Received: 31 May 2019 | Accepted: 4 July 2019

DOI: 10.1002/pamm.201900437

Model order reduction applied to ALE-fluid dynamics

Davide Baroli[1,*] and **Andreas Zilian**[1]

[1] University of Luxembourg, Faculty of Science, Technology and Communication, Institute of Computational Engineering, 6 Avenue de la Fonte, L-4362 Esch-sur-Alzette, Luxembourg

The Arbitrary Lagrangian Eulerian formulation allows the description of fluid dynamics involving moving and deforming fluid domains as they are found in many fluid-structure interaction problems. This paper discusses a reduced order approach for ALE incompressible Navier-Stokes flow formulated on a fixed reference configuration. Based on the proper orthogonal decomposition (POD) for generation of the reduced basis, the supermizer technique is adapted to the formulation involving a non-affine term and used to ensure the fulfilment of the inf-sup condition among the velocity and pressure basis. The approach is studied for an extended version of the classical driven-cavity setup involving a non-affine parameterization of the prescribed mesh deformation and a time-dependent non-homogeneous lid-flow boundary condition.

1 ALE-Fluid Dynamics

Many problems in fluid-structure interaction involve a fluid flow on a deforming domain due to a deformation of the structure or moving fluid boundaries. Energy harvesting devices driven by flow-induced vibrations [1] are an exemplary engineering application. However, proper representation of the multi-physics problem requires high-fidelity unsteady simulations which are computationally demanding. This is particularly the case when design studies require parameterization of the problem on the material parameters, boundary conditions and geometry.

An incompressible flow is considered, described by the Navier-Stokes equations in velocity-pressure form. Let Ω_f be the spatial domain occupied by the current fluid body, the formulation in the ALE framework reads

$$\rho\frac{\partial \mathbf{v}}{\partial t} + ((\mathbf{v}-\mathbf{w})\cdot\nabla)\mathbf{v} - \nabla\cdot\mathbf{T} = \mathbf{f}, \quad \text{on} \quad \Omega_f \times (0,T) \tag{1}$$

$$\nabla\cdot\mathbf{v} = 0 \quad \text{on} \quad \Omega_f \times (0,T) \tag{2}$$

where $\mathbf{v}$ is the velocity and ρ the density. The advective velocity $\mathbf{v}-\mathbf{w}$ takes into account the relative motion of fluid and discretization, where $\mathbf{w}$ denotes the mesh velocity, which is assumed to be known in order to simplify the present study. The stress tensor $\mathbf{T}$ of a Newtonian fluid is defined as $\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D}(\mathbf{v})$ with $\mathbf{D}(u) = \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^T)$, where p denotes the pressure and μ the dynamic viscosity. The functional spaces for the state variables velocity and pressure are defined on the (fixed) reference domain $\Omega_r := \chi^{-1}(\Omega_f)$ by $\mathcal{V}_g = \{\mathbf{v}|\mathbf{v}\in H^1(\Omega_r), \mathbf{v} = \mathbf{g}_v \,\text{on}\, \Gamma_u^r\}$; $\mathcal{W} = \{\mathbf{w}|\mathbf{w}\in H^1(\Omega_r), \mathbf{w} = \mathbf{g}_w \,\text{on}\, \Gamma_w\}$, $\mathcal{Q}_g = \{p|p\in L^2(\Omega_r), p = \delta(\mathbf{x}-\mathbf{x}_c) \,\text{on}\, \Gamma_p^r\}$. Introducing the weak parameterized formulation, the incompressible Navier-Stokes in ALE formulation in the reference configuration reads: find $(\mathbf{v},p)\in\mathcal{V}_g\times\mathcal{Q}_g$ such that

$$\int_{\Omega_r} \delta\mathbf{v}\rho J\cdot\frac{\partial\mathbf{v}}{\partial t}\cdot d\mathbf{x} + \int_{\Omega_r} \nabla\delta\mathbf{v} : J(\mu\cdot(\mathbf{F}^{-T}\cdot(\nabla\mathbf{v})^T + \nabla\mathbf{v}\cdot\mathbf{F}^{-1}))\cdot\mathbf{F}^{-T}\cdot d\mathbf{x} \tag{3}$$

