ON THE XIAO CONJECTURE FOR PLANE CURVES

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ABSTRACT. Let $f : S \longrightarrow B$ be a non-trivial fibration from a complex projective smooth surface S to a smooth curve B of genus b. Let c_f the Clifford index of the general fibre F of f. In [BGN16] it is proved that the relative irregularity of f, $q_f = h^{1,0}(S) - b$ is less or equal than or equal to $g(F) - c_f$. In particular this proves the (modified) Xiao's conjecture: $q_f \leq \frac{g(F)}{2} + 1$ for fibrations of general Clifford index. In this short note we assume that the general fiber of f is a plane curve of degree $d \geq 5$ and we prove that $q_f \leq g(F) - c_f - 1$. In particular we obtain the conjecture for families of quintic plane curves. This theorem is implied for the following result on infinitesimal deformations: let F a smooth plane curve of degree $d \geq 5$ and let ξ be an infinitesimal deformation of F preserving the planarity of the curve. Then the rank of the cup-product map $H^0(F, \omega_F) \xrightarrow{\cdot \xi} H^1(F, O_F)$ is at least $d - 3$. We also show that this bound is sharp.

1. Introduction

Let $f : S \longrightarrow B$ be a non-isotrivial fibration from a complex projective smooth surface S to a smooth curve B of genus b. A natural question is trying to understand the relation between the invariants of the surface, the base curve B and of the general fibre F. Non-isotrivial means that the smooth fibres are not isomorphic to each other, in other words: the natural modular map $B^0 \longrightarrow \mathcal{M}_g$ in the moduli space of curves of genus $g = g(F)$ is not constant, where B^0 is the open set of B with smooth fibres. The invariants we consider are the relative irregularity $0 \leq q_f := h^{1,0}(S) - g(B)$ and the genus g of the general fibre. Xiao proved in [X87] that for non-isotrivial fibrations the inequality

$$q_f \leq \frac{5g+1}{6}$$

holds, and he formulated in [X88] the conjecture

$$q_f \leq \frac{g+1}{2}.$$

After some counterexamples found by Pirola in [P92] (see also [AP16]) the conjecture is nowadays reformulated as follows

Modified Xiao's conjecture: For non-isotrivial fibrations it holds that

$$q_f \leq \frac{g}{2} + 1$$

Observe that it is equivalent to the initial conjecture for g odd.

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There are some evidences for this conjecture: Xiao proved that it holds if $b = 0$ and it is known to be true when F is hyperelliptic (see [C95]). Moreover in [BGN16] a upper bound of q_f is found in terms of the Clifford index c_f of the general fibre. Remember that the Clifford index of the curve F is defined as:

$$\text{Cliff}(F) := \min\{deg(L) - 2(h^0(F, \mathcal{O}_F(L)) - 1) \mid h^0(F, \mathcal{O}_F(L)) \geq 2, h^1(F, \mathcal{O}_F(L)) \geq 2\}.$$

Clifford's Theorem states that $\text{Cliff}(F) \geq 0$ and it is 0 if and only if F is hyperelliptic. It is known that $\text{Cliff}(F) = 1$ if and only if either F is trigonal or isomorphic to a smooth quintic plane curve. A smooth plane curve of degree d has Clifford index $d - 4$ and a general curve in $\mathcal{M}_g$ has Clifford index $\lfloor \frac{g-1}{2} \rfloor$.

The main theorem in [BGN16] says that for a non-isotrivial fibration it holds

$$q_f \leq g - c_f.$$

In particular the conjecture is true when the general fiber has general Clifford index. Combining the results given above one easily checks that the conjecture is true for $g \leq 4$ and that the first open case corresponds to $g \geq 5$ and $c_f = 1$. In this paper we take care of the quintic plane curve case. More generally we consider families of smooth plane curves of degree $d \geq 5$. Our main theorem is the following:

Theorem 1.1. *Let $f : S \longrightarrow B$ be a fibration such that the general fibre is a plane curve of degree $d \geq 5$. Then*

$$q_f \leq g - c_f - 1 = g - (d - 4) - 1 = g - d + 3$$

In the case $d = 5$, hence $g = 6$, we obtain $q_f \leq 4$ which is the predicted bound. Hence we obtain:

