Analysis and Dynamics on the Cone of Discrete Radon Measures

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Summary

In this thesis, we develop analysis and study models on the cone of positive discrete Radon measures $\mathbb{K}(\mathbb{R}^d)$. This is a rather new but natural approach to model interacting particle systems on a continuous state space. The thesis is structured as follows:

Section 1 gives a historic overview over the subject of interacting particle systems. Furthermore, some external motivations for the consideration of the cone are given.

Part 2 establishes the preliminaries needed to develop analysis on the cone. Here, we introduce the cone and draw the connection to homogeneous configuration spaces. Heuristically, this connection can be explained by Plato's theory of forms. Furthermore, we also give a rigorous mathematical description. Topological and measurable strucures on the cone are introduced. Also, we discuss harmonic analysis and the relevant notions of Markov evolution in this chapter.

In Chapter 3, we discuss geometry on the cone $\mathbb{K}(\mathbb{R}^d)$. We introduce the notions of a gradient and Laplacian. Furthermore, we compare these notions to the so-called Plato space introduced in Section 2. A short part of this chapter is devoted to the so-called Umbral calculus, which is concerned with the analysis of polynomials. Also, we introduce a new notion of geometry on the cone. Namely, the so-called difference calculus, which considers discrete differences instead of infinitesimal objects. We also discuss some commutation relations, a connection to Umbral calculus as well as the notion of a discrete Laplacian.

Chapter 4 is concerned with the study of concrete particle systems on the cone $\mathbb{K}(\mathbb{R}^d)$. We consider the following three models:

- Glauber Dynamics
- Continuous Contact Model
- Bolker-Dieckmann-Law-Pacala Model

All three models belong to the class of so-called birth-and-death models. Here, stationary particles appear and disappear according to some rates depending on the model. This variety illustrates different challenges present for each model, which have to be solved using different techniques. We show the existence of the different dynamics as well as some additional properties typical for each model.

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1 Introduction

1.1 General Setting

During the last century, interacting particle systems have become an integral part of mathematical modelling. When describing systems of large quantities of particles or agents, one needs to take into account the interactions between these agents. The applications range from physics and biology to economics and social systems. On the level of individually interacting entities, the description is called microscopic. Depending on the context, the number of particles falls in the range of 10^4 (large biological systems) to 10^{23}(molecules). Due to this large size, it is practically impossible to track the motion and development of each single agent in the system. Furthermore, for a more realistic approach, spatial models had to be considered, i.e. models that take into account positions of the entities of the system. Therefore, mathematical methods from functional analysis and probability theory had to be developed to describe such processes.

The first models taking into account the spatial structure of a system were developed by Preston [49] in 1975, where he used Markov semigroup methods to describe spatial models. The considered space was a bounded region of $\mathbb{R}^d$ or a finite set, and a finite number of particles was considered. Other works were focussed on a discrete state space to open up more possibilities regarding other aspects of the system, see [44] and the references therein.

Later, systems on an unbounded state space (e.g. $\mathbb{R}^d$) were considered. For the development of the necessary analysis and geometry, see [2, 4]. Furthermore, to have a non-trivial density on the space, an infinite number of particles had to be considered. During the last years, these systems have been extended by considering multi-component systems [22, 23], interaction with a random environment [7, 34, 35] or spatially dependent rates [20, 40].

When choosing a model, one needs to take into account different features which are relevant for the behaviour and properties of the system:

- Discrete vs. continuous: The considered state space can be chosen as a discrete set, e.g. $\mathbb{Z}^d$ or some other connected graph, or continuous, such as $\mathbb{R}^d$ or more generally, a Riemannian manifold X. While discrete models are easier to analyse (e.g. [44]) and yield more results, a continuous state space models a physical system more realistically.

- Bounded region vs. unbounded region/state space: A bounded region makes more sense from a modelling point of view. On the other hand, one needs to take into account the interaction of particles with the boundary. A way to circumvent this is by considering an unbounded region and restricting the system after analysing the model. The kind of region also determines whether a finite or an infinite amount of particles should be considered.

  Another advantage of an unbounded region with an infinite number of particles is that phase transitions may be observed since invariant measures may not be uniquely determined. For examples, see [12] and the references therein.

- Birth-and-Death models vs. diffusion vs. jump-type processes: Different mechanisms yield different behaviours of the system. This choice of course depends on the desired phenomenon which is to be modelled. For instance, the description of hopping particles on configuration spaces was analysed in [5].

There are some additional options which were already mentioned above. For our situation, we choose a specific version of a continuous particle system with unbounded state space $\mathbb{R}^d$. Futhermore, the considered models are only of birth-and-death type. Instead of considering a homogeneous configuration space, the particle system comes from the cone of positive discrete Radon measures. One specific property of this object is that particles in the space $\mathbb{R}^d$ are assigned a positive number, or "mark", which represents a property of the particle such as weight. Some general analytic and geometric considerations for models on the cone of Radon measures have been carried out in [27, 28, 38]. This thesis takes the results from these works to expand on them, especially in the direction of specific models.

Note that this approach differs from the so-called marked configuration spaces considered in [1, 39]. On the other hand, there is a direct relation to the extended configuration space $\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$ which is explored in later chapters. While the analysis and dynamics on the cone are of special interest and the modelling possibilities of the cone are useful in applications, one may also give some motivations for this object without referring to these analytical properties or configuration spaces in general. The next section explains three motivations from theoretical biology, probability theory and representation theory.

1.2 Motivation for the Cone

The mathematical object of interest for us is the cone of positive discrete Radon measures, defined by

$$\mathbb{K}(\mathbb{R}^d) := \left\{ \eta = \sum_i s_i \delta_{x_i} \in \mathbb{M}(\mathbb{R}^d) \,\middle|\, s_i \in (0,\infty), x_i \in \mathbb{R}^d \right\}$$

where by convention, the zero measure $0 \in \mathbb{K}(\mathbb{R}^d)$ is included. This thesis is concerned with covering analytic properties of the cone. On the other hand, there are three approaches which justify the use of this object without even considering its analytical properties. For one, there is the aspect of modelling biological systems. Second, the cone appears naturally when considering certain generalised stochastic processes. Third, the cone is given as the space where Gamma measures are localised, which emerge from representation theory for current groups. These three motivations will be explained in this chapter.

1.2.1 Applications to Biological Models

There is an external non-mathematical motivation to study particle systems realised as elements of the cone. Namely, Vladimir Vernadsky (1998) wrote the following:

- "Organisms [...] are always separated from the surrounding inert matter by a clear and firm boundary." [58, p. 56]
- "Living matter [...] is spread over the entire surface of the Earth in a manner analogous to a gas [...]."[58, p. 59]
- "In the course of time, living matter clothes the whole terrestrial globe with a continuous envelope [...]." [58, p. 60]

This can be interpreted in the sense that system of living matter should possess two properties: For one, the system should have a discrete nature. Furthermore, there is living matter everywhere in the system. In mathematical terms, this means that the support of this system should be dense in the underlying position space. Lastly, to be realistic, the system should have finite local mass due to the physical limitations of our world. The mathematical realisation of these properties is given by the cone.

1.2.2 Probability Theory

The second motivation comes from the theory of generalised stochastic processes, i.e. processes on the space $\mathcal{D}'(\mathbb{R}^d)$ of generalised functions. By [54, Thm. 3.3.24], infinitely divisible processes on $\mathcal{D}'(\mathbb{R}^d)$ are actually concentrated on the subspace $\mathbb{K}(\mathbb{R}^d)$. Note that this result holds independently of the topological and analytical considerations done in later chapters. For a subclass of measures, the so-called Gamma measures, we will also show a direct proof of this statement.

1.2.3 Representation Theory for Current Groups

Measures supported on $\mathbb{K}(\mathbb{R}^d)$ naturally appear in the study of representations for current groups. Namely, when studying so-called commutative models of representations of $(\mathrm{SL}(2,\mathbb{R}))^{\mathbb{R}^d}$. When considering representations with respect to the unipotent subgroup of $(\mathrm{SL}(2,\mathbb{R}))^{\mathbb{R}^d}$, we arrive at spectral measures which are defined on the space $\mathcal{D}'(\mathbb{R}^d)$ and supported on $\mathbb{K}(\mathbb{R}^d)$. Furthermore, these measures show some invariance properties. These considerations were first done by Gelfand, Graev and Vershik [24]. Later, Tsilevich, Vershik and Yor [57] used this as a starting point to further analyse so-called Gamma processes.

As seen here, these measures supported on the cone $\mathbb{K}(\mathbb{R}^d)$ appear naturally without any *a priori* restiction of the spaces or aspects of modelling.

1.2.4 Analytical Motivation

There is another mathematical explanation why it makes sense to consider $\mathbb{K}(\mathbb{R}^d)$. If we take the class of Gamma-Poisson-measures on the extended configuration space $\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$, we see that these measures assign full mass to the subset of configurations with finite local mass, or Plato configurations. These configurations can be identified with objects in the cone, i.e. there exists a one-to-one correspondence between the so-called Plato space $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and the cone $\mathbb{K}(\mathbb{R}^d)$. Later, we give an explicit proof of this statement.

1.3 Description of Results/Outline of Thesis

We give a short outline of the thesis to guide the reader to the main results.

1.3.1 Connection to Configuration Spaces

There exists a natural connection of the cone $\mathbb{K}(\mathbb{R}^d)$ to the so-called extended configuration space $\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$. Namely, there exists a bijection $\mathcal{R}\colon \Gamma_{\mathrm{pf}}(\mathbb{R}_+^* \times \mathbb{R}^d) \to \mathbb{K}(\mathbb{R}^d)$, where $\Gamma_{\mathrm{pf}}(\mathbb{R}_+^* \times \mathbb{R}^d)$ is the space of pinpointing configurations with finite local mass, also called Plato space $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ below. Our considerations will heavily rely on this bijection, since the theory on $\Gamma(Y)$ for general Y is well-established. On the other hand, the use of $\mathbb{K}(\mathbb{R}^d)$ generates some new phenomena explored in this work. In Section 2, we introduce all necessary background related to configuration spaces $\Gamma(Y)$, specifically the Plato space $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and establish the aforementioned connection to the cone $\mathbb{K}(\mathbb{R}^d)$.

1.3.2 Topological and Measure-Theoretical Considerations

To analyse the dynamics on the cone $\mathbb{K}(\mathbb{R}^d)$, we first need to establish the topological and measurable structures on this object in Chapters 2.6 and 2.7. We use the aforementioned connection to configuration spaces to define a topology on $\mathbb{K}(\mathbb{R}^d)$. Furthermore, we analyse the generated Borel-σ-algebra. In Chapter 2.8, we define classes of probability measures on $\mathbb{K}(\mathbb{R}^d)$. Namely, on $\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$, Poisson measures assign full mass to the space $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$. We recall this fact including the proof in Theorem 2.24. Therefore, we define probability measures on $\mathbb{K}(\mathbb{R}^d)$ as image measures of such measures. Furthermore, we construct Gibbs measures on $\mathbb{K}(\mathbb{R}^d)$ using the well-known DLR approach, see also [29].

1.3.3 Harmonic Analysis and Markov Evolution

In Sections 2.9 and 2.10, we adapt the well-known theory of harmonic analysis on configuration spaces to the case of $\mathbb{K}(\mathbb{R}^d)$. More precisely, we define the set of finitely supported Radon measures $\mathbb{K}_0(\mathbb{R}^d)$, which is connected to $\mathbb{K}(\mathbb{R}^d)$ via the so-called K-transform. Furthermore, we discuss the connections between measures on $\mathbb{K}(\mathbb{R}^d)$ and its correlation functions, which is needed for the analysis of the dynamics. In Chapter 2.11, we use this connection to define various equivalent evolution equations describing these dynamics.

1.3.4 Calculus

Chapter 3 is concerned with the establishment of various analytic structures on $\mathbb{K}(\mathbb{R}^d)$. In Chapters 3.1 and 3.2, we consider the differential calculus established in [29]. Here, continuous derivatives, an integration by parts formula and a continuous Laplacian are introduced with respect to some underlying Lie group, i.e. the group of currents. Furthermore, the results are compared to the case of $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and we show a direct correspondence between the formulae on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\mathbb{K}(\mathbb{R}^d)$ in Chapter 3.3.

Next, we take a short look at the umbral calculus on $\mathbb{K}(\mathbb{R}^d)$, i.e. the calculus of polynomials on this space. We concentrate our considerations to the special sequence of so-called fake falling factorials. This sequence has a nice connection to the aforementioned K-transform, which we will state here. These considerations can be found in Chapter 3.4.

As a third part of the calculus (Chapter 3.5), we make use of the discrete structure of the elements in $\mathbb{K}(\mathbb{R}^d)$ to examine difference calculus on $\mathbb{K}(\mathbb{R}^d)$. Here, instead of looking at infinitesimal differences, we consider discrete differences. We define birth and death gradients as well as a related integration by parts formula. Furthermore, we consider the corresponding discrete Laplacian, which yields a jump-type Markov operator in this case.

1.3.5 Dynamics and Considered Models

For the last part of this work, we analyse three different birth-and-death models on the cone.

Glauber Dynamics (Chapter 4.1): This model can be obtained by considering a discrete Dirichlet form on $\mathbb{K}(\mathbb{R}^d)$ with respect to a Gibbs measure. In this model, particles disappear with constant rate, while they appear according to some pair potential in relation to all particles in a neighbourhood specified by the potential. Usually, the Glauber dynamics are used to describe a homogeneous gas with a given potential.

We show the existence of the dynamics for this model and calculate the corresponding operators for statistical dynamics and obtain the hierarchical structure of the correlation functions.

Continuous Contact Model (Chapter 4.2): The generator of this model is constructed explicitly by considering the desired heuristics: Particles disappear according to the death rate which may depend on the mark of a particle. On the other hand, each existing particle may spawn a new particle according to a given birth rate which may also depend on the mark. The spawning procedure is independent of all other particles. This model can be used to describe infection spreading, plant growth and similar processes.

We establish the existence of the dynamics using the hierarchical system of correlation functions. Furthermore, we establish *a priori* estimates for each order of correlation as well as estimates which are uniform in the order of correlation. Also, we show that the contact model admits clustering. Lastly, we show the existence of invariant measures of the contact model under some conditions.

Bolker-Dieckmann-Law-Pacala Model (Chapter 4.3): This model can be seen as a modified contact model by adding a competition term: The mortality rate is increased if particles are clustered together. This way, the clustering experienced by the contact model can be prevented.

The density-dependent mortality also has a technical advantage. Namely, we are able to use perturbation methods to show the existence of the dynamics. Furthermore, we calculate the evolution equation for the statistical dynamics and comment on the non-clustering behaviour of the system.

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2 Preliminaries

This section will include basic concepts for self-containedness as well as some fundamental concepts from configuration spaces $\Gamma(Y)$, the cone $\mathbb{K}(\mathbb{R}^d)$ and the connection between these two. Furthermore, we recall the topological and measurable structures on configuration spaces and establish similar structures on the cone $\mathbb{K}(\mathbb{R}^d)$. We also introduce measures on these spaces as well as harmonic analysis and explain how the evolution of dynamics can be described using various types of evolution equations.

2.1 The Cone of Positive Discrete Radon Measures

We start the preliminary chapter by the introduction of the cone of positive discrete Radon measures. Furthermore, the notion of the support of a measure and relations between elements in $\mathbb{K}(\mathbb{R}^d)$ are defined. Recall that by Vernadsky's theory of living matter, a system should be dense everywhere, discrete and have finite local mass.

One more property which we want from our system is that its elements are indistinguishable in the sense that the system given by $(s_i, x_i)_{i\in I}$ and $(s_{\pi(i)}, x_{\pi(i)})_{i\in I}$ behave the same, where I is some countable index set and π an arbitrary permutation of I. One possibility is to realise our system as sums of point masses δ_y, where y is either the mark and position, or just the position of a particle, depending on the setup. This automatically yields a discrete particle system. To obtain the other two properties, it is useful to let y represent the position of a particle, while the mark is considered as a weight of the point mass. These properties become clear when we consider a certain class of measures, namely, Gamma measures. The properties are then proven in Proposition 2.30 and Theorem 2.24, respectively.

Definition 2.1. *1. The cone of nonnegative discrete Radon measures is defined as follows:*

$$\mathbb{K}(\mathbb{R}^d) := \left\{ \eta = \sum_i s_i \delta_{x_i} \in \mathbb{M}(\mathbb{R}^d) \middle| s_i \in (0, \infty), x_i \in \mathbb{R}^d \right\}$$

By convention, the zero measure $\eta = 0$ is included in $\mathbb{K}(\mathbb{R}^d)$.

2. *We denote the support of $\eta \in \mathbb{K}(\mathbb{R}^d)$ by*

$$\tau(\eta) := \{x \in \mathbb{R}^d \mid 0 < \eta(\{x\}) =: s_x(\eta)\}.$$

If η is fixed, we write $s_x := s_x(\eta)$.

3. *For $\eta, \xi \in \mathbb{K}(\mathbb{R}^d)$ we write $\xi \subset \eta$ if $\tau(\xi) \subset \tau(\eta)$ and $s_x(\xi) = s_x(\eta)$ for all $x \in \tau(\xi)$. If additionally $|\tau(\xi)| < \infty$, we write $\xi \Subset \eta$.*

4. *For a function $f \in C_c(\mathbb{R}^d)$, denote the pairing with an element $\eta \in \mathbb{K}(\mathbb{R}^d)$ by*

$$\langle f, \eta \rangle := \sum_{x\in\tau(\eta)} s_x f(x).$$

While $\mathbb{K}(\mathbb{R}^d)$ can be viewed as a subset of the space of positive Radon measures $\mathbb{M}(\mathbb{R}^d)$, it is not advisable to consider it as a subset topologically. This method works for the space $\Gamma(Y)$ introduced below, as will be explained later. For $\mathbb{K}(\mathbb{R}^d)$, it does not yield satisfactory topological results. Instead, we keep Plato's theory in mind and see $\mathbb{K}(\mathbb{R}^d)$ as the real-world projection of another space, called the Plato space $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$. It is introduced in Chapter 2.5.

2.2 The Group $(\mathbb{R}_+^*, \cdot)$

The set $\mathbb{R}_+^*$ plays a special role for the analysis on the cone. Since we want to consider harmonic analysis on $\mathbb{R}_+^*$, we need to establish integration theory with respect to the Haar measure, i.e. we need to consider the group structure on $\mathbb{R}_+^*$.

Set $\mathbb{R}_+^* := (0, \infty)$ and consider the Abelian group $(\mathbb{R}_+^*, \cdot)$. There is a natural bijection between $(\mathbb{R}, +)$ and $(\mathbb{R}_+^*, \cdot)$ given by the exponential function $x \mapsto e^x$. The measure which is invariant under the group operation (also known as the Haar measure) is given by

$$h(ds) = \frac{1}{s} ds$$

where ds denotes the Lebesgue measure on $\mathbb{R}_+^*$.

Using the bijection mentioned above, we may introduce a metric on $(\mathbb{R}_+^*, \cdot)$. Consider the Euclidean metric on $(\mathbb{R}, +)$, i.e.

$$d(x, y) = |x - y|, \ x, y \in \mathbb{R}.$$

For any $u, v \in \mathbb{R}_+^*$, there exist $x, y \in \mathbb{R}$ such that $u = e^x, v = e^y$. Then we may define the corresponding metric on $\mathbb{R}_+^*$ the following way:

$$\rho(u, v) = d(x, y) = |x - y| = |\log u - \log v| = \left|\log \frac{u}{v}\right|$$

For transformations considered later, it is interesting to consider the unitary characters of $(\mathbb{R}_+^*, \cdot)$, i.e. group homomorphisms to the unit sphere $S^1 \subset \mathbb{C}^*$. These characters are given by mappings of the form

$$f_\lambda(u) = e^{i\lambda \log u}$$

where $\lambda \in \mathbb{R}$. Therefore, the dual group to $(\mathbb{R}_+^*, \cdot)$ in the Pontrjagin sense is again $(\mathbb{R}, +)$.

Analogously to the Fourier transform on $(\mathbb{R}, +)$, we may use the above considerations to introduce a transform on functions on $(\mathbb{R}_+^*, \cdot)$.

Definition 2.2. *Let $f \colon \mathbb{R}_+^* \to \mathbb{R} \in L^1(\mathbb{R}_+^*, h)$. The Fourier transform of f is defined as*

$$\mathcal{F}_{\mathbb{R}_+^*} f(\lambda) = \int_{\mathbb{R}_+^*} f(u) e^{-i\lambda \log u} h(ds)$$

Remark 2.3. *Note the similarity to the Mellin transform on $\mathbb{R}_+^*$: It is defined as*

$$M_f(\alpha) = \int_{\mathbb{R}_+^*} f(s) s^{\alpha - 1} ds = \int_{\mathbb{R}_+^*} f(s) s^\alpha h(ds).$$

If we write $s^\alpha = e^{\alpha \log s}$, we obtain the following form:

$$M_f(\alpha) = \int_{\mathbb{R}_+^*} f(s) e^{\alpha \log s} h(ds)$$

Setting $\alpha = -i\lambda$, we see that

$$M_f(-i\lambda) = \mathcal{F}_{\mathbb{R}_+^*} f(\lambda).$$

There is a direct connection to the Fourier transform on $\mathbb{R}$: Let $f\colon \mathbb{R} \to \mathbb{R}$ such that its Fourier transform exists. Then $f \circ \log\colon \mathbb{R}_+^* \to \mathbb{R}$ and

$$[\mathcal{F}_{\mathbb{R}} f](p) = \int_{-\infty}^{\infty} f(x) e^{-ipx} dx = \int_0^{\infty} f(\log s) e^{-ip\log s} \frac{ds}{s} = \left[\mathcal{F}_{\mathbb{R}_+^*}(f \circ \log)\right](p)$$

On the other hand, we can calculate the inverse Fourier transform on $\mathbb{R}_+^*$ the same way: For a function $g\colon \mathbb{R} \to \mathbb{R}$,

$$\left[\mathcal{F}_{\mathbb{R}}^{-1} g\right](x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipx} g(p) dp = \frac{1}{2\pi} \int_0^{\infty} e^{ix\log z} g(\log z) \frac{dz}{z} = \left[\mathcal{F}_{\mathbb{R}_+^*}^{-1}(g \circ \log)\right](x)$$

In other words, this implies for a function $\psi\colon \mathbb{R}_+^* \to \mathbb{R}$,

$$\left[\mathcal{F}_{\mathbb{R}_+^*}^{-1} \psi\right](s) = \frac{1}{2\pi} \int_0^{\infty} e^{is\log z} \psi(z) \frac{dz}{z}$$

provided, the expression exists. We denote $\mathcal{F} := \mathcal{F}_{\mathbb{R}_+^*} \circ \mathcal{F}_{\mathbb{R}^d}$ for functions from $\mathbb{R}_+^* \times \mathbb{R}^d$ to $\mathbb{R}$, where $\mathcal{F}_{\mathbb{R}^d}$ denotes the Fourier transform on $\mathbb{R}^d$.

Lemma 2.4. *The following relations hold:*

$$\mathcal{F}_{\mathbb{R}_+^*}\left[Q\left(\frac{\cdot}{s}\right)\right](z) = e^{-iz\log s}\left(\mathcal{F}_{\mathbb{R}_+^*} Q\right)(z)$$
$$\mathcal{F}_{\mathbb{R}^d}\left[a(\cdot - x)\right](p) = e^{-i(p,x)}\left(\mathcal{F}_{\mathbb{R}^d} a\right)(p)$$

Proof. Use variable substitution in the integral terms. □

The following estimate will be useful in later calculations.

Lemma 2.5. *Let $\psi\colon \mathbb{R}_+^* \times \mathbb{R}^d \to \mathbb{R}, \psi \in L^1(\mathbb{R}_+^* \times \mathbb{R}^d, h(dz) \otimes dp)$. Then*

$$\left|\mathcal{F}^{-1}\psi\right|(s,x) \leq \frac{1}{(2\pi)^{d+1}} \int_0^{\infty} \int_{\mathbb{R}^d} |\psi(z,p)| dp h(dz)$$

Proof. Direct calculation using the above definition. □

The following Lemma is also needed to close our arguments.

Lemma 2.6. *Let $Q\colon \mathbb{R}_+^* \to \mathbb{R}$ be an even function in the sense that*

$$Q(s) = Q(s^{-1}) \ \forall s \in \mathbb{R}_+^*.$$

Then its Fourier transform $\mathcal{F}_{\mathbb{R}_+^*} Q$ is real-valued, provided, it exists.

Proof. We use Euler's identity to rewrite the integral:

$$(\mathcal{F}_{\mathbb{R}_+^*} Q)(z) = \int_0^{\infty} \cos(z\log s) Q(s) h(ds) + i \int_0^{\infty} \sin(z\log s) Q(s) h(ds)$$

We need to show that the second expression equals zero. We use the variable transform $s \mapsto \frac{1}{t}$ and see that

$$\begin{aligned}\int_0^{\infty} \sin(z\log s) Q(s) h(ds) &= -\int_0^{\infty} \sin(-z\log s) Q(s) \frac{1}{s} ds \\ &= -\int_0^{\infty} \sin(z\log t) Q\left(\frac{1}{t}\right) t \frac{dt}{t^2} = -\int_0^{\infty} \sin(z\log t) Q(t) h(dt)\end{aligned}$$

which proves the claim. □

The same statement holds for the function a on $\mathbb{R}^d$. For applications in later chapters, the relation of the Fourier transform to the convolution is useful. To this end, we define the convolution on $\mathbb{R}_+^*$ analogously to the case of $\mathbb{R}$.

Definition 2.7. *Let $f, g \in L^1(\mathbb{R}_+^*, h)$. The convolution on $\mathbb{R}_+^*$ is defined as*

$$(f * g)(u) = \int_{\mathbb{R}_+^*} f(v) g\left(\frac{u}{v}\right) h(dv)$$

The expected result also holds on $\mathbb{R}_+^*$:

Proposition 2.8. *The following relation holds for two functions $f, g \colon \mathbb{R}_+^* \to \mathbb{R}$:*

$$\mathcal{F}_{\mathbb{R}_+^*}(f * g) = \mathcal{F}_{\mathbb{R}_+^*} f \cdot \mathcal{F}_{\mathbb{R}_+^*} g$$

2.3 Plato's theory

As stated in the introduction, the cone $\mathbb{K}(\mathbb{R}^d)$ is a suitable object to describe particle systems in the real world. On the other hand, the question arises how to define and interpret mathematical structures on the space $\mathbb{K}(\mathbb{R}^d)$. As a motivation, we give a short overview of Plato's theory of forms.

In the theory, Plato stated that observations in the real world are mere projections of higher forms or ideas. One way to picture this is the so-called cave allegory, which was recited by Ross (1951) as follows: "A company of men is imprisoned in an underground cave, with their heads fixed so that they can look only at the back wall of the cave. Behind them across the cave runs a wall behind which men pass, carrying all manner of vessels and statues which overtop the wall. Behind these again is a fire. The prisoners can only see the shadows [...] of the things carried behind the wall, and must take these to be the only realities" [52, P. 69].

Applied to our setting, the space $\mathbb{K}(\mathbb{R}^d)$ is interpreted as the shadows projected onto the cave wall. On the other hand, the space $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ which will be introduced below is the space of forms or ideas, represented by the objects carried in front of the fire. While the space $\mathbb{K}(\mathbb{R}^d)$ is taken to be our reality, we use the space $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ to define mathematical operations. The spaces are connected via the bijection $\mathcal{R} \colon \Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \to \mathbb{K}(\mathbb{R}^d)$ introduced below. In accordance with the cave allegory, $\mathcal{R}$ is also called reflection mapping.

2.4 Configuration Spaces

As we will see in the next chapter, the Plato space $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ is a very specific subset of the so-called configuration space $\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$, which will fulfill the assumptions stated heuristically in Chapter 2.1.

In general, the space of locally finite configurations $\Gamma(Y)$ is the space of all subsets of Y which are finite in any compact set $\Lambda \subset Y$. The following definition makes this notion more precise.

Definition 2.9. *Let Y be a locally compact Hausdorff space. The space of locally finite configurations over Y is defined as*

$$\Gamma(Y) = \{\gamma \subset Y \colon |\gamma \cap \Lambda| < \infty \ \forall \Lambda \subset Y \text{ compact}\}$$

where $|\cdot|$ denotes the number of elements of a set.

From a physical perspective, Y is considered as phase space of an interacting particle system. A configuration $\gamma \in \Gamma(Y)$ represents a set of indistinguishable agents (e.g. particles, plants) which may interact with each other. In our considerations, we always consider $Y = \mathbb{R}_+^* \times \mathbb{R}^d$. More generally, $\mathbb{R}^d$ could be replaced by some more general locally comapct space X. In this chapter, we recall some properties of $\Gamma(Y)$ which will form the basis for the Plato space $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$.

2.4.1 Topology and Measurable Structure of $\Gamma(Y)$

There exists a natural embedding of $\Gamma(Y)$ into the space of Radon measures $\mathbb{M}(Y)$ on Y, namely

$$\Gamma(Y) \ni \gamma \mapsto \sum_{y \in \gamma} \delta_y \in \mathbb{M}(Y)$$

where δ_y denotes the Dirac measure at point $y \in Y$. Note that we use the notion of γ as a subset of Y and as a measure on Y interchangably. We equip $\Gamma(Y)$ with the vague topology induced by $\mathbb{M}(Y)$, i.e. the coarsest topology such that the following mappings are continuous for all $f \in C_c(Y)$, where $C_c(Y)$ denotes the space of continuous functions with compact support:

$$\Gamma(Y) \ni \gamma \mapsto \langle f, \gamma \rangle = \sum_{y \in \gamma} f(y)$$

In fact, $\Gamma(Y)$ equipped with this topology is a Polish space. A more detailed analysis of the topological properties of $\Gamma(Y)$ can be found in [32].

The construction of a topology enables us to consider the Borel-σ-algebra $\mathcal{B}(\Gamma(Y))$. It should be noted that this σ-algebra coincides with the σ-algebra generated by the following mappings:

$$N_\Lambda \colon \Gamma(Y) \to \mathbb{N}_0, \gamma \mapsto N_\Lambda(\gamma) = |\gamma \cap \Lambda|, \ \Lambda \in \mathcal{B}_c(Y)$$

where $\mathcal{B}_c(Y)$ denotes all precompact Borel subsets of Y, see e.g. [30].

We give another construction of the measurable space $(\Gamma(Y), \mathcal{B}(\Gamma(Y))$ which will be useful for other considerations. For $\Lambda \in \mathcal{B}_c(Y)$, we define the space of configurations supported in Λ.

$$\Gamma(\Lambda) := \{\gamma \in \Gamma(Y) \colon \gamma \cap \Lambda = \gamma\}.$$

Furthermore, for $n \in \mathbb{N}$, consider the set of n-point-configurations supported in Λ:

$$\Gamma^{(n)}(\Lambda) := \{\gamma \in \Gamma(\Lambda) \colon |\gamma| = n\}, \Gamma^{(0)}(\Lambda) := \{\emptyset\}$$

Since $\gamma \in \Gamma(Y)$ is locally finite, the elements of $\Gamma(\Lambda)$ are finite and we have the disjoint decomposition

$$\Gamma(\Lambda) = \bigcup_{n=0}^{\infty} \Gamma^{(n)}(\Lambda). \tag{1}$$

We can represent $\Gamma^{(n)}(\Lambda)$ via symmetrization of the underlying space:

$$\tilde{\Lambda}^n / S_n \simeq \Gamma^{(n)}(\Lambda) \tag{2}$$

where

$$\tilde{\Lambda}^n := \{(x_1, \ldots, x_n) \in \Lambda^n \mid x_i \neq x_j \ \forall i \neq j\}$$

the off-diagonals and S_n the symmetric group of n elements. Denote the bijection (2) by sym_n. This way, $\Gamma^{(n)}(\Lambda)$ can be equipped with the topology induced via Λ^n. Furthermore, $\Gamma(\Lambda)$ is equipped with the topology of disjoint unions. Hence, we can define the Borel-σ-algebra $\mathcal{B}(\Gamma(\Lambda))$ given by this topology.

For two sets $\Lambda_1, \Lambda_2 \in \mathcal{B}(Y), \Lambda_2 \subset \Lambda_1$, define the projection mapping

$$p_{\Lambda_1,\Lambda_2} \colon \Gamma(\Lambda_1) \to \Gamma(\Lambda_2), \gamma \mapsto \gamma \cap \Lambda_2$$

where we set $p_{\Lambda_2} := p_{Y,\Lambda_2}$. It was shown in e.g. [53] that $(\Gamma(Y), \mathcal{B}(\Gamma(Y))$ is the projective limit of the spaces $(\Gamma(\Lambda), \mathcal{B}(\Gamma(\Lambda))$ for $\Lambda \in \mathcal{B}_c(Y)$. This especially implies that the mappings p_Λ are $\mathcal{B}(\Gamma(Y))$-$\mathcal{B}(\Gamma(\Lambda))$-measurable. The construction of $\mathcal{B}(\Gamma(Y))$ via projections will play an important role in the construction of measures on $\Gamma(Y)$.

2.4.2 The Space of Finite Configurations

For mathematical purposes, it is important to also consider the space $\Gamma_0(Y)$ of finite configurations, i.e.

$$\Gamma_0(Y) := \{\gamma \in \Gamma(Y) \colon |\gamma| < \infty\}$$

where $|\cdot|$ denotes the number of elements of a set. While the definition implies that $\Gamma_0(Y)$ is a subset of $\Gamma(Y)$, the interpretation is a different one: $\Gamma_0(Y)$ serves as a mathematical counterpart to the physical space $\Gamma(Y)$. Also, the spaces $\Gamma(Y)$ and $\Gamma_0(Y)$ are topologically different: While $\Gamma(Y)$ is seen as a subspace of $\mathbb{M}(Y)$ with the inherited topology, we use a different approach for $\Gamma_0(Y)$ which will be explained in this chapter. The approach is similar to the one used in Chapter 2.4.1, but yields different results. We set

$$\Gamma_0^{(n)}(\Lambda) := \Gamma^{(n)}(\Lambda)$$

where Λ is an arbitrary Borel subset of Y. Since we only deal with finite configurations, we may use decomposition (1) for $\Lambda = Y$, i.e.

$$\Gamma_0(Y) = \bigsqcup_{n=0}^{\infty} \Gamma_0^{(n)}(Y).$$

Furthermore, we may consider the symmetrization (2) to obtain

$$\tilde{Y}^n / S_n \simeq \Gamma^{(n)}(Y).$$

For $\Gamma^{(n)}(Y)$, we choose the topology induced by the space Y^n. For $\Gamma_0(Y)$, we may use the topology of disjoint unions. For a more detailed description of the topology used here, we refer to [30].

