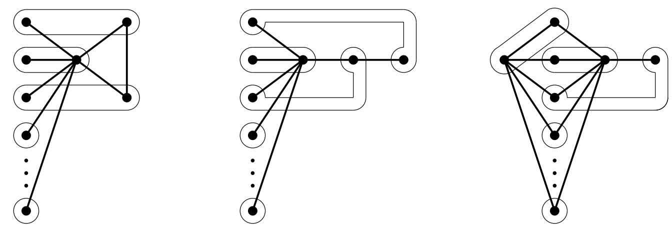
The solution for $\beta_p(G) = n-2$ is also almost immediately derived.

Theorem 23. *If G is a graph of order $n \geq 7$, then $\beta_p(G) = n-2$ if and only if $G \in \Lambda_n \setminus \{H_{12}, H_{17}\}$.*

Proof. If $G \in \Lambda_n \setminus \{H_{12}, H_{17}\}$ then, according to Proposition 16, $\beta_p(G) = n-2$.
Conversely, let G be a graph of order $n \geq 7$ such that $\beta_p(G) = n-2$. Thus, $\eta_p(G) = n-2$, since by Theorem 1 and Theorem 15 we know that $\beta_p(G) \geq n-1$ if and only if $\eta_p(G) \geq n-1$. Hence, by Theorem 22, we derive that $G \in \Lambda_n$. Finally, $\beta_p(G) = n-3$ if $G \in \{H_{12}, H_{17}\}$. Indeed, in such a case, $\beta_p(G) \geq n-3$, because G contains a twin set of cardinality $n-3$, and a resolving partition of cardinality $n-3$ for H_{12} and H_{17} is shown in Figure 9. □

Remark 24. *Theorem 23 corrects an inaccurate result shown in [28] (Theorem 3.2).*

A graph G is called *doubly-connected* if both G and its complement $\overline{G}$ are connected. We finally show a couple of Nordhaus-Gaddum-type results, which are a straightforward consequence of Theorems 22 and 23.



H_{12} $\quad$ H_{17} $\quad$ $\overline{H_{16}}$

Figure 9: Resolving partitions of cardinality $n-3$ of H_{12}, H_{17} and $\overline{H_{16}}$.

Theorem 25. *If G is a doubly-connected graph of order $n \geq 3$, then*

(1) $6 \leq \eta_p(G) + \eta_p(\overline{G}) \leq 2n-4$.

(2) *The equality $\eta_p(G) + \eta_p(\overline{G}) = 6$ is attained, among others, by P_4 and C_5.*

(3) *If $n \geq 7$, then $\eta_p(G) + \eta_p(\overline{G}) = 2n-4$ if and only if $G \in \{H_{15}, H_{17}\}$.*

Proof. (1) Note that $\eta_p(G) = 2$ if and only if $G \cong P_2$, but in this case $\overline{G}$ is not connected. Thus, if G is a doubly-connected graph of order n, then $\eta_p(G) \geq 3$ and $\eta_p(\overline{G}) \geq 3$, and the lower bound holds. On other hand, by Theorem 15, if $\eta_p(G) \geq n-1$, then $\overline{G}$ is not connected. Thus, $\eta_p(G) + \eta_p(\overline{G}) \leq 2n-4$.

(2) We know that $\overline{P_4} = P_4$ and $\overline{C_5} = C_5$, and it is easily verified that $\eta_p(P_4) = 3$ and $\eta_p(C_5) = 3$. Hence, P_4 and C_5 satisfy the given equality.

(3) Finally, a doubly-connected graph G of order at least 7 attaining the upper bound must satisfy $\eta_p(G) = \eta_p(\overline{G}) = n-2$. Therefore, the equality $\eta_p(G) + \eta_p(\overline{G}) = 2n-4$ is attained if and only if $\{G, \overline{G}\} \subseteq \{H_1, \ldots, H_{17}\}$ (see Theorem 22). It is easy to check that this is satisfied if and only if $G \in \{H_{15}, H_{17}\}$ (observe that $\overline{H_{15}} = H_{17}$).

□

Theorem 26. *If G is a doubly-connected graph of order $n \geq 3$, then*

(1) $4 \leq \beta_p(G) + \beta_p(\overline{G}) \leq 2n-5$.

(2) $\beta_p(G) + \beta_p(\overline{G}) = 4$ *if and only if* $G = P_4$.

(3) *If $n \geq 7$, then $\beta_p(G) + \beta_p(\overline{G}) = 2n-5$ if and only if $G \in \{H_{15}, H_{16}, H_{17}\}$.*

Proof. (1) Every graph G of order at least 3 satisfies $\beta_p(G) \geq 2$. Hence, the lower bound holds. By Theorem 1, if a graph G satisfies $\beta_p(G) \geq n-1$, then $\overline{G}$ is not connected. Therefore, any doubly-connected graph G satisfies $\beta_p(G) \leq n-2$. By Theorem 23, the graphs G satisfying $\beta_p(G) = n-2$ are those from $\Lambda_n \setminus \{H_{12}, H_{17}\}$. It is easy to check that the only doubly-connected graphs of this set are H_{15} and H_{16}. Their complements are $\overline{H_{15}} = H_{17}$, and $\overline{H_{16}}$ is shown in Figure 9. On the one hand, we have seen in the proof of Theorem 23 that $\beta_p(H_{17}) = n-3$. On the other hand, we have that $\beta_p(H_{16}) = n-3$. Indeed, $\beta(H_{16}) \leq n-3$ because H_{16} has a twin set of cardinality $n-3$, and a resolving partition of cardinality $n-3$ is given in Figure 9. Hence, $\beta_p(G) + \beta_p(\overline{G}) \leq 2n-5$ if G is doubly-connected.

(2) We know that $\beta_p(G) = 2$ if and only if G is the path P_n, and $\overline{P_n}$ is a path if and only if $n = 4$. Hence, the equality $\beta_p(G) + \beta_p(\overline{G}) = 4$ holds if and only if $G \cong P_4$.

(3) This equality is satisfied if and only if G is a doubly-connected graph such that $\{\beta_p(G), \beta_p(\overline{G}\} = \{n-2, n-3\}$, and as we have seen in the proof of item i), it happens if and only if $G \in \{H_{15}, H_{16}, H_{17}\}$.

□

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