Corollary 1.2. *The modified Xiao's conjecture holds for fibrations with general fibre a quintic plane curve.*

The idea of the proof of (1.1) is as follows: let us fix a general point of B and let F the fibre at this point, then f induces an infinitesimal deformation $\xi \in H^1(F, T_F)$. The kernel W_ξ of the cup-product map -

$$H^0(F, \omega_F) \xrightarrow{\cdot\xi} H^1(F, \mathcal{O}_F)$$

contains the vector space $H^0(S, \Omega^1_S)/f^*H^0(B, \omega_B)$ (see [BGN16, Section 2] for the details). Therefore $q_f \leq \dim W_\xi = g - rank(\cdot\xi)$. Thus it is enough to find a lower bound of the rank of the map given by the cup-product with ξ. Then the next Theorem immediately implies Theorem (1.1):

Theorem 1.3. *Let ξ be an infinitesimal deformation of F as smooth plane curve, then the rank of the map $H^0(F, \omega_F) \xrightarrow{\cdot\xi} H^1(F, \mathcal{O}_F)$ is at least $d - 3$, and this bound is realized for the Fermat curve.*

The rest of the paper is devoted to the proof of Theorem (1.3). In the next section we recall how to use the Jacobian ring of a plane curve in order to understand the cohomology of the curve F and the cup-product maps. We also state two theorems of Green on multiplication and restriction maps of polynomials. In section 3 we use these facts combined with the classical Macaulay's Theorem to prove Theorem (1.3).

2. Preliminaries

Let F be a smooth curve of genus $g \geq 3$ and an let $\xi \in H^1(F, T_F)$ be a non-trivial infinitesimal deformation of F. To simplify notations we call *rank* of ξ to the rank of the cup-product map

$$H^0(F, \omega_F) \xrightarrow{\cdot\xi} H^1(F, \mathcal{O}_F)$$

considered in the Introduction. We keep the notation W_ξ for the kernel of this map.
Let F be a smooth plane curve defined as the zero locus of the homogeneous polynomial $f \in \mathbb{C}[x, y, z] = S$ of degree d. Its *Jacobian ring* R is the graded ring

$$R = \bigoplus_{n \geq 0} R^n = \bigoplus_{n \geq 0} (S^n / J^n).$$

Here J^n is the degree n part of the *Jacobian ideal* $J = (f_x, f_y, f_z)$. As F is smooth, $\{f_x, f_y, f_z\}$ is a regular sequence and this implies that R satisfies the following properties:

Theorem 2.1 (Macaulay)**.** *Let N be $3(d-2)$. Then R^N has dimension 1 and, for every k such that $0 \leq k \leq N$ we have that the multiplication map*

$$R^k \otimes R^{N-k} \to R^N$$

is a perfect pairing. Moreover $R^k = 0$ for $k > N$ or $k < 0$ and the dimension of R^k for $0 \leq k \leq N$ is determined only by d.

In addition to these, Griffiths proved that one can read canonically several pieces of the Hodge structure of F in R. More precisely

Theorem 2.2. *Let R be the Jacobian ring of a smooth plane curve of degree d. Then*

- $H^0(F, \omega_F) \simeq S^{d-3} = R^{d-3}$;
- $H^1(F, \mathcal{O}_F) \simeq R^{N-d+3} = R^{2d-3}$;
- *the subspace of $H^1(F, T_F)$ of all the infinitesimal deformations that preserve the planarity of F is isomorphic to R^d;*
- *multiplication in R induces, using the previous identifications, cup product of the corresponding elements.*

We refer to [V03] for a proof of these facts.

Next we quote two theorems of Green concerning properties of $S = \mathbb{C}[x, y, z]$. We stress that they are valid in any dimension although we state (and use) them only in dimension $n = 2$. The first theorem can be found in [G94, Lecture 7, page 74], putting $p = 0$:

Theorem 2.3 (Green)**.** *Let W be a subspace of S^a of codimension c and assume that $|W|$ is a base point free linear system on $\mathbb{P}^2$. Then, for any $m \geq c$ one has that the multiplication map*

$$W \otimes S^m \to S^{m+a}$$

is surjective.