Remark 2.10. *The purpose of the space of finite configurations will become clearer once we examine specific models. Since the models are introduced on the cone, we postpone this discussion until after we have introduced the relevant spaces related to $\mathbb{K}(\mathbb{R}^d)$.*

2.5 Relation Between $\mathbb{K}(\mathbb{R}^d)$ and $\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$: The Plato Space $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$

In this section, we want to establish the connection between the configuration space $\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$ and the cone $\mathbb{K}(\mathbb{R}^d)$. Our goal is to define a certain subspace $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \subset$

$\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$ such that there exists a one-to-one-correspondence between $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\mathbb{K}(\mathbb{R}^d)$ in the following form:

$$\mathcal{R}\colon \Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \to \mathbb{K}(\mathbb{R}^d), \gamma = \sum_{(s,x)\in\gamma} \delta_{(s,x)} \mapsto \sum_{(s,x)\in\gamma} s\delta_x.$$

In terms of Plato's theory, this mapping takes ideas $\gamma \in \Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and projects (or reflects) them to real-world objects $\eta \in \mathbb{K}(\mathbb{R}^d)$. Obviously, $\mathcal{R}$ is not defined on the whole space $\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$. Therefore, we need to construct a suitable subspace. In other terms, the Plato space constructed below is also known as the set of pinpointing configurations with finite local mass, denoted by $\Gamma_{\mathrm{pf}}(\mathbb{R}_+^* \times \mathbb{R}^d)$. We explore these two properties in more detail below.

Define the set of pinpointing configurations $\Gamma_{\mathrm{p}}(\mathbb{R}_+^* \times \mathbb{R}^d) \subset \Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$ as all configurations such that if $(s_1, x_1), (s_2, x_2) \in \gamma$ with $x_1 = x_2$, then $s_1 = s_2$.

Remark 2.11. *The pinpointing property ensures that there are no two elements of a system at the same position. Due to the shape of elements in $\mathbb{K}(\mathbb{R}^d)$, it is obvious that this would not be possible.*

Let us now take into account the second property of $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$. To this end, we define the local mass of a configuration.

Definition 2.12. *For a configuration $\gamma \in \Gamma_{\mathrm{p}}(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\Lambda \subset \mathbb{R}^d$ compact, set the local mass as*

$$\gamma(\Lambda) = \int_{\mathbb{R}_+^* \times \mathbb{R}^d} s\mathbb{1}_\Lambda(x)\ d\gamma(s,x) = \sum_{(s,x)\in\gamma} s\mathbb{1}_\Lambda(x) \in [0,\infty]$$

This notion enables us to define the Plato space.

Definition 2.13. *The Plato space $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \subset \Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$ is defined as the space of all pinpointing configurations with finite local mass, i.e.*

$$\Pi(\mathbb{R}_+^* \times \mathbb{R}^d) := \Gamma_{\mathrm{pf}}(\mathbb{R}_+^* \times \mathbb{R}^d) = \{\gamma \in \Gamma_p \mid \gamma(\Lambda) < \infty \textit{ for all } \Lambda \subset \mathbb{R}^d \textit{ compact}\}.$$

Remark 2.14.

1. *The property of finite local mass accounts for the third property stated in Chapter 2.1. It ensures that the system only has finite mass in any bounded volume, which makes it physically viable.*
2. *The pinpointing property as well as the finiteness of local mass are sufficient to make $\mathcal{R}\colon \Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \to \mathbb{K}(\mathbb{R}^d)$ bijective.*
3. *The state space needs to be of the specific form $Y = \mathbb{R}_+^* \times X$ for the notion of pinpointing configurations to make sense.*

To establish a viable connection via $\mathcal{R}$, we need to examine its measurability. First, let us show that the set of pinpointing configurations on each compact $\Lambda \in \mathcal{B}(\mathbb{R}_+^* \times \mathbb{R}^d)$ is measurable.

Lemma 2.15 ([27])**.** *Define the set of pinpointing configurations in $\Lambda \in \mathcal{B}(\mathbb{R}_+^* \times \mathbb{R}^d)$ as*

$$\Gamma_{\mathrm{p}}(\Lambda) := \{\gamma \in \Gamma(\Lambda)\colon (s_1, x_1), (s_2, x_2) \in \gamma, x_1 = x_2 \Rightarrow s_1 = s_2\}.$$

Then $\Gamma_{\mathrm{p}}(\Lambda) \in \mathcal{B}(\Gamma(\Lambda))$. In particular, this holds for $\Lambda = \mathbb{R}_+^* \times \mathbb{R}^d$.

Proof. We recall the proof from [27]. For the first case, consider a bounded set $\Lambda \in \mathcal{B}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$. Set $D = \{(x,x) \colon x \in \mathbb{R}^d\} \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d)$ the diagonal set of $\mathbb{R}^d$. Then the complement of $\Gamma_p(\Lambda)$ admits the following representation:

$$\Gamma_{\mathrm{p}}(\Lambda)^c = \{\gamma \in \Gamma(\Lambda) \colon \exists (s_1, x_1), (s_2, x_2) \in \gamma \colon (x_1, x_2) \in D\} = \bigcup_{k=0}^{\infty} A_k(\Lambda) \tag{3}$$

where $A_k(\Lambda) \subset \Gamma^{(k)}(\Lambda)$ is defined as

$$A_k(\Lambda) := \{\gamma \in \Gamma^{(k)}(\Lambda) \colon |\gamma| = k, \exists (s_1, x_1), (s_2, x_2) \in \gamma \colon (x_1, x_2) \in D\}$$

By definition of of $\Gamma^{(k)}(\Lambda)$, there exists $\tilde{A}_k \in \mathcal{B}(\Lambda^k)$ such that

$$\tilde{A}_k = \mathrm{sym}_k^{-1}(A_k)$$

Since sym_k is measurable, we have $A_k(\Lambda) \in \mathcal{B}(\Gamma(\Lambda))$ and hence, $\Gamma_{\mathrm{p}}(\Lambda) \in \mathcal{B}(\Gamma(\Lambda))$.

Let us now consider a general $\Lambda \in \mathcal{B}(\mathbb{R}_+^* \times \mathbb{R}^d)$. We can find a covering $\{\Lambda_n\}_{n=1}^{\infty}$ of Λ with Λ_n compact and $\Lambda_n \subset \Lambda_{n+1}$ for all $n \in \mathbb{N}$. Then

$$\Gamma_{\mathrm{p}}(\Lambda) = \bigcap_{n=1}^{\infty} \{\gamma \in \Gamma(\Lambda) \colon \gamma_{\Lambda_n} \in \Gamma_{\mathrm{p}}(\Lambda_n)\} = \bigcap_{n=1}^{\infty} p_{\Lambda,\Lambda_n}^{-1}(\Gamma_p(\Lambda_n)). \tag{4}$$

Since p_{Λ,Λ_n} is $\mathcal{B}(\Gamma(\Lambda))$-$\mathcal{B}(\Gamma(\Lambda_n))$-measurable, we have $\Gamma_{\mathrm{p}}(\Lambda) \in \mathcal{B}(\Gamma(\Lambda))$. $\square$

We will see later that for the class of Poisson measures on $\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$, we have $\pi(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)) = 1$, which gives another justification that $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ is suitable for our considerations. Let us introduce a pairing on $\mathbb{K}(\mathbb{R}^d)$ which uses the reflection mapping.

Definition 2.16. *Let $f \in C_c(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\eta \in \mathbb{K}(\mathbb{R}^d)$. Define the following pairing:*

$$\langle\langle f, \eta \rangle\rangle := \langle f, \mathcal{R}^{-1}\eta \rangle = \sum_{(s,x) \in \mathcal{R}^{-1}\eta} f(s,x)$$

2.6 Topological and Metric Structures on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$

The Plato space $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ naturally inherits the topological structure of $\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$, i.e. the topology is given by the vague topology induced from the space of Radon measures $\mathbb{M}(\mathbb{R}_+^* \times \mathbb{R}^d)$. For a detailed description of topological and metric characterizations, see e.g. [32].

Remark 2.17. *The space $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ is not complete: Take for example some $x_0 \in \mathbb{R}^d$ and $s_1 \neq s_2 \in \mathbb{R}_+^*$. Furthermore, consider sequences $s_i^{(n)}, x_i^{(n)}, i = 1, 2$ with $s_1^{(n)} \neq s_2^{(n)}, x_1^{(n)} \neq x_2^{(n)}$ for all $n \in \mathbb{N}$ and*

$$s_i^{(n)} \to s_i, x_i^{(n)} \to x_i,\ n \to \infty, i = 1, 2.$$

Set

$$\begin{aligned} \gamma^{(n)} &:= \{(s_1^{(n)}, x_1^{(n)}), (s_2^{(n)}, x_2^{(n)})\} \in \Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \\ \gamma &:= \{(s_1, x_0), (s_2, x_0)\} \in \Gamma(\mathbb{R}_+^* \times \mathbb{R}^d) \setminus \Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \end{aligned}$$

Let $f \in C_c(\mathbb{R}_+^* \times \mathbb{R}^d)$. Then

$$\begin{aligned} |\langle f, \gamma^{(n)} \rangle - \langle f, \gamma \rangle| &= |f(s_1^{(n)}, x_1^{(n)}) + f(s_2^{(n)}, x_2^{(n)}) - f(s_1, x_0) - f(s_2, x_0)| \\ &\leq |f(s_1^{(n)}, x_1^{(n)}) - f(s_1, x_0)| + |f(s_2^{(n)}, x_2^{(n)}) - f(s_2, x_0)| \\ &\to 0,\ n \to \infty. \end{aligned}$$

Therefore, $\gamma^{(n)} \to \gamma,\ n \to \infty$ in $\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ is not complete.

2.7 Topology on the Cone $\mathbb{K}(\mathbb{R}^d)$.

From a naive point of view, it seems to make sense to consider the embedding $\mathbb{K}(\mathbb{R}^d) \subset \mathbb{M}(\mathbb{R}^d)$ of the cone into the space of Radon measures, equipped with the vague topology. Unfortunately, this topology has no relation to the vague topology introduced above on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$. In the spirit of Plato's theory of ideas, the connection between $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\mathbb{K}(\mathbb{R}^d)$ is essential. Therefore, we consider the final topology on $\mathbb{K}(\mathbb{R}^d)$ induced by the reflection mapping $\mathcal{R}$, i.e. the finest topology such that the mapping

$$\mathcal{R}\colon \Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \to \mathbb{K}(\mathbb{R}^d), \gamma = \sum_{(s_x,x)\in\gamma} \delta_{(s_x,x)} \mapsto \sum_{x\in\tau(\gamma)} s_x\delta_x$$

is continuous. Here, we set for $\gamma \in \Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$,

$$\tau(\gamma) := \{x \in \mathbb{R}^d \mid \exists s \in \mathbb{R}_+^*\colon (s,x) \in \gamma\}$$

the support of γ. The usage of this topology has the obvious side effect that $\mathcal{R}$ becomes a homeomorphism, which is helpful in and of itself in other regards. Some comments on the final topology can be found in Appendix A.

2.8 Measures on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\mathbb{K}(\mathbb{R}^d)$

The following chapter is devoted to the construction of a class of probability measures on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$, namely, Poisson measures. The construction is done on the larger space $\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$. For the class of Poisson measures, we show that they assign full mass to the Plato space $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$.

To obtain measures on $\mathbb{K}(\mathbb{R}^d)$, we use the pushforward of measures on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ under the mapping $\mathcal{R}$. A certain subclass of specific interest is the class of Gamma measures which will be introduced below. One technical step will be to show the compatibility of measurable structures on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\mathbb{K}(\mathbb{R}^d)$. Finally, we construct Gibbs measures on $\mathbb{K}(\mathbb{R}^d)$ as perturbations of Gamma measures.

The general construction of Poisson measures can also be found in e.g. [4]. For the construction of Gibbs measures, see [3] for homogeneous configuration spaces and [29] for the case considered here.

2.8.1 Construction of Poisson Measures on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$

In this section, we explicitly construct the class of Poisson measures on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$. This is done by constructing measures on $\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$ and restricting these measures to the subspace $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$. We also define a specific subclass, known as Gamma-Poisson measures. Furthermore, we show that this class assigns full mass to the Plato space, i.e. $\pi(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)) = 1$.

The general approach is to explicitly define a finite measure on $\Gamma(\Lambda)$ for any bounded volume $\Lambda \in \mathcal{B}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$, which is then normalised to obtain a probability measure on $\Gamma(\Lambda)$. Next, we use the consistency property of this family to show the existence of a probability measure on $\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$. This approach is well-known in literature, see e.g. [4].

Let ν be a Radon measure on the space $\mathbb{R}_+^*$ and σ a nonatomic Radon measure on $\mathbb{R}^d$ (e.g. the Lebesgue measure). Recall that by decomposition (1), we have

$$\Gamma(\Lambda) = \bigcup_{n=0}^{\infty} \Gamma^{(n)}(\Lambda) \simeq \bigcup_{n=0}^{\infty} \tilde{\Lambda}^n/S_n.$$

Since $(\nu \otimes \sigma)^{\otimes n}$ defines a measure on $\tilde{\Lambda}^n$, we see that $\varkappa_{n,\Lambda}^{\nu} := (\nu \otimes \sigma)^{\otimes n} \circ \mathrm{sym}_n^{-1}$ defines a measure on $\Gamma^{(n)}(\Lambda)$. We then proceed to define a finite measure on $\Gamma(\Lambda)$ as follows:

$$\lambda_{\nu,\sigma}^{\Lambda} := \sum_{n=0}^{\infty} \frac{1}{n!} \varkappa_{n,\Lambda}^{\nu},$$

where we set $\varkappa_{0,\Lambda}^{\nu} = \delta_{\{\emptyset\}}$. The measure defined this way is called the Lebesgue-Poisson measure with intensity $\nu \otimes \sigma$. It is easy to see that the full mass of $\lambda_{\nu,\sigma}^{\Lambda}$ is equal to $e^{\nu \otimes \sigma(\Lambda)}$. Hence, we obtain a probablity measure on $\Gamma(\Lambda)$ by setting

$$\pi_{\nu,\sigma}^{\Lambda} := e^{-\nu \otimes \sigma(\Lambda)} \lambda_{\nu,\sigma}^{\Lambda}.$$

This measure is also known as the Poisson measure on $\Gamma(\Lambda)$. If the intensity measures ν, σ are fixed, we may omit them in the notation. It can be shown that the family $\{\pi^{\Lambda}\}_{\Lambda \in \mathcal{B}_c(\mathbb{R}^d)}$ is consistent, i.e. for $\Lambda_2 \subset \Lambda_1, \Lambda_i \in \mathcal{B}_c(\mathbb{R}^d)$,

$$\pi^{\Lambda_2} = \pi^{\Lambda_1} \circ p_{\Lambda_1,\Lambda_2}^{-1}$$

By Kolmogorov's theorem for projective limits (see e.g. [48]), we obtain the existence of a probability measure for $\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$, which is the projective limit we constructed in Chapter 2.4.

Definition 2.18. *The measure given by the projective limit of the family* $\{\pi_{\nu,\sigma}^{\Lambda}\}_{\Lambda \in \mathcal{B}_c(\mathbb{R}^d)}$ *is called the Poisson measure on* $\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$ *with intensity measure* $\nu \otimes \sigma$*. It is denoted by*

$$\pi = \pi_{\nu} = \pi_{\nu,\sigma} = \pi_{\nu \otimes \sigma}.$$

Remark 2.19. *There exists an alternative definition of the Poisson measure given via its Laplace transform:* $\pi_{\nu \otimes \sigma}$ *is the unique measure such that the following holds for all functions* $\varphi \in C_c(\mathbb{R}_+^* \times \mathbb{R}^d)$*:*

$$\int_{\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)} e^{\langle \varphi, \gamma \rangle} \pi_{\nu \otimes \sigma}(d\gamma) = \exp\left(\int_{\mathbb{R}_+^* \times \mathbb{R}^d} e^{\varphi(s,x)} - 1 (\nu \otimes \sigma)(ds, dx) \right) \tag{5}$$

see e.g. [4].

There exists another characterization of the Poisson measure which is useful for the construction of measures on $\mathbb{K}(\mathbb{R}^d)$, since it yields a similar identity there. Denote by $\mathbb{M}_+(\mathbb{R}_+^* \times \mathbb{R}^d)$ the set of all nonnegative Radon measures on $\mathbb{R}_+^* \times \mathbb{R}^d$.

Proposition 2.20 (Mecke identity, [46, Satz 3.1])**.** *Let* μ *be a probability measure on* $\mathbb{M}_+(\mathbb{R}_+^* \times \mathbb{R}^d)$ *such that its local first moments exist, i.e.*

$$\int_{\mathbb{M}_+(\mathbb{R}_+^* \times \mathbb{R}^d)} \langle \mathbb{1}_{\Lambda}, \gamma \rangle \mu(d\gamma) = \int_{\mathbb{M}_+(\mathbb{R}_+^* \times \mathbb{R}^d)} \int_{\Lambda} 1 d\gamma \mu(d\gamma) < \infty \text{ for all } \Lambda \in \mathcal{B}_c(\mathbb{R}_+^* \times \mathbb{R}^d).$$

Then μ *is the Poisson measure with intensity* $\nu \otimes \sigma$ *if and only if the following equation holds for all nonnegative measurable functions* $F\colon \mathbb{R}_+^* \times \mathbb{R}^d \times \mathbb{M}_+(\mathbb{R}_+^* \times \mathbb{R}^d) \to \mathbb{R}_+$*:*

$$\int_{\mathbb{M}_+(\mathbb{R}_+^* \times \mathbb{R}^d)} \int_{\mathbb{R}_+^* \times \mathbb{R}^d} F(s,x,\gamma) \gamma(ds, dx) \pi_{\nu \otimes \sigma}(d\gamma) =$$
$$= \int_{\mathbb{R}_+^* \times \mathbb{R}^d} \int_{\mathbb{M}_+(\mathbb{R}_+^* \times \mathbb{R}^d)} F(s,x,\gamma + \delta_{(s,x)}) \pi_{\nu \otimes \sigma}(d\gamma) (\nu \otimes \sigma)(ds, dx)$$

Remark 2.21. *1. From now on, we assume that the first moment of ν exists, i.e.*

$$\int_{\mathbb{R}_+^*} s\nu(ds) < \infty. \tag{6}$$

2. We sometimes denote the intensity measure on $\mathbb{R}_+^* \times \mathbb{R}^d$ by $\varkappa(ds, dx) = \nu(ds) \otimes \sigma(dx)$.

The next definition gives a special subclass of Poisson measures, which will be examined in more detail.

Definition 2.22. *Consider the intensity measure on $\mathbb{R}_+^*$ given by the following expression:*

$$\nu_\theta(ds) = \theta \frac{1}{s} e^{-s} ds$$

where $\theta > 0$ is a fixed shape parameter. Obviously, ν_θ fufills assumption (6)*. The corresponding Poisson measure is called the Gamma-Poisson measure, denoted by π_θ. As above, we denote the projection to $\Gamma(\Lambda), \Lambda \in \mathcal{B}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$ by*

$$\pi_{\Lambda,\theta} := \pi_\theta \circ p_\Lambda^{-1}$$

Before we turn our attention to the specific properties of Gamma-Poisson measures, let us show a useful statement that holds for the general class of Poisson measures. Namely, a Poisson measure assign full mass to the set $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$. An explicit proof was given in e.g. [27]. We include the proof for completeness. We start by showing this property for the set of pinpointing configurations $\Gamma_{\mathrm{p}}(\mathbb{R}_+^* \times \mathbb{R}^d)$.

Theorem 2.23 ([27])**.** *Let π_ν be a Poisson measure such that ν fulfills* (6)*. Then we have $\pi_\nu(\Gamma_{\mathrm{p}}(\mathbb{R}_+^* \times \mathbb{R}^d)) = 1$.*

Proof. As in the proof of Lemma 2.15, denote by $D = \{(x,x) \colon x \in \mathbb{R}^d\} \in \mathcal{B}(\mathbb{R}^{2d})$ the diagonal set in $\mathbb{R}^{2d}$. Note that $(\sigma \otimes \sigma)(D) = 0$. Again, let $\{\Lambda_n\}_{n=1}^\infty$ be a covering of $\mathbb{R}_+^* \times \mathbb{R}^d$ with Λ_n compact and $\Lambda_n \subset \Lambda_{n+1}$ for all $n \in \mathbb{N}$. Furthermore, assume that $\Lambda_n = \Lambda_{n,\mathbb{R}_+^*} \times \Lambda_{n,\mathbb{R}^d}$ for each n. Using representation (4), we obtain

$$\pi_\nu(\Gamma_{\mathrm{p}}^c(\mathbb{R}_+^* \times \mathbb{R}^d)) = \pi_\nu\left(\bigcup_{n\in\mathbb{N}} p_{\Lambda_n}^{-1}(\Gamma_{\mathrm{p}}(\Lambda_n)^c)\right) \leq \sum_{n=1}^\infty \pi_\nu\left(p_{\Lambda_n}^{-1}(\Gamma_{\mathrm{p}}(\Lambda_n)^c)\right).$$

Since for any compact Λ, we have

$$\pi_\nu\left(p_\Lambda^{-1}(\Gamma_{\mathrm{p}}(\Lambda)^c)\right) = e^{-\nu\otimes\sigma(\Lambda)} \lambda_\nu\left(\Gamma_{\mathrm{p}}(\Lambda)^c\right),$$

it is enough to show that for any $\Lambda \in \mathcal{B}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$ with above assumptions, we have

$$\lambda_\nu\left(\Gamma_{\mathrm{p}}(\Lambda)^c\right) = 0.$$

Using decomposition (3) and the product structure of Λ, we see

$$\begin{aligned}
\lambda_\nu\left(\Gamma_{\mathrm{p}}(\Lambda)^c\right) &\leq \sum_{n=1}^\infty \lambda_\nu\left(\{\gamma \in \Gamma^{(k)}(\Lambda) \mid \exists (s_1, x_1), (s_2, x_2) \in \gamma \colon (x_1, x_2) \in D\}\right) \\
&= \sum_{k=0}^\infty \frac{1}{k!} (\nu \otimes \sigma)^{\otimes k}\left(\{(s_i, x_i)\}_{i=1}^k \subset \Lambda \mid \exists i \neq j \colon (x_i, x_j) \in D\}\right) \\
&\leq \sum_{k=2}^\infty \frac{1}{k!} \binom{k}{2} \left[(\nu \otimes \sigma)^{\otimes(k-2)}(\Lambda^{k-2})\right] \cdot \left[(\sigma \otimes \sigma)(D)\right] \cdot \nu(\Lambda_{\mathbb{R}_+^*}) \\
&= 0
\end{aligned}$$

$\square$

Next, let us show that $\gamma(\Lambda) < \infty$ for π_ν-almost all $\gamma \in \Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$, which implies $\pi_\nu(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)) = 1$.

Theorem 2.24 ([27])**.** *For π_ν as above, $\pi_\nu(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)) = 1$.*

Proof. For any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, we have

$$\begin{aligned}\int_{\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)} \gamma(\Lambda)\pi_\nu(d\gamma) &= \int_{\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)} \langle s \otimes \mathbb{1}_\Lambda(x), \gamma\rangle \pi_\nu(d\gamma) \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}_+^*} s\mathbb{1}_\Lambda(x)\nu(ds)\sigma(dx) = \sigma(\Lambda)\int_{\mathbb{R}^d} s\nu(ds) < \infty\end{aligned}$$

which implies our claim. Note that it is essential that $(s \mapsto s) \in L^1(\mathbb{R}_+^*, d\nu)$. $\square$

Considering the above result, we may consider π_ν as a probability measure on the space $(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d), \mathcal{B}(\Pi))$, where $\mathcal{B}(\Pi) = \mathcal{B}(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d))$ is the trace-σ-algebra, i.e.

$$\mathcal{B}(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)) := \left\{A \cap \Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \mid A \in \mathcal{B}(\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d))\right\}.$$

2.8.2 Probability Measures on $\mathbb{K}(\mathbb{R}^d)$

We are now ready to consider two important classes of probability measures on $\mathbb{K}(\mathbb{R}^d)$. First, we introduce the class of Gamma measures, denoted by $\mathcal{G}_\theta$, which are the image of the Gamma-Poisson measures on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ under the mapping $\mathcal{R}$. Next, we consider Gibbs measures on $\mathbb{K}(\mathbb{R}^d)$, which are given through perturbations of said Gamma measures.

In the previous chapter, we established the relation between $\mathbb{K}(\mathbb{R}^d)$ and $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$. Therefore, it makes sense to further investigate the relation given via the mapping $\mathcal{R}$. For instance, we may show the relation between the σ-algebras $\mathcal{B}(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d))$ and $\mathcal{B}(\mathbb{K}(\mathbb{R}^d))$.

Theorem 2.25. *The image σ-algebra of $\mathcal{B}(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d))$ under $\mathcal{R}$ and $\mathcal{B}(\mathbb{K}(\mathbb{R}^d))$ coincide, i.e.*

$$\mathcal{B}(\mathbb{K}(\mathbb{R}^d)) = \left\{\mathcal{R}(A \cap \Pi(\mathbb{R}_+^* \times \mathbb{R}^d)) \mid A \in \mathcal{B}(\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d))\right\}$$

Proof. The proof is a direct consequence of the topological considerations found in Appendix A. $\square$

Since we established the connection between $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\mathbb{K}(\mathbb{R}^d)$, let us come to the definition of Gamma measures on $\mathbb{K}(\mathbb{R}^d)$.

Definition 2.26. *Let π_θ be a Gamma-Poisson measure on the space $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$. Then the Gamma measure on $\mathbb{K}(\mathbb{R}^d)$ is defined as the image measure of π_θ, i.e. for any bounded and measurable function $F\colon \mathbb{K}(\mathbb{R}^d) \to \mathbb{R}$,*

$$\int_{\mathbb{K}(\mathbb{R}^d)} F(\eta)\mathcal{G}_\theta(d\eta) = \int_{\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)} F(\mathcal{R}\gamma)\pi_\theta(d\gamma)$$

we denote the Gamma measure by $\mathcal{G} = \mathcal{G}_\nu = \mathcal{G}_\theta$.

Remark 2.27. *1. For a class of cylindrical functions, we get the following explicit formula: Let $g \in C_b^\infty(\mathbb{R}^N)$ for some $N \in \mathbb{N}$ and $\varphi_1, \dots, \varphi_N \in C_c^\infty(\mathbb{R}^d)$. Then*

$$\int_{\mathbb{K}(\mathbb{R}^d)} g\left(\langle \varphi_1, \eta\rangle, \dots, \langle \varphi_N, \eta\rangle\right) \mathcal{G}_\theta(d\eta)$$
$$= \int_{\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)} g\left(\langle \mathrm{id} \otimes \varphi_1, \gamma\rangle, \dots, \langle \mathrm{id} \otimes \varphi_N, \gamma\rangle\right) \pi_\theta(d\gamma),$$

where $\mathrm{id} \otimes \varphi(s,x) := s\varphi(x)$.

2. Similar to the case of $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$, there is an alternative characterization of $\mathcal{G}_\theta$ via its Laplace transform: For any $\varphi \in C_c(\mathbb{R}^d)$,

$$\int_{\mathbb{K}(\mathbb{R}^d)} e^{\langle \varphi, \eta\rangle} \mathcal{G}_\theta(d\eta) = \exp\left(\int_{\mathbb{R}_+^* \times \mathbb{R}^d} e^{s\varphi(x)} - 1(\nu \otimes \sigma)(ds, dx)\right)$$

this is easily seen by using Remark 2.19 and the definition of the Gamma measure.

Also, the Gamma measures admit a Mecke-type characterization similar to Proposition 2.20.

Proposition 2.28 ([29])**.** *Let μ be a probability measure on $\mathbb{M}_+(\mathbb{R}^d)$ which has finite first local moments, i.e. for any $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$, we have*

$$\int_{\mathbb{M}_+(\mathbb{R}^d)} \eta(\Delta)\mu(d\eta) < \infty. \tag{7}$$

Then $\mu = \mathcal{G}_\theta$ if and only if for any measurable function $F\colon \mathbb{R}^d \times \mathbb{M}_+(\mathbb{R}^d) \to \mathbb{R}_+$, we have

$$\int_{\mathbb{M}_+(\mathbb{R}^d)} \int_{\mathbb{R}^d} F(x,\eta)\eta(dx)\mu(d\eta) = \int_{\mathbb{M}_+(\mathbb{R}^d)} \int_{\mathbb{R}^d} \int_{\mathbb{R}_+^*} sF(x, \eta + s\delta_x)\nu_\theta(ds)\sigma(dx)\mu(d\eta). \tag{8}$$

Proof. "$\Rightarrow$": As a first step, let us consider functions of the form

$$F(x,\eta) = f(x) g\left(\langle \varphi_1, \eta\rangle, \dots, \langle \varphi_N, \eta\rangle\right), \tag{9}$$

where $f, \varphi_1, \dots, \varphi_N \in C_c(\mathbb{R}^d)$ and $g \in C_c(\mathbb{R}^N)$. We then rewrite the left-hand-side of (8) recalling that $\mu = \mathcal{G}_\theta$:

$$\begin{aligned}
&\int_{\mathbb{M}_+(\mathbb{R}^d)} \int_{\mathbb{R}^d} F(x,\eta)\eta(dx)\mu(d\eta) = \\
&\quad= \int_{\mathbb{K}(\mathbb{R}^d)} \int_{\mathbb{R}^d} f(x) g(\langle \varphi_1, \eta\rangle, \dots, \langle \varphi_N, \eta\rangle)\eta(dx)\mathcal{G}_\theta(d\eta) \\
&\quad= \int_{\mathbb{K}(\mathbb{R}^d)} \langle f, \eta\rangle g(\langle \varphi_1, \eta\rangle, \dots, \langle \varphi_N, \eta\rangle)\mathcal{G}_\theta(d\eta) \\
&\quad= \int_{\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)} \langle \mathrm{id} \otimes f, \gamma\rangle g(\langle \mathrm{id} \otimes \varphi_1, \gamma\rangle, \dots, \langle \mathrm{id} \otimes \varphi_N, \gamma\rangle)\pi_\theta(d\gamma) \\
&\quad= \int_{\mathbb{R}^d} \int_{\mathbb{R}_+^*} \int_{\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)} sf(x) \times \\
&\qquad \times g(\langle \mathrm{id} \otimes \varphi_1, \gamma + \delta_{(s,x)}\rangle, \dots, \langle \mathrm{id} \otimes \varphi_N, \gamma + \delta_{(s,x)}\rangle)\pi_\theta(d\gamma)\nu_\theta(ds)\sigma(dx) \\
&\quad= \int_{\mathbb{R}^d} \int_{\mathbb{R}_+^*} \int_{\mathbb{K}(\mathbb{R}^d)} sf(x) g(\langle \varphi_1, \eta + s\delta_x\rangle, \dots, \langle \varphi_N, \eta + s\delta_x\rangle)\mathcal{G}_\theta(d\gamma)\nu_\theta(ds)\sigma(dx) \\
&\quad= \int_{\mathbb{K}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \int_{\mathbb{R}_+^*} sF(x, \eta + s\delta_x)\nu_\theta(ds)\sigma(dx)\mathcal{G}_\theta(d\eta)
\end{aligned}$$

By the monotone class theorem and approximation arguments, this identity can be extended to all nonnegative measurable functions.

"$\Leftarrow$": For the other direction, we show that the Laplace transform of ν_θ with (8) coincides with (5). To this end, let $\varphi \in C_c(\mathbb{R}^d)$ with $\varphi \geq 0$ and set

$$L(t) := \int_{\mathbb{M}_+(\mathbb{R}^d)} \exp\left(-t\langle \varphi, \eta\rangle\right) \mu(d\eta).$$

Note the following two properties of this function:

L **is strictly positive:** By Jensen's inequality, we see using (7) that

$$\begin{aligned}
\int_{\mathbb{M}_+(\mathbb{R}^d)} \exp(-t\langle \varphi, \eta\rangle)\mu(d\eta) &\geq \exp\left(-t \int_{\mathbb{M}_+(\mathbb{R}^d)} \langle \varphi, \eta\rangle \mu(d\eta)\right) \\
&\geq \exp\left(-t \int_{\mathbb{M}_+(\mathbb{R}^d)} \|\varphi\|_\infty \eta(\operatorname{supp}\varphi)\mu(d\eta)\right) \\
&> 0
\end{aligned}$$

Differentiability of L: L is continuous on $[0, \infty)$ and continuously differentiable on $(0, \infty)$. Since μ is a probability measure and $\exp(-t\langle \varphi, \cdot\rangle)$ is bounded, this assertion follows by Lebesgue's Theorem.

If we differentiate $L(t)$, we obtain

$$\begin{aligned}
\frac{d}{dt} L(t) &= -\int_{\mathbb{M}_+(\mathbb{R}^d)} \langle \varphi, \eta\rangle \exp(-t\langle \varphi, \eta\rangle)\mu(d\eta) \\
&\overset{(8)}{=} -\int_{\mathbb{R}^d}\int_{\mathbb{R}_+^*}\int_{\mathbb{M}_+(\mathbb{R}^d)} s\varphi(x) \exp(-t\langle \varphi, \eta + s\delta_x\rangle)\mu(d\eta)\nu_\theta(ds)\sigma(dx) \\
&= -\int_{\mathbb{R}^d}\int_{\mathbb{R}_+^*} s\varphi(x) e^{-ts\varphi(x)} \int_{\mathbb{M}_+(\mathbb{R}^d)} \exp(-t\langle \varphi, \eta\rangle)\mu(d\eta)\nu_\theta(ds)\sigma(dx)
\end{aligned}$$

Therefore, L satisfies the initial value problem

$$\begin{cases} \frac{d}{dt} L(t) & = -C(t)L(t) \\ L(0) & = 1 \end{cases} \tag{10}$$

where

$$C(t) = \int_{\mathbb{R}^d}\int_{\mathbb{R}_+^*} s\varphi(x) e^{-ts\varphi(x)} \nu_\theta(ds)\sigma(dx).$$

The ODE (10) has the following unique solution:

$$L(t) = L(0) \exp\left(-\int_0^t C(u)du\right) = \exp\left(-\int_{\mathbb{R}^d}\int_{\mathbb{R}_+^*} e^{-ts\varphi(x)}\nu_\theta(ds)\sigma(dx)\right)$$

Setting $t = 1$, we see that $L(1)$ is exactly the Laplace transform of $\mathcal{G}_\theta$. This holds for all $\varphi \in C_c(\mathbb{R}^d)$ with $\varphi \geq 0$, and therefore, $\mu = \mathcal{G}_\theta$.