$$+\int_{\Omega_r} \delta\mathbf{v}\cdot\rho\, J\cdot\nabla\mathbf{v}\cdot\mathbf{F}^{-1}\cdot(\mathbf{v}-\mathbf{w})\cdot d\mathbf{x} + \int_{\Omega_r}\delta p\cdot\nabla\cdot(J\mathbf{F}^{-1}\mathbf{v})\cdot d\mathbf{x} - \int_{\Omega_r}\delta\mathbf{v}\cdot J\cdot\mathbf{f}\cdot d\mathbf{x} \tag{4}$$

$$+\int_{\Omega_r}\nabla\delta\mathbf{v} : J(-p\mathbf{I})\cdot\mathbf{F}^{-T}d\mathbf{x} = 0 \tag{5}$$

where ρ and μ are constant and the presence of J in terms of the prescribed mesh deformation leads to non-affinity of the parameterization. The mesh displacement $\mathbf{d}(\mathbf{x},t)$ provides mesh velocity $\mathbf{w} = \frac{\partial\mathbf{d}}{\partial t}$ and $\mathbf{F} = (\mathbf{I}+\nabla\mathbf{d})$ together with $J = \det\mathbf{F}$.

2 Reduced order method: POD-EIM

The fundamental idea of projection-based reduced order method relies on the mapping of a high dimensional system to a low-dimensional subspace. Among different techniques described in [2], in this study the proper-orthogonal decomposition method (POD) is used to define the basis that span the sub-space manifold. The computational effort of the reduced order method is separated in offline stage, where snapshots of velocity and pressure are collected and the POD basis is generated; and the online stage where the main operations are the projection of pre-computed discrete full-order system and the reconstruction of the solution in the high-dimensional manifold.

* Corresponding author: Andreas Zilian andreas.zilian@uni.lu

In this study, the snapshot matrix has dimension $(M \times S) \times \mathcal{N}$, with M the number of parameter realisations (e.g. Reynolds number), S the number of time-steps and $\mathcal{N}$ the number of degree of freedom of full-order discretization. In addition, to satisfy the inf-sup condition for the reduced velocity-pressure basis, the velocity POD basis is enriched by the supermizer approach, introduced in [3], before the online projection. The enriched velocity space is defined as $V = \langle \{\mathbf{v}_i\}_{i=1}^{N_\mathbf{v}} \rangle \oplus POD(\{S^\mu p(\mu_i); N_p\}_{i=1}^{N_p})$ where the *inner pressure supermizer operator* is defined as

$$(S^\mu q, \mathbf{v})_{H_0^1} = J(\mathbf{d}(\mathbf{x},t)) \cdot (q, \nabla \cdot (J(\mathbf{d}(\mathbf{x},t))\mathbf{F}^{-T}(\mathbf{d}(\mathbf{x},t))\mathbf{v})) \quad \forall \mathbf{v} \in POD(\{v_i\}_{i=1}^{N_\mathbf{v}}; N_\mathbf{v}). \tag{6}$$

3 Numerical Test

The ALE fluid discussed here is investigated for the driven-cavity test-case to show the challenges associated with the POD approach. The time-dependent displacement of the mesh $\mathbf{d}(\mathbf{x},t) = [0.1 \cdot \sin(4 \cdot \pi \cdot \mathbf{x}_0) \cdot \sin(\pi \cdot t), 0]$ is prescribed at each node. The pressure level is prescribed at the upper left corner. An additional source of non-affine and non-linear term arises with the lid velocity $v(t) = \tanh(t)$ boundary condition. The development of the cavity vortex is studied in the time interval $[0, 2]$ seconds for different Reynolds numbers (10-100) as problem parameterization. The implementation uses the open-source finite element library FEniCS [4]; the reduced order method is implemented with the RBniCS package [5]. The number of degree of freedom of the full-order model is 20402 for velocity and 2601 for pressure.

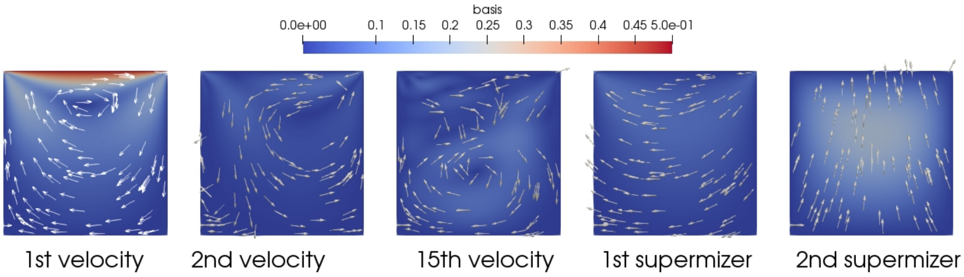

Fig. 1: Magnitude and vector field of the velocity basis (3 left) and enriching supermizer POD basis (2 right) for the driven cavity problem.

A selection of basis vectors for the velocity state are given in Figure 1 together with the basis enrichment obtained with the supermizer approach. The POD-generated velocity basis captures the complexity of the developing cavity vortex induced by the lid velocity, while the supermizer basis contributes the complementary components required for stability of the discretized velocity-pressure fluid formulation. Figure 2 shows the relative error with respect to the number of modes used in the reduced model. The non-monotone pressure error is assumed to be caused by the mesh motion and requires further investigation.

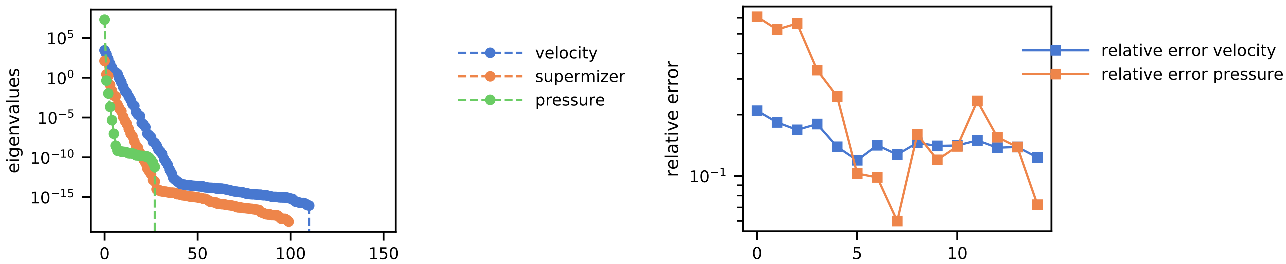

Fig. 2: POD eigenvalues and relative error in norm H_0^1 and L^2 for velocity and pressure with respect to the FEM solution.

Acknowledgements We acknowledge the financial support provided through the FSI-HARVEST (EU 322151) and MACAFIVE project and the support of F. Ballarin, developer of RBniCS.

References

[1] S. Ravi and A. Zilian. Simultaneous finite element analysis of circuit-integrated piezoelectric energy harvesting from fluid-structure interaction. Mechanical Systems & Signal Processing, 114, 2019.
[2] J. Hestaven, G. Rozza, B. Stamm, Certified reduced basis for parametrized partial differential equations, 2016, Springer.
[3] G. Rozza and K. Veroy, On the stability of the reduced basis method for Stokes equations in parametrized domains. CMAME **196**, p. 1244–1260, 2007.
[4] A. Logg, K.-A. Mardal, G. Wells, Automated solution of differential equation by the finite element method: the FEniCS book, **84**, 2012, Springer Science & Business Media.
[5] F. Ballarin, A. Sartori and G. Rozza, RBniCS-reduced order modelling in FEniCS. 2015. http://mathlab.sissa.it/rbnics, 2015.