In order to state the second theorem of Green we need some notation. Given a positive integer a, for any $c \geq 0$ there is a unique expression

$$c = \binom{k_a}{a} + \binom{k_{a-1}}{a-1} + \cdots + \binom{k_1}{1} = \binom{k_a}{a} + \binom{k_{a-1}}{a-1} + \cdots + \binom{k_\delta}{\delta},$$

such that $k_a > k_{a-1} > \cdots > k_\delta \geq \delta > 0$. The numbers $(k_a, k_{a-1}, \ldots, k_\delta)$ are uniquely identified by this definition and are called *Macaulay's Coefficients of c with respect to a.*

If $(k_a, k_{a-1}, \dots, k_\delta)$ are the Macaulay coefficients with respect to a, we denote by $c_{\langle a\rangle}$ the number

$$c_{\langle a\rangle} = \binom{k_a - 1}{a} + \binom{k_{a-1} - 1}{a-1} + \cdots + \binom{k_\delta - 1}{\delta},$$

where $\binom{m}{n} = 0$ if $m < n$. Keeping this terminology we can state the following theorem:

Theorem 2.4 (Green, [G89])**.** *Let $W \subset S^a$ be a linear system with codimension c. Let H be a general line in $\mathbb{P}^2$. Then the codimension c_H of the image of the restriction map*

$$W \longrightarrow H^0(H, \mathcal{O}_H(a))$$

satisfies $c_H \le c_{\langle a\rangle}$.

Corollary 2.5. *Under the assumption of Theorem 2.4, if $c < a$ then $c_H = 0$, i.e. the restriction map $W \to H^0(H, \mathcal{O}_H(a))$ is surjective.*

Proof. If $c \le a$ we can write $c = a - r$ for some r such that $0 \le r \le a$. As

$$c = a - r = \binom{a}{a} + \binom{a-1}{a-1} + \cdots + \binom{r+1}{r+1}$$

we have that $(a, a-1, \dots, r+1)$ are the Macaulay's coefficients of c with respect to a. Therefore

$$c_{\langle a\rangle} = \binom{a-1}{a} + \binom{a-2}{a-1} + \cdots + \binom{r}{r+1} = 0.$$

By definition of c_H we have that the restriction is surjective as claimed. □

3. Proof of the Theorem (1.3)

The Theorem (1.3) in the introduction asserts that given a smooth curve F of degree d, the rank of a non trivial infinitesimal deformation of F which preserves the planarity of F is bounded below by $d-3$. By using the identifications provided by Griffiths' results in (2.2) this translates into the following statement:

Theorem 3.1. *Let F be a smooth plane curve of degree $d \ge 5$ and let R be its Jacobian ring. Let $\xi \in R^d \setminus \{0\}$, then the rank of the map*

$$S^{d-3} = R^{d-3} \xrightarrow{\cdot\xi} R^{2d-3}$$

is at least $d-3$.

Proof. Denote by $W = W_\xi \subset S^{d-3}$ the kernel of the multiplication map

$$\cdot\xi : S^{d-3} \to R^{2d-3}.$$

Note that the codimension of W in S^{d-3} is equal to the rank of ξ. The proof is divided in two cases depending of the existence or not of a fixed loci.

Case 1: $|W|$ is base-point-free.
We are going to see that $\mathrm{rk}(\xi) \ge d-2$. Assume that the opposite is true, i.e. that $\mathrm{rk}(\xi) \le d-3$ holds. Then, by Green's Theorem, for every $m \ge \mathrm{cod}_{S^{d-3}}(W) = \mathrm{rk}(\xi)$ we have that the multiplication map

$$\mu_m : W \otimes S^m \to S^{m+d-3}$$

is surjective. In particular, as $\mathrm{rk}(\xi) \le d-3$ we can take $m = d-3$. Hence we have

$$\mu_{d-3} : W \otimes S^{d-3} \to S^{2d-6}$$

is surjective and the same holds for the map obtained by passing to the quotient, i.e. to the Jacobian ring:

$$\mu_{d-3} : W \otimes R^{d-3} \to R^{2d-6}.$$

But this is impossible since by the definition of W, the image of μ_{d-3} is killed by $\xi \neq 0$, hence the pairing

$$R^{2d-6} \otimes R^d \to R^{3d-6} = R^N$$

is degenerated contradicting Macaulay's Theorem. Hence we have necessarily $\mathrm{rk}(\xi) \geq d-2$ as claimed.

Case 2: $|W|$ is not base-point-free.
First we observe that we can assume that there are no base components. Indeed, assume that there exists a curve C of degree $0 < d' < d-3$ in the fixed part of $|W|$. Then we have that $W \subset C \cdot S^{d-d'-3}$ and therefore $\dim W \leq \dim S^{d-d'-3} \leq \dim S^{d-4}$, hence

$$\operatorname{codim} W \geq \binom{d-1}{2} - \binom{d-2}{2} = d-2,$$

as wanted.
Hence now on we assume that $|W|$ has only isolated base points. We proceed by contradiction, so assume that $\mathrm{rk}(\xi) \leq d-4$ holds.

Let Z be the base locus of $|W|$ and denote by $\mathcal{I}_Z$ the ideal sheaf of Z as subscheme of $\mathbb{P}^2$. Then the evaluation induces a surjection

$$W \otimes \mathcal{O}_{\mathbb{P}^2} \twoheadrightarrow \mathcal{I}_Z(d-3).$$

Denoting by M_W its kernel we have the short exact sequence

$$0 \longrightarrow M_W \longrightarrow W \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{ev} \mathcal{I}_Z(d-3) \longrightarrow 0. \tag{1}$$

Let s be a general element in $S^1 = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. As the base points are isolated, we can assume that the line $L = \{s = 0\}$ is disjoint with Z. By considering the multiplication by s, the short exact sequence (1) induces the following commutative diagram with exact rows and columns

(2)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_W & \longrightarrow & W \otimes \mathcal{O}_{\mathbb{P}^2} & \xrightarrow{ev} & \mathcal{I}_Z(d-3) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \cdot s} & & \downarrow{\scriptstyle \cdot s} & & \downarrow{\scriptstyle \cdot s} & & \\
0 & \longrightarrow & M_W(1) & \longrightarrow & W \otimes \mathcal{O}_{\mathbb{P}^2}(1) & \xrightarrow{ev_1} & \mathcal{I}_Z(d-2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_L & \longrightarrow & W \otimes \mathcal{O}_L(1) & \xrightarrow{ev_L} & \mathcal{O}_L(d-2) & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The sheaf M_L in the third row is by the definiton the kernel of ev_L. Notice that the rightmost sheaf of the last row is $\mathcal{O}_L(d-2)$ because we are assuming that L is disjoint from Z.

Claim: The vanishing $H^1(L, M_L) = 0$ holds.

Proof. (of the claim). Indeed, under our hypothesis, Corollary 2.5 implies that the restriction map

$$W \to H^0(L, \mathcal{O}_L(d-3))$$

is surjective. Tensoring with $H^0(L, \mathcal{O}_L(1))$ we get that also the evaluation map

$$ev_l : W \otimes H^0(L, \mathcal{O}_L(1)) \longrightarrow H^0(L, \mathcal{O}_L(d-3)) \otimes H^0(L, \mathcal{O}_L(1)) \twoheadrightarrow H^0(L, \mathcal{O}_L(d-2))$$

is surjective. Then the cohomology sequence of the third row in the diagram gives the vanishing. □

By taking cohomology in the second row we have a map:

$$W \otimes S^1 \xrightarrow{\eta_1} H^0(\mathbb{P}^2, \mathcal{I}_Z(d-2)) \longrightarrow H^1(\mathbb{P}^2, M_W(1)) \longrightarrow 0$$

and denote with W_1 the image of η_1. Our strategy is to replace W by $W_1 \subset S^{d-2}$ and apply again the same argument in order to finally reach a subspace of $W_2 \subset S^{d-1}$ where it is easier to finish the proof as we will see below. Hence we need to show that the codimension of W_1 in S^{d-2} is lower than or equal to the codimension of W in S^{d-3}. To prove this we consider the cohomology exact sequences of the first two rows of the diagram (2):

$$\begin{array}{ccccccc}
W & \hookrightarrow & H^0(\mathbb{P}^2, \mathcal{I}_Z(d-3)) & \longrightarrow & H^1(\mathbb{P}^2, M_W) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \cdot s} & & \downarrow{\scriptstyle \cdot s} & & \downarrow{\scriptstyle \cdot s} & & \\
W \otimes S^1 & \xrightarrow{\eta_1} & H^0(\mathbb{P}^2, \mathcal{I}_Z(d-2)) & \longrightarrow & H^1(\mathbb{P}^2, M_W(1)) & \longrightarrow & 0
\end{array}$$

The last vertical arrow is surjective due to the claim. Therefore

$$\operatorname{codim}_{H^0(\mathbb{P}^2, \mathcal{I}_Z(d-2))} W_1 = h^1(\mathbb{P}^2, M_W(1)) \leq h^1(\mathbb{P}^2, M_W) = \operatorname{codim}_{H^0(\mathbb{P}^2, \mathcal{I}_Z(d-3))} W. \tag{3}$$

Now we need to compare $\operatorname{codim}_{S^{d-2}} H^0(\mathbb{P}^2, \mathcal{I}_Z(d-2))$ with $\operatorname{cod}_{S^{d-3}} H^0(\mathbb{P}^2, \mathcal{I}_Z(d-3))$. As we will observe in a moment, they coincide as a consequence of the vanishing of $H^1(L, M_L) = 0$.

Claim: We have the equality:

$$\operatorname{codim}_{S^{d-2}} H^0(\mathbb{P}^2, \mathcal{I}_Z(d-2)) = \operatorname{codim}_{S^{d-3}} H^0(\mathbb{P}^2, \mathcal{I}_Z(d-3)).$$

Proof. (of the claim). We start again with the diagram (2). The cohomology exact sequence of the last two columns gives

$$\begin{array}{ccc}
W & \longrightarrow & H^0(\mathbb{P}^2, \mathcal{I}_Z(d-3)) \\
\downarrow{\scriptstyle \cdot s} & & \downarrow{\scriptstyle \cdot s} \\
W \otimes S^1 & \xrightarrow{\eta_1} & H^0(\mathbb{P}^2, \mathcal{I}_Z(d-2)) \\
\downarrow{\scriptstyle res_1} & & \downarrow{\scriptstyle res_2} \\
W \otimes H^0(L, \mathcal{O}_L(1)) & \xrightarrow{\alpha} & H^0(L, \mathcal{O}_L(d-2)) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & H^1(\mathbb{P}^2, \mathcal{I}_Z(d-3)).
\end{array}$$

Since $H^1(L, M_L) = 0$ both α and $\alpha \circ res_1$ are surjective. Therefore res_2 is also surjective. This implies the isomorphism

$$H^1(\mathbb{P}^2, \mathcal{I}_Z(d-3)) \xrightarrow{\cong} H^1(\mathbb{P}^2, \mathcal{I}_Z(d-2)).$$

Now consider the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}_Z(d-3) & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(d-3) & \longrightarrow & \mathcal{O}_Z(d-3) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \cong & & \\ 0 & \longrightarrow & \mathcal{I}_Z(d-2) & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(d-2) & \longrightarrow & \mathcal{O}_Z(d-2) & \longrightarrow & 0. \end{array}$$

Implementing the isomorphism above we obtain the claim. □

Combining the inequality (3) with the claim we have:

$$\operatorname{cod}(W_1) \leq \operatorname{cod}(W) \leq d-4.$$

Observe that $|W_1|$ has, by construction, the same base locus of $|W|$ and, as we have just proven, $\operatorname{cod}(W_1) \leq d-4$. Hence, we can apply the same argument using W_1 instead of W starting with the surjection

$$W_1 \otimes \mathcal{O}_{\mathbb{P}^2} \to \mathcal{I}_Z(d-2).$$

By doing this we obtain $W_2 \subset S^{d-1}$ which satisfy

$$\operatorname{cod}(W_2) \leq \operatorname{cod}(W_1) \leq \operatorname{cod}(W) \leq d-4.$$

To finish the proof we consider the subspace

$$\tilde{W} := W_2 + \langle f_x, f_y, f_z \rangle = W_2 + J^{d-1}.$$

Observe that in $\tilde{W}$ there is a section which doesn't vanish on at least one point (we are assuming that Z is not empty) of the base locus, as the partial derivatives cannot all vanish in a point. Hence the dimension of $\tilde{W}$ has increased at least by 1 and

$$\operatorname{cod}(\tilde{W}) \leq \operatorname{cod}(W_2) - 1 \leq d-5.$$

Moreover $\tilde{W}$ is base point free by construction so we can apply Green's Theorem (2.3) again with $m = d-5$ and we obtain that the multiplication map

$$\tilde{W} \otimes S^{d-5} \to S^{d-5+d-1} = S^{2d-6}$$

is surjective.

Passing to the quotient by the Jacobian ideal we have a surjective map

$$W_2 \otimes R^{d-5} \to R^{2d-6}.$$

This implies by the definition of W_2 (image of $W_1 \otimes S^1$) and W_1 (image of $W \otimes S^1$) that all the elements in R^{2d-6} are orthogonal to ξ which contradicts Macaulay's Theorem. This finishes the prove of the bound.

Finally we see that the bound is sharp. Fix $d \geq 4$ and let F be the Fermat curve of degree d in $\mathbb{P}^2$, i.e. the zero locus of the polynomial $f = x^d + y^d + z^d$. In this case, the Jacobian ideal is simply

$$J = (x^{d-1}, y^{d-1}, z^{d-1})$$

and one can easily prove that $R^N = \langle (xyz)^{d-2} \rangle$. Consider the element $[x^{d-2}y^2] \in R^d$ and denote it by ξ. Observe, moreover, that this is not zero as $x^{d-2}y^2 \notin J$. If $m = x^a y^b z^c$

with $a+b+c=d-3$, we have $m\cdot\xi\neq 0$ if and only if $a=0$ and $0\leq b\leq d-4$. Hence the image of the multiplication by ξ is generated by the $d-3$ elements of

$$\{[x^{d-2}y^{2+a}z^{d-3-a}]\,|\,0\leq a\leq d-4\}.$$

As they are independent we have $\mathrm{rk}(\xi)=d-3$. Notice that W is generated by monomials and $x^{d-3},y^{d-3}\in W$ but $z^{d-3}\notin W$, hence the base locus of $|W|$ is $P_2=(0:0:1)$. □

Remark 3.2. *In Theorem 3.1 we have proven that the bound* $\mathrm{rk}(\cdot\xi)\geq d-3$ *is sharp by showing that for the Fermat curve of degree d there exists an element in* R^d *whose rank is exactly* $d-3$. *There are other curves that have this property. For example, denote by* $f_\lambda\in S^5$ *the polynomial* $x^5+y^5+z^5+5\lambda x^3y^2$ *with* $\lambda\in\mathbb{C}$ *and consider the quintic* $F_\lambda=\{f_\lambda=0\}$. *Notice that* F_0 *is the Fermat quintic so the general quintic of this type is indeed smooth. Moreover, as* x^3y^2 *is not zero in the Jacobian ideal of the Fermat quintic, we know that* F_λ *is not biholomorphic to* F_0 *for* λ *general. If we denote by* R_λ *the Jacobian ideal of* F_λ *it is easy to see that* $\xi_\lambda=[x^3y^2]$ *is not zero in* R_λ. *Now we want to prove that* $\mathrm{rk}(\xi_\lambda)=2$ *for all* $\lambda\in\mathbb{C}$. *As we have already seen,* ξ_λ *is non trivial so, by Theorem 1.3 we know that* $\mathrm{rk}(\xi_\lambda)\geq 2$. *In order to see that the equality holds it is enough to prove that* $\dim W_{\xi_\lambda}\geq 4$ *(as* $\dim R^2_\lambda=6$*). With a little bit of effort one can show that*

$$x^2, xy, y^2, xz+18\lambda^3yz\in W_{\xi_\lambda}$$

thus proving the claim.

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