$\square$

Remark 2.29. *Since it is needed in the construction of Gibbs measures, we want to mention the independence property of Gamma measures: For a collection of disjoint sets* $\Delta_1, \ldots, \Delta_N \in \mathcal{B}_c(\mathbb{R}^d)$*, the random variables* $\eta(\Delta_1), \ldots, \eta(\Delta_N)$ *are independent. In other words,*

$$\int_{\mathbb{K}(\mathbb{R}^d)} \prod_{i=1}^{N} \varphi_i\left(\eta(\Delta_i)\right) \mathcal{G}_\theta(d\eta) = \prod_{i=1}^{N} \int_{\mathbb{K}(\mathbb{R}^d)} \varphi_i(\eta(\Delta_i)) \mathcal{G}_\theta(d\eta) \tag{11}$$

for any $\varphi_i \in L^\infty(\mathbb{R}^d)$*,* $i = 1, \ldots, N$*.*

Typical configurations under $\mathcal{G}_\theta$ have the interesting property of having dense support $\tau(\eta)$.

Proposition 2.30. *For any compact set* $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$ *with* $\sigma(\Delta) > 0$*, we have*

$$\mathcal{G}_\theta(\{\eta \in \mathbb{K}(\mathbb{R}^d) \colon |\tau(\eta) \cap \Delta| = n\}) = 0 \ \forall n \in \mathbb{N}_0$$

Proof. For the proof, recall that the Poisson measure π_θ on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ with intensity measure $\varkappa_\theta = \nu_\theta \otimes \sigma$ has the following representation:

$$\pi_\theta(\{\gamma \in \Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \colon |\gamma \cap \tilde{\Lambda}| = n\}) = \frac{\varkappa_\theta(\tilde{\Lambda})}{n!} e^{-\varkappa_\theta(\tilde{\Lambda})} \tag{12}$$

for any compact set $\tilde{\Lambda} \in \mathcal{B}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$. Now, set $\Lambda = \mathbb{R}_+^* \times \Delta$. We have the following relation between $\mathcal{G}_\theta$ and π_θ:

$$\begin{aligned}
\mathcal{G}_\theta(\{\eta \in \mathbb{K}(\mathbb{R}^d) \colon |\tau(\eta_\Delta)| = n\}) &= \int_{\mathbb{K}(\mathbb{R}^d)} \mathbb{1}_{\{|\tau(\eta_\Delta)| = n\}}(\eta) \mathcal{G}_\theta(d\eta) \\
&= \int_{\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)} \mathbb{1}_{\{|\gamma_\Lambda| = n\}}(\gamma) \pi_\theta(d\gamma) \\
&= \pi_\theta(\underbrace{\{\gamma \in \Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \colon |\gamma \cap \Lambda| = n\}}_{=:A})
\end{aligned}$$

We have the following decomposition of A:

$$A = \bigcup_{j_0 \in \mathbb{N}} \bigcap_{j \geq j_0} \{\gamma \in \Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \colon |\gamma \cap \Lambda_j| = n\}$$

where we set

$$\Lambda_j = \Delta \times \left[\frac{1}{j}, j\right] \in \mathcal{B}_c(\mathbb{R}_+^* \times \mathbb{R}^d).$$

Applying (12), we obtain

$$\begin{aligned}
\pi_\theta(A) &= \pi_\theta\left(\bigcup_{j_0 \in \mathbb{N}} \bigcap_{j \geq j_0} \{\gamma \in \Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \colon |\gamma \cap \Lambda_j| = n\}\right) \\
&= \lim_{j_0 \to \infty} \pi_\theta\left(\bigcap_{j \geq j_0} \{\gamma \in \Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \colon |\gamma \cap \Lambda_j| = n\}\right) \\
&\leq \lim_{j_0 \to \infty} \pi_\theta\left(\{\gamma \in \Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \colon |\gamma \cap \Lambda_j| = n\}\right) \\
&= \lim_{j_0 \to \infty} \frac{\varkappa_\theta(\Lambda_j)}{n!} e^{-\varkappa_\theta(\Lambda_j)} = 0
\end{aligned}$$

Note that the property $\nu_\theta(\mathbb{R}_+^*) = \infty$ is crucial to obtain the density of $\tau(\eta)$. $\square$

2.8.3 Gibbs Measures on $\mathbb{K}(\mathbb{R}^d)$

Gibbs measures play an important role in the analysis of a given particle system. They indicate invariant states and show the existence of phase transitions within the system. For some basic notions related to Gibbs measures, see e.g. the works [25, 26].

Since we are working with particle systems on an infinite volume, there are some technical steps needed in the construction of Gibbs measures. This chapter provides an overview of the construction of Gibbs measures as perturbations of Gamma measures on the cone. Furthermore, some technical differences to the classical construction on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ are discussed. The main source of this chapter is [29]. For a more detailed discussion of Gibbs measures on configuration spaces and the cone, see [10, 11]. The general construction will follow the approach of Dobrushin, Lanford and Ruelle, see e.g. [26].

For the construction of Gibbs measures, we need to consider a class of admissible pair potentials. Consider a pair potential

$$\phi\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$$

which is assumed to be measurable, symmetric and bounded. Set the following:

$$\begin{aligned}
\|\phi^-\|_\infty &:= \sup_{x,y\in\mathbb{R}^d} \left(\max\{-\phi(x,y), 0\}\right) \leq 0 \\
\|\phi\|_\infty &:= \sup_{x,y\in\mathbb{R}^d} |\phi(x,y)| < \infty
\end{aligned}$$

We impose the following conditions on the potential ϕ:

Finite range condition (FR): There exists $R > 0$ such that

$$\phi(x,y) = 0 \text{ if } |x-y| > R$$

Repulsion condition (RC): There exists $\delta > 0$ such that

$$\inf_{|x-y|\leq\delta} \phi(x,y) > 2m_\delta^\phi \|\phi^-\|$$

where $m_\delta^\phi > 0$ is an explicitly given constant. Heuristically, (RC) means that the repulsion of two particles close to each other is stronger than the global attraction.

Under these conditions, we can define the relative Hamiltonian of the system.

Definition 2.31. *Fix $\eta, \xi \in \mathbb{K}(\mathbb{R}^d)$ and $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$. Then the relative energy or Hamiltonian is given by*

$$\begin{aligned}
H_\Delta(\eta \mid \xi) &= \int_\Delta \int_\Delta \phi(x,y)\eta(dx)\eta(dy) + 2\int_{\Delta^c}\int_\Delta \phi(x,z)\eta(dx)\xi(dz) \\
&= \sum_{x,y\in\tau(\eta)\cap\Delta} \phi(x,y)s_x s_y + 2\sum_{\substack{x\in\tau(\eta)\cap\Delta \\ z\in\tau(\xi)\cap\Delta^c}} \phi(x,z)s_x s_z
\end{aligned}$$

The element ξ may bee seen as the boundary conditions for the localised particle system $\eta \cap \Delta$.

Remark 2.32. *The reasoning why the conditions (FR) and (RC) yield a suitable potential is quite technical and involves a partition of $\mathbb{R}^d$ in appropriate cubes related to the interaction of ϕ. This partition is then used to show finiteness and boundedness from below of H_Δ. A detailed anaylsis is given in [29].*

The construction starts locally as a perturbation of Gamma measures. Therefore, we need to localise the notion of Gamma measures. Furthermore, we need to normalise the constructed measure using the corresponding partition function. After constructing a local specification, we may extend the underlying set to obtain a Gibbs measure on the whole space.

Definition 2.33. *Let $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$. For a fixed Gamma measure $\mathcal{G}_\theta$, define the corresponding local Gamma measure on Δ as*

$$\mathcal{G}_{\Delta,\theta} := \mathcal{G}_\theta \circ p_\Delta^{-1}$$

where $p_\Delta \colon \mathbb{K}(\mathbb{R}^d) \to \mathbb{K}(\Delta)$ is defined via

$$\eta = \sum_{x \in \tau(\eta)} s_x \delta_x \mapsto \sum_{x \in \tau(\eta) \cap \Delta} s_x \delta_x.$$

For $\xi \in \mathbb{K}(\mathbb{R}^d)$, define the partition function as

$$Z_\Delta(\xi) := \int_{\mathbb{K}(\Delta)} \exp\left[-H_\Delta(\eta \mid \xi)\right] \mathcal{G}_{\Delta,\theta}(d\eta)$$

The partition function will serve as the normalising constant for our local Gibbs measure. Furthermore, note that the definition depends on the choice of Gamma measure. It can be shown that under the assumptions (FR) and (RC), the partition function serves the right purpose:

Lemma 2.34 ([29, Lemma 3.3])**.** *Under assumptions (FR) and (RC), for any $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$ and $\xi \in \mathbb{K}(\mathbb{R}^d)$, we have*

$$0 < Z_\Delta(\xi) < \infty$$

where $Z_\Delta(\xi) \leq 1$, if additionally $\phi \geq 0$.

Given the partition function, we can now proceed to define the local Gibbs measures.

Definition 2.35. *For any $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$, define the local Gibbs measure with boundary condition $\xi \in \mathbb{K}(\mathbb{R}^d)$ as*

$$\mu_\Delta(d\eta \mid \xi) := \frac{1}{Z_\Delta(\xi)} e^{-H_\Delta(\eta \mid \xi)} \mathcal{G}_{\Delta,\theta}(d\eta)$$

Remark 2.36.

- *Note that Lemma 2.34 is needed to make sense of this definition.*
- *For fixed $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$ and $\xi \in \mathbb{K}(\mathbb{R}^d)$, the measure $\mu_\Delta(d\eta \mid \xi)$ is a probability measure on $\mathbb{K}(\Delta)$.*

For the construction of the desired Gibbs measure, we need to "lift" the family of measures μ_Δ to the space $\mathbb{K}(\mathbb{R}^d)$. This is done in the following definition.

Definition 2.37. *Define the local specification* $\Pi = \{\pi_\Delta\}_{\Delta \in \mathcal{B}_c(\mathbb{R}^d)}$ *as a family of stochastic kernels*

$$\pi_\Delta\colon \mathcal{B}(\mathbb{K}(\mathbb{R}^d)) \times \mathbb{K}(\mathbb{R}^d) \to [0,1]$$
$$\pi_\Delta(B \mid \xi) := \mu_\Delta(B_{\Delta,\xi} \mid \xi)$$

where

$$B_{\Delta,\xi} := \{\eta \in \mathbb{K}(\Delta) \mid \eta + \xi_{\Delta^c} \in B\}$$

By using the structure of H_Δ and Property (11), we see that the family Π is consistent:

$$\int_{\mathbb{K}(\mathbb{R}^d)} \pi_{\tilde{\Delta}}(B \mid \eta)\pi_\Delta(d\eta \mid \xi) = \pi_\Delta(B \mid \xi),\ \Delta, \tilde{\Delta} \in \mathcal{B}_c(\mathbb{R}^d),\ \tilde{\Delta} \subset \Delta$$

Heuristically, if we let Δ "grow" to the whole space $\mathbb{R}^d$, the boundary condition disappears and we obtain the Dobrushin-Lanford-Ruelle (DLR) equation as definition for Gibbs measures.

Definition 2.38. *A probability measure* μ *on* $\mathbb{K}(\mathbb{R}^d)$ *is called a Gibbs measure with pair potential* ϕ *if it satisfies the DLR equilibrium equation*

$$\int_{\mathbb{K}(\mathbb{R}^d)} \pi_\Delta(B \mid \eta)\mu(d\eta) = \mu(B)$$

for all $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$ *and* $B \in \mathcal{B}(\mathbb{K}(\mathbb{R}^d))$*. The set of all Gibbs measures with respect to a given potential* ϕ *is denoted by* $G(\phi)$*.*

The rest of this subchapter is devoted to the outline of the proof of existence of such Gibbs measures given in [29]. In fact, it can even be shown that there exists a special subclass of $G(\phi)$, known as tempered Gibbs measures. It is defined as follows:

Consider the following space

$$\mathbb{K}_\alpha(\mathbb{R}^d) := \{\eta \in \mathbb{K}(\mathbb{R}^d) \mid M_\alpha(\eta) < \infty\},$$

where

$$M_\alpha(\eta) := \left(\sum_{k \in \mathbb{Z}^d} \eta(Q_k)^2 e^{-\alpha|k|}\right)^{\frac{1}{2}}$$

and Q_k is the cube centered at $k \in \mathbb{Z}^d$ with edge length $\delta/\sqrt{d} > 0$, where δ was given by (RC).

Definition 2.39. *Define the set of all tempered discrete Radon measures by*

$$\mathbb{K}^t(\mathbb{R}^d) := \bigcap_{\alpha > 0} \mathbb{K}_\alpha(\mathbb{R}^d).$$

Then the set of all tempered Gibbs measures are all Gibbs measures concentrated on $\mathbb{K}^t(\mathbb{R}^d)$*, i.e.*

$$G^t(\phi) := G(\phi) \cap \mathcal{P}(\mathbb{K}^t(\mathbb{R}^d))$$

where $\mathcal{P}(\mathbb{K}^t(\mathbb{R}^d))$ *denotes the set of probability measures over* $\mathbb{K}^t(\mathbb{R}^d)$*.*

Remark 2.40. *1. The idea of the existence proof relies on finding an appropriate topology in which we can find a sequence $\{\pi_{\Delta_N}(\cdot \mid \xi)\}_{N\in\mathbb{N}}$ from the local specification Π which converges to a probability measure on $\mathbb{K}^t(\mathbb{R}^d)$. Secondly, the DLR equation needs to be verified.*

2. Note that classic approaches like the one Ruelle used are not applicable here due to the structure of our underlying space. Of course, the existence of Gibbs measures on $\mathbb{K}^t(\mathbb{R}^d)$ can be translated to the existence on $\Gamma(Y)$ via the mapping $\mathcal{R}$. But the techniques used for $\Gamma(Y)$ rely on, for example, a uniform integrability condition which is not given here (see e.g. [3]): For $\Gamma(Y)$, our potential translates to

$$\phi_\Gamma(s,x,t,y) = ts\phi(x,y),$$

for which we have

$$\operatorname*{ess\,sup}_{(s,x)\in\mathbb{R}_+^*\times\mathbb{R}^d} \int_{\mathbb{R}_+^*}\int_{\mathbb{R}^d} |e^{-st\phi(x,y)} - 1|\nu_\theta(dt)\otimes\sigma(dy) = \infty$$

The following identity is useful for calculations regarding Gibbs measures. It first appeared in [25] and [47]. The version used in this work can be found in [29].

Proposition 2.41 (Georgii-Nguyen-Zessin identity, GNZ)**.** *Let $F\colon \mathbb{R}^d\times\mathbb{K}(\mathbb{R}^d)\to\mathbb{R}_+$ measurable and $\mu\in G(\phi)$. Then*

$$\begin{aligned}\int_{\mathbb{K}(\mathbb{R}^d)}\int_{\mathbb{R}^d} & F(x,\eta)\eta(dx)\mu(d\eta)\\ &= \int_{\mathbb{K}(\mathbb{R}^d)}\int_{\mathbb{R}_+^*\times\mathbb{R}^d} F(x,\eta+s\delta_x)e^{-\Phi((s,x);\eta)}s\nu_\theta(ds)\sigma(dx)\mu(d\eta)\end{aligned} \tag{13}$$

where for $\eta := (s_y,y)_{y\in\tau(\eta)}\in\mathbb{K}(\mathbb{R}^d)$

$$\Phi\left((s,x);\eta\right) := 2s\sum_{y\in\tau(\eta)} s_y\phi(x,y)$$

Proof. As before, we consider functions of the form (9). For short, we just write $F(x,\eta) = f(x)g(\eta)$. Assume that $\operatorname{supp} f \subset \operatorname{supp}\varphi_i \subset \Delta$ for all i and some $\Delta\in\mathcal{B}_c(\mathbb{R}^d)$. Since the Gibbs measures are defined via Gamma measures and the DLR equations, it makes sense to make use of that in the proof. Therefore, for μ Gibbs measure,

$$\begin{aligned}\int_{\mathbb{K}(\mathbb{R}^d)}\int_{\mathbb{R}^d} & F(x,\eta)\eta(dx)\mu(d\eta)\\ &= \int_{\mathbb{K}(\mathbb{R}^d)} \langle f,\eta_\Delta\rangle g(\eta_\Delta)\mu(d\xi)\\ &= \int_{\mathbb{K}(\mathbb{R}^d)}\int_{\mathbb{K}(\mathbb{R}^d)} \langle f,\eta_\Delta\rangle g(\eta_\Delta)\pi_\Delta(d\eta\mid\xi)\mu(d\xi)\\ &= \int_{\mathbb{K}(\mathbb{R}^d)}\int_{\mathbb{K}(\Delta)} \langle f,\eta_\Delta\rangle g(\eta_\Delta)\frac{1}{Z_\Delta(\xi)}e^{-H(\eta_\Delta|\xi_{\Delta^c})}\mathcal{G}_{\Delta,\theta}(d\eta_\Delta)\mu(d\xi)\\ &= \int_{\mathbb{K}(\mathbb{R}^d)}\int_{\mathbb{K}(\Delta)}\int_\Delta\int_{\mathbb{R}_+^*} f(x)g(\eta_\Delta+s\delta_x)\frac{1}{Z_\Delta(\xi)}\times\end{aligned}$$

$$
\times e^{-H(\eta_\Delta + s\delta_x | \xi_{\Delta^c})} s\nu_\theta(ds)\sigma(dx)\mathcal{G}_{\Delta,\theta}(d\eta)\mu(d\xi)
$$

$$
= \int_\Delta \int_{\mathbb{R}_+^*} \int_{\mathbb{K}(\mathbb{R}^d)} \int_{\mathbb{K}(\Delta)} F(x, \eta_\Delta + s\delta_x) \frac{1}{Z_\Delta(\xi)} e^{-H(\eta_\Delta | \xi_{\Delta^c})} \times
$$

$$
\times e^{-\Phi((s,x);\eta_\Delta + \xi_{\Delta^c})} \mathcal{G}_{\Delta,\theta}(d\eta_\Delta)\mu(d\xi) s\nu_\theta(ds)\sigma(dx)
$$

$$
= \int_\Delta \int_{\mathbb{R}_+^*} \int_{\mathbb{K}(\mathbb{R}^d)} \int_{\mathbb{K}(\mathbb{R}^d)} F(x, \eta + s\delta_x) e^{-\Phi((s,x);\eta)} \pi_\Delta(\eta \mid \xi)\mu(d\xi) s\nu_\theta(ds)\sigma(dx)
$$

$$
= \int_\Delta \int_{\mathbb{R}_+^*} \int_{\mathbb{K}(\mathbb{R}^d)} F(x, \eta + s\delta_x) e^{-\Phi((s,x);\eta)} \mu(d\xi) s\nu_\theta(ds)\sigma(dx)
$$

$$
= \int_{\mathbb{K}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \int_{\mathbb{R}_+^*} F(x, \eta + s\delta) e^{-\Phi((s,x);\eta)} s\nu_\theta(ds)\sigma(dx)\mu(d\eta)
$$

The claim for general F follows again by a monotone class argument and approximation.

□

2.9 Harmonic Analysis on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$

Due to the infinite-dimensional nonlinear structure of the considered spaces, the dynamics modeled on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ are rather difficult to analyse directly. Instead, we intend to rewrite equations on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ to the space of finite configurations $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d) \subset \Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$. This can be done using the so-called K-transform, which will be introduced in this chapter. Furthermore, we show relations between functions on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$. The approach used here is well-known in the theory of statistical physics. In the homogeneous situation, the theory can be found in [30]. We start with the space $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$.

2.9.1 The K-Transform

As noted above, we want to introduce our auxiliary space which is related to the Plato space via the K-transform.

Definition 2.42. *The Plato space of finite configurations $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$ is defined as*

$$
\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d) := \{\gamma \in \Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \mid |\gamma| < \infty\}
$$

where $|\cdot|$ denotes the number of elements in a set. Its topology is induced by the set $\Gamma_0(\mathbb{R}_+^* \times \mathbb{R}^d)$, see Chapter 2.4.2.

Remark 2.43. *While $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d) \subset \Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ as a set, it fulfills a different role than $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$. $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ is the space which stands for the ideas or forms of the "real" physical system, while the set $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$ is seen as a mathematical construct besides $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$. Furthermore, the topological properties are entirely different, which will become clear in this chapter.*

For technical purposes, we may introduce subspaces of $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$ which are used to decompose the space.

Definition 2.44. *1. For $n \in \mathbb{N}_0$, the set of n-point configurations is defined as*

$$
\Pi_0^{(n)}(\mathbb{R}_+^* \times \mathbb{R}^d) := \left\{\gamma \in \Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d) \colon |\gamma| = n\right\}
$$

2. *For a set $\Lambda \subset \mathbb{R}_+^* \times \mathbb{R}^d$, the set of all configurations supported in Λ is defined as*

$$\Pi_0(\Lambda) := \left\{\gamma \in \Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d) \colon \gamma \subset \Lambda\right\}$$

3. *A Borel set $A \subset \Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$ is called bounded if there exists $\Lambda \subset \mathbb{R}_+^* \times \mathbb{R}^d$ compact and $N \in \mathbb{N}$ such that*

$$A \subset \bigcup_{n=0}^{N} \Pi_0^{(n)}(\Lambda).$$

Denote the system of all such sets by $\mathcal{B}_b(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$.

Note that we have the following decompositions:

$$\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d) = \bigsqcup_{n=0}^{\infty} \Pi_0^{(n)}(\mathbb{R}_+^* \times \mathbb{R}^d) = \bigcup_{\Lambda \in \mathcal{B}_c(\mathbb{R}_+^* \times \mathbb{R}^d)} \Pi_0(\Lambda)$$

where the first union is disjoint and $\mathcal{B}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$ denotes all Borel subsets of $\mathbb{R}_+^* \times \mathbb{R}^d$ with compact closure.

The next step is to introduce the K-transform between $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$. To this end, we need to define classes of functions on which this transform is well-defined. Furthermore, for the extension of the K-transform to a wider class of functions, we introduce a specific class of measures.

Definition 2.45. 1. *A function $G \colon \Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d) \to \mathbb{R}$ is said to be bounded with local support if there exist $C > 0$ and $\Lambda \in \mathcal{B}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$ such that the following estimate holds for all $\eta \in \mathbb{K}_0(\mathbb{R}^d)$:*

$$|G(\gamma)| \leq C \mathbb{1}_{\Pi_0(\Lambda)}(\gamma) \tag{14}$$

note that this implies that $G(\gamma) = 0$ if $\gamma \cap \Lambda^c \neq \emptyset$. We denote by $B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$ all functions $G \colon \Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d) \to \mathbb{R}$ which are bounded with local support.

2. *A function $G \colon \Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d) \to \mathbb{R}$ is called bounded with bounded support if there exists $\Lambda \in \mathcal{B}_c(\mathbb{R}_+^* \times \mathbb{R}^d), N \in \mathbb{N}$ and $C > 0$ such that*

$$|G(\gamma)| \leq C \mathbb{1}_{\Pi_0(\Lambda)}(\gamma) \mathbb{1}_{\{|\gamma| \leq N\}}(\gamma), \tag{15}$$

i.e. $G(\gamma) = 0$ whenever $|\gamma| > N$ or $\gamma \cap \Lambda^c \neq \emptyset$. Denote the space of all such functions by $B_{\mathrm{bs}}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$. Obviously, we have $B_{\mathrm{bs}}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)) \subset B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$.

3. *A measure ρ on $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$ is called locally finite if for any $\Lambda \in \mathcal{B}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$ and for any $m \in \mathbb{N}_0$, the value of $\rho(\Pi_0^{(m)}(\Lambda))$ is finite. Equivalently, $\rho(A)$ is finite for all bounded measurable sets $A \subset \Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$. The space of all locally finite measures on $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$ is denoted by $\mathcal{M}_{\mathrm{lf}}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$.*

We now have enough preparation to explicitly define the K-transform and show important properties.

Definition 2.46 ([30])**.** *Let $G \in B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$. The K-transform of G is the function $KG \colon \Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \to \mathbb{R}$ defined by*

$$(K_\Pi G)(\gamma) = (KG)(\gamma) := \sum_{\xi \Subset \gamma} G(\xi)$$

where the inclusion $\xi \Subset \gamma$ means that the sum is taken over all finite subsets of γ. The dependence on Π is omitted if no confusion can arise. Note that by the definition of $B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$, the K-transform is well-defined on such functions.

Let us recall some results which can be taken directly from the theory of homogeneous configuration spaces.

Proposition 2.47 ([30]). *1. The K-transform maps functions from $B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$ to cylinder functions $\mathcal{F}L^0(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d))$, i.e. for $G \in B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$, there exists $\Lambda \in \mathcal{B}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$ such that*

$$(KG)(\gamma) = (KG)(\gamma \cap \Lambda)$$

for all $\gamma \in \Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$.

2. *The K-transform maps $B_{\mathrm{bs}}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$ to polynomially bounded functions, i.e. for $G \in B_{\mathrm{bs}}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$, there exist $\Lambda \in \mathcal{B}_c(\mathbb{R}_+^* \times \mathbb{R}^d), N \in \mathbb{N}$ and $C > 0$ such that*
$$|KG|(\gamma) \leq C(1 + |\gamma \cap \Lambda|)^N.,\ \gamma \in \Pi(\mathbb{R}_+^* \times \mathbb{R}^d).$$

3. *The mapping $K\colon B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)) \to \mathcal{F}L(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d))$ is invertible with*
$$K^{-1}F(\gamma) = \sum_{\xi \subset \gamma} (-1)^{|\gamma \setminus \xi|} F(\xi),\ \gamma \in \Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d).$$

4. *K is linear and positivity preserving.*

5. *If $G \in B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$ and continuous, then KG is also continuous.*

Let us consider an example from statistical mechanics. Namely, the so-called coherent states.

Example 1 ([30]). *For a function $f \in C_c(\mathbb{R}_+^* \times \mathbb{R}^d)$, define the coherent state or Lebesgue-Poisson exponent as*

$$e_\lambda(f)\colon \Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d) \to \mathbb{R},\ \gamma \mapsto e_\lambda(f, \gamma) := \prod_{(s,x) \in \gamma} f(s, x).$$

Then $e_\lambda(f) \in B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$. Its K-transform is given by

$$(Ke_\lambda(f))(\gamma) = \prod_{(s,x) \in \gamma} (1 + f(s, x)),\ \gamma \in \Pi(\mathbb{R}_+^* \times \mathbb{R}^d).$$

Let us finish this part by introducing the $\star$-convolution, which is related to the K-transform as the standard convolution on $\mathbb{R}^d$ to the Fourier transform.

Definition 2.48. *Let $F, G \in B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$. Define the $\star$-convolution as*

$$(F \star G)(\gamma) := \sum_{(\xi_1, \xi_2, \xi_3) \in \mathcal{P}_\emptyset^3(\gamma)} F(\xi_1 \cup \xi_2) G(\xi_2 \cup \xi_3),\ \gamma \in \Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d).$$

where $\mathcal{P}_\emptyset^3(\gamma)$ denotes all partitions of γ into three parts, where the parts may be empty.

As stated before, the following relation holds:

Proposition 2.49 ([30]). *Let $F, G \in B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$ be given. Then*

$$K(F \star G) = KF \cdot KG.$$

2.9.2 Correlation Measures on $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$

To consider dynamics, it is essential to work on Banach spaces such as L^1-type spaces. To this end, we need to introduce suitable classes of measures. Furthermore, these measures correspond to probability measures on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$, which are used to model the dynamics of our system. Also, the class of considered measures enables us to extend the K-transform to the aforementioned L^1-spaces. The method is based on [30, 43].

To extend the K-transform, we introduce an integration kernel based on this mapping.

Definition 2.50. *Define the following pre-kernel based on the K-transform by*

$$\begin{aligned} \mathcal{K}\colon \mathcal{B}_b(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)) \times \Pi(\mathbb{R}_+^* \times \mathbb{R}^d) &\to [0, \infty) \\ (A, \gamma) \mapsto \mathcal{K}(A, \gamma) &:= (K\mathbb{1}_A)(\gamma) \end{aligned} \tag{16}$$

Let us prove that $\mathcal{K}$ is in fact a pre-kernel. The property $\mathcal{K}(\emptyset, \gamma) = 0$ for any $\gamma \in \Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$ is clear. For σ-additivity, let $A_i \in \mathcal{B}_b(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$, $i \in \mathbb{N}$ disjoint such that the countable union is again in $\mathcal{B}_b(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$. Then there exist $N \in \mathbb{N}$ and $\Lambda \in \mathcal{B}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$ such that

$$\bigcup_{i=1}^{\infty} A_i \subset \bigcup_{k=0}^{N} \Pi_0^{(k)}(\Lambda).$$

This means for $\gamma \in \Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$,

$$\mathcal{K}\left(\bigcup_{i=1}^{\infty} A_i, \gamma\right) = \sum_{\xi \Subset \gamma} \sum_{i=1}^{\infty} \mathbb{1}_{A_i}(\xi) = \sum_{\substack{\xi \Subset \gamma \\ |\gamma| \le N}} \sum_{i=1}^{\infty} \mathbb{1}_{A_i}(\xi) \sum_{i=1}^{\infty} \sum_{\substack{\xi \Subset \gamma \\ |\xi| \le N}} \mathbb{1}_{A_i}(\xi) = \sum_{i=1}^{\infty} \mathcal{K}(A, \gamma),$$

which shows the claim. Furthermore, $\mathcal{K}$ can in fact be extended:

Lemma 2.51. *The pre-kernel $\mathcal{K}$ has a unique extension to a kernel on $\mathcal{B}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)) \times \Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$.*

Proof. Since $\mathcal{B}_b(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$ is a ring, we only need to show σ-finiteness of $\mathcal{K}(\cdot, \gamma)$ to obtain a unique extension to $\mathcal{B}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$. But for $A \in \mathcal{B}_b(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$, the sum

$$\mathcal{K}(A, \gamma) = \sum_{\xi \Subset \gamma} \mathbb{1}_A(\xi)$$

is finite. Therefore, by Carathéodory's theorem, $\mathcal{K}$ can be extended uniquely to a kernel on $\mathcal{B}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)) \times \Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$. □

We may also extend Relation (16) to more general functions.

Proposition 2.52. *Let $G\colon \Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d) \to \mathbb{R}$ be a measurable function with $G \geq 0$ or $G \in B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$. Then*

$$\int_{\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)} G(\xi)\mathcal{K}(d\xi, \gamma) = \sum_{\xi \Subset \gamma} G(\xi) = (KG)(\eta).$$

Proof. Note that G may be approximated by a sequence of simple functions, i.e.

$$G(\gamma) = \sum_{k=1}^{\infty} a_k \mathbb{1}_{A_k}(\gamma)$$

where $a_k \in \mathbb{R}, A \in \mathcal{B}_b(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)), \gamma \in \Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$. The identity can then be obtained by monotone limits. For more details, see [30, 43]. □

We may now use the kernel $\mathcal{K}$ to construct measures on $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$ corresponding to probability measures on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$.

Definition 2.53. *Let μ be a probability measure on $(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d), \mathcal{B}(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)))$. The corresponding correlation measure is defined on $(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d), \mathcal{B}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)))$ by the relation*

$$\rho_\mu(A) := \int_{\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)} \mathcal{K}(A, \gamma)\mu(d\gamma).$$

The class of locally finite measures on $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$ introduced above forms an interesting class of measures in applications. Therefore, it is of interest to characterize locally finite correlation measures via its underlying probability measure on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$. Namely, the class of probability measures with finite local moments. We define this class of probability measures and state the relation to locally finite measures.

Proposition 2.54. *Let μ be a probability measure on $(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d), \mathcal{B}(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)))$. Then the corresponding correlation measure ρ_μ is locally finite if and only if the following holds: For any $\Lambda \in \mathcal{B}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $N \in \mathbb{N}$,*

$$\int_{\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)} |\gamma \cap \Lambda|^N \mu(d\gamma) < \infty. \tag{17}$$

Definition 2.55. *A measure μ with property* (17) *is said to have finite local moments of all order. The space of all such measures is denoted by $\mathcal{M}^1_{\mathrm{fm}}(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d))$.*

Proof of Proposition 2.54. The proof works analogously to the case of classical configuration spaces, see [30]. It is repeated for convenience in our case.

"$\Leftarrow$": Let $A \in \mathcal{B}_b(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$. Then there exist $\Lambda \in \mathcal{B}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $N \in \mathbb{N}$ such that

$$A \subset \bigcup_{k=0}^{N} \Pi_0^{(k)}(\Lambda).$$

Then

$$\begin{aligned}\rho_\mu(A) &= \int_{\mathbb{K}(\mathbb{R}^d)} \mathcal{K}(A, \gamma)\mu(d\gamma)\\ &\leq \sum_{k=0}^{N} \int_{\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)} \mathcal{K}(\Pi_0^{(k)}(\Lambda), \gamma)\mu(d\gamma) = \sum_{k=0}^{N} \int_{\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)} \binom{|\gamma \cap \Lambda|}{k} \mu(d\gamma)\end{aligned} \tag{18}$$

where we used that

$$\mathcal{K}(\Pi_0^{(k)}(\Lambda), \gamma) = \sum_{\xi \Subset \gamma} \mathbb{1}_{\Pi_0^{(k)}(\Lambda)}(\xi) = \sum_{\substack{\xi \Subset \gamma \\ |\gamma| = k}} \mathbb{1}_{\Pi_0^{(k)}(\Lambda)}(\xi) = \binom{|\gamma \cap \Lambda|}{k}$$

The last expression of (18) consists of a finite linear combination of powers of $|\gamma \cap \Lambda|$ and is therefore finite by assumption.

"$\Rightarrow$": Let $N \in \mathbb{N}$ and $\Lambda \in \mathcal{B}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$. The expression $|\gamma \cap \Lambda|$ can be written as a linear combination of binomial coefficients $\binom{|\gamma \cap \Lambda|}{k}$ for $k \leq n$ and therefore, by a calculation similar to (18), we obtain the result. $\square$

For the class $\mathcal{M}^1_{\mathrm{fm}}(\Pi(\mathbb{R}^*_+ \times \mathbb{R}^d))$, we may now extend the K-transform to L^1-spaces related to these measures.

Proposition 2.56 ([30]). *Let $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Pi(\mathbb{R}^*_+ \times \mathbb{R}^d))$ be given. For all functions $G \in B_{\mathrm{bs}}(\Pi_0(\mathbb{R}^*_+ \times \mathbb{R}^d))$, we have $G \in L^1(\Pi_0(\mathbb{R}^*_+ \times \mathbb{R}^d), \rho_\mu)$. Furthermore, if $G \geq 0$ or $G \in B_{\mathrm{bs}}(\Pi_0(\mathbb{R}^*_+ \times \mathbb{R}^d))$, then*

$$\int_{\Pi_0(\mathbb{R}^*_+ \times \mathbb{R}^d)} G(\gamma)\rho_\mu(d\gamma) = \int_{\Pi(\mathbb{R}^*_+ \times \mathbb{R}^d)} (KG)(\gamma)\mu(d\gamma) \tag{19}$$

Proof. The proof works directly as in [30]. Since $\mu(\Pi(\mathbb{R}^*_+ \times \mathbb{R}^d)) = 1$, the restriction from $\Gamma(\mathbb{R}^*_+ \times \mathbb{R}^d)$ to $\Pi(\mathbb{R}^*_+ \times \mathbb{R}^d)$ makes no difference for the identity. □

Remark 2.57. *For a measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Pi(\mathbb{R}^*_+ \times \mathbb{R}^d))$, we may define the correlation measure without using the kernel $\mathcal{K}$ directly via*

$$\rho_\mu(A) := \int_{\Pi(\mathbb{R}^*_+ \times \mathbb{R}^d)} K\mathbb{1}_A(\gamma)\mu(d\gamma), \; A \in \mathcal{B}_b(\Pi_0(\mathbb{R}^*_+ \times \mathbb{R}^d)).$$

This is due to Proposition 2.54, since $K\mathbb{1}_A \in L^1(\mu)$ for $A \in \mathcal{B}_b(\Pi_0(\mathbb{R}^*_+ \times \mathbb{R}^d))$.

Definition 2.58. *The remark above enables us to explicitly define the dual operator of K, i.e.*

$$\begin{aligned} K^* \colon \mathcal{M}^1_{\mathrm{fm}}(\Pi(\mathbb{R}^*_+ \times \mathbb{R}^d)) &\to \mathcal{M}_{\mathrm{lf}}(\Pi_0(\mathbb{R}^*_+ \times \mathbb{R}^d)) \\ \mu &\mapsto K^*\mu := \rho_\mu \end{aligned}$$

To finally extend the K-transform, we need one more continuity result of this mapping.

Lemma 2.59 ([30]). *Let $\{G_n\}_{n\in\mathbb{N}} \subset B_{\mathrm{bs}}(\Pi_0(\mathbb{R}^*_+ \times \mathbb{R}^d))$ be a sequence converging in $L^1(\Pi_0(\mathbb{R}^*_+ \times \mathbb{R}^d), \rho_\mu)$ for some measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Pi(\mathbb{R}^*_+ \times \mathbb{R}^d))$. Then $\{KG_n\}_{n\in\mathbb{N}}$ converges in $L^1(\Pi(\mathbb{R}^*_+ \times \mathbb{R}^d), \mu)$.*

Proof. Calculation using the triangle inequality $|KG| \leq K|G|$. □

We may now prove the extension result for the K-transform on L^1-spaces.

Theorem 2.60 ([30]). *Let $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Pi(\mathbb{R}^*_+ \times \mathbb{R}^d))$. For any $G \in L^1(\Pi_0(\mathbb{R}^*_+ \times \mathbb{R}^d), \rho_\mu)$, define*

$$KG(\gamma) := \sum_{\xi \Subset \gamma} G(\xi)$$

where the series converges absolutely μ-almost surely. Furthermore, we have the following estimate:

$$\|KG\|_{L^1(\mu)} \leq \|K|G|\|_{L^1(\mu)} = \|G\|_{L^1(\rho_\mu)}$$

which implies that $KG \in L^1(\mu)$. Also, identity (19) holds for all $G \in L^1(\rho_\mu)$.

Proof. The statement for non-negative functions follows from calculations using the previous lemma as well as Fatou's lemma. The extension to general functions is done by decomposing the function into a positive and negative part. □

One important identity related to the K-transform concerns the cominatorial convolution given by Definition 2.48. The identity given by Proposition 2.49 can also be extended under some conditions.

Proposition 2.61 ([30]). *Let $F, G \in B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$ and let $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d))$. Assume one of the following conditions:*

1. $F, G \geq 0$
2. $|F| \star |G| \in L^1(\rho_\mu)$ *(and consequentially $K(F \star G) \in L^1(\mu)$)*
3. $F, G \in L^1(\rho_\mu)$.

Then the following identity holds μ-almost surely:

$$K(F \star G) = KF \cdot KG.$$

Proof. Direct consequence of Theorem 2.60. □

2.9.3 Correlation Functions on $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$

For a certain subclass of measures of $\mathcal{M}^1_{\mathrm{fm}}(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d))$, we may show existence of a density function for the correlation measure ρ_μ. This density is known as the correlation function of μ. Since the representation of dynamics is easier in terms of correlation functions, it is useful in applications to find out when these fucntions exist. In this chapter, we want to discuss necessary conditions. Namely, the property of local absolute continuity of a measure with respect to the Poisson measure introduced in Chapter 2.8.1.

Definition 2.62. *We call a measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d))$ locally absolutely continuous with respect to the Poisson measure π if the measure μ^Λ is absolutely continuous with respect to π^Λ for all $\Lambda \in \mathcal{B}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$, where $\mu^\Lambda := \mu \circ p_\Lambda^{-1}$ as in Chapter 2.8.1.*

This property can be transferred to the corresponding correlation measure. Furthermore, it implies the existence of a density function:

Proposition 2.63 ([30]). *For a measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d))$ which is locally absolutely continuous with respect to π, the correlation measure ρ_μ is absolutely continuous with respect to the Lebesgue-Poisson measure λ introduced in Chapter 2.8.1. The density function has the following representation for any $\gamma \in \Pi_0(\Lambda)$:*

$$k_\mu(\gamma) = \frac{d\rho_\mu}{d\lambda} = \int_{\Pi(\Lambda)} \frac{d\mu^\Lambda}{d\pi^\Lambda}(\gamma \cup \xi)\pi^\Lambda(d\xi)$$

Definition 2.64. *The function $k_\mu \colon \Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d) \to \mathbb{R}$ defined by the previous proposition is called the correlation function corresponding to μ. Furthermore, we have the decomposition $k_\mu \simeq \{k_\mu^{(n)}\}_{n=0}^\infty$, where for any $n \in \mathbb{N}$, $k_\mu^{(n)} : (\mathbb{R}_+^* \times \mathbb{R}^d)^n \to \mathbb{R}$ is a symmetric function with*

$$k_\mu^{(n)}(s_1, x_1, \ldots, s_n, x_n) := \begin{cases} k_\mu(\{(s_1, x_1), \ldots, (s_n, x_n)\}), & \text{if } |\{(s_1, x_1), \ldots, (s_n, x_n)\}| = n \\ 0, & \text{otherwise} \end{cases}$$

The functions $k_\mu^{(n)}$ are called n-point correlation functions.

We want to mention a relation which will be helpful when introducing the notion of a correlation function on the cone $\mathbb{K}(\mathbb{R}^d)$. Namely, the so-called Bogoliubov functional, which can be set in relation with the correlation function of a measure. A more detailed discussion can be found in [31].

Definition 2.65. *Let $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Pi(\mathbb{R}^*_+ \times \mathbb{R}^d))$. The Bogoliubov functional L^Π_μ corresponding to μ is defined as*

$$L^\Pi_\mu(\varphi) := \int_{\Pi(\mathbb{R}^*_+ \times \mathbb{R}^d)} \prod_{(s,x)\in\gamma} (1+\varphi(s,x))\mu(d\gamma)$$

for any measurable function $\varphi\colon \mathbb{R}^*_+ \times \mathbb{R}^d \to \mathbb{R}$, provided, the right-hand side exists for $|\varphi|$.

We only provide a heuristic version of the following proposition, since it only serves as a motivation for the correct shape of the correlation functions on $\mathbb{K}(\mathbb{R}^d)$ later.

Proposition 2.66 ([31])**.** *Under some assumptions, the Bogoliubov functional is the generating functional of the correlation function. In other words, for any $\varphi\colon \mathbb{R}^*_+ \times \mathbb{R}^d \to \mathbb{R}$ such that $L^\Pi_\mu(\varphi)$ is well-defined, we have*

$$\begin{aligned} L^\Pi_\mu(\varphi) = \sum_{n=0}^\infty \frac{1}{n!} \int_{\mathbb{R}^*_+ \times \mathbb{R}^{d^n}} & \varphi(s_1,x_1)\cdots\varphi(s_n,x_n)\times \\ & \times k^{(n)}_\mu(s_1,\dots,x_n)\nu(ds_1)\sigma(dx_1)\dots\nu(ds_n)\sigma(dx_n) \end{aligned}$$

2.10 Harmonic Analysis on $\mathbb{K}(\mathbb{R}^d)$

Now that we have established the framework of harmonic analysis on the underlying Plato space $\Pi(\mathbb{R}^*_+ \times \mathbb{R}^d)$, we want to transfer these notions to the physical space of the cone $\mathbb{K}(\mathbb{R}^d)$. Let us start by introducing analogous notions of the auxiliary spaces introduced above as well as classes of functions on these spaces needed for the analysis.

Definition 2.67.

1. *The set of Radon measures with finite support is defined as*

$$\mathbb{K}_0(\mathbb{R}^d) := \left\{\eta \in \mathbb{K}(\mathbb{R}^d)\colon |\tau(\eta)| < \infty\right\}$$

2. *For $n \in \mathbb{N}_0$, the set of n-point measures is defined as*

$$\mathbb{K}^{(n)}_0(\mathbb{R}^d) := \left\{\eta \in \mathbb{K}_0(\mathbb{R}^d)\colon |\tau(\eta)| = n\right\},\ n \in \mathbb{N}$$

and $\mathbb{K}^{(0)}_0(\mathbb{R}^d) = \{0\}$ the set consisting of the zero measure.

3. *For a compact set $\Lambda \subset \mathbb{R}^d$, the set of all measures supported in Λ is defined as*

$$\mathbb{K}_0(\Lambda) := \left\{\eta \in \mathbb{K}_0(\mathbb{R}^d)\colon \tau(\eta) \subset \Lambda\right\}$$

4. *A set $A \subset \mathbb{K}_0(\mathbb{R}^d)$ is called bounded if there exists a compact set $\Lambda \subset \mathbb{R}^d$ and $N \in \mathbb{N}$ such that*

$$A \subset \bigcup_{n=0}^N \mathbb{K}^{(n)}_0(\Lambda).$$

Denote the collection of all bounded subsets of $\mathbb{K}_0(\mathbb{R}^d)$ by $\mathcal{B}_b(\mathbb{K}_0(\mathbb{R}^d))$.

5. *A bounded set $A \subset \mathbb{K}_0(\mathbb{R}^d)$ is said to have compact marks if additionally, there exists a compact set $I \subset \mathbb{R}_+^*$ such that*

$$A \cap \{\eta \in \mathbb{K}_0(\mathbb{R}^d) \mid \exists x \in \tau(\eta)\colon s_x \notin I\} = \emptyset$$

Denote the collection of all such sets by $\mathcal{B}_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$.

Note that we have

$$\mathbb{K}_0(\mathbb{R}^d) = \bigsqcup_{n=0}^{\infty} \mathbb{K}_0^{(n)}(\mathbb{R}^d)$$
$$\mathbb{K}_0(\mathbb{R}^d) = \bigcup_{\Lambda \in \mathcal{B}_c(\mathbb{R}^d)} \mathbb{K}_0(\Lambda)$$

where the first union is disjoint.

Remark 2.68. *Even though the spaces $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\mathbb{K}(\mathbb{R}^d)$ appear simultaneously, the notions introduced above pose a small difference: For $\mathbb{K}_0(\Lambda)$, we do not assume that the set of marks is concentrated in a compact set. Therefore, a second notion of boundedness is introduced via $\mathcal{B}_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$.*

Let us relate the subspaces of $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\mathbb{K}(\mathbb{R}^d)$ via the reflection mapping $\mathcal{R}$ introduced in Chapter 2.5.

Proposition 2.69. *The following relations hold:*

1. $\mathcal{R}\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d) = \mathbb{K}_0(\mathbb{R}^d)$
2. $\mathcal{R}\Pi_0^{(n)}(\mathbb{R}_+^* \times \mathbb{R}^d) = \mathbb{K}_0^{(n)}(\mathbb{R}^d)$ *for any* $n \in \mathbb{N}_0$.
3. $\mathcal{R}\Pi_0(\mathbb{R}_+^* \times \Lambda) = \mathbb{K}_0(\Lambda)$ *for any set* $\Lambda \subset \mathbb{R}^d$.
4. *For any $A \in \mathcal{B}_b(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$, we have $\mathcal{R}A \in \mathcal{B}_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$ and vice versa.*

Proof. Let us prove the first statement. The other statements follow analogously.

"$\subset$": Let $\gamma \in \Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$, then there exists a representation $\gamma = \sum_{i=1}^n \delta_{(s_i,x_i)}$. This yields $\mathcal{R}\gamma = \sum_{i=1}^n s_i\delta_{x_i} \in \mathbb{K}_0(\mathbb{R}^d)$.

"$\supset$": Let $\eta \in \mathbb{K}_0(\mathbb{R}^d)$. Again, we have $\eta = \sum_{i=1}^n s_i\delta_{x_i}$. By setting $\gamma = \sum_{i=1}^n \delta_{(s_i,x_i)}$, we obtain $\gamma \in \Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\mathcal{R}\gamma = \eta$. □

We continue by introducing the analogue of function spaces on $\mathbb{K}_0(\mathbb{R}^d)$. There are some function spaces which are specifically using the mark structure of $\mathbb{K}_0(\mathbb{R}^d)$. This enables us to consider L^1-spaces corresponding to measures with mark weights such as in Definition 2.22.

Definition 2.70. 1. *A function $G\colon \mathbb{K}_0(\mathbb{R}^d) \to \mathbb{R}$ is said to be bounded with local support if there exist $C > 0$ and $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ such that the following estimate holds for all $\eta \in \mathbb{K}_0(\mathbb{R}^d)$:*

$$|G(\eta)| \le C\mathbb{1}_{\mathbb{K}_0(\Lambda)}(\eta) \prod_{x \in \tau(\eta)} s_x \tag{20}$$

note that this implies that $G(\eta) = 0$ if $\tau(\eta) \cap \Lambda^c \neq \emptyset$. We denote by $B_{\mathrm{ls}}(\mathbb{K}_0(\mathbb{R}^d))$ all functions $G\colon \mathbb{K}_0(\mathbb{R}^d) \to \mathbb{R}$ which are bounded with local support.

2. *A function $G\colon \mathbb{K}_0(\mathbb{R}^d) \to \mathbb{R}$ is called bounded with bounded support if there exists $\Lambda \in \mathcal{B}_c(\mathbb{R}^d), N \in \mathbb{N}$ and $C > 0$ such that*

$$|G(\eta)| \leq C \mathbb{1}_{\mathbb{K}_0(\Lambda)}(\eta) \mathbb{1}_{\{|\tau(\eta)| \leq N\}}(\eta),$$

i.e. $G(\eta) = 0$ whenever $|\tau(\eta)| > N$ or $\tau(\eta) \cap \Lambda^c \neq \emptyset$. Denote the space of all such functions by $B_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$.

3. *Let us also define a modified version of $B_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$ which takes into account the effect of the marks as above. We define the space $\widetilde{B}_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$ as all functions $G\colon \mathbb{K}_0(\mathbb{R}^d) \to \mathbb{R}$ which satisfy the bound*

$$|G(\eta)| \leq C \mathbb{1}_{\mathbb{K}_0(\Lambda)}(\eta) \mathbb{1}_{\{|\tau(\eta)| \leq N\}}(\eta) \prod_{x \in \tau(\eta)} s_x$$

for some $\Lambda \in \mathcal{B}_c(\mathbb{R}^d), N \in \mathbb{N}$ and $C > 0$. Obviously, we have $\widetilde{B}_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d)) \subset B_{\mathrm{ls}}(\mathbb{K}_0(\mathbb{R}^d))$.

4. *Define the space of bounded functions with compact mark support as all functions $G \in B_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$ such that there exists a compact set $I \subset \mathbb{R}_+^*$ such that*

$$|G(\eta)| \leq C \mathbb{1}_{\mathbb{K}_0(\Lambda)} \mathbb{1}_{\{|\tau(\eta)| \leq N\}} \prod_{x \in \tau(\eta)} \mathbb{1}_I(s_x) \tag{21}$$

where Λ, C and N are as above. One class of compact sets of special interest will be $I = [a, b]$ with $0 < a < b < \infty$. Denote the space of bounded functions with compact marks by $B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$.

5. *A measure ρ on $\mathbb{K}_0(\mathbb{R}^d)$ is called locally finite if for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and for any $m \in \mathbb{N}_0$, the value of $\rho(\mathbb{K}_0^{(m)}(\Lambda))$ is finite. Equivalently, $\rho(A)$ is finite for all bounded measurable sets $A \subset \mathbb{K}_0(\mathbb{R}^d)$. The space of all locally finite measures on $\mathbb{K}_0(\mathbb{R}^d)$ is denoted by $\mathcal{M}_{\mathrm{lf}}(\mathbb{K}_0(\mathbb{R}^d))$.*

6. *A measure ρ on $\mathbb{K}_0(\mathbb{R}^d)$ is called mark-locally finite if $\rho(A) < \infty$ for all $A \in \mathcal{B}_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$. Obviously, a locally finite measure ρ is also mark-locally finite.*

Let us now relate the function spaces on $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$ and on $\mathbb{K}_0(\mathbb{R}^d)$. We use the reflection mapping $\mathcal{R}$ to map functions $\mathcal{F}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$ on $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$ to functions $\mathcal{F}(\mathbb{K}_0(\mathbb{R}^d))$ on $\mathbb{K}_0(\mathbb{R}^d)$ the following way:

Definition 2.71. *Define the pushforward of functions on $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$ to $\mathbb{K}_0(\mathbb{R}^d)$ as follows:*

$$\begin{aligned} \mathcal{R}\colon \mathcal{F}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)) &\to \mathcal{F}(\mathbb{K}_0(\mathbb{R}^d)) \\ F &\mapsto \mathcal{R}F := F \circ \mathcal{R}^{-1} \end{aligned}$$

analogously, we may define the inverse mapping $\mathcal{R}^{-1}\colon \mathcal{F}(\mathbb{K}_0(\mathbb{R}^d)) \to \mathcal{F}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$.

The main difference between function spaces on $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$ and on $\mathbb{K}_0(\mathbb{R}^d)$ is that the definitions on $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$ *a priori* require compactness of mark support, while functions on $\mathbb{K}_0(\mathbb{R}^d)$ need to be bounded in the mark variables as in Definition 2.70 above. The next Proposition shows that one needs to be careful when comparing locally supported functions on $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\mathbb{K}_0(\mathbb{R}^d)$.

Proposition 2.72. *For the above spaces, the following relations hold:*

1. $\mathcal{R}B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)) \not\subset B_{\mathrm{ls}}(\mathbb{K}_0(\mathbb{R}^d))$ *and* $B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)) \not\supset \mathcal{R}^{-1}B_{\mathrm{ls}}(\mathbb{K}_0(\mathbb{R}^d))$.

2. $\mathcal{R}B_{\mathrm{bs}}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)) = B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$.

Proof. 1. Let $G \in B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$ such that for some $0 < a < 1$, we have $[a,b] \times \Lambda' \subset \Lambda$, where Λ is as in Definition 2.45 and $\Lambda' \subset \mathbb{R}^d$ compact. We require the estimate

$$C\mathbb{1}_{\Pi_0(\Lambda)}(\mathcal{R}^{-1}\eta) \leq C_1\mathbb{1}_{\mathbb{K}_0(\Lambda')}(\eta) \prod_{x\in\tau(\eta)} s_x$$

for some $C, C_1 > 0$. But since $a < s_x < 1$ is possible and the number of points in η is arbitrary, the right-hand side can be arbitrarily small. On the other hand, let $G \in B_{\mathrm{ls}}(\mathbb{K}_0(\mathbb{R}^d))$. To show $\mathcal{R}^{-1}G \in B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$, we require

$$C\mathbb{1}_{\mathbb{K}_0(\Lambda')}(\mathcal{R}\gamma) \prod_{x\in\tau(\mathcal{R}\gamma)} s_x \leq C_1\mathbb{1}_{\Pi_0(\Lambda)}(\mathcal{R}\gamma)$$

for some $C, C_1 > 0$ and Λ, Λ' as in the definitions above. Since there is no compactness requirement on the marks in $B_{\mathrm{ls}}(\mathbb{K}_0(\mathbb{R}^d))$, the left-hand side can be arbitrarily large.

2. Let $G \in B_{\mathrm{bs}}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$. Then there exist $\Lambda \in \mathcal{B}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$ compact, $N \in \mathbb{N}$ and $C > 0$ such that (15) holds. Since Λ is compact, there exist $0 < a < b$ such that $\Lambda \subset [a,b] \times \Lambda'$ for some $\Lambda' \in \mathcal{B}_c(\mathbb{R}^d)$. Then

$$\begin{aligned} G(\mathcal{R}^{-1}\eta) &\leq C\mathbb{1}_{\Pi_0(\Lambda)}(\mathcal{R}^{-1}\eta)\mathbb{1}_{\{|\mathcal{R}^{-1}\eta|\leq N\}}(\mathbb{R}^{-1}\eta) \\ &\leq C\mathbb{1}_{\Pi_0([a,b]\times\Lambda')}(\mathcal{R}^{-1}\eta)\mathbb{1}_{\{|\mathcal{R}^{-1}\eta|\leq N\}}(\mathcal{R}^{-1}\eta) \\ &= C\mathbb{1}_{\mathbb{K}_0(\Lambda')}(\eta)\mathbb{1}_{\{|\tau(\eta)|\leq N\}}(\eta) \prod_{x\in\tau(\eta)} \mathbb{1}_{[a,b]}(s_x) \end{aligned}$$

which shows the first inclusion. On the other hand, let $G \in B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$. Then there exist $\Lambda' \in \mathcal{B}_c(\mathbb{R}^d)$, $I \in \mathcal{B}_c(\mathbb{R}_+^*)$, $N \in \mathbb{N}$ and $C > 0$ such that (21) holds. Also, $I \times \Lambda' \in \mathcal{B}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$ and since

$$\mathbb{1}_{\mathbb{K}_0(\Lambda')}(\eta) \prod_{x\in\tau(\eta)} \mathbb{1}_I(s_x) = \mathbb{1}_{\Pi_0(I\times\Lambda')}(\mathcal{R}^{-1}\eta),$$

the claim follows.

□

Analogously to the case of $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$, we may define the K-transform. The definition is exactly the same. The differing function spaces offer different estimates on the transformed function, though.

Definition 2.73. *Let $G \in B_{\mathrm{ls}}(\mathbb{K}_0(\mathbb{R}^d))$. The K-transform of G is defined as the function $KG\colon \mathbb{K}(\mathbb{R}^d) \to \mathbb{R}$ of the form*

$$(K_{\mathbb{K}}G)(\eta) = (KG)(\eta) := \sum_{\xi\Subset\eta} G(\xi)$$

where the inclusion $\xi \Subset \eta$ is meant in the sense of Definition 2.1. Again, the dependence on $\mathbb{K}$ is dropped if it is clear from context.

Lemma 2.74. *For any $G \in B_{\mathrm{ls}}(\mathbb{K}_0(\mathbb{R}^d))$, the K-transform is well-defined and the following estimate holds:*

$$|(KG)(\eta)| \leq C \prod_{x \in \tau(\eta) \cap \Lambda} (1 + s_x)$$

where C and Λ are as in Definition 2.70.

Proof. We have

$$|(KG)(\eta)| \leq \sum_{\xi \Subset \eta} |G(\eta)| \leq C \sum_{\substack{\xi \in \mathbb{K}_0(\Lambda) \\ \tau(\xi) \subset \tau(\eta)}} \prod_{x \in \tau(\xi)} s_x = C \prod_{x \in \tau(\eta) \cap \Lambda} (1 + s_x)$$

where the product in the last expression is finite if and only if the sum

$$\sum_{x \in \tau(\eta) \cap \Lambda} s_x$$

is finite. Since the latter is true by definition of $\eta \in \mathbb{K}(\mathbb{R}^d)$, the claim follows. □

The following example is the $\mathbb{K}_0(\mathbb{R}^d)$-analogue of coherent states. Furthermore, it relates these states to the coherent states on $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$.

Example 2. *For a function $\varphi \in C_c(\mathbb{R}^d)$, we define the coherent state as the function $e_{\mathbb{K}}(\varphi) \colon \mathbb{K}_0(\mathbb{R}^d) \to \mathbb{R}$ by*

$$e_{\mathbb{K}}(\varphi, \eta) := \prod_{x \in \tau(\eta)} s_x \varphi(x)$$

since φ is bounded, $e_{\mathbb{K}}(\varphi)$ fulfills bound (20). We can calculate its K-transform:

$$(K e_{\mathbb{K}}(\varphi))(\eta) = \prod_{x \in \tau(\eta)} (1 + s_x \varphi(x))$$

For the right-hand-side to be well-defined, the series $\sum_{x \in \tau(\eta)} s_x \varphi(x)$ needs to be convergent. Since φ is compactly supported, this is given in our case.

For $f_\varphi(s, x) := s\varphi(x)$, consider the Lebesgue-Poisson exponent $e_\lambda(f_\varphi)$ from Example 1, we see that

$$e_{\mathbb{K}}(\varphi, \mathcal{R}\gamma) = e_\lambda(f_\varphi, \gamma), \;\; \gamma \in \Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$$

We can relate the K-transform on $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$ and on $\mathbb{K}_0(\mathbb{R}^d)$ in the following way:

Proposition 2.75. *For $G \in B_{\mathrm{ls}}(\mathbb{K}_0(\mathbb{R}^d)) \cap \mathcal{R}B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$ and $\eta \in \mathbb{K}(\mathbb{R}^d)$, the following holds:*

$$(K_{\mathbb{K}} G)(\eta) = (K_\Pi(\mathcal{R}^{-1} G))(\mathcal{R}^{-1}\eta)$$

Below, we will show that $B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$ is dense in various L^1-spaces. Furthermore, we know that $B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d)) \subset B_{\mathrm{ls}}(\mathbb{K}_0(\mathbb{R}^d)) \cap \mathcal{R}B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d))$ by Proposition 2.72. Therefore, the above relation holds on a nontrivial set of functions in $L^1(\mathbb{K}_0, \rho)$ for some class of measures ρ appearing below.

Proof of Proposition 2.75. Let $\eta = \sum_{i\in I} s_i \delta_{x_i}$, where $I \subset \mathbb{N}$. Then

$$
\begin{aligned}
(K_{\mathbb{K}}G)(\eta) &= \sum_{n=0}^{\infty} \sum_{\{i_1,\dots,i_n\}\subset I} G\left(\sum_{i=1}^{n} s_{i_k}\delta_{x_{i_k}}\right) = \sum_{n=0}^{\infty} \sum_{\{i_1,\dots,i_n\}\subset I} G\left(\mathcal{R}\left[\sum_{i=1}^{n} \delta_{(x_{i_k},s_{i_k})}\right]\right) \\
&= \sum_{n=0}^{\infty} \sum_{\{i_1,\dots,i_n\}\subset I} (\mathcal{R}^{-1}G)\left(\sum_{i=1}^{n} \delta_{(x_{i_k},s_{i_k})}\right) \\
&= \sum_{n=0}^{\infty} \sum_{\{i_1,\dots,i_n\}\subset I} (\mathcal{R}^{-1}G)\left(\mathcal{R}^{-1}\left[\sum_{i=1}^{n} s_{i_k}\delta_{x_{i_k}}\right]\right) \\
&= (K_{\Pi}(\mathcal{R}^{-1}G))(\mathcal{R}^{-1}\eta)
\end{aligned}
$$

□

For the calculations later in this work, we also need some identities regarding the K-transform. The following Lemma simplifies the combinatorial calculations related to the K-transform.

Lemma 2.76. *Let $G, G_1, G_2 \in B_{\mathrm{ls}}(\mathbb{K}_0(\mathbb{R}^d))$.*

1. *The K-transform has the following properties:*

$$
\begin{aligned}
KG(\eta - s_x\delta_x) - KG(\eta) &= -(KG(\cdot + s_x\delta_x))(\eta - s_x\delta_x) \\
KG(\eta + s_x\delta_x) - KG(\eta) &= (KG(\cdot + s_x\delta_x))(\eta)
\end{aligned}
$$

2. *The K-transform and the $\star$-convolution have the following relation:*

$$K(G_1 \star G_2) = KG_1 \cdot KG_2$$

Proof. These can be shown by direct calculations using combinatorial arguments. A similar proof for the case $\Gamma(Y)$ can be found in [30]. □

For calculations on the space of finite measures, we need the following identity, also known as Minlos Lemma. The proof is identical to the case of $\Gamma(Y)$.

Lemma 2.77 ([36])**.** *Let $\lambda_\varkappa$ be the Lebesgue-Poisson measure on $\mathbb{K}_0(\mathbb{R}^d)$ associated with some intensity measure $\varkappa = \nu \otimes \sigma$.*

1. *Let $G\colon \mathbb{K}_0(\mathbb{R}^d) \to \mathbb{R}$, $H\colon (\mathbb{K}_0(\mathbb{R}^d))^2 \to \mathbb{R}$. Then*

$$
\begin{aligned}
&\int_{\mathbb{K}_0(\mathbb{R}^d)} \int_{\mathbb{K}_0(\mathbb{R}^d)} G(\xi_1 + \xi_2) H(\xi_1, \xi_2) \lambda_\varkappa(d\xi_1)\lambda_\varkappa(d\xi_2) \\
&\qquad = \int_{\mathbb{K}_0(\mathbb{R}^d)} G(\eta) \sum_{\xi\subset\eta} H(\xi, \eta - \xi)\lambda_\varkappa(d\eta)
\end{aligned}
$$

2. *Let $H\colon \mathbb{K}_0(\mathbb{R}^d) \times \mathbb{R}_+^* \times \mathbb{R}^d \to \mathbb{R}$. Then*

$$
\begin{aligned}
&\int_{\mathbb{K}_0(\mathbb{R}^d)} \sum_{x\in\tau(\eta)} H(\eta, s_x, x)\lambda_\varkappa(d\eta) \\
&\qquad = \int_{\mathbb{K}_0(\mathbb{R}^d)} \int_{\mathbb{R}_+^*\times\mathbb{R}^d} H(\eta + s\delta_x, s, x)\varkappa(ds, dx)\lambda_\varkappa(d\eta)
\end{aligned}
$$

provided, at least one side of the equation exists.

Our goal now is to extend the K-transform to the whole space $L^1(\rho)$ for (mark-)locally finite measures ρ on $\mathbb{K}_0(\mathbb{R}^d)$.

The rest of this chapter is devoted to density statements of the spaces introduced above. Especially when considering semigroup theory and dynamics in general, for technical reasons, it is essential to consider dense subspaces of L^1-type spaces. Especially $B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$ will be of interest.

Lemma 2.78. *For any locally finite measure ρ, the space $B_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$ is dense in $L^1(\rho)$.*

Proof. Let $G \in L^1(\mathbb{K}_0(\mathbb{R}^d), \rho)$ for some measure ρ on $\mathbb{K}_0(\mathbb{R}^d)$. Let us first approximate unbounded functions with bounded support. Define

$$\begin{aligned} G_n(\eta) &:= \left[G(\eta)\mathbb{1}_{\mathbb{K}_0(B_n)}(\eta)\mathbb{1}_{\{|\tau(\eta)|\leq n\}}(\eta)\right] \wedge n \\ G_n'(\eta) &:= \left[G(\eta)\mathbb{1}_{\mathbb{K}_0(B_n)}(\eta)\mathbb{1}_{\{|\tau(\eta)|\leq n\}}(\eta)\right] \end{aligned}$$

where $B_n \subset \mathbb{R}^d$ is the ball with radius n centered at 0. Then $G_n \in B_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$ and

$$\begin{aligned} \|G_n(\eta) - G_n'(\eta)\|_{L^1(\rho)} &= \int_{\mathbb{K}_0(\mathbb{R}^d)} |G_n(\eta) - G_n'(\eta)|\rho(d\eta) \\ &= \int_{\mathbb{K}_0(\mathbb{R}^d)} |G_n'(\eta)|\mathbb{1}_{\{|G_n(\eta)|\geq n\}}(\eta)\rho(d\eta) \\ &\leq \int_{\mathbb{K}_0(\mathbb{R}^d)} |G(\eta)|\mathbb{1}_{\{|G(\eta)|\geq n\}}(\eta)\rho(d\eta) \end{aligned}$$

since $G \in L^1(\mathbb{K}_0(\mathbb{R}^d), \rho)$, the last term converges to 0 for $n \to \infty$. Next, define

$$G_n''(\eta) = G(\eta)\mathbb{1}_{\mathbb{K}_0(B_n)}(\eta)$$

Recall that we have the disjoint decomposition of $\mathbb{K}_0(\mathbb{R}^d)$ into n-point-configurations, i.e.

$$\mathbb{K}_0(\mathbb{R}^d) = \bigcup_{m=0}^{\infty} \mathbb{K}_0^{(m)}(\mathbb{R}^d),\ \mathbb{K}_0^{(m)}(\mathbb{R}^d) = \left\{\eta \in \mathbb{K}_0(\mathbb{R}^d) \colon |\tau(\eta)| = m\right\}$$

Using this decomposition, we get

$$\begin{aligned} \|G_n'(\eta) - G_n''(\eta)\| &= \int_{\mathbb{K}_0(\mathbb{R}^d)} |G_n'(\eta) - G_n''(\eta)|\rho(d\eta) \\ &= \sum_{m=n+1}^{\infty} \int_{\mathbb{K}_0^{(n)}(\mathbb{R}^d)} |G_n''(\eta)|\rho(d\eta) \leq \sum_{m=n+1}^{\infty} \int_{\mathbb{K}_0^{(n)}(\mathbb{R}^d)} |G(\eta)|\rho(d\eta) \end{aligned}$$

and since $G \in L^1(\mathbb{K}_0(\mathbb{R}^d))$, the series is absolutely convergent. Therefore, the last expression converges to 0 for $n \to \infty$. For the last step, note that the increasing sequence $\{\mathbb{K}_0(B_n)\}_{n=1}^{\infty}$ approximates $\mathbb{K}_0(\mathbb{R}^d)$ and therefore

$$\|G_n'' - G\|_{L^1} = \int_{\mathbb{K}_0(\mathbb{R}^d)} |G(\eta)|\rho(d\eta) - \int_{\mathbb{K}_0(B_n)} |G(\eta)|\rho(d\eta)$$

which converges to 0 as $n \to \infty$ by the argument above. $\square$

A similar result can be obtained for $\widetilde{B}_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$ with a modified measure:

Corollary 2.79. *Define the density function*

$$f(\eta) = \prod_{x\in\tau(\eta)} \frac{1}{s_x}.$$

Then the space $\widetilde{B}_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$ is dense in $L^1(\mathbb{K}_0(\mathbb{R}^d), f\rho)$.

Proof. Let $G \in L^1(f\rho)$. Then by definition, we have

$$\|G\|_{L^1(f\rho)} = \int_{\mathbb{K}_0(\mathbb{R}^d)} |G(\eta)| \prod_{x\in\tau(\eta)} \frac{1}{s_x}\rho(d\eta) < \infty$$

This implies that $G\cdot f \in L^1(\rho)$. Since $B_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$ is dense in $L^1(\rho)$, there exists a sequence $\{G_n\}_{n=1}^{\infty} \subset B_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$ such that

$$\|G_n - G\cdot f\|_{L^1(\rho)} \to 0,\ n\to\infty.$$

on the other hand, the sequence $\{\widetilde{G}_n\}_{n=1}^{\infty}$ is in $\widetilde{B}_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$, where $\widetilde{G}_n := \frac{G_n}{f}$:

$$\widetilde{G}_n(\eta) = \frac{G_n(\eta)}{f(\eta)} \le C\mathbb{1}_{\mathbb{K}_0(\Lambda)}(\eta)\mathbb{1}_{\{|\tau(\eta)|\le N\}}(\eta) \prod_{x\in\tau(\eta)} s_x$$

Furthermore, it converges to G in $L^1(f\rho)$:

$$\begin{aligned}\|\widetilde{G}_n - G\|_{L^1(f\rho)} &= \int_{\mathbb{K}_0(\mathbb{R}^d)} \left|\frac{G_n}{f} - G\right| f d\rho \\ &= \int_{\mathbb{K}_0(\mathbb{R}^d)} |G_n - Gf| d\rho = \|G_n - Gf\|_{L^1(\rho)} \to 0\ n\to\infty.\end{aligned}$$

which completes the proof. □

One typical example of a locally finite measure on $\mathbb{K}_0(\mathbb{R}^d)$ is derived from the Lebesgue-Poisson measure on $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$. Since we are only working on $\mathbb{K}_0(\mathbb{R}^d)$ in later chapters, we use the same notation for the measures on $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\mathbb{K}_0(\mathbb{R}^d)$. If we need to distinguish between those two, we will remark this accordingly.

Example 3. *Let ν be a finite measure on $\mathbb{R}_+^*$ and σ a non-atomic measure on $\mathbb{R}^d$ (e.g. the Lebesgue measure). Define the measure $\lambda = \lambda_{\nu\otimes\sigma}$ as*

$$\begin{aligned}&\int_{\mathbb{K}_0(\mathbb{R}^d)} F(\eta)\lambda(d\eta) = \\ &\qquad = F(0) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}_+^*\times\mathbb{R}^d)^n} F\left(\sum_{i=1}^{n} s_i\delta_{x_i}\right) \nu(ds_1)\dots\nu(ds_n)\sigma(dx_1)\dots\sigma(dx_n)\end{aligned}$$

Where $F\colon \mathbb{K}_0(\mathbb{R}^d) \to \mathbb{R}$ such that the above expression exists and 0 denotes the zero measure. Then λ is locally finite.

Proof. Let $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and $m \in \mathbb{N}_0$. For $m = 0$, the statement is clear. Let $m \geq 1$. Then

$$\begin{aligned}\lambda\left(\mathbb{K}_0^{(m)}(\Lambda)\right) &= \frac{1}{m!}\int_{(\mathbb{R}_+^* \times \mathbb{R}^d)^m} \mathbb{1}_{\mathbb{K}_0^{(m)}(\Lambda)}\left(\sum_{i=1}^m s_i\delta_{x_i}\right)\nu(ds_1)\ldots\nu(ds_m)\sigma(dx_1)\ldots\sigma(dx_m)\\ &= \frac{1}{m!}\int_{(\mathbb{R}_+^* \times \mathbb{R}^d)^m}\prod_{i=1}^m \mathbb{1}_\Lambda(x_i)\nu(ds_1)\ldots\nu(ds_m)\sigma(dx_1)\ldots\sigma(dx_m)\\ &= \frac{\nu(\mathbb{R}_+^*)^m\sigma(\Lambda)^m}{m!} < \infty\end{aligned}$$

□

Example 4. *One specific example is the measure based on $\nu_\theta(ds)$ from Definition 2.22. We fix $\theta > 0$ and omit the dependence for convenience. Then $\lambda_{\nu\otimes\sigma}$ can be rewritten as*

$$\lambda_{\nu\otimes\sigma}(d\eta) = f(\eta)\lambda_{\tilde{\nu}\otimes\sigma}(d\eta),$$

where f is the density function from Corollary 2.79 and $\tilde{\nu}(ds) = \theta e^{-s}ds$. Since $\tilde{\nu}$ is a finite measure on $\mathbb{R}_+^*$, we see by Example 3 that $\lambda_{\tilde{\nu}\otimes\sigma}$ is locally finite. Therefore, $B_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$ is dense in $L^1(\lambda_{\tilde{\nu}\otimes\sigma})$ and $\widetilde{B}_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$ is dense in $L^1(\lambda_{\nu\otimes\sigma})$.

As stated above, it also makes sense to state the density of $B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$ in the L^1-spaces.

Proposition 2.80. *Let $\rho \in \mathcal{M}_{\mathrm{lf}}(\mathbb{K}_0(\mathbb{R}^d))$. The space $B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$ is dense in $L^1(\rho)$ as well as $L^1(f\rho)$, where f is the density function from Corollary 2.79.*

Proof. By Lemma 2.78 and Corollary 2.79, it suffices to show that functions in the spaces $B_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$ and $\widetilde{B}_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$ can be approximated by functions in $B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$, where the convergence is taken with respect to $L^1(\rho)$ and $L^1(f\rho)$ respectively. Consider $G \in B_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$. Define the sequence $\{G_n\}_{n=0}^\infty$ by

$$G_n(\eta) := G(\eta)\cdot\prod_{x\in\tau(\eta)}\mathbb{1}_{[\frac{1}{n},n]}(s_x),\ \eta \in \mathbb{K}_0(\mathbb{R}^d).$$

Then

$$|G_n(\eta)| \leq C\mathbb{1}_{\mathbb{K}_0(\Lambda)}(\eta)\mathbb{1}_{\{|\tau(\eta)|\leq N\}}\prod_{x\in\tau(\eta)}\mathbb{1}_{[\frac{1}{n},n]}(s_x)$$

where C, Λ and N are given as in Definition 2.70. This shows that $G_n \in B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$. Let us show that G can be approximated by such functions:

$$\begin{aligned}\|G_n - G\|_{L^1} &= \int_{\mathbb{K}_0(\mathbb{R}^d)}|G_n(\eta) - G(\eta)|\rho(d\eta)\\ &= \int_{\mathbb{K}_0(\mathbb{R}^d)}|G(\eta)|\left|1 - \prod_{x\in\tau(\eta)}\mathbb{1}_{[\frac{1}{n},n]}(s_x)\right|\rho(d\eta)\\ &= \int_{\mathbb{K}_0(\mathbb{R}^d)}|G(\eta)|\rho(d\eta) - \int_{\mathbb{K}_0(\mathbb{R}^d)}|G(\eta)|\prod_{x\in\tau(\eta)}\mathbb{1}_{[\frac{1}{n},n]}(s_x)\rho(d\eta)\end{aligned}$$

Since $G \in L^1(\rho)$, by Lebesgue's theorem, it suffices to show $G_n \to G$ pointwisely. Fix $\eta \in \mathbb{K}_0(\mathbb{R}^d)$. Since $\tau(\eta)$ is finite, there exists $n_0 \in \mathbb{N}$ such that $s_x \in [\frac{1}{n}, n]\ \forall x \in \tau(\eta)$ for all $n \geq n_0$. Therefore,

$$\prod_{x\in\tau(\eta)} \mathbb{1}_{[\frac{1}{n},n]}(s_x) = 1\ \forall n \geq n_0$$

which means that $G_n(\eta) = G(\eta)$ for all $n \geq n_0$, i.e. $G_n \to G$ pointwisely. By above arguments, this implies $G_n \to G$ in L^1, which completes the proof.

The proof that $B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$ is dense in $L^1(f\rho)$ follows the same scheme. Note that the estimate for G_n as above for $G \in \widetilde{B}_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$ reads

$$\begin{aligned}|G_n(\eta)| &\leq C\mathbb{1}_{\mathbb{K}_0(\Lambda)}(\eta)\mathbb{1}_{\{|\tau(\eta)|\leq N\}} \prod_{x\in\tau(\eta)} s_x \mathbb{1}_{[\frac{1}{n},n]}(s_x)\\ &\leq Cn^N \mathbb{1}_{\mathbb{K}_0(\Lambda)}(\eta)\mathbb{1}_{\{|\tau(\eta)|\leq N\}} \prod_{x\in\tau(\eta)} \mathbb{1}_{[\frac{1}{n},n]}(s_x)\end{aligned}$$

which also implies $G_n \in B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$. □

Similar to the case of $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$, we now want to consider the extension of the K-transform to L^1-type spaces.

2.10.1 Correlation Measures on $\mathbb{K}_0(\mathbb{R}^d)$

In this section, we introduce the correlation measures related to a probability measure μ on $(\mathbb{K}(\mathbb{R}^d), \mathcal{B}(\mathbb{K}(\mathbb{R}^d)))$, similarly to $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$. We then extend the K-transform to L^1-type spaces. Note that due to the structure of the spaces $\mathbb{K}(\mathbb{R}^d)$ and $\mathbb{K}_0(\mathbb{R}^d)$, we need to pay attention to the properties of measures with respect to its marks. We proceed as before:

Definition 2.81. *Define the pre-kernel derived from the K-transform by*

$$\begin{aligned}\mathcal{K}\colon \mathcal{B}_b(\mathbb{K}_0(\mathbb{R}^d)) \times \mathbb{K}(\mathbb{R}^d) &\to [0,\infty)\\ (A,\eta) &\mapsto \mathcal{K}(A,\eta) := (K\mathbb{1}_A)(\eta)\end{aligned}$$

In a similar fashion to Definition 2.50, we can show that $\mathcal{K}$ is a pre-kernel. Furthermore, the same extension result holds:

Lemma 2.82. *The pre-kernel $\mathcal{K}$ can be uniquely extended to a kernel on $\mathcal{B}(\mathbb{K}_0(\mathbb{R}^d)) \times \mathbb{K}(\mathbb{R}^d)$.*

Proof. Similar to the proof of Lemma 2.51 □

The following proposition relates the K-transform to the kernel defined above.

Proposition 2.83. *Let $G\colon \mathbb{K}_0(\mathbb{R}^d) \to \mathbb{R}$ be a measurable function with $G \geq 0$ or $G \in B_{\mathrm{ls}}(\mathbb{K}_0(\mathbb{R}^d))$. Then*

$$\int_{\mathbb{K}_0(\mathbb{R}^d)} G(\xi)\mathcal{K}(d\xi,\eta) = \sum_{\xi\Subset\eta} G(\xi) = (KG)(\eta)$$

Proof. Similar to the proof of Proposition 2.52 □

Now that $\mathcal{K}$ is defined as a kernel on $(\mathbb{K}(\mathbb{R}^d), \mathcal{B}(\mathbb{K}(\mathbb{R}^d)))$, we may use it to construct measures on $\mathbb{K}_0(\mathbb{R}^d)$.

Definition 2.84. *Let μ be a probability measure on the space $(\mathbb{K}(\mathbb{R}^d), \mathcal{B}(\mathbb{K}(\mathbb{R}^d)))$. The corresponding correlation measure is defined as a measure on $(\mathbb{K}_0(\mathbb{R}^d), \mathcal{B}(\mathbb{K}_0(\mathbb{R}^d)))$ by the relation*

$$\rho_\mu(A) := \int_{\mathbb{K}(\mathbb{R}^d)} \mathcal{K}(A, \eta)\mu(d\eta)$$

As we have seen before, it may be of interest to know if a correlation measure ρ_μ is locally finite. One should note though, that the measures on $\mathbb{K}_0(\mathbb{R}^d)$ used in applications will not be locally finite as defined above, cf. Example 4. Such measures are mark-locally finite, though. Since this measure was constructed using a locally finite measure, it still makes sense to examine the class of locally finite measures. The connection to properties of μ is stated in the next proposition.

Proposition 2.85. *Let μ be a probability measure on $(\mathbb{K}(\mathbb{R}^d), \mathcal{B}(\mathbb{K}(\mathbb{R}^d)))$. Then the corresponding correlation measure ρ_μ is locally finite if and only if the following holds: For any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and $N \in \mathbb{N}$,*

$$\int_{\mathbb{K}(\mathbb{R}^d)} |\tau(\eta) \cap \Lambda|^N \mu(d\eta) < \infty$$

A measure μ with the above property is said to have finite local moments of all order. The space of all such measures is denoted by $\mathcal{M}^1_{\mathrm{fm}}(\mathbb{K}(\mathbb{R}^d))$.

Remark 2.86. *It should be stressed again that typical measures from applications do not fulfill the above property: As seen in Proposition 2.30, we have $|\tau(\eta) \cap \Lambda| = \infty$ for any compact set $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ due to the fact that Λ does not take into account the mark variable. From a constructive standpoint, it is still crucial to consider these measures.*

Proof of Proposition 2.85. The proof works analogously to the one of Proposition 2.54. □

We want to turn our attention to measures on $\mathbb{K}_0(\mathbb{R}^d)$ which are not locally finite, but at least mark-locally finite. These measures are the analogue of locally finite measures on $\Pi_0(\mathbb{R}^*_+ \times \mathbb{R}^d)$. More precisely, we want to examine a certain type of measures mentioned already: For a measure ρ on $\mathbb{K}_0(\mathbb{R}^d)$, set

$$\widetilde{\rho}(d\eta) := f(\eta)\rho(d\eta)$$

where $f \colon \mathbb{K}_0(\mathbb{R}^d) \to (0, \infty)$ is the density function defined as

$$f(\eta) = \prod_{x \in \tau(\eta)} \frac{1}{s_x}.$$

Lemma 2.87. *Let ρ be a locally finite measure. Then $\widetilde{\rho}$ is mark-locally finite.*

Proof. Let $A \in \mathcal{B}_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$. Then $A \in \mathcal{B}_b(\mathbb{K}_0(\mathbb{R}^d))$. Furthermore, there exists $a > 0$ such that for all $\eta \in A$, we have $s_x \geq a$ for all $x \in \tau(\eta)$. Then

$$\begin{aligned}
\widetilde{\rho}(A) &= \int_{\mathbb{K}_0(\mathbb{R}^d)} \mathbb{1}_A(\eta)\widetilde{\rho}(d\eta) = \int_{\mathbb{K}_0(\mathbb{R}^d)} \mathbb{1}_A(\eta) f(\eta)\rho(d\eta) \\
&\leq \int_{\mathbb{K}_0(\mathbb{R}^d)} \mathbb{1}_A(\eta) \max\left(1, \frac{1}{a^N}\right) \rho(d\eta) = \max\left(1, \frac{1}{a^N}\right) \rho(A) < \infty
\end{aligned}$$

□

2.10.2 Correlation Functions on $\mathbb{K}_0(\mathbb{R}^d)$

For the correlation measures on $\mathbb{K}_0(\mathbb{R}^d)$, we now want to analyse the question of existence of a density function. From the point of view of applications, the considered correlation measures are usually mark-locally finite, but not locally finite. Therefore, we concentrate our efforts on this class of measures. We analyse such measures by pulling them back to the Plato space $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and using the results from the previous chapters. To compare measures on $\mathbb{K}(\mathbb{R}^d)$ and $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$, we need the notion of mark-local absolute continuity.

Definition 2.88. *Let μ be a probability measure on $(\mathbb{K}(\mathbb{R}^d), \mathcal{B}(\mathbb{K}(\mathbb{R}^d)))$.*

1. *Let $\Lambda \subset \mathbb{R}_+^* \times \mathbb{R}^d$. For $\eta \in \mathbb{K}(\mathbb{R}^d)$ of the form $\eta = \sum_{x\in\tau(\eta)} s_x\delta_x$, define the projection with marks as*
$$p_\Lambda(\eta) = \sum_{\substack{x\in\tau(\eta)\\(s_x,x)\in\Lambda}} s_x\delta_x$$
The projection measure is defined as
$$\mu^\Lambda := \mu \circ p_\Lambda^{-1}.$$
2. *The measure μ is called mark-locally absolutely continuous with respect to the gamma measure $\mathcal{G}_\theta$ if for any $\Lambda \subset \mathbb{R}_+^* \times \mathbb{R}^d$ compact, the measure μ^Λ is absolutely continuous with respect to $\mathcal{G}_\theta^\Lambda$.*

Proposition 2.89. *A correlation measure ρ_μ on $\mathbb{K}_0(\mathbb{R}^d)$ corresponding to a probability measure μ on $\mathbb{K}(\mathbb{R}^d)$ is mark-locally finite if and only if the measure $\rho_{\mu_{\mathcal{R}^{-1}}}$ on $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$ is locally finite.*

Proof. Let ρ_μ be the correlation measure of a measure μ on $\mathbb{K}(\mathbb{R}^d)$. Then for a set $A \in \mathcal{B}_{\mathrm{cm}}(\mathbb{K}_0)$,

$$\begin{aligned}
\rho_\mu(A) &= \int_{\mathbb{K}(\mathbb{R}^d)} \mathcal{K}(A,\eta)\mu(d\eta) = \int_{\mathbb{K}(\mathbb{R}^d)} (K_{\mathbb{K}}\mathbb{1}_A)(\eta)\mu(d\eta)\\
&= \int_{\mathbb{K}(\mathbb{R}^d)} \left[K_\Pi(\mathcal{R}^{-1}\mathbb{1}_A)\right](\mathcal{R}^{-1}\eta)\mu(d\eta) = \int_{\mathbb{K}(\mathbb{R}^d)} \left[K_\Pi \mathbb{1}_{\mathcal{R}^{-1}A}\right](\mathcal{R}^{-1}\eta)\mu(d\eta)\\
&= \int_{\Pi(\mathbb{R}_+^*\times\mathbb{R}^d)} (K_\Pi \mathbb{1}_{\mathcal{R}^{-1}A})(\gamma)\mu_{\mathcal{R}^{-1}}(d\gamma)\\
&= \rho_{\mu_{\mathcal{R}^{-1}}}(\mathcal{R}^{-1}A)
\end{aligned}$$

where $\mu_{\mathcal{R}^{-1}}$ is the pullback measure of μ under $\mathcal{R}$. Reversing the calculations yields the converse result. $\square$

The following Lemma is needed to compare measures on $\mathbb{K}(\mathbb{R}^d)$ and $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$:

Lemma 2.90. *Let μ as a probability measure on $\mathbb{K}(\mathbb{R}^d)$ be mark-locally absolutely continuous with respect to $\mathcal{G}_\theta$. Then $\mu_{\mathcal{R}^{-1}}$ on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ is locally absolutely continuous with respect to π_θ.*

Proof. Consider $\Lambda \in \mathcal{B}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$ and let $A \in \Pi(\Lambda)$ with $\pi_\theta^\Lambda(A) = 0$. Then

$$0 = \pi_\theta^\Lambda(A) = \mathcal{G}_\theta^\Lambda(\mathcal{R}A)$$

since μ^Λ is absolutely continuous with respect to $\mathcal{G}_\theta^\Lambda$, this implies

$$0 = \mu^\Lambda(\mathcal{R}A) = \mu^\Lambda_{\mathcal{R}^{-1}}(A)$$

which implies the claim. □

These two statements together yield the following:

Proposition 2.91. *Let μ be a probability measure on $\mathbb{K}(\mathbb{R}^d)$ which is mark-locally absolutely continuous with respect to $\mathcal{G}_\theta$ and its associated correlation measure ρ_μ is mark-locally finite. Then $\mu_{\mathcal{R}^{-1}} \in \mathcal{M}^1_{\mathrm{fm}}(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d))$ and it is locally absolutely continuous with respect to π_θ. Furthermore, $\rho_{\mu_{\mathcal{R}^{-1}}}$ is absolutely continuous with respect to λ_θ and its correlation function exists.*

Proof. The statement follows with Proposition 2.89, Lemma 2.90 and Proposition 2.63. □

Altogether, we obtain the desired result:

Theorem 2.92. *Assume the conditions of Proposition 2.91. Then the correlation function of μ exists, i.e. a function $k_\mu \colon \mathbb{K}_0(\mathbb{R}^d) \to \mathbb{R}$ such that k_μ is the density function of ρ_μ with respect to $\lambda_{\theta,\mathcal{R}}$.*

Proof. By the above proposition, we obtain the existence of a correlation function for $\rho_{\mu_{\mathcal{R}^{-1}}}$ on $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$ with respect to λ_θ. The claim follows by transferring this function via the reflection mapping $\mathcal{R}$. □

There are some more considerations related to correlation functions useful in applications, which should be mentioned here. Namely, the so-called hierarchical structure associated with a function $k \colon \mathbb{K}_0(\mathbb{R}^d) \to \mathbb{R}$. This way, we may replace a function on $\mathbb{K}_0(\mathbb{R}^d)$ by a sequence of functions on $(\mathbb{R}_+^* \times \mathbb{R}^d)^n$. This is useful in applications, since we may replace an evolution equation on an infinite-dimensional space by a sequence of equations on finite-dimensional spaces. A similar notion was introduced in Definition 2.64.

Definition 2.93. *Let $k \colon \mathbb{K}_0(\mathbb{R}^d) \to \mathbb{R}$. The hierarchical structure corresponding to k is defined as the sequence of symmetric functions $\{k^{(n)}\}_{n=0}^\infty$, $k^{(n)} \colon (\mathbb{R}_+^* \times \mathbb{R}^d)^n \to \mathbb{R}$ by*

$$k^{(n)}(s_1, x_1, \ldots, s_n, x_n) := \begin{cases} k(\sum_{i=1}^n s_i \delta_{x_i}), & \text{if } \eta = \sum_{i=1}^n s_i \delta_{x_i} \in \mathbb{K}_0^{(n)}(\mathbb{R}^d) \\ 0, & \text{otherwise} \end{cases}$$

If associated with a correlation function k_μ, the function $k_\mu^{(n)}$ is called the n-point correlation function of μ. We also write

$$k^{(n)}(s_1, \ldots, x_n) := k^{(n)}(s_1, x_1, \ldots, s_n, x_n)$$

for convenience.

There is a related definition which may also be seen as correlation function on the space $\mathbb{K}_0(\mathbb{R}^d)$. Namely, we use the relation to the Bogoliubov functional on $\mathbb{K}(\mathbb{R}^d)$. It is defined the following way:

Definition 2.94. *For a function $\psi \in C_c(\mathbb{R}^d)$, define the Bogoliubov functional associated with a probability measure μ on $\mathbb{K}(\mathbb{R}^d)$ as*

$$L_\mu^{\mathbb{K}}(\psi) = \int_{\mathbb{K}(\mathbb{R}^d)} \prod_{x\in\tau(\eta)} (1 + s_x\psi(x))\mu(d\eta).$$

Using Proposition 2.66, we see that for a function $\psi \in C_c(\mathbb{R}^d)$, we obtain

$$\begin{aligned} L_\mu^{\mathbb{K}}(\psi) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}_+^* \times \mathbb{R}^d)^n} & s_1 \cdots s_n \psi_1(x_1) \cdots \psi_n(x_n) \times \\ & \times k_\mu^{(n)}(s_1, \dots, x_n)\nu(ds_1) \dots \nu(ds_n)\sigma(dx_1) \dots \sigma(dx_n) \\ = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} & \psi(x_1) \cdots \psi_n(x_n) \times \\ & \times \left[\int_{(\mathbb{R}_+^*)^n} s_1 \cdots s_n k_\mu^{(n)}(s_1, \dots, x_n)\nu(ds_1) \dots \nu(ds_n) \right] \sigma(dx_1) \dots \sigma(dx_n) \end{aligned}$$

Therefore, we may define the following notion of a correlation function:

Definition 2.95. *The n-point correlation function on $\mathbb{K}_0(\mathbb{R}^d)$ with respect to positions is defined as*

$$\widetilde{k}_\mu^{(n)}(x_1, \dots, x_n) := \int_{(\mathbb{R}_+^*)^n} s_1, \cdots s_n k_\mu^{(n)}(s_1, \dots, x_n)\nu(ds_1) \dots \nu(ds_n).$$

where $k^{(n)}$ is the n-point correlation function introduced in Definition 2.93.

Remark 2.96. *Since the function $\widetilde{k}^{(n)}$ can be obtained via integration of $k^{(n)}$, we proceed by only analysing the latter.*

2.11 Markov Evolution

It is well-known that there is a link between functional analysis and probability relating semigroups and stochastic processes. We want to briefly explain this connection, since it forms the foundation for the construction of our dynamics. Namely, we consider jump-type operators on the space of functions on $\mathbb{K}(\mathbb{R}^d)$ to define our models.

As seen in [15, Chapter 4.2], we may explicitly construct a Markov process given a bounded operator of the type

$$Af(x) = \kappa \int_E f(y) - f(x)\mu(x, dy) \tag{22}$$

where $\kappa > 0$ is an intensity parameter and $\mu\colon E \times \mathcal{B}(E)$ a transition function, where E is a Banach space. The operator A is defined on the space $B(E)$ of bounded functions on E and corresponds to a Markov process with $\exp(\kappa)$-distributed jump times. The jumps occur from x to y with distribution $\mu(x, dy)$. An explicit construction of this process can be found in the aforementioned work. Instead of analysing the stochastic process, one may also analyse the evolution equation

$$\begin{aligned} \frac{\partial}{\partial t} u_t(x) &= Au_t(x) \\ u_t(x)_{|t=0} &= u_0(x). \end{aligned}$$

We take this idea and apply it to our setting.

In Chapter 4, we will define operators of the shape (22) as starting point. Note that in our case, the whole particle system $\eta \in \mathbb{K}(\mathbb{R}^d)$ "jumps" to $\eta + s\delta_y$ or $\eta - s_x\delta_x$, corresponding to birth or death of a particle. Due to the complex structure of the underlying particle system, a direct analysis of this operator is usually not possible. Below, we describe alternative ways to describe and analyse the dynamical systems on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\mathbb{K}(\mathbb{R}^d)$.

2.11.1 Markov Evolution on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$

As seen in the previous chapter, there exists a correspondence between classes of functions on $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$. The analysis of physical systems on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ or $\mathbb{K}(\mathbb{R}^d)$ poses various problems: For one, the number of particles is infinite. Furthermore, the underlying space is infinite-dimensional which restricts the number of available tools.

On the other hand, the K-transform gives us a possibility to analyse the dynamics by transforming evolution equations on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ to equations on $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$. This chapter describes the equations which can be used to describe the dynamics in this way. The scheme described here was also used in the homogeneous case in [30].

From a physical perspective, we start with the evolution of measures, or states. The system of infinitely many particles is too complex to describe it pathwisely, i.e. "track" the motion of each particle. Instead, a statistical approach is more suitable. The evolution is governed by the so-called Fokker-Planck equation, also known as forward Kolmogorov equation:

$$\frac{\partial}{\partial t}\langle F, \mu_t\rangle = \langle F, L^*\mu_t\rangle$$
$$\mu_{t|t=0} = \mu_0$$

where μ_0 is a probability measure on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$, L^* a linear operator on the class of probability measures, and F from a suitable class of test functions.

From a modelling perspective, the dynamics are usually defined via the so-called backward Kolmogorov equation

$$\frac{\partial}{\partial t}F_t(\gamma) = LF_t(\gamma),\ \gamma \in \Pi(\mathbb{R}_+^* \times \mathbb{R}^d).$$
$$F_{t|t=0} = F_0$$

where L is the pre-dual operator to L^*. This way, the evolution can be described explicitly via the Markov-type operator L. The functions $F\colon \Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \to \mathbb{R}$ are called observables. They represent physical quantities of a dynamical system, such as energy. The duality mentioned above is given by

$$\langle F, \mu\rangle = \int_{\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)} F(\gamma)\mu(d\gamma),$$

where μ is a probability measure on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$. Typically, we use the Poisson measure here. We will see how to describe the dynamics explicitly via the operator L in later chapters.

As stated before, the analysis of the infinite-dimensional system above poses some difficulties. Instead, we use the K-transform to define new operators on functions on

$\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$. As we saw in Chapter 2.9.3, under some conditions, the correlation measure ρ_{μ_0} admits a correlation function k_0. The evolution of this function is described by the so-called quasi-Fokker-Planck equation

$$\begin{aligned} \frac{\partial}{\partial t} k_t(\gamma) &= L^{\triangle} k_t(\gamma), \ \gamma \in \Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d) \\ k_{t|t=0} &= k \end{aligned} \tag{23}$$

Here, the operator $L^{\triangle}$ is given as the dual K-transform of L^*, i.e. $L^{\triangle} = K^* L^* K^{*-1}$. One related question is of course if the evolution k_t is again the correlation function of a measure μ_t on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$.

The fourth way to describe the dynamics of our system is via the quasi-Kolmogorov equation. We consider the pre-dual operator of $L^{\triangle}$ with respect to the duality

$$\langle\langle G, k \rangle\rangle = \int_{\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)} G(\gamma) k(\gamma) \lambda(d\gamma). \tag{24}$$

where λ is a locally finite measure on $\Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d)$. We usually consider the Lebesgue-Poisson measure here. The operator is denoted by $\hat{L}$ and can be explicitly calculated as $\hat{L} = K^{-1} L K$. It is also referred to as the symbol of L. The equation has the following form:

$$\begin{aligned} \frac{\partial}{\partial t} G_t(\gamma) &= \hat{L} G_t(\gamma), \ \gamma \in \Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d) \\ G_{t|t=0} &= G_0 \end{aligned}$$

2.11.2 Markov Evolution on $\mathbb{K}(\mathbb{R}^d)$

The description of evolutions can also be done on the spaces $\mathbb{K}(\mathbb{R}^d)$ and $\mathbb{K}_0(\mathbb{R}^d)$. Similarly, the physical description of a particle system is given by the Fokker-Planck equation

$$\begin{aligned} \frac{\partial}{\partial t} \langle F, \mu_t \rangle &= \langle F, L^* \mu_t \rangle \\ \mu_{t|t=0} &= \mu_0 \end{aligned}$$

where μ_0 is a probability measure on $\mathbb{K}(\mathbb{R}^d)$ and F from a suitable class of test functions. Again, from a modelling perspective, the (backward) Kolmogorov equation is appropriate:

$$\begin{aligned} \frac{\partial}{\partial t} F_t(\eta) &= L F_t(\eta), \ \eta \in \mathbb{K}(\mathbb{R}^d). \\ F_{t|t=0} &= F_0 \end{aligned}$$

where L is a Markov (pre-)generator. The functions $F\colon \mathbb{K}(\mathbb{R}^d) \to \mathbb{R}$ are called observables. They represent physical quantities of a dynamical system, such as energy. The duality is similarly given by

$$\langle F, \mu \rangle = \int_{\mathbb{K}(\mathbb{R}^d)} F(\eta) \mu(d\eta).$$

Also, we have corresponding equations on the spaces of Radon measures with finite support $\mathbb{K}_0(\mathbb{R}^d)$, also named quasi-Kolmogorov equation and quasi-Fokker-Planck equation:

$$\begin{aligned} \frac{\partial}{\partial t} G_t(\eta) &= \hat{L} G(\eta), \ \eta \in \mathbb{K}_0(\mathbb{R}^d) \\ G_{t|t=0} &= G_0 \end{aligned}$$

and

$$\frac{\partial}{\partial t}k_t(\eta) = L^{\triangle}k_t(\eta),\ \eta \in \mathbb{K}_0(\mathbb{R}^d) \\ k_{t|t=0} = k_0 \tag{25}$$

As before, the operator $\hat{L} = K^{-1}LK$ is called the symbol of L. Equation (25) of course only makes sense if the evolution of measures on $\mathbb{K}_0(\mathbb{R}^d)$ admits a density function as explained in Chapter 2.10. The duality used for functions on $\mathbb{K}_0(\mathbb{R}^d)$ is given by

$$\langle\langle G, k\rangle\rangle = \int_{\mathbb{K}_0(\mathbb{R}^d)} G(\eta)k(\eta)\lambda(d\eta). \tag{26}$$

where λ is a mark-locally finite measure on $\mathbb{K}_0(\mathbb{R}^d)$, usually the image measure of the Lebesgue-Poisson measure under $\mathcal{R}$. The approach used for the analysis of the models in Chapter 4 is as follows: We set up the specific generator on functions on $\mathbb{K}(\mathbb{R}^d)$. This way, we explicitly see the heuristic dynamics of the system. Next, we calculate the corresponding operator $\hat{L}$ on quasi-observables using the K-transform and some combinatorial arguments. Since the calculations are quite similar for all models, we do not carry out the calculations for all models. If appropriate, we use $\hat{L}$ to prove the existence of the dynamics.

The next step is to calculate the operator $L^{\triangle}$ of the statistical dynamics. As stated in Chapter 2.10, the correlation functions has a direct relation to the underlying probability measure μ. Therefore, also the evolution of correlation functions may have this connection to the evolution of states. If appropriate, we take the evolution of correlation functions to show specific properties of the model. At this point, the representation of the correlation functions via the hierarchical structure is very useful, since it lets us see dependencies within the particle system.

After considering the evolution of (25), one important question is the existence of a probability measure on $\mathbb{K}(\mathbb{R}^d)$ such that the solution k_t is the correlation function of the measure μ. This question can be answered using a classical result by Lenard, [43].

Theorem 2.97 (Cf. [30,43])**.** *Assume that the function $k\colon \mathbb{K}_0(\mathbb{R}^d) \to \mathbb{R}$ and its associated sequence $\{k^{(n)}\}_{n=0}^{\infty}$ satisfy the following conditions:*

1. *Normalisation: $k^{(0)} \equiv 1$*
2. *Lenard positivity: For any $G\colon \mathbb{K}_0(\mathbb{R}^d) \to \mathbb{R}$ with $KG \geq 0$, we have*

$$\int_{\mathbb{K}_0(\mathbb{R}^d)} G(\eta)k(\eta)\lambda(d\eta) \geq 0$$

3. *Moment growth: For any bounded set $\Lambda \subset \mathbb{R}_+^* \times \mathbb{R}^d$ and $j \geq 0$, the following holds:*

$$\sum_{n=0}^{\infty}(m_{n+j}^{\Lambda})^{-\frac{1}{n}} = \infty$$

where the moments are defined as

$$m_n^{\Lambda} := (n!)^{-1}\int_{\Lambda^n} k^{(n)}(s_1,\dots,x_n)\nu(ds_1)\dots\nu(ds_n)\sigma(dx_1)\dots\sigma(dx_n)$$

Then there exists a unique measure μ on $(\mathbb{K}(\mathbb{R}^d), \mathcal{B}(\mathbb{K}(\mathbb{R}^d)))$ such that k is the correlation function associated with μ.

Remark 2.98. *The theorem in [30] was originally formulated on the space $\Gamma(Y)$. Nevertheless, the way it is stated above, it also works for $\mathbb{K}(\mathbb{R}^d)$ by transferring it to $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$. This is due to the fact that the existence of a correlation function as above suffices to have a measure on $\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$ which assigns full mass to the Plato space $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$.*

3 Calculus on the Spaces $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\mathbb{K}(\mathbb{R}^d)$

The spaces $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\mathbb{K}(\mathbb{R}^d)$ open up different possibilities to define calculus of functions on these spaces. For one, these spaces have a continuous structure by the nature of the underlying space $\mathbb{R}^d$. This gives rise to the classical notion of differential calculus. The space $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ naturally inherits the differential structure from the superset $\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$. Since the restriction conserves full mass with respect to typical measures, the process is straightforward and is explained in the first part of this chapter. In the general case of $\Gamma(X)$ for some Riemannian manifold X, the theory was already established in [4]. We may take the notions of e.g. the Laplacian and apply it to our case directly.

The cone $\mathbb{K}(\mathbb{R}^d)$ gives rise to a slightly altered differential structure. Due to the asymmetry in marks, we need to introduce a specific group of flows when transitioning from $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ to $\mathbb{K}(\mathbb{R}^d)$. In the end, we will show that these construction describe the same phenomenon, though, at least for some special cases. This way, we may also define objects such as the Laplacian on $\mathbb{K}(\mathbb{R}^d)$, see e.g. [28] or [38]. One should note, though, that there are some technical steps to be considered, since the underlying measure μ on $\mathbb{K}(\mathbb{R}^d)$ is not quasi-invariant with respect to the group action generating the differential calculus.

On the other hand, the discrete nature of elements $\gamma \in \Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\eta \in \mathbb{K}(\mathbb{R}^d)$ give rise to a different kind of calculus, namely, difference calculus. This way, we may combine the continuous structure of the state space $\mathbb{R}^d$ with the discrete structure of elements in $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\mathbb{K}(\mathbb{R}^d)$. The associated discrete Laplacian is introduced and analysed.

A short part of this chapter is also devoted to the so-called umbral calculus. This theory is concerned with the analysis of certain types of polynomials and acts as a generalisation of the theory of combinatorics related to the binomial coefficient and gives a nice relation to the combinatorial structure of difference calculus and the K-transform. For a general introduction to umbral calculus, see e.g. [51]. For a more detailed view on the infinite-dimensional case, see [21] and the references therein.

3.1 Differential Calculus on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$

To properly introduce dynamics on the space $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$, we should analyse the differential structure of the space. The theory has been established on $\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$ in works such as [2, 4]. See [3] for the case where Gibbs measures are considered as underlying measures on the space. Since typical probability measures on $\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$ assign full mass to $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$, the theory remains unchanged when considering $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ instead of $\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$. The goal of this chapter is to introduce typical notions of differential geometry on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$. We define a gradient and show an integration by parts formula as well as a Laplace operator on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$. Note that the theory can be easily extended to a more general Riemannian manifold instead of $\mathbb{R}^d$.

From now on, assume that the intensity measure $\varkappa$ of the Poisson measure $\pi_\varkappa$ on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d), \mathcal{B}(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d))$ admits a density function with respect to the Lebesgue measure.

3.1.1 The Group of Diffeomorphisms

The differential geometry on configuration spaces may be defined as a lifting of the differential geometry on the underlying manifold, in our case, $\mathbb{R}^d$. To define directional derivatives, we consider the group $\mathrm{Diff}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$ of all diffeomorphisms on Y which are equal to the identity outside of a compact set, where the group operation is the usual composition. We now want to lift this group to $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$.

Definition 3.1. *Let $\varphi \in \mathrm{Diff}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$. Define a function on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ as*

$$\varphi\colon \Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \to \Pi(\mathbb{R}_+^* \times \mathbb{R}^d),\ \varphi(\gamma) := \{\varphi(s,x) \mid (s,x) \in \gamma\}$$

The choice of $\mathrm{Diff}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$ implies that there are only finitely many points of γ which are changed by the mapping φ.

Let us recall some results from [4]. The properties are especially essential for our case, since it implies that the diffeomorphisms preserve the mass of the space $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$.

Proposition 3.2 ([4], Proposition 2.1, 2.2)**.** *Let $\varphi \in \mathrm{Diff}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\pi_\varkappa$ the Poisson measure with intensity measure $\varkappa$. Then the following holds:*

1. *Set $\varphi^*\pi_\varkappa := \pi_\varkappa \circ \varphi^{-1}$. Then*
$$\varphi^*\pi_\varkappa = \pi_{\varphi^*\varkappa}$$

2. *The measure $\pi_\varkappa$ is quasi-invariant with respect to $\mathrm{Diff}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$ and we have*
$$\frac{d(\varphi^*\pi_\varkappa)}{d\pi_\varkappa}(\gamma) = \prod_{(s,x)\in\gamma} p_\varphi^\varkappa(s,x) \exp\left(\int_{\mathbb{R}_+^* \times \mathbb{R}^d} (1 - p_\varphi^\varkappa(s,x))\varkappa(ds,dx)\right)$$
where $p_\varphi^\varkappa$ depends on the measure $\varkappa$ and the transformation φ and is given explicitly.

3.1.2 The Gradient on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$

We may now introduce the notion of derivatives of functions on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$. To this end, we consider the connection between the group $\mathrm{Diff}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$ and its corresponding Lie algebra. Set $V_c(\mathbb{R}_+^* \times \mathbb{R}^d)$ to be the set of all C^∞-vector fields $v_\Pi\colon \Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \to \Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ such that there exists a compact set $K \subset \mathbb{R}_+^* \times \mathbb{R}^d$ such that $v_\Pi(s,x) = 0$ if $(s,x) \notin K$. In other words, we consider smooth vector fields with compact support. For a fixed $v \in V_c(\mathbb{R}_+^* \times \mathbb{R}^d)$, we may consider the associated flow, i.e. the subgroup $\{\varphi_t^v\}_{t\in\mathbb{R}}$ of $\mathrm{Diff}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$ such that the following equation is solved for any $(s,x) \in \mathbb{R}_+^* \times \mathbb{R}^d$:

$$\begin{aligned} \frac{d}{dt}\varphi_t^v(s,x) &= v(\varphi_t^v(s,x)),\ t \in \mathbb{R} \\ \varphi_0^v(s,x) &= (s,x) \end{aligned}$$

This enables us to define a directional derivative for functions on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$.

Definition 3.3. *Let $F\colon \Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \to \mathbb{R}$ and $v_\Pi \in V_c(\mathbb{R}_+^* \times \mathbb{R}^d)$. Define the directional derivative along v_Π as*

$$(\nabla_{v_\Pi}^\Pi F)(\gamma) := \frac{d}{dt} F(\varphi_t^{v_\Pi}(\gamma))_{|t=0}$$

provided it exists.

For a special class of functions, we may give a more explicit representation of the derivative.

Definition 3.4. *Let $\mathcal{D} = C_c^\infty(\mathbb{R}_+^* \times \mathbb{R}^d)$ be the space of all infinitely differentiable functions $\varphi\colon Y \to \mathbb{R}$ with compact support and $C_b^\infty(\mathbb{R}^N)$, $N \in \mathbb{N}$ all infinitely differentiable bounded functions $g\colon \mathbb{R}^n \to \mathbb{R}$. The space $\mathcal{F}C_b^\infty(\mathcal{D}, \Pi(\mathbb{R}_+^* \times \mathbb{R}^d))$ is defined as the space of all functions $F\colon \Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \to \mathbb{R}$ which are of the form*

$$F(\gamma) = g(\langle \varphi_1, \gamma \rangle, \dots, \langle \varphi_N, \gamma \rangle)$$

where $g \in C_b^\infty(\mathbb{R}^N)$ and $\varphi_i \in \mathcal{D}$, $i = 1, \dots, N$. The pairing of $\varphi \in \mathcal{D}$ and $\gamma \in \Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ is given by

$$\langle \varphi, \gamma \rangle = \sum_{(s,x) \in \gamma} \varphi(s, x).$$

Remark 3.5. *A function $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Pi(\mathbb{R}_+^* \times \mathbb{R}^d))$ belongs to the class of so-called cylinder functions, i.e. it only depends on elements of γ in some compact set $\Lambda \subset \mathbb{R}_+^* \times \mathbb{R}^d$.*

Proposition 3.6 ([4])**.** *Let $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Pi(\mathbb{R}_+^* \times \mathbb{R}^d))$. Then the directional derivative is given by*

$$(\nabla_{v_\Pi}^\Pi F)(\gamma) = \sum_{j=1}^N \partial_j g\left(\langle \varphi_1, \gamma \rangle, \dots, \langle \varphi_N, \gamma \rangle\right) \langle \nabla_{v_\Pi} \varphi_j, \gamma \rangle$$

where ∇_{v_Π} denotes the directional derivative along v_Π on $\mathbb{R}_+^* \times \mathbb{R}^d$.

To introduce the gradient, we also need to define the appropriate tangent space. This allows us to use Riesz' representation theorem to obtain the existence of the gradient.

Definition 3.7. *We define the tangent space $T_\gamma(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d))$ at a configuration $\gamma \in \Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ as*

$$T_\gamma(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)) = L^2(\mathbb{R}_+^* \times \mathbb{R}^d \to T(\mathbb{R}_+^* \times \mathbb{R}^d), d\gamma)$$

i.e. the space of vector fields $V\colon \mathbb{R}_+^* \times \mathbb{R}^d \to T(\mathbb{R}_+^* \times \mathbb{R}^d)$ which are square-summable with respect to γ. The corresponding scalar product is given by

$$\begin{aligned}
\langle V_\gamma^1, V_\gamma^2 \rangle_{T_\gamma(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d))} &:= \int_{\mathbb{R}_+^* \times \mathbb{R}^d} \langle V_\gamma^1(s,x), V_\gamma^2(s,x) \rangle_{T_{(s,x)}(\mathbb{R}_+^* \times \mathbb{R}^d)} \gamma(ds, dx) \\
&= \sum_{(s,x) \in \gamma} \langle V_\gamma^1(s,x), V_\gamma^2(s,x) \rangle
\end{aligned}$$

The scalar product on $T_{(s,x)}(\mathbb{R}_+^* \times \mathbb{R}^d)$ is just the Euclidean scalar product. Furthermore, the space $T_\gamma(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d))$ equipped with $\langle \cdot, \cdot \rangle_{T_\gamma(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d))}$ is a Hilbert space. We see that $v_\Pi \in V_c(\mathbb{R}_+^* \times \mathbb{R}^d)$ can be seen as a vector field $v_\Pi \in T_\gamma(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d))$ which is constant in $\gamma \in \Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$.

The introduction of a tangent space enables us to also define the gradient.

Definition 3.8. *Let $F\colon \Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \to \mathbb{R}$. We define the gradient of F as the mapping*

$$\nabla^\Pi F\colon \Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \to T_\gamma(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)),\ \gamma \mapsto (\nabla^\Pi F)(\gamma)$$

such that for any $v_\Pi \in V_c(\mathbb{R}_+^* \times \mathbb{R}^d)$, the following holds:

$$(\nabla_{v_\Pi}^\Pi F)(\gamma) = \langle \nabla^\Pi F, v_\Pi \rangle_{T_\gamma(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d))}.$$

Similar to Proposition 3.6, we have an explicit representation of the gradient for smooth cylinder functions:

Proposition 3.9. *Let $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Pi(\mathbb{R}_+^* \times \mathbb{R}^d))$. Then for any $\gamma \in \Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $(s,x) \in \mathbb{R}_+^* \times \mathbb{R}^d$, we have*

$$(\nabla^\Pi F)(\gamma; s, x) = \sum_{j=1}^N \partial_j g(\langle \varphi_1, \gamma\rangle, \dots, \langle \varphi_N, \gamma\rangle) \nabla \varphi(s,x) \tag{27}$$

where ∇ is the gradient on $\mathbb{R}_+^* \times \mathbb{R}^d$.

3.1.3 Integration by Parts Formula on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$

After the definition of the gradient on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$, we want to state the corresponding integration by parts formula. In other words, we calculate the adjoint $(\nabla^\Pi)^*$ of the gradient ∇^Π. Here, we only consider the case where the probability measure on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ is Poissonian as defined in Chapter 2.8.1. We denote by $\mathrm{div}^{(s,x)}$ the divergence operator in $(s,x) \in \mathbb{R}_+^* \times \mathbb{R}^d$.

Theorem 3.10 ([4], Theorem 3.1)**.** *Let $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Pi(\mathbb{R}_+^* \times \mathbb{R}^d))$ and a vector field $v \in V_c(\mathbb{R}_+^* \times \mathbb{R}^d)$. Then, the following holds:*

$$\begin{aligned}\int_{\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)} (\nabla_v^\Pi F)(\gamma) G(\gamma) \pi_\varkappa(d\gamma) = &- \int_{\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)} F(\gamma)(\nabla_v^\Pi G)(\gamma) \pi_\varkappa(d\gamma) \\ &- \int_{\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)} F(\gamma) G(\gamma) B_v^{\pi_\varkappa}(\gamma) \pi_\varkappa(d\gamma)\end{aligned}$$

where $B_v^{\pi_\varkappa}$ plays the role of a logarithmic derivative of the classical integration by parts formula and is given by the following expression:

$$B_v^{\pi_\varkappa}(\gamma) = \langle \beta_v^\varkappa, \gamma\rangle = \sum_{(s,x)\in\gamma} \langle \beta^\varkappa(s,x), v(s,x)\rangle + \mathrm{div}^{(s,x)} v(s,x)$$

where

$$\beta^\varkappa(s,x) := \frac{\nabla \rho(x)}{\rho(x)} \tag{28}$$

and ρ is the density function of $\varkappa$ with respect to the Lebesgue measure.

Remark 3.11.

1. *As also seen in Proposition 3.6, the shape of the logarithmic derivative hints at the fact that the differential geometry on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ can be seen as a lifting of the geometry of the underlying space Y. For a more detailed explanation, see [4].*

2. *The operator ∇^Π can be extended to a domain $D(\mathcal{E}_{\pi_\varkappa}^\Pi)$, where $D(\mathcal{E}_{\pi_\varkappa}^\Pi)$ will be specified in the next chapter.*

3.1.4 The Continuous Laplacian on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$

One more important object from differential calculus is the Laplace operator, especially when considering stochastic processes. The Laplacian will be defined using the theory of Dirichlet forms. To this end, we define the following pre-Dirichlet form associated with the gradient introduced before.

Definition 3.12. *Let $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Pi(\mathbb{R}_+^* \times \mathbb{R}^d))$. Define the bilinear form*

$$\mathcal{E}_{\pi_\varkappa}^\Pi(F,G) := \int_{\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)} \langle \nabla^\Pi F(\gamma), \nabla^\Pi G(\gamma) \rangle_{T_\gamma(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d))} \pi_\varkappa(d\gamma).$$

$\mathcal{E}_{\pi_\varkappa}^\Pi$ is also called the intrinsic pre-Dirichlet form corresponding to $\pi_\varkappa$ on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$.

We proceed to define a differential operator which will turn out to be the "Laplacian" associated with the bilinear form introduced above.

Definition 3.13. *Let $F \in \mathcal{F}_b^\infty(\mathcal{D}, \Pi(\mathbb{R}_+^* \times \mathbb{R}^d))$ of the form*

$$F(\gamma) = g(\langle \varphi_1, \gamma \rangle, \dots, \langle \varphi_N, \gamma \rangle)$$

with all objects as defined before. Define an operator $(H_{\pi_\varkappa}^\Pi, \mathcal{F}C_b^\infty(\mathcal{D}, \Pi(\mathbb{R}_+^* \times \mathbb{R}^d)))$ as

$$\begin{aligned}(H_{\pi_\varkappa}^\Pi F)(\gamma) = &- \sum_{i,j=1}^N \partial_i \partial_j g(\langle \varphi_1, \gamma \rangle, \dots, \langle \varphi_N, \gamma \rangle) \sum_{(s,x)\in\gamma} \langle \nabla \varphi_i(s,x), \nabla \varphi_j(s,x) \rangle \\ &- \sum_{j=1}^N \partial_j g(\langle \varphi_1, \gamma \rangle, \dots, \langle \varphi_N, \gamma \rangle) \sum_{(s,x)\in\gamma} \Delta \varphi_j(s,x) \\ &- \sum_{j=1}^N \partial_j g(\langle \varphi_1, \gamma \rangle, \dots, \langle \varphi_N, \gamma \rangle) \sum_{(s,x)\in\gamma} \langle \nabla \varphi_j(s,x), \beta^\varkappa(s,x) \rangle\end{aligned}$$

where $\beta^\varkappa$ was defined in (28) and Δ is the Laplacian on $\mathbb{R}_+^* \times \mathbb{R}^d$.

As similar calculations will be done later for the difference calculus on $\mathbb{K}(\mathbb{R}^d)$, we only state the results and omit the calculations here. We refer to [4] for further details.

Theorem 3.14 ([4], Theorem 4.1, Corollary 4.1)**.** *The operator $H_{\pi_\varkappa}^\Pi$ is associated with the Dirichlet form $\mathcal{E}_{\pi_\varkappa}^\Pi$. More precisely, for any $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Pi(\mathbb{R}_+^* \times \mathbb{R}^d))$, we have*

$$\mathcal{E}_{\pi_\varkappa}^\Pi(F,G) = (H_{\pi_\varkappa}^\Pi F, G)_{L^2(\pi_\varkappa)}.$$

Moreover, the form $(\mathcal{E}_{\pi_\varkappa}^\Pi, \mathcal{F}C_b^\infty(\mathcal{D}, \Pi(\mathbb{R}_+^* \times \mathbb{R}^d)))$ is closable on $L^2(\pi_\varkappa)$. Its closure, which is denoted by $(\mathcal{E}_{\pi_\varkappa}^\Pi, D(\mathcal{E}_{\pi_\varkappa}^\Pi))$, is associated with the positive definite, self-adjoint Friedrichs' extension $(H_{\pi_\varkappa}^\Pi, D(H_{\pi_\varkappa}^\Pi))$ of the operator $(H_{\pi_\varkappa}^\Pi, \mathcal{F}C_b^\infty(\mathcal{D}, \Pi(\mathbb{R}_+^* \times \mathbb{R}^d)))$.

We may also extend the gradient introduced before.

Proposition 3.15 ([4], Corollary 4.2)**.** *The operator ∇^Π can be extended to the domain $D(\mathcal{E}_{\pi_\varkappa}^\Pi)$. Moreover, let $F\colon \Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \to \mathbb{R}$ of the form*

$$F(\gamma) = g(\langle \varphi_1, \gamma \rangle, \dots, \langle \varphi_N, \gamma \rangle)$$

where this time, $g \in C_b^\infty(\mathbb{R}^N)$ and $\varphi_i \in H_0^1(\mathbb{R}_+^* \times \mathbb{R}^d, \varkappa)$ from the Sobolev space of order 1. Then $F \in D(\mathcal{E}_{\pi_\varkappa}^\Pi)$ and (27) also holds for F (almost everywhere).

3.2 Differential Calculus on $\mathbb{K}(\mathbb{R}^d)$

We now proceed to introduce differential structures on $\mathbb{K}(\mathbb{R}^d)$. Note that one needs to account for the asymmetry arising through the difference between the weights and the positions of particles. This gives rise to the definition of an extrinsic and intrinsic gradient. The goal is to combine these two to obtain a gradient for functions on $\mathbb{K}(\mathbb{R}^d)$. Unfortunately, the asymmetric structure of the underlying space prevents certain invariance properties usually given on other spaces. More precisely, we usually want to consider an infinite measure on the marks from $\mathbb{R}_+^*$, i.e.

$$\int_{\mathbb{R}_+^*} \nu(ds) = \infty. \tag{29}$$

Therefore, it is not possible to obtain a unitary representation of the Lie group corresponding to the gradient. It is still possible to define a Laplace operator. Note the comparison with the space $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$, where the mark variable is treated the same as all position variables.

Due to the special structure of $\mathbb{K}(\mathbb{R}^d)$, we need to construct the gradients for the marks and position separately. The gradients corresponding to the marks are called extrinsic gradients, since these operate on the external structure of a fixed element $\eta \in \mathbb{K}(\mathbb{R}^d)$. The gradients with respect to the positions are called intrinsic, since these are related to the underlying spatial and differential structure of the state space $\mathbb{R}^d$. After constructing these gradients separately, we may combine them to obtain a joint gradient for functions on $\mathbb{K}(\mathbb{R}^d)$. To this end, we define a semidirect product of groups corresponding to the different types of gradients.

The construction of the extrinsic and intrinsic gradient follows the same scheme as the construction of the gradient on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$. In general, this chapter follows the considerations given in [27] and [38].

3.2.1 Extrinsic Gradient

Let us start by considering the extrinsic gradient. To this end, consider the group of so-called currents. It is defined as the set of all continuous functions $\theta\colon \mathbb{R}^d \to \mathbb{R}_+^*$ which are equal to one outside a compact set. The group operation is given by pointwise multiplication of two such functions. Denote this group by $C_c(\mathbb{R}^d \to \mathbb{R}_+^*)$. It acts on the space of Radon measures $\mathbb{M}(\mathbb{R}^d)$ in the following way: If we take $\theta \in C_c(\mathbb{R}^d \to \mathbb{R}_+^*)$ and a measure $\eta \in \mathbb{M}(\mathbb{R}^d)$, the modified measure $\theta\eta$ is defined as

$$\theta\eta(dx) = \theta(x)\eta(dx).$$

in particular, for $\eta = \sum_{x\in\tau(\eta)} s_x\delta_x \in \mathbb{K}(\mathbb{R}^d)$, this means

$$\theta(x)\eta = \sum_{x\in\tau(\eta)} \theta(x)s_x\delta_x.$$

We can now use this action of $C_c(\mathbb{R}^d \to \mathbb{R}_+^*)$ on $\mathbb{M}(\mathbb{R}^d)$ to define the extrinsic derivative.

Definition 3.16. *Let $h \in C_c(\mathbb{R}^d)$ and consider the corresponding one-parameter subgroup of $C_c(\mathbb{R}^d \to \mathbb{R}_+^*)$ given by $(e^{th})_{t\in\mathbb{R}}$. For a function $F\colon \mathbb{M}(\mathbb{R}^d) \to \mathbb{R}$, an extrinsic derivative in direction h is defined the following way:*

$$(\nabla_h^{\text{ext}} F)(\eta) = \frac{d}{dt}F(e^{th}\eta)_{|t=0}, \;\; \eta \in \mathbb{M}(\mathbb{R}^d).$$

provided, the derivative exists. The corresponding extrinsic tangent space is given by

$$T_\eta^{\mathrm{ext}}(\mathbb{M}) = L^2(\mathbb{R}^d, \eta)$$

which becomes a Hilbert space with the standard L^2-scalar product. The extrinsic gradient of F at the point η is defined as $\nabla^{\mathrm{ext}} F \colon \mathbb{K}(\mathbb{R}^d) \to T_\eta^{\mathrm{ext}}(\mathbb{M})$ via the following relation:

$$(\nabla_h^{\mathrm{ext}} F)(\eta) = \langle (\nabla^{\mathrm{ext}} F)(\eta), h \rangle_{T_\eta^{\mathrm{ext}}(\mathbb{M})} = \int_{\mathbb{R}^d} (\nabla^{\mathrm{ext}} F)(\eta) h(x) \eta(dx) \ \forall h \in C_c(\mathbb{R}^d).$$

3.2.2 Intrinsic Gradient

Let us continue with the notion of an intrinsic gradient. Again, we define a group which acts on the space $\mathbb{M}(\mathbb{R}^d)$. Let $\mathrm{Diff}_c(\mathbb{R}^d)$ denote the group of diffeomorphisms with compact support. In this case, the notion of compact support means that a diffeomorphism $\psi \in \mathrm{Diff}_c(\mathbb{R}^d)$ is equal to the identity outside of a compact set. The group operation is the usual composition of mappings. Furthermore, the action of $\mathrm{Diff}_c(\mathbb{R}^d)$ on the space $\mathbb{M}(\mathbb{R}^d)$ is given as follows: For an element $\psi \in \mathrm{Diff}_c(\mathbb{R}^d)$ and $\eta \in \mathbb{M}(\mathbb{R}^d)$, and a Borel set $\Lambda \in \mathcal{B}(\mathbb{R}^d)$, we set

$$(\psi^* \eta)(\Lambda) := \eta(\psi^{-1} \Lambda),$$

i.e. $\psi^* \eta$ is the pushforward measure of η under ψ. In the special case where $\eta \in \mathbb{K}(\mathbb{R}^d)$, $\eta = \sum_{x \in \tau(\eta)} s_x \delta_x$, we obtain

$$\psi^* \eta(\Lambda) = \sum_{x \in \tau(\eta)} s_x \delta_x \left(\psi^{-1}(\Lambda) \right) = \sum_{x \in \tau(\eta)} s_x \delta_{\psi(x)}(\Lambda).$$

since the total mass is conserved and ψ is a diffeomorphism, we see that $\psi^* \eta \in \mathbb{K}(\mathbb{R}^d)$.

Similar to the differential geometry on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$, we consider the set $V_c(\mathbb{R}^d)$ of all compactly supported vector fields on $\mathbb{R}^d$. Each $v \in V_c(\mathbb{R}^d)$ can be associated with a one-parameter subgroup of diffeomorphisms $\{\psi_t^v\}_{t \in \mathbb{R}} \subset \mathrm{Diff}_c(\mathbb{R}^d)$ as the solution to the family of differential equations

$$\frac{d}{dt} \psi_t(x) = v(\psi_t(x))$$
$$\psi_0(x) = x$$

for $x \in \mathbb{R}^d$. We call $\{\psi_t^v\}_{t \in \mathbb{R}}$ the flow associated with v.

The directional derivative can again be defined in the sense of Lie derivatives.

Definition 3.17. *Let $v \in V_c(\mathbb{R}^d)$ and $\{\psi_t^v\}_{t \in \mathbb{R}} \subset \mathrm{Diff}_c(\mathbb{R}^d)$ the associated flow. For a function $F \colon \mathbb{K}(\mathbb{R}^d) \to \mathbb{R}$, the intrinsic derivative along v is defined by*

$$(\nabla_v^{\mathrm{int}} F)(\eta) = \frac{d}{dt} F(\psi_t^{v*} \eta)_{|t=0}$$

provided, it exists. The corresponding intrinsic tangent space is set as

$$T_\eta^{\mathrm{int}}(\mathbb{K}(\mathbb{R}^d)) := L^2(\mathbb{R}^d \to T(\mathbb{R}^d), \eta)$$

with the scalar product

$$\langle V_\eta^1, V_\eta^2 \rangle_{T_\eta^{\mathrm{int}}(\mathbb{K}(\mathbb{R}^d))} := \sum_{x \in \tau(\eta)} s_x \langle V_\eta^1(x), V_\eta^2(x) \rangle_{T_x(\mathbb{R}^d)}$$

where the scalar product on $T_x(\mathbb{R}^d)$ is again just the Euclidean scalar product.

We define the intrinsic gradient of such a function F as the mapping $\nabla^{\text{int}} F\colon \mathbb{K}(\mathbb{R}^d) \to T_\eta^{\text{int}}(\mathbb{K}(\mathbb{R}^d))$ which satisfies the following equation for all $v \in V_c(\mathbb{R}^d)$:

$$(\nabla_v^{\text{int}} F)(\eta) = \langle (\nabla^{\text{int}} F)(\eta), v \rangle_{T_\eta^{\text{int}}(\mathbb{K}(\mathbb{R}^d))}.$$

3.2.3 The Group $\mathfrak{G}$ on $\mathbb{K}(\mathbb{R}^d)$

We want to define the group acting on $\mathbb{K}(\mathbb{R}^d)$ as the analogue of $\text{Diff}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$ on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$. To this end, we combine the actions used in the extrinsic and intrinsic gradient above. The joint action of $g = (\theta, \psi) \in C_c(\mathbb{R}^d \to \mathbb{R}_+^*) \times \text{Diff}_c(\mathbb{R}^d)$ on an element $\eta = \sum_{x \in \tau(\eta)} s_x \delta_x \in \mathbb{K}(\mathbb{R}^d)$ is given by

$$(g\eta) = \theta \cdot \psi^* \eta = \sum_{x \in \tau(\eta)} \theta(x) s_x \delta_{\psi(x)}$$

where $\psi^* \eta$ denotes the pushforward measure.

For two elements $g_1 = (\theta_1, \psi_1)$ and $g_2 = (\theta_2, \psi_2)$, the composition acts as

$$(g_2 \circ g_1)(\eta) = g_2 g_1 \eta = g_2 \left[\sum_{x \in \tau(\eta)} h_1(x) s_x \delta_{\psi_1(x)} \right] = \sum_{x \in \tau(\eta)} h_2(x) h_1(\psi_2^{-1}(x)) s_x \delta_{\psi_2 \circ \psi_1(x)}$$

Therefore, the product on the group $\mathfrak{G}$ is given by

$$g_2 \circ g_1 = (\theta_2 \cdot \theta_1 \circ \psi_2^{-1}, \psi_2 \circ \psi_1).$$

Group-theoretically speaking, the group $\mathfrak{G}$ equipped with this product becomes the semi-direct product of the aforementioned groups:

$$\mathfrak{G} = C_c(\mathbb{R}^d \to \mathbb{R}_+^*) \rtimes \text{Diff}_c(\mathbb{R}^d)$$

Later, we will compare the actions of the groups on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\mathbb{K}(\mathbb{R}^d)$.

3.2.4 Joint Gradient

The extrinsic and intrinsic gradient can be merged together to obtain a joint gradient for functions on $\mathbb{K}(\mathbb{R}^d)$. After introducing the joint gradient, we proceed by explicitly calculating the action for a class of cylinder functions.

Definition 3.18. *Let $h \in C_c(\mathbb{R}^d)$ and $v \in V_c(\mathbb{R}^d)$. The directional derivative of a function $F\colon \mathbb{K}(\mathbb{R}^d) \to \mathbb{R}$ is given as the sum of the extrinsic and intrinsic derivative, i.e.*

$$(\nabla_{(h,v)}^{\mathbb{K}} F)(\eta) := (\nabla_h^{\text{ext}} F)(\eta) + (\nabla_v^{\text{int}} F)(\eta).$$

The corresponding tangent space is given by

$$T_\eta(\mathbb{K}(\mathbb{R}^d)) = T_\eta^{\text{ext}}(\mathbb{K}(\mathbb{R}^d)) \oplus T_\eta^{\text{int}}(\mathbb{K}(\mathbb{R}^d)).$$

The gradient can be defined directly as

$$\nabla^{\mathbb{K}} := (\nabla^{\text{ext}}, \nabla^{\text{int}}).$$

Let us now consider concrete functions. We define the following set of cylinder functions on $\mathbb{K}(\mathbb{R}^d)$ where we are able to give an explicit expression for the directional derivative.

Definition 3.19. *Consider the following set of functions:*

$$\mathcal{F}C(\mathbb{K}(\mathbb{R}^d)) := \mathcal{R}\left[\mathcal{F}C_b^\infty(\mathcal{D}(\mathbb{R}_+^* \times \mathbb{R}^d), \Pi(\mathbb{R}_+^* \times \mathbb{R}^d))\right]$$

In other words, $F \in \mathcal{F}C(\mathbb{K}(\mathbb{R}^d))$ iff there exists a function $G\colon \Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \to \mathbb{R}$,

$$G(\gamma) = g(\langle \varphi_1, \gamma\rangle, \ldots, \langle \varphi_N, \gamma\rangle),\ \ g \in C_b^\infty(\mathbb{R}^N), \varphi_i \in \mathcal{D}(\mathbb{R}_+^* \times \mathbb{R}^d)$$

such that $F(\eta) = (\mathcal{R}G)(\eta) = G(\mathcal{R}^{-1}\eta)$. We may also write

$$F(\eta) = g(\langle\langle \varphi_1, \eta\rangle\rangle, \ldots, \langle\langle \varphi_N, \eta\rangle\rangle), \tag{30}$$

where $\langle\langle \varphi, \eta\rangle\rangle = \langle \varphi, \mathcal{R}^{-1}\eta\rangle$.

Let us calculate the directional derivatives considered above for such functions.

Proposition 3.20. *Let $h \in C_c(\mathbb{R}^d), v \in V_c(\mathbb{R}^d)$ and $F \in \mathcal{F}C(\mathbb{K}(\mathbb{R}^d))$. Then the directional derivative has the following representation:*

$$\begin{aligned}&(\nabla^{\mathbb{K}}_{(h,v)}F)(\eta) = \\ &\quad = \sum_{i=1}^N \partial_j g(\langle\langle \varphi_1, \eta\rangle\rangle, \ldots, \langle\langle \varphi_N, \eta\rangle\rangle) \sum_{x\in\tau(\eta)} \left[\partial_s \varphi_j(s_x, x)h(x)s_x + \nabla^{\mathbb{R}^d}\varphi_j(s_x,x)v(x)\right]\end{aligned}$$

Proof. Let $h \in C_c(\mathbb{R}^d), F \in \mathcal{F}C(\mathbb{K}(\mathbb{R}^d))$. We calculate the extrinsic and intrinsic gradient separately. Start with the calculation of

$$(\nabla^{\mathbb{K}}_h F)(\eta) = \frac{d}{dt}\left[g(\langle\langle \varphi_1, e^{th}\eta\rangle\rangle, \ldots, \langle\langle \varphi_N, e^{th}\eta\rangle\rangle)\right]_{|t=0}.$$

To this end, let $\varphi \in C_c(\mathbb{R}_+^* \times \mathbb{R}^d)$. Then

$$\langle\langle \varphi, e^{th}\eta\rangle\rangle = \sum_{x\in\tau(\eta)} \varphi(e^{th(x)}s_x, x) = \sum_{\substack{(s_x,x)\in\mathcal{R}^{-1}\eta \\ (s_x,x)\in\operatorname{supp}\varphi}} \varphi(e^{th(x)}s_x, x)$$

Since φ has compact support, the sum is finite and we can calculate the derivative in t:

$$\frac{d}{dt}\langle\langle \varphi, e^{th}\eta\rangle\rangle = \sum_{x\in\tau(\eta)} \frac{d}{dt}\varphi(e^{th(x)}s_x, x) = \sum_{x\in\tau(\eta)} h(x)e^{th(x)}s_x\partial_s\varphi(e^{th(x)}s_x, x)$$

For the derivative of F, this means

$$\begin{aligned}&\frac{d}{dt}\left[g(\langle\langle \varphi_1, e^{th}\eta\rangle\rangle, \ldots, \langle\langle \varphi_N, e^{th}\eta\rangle\rangle)\right] \\ &\quad = \sum_{j=1}^N \partial_j g(\langle\langle \varphi_1, e^{th}\eta\rangle\rangle, \ldots, \langle\langle \varphi_N, e^{th}\eta\rangle\rangle)\frac{d}{dt}\langle\langle \varphi_j, e^{th}\eta\rangle\rangle \\ &\quad = \sum_{j=1}^N \partial_j g(\langle\langle \varphi_1, e^{th}\eta\rangle\rangle, \ldots, \langle\langle \varphi_N, e^{th}\eta\rangle\rangle) \sum_{x\in\tau(\eta)} h(x)e^{th(x)}s_x\partial_s\varphi(e^{th(x)}s_x, x)\end{aligned}$$

At $t=0$, this means

$$(\nabla^{\mathbb{K}}_h F)(\eta) = \sum_{j=1}^N \partial_j g(\langle\langle \varphi_1, \eta\rangle\rangle, \ldots, \langle\langle \varphi_N, \eta\rangle\rangle) \sum_{x\in\tau(\eta)} h(x) s_x \partial_s \varphi(s_x, x)$$
$$= \sum_{j=1}^N \partial_j g(\langle\langle \varphi_1, \eta\rangle\rangle, \ldots, \langle\langle \varphi_N, \eta\rangle\rangle) \langle\langle \partial_s\varphi \cdot h, \eta\rangle\rangle.$$

Let us now look at the derivative in the space variable, i.e.

$$(\nabla^{\mathbb{K}}_v F)(\eta) = \frac{d}{dt}\left[g(\langle\langle \varphi_1, \psi^v_t(\eta)\rangle\rangle, \ldots, \langle\langle \varphi_N, \psi^v_t(\eta)\rangle\rangle)\right]_{|t=0}.$$

Let $v \in V_c(\mathbb{R}^d)$ and $\{\psi^v_t\}_{t\in\mathbb{R}} \subset \mathrm{Diff}_c(\mathbb{R}^d)$ the associated flow. Furthermore, let $\varphi \in C_c(\mathbb{R}^*_+ \times \mathbb{R}^d)$. Then

$$\frac{d}{dt}\langle\langle \varphi, \psi^v_t(\eta)\rangle\rangle = \sum_{x\in\tau(\eta)} \frac{d}{dt}\varphi(s_x, \psi^v_t(x))$$
$$= \sum_{x\in\tau(\eta)} \nabla^{\mathbb{R}^d}\varphi(s_x, \psi^v_t(x)) \cdot \frac{d}{dt}\psi^v_t(x)$$
$$= \sum_{x\in\tau(\eta)} \nabla^{\mathbb{R}^d}\varphi(s_x, \psi^v_t(x)) \cdot v(\psi^v_t(x))$$

This implies

$$\frac{d}{dt}\left[g(\langle\langle \varphi_1, \psi^v_t(\eta)\rangle\rangle, \ldots, \langle\langle \varphi_N, \psi^v_t(\eta)\rangle\rangle)\right]$$
$$= \sum_{j=1}^N \partial_j g(\langle\langle \varphi_1, \psi^v_t(\eta)\rangle\rangle, \ldots, \langle\langle \varphi_N, \psi^v_t(\eta)\rangle\rangle) \sum_{x\in\tau(\eta)} \left[\nabla^{\mathbb{R}^d}\varphi_j(s_x, \psi^v_t(x))\right] \cdot v(\psi^v_t(x))$$

where $\nabla^{\mathbb{R}^d}$ denotes the gradient on $\mathbb{R}^d$. At $t=0$, we obtain

$$(\nabla^{\mathbb{K}}_v F)(\eta) = \sum_{j=1}^N \partial_j g(\langle\langle \varphi_1, \eta\rangle\rangle, \ldots, \langle\langle \varphi_N, \eta\rangle\rangle) \sum_{x\in\tau(\eta)} \nabla^{\mathbb{R}^d}\varphi_j(s_x, x) \cdot v(x)$$
$$= \sum_{j=1}^N \partial_j g(\langle\langle \varphi_1, \eta\rangle\rangle, \ldots, \langle\langle \varphi_N, \eta\rangle\rangle) \sum_{x\in\tau(\eta)} \nabla^{\mathbb{R}^d}_v\varphi_j(s_x, x)$$

where $\nabla^{\mathbb{R}^d}_v$ denotes the directional derivative on $\mathbb{R}^d$ along v.

Let us combine these calculations. Let $h \in C_c(\mathbb{R}^d)$ and $v \in V_c(\mathbb{R}^d)$. Then

$$(\nabla^{\mathbb{K}}_{(h,v)} F)(\eta) = (\nabla^{\mathbb{K}}_h F)(\eta) + (\nabla^{\mathbb{K}}_v F)(\eta)$$
$$= \sum_{j=1}^N \partial_j g(\langle\langle \varphi_1, \eta\rangle\rangle, \ldots, \langle\langle \varphi_N, \eta\rangle\rangle) \sum_{x\in\tau(\eta)} \partial_s\varphi_j(s_x, x) h(x) s_x$$
$$+ \sum_{j=1}^N \partial_j g(\langle\langle \varphi_1, \eta\rangle\rangle, \ldots, \langle\langle \varphi_N, \eta\rangle\rangle) \sum_{x\in\tau(\eta)} \nabla^{\mathbb{R}^d}\varphi_j(s_x, x) v(x)$$
$$= \sum_{i=1}^N \partial_j g(\ldots) \sum_{x\in\tau(\eta)} \left[\partial_s\varphi_j(s_x, x) h(x) s_x + \nabla^{\mathbb{R}^d}\varphi_j(s_x, x) v(x)\right]$$

□

3.2.5 Integration by Parts on $\mathbb{K}(\mathbb{R}^d)$

The next step is to introduce an integration by parts formula on $\mathbb{K}(\mathbb{R}^d)$. There are some technical details which need to be considered, especially if we want to assume that (29) holds. This property breaks the quasi-invariance of measures $\mathcal{G}_\nu$ on $\mathbb{K}(\mathbb{R}^d)$ with respect to the underlying group of motions $\mathfrak{G}$. To state the integration by parts formula, we need to impose some assumptions on the measure ν:

1. Assume that (29) holds, or in other words, $\nu(\mathbb{R}_+^*) = \infty$.
2. Assume that ν has a representation
$$\nu(ds) = \frac{l(s)}{s} ds \tag{31}$$
where $l\colon \mathbb{R}_+^* \to (0, \infty)$. Note that we assume $l > 0$.
3. Assume that l fulfills the following integrability condition:
$$\int_{\mathbb{R}_+^*} l(s) \min\{1, s^{-1}\} ds < \infty \tag{32}$$

For technical reasons, one needs to be more careful regarding measurability. Therefore, for $n \in \mathbb{N}$, we introduce $\mathcal{B}_n(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d))$ as the smallest σ-algebra on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ such that the following mappings are measurable:

$$\gamma \mapsto |\gamma \cap \Lambda|$$

where $\Lambda \subset Y$ is compact and additionally, $\Lambda \subset \left[\frac{1}{n}, \infty\right) \times \mathbb{R}^d$.

On $\mathbb{K}(\mathbb{R}^d)$, we may introduce the image-σ-algebra

$$\mathcal{B}_n(\mathbb{K}(\mathbb{R}^d)) := \{\mathcal{R}A \mid A \in \mathcal{B}_n(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d))\}.$$

Note that for any function $F \in \mathcal{F}C(\mathbb{K}(\mathbb{R}^d))$, due to the compact support of the functions φ_i, there exists some $n \in \mathbb{N}$ such that F is $\mathcal{B}_n(\mathbb{K}(\mathbb{R}^d))$-measurable.

Under these considerations, we may now state the integration by parts result on $\mathbb{K}(\mathbb{R}^d)$. Denote by div^x the divergence operator in $x \in \mathbb{R}^d$.

Theorem 3.21 ([38], Theorem 14)**.** *Assume that* (29)*,* (31) *and* (32) *hold. Furthermore, assume that $l \in C^1(\mathbb{R}_+^*)$ and $l' \in L^1(\mathbb{R}_+^*, ds)$. Let $h \in C_c(\mathbb{R}^d)$ and $v \in V_c(\mathbb{R}^d)$. Then for any $F, G \in \mathcal{F}C(\mathbb{K}(\mathbb{R}^d))$ such that F, G are $\mathcal{B}_n(\mathbb{K}(\mathbb{R}^d))$-measurable, the following holds:*

$$\begin{aligned}\int_{\mathbb{K}(\mathbb{R}^d)} (\nabla^{\mathbb{K}}_{(h,v)} F)(\eta) G(\eta) \mathcal{G}_\nu(d\eta) = &- \int_{\mathbb{K}(\mathbb{R}^d)} F(\eta) (\nabla^{\mathbb{K}}_{(h,v)} G)(\eta) \mathcal{G}_\nu(d\eta) \\ &- \int_{\mathbb{K}(\mathbb{R}^d)} F(\eta) G(\eta) B^{(n)}_{(h,v)}(\eta) \mathcal{G}_\nu(d\eta)\end{aligned}$$

where the logarithmic derivative is defined as follows:

$$\begin{aligned} B^{(n)}_{(h,v)} &= B_h + B^{(n)}_v \\ B_h(\eta) &= \sum_{x \in \tau(\eta)} s_x \frac{l'(s_x)}{l(s_x)} h(x) + l(0) \int_{\mathbb{R}^d} h(x) dx \\ B^{(n)}_v(\eta) &= \sum_{\substack{x \in \tau(\eta) \\ s_x \geq 1/n}} \operatorname{div}^x v(x) \end{aligned}$$

where $l(0) := \lim_{s \to 0} l(s)$.

3.2.6 The Continuous Laplacian on $\mathbb{K}(\mathbb{R}^d)$

This section is devoted to the introduction of a Laplace operator related to the gradient introduced above. Again, we proceed using Dirichlet form theory. The corresponding Laplace operator can be given explicitly as lifting. It can then be shown that this lifted operator is the operator associated with the Dirichlet form.

Definition 3.22. *Assume that* (31) *and* (32) *hold. For* $F, G \in \mathcal{F}C(\mathbb{K}(\mathbb{R}^d))$*, define the bilinear form*

$$\mathcal{E}_\nu^{\mathbb{K}}(F,G) := \frac{1}{2}\int_{\mathbb{K}(\mathbb{R}^d)} \langle \nabla^{\mathbb{K}} F(\eta), \nabla^{\mathbb{K}} G(\eta)\rangle_{T_\eta(\mathbb{K}(\mathbb{R}^d))} \mathcal{G}_\nu(d\eta)$$

To introduce the operator which will turn out to be the Laplacian, we need to introduce some auxiliary differential operators.

Definition 3.23. *Let* $F \in \mathcal{F}C(\mathbb{K}(\mathbb{R}^d)), \eta \in \mathbb{K}(\mathbb{R}^d)$ *and* $x \in \tau(\eta)$*. Set*

$$(\Delta_x^{\mathbb{R}^d} F)(\eta) := \Delta^{\mathbb{R}^d} F(\eta - s_x\delta_x + s_x\delta_y)_{|y=x}$$
$$(\Delta_x^{\mathbb{R}_+^*} F)(\eta) := \Delta_u^{\mathbb{R}_+^*} F(\eta - s_x\delta_x + u\delta_x)_{|u=s}$$

where $\Delta^{\mathbb{R}^d}$ *is the Laplacian on* $\mathbb{R}^d$ *and*

$$(\Delta^{\mathbb{R}_+^*} f)(s) := s^2 f''(s) + s f'(s) + s^2 \frac{l'(s)}{l(s)} f'(s), \;\; s \in \mathbb{R}_+^*.$$

These differential operators can be seen as a "lifting" of operators on $\mathbb{R}^d$ *and* $\mathbb{R}_+^*$*, respectively.*

Let us define the Laplacian and state the connection to the Dirichlet integral defined above.

Theorem 3.24 ([38], Theorem 16)**.** *Assume* (31) *and* (32)*. Furthermore, assume that* $l \in C^1(\mathbb{R}_+^*)$*. For* $F \in \mathcal{F}C(\mathbb{K}(\mathbb{R}^d))$*, define the operator*

$$(L_\nu^{\mathbb{K}} F)(\eta) := \frac{1}{2}\sum_{x\in\tau(\eta)} \Big[(\Delta_x^{\mathbb{R}^d} F)(\eta) + (\Delta_x^{\mathbb{R}_+^*} F)(\eta)\Big]$$

Then $(L_\nu^{\mathbb{K}}, \mathcal{F}C(\mathbb{K}(\mathbb{R}^d)))$ *is a symmetric operator on* $L^2(\mathbb{K}(\mathbb{R}^d), \mathcal{G}_\nu)$ *and the following holds:*

$$\mathcal{E}_\nu^{\mathbb{K}}(F,G) = (-L_\nu^{\mathbb{K}} F, G)_{L^2(\mathbb{K}(\mathbb{R}^d),\mathcal{G}_\nu)}$$

Furthermore, $(\mathcal{E}_\nu^{\mathbb{K}}, \mathcal{F}C(\mathbb{K}(\mathbb{R}^d)))$ *is closable on* $L^2(\mathbb{K}(\mathbb{R}^d), \mathcal{G}_\nu)$*. Denote the closure of this form by* $(\mathcal{E}_\nu^{\mathbb{K}}, D(\mathcal{E}_\nu^{\mathbb{K}}))$*. Also, the operator* $(L_\nu^{\mathbb{K}}, \mathcal{F}C(\mathbb{K}(\mathbb{R}^d)))$ *has Friedrichs' extension which we denote by* $(L_\nu^{\mathbb{K}}, D(L_\nu^{\mathbb{K}}))$*. It is the operator associated with the form* $(\mathcal{E}_\nu^{\mathbb{K}}, D(\mathcal{E}_\nu^{\mathbb{K}}))$*.*

3.3 Comparing the Differential Calculus of $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\mathbb{K}(\mathbb{R}^d)$

Since the spaces $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\mathbb{K}(\mathbb{R}^d)$ are related via the reflection and should represent different viewpoints of the same situation, it makes sense that we should be able to arrive at the same results comparing the differential geometry of those spaces. The different nature of these spaces gives rise to different generalisations, but in some special cases, we may see that the results coincide. We start by comparing the group actions on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\mathbb{K}(\mathbb{R}^d)$, which motivates a certain explicit relation. Next, we compare the corresponding Lie algebras to show that the relation gives rise to well-defined flows, even though the element on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ related to an algebra element on $\mathbb{K}(\mathbb{R}^d)$ is not in $\mathrm{Diff}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$. Afterwards, we compare the notions of gradients and integration by parts to show that this correspondence in fact yields the same results on both spaces.

3.3.1 Comparing $\mathrm{Diff}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $C_c(\mathbb{R}^d \to \mathbb{R}_+^*) \rtimes \mathrm{Diff}_c(\mathbb{R}^d)$

As a first step, we motivate a correspondence between $\mathrm{Diff}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\mathfrak{G}$. Due to the nature of the considered groups, we do not obtain a one-to-one-correspondence. Nevertheless, we may still compare the action of the groups $\mathrm{Diff}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\mathfrak{G}$.

Let us start at the group $\mathfrak{G}$ on $\mathbb{K}(\mathbb{R}^d)$. Let $\eta \in \mathbb{K}(\mathbb{R}^d)$ and $g = (\theta, \psi) \in \mathfrak{G}$. Then as before,

$$g\eta = \sum_{x \in \tau(\eta)} \theta(x) s_x \delta_{\psi(x)}.$$

Taking the reverse reflection, we obtain

$$\mathcal{R}^{-1}(g\eta) = \sum_{(s,x) \in \mathcal{R}^{-1}\eta} \delta_{(\theta(x)s_x, \psi(x))}.$$

On $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$, this corresponds to the mapping

$$\varphi(s, x) = (\theta(x)s, \psi(x)).$$

This mapping is invertible with the inverse

$$\varphi^{-1}(s, x) = \left(\frac{1}{\theta(\psi^{-1}(x))} s, \psi^{-1}(x) \right).$$

Also, the inverse mapping φ^{-1} has the form $\varphi^{-1}(s, x) = (\vartheta(x)s, \phi(x))$ for some $\vartheta \in C_c(\mathbb{R}^d \to \mathbb{R}_+^*)$ and $\phi \in \mathrm{Diff}_c(\mathbb{R}^d)$, namely $\vartheta(x) = 1/\theta(\psi^{-1}(x))$ and $\phi(x) = \psi^{-1}$. This fact can also easily be seen considering the group structure of $\mathfrak{G}$.

Of course, in general, we still do not have $\varphi \in \mathrm{Diff}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$, since θ may not be differentiable and $s \cdot \theta(x)$ is not compactly supported in $\mathbb{R}_+^* \times \mathbb{R}^d$.

3.3.2 Explicit Relation: Lie Algebras

The relation which was motivated above can also be checked on the level of corresponding algebras. As noted before, we have the following setting:

The algebra corresponding to the group $\mathrm{Diff}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$ on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ is given by $V_c(\mathbb{R}_+^* \times \mathbb{R}^d)$, which is the set of all vector fields on $\mathbb{R}_+^* \times \mathbb{R}^d$ with compact support. The correspondence is given as above via the associated flow.

On $\mathbb{K}(\mathbb{R}^d)$, the corresponding set to the group $\mathfrak{G}$ is given by the pair (h, v), where $h \in C_c(\mathbb{R}^d)$ and $v \in V_c(\mathbb{R}^d)$. The corresponding flow is given by (e^{th}, ψ_t^v), where ψ_t^v is the flow associated with v.

Let us start with an element $(h, v) \in C_c(\mathbb{R}^d) \times V_c(\mathbb{R}^d)$. Our aim is to find a vector field on $\mathbb{R}_+^* \times \mathbb{R}^d$ corresponding to the pair (h, v) such that relations on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\mathbb{K}(\mathbb{R}^d)$ coincide.

As seen in the previous chapter, for an element $g = (\theta, \psi) \in \mathfrak{G}$, we have a corresponding element $(s \cdot \theta, \psi)$ on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$. More precisely, consider a flow $\{(e^{th(x)}, \psi_t^v(x))\}_{t\in\mathbb{R}} \subset \mathfrak{G}$ on $\mathbb{K}(\mathbb{R}^d)$. Then the corresponding flow on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ is obtained as

$$\varphi_t(s, x) = (s\theta_t(x), \psi_t^v(x)).$$

The vector field corresponding to $\{\varphi_t\}_{t\in\mathbb{R}}$ is given by

$$\frac{d}{dt}\varphi_t(s, x)_{|t=0} = (s \cdot h(x), v(x)) =: v_\Pi(s, x)$$

We want to consider this relation to show that the differential objects on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\mathbb{K}(\mathbb{R}^d)$ coincide. Note that we do not have $v_\Pi \in V_c(\mathbb{R}_+^* \times \mathbb{R}^d)$. Nevertheless, v_Π corr-responds to a subgroup of $\mathrm{Diff}(\mathbb{R}_+^* \times \mathbb{R}^d)$ which can be constructed in the same way as for the associated flow before: We are searching for a flow $\{\varphi_t\}_{t\in\mathbb{R}} \subset \mathrm{Diff}(\mathbb{R}_+^* \times \mathbb{R}^d)$ such that

$$\begin{aligned}\frac{d}{dt}\varphi_t(s, x) &= v_\Pi(\varphi_t(s, x))\\ \varphi_0(s, x) &= (s, x)\end{aligned}$$

If we write $\varphi_t(s, x) = (\alpha_t(s, x), \psi_t(s, x))$ for some functions $\alpha_t \colon Y \to \mathbb{R}_+^*$ and $\psi_t \colon Y \to \mathbb{R}^d$, the equation becomes

$$\begin{aligned}\frac{d}{dt}\alpha_t(s, x) &= \alpha_t(s, x)h(\psi_t(s, x))\\ \frac{d}{dt}\psi_t(s, x) &= v(\psi_t(s, x))\end{aligned}$$

The second equation is independent of the first, which implies $\psi_t(s, x) = \psi_t^v(x)$, where ψ_t^v is the flow associated with v. For the first equation, this means

$$\frac{d}{dt}\alpha_t(s, x) = \alpha_t(s, x)h(\psi_t^v(x))$$

which can be solved using standard ODE techniques in t. The solution can be represented in the form

$$\alpha_t(s, x) = \exp\left(\int_0^t h(\psi_t^v(x))dt\right)\alpha_0(s, x) = \exp\left(\int_0^t h(\psi_t^v(x))dt\right)s$$

which is well-defined for any $(s, x) \in \mathbb{R}_+^* \times \mathbb{R}^d$, since h and ψ_t^v are continuous. Therefore, even though the mapping $(h, v) \mapsto (s \cdot h, v)$ is not into $\mathrm{Diff}_c(\mathbb{R}_+^* \times \mathbb{R}^d)$, we have a well-defined flow on $\mathbb{R}_+^* \times \mathbb{R}^d$.

In what follows, we want to compare the notions of the derivative and integration by parts formula in the case outlined above.

3.3.3 Comparing Derivatives on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\mathbb{K}(\mathbb{R}^d)$

For the space of cylinder functions, we have already calculated the directional derivatives on both spaces $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\mathbb{K}(\mathbb{R}^d)$. Recall Propositions 3.6 and 3.20:

Reminder. *1. Let $v_\Pi \in V_c(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Pi(\mathbb{R}_+^* \times \mathbb{R}^d))$. Then the directional derivative is given by*

$$(\nabla_{v_\Pi}^{\Pi} F)(\gamma) = \sum_{j=1}^{N} \partial_j g\left(\langle \varphi_1, \gamma\rangle, \dots, \langle \varphi_N, \gamma\rangle\right) \langle \nabla_{v_\Pi} \varphi_j, \gamma\rangle \tag{33}$$

where ∇ denotes the gradient on $\mathbb{R}_+^* \times \mathbb{R}^d$.

2. *Let $h \in C_c(\mathbb{R}^d), v \in V_c(\mathbb{R}^d)$ and $F \in \mathcal{F}C(\mathbb{K}(\mathbb{R}^d))$. Then the directional derivative has the following representation:*

$$(\nabla_{(h,v)}^{\mathbb{K}} F)(\eta) =$$
$$= \sum_{i=1}^{N} \partial_j g(\langle\langle \varphi_1, \eta\rangle\rangle, \dots, \langle\langle \varphi_N, \eta\rangle\rangle) \sum_{x \in \tau(\eta)} \Big[\partial_s \varphi_j(s_x, x) h(x) s_x + \nabla^{\mathbb{R}^d} \varphi_j(s_x, x) v(x)\Big]$$

We show that in the case $v_\Pi = (s \cdot h, v)$, the representations coincide. The interesting part of (33) is the directional derivative in φ_j. For the special form of v_Π, this becomes

$$\begin{aligned}\nabla_{v_\Pi} \varphi_j = \langle \nabla \varphi_j, v_\Pi\rangle &= \partial_s \varphi_j(s, x^1, \dots, x^d) s h(x) + \sum_{i=1}^{d} \partial_{x_i} \varphi_j(s, x^1, \dots, x^d) v^i(x) \\ &= \partial_s \varphi_j(s, x^1, \dots, x^d) s h(x) + (\nabla^{\mathbb{R}^d} \varphi_j)(s, x^1, \dots, x^d) \cdot v(x)\end{aligned}$$

where $x = (x^1, \dots, x^d) \in \mathbb{R}^d$ and $v(x) = (v^1(x), \dots, v^d(x))$. In total, the directional derivative has the form

$$(\nabla_{v_\Pi}^{\Pi} F)(\gamma) = \sum_{j=1}^{N} \partial_j g(\langle \varphi_1, \gamma\rangle, \dots, \langle \varphi_N, \gamma\rangle) \sum_{(s,x) \in \gamma} \Big[\partial_s \varphi_j(s, x) s h(x) + (\nabla^{\mathbb{R}^d} \varphi_j)(s, x) \cdot v(x)\Big].$$

Keeping in mind the definition of the pairing $\langle\langle \cdot, \cdot \rangle\rangle$, we see that the derivatives coincide.

3.3.4 Comparing the Integration by Parts Formulae

Next, we compare the integration by parts formulae on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\mathbb{K}(\mathbb{R}^d)$. Note that this comparison assumes that $\nu(\mathbb{R}_+^*) < \infty$, which is not necessary for the theory on $\mathbb{K}(\mathbb{R}^d)$. While the rest of the formula is similar, the interesting part here is the logarithmic derivative. While in the general case, it is difficult to compare these parts, we may look at a special case where these functions on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\mathbb{K}(\mathbb{R}^d)$ coincide.

Reminder. *The logarithmic derivatives for the integration by parts formulae are given as follows:*

1. *For the case $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$:*

$$\begin{aligned}B_{v_\Pi}^{\pi_\varkappa}(\gamma) := \langle \beta_{v_\Pi}^{\varkappa}, \gamma\rangle &= \int_{\mathbb{R}_+^* \times \mathbb{R}^d} \Big[\langle \beta^\varkappa(y), v_\Pi(y)\rangle_{T_y(\mathbb{R}_+^* \times \mathbb{R}^d)} + \operatorname{div}^{(s,x)} v_\Pi(y)\Big] \gamma(dy) \\ &= \sum_{(s,x) \in \gamma} \Big[\langle \beta^\varkappa(s,x), v_\Pi(s,x)\rangle_{T_{(s,x)}(\mathbb{R}_+^* \times \mathbb{R}^d)} + \operatorname{div}^{(s,x)} v_\Pi(s,x)\Big]\end{aligned}$$

with $y = (s, x)$ and $\beta^\varkappa$ being the logarithmic derivative of the density function ρ, i.e.

$$\beta_{v_\Pi}^{\varkappa} = \left\langle \frac{\nabla \rho}{\rho}, v_\Pi \right\rangle, \ \ \varkappa(dy) = \rho(y) dy$$

2. For the case $\mathbb{K}(\mathbb{R}^d)$*:*

$$B^{(n)}_{(h,v)}(\eta) = \int_{\mathbb{R}^d} \frac{l'(s_x)}{l(s_x)} h(x)\eta(dx) + l(0)\int_{\mathbb{R}^d} h(x)dx + \sum_{\substack{x\in\tau(\eta)\\ s_x\geq \frac{1}{n}}} \operatorname{div}^x v(x) \tag{34}$$

Remark 3.25. *The additional modification using the parameter* n *is needed since* $\mathcal{G}_\nu$ *is not quasi-invariant with respect to the group* $\mathfrak{G}$ *if*

$$\nu(\mathbb{R}^*_+) = \infty, \ \nu(ds) = \frac{l(s)}{s}ds.$$

If we assume that $\nu(\mathbb{R}^*_+) < \infty$, we find that $l(0) = 0$ and we can (heuristically) take $n \to \infty$ in (34). The expression for the logarithmic derivative becomes

$$B^{(\infty)}_{(h,v)}(\eta) = \sum_{x\in\tau(\eta)} s_x \frac{l'(s_x)}{l(s_x)} h(x) + \sum_{x\in\tau(\eta)} \operatorname{div}^x v(x)$$

Let us consider a special case on $\Pi(\mathbb{R}^*_+ \times \mathbb{R}^d)$ to show that the above formulae coincide. In our construction of Gamma measures on $\mathbb{K}(\mathbb{R}^d)$, we only consider the case where the underlying intensity measure has the form

$$\varkappa(ds, dx) = \frac{l(s)}{s} ds dx.$$

The density function of $\varkappa$ is then given by

$$\rho(s, x) = \frac{l(s)}{s}.$$

Therefore, its logarithmic derivative is

$$\beta^{\varkappa}(s, x) = \left(\frac{l'(s)}{l(s)} - \frac{1}{s}, 0, \dots, 0\right)$$

For our choice of v_Π, we obtain for the logarithmic derivative on $\Pi(\mathbb{R}^*_+ \times \mathbb{R}^d)$,

$$\begin{aligned} B^{\pi_\varkappa}_{v_\Pi}(\gamma) &= \sum_{(s,x)\in\gamma} \left(\frac{l'(s)}{l(s)} - \frac{1}{s}\right) sh(x) + \partial_s(sh(x)) + \sum_{i=1}^d \partial_{x_i} v^i(x) \\ &= \sum_{(s,x)} \frac{l'(s)}{l(s)} sh(x) + \operatorname{div}^x v(x) \end{aligned}$$

which coincides with the logarithmic derivative on $\mathbb{K}(\mathbb{R}^d)$, and therefore, the integration by parts formulae coincide.

3.4 Umbral Calculus

The theory of umbral calculus is concerned with studying sequences of certain polynomials. The objects of this theory can be seen as generalisations of the monomial sequence $p_n(x) = x^n$. In this case, the sequence fulfills the binomial theorem

$$p_n(x + y) = \sum_{i=0}^n p_i(x)p_{n-i}(y), \tag{35}$$

but there are other sequences of polynomials for which this or a similar identity holds. In the one-dimensional case, the theory was extensively studied by e.g. Roman and Rota, see [51] and the references therein.

On the other hand, it is also possible to define similar sequences of polynomials on infinite-dimensional spaces such as the space of Radon measures. The most straightforward way to define polynomials on $\mathbb{K}(\mathbb{R}^d)$ is to consider polynomials on the larger space $\mathbb{M}(\mathbb{R}^d)$ and simply restrict the mappings to the smaller space. However, there are some sequences which need to be defined directly on $\mathbb{K}(\mathbb{R}^d)$. The theory is mentioned here because there exist some nice applications related to infinite-dimensional combinatorics introduced via the K-transform earlier. For a more detailed picture in the case $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ (or rather $\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$), see [21].

In this work, we only illustrate the theory via the example of falling factorials. In one dimension, this sequence is defined as

$$p_n(x) := (x)_n = \prod_{k=1}^{n}(x-k+1) = x(x-1)\cdots(x-n+1),$$

which also fulfill the binomial identity (35). For this sequence, the so-called generating functional equals

$$\sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!} = (1+t)^x = \exp\left(x\log(1+t)\right).$$

Also, note that we have the relation

$$\binom{x}{n} = \frac{1}{n!}(x)_n.$$

where the left-hand side is the generalised binomial coefficient.

3.4.1 Umbral Calculus on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$

To define polynomials on the space $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$, we embed it in the space of generalised functions $\mathcal{D}(\mathbb{R}_+^* \times \mathbb{R}^d)'$, where as before, $\mathcal{D} = \mathcal{D}(\mathbb{R}_+^* \times \mathbb{R}^d) = C_c^\infty(\mathbb{R}_+^* \times \mathbb{R}^d)$. Therefore, we proceed by defining polynomials on the space $\mathcal{D}'$. For technical reasons, this step is essential, since $\mathcal{D}$ is a so-called nuclear space, which is found in the Gel'fand triple

$$\mathcal{D} \subset L^2(\mathbb{R}_+^* \times \mathbb{R}^d, d\varkappa) \subset \mathcal{D}'.$$

For a more detailed description of the theoretical background, see [21]. We set $\mathcal{D}^{\odot k}$ to be the space of all symmetric functions $f \in \mathcal{D}^{\otimes k}$ for $k \in \mathbb{N}_0$.

Definition 3.26 ([21])**.** *A function $P\colon \mathcal{D}' \to \mathbb{R}$ is called a polynomial of degree $n \in \mathbb{N}$ on $\mathcal{D}'$ if it has the form*

$$P(\eta) = \sum_{k=0}^{n}\langle f^{(k)}, \eta^{\otimes k}\rangle$$

where $f^{(k)} \in \mathcal{D}^{\odot k}$ for $k = 0, \dots, n, f^{(n)} \neq 0$ and $\eta^{\otimes 0} = 1$. The notation $\langle \cdot, \cdot \rangle$ denotes the dual pairing between $\mathcal{D}'^{\otimes k}$ and $\mathcal{D}^{\otimes k}$. Denote the space of all polynomials on $\mathcal{D}'$ by $\mathcal{P}(\mathcal{D}')$.

By lifting the theory from one dimension to the infinite-dimensional case, we arrive at the following representation. In analogy of the one-dimensional case, we write $(\cdot)_n$ for the sequence of falling factorials.

Proposition 3.27 ([21]). *The falling factorials on $\mathcal{D}'$ have the following explicit form:*

$$(\gamma)_0 = 1$$
$$(\gamma)_1 = \gamma$$
$$(\gamma)_n(x_1, \dots, x_n) = \gamma(x_1)(\gamma(x_2) - \delta_{x_1}(x_2)) \cdots (\gamma(x_n) - \delta_{x_1} - \cdots - \delta_{x_{n-1}}(x_n))$$

where $\gamma(x) := \gamma(\{x\})$. This sequence of polynomials is interesting for our theory, since it encompasses the combinatorics of the K-transform introduced before.

Corollary 3.28 ([21]). *Let $\gamma = \sum_{i=1}^{\infty} \delta_{x_i} \in \Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}_+^* \times \mathbb{R}^d)$ and set $\binom{\gamma}{n} := \frac{1}{n}(\gamma)_n$. Then, the following formula holds:*

$$\binom{\gamma}{n} = \sum_{\{i_1, \dots, i_n\} \subset \mathbb{N}} \delta_{x_{i_1}} \odot \cdots \odot \delta_{x_{i_n}} \tag{36}$$

where $\odot$ denotes the symmetric tensor product.

Remark 3.29. *The relation directly links to the K-transform: For $\gamma = \{x_i\}_{i=1}^{\infty}$ and $G\colon \Pi_0(\mathbb{R}_+^* \times \mathbb{R}^d) \to \mathbb{R}$, we have*

$$(K^{\Pi}G)(\gamma) = \sum_{n=0}^{\infty} \left\langle G^{(n)}, \binom{\gamma}{n} \right\rangle$$

whenever the transform is well-defined for G, and $\{G^{(n)}\}_{n=0}^{\infty}$ is the sequence of symmetric functions $G^{(n)}\colon (\mathbb{R}_+^* \times \mathbb{R}^d)^n \to \mathbb{R}$ associated with G.

In terms of umbral calculus, sequences of polynomials may be introduced via its generating functional. For the falling factorials, it is given for $\varphi \in C_c(\mathbb{R}_+^* \times \mathbb{R}^d)$, by the following expression:

$$E_\varphi(\gamma) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \varphi^{\otimes n}, (\gamma)_n \rangle$$

There are more relations to combinatorics and point processes connected to the theory of umbral calculus which would exceed the scope of this work which can be found in the aforementioned paper.

3.4.2 Umbral Calculus on $\mathbb{K}(\mathbb{R}^d)$

We may also introduce polynomials on the space $\mathbb{K}(\mathbb{R}^d)$. For a large class of polynomials, as above, it is possible to define polynomials on the space $\mathcal{D}'(\mathbb{R}^d) \supset \mathbb{K}(\mathbb{R}^d)$ and restrict these mappings to $\mathbb{K}(\mathbb{R}^d)$. Note, however, there are some classes of polynomials which can only be defined on $\mathbb{K}(\mathbb{R}^d)$ directly. This is especially interesting in the case of falling factorials: While we may introduce the falling factorials in the same way as on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$, a more interesting class is the sequence of so-called fake falling factorials, which are the image of the falling factorials on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$. They can be introduced explicitly using a similar representation as Proposition 3.27.

Definition 3.30. *For each $\eta \in \mathbb{K}(\mathbb{R}^d)$, define the sequence of fake falling factorials as*

$$P^{(0)}(\eta) = 1$$
$$P^{(1)}(\eta) = \eta$$
$$\begin{aligned} P^{(n)}(\eta)(x_1, \dots, x_n) = \eta(x_1)(\eta(x_2) - s_1\delta_{x_1}(x_2)) \times \\ \times \cdots (\eta(x_n) - s_1\delta_{x_1}(x_n) - \cdots - s_{n-1}\delta_{x_{n-1}}(x_n)) \end{aligned}$$

where $\eta = \sum_{x \in \tau(\eta)} s_x \delta_x$ and $\eta(x) = \eta(\{x\})$.

As this definition relies explicitly on the coefficients of $\eta \in \mathbb{K}(\mathbb{R}^d)$, this sequence can not be extended to the whole space $\mathcal{D}'(\mathbb{R}^d)$. On the other hand, we have a similar representation as in Corollary 3.28: For $\eta \in \mathbb{K}(\mathbb{R}^d)$, we have

$$\frac{1}{n!}P^{(n)}(\eta) = \sum_{\{i_1,\ldots,i_n\}\subset\mathbb{N}} s_{x_{i_1}} \cdots s_{x_{i_n}} \delta_{x_1} \odot \cdots \odot \delta_{x_n} \tag{37}$$

where $\odot$ represents the symmetric tensor product and $\eta = \sum_{i\in\mathbb{N}} s_{x_i}\delta_{x_i}$.

As stated above, the fake falling factorials also arise as the image of falling factorials on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ under the reflection mapping.

Proposition 3.31. *Let $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ be a measurable function with compact support and set $f_\varphi(s,x) := s\varphi(x)$ for $(s,x) \in \mathbb{R}_+^* \times \mathbb{R}^d$. Then for $\eta \in \mathbb{K}(\mathbb{R}^d)$ and all $n \in \mathbb{N}_0$, the following holds:*

$$\langle \varphi^{\otimes n}, P^{(n)}(\eta)\rangle = \langle\langle f_\varphi^{\otimes n}, (\mathcal{R}^{-1}\eta)_n\rangle\rangle$$

where $(\cdot)_n$ denotes the falling factorials on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ as introduced in Proposition 3.27.

Proof. The relation follows immediately when comparing formulae (36) and (37). □

Remark 3.32. *The reflection mapping $\mathcal{R}$ allows us to transfer other classes of polynomials such as Charlier or Hermite polynomials from $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ to $\mathbb{K}(\mathbb{R}^d)$ as well. As we only focus on the applicability of polynomials to infinite-dimensional combinatorics here, these other classes will not be discussed in this work.*

We want to relate the fake falling factorials to the K-transform on $\mathbb{K}(\mathbb{R}^d)$, which is the reason why this class is more relevant for us than the "real" falling factorials on $\mathbb{K}(\mathbb{R}^d)$. Consider a symmetric function with compact support $\varphi^{(n)} \in \mathcal{D}^{\odot n}(\mathbb{R}^d)$ and set

$$G_{\varphi^{(n)}}(\eta) := \begin{cases} s_1 \cdot \ldots \cdot s_n \varphi^{(n)}(x_1,\ldots,x_n), & \text{if } \eta = \sum_{i=1}^n s_i\delta_{x_i} \in \mathbb{K}_0^{(n)}(\mathbb{R}^d) \\ 0, & \text{otherwise} \end{cases}$$

Then $G_{\varphi^{(n)}} \in \widetilde{B}_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$ and the following holds:

$$(K^{\mathbb{K}} G_{\varphi^{(n)}})(\eta) = \frac{1}{n!}\langle \varphi^{(n)}, P^{(n)}(\eta)\rangle$$

Note that this relation is valid for the specific class of functions defined above. Nevertheless, such functions play an important role as L^1-functions considering the shape of the intensity measure on $\mathbb{R}_+^*$ introduced in Definition 2.22.

The fake falling factorials will also appear in the next chapter when we talk about difference calculus on the cone.

3.5 Difference Calculus

We now want to take into account the discrete structure of the configurations themselves. This enables us to define an entirely different kind of calculus, namely, difference calculus. Here, we consider discrete differences between a function evaluated at two different points. The discrete nature of our configurations dictates how these differences will look like. While the differential calculus introduces above corresponds to diffusion processes on the state spaces, the difference calculus can be intepreted as corresponding jump-type processes. These kind of expressions will also appear later when we consider birth-and-death models.

As before, we first consider the situation in the Plato space $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$. Furthermore, there are some identities related to umbral calculus which are stated in this chapter. We also point out difficulties which arise when transferring the identities from $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ to $\mathbb{K}(\mathbb{R}^d)$. For the general case on $\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$ or $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$, we refer to [21] for a more detailed analysis.

3.5.1 Difference Calculus on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$

Let us start with the case on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$. In this case, configurations $\gamma \in \Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ are described only by the positions of their elements and marks are treated the same way. We introduce the elementary discrete gradients as well as the derived directional derivatives.

Definition 3.33. *Let $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Pi(\mathbb{R}_+^* \times \mathbb{R}^d))$. Introduce the following discrete gradients:*

1. *Let $\gamma \in \Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $(s,x) \in \gamma$. The elementary death gradient is defined as*

$$D^-_{(s,x)}F(\gamma) = F(\gamma - \delta_{(s,x)}) - F(\gamma).$$

The corresponding tangent space is set as $T_\gamma^-(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)) = L^2(\mathbb{R}_+^* \times \mathbb{R}^d, \gamma)$.

2. *For a function $\psi \in C_c(\mathbb{R}_+^* \times \mathbb{R}^d)$, we define the directional derivative as*

$$D^-_\psi F(\gamma) = \sum_{(s,x)\in\gamma} \psi(s,x) D^-_{(s,x)}F(\gamma).$$

3. *For $(s,x) \in \mathbb{R}_+^* \times \mathbb{R}^d$, we define the elementary birth derivative as*

$$D^+_x F(\gamma) = F(\gamma + \delta_{(s,x)}) - F(\gamma)$$

with the corresponding tangent space $T_\gamma^+(\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)) = L^2(\mathbb{R}_+^* \times \mathbb{R}^d, d\varkappa)$. Note that $\gamma \in \Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ is a set of zero Lebesgue measure and hence, the expression above is well-defined almost everywhere.

4. *For a function $\varphi \in C_c(\mathbb{R}_+^* \times \mathbb{R}^d)$, the directional (birth) derivative is defined as*

$$D^+_\varphi F(\gamma) = \int_{\mathbb{R}_+^* \times \mathbb{R}^d} \varphi(s,x) D^+_{(s,x)}F(\gamma) \varkappa(ds,dx).$$

We now want to give some connections of difference calculus to umbral calculus, namely, the connection between the gradients defined above and the falling factorials on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$.

Proposition 3.34. *Let $\psi \in L^2(\mathbb{R}_+^* \times \mathbb{R}^d, d\varkappa)$ and $\varphi \in C_c(\mathbb{R}_+^* \times \mathbb{R}^d)$. Then, the generating functional of the falling factorials on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ fulfills the following relation:*

$$D^+_\psi E_\varphi(\gamma) = \langle \varphi\psi \rangle E_\varphi(\gamma)$$

where $\langle \varphi \rangle$ denotes the expected value (integral) of a function φ.

Proof. Since

$$(D_\psi^+ E_\varphi)(\gamma) = D_\psi^+ \left[\sum_{n=0}^\infty \frac{1}{n!}\langle \varphi^{\otimes n}, (\cdot)_n\rangle\right](\gamma) = \sum_{n=0}^\infty \frac{1}{n!} D_\psi^+(\langle \varphi^{\otimes n}, (\cdot)_n\rangle)(\gamma)$$

we can fix $n \geq 1$ and consider one summand: Denote $y = (s, x)$ and $y_i = (s_i, x_i)$.

$$\begin{aligned}
&D_\psi^+(\langle \varphi^{\otimes n}, (\cdot)_n\rangle)(\gamma)\\
&\quad= \int_{\mathbb{R}_+^* \times \mathbb{R}^d} \psi(y)[\langle \varphi^{\otimes n}, (\gamma + \delta_y)_n\rangle - \langle \varphi^{\otimes n}, (\gamma)_n\rangle]\varkappa(dy)\\
&\quad= \int_{\mathbb{R}_+^* \times \mathbb{R}^d} \psi(y)\left[\sum_{\{y_1,\ldots,y_n\}\subset\gamma+\delta_y} \varphi(y_1)\cdots\varphi(y_n) - \sum_{\{y_1,\ldots,y_n\}\subset\gamma} \varphi(y_1)\cdots\varphi(y_n)\right]\varkappa(dy)\\
&\quad= \int_{\mathbb{R}_+^* \times \mathbb{R}^d} \psi(y)n\varphi(y) \sum_{\{y_1,\ldots,y_{n-1}\}\subset\gamma} \varphi(y_1)\cdots\varphi(x_{n-1})\varkappa(dy)\\
&\quad= n\int_{\mathbb{R}_+^* \times \mathbb{R}^d} \psi(y)\varphi(y)\varkappa(dy)\cdot\langle \varphi^{\otimes(n-1)}, (\gamma)_{n-1}\rangle
\end{aligned}$$

For $n = 0$, we get $D_\psi^+(\langle \varphi^{\otimes 0}, (\cdot)_0\rangle)(\gamma) = 0$. Together, this means

$$(D_\psi^+ E_\varphi)(\gamma) = \langle \psi\varphi\rangle \sum_{n=1}^\infty \frac{1}{(n-1)!}\langle \varphi^{\otimes(n-1)}, (\gamma)_{n-1}\rangle = \langle \varphi\psi\rangle E_\varphi(\gamma)$$

□

3.5.2 Difference Calculus on $\mathbb{K}(\mathbb{R}^d)$

By treating the mark space $\mathbb{R}_+^*$ as a separate entity as it is done on $\mathbb{K}(\mathbb{R}^d)$ and not as another variable as on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$, we may refine the definition of the birth and death gradients for functions on $\mathbb{K}(\mathbb{R}^d)$. We redefine these gradients taking into account these marks. Furthermore, we show a discrete integration by parts formula and define a Dirichlet form and Laplacian.

Definition 3.35. *Let $F \in \mathcal{F}C(\mathbb{K}(\mathbb{R}^d))$.*

1. *The discrete death gradient of F is given by*

$$(D_x^- F)(\eta) := F(\eta - s_x\delta_x) - F(\eta)$$

where $x \in \tau(\eta)$. Corresponding to this, we define a tangent space as $T_\eta^-(\mathbb{K}(\mathbb{R}^d)) := L^2(\mathbb{R}^d, \eta)$.

2. *For a function $h \in C_c(\mathbb{R}^d)$, define the directional derivative along h as*

$$\begin{aligned}
(D_h^- F)(\eta) &:= \langle (D^- F)(\eta), h\rangle_{T_\eta^-(\mathbb{K}(\mathbb{R}^d))}\\
&= \int_{\mathbb{R}^d} (D_x^- F)(\eta)h(x)\eta(dx) = \sum_{x\in\tau(\eta)} s_x h(x)(D_x^- F)(\eta)
\end{aligned}$$

3. *The discrete birth gradient is defined by*

$$(D^+_{(s,x)}F)(\eta) := F(\eta + s\delta_x) - F(\eta)$$

where $(s,x) \in \mathbb{R}^*_+ \times \mathbb{R}^d, x \notin \tau(\eta)$*. Here, the corresponding tangent space is defined as* $T^+_\eta(\mathbb{K}(\mathbb{R}^d)) = L^2(\mathbb{R}^*_+ \times \mathbb{R}^d, \nu \otimes \sigma)$*. For a function* $g \in L^2(\mathbb{R}^*_+ \times \mathbb{R}^d, \nu \otimes \sigma)$*, we may define the directional derivative as*

$$(D^+_g F)(\eta) := \langle (D^+F)(\eta), g\rangle_{T^+_\eta(\mathbb{K}(\mathbb{R}^d))} = \int_{\mathbb{R}^*_+ \times \mathbb{R}^d} (D^+_{(s,x)}F)(\eta) g(s,x)(\nu \otimes \sigma)(ds, dx).$$

There is an adjoint-like relation between the two derivatives defined above, which can be expressed via the following integration by parts-type equation.

Proposition 3.36 (Discrete integration by parts)**.** *For the measure* ν *on* $\mathbb{R}^*_+$*, assume the moment condition* (6)*. For any* $F, G \in \mathcal{FC}(\mathbb{K}(\mathbb{R}^d))$ *and* $h \in C_c(\mathbb{R}^d)$*, we have*

$$\begin{aligned}\int_{\mathbb{K}(\mathbb{R}^d)} (D^-_h F)(\eta) G(\eta) \mathcal{G}(d\eta) = &\int_{\mathbb{K}(\mathbb{R}^d)} F(\eta)(D^+_{\mathrm{id}\otimes h} G)(\eta)\mathcal{G}(d\eta) \\ &- \int_{\mathbb{K}(\mathbb{R}^d)} F(\eta)G(\eta)B_{\mathcal{G},h}(\eta)\mathcal{G}(d\eta)\end{aligned}$$

where $(\mathrm{id} \otimes h)(s,x) := sh(x)$ *and*

$$\begin{aligned}B_{\mathcal{G},h}(\eta) &= \int_{\mathbb{R}^d} h d\eta - \int_{\mathbb{R}^*_+ \times \mathbb{R}^d} (\mathrm{id} \otimes h) d(\nu \otimes \sigma) \\ &= \sum_{x \in \tau(\eta)} s_x h(x) - \int_{\mathbb{R}^*_+ \times \mathbb{R}^d} sh(x)\nu(ds)\sigma(dx)\end{aligned}$$

Proof. It is enough to consider the term containing $F(\eta - s_x\delta_x)$. For this term, we obtain by using Mecke's formula (8)

$$\begin{aligned}&\int_{\mathbb{K}(\mathbb{R}^d)} \int_{\mathbb{R}^d} h(x) F(\eta - s_x\delta_x) G(\eta) \eta(dx) \mathcal{G}(d\eta) = \\ &\quad= \int_{\mathbb{K}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^*_+} sh(x) F(\eta) \underbrace{G(\eta + s_x\delta_x)}_{=G(\eta+s_x\delta_x)-G(\eta)+G(\eta)} \nu(ds)\sigma(dx)\mathcal{G}(d\eta) \\ &\quad= \int_{\mathbb{K}(\mathbb{R}^d)} F(\eta) \int_{\mathbb{R}^d} \int_{\mathbb{R}^*_+} sh(x) D^+_{(s,x)} G(\eta)\nu(ds)\sigma(dx)\mathcal{G}(d\eta) \\ &\qquad+ \int_{\mathbb{K}(\mathbb{R}^d)} F(\eta)G(\eta) \int_{\mathbb{R}^d} \int_{\mathbb{R}^*_+} sh(x)\nu(ds)\sigma(dx)\mathcal{G}(d\eta) \\ &\quad= \int_{\mathbb{K}(\mathbb{R}^d)} F(\eta)(D^+_{\mathrm{id}\otimes h} G)(\eta)\mathcal{G}(d\eta) \\ &\qquad+ \int_{\mathbb{K}(\mathbb{R}^d)} F(\eta)G(\eta) \int_{\mathbb{R}^d} \int_{\mathbb{R}^*_+} sh(x)\nu(ds)\sigma(dx)\mathcal{G}(d\eta)\end{aligned}$$

Now adding the remaining term, we obtain the statement of the proposition. ◻

3.5.3 Discrete Laplacian

There is a Laplacian-type operator associated with the above gradients. The definition is straightforward using the Dirichlet integral for the class of cylinder functions $F, G \in \mathcal{F}C(\mathbb{K}(\mathbb{R}^d))$,

$$\mathcal{E}^{\mathrm{dis}}(F,G) := \int_{\mathbb{K}(\mathbb{R}^d)} \langle D^- F, D^- G\rangle_{T^-_\eta(\mathbb{K}(\mathbb{R}^d))} \mathcal{G}(d\eta).$$

As it turns out, using the discrete birth gradient yields the same bilinear form:

Lemma 3.37. *Let $F, G \in \mathcal{F}C(\mathbb{K}(\mathbb{R}^d))$. Then*

$$\mathcal{E}^{\mathrm{dis}}(F,G) = \int_{\mathbb{K}(\mathbb{R}^d)} \langle D^+ F, D^+ G\rangle_{T^+_\eta(\mathbb{K}(\mathbb{R}^d))} \mathcal{G}(d\eta)$$

Proof. Using Mecke's formula (8), we obtain

$$\begin{aligned}
\mathcal{E}^{\mathrm{dis}}(F,G) &= \int_{\mathbb{K}(\mathbb{R}^d)} \sum_{x\in\tau(\eta)} D^-_x F(\eta) D^-_x G(\eta) \mathcal{G}(d\eta) \\
&= \int_{\mathbb{K}(\mathbb{R}^d)} \sum_{x\in\tau(\eta)} [F(\eta - s_x\delta_x) - F(\eta)][G(\eta - s_x\delta_x) - G(\eta)]\mathcal{G}(d\eta) \\
&= \int_{\mathbb{K}(\mathbb{R}^d)} \int_{\mathbb{R}^*_+\times\mathbb{R}^d} s[F(\eta) - F(\eta + s\delta_x)][G(\eta) - G(\eta + s\delta_x)]\nu(ds)\sigma(dx)\mathcal{G}(d\eta) \\
&= \int_{\mathbb{K}(\mathbb{R}^d)} \int_{\mathbb{R}^*_+\times\mathbb{R}^d} s(D^+_{(s,x)}F)(\eta)(D^+_{(s,x)}G)(\eta)\nu(ds)\sigma(dx)\mathcal{G}(d\eta) \\
&= \int_{\mathbb{K}(\mathbb{R}^d)} \langle D^+ F, D^+ G\rangle_{T^+_\eta(\mathbb{K}(\mathbb{R}^d))} \mathcal{G}(d\eta)
\end{aligned}$$

□

Proposition 3.38. *The mapping $(\mathcal{E}^{\mathrm{dis}}, \mathcal{F}C(\mathbb{K}(\mathbb{R}^d)))$ is a well-defined symmetric bilinear form on $L^2(\mathbb{K}(\mathbb{R}^d), \mathcal{G})$.*

Proof. The symmetry and bilinearity are clear. We need to show that the form $\mathcal{E}^{\mathrm{dis}}$ gives the same result for elements from the same equivalence class. Therefore, consider $F, G \in \mathcal{F}C(\mathbb{K}(\mathbb{R}^d))$ with $F = 0$ $\mathcal{G}$-a.e. Then by Mecke's formula (2.20), we get for any $A \in \mathcal{B}_c(\mathbb{R}^d)$,

$$\int_{\mathbb{K}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^*_+} |F(\eta + s\delta_x)| \mathbb{1}_A(x)\nu(ds)\sigma(dx)\mathcal{G}(d\eta) = \int_{\mathbb{K}(\mathbb{R}^d)} |F(\eta)|\eta(A)\mathcal{G}(d\eta) = 0$$

which implies that $F(\eta + s\delta_x) = 0$ $d\mathcal{G}d\nu d\sigma$-almost everywhere on $\mathbb{K}(\mathbb{R}^d) \times \mathbb{R}^*_+ \times \mathbb{R}^d$. Plugging this in, we see

$$\mathcal{E}^{\mathrm{dis}}(F,G) = \int_{\mathbb{K}(\mathbb{R}^d)} \int_{\mathbb{R}^*_+\times\mathbb{R}^d} [F(\eta + s\delta_x) - F(\eta)][G(\eta + s\delta_x) - G(\eta)]\nu(ds)\sigma(dx)\mathcal{G}(d\eta) = 0$$

□

The discrete Laplacian is now given by the Markov generator associated with the above form.

Proposition 3.39. *For each $F \in \mathcal{F}C(\mathbb{K}(\mathbb{R}^d))$, we set*

$$
\begin{aligned}
(L^{\mathrm{dis}}F)(\eta) &= \int_{\mathbb{R}^d} (D_x^- F)(\eta)\eta(dx) + \int_{\mathbb{R}_+^* \times \mathbb{R}^d} (D_{(s,x)}^+ F)(\eta)(\nu \otimes \sigma)(ds, dx) \\
&= \sum_{x \in \tau(\eta)} s_x 5[F(\eta - s_x\delta_x) - F(\eta)] + \int_{\mathbb{R}_+^* \times \mathbb{R}^d} s[F(\eta + s\delta_x) - F(\eta)](\nu \otimes \sigma)(ds, dx)
\end{aligned}
$$

Then $(L^{\mathrm{dis}}, \mathcal{F}C(\mathbb{K}(\mathbb{R}^d)))$ is a symmetric operator in $L^2(\mathbb{K}(\mathbb{R}^d), \mathcal{G})$. Also, it is associated to the form introduced above, i.e.

$$
\mathcal{E}^{\mathrm{dis}}(F, G) = (-L^{\mathrm{dis}}F, G)_{L^2(\mathbb{K}(\mathbb{R}^d), \mathcal{G})}, \ F, G \in \mathcal{F}C(\mathbb{K}(\mathbb{R}^d)). \tag{38}
$$

Furthermore, the form $(\mathcal{E}^{\mathrm{dis}}, \mathcal{F}C(\mathbb{K}(\mathbb{R}^d)))$ is closable on $L^2(\mathbb{K}(\mathbb{R}^d), \mathcal{G})$. Also, for the operator $(L^{\mathrm{dis}}, \mathcal{F}C(\mathbb{K}(\mathbb{R}^d)))$, there exists the Friedrichs' extension. Denote these extensions by $(\mathcal{E}^{\mathrm{dis}}, D(\mathcal{E}^{\mathrm{dis}}))$ and $(L^{\mathrm{dis}}, D(L^{\mathrm{dis}}))$, respectively. Then the operator $(L^{\mathrm{dis}}, D(L^{\mathrm{dis}}))$ is the generator of the bilinear form $(\mathcal{E}^{\mathrm{dis}}, D(\mathcal{E}^{\mathrm{dis}}))$.

Remark 3.40. *The operator L^{dis} defined above is the $\mathbb{K}(\mathbb{R}^d)$-analogue of the operator on $\Gamma(\mathbb{R}_+^* \times \mathbb{R}^d)$ which is the generator of the so-called Surgailis process. This operator models independent birth and death of particles on the underlying state space. It was studied in e.g. [16, 55, 56].*

For the proof of the proposition, recall the following result:

Theorem 3.41 ([50, Theorem X.23])**.** *Let A be a positive symmetric operator and let $\mathcal{E}(F, G) = (AF, G)$ for $F, G \in D(A)$. Then $\mathcal{E}$ is a closable quadratic form and its closure $\hat{\mathcal{E}}$ is the quadratic form of a unique self-adjoint operator $\hat{A}$.*

Proof of Proposition 3.39. Let us start by showing identity (38). For functions $F, G \in \mathcal{F}C(\mathbb{K}(\mathbb{R}^d))$, we have using Mecke's formula,

$$
\begin{aligned}
\mathcal{E}^{\mathrm{dis}}(F, G) &= \int_{\mathbb{K}(\mathbb{R}^d)} \sum_{x \in \tau(\eta)} (D_x^- F)(\eta) G(\eta - s_x\delta_x) \mathcal{G}(d\eta) \\
&\quad - \int_{\mathbb{K}(\mathbb{R}^d)} \sum_{x \in \tau(\eta)} (D_x^- F)(\eta) G(\eta) \mathcal{G}(d\eta) \\
&= - \int_{\mathbb{K}(\mathbb{R}^d)} \int_{\mathbb{R}_+^* \times \mathbb{R}^d} s(D_{(s,x)}^+ F)(\eta) G(\eta) \nu(ds)\sigma(dx) \\
&\quad - \int_{\mathbb{K}(\mathbb{R}^d)} \sum_{x \in \tau(\eta)} (D_x^- F)(\eta) G(\eta) \mathcal{G}(d\eta) \\
&= (-L^{\mathrm{dis}}F, G)_{L^2(\mathbb{K}(\mathbb{R}^d), \mathcal{G})}
\end{aligned}
$$

The symmetry of L^{dis} is now clear since $\mathcal{E}^{\mathrm{dis}}$ is symmetric. Let us check that L^{dis} actually maps functions $F \in \mathcal{F}C(\mathbb{K}(\mathbb{R}^d))$ to $L^2(\mathbb{K}(\mathbb{R}^d), \mathcal{G})$. As it is easily seen by representation (30), there exists a compact set $\Lambda \subset \mathbb{R}_+^* \times \mathbb{R}^d$ and a constant $C > 0$ such that

$$
\begin{aligned}
(D_x^- F)(\eta) &\le C\mathbb{1}_\Lambda(s_x, x), \ x \in \tau(\eta) \\
(D_{(s,x)}^+ F)(\eta) &\le C\mathbb{1}_\Lambda(s, x), \ (s, x) \in \mathbb{R}_+^* \times \mathbb{R}^d \setminus \tau(\eta).
\end{aligned}
$$

We may also decompose our expression using the triangle inequality:

$$\left\|L^{\mathrm{dis}}F\right\|_{L^2(\mathbb{K}(\mathbb{R}^d),\mathcal{G})} \leq \left\|\sum_{x\in\tau(\cdot)} s_x D_x^- F\right\|_{L^2(\mathbb{K}(\mathbb{R}^d),\mathcal{G})} + \left\|\int_{\mathbb{R}_+^*\times\mathbb{R}^d} s(D_{(s,x)}^+ F)\nu(ds)\sigma(dx)\right\|_{L^2(\mathbb{K}(\mathbb{R}^d),\mathcal{G})}$$

Let us consider the first term and show that it is finite: We split the expression into diagonal and off-diagonal terms and use Mecke's identity:

$$\begin{aligned}
&\left\|\sum_{x\in\tau(\cdot)} s_x D_x^- F\right\|^2_{L^2(\mathbb{K}(\mathbb{R}^d),\mathcal{G})} = \\
&= \int_{\mathbb{K}(\mathbb{R}^d)} \left[\sum_{x\in\tau(\eta)} (D_x^- F)(\eta)\right]^2 \mathcal{G}(d\eta) \\
&\leq C^2 \int_{\mathbb{K}(\mathbb{R}^d)} \left[\sum_{x\in\tau(\eta)} s_x \mathbb{1}_\Lambda(s_x, x)\right]^2 \mathcal{G}(d\eta) \\
&= C^2 \int_{\mathbb{K}(\mathbb{R}^d)} \sum_{x\in\tau(\eta)} \sum_{y\in\tau(\eta - s_x\delta_x)} s_x s_y \mathbb{1}_\Lambda(s_x, x)\mathbb{1}_\Lambda(s_y, y)\mathcal{G}(d\eta) \\
&\quad + C^2 \int_{\mathbb{K}(\mathbb{R}^d)} \sum_{x\in\tau(\eta)} s_x^2 \mathbb{1}_\Lambda(s_x, x)\mathcal{G}(d\eta) \\
&= C^2 \int_{\mathbb{K}(\mathbb{R}^d)} \int_{\mathbb{R}_+^*\times\mathbb{R}^d} \int_{\mathbb{R}_+^*\times\mathbb{R}^d} s_1 s_2 \mathbb{1}_\Lambda(s_1, x_1)\mathbb{1}_\Lambda(s_2, x_2)\nu(ds_1)\nu(ds_2)\sigma(dx_1)\sigma(dx_2)\mathcal{G}(d\eta) \\
&\quad + C^2 \int_{\mathbb{K}(\mathbb{R}^d)} \int_{\mathbb{R}_+^*\times\mathbb{R}^d} s^2 \mathbb{1}_\Lambda(s, x)\nu(ds)\sigma(dx)\mathcal{G}(d\eta) \\
&< \infty
\end{aligned}$$

The expression is finite since the inner integral is bounded and $\mathcal{G}$ is a probability measure. For the second expression of $L^{\mathrm{dis}}F$, we may proceed in the same way as above to obtain finiteness. Therefore, we obtain $L^{\mathrm{dis}}F \in L^2(\mathbb{K}(\mathbb{R}^d), \mathcal{G})$.

The Friedrichs' extension is obtained by showing that $-L^{\mathrm{dis}}$ is a positive operator. Let $F \in \mathcal{F}C(\mathbb{K}(\mathbb{R}^d))$. Then by Mecke's identity,

$$\begin{aligned}
(L^{\mathrm{dis}}F, F) &= \int_{\mathbb{K}(\mathbb{R}^d)} \sum_{x\in\tau(\eta)} s_x(D_x^- F)(\eta)F(\eta) + \int_{\mathbb{R}_+^*\times\mathbb{R}^d} s(D_{(s,x)}^+ F)(\eta)F(\eta)\nu(ds)\sigma(dx)\mathcal{G}(d\eta) \\
&= \int_{\mathbb{K}(\mathbb{R}^d)} \int_{\mathbb{R}_+^*\times\mathbb{R}^d} s[F(\eta) - F(\eta + s\delta_x)]F(\eta + s\delta_x)\nu(ds)\sigma(dx) \\
&\quad + \int_{\mathbb{R}_+^*\times\mathbb{R}^d} s[F(\eta + s\delta_x) - F(\eta)]F(\eta)\nu(ds)\sigma(dx)\mathcal{G}(d\eta) \\
&= \int_{\mathbb{K}(\mathbb{R}^d)} \int_{\mathbb{R}_+^*\times\mathbb{R}^d} - \left[(D_{(s,x)}^+ F)(\eta)\right]^2 \nu(ds)\sigma(dx)\mathcal{G}(d\eta) \leq 0
\end{aligned}$$

Therefore, L^{dis} is negative and $-L^{\mathrm{dis}}$ is a positive operator. □

Next, we establish the connection between the falling factorials and difference calculus on $\mathbb{K}(\mathbb{R}^d)$. The following relations hold between the discrete gradients and the falling factorials:

Proposition 3.42. *Let φ be a measurable and compactly supported function. Then, the following relations hold:*

1. *Let $\eta \in \mathbb{K}(\mathbb{R}^d)$ and $x \in \tau(\eta)$. Then*
$$D_x^- \langle \varphi^{\otimes n}, P^{(n)}(\cdot)\rangle(\eta) = -ns_x\varphi(x)\langle \varphi^{\otimes n-1}, P^{(n-1)}(\cdot - s_x\delta_x)\rangle(\eta)$$
2. *For $h \in C_c(\mathbb{R}^d)$, we have*
$$D_h^- \langle \varphi^{\otimes n}, P^{(n)}(\cdot)\rangle(\eta) = -n \sum_{x\in\tau(\eta)} s_x h(x) s_x\varphi(x)\langle \varphi^{\otimes n-1}, P^{(n-1)}(\cdot - s_x\delta_x)\rangle(\eta)$$
3. *For $(s,x) \in \mathbb{R}_+^* \times \mathbb{R}^d$, $x \notin \tau(\eta)$, we get*
$$D_{(s,x)}^+ \langle \varphi^{\otimes n}, P^{(n)}(\cdot)\rangle(\eta) = ns\varphi(x)\langle \varphi^{\otimes n-1}, P^{(n-1)}(\cdot)\rangle(\eta)$$
4. *For $g \in L^2(\mathbb{R}_+^* \times \mathbb{R}^d)$, the following holds:*
$$\begin{aligned} D_g^+ \langle \varphi^{\otimes n}, P^{(n)}(\cdot)\rangle(\eta) &= n\int_{\mathbb{R}_+^*\times\mathbb{R}^d} s\varphi(x)g(s,x)\langle \varphi^{\otimes n-1}, P^{(n-1)}(\eta)\rangle(\sigma\otimes l)(dx,ds) \\ &= n\langle f_\varphi, g\rangle_{L^2(\mathbb{R}_+^*\times\mathbb{R}^d)}\langle \varphi^{\otimes n-1}, P^{(n-1)}(\eta)\rangle \end{aligned}$$

Proof. 1. Using equation (37), we obtain
$$\begin{aligned} D_x^- \langle \varphi^{\otimes n}, P^{(n)}(\cdot)\rangle(\eta) &= \sum_{\substack{\{i_1,\dots,i_n\}\subset\mathbb{N} \\ x_{i_j}\neq x \forall j}} s_{i_1}\cdots s_{i_n}\varphi(x_{i_1})\cdots\varphi(x_{i_n}) \\ &\quad - \sum_{\{i_1,\dots,i_n\}\subset\mathbb{N}} s_{i_1}\cdots s_{i_n}\varphi(x_{i_1})\cdots\varphi(x_{i_n}) \\ &= -\sum_{\substack{\{i_1,\dots,i_n\}\subset\mathbb{N} \\ \exists j:\, x_{i_j}=x}} s_{i_1}\cdots s_{i_n}\varphi(x_{i_1})\cdots\varphi(x_{i_n}) \\ &= -ns_x\varphi(x)\sum_{\substack{\{i_1,\dots,i_{n-1}\}\subset\mathbb{N} \\ \forall j:\, x_{i_j}\neq x}} s_{i_1}\cdots s_{i_{n-1}}\varphi(x_{i_1})\cdots\varphi(x_{i_{n-1}}) \\ &= -ns_x\varphi(x)\langle \varphi^{\otimes n-1}, P^{(n-1)}(\cdot - s_x\delta_x)\rangle(\eta) \end{aligned}$$

2. The second part follows directly by part 1 and the definition.

3. For the third statement, we obtain
$$\begin{aligned} D_{(s,x)}^+ \langle \varphi^{\otimes n}, P^{(n)}(\cdot)\rangle(\eta) &= D_{(s,x)}^+ \sum_{\{i_1,\dots,i_n\}\subset\mathbb{N}} s_{i_1}\cdots s_{i_n}\varphi(x_{i_1})\cdots\varphi(x_{i_n}) \\ &= \sum_{\substack{\{i_1,\dots,i_n\}\subset\mathbb{N} \\ \exists j:\, x_{i_j}=x}} s_{i_1}\cdots s_{i_n}\varphi(x_{i_1})\cdots\varphi(x_{i_n}) \\ &= n\sum_{\{i_1,\dots,i_{n-1}\}\subset\mathbb{N}} s_{i_1}\cdots s_{i_{n-1}}\varphi(x_{i_1})\cdots\varphi(x_{i_{n-1}})s\varphi(x) \\ &= ns\varphi(x)\langle \varphi^{\otimes n-1}, P^{(n-1)}(\cdot)\rangle(\eta) \end{aligned}$$

4. The fourth statement follows by using part 3 as well as the definition of the birth gradient.

□

We may apply the above result to the mapping which is defined analogously to the generating functional on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$.

Definition 3.43. *On $\mathbb{K}(\mathbb{R}^d)$, define the following mapping:*

$$E_\varphi(\eta) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \varphi^{\otimes n}, P^{(n)} \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \langle\langle f_\varphi^{\otimes n}, (\mathcal{R}^{-1}\eta)_n \rangle\rangle$$

where $\varphi \in C_c(\mathbb{R}^d)$ and $f_\varphi(s,x) = s\varphi(x)$.

Corollary 3.44. *Let $\psi \in L^2(\mathbb{R}_+^* \times \mathbb{R}^d, d\varkappa)$ and $\varphi \in C_c(\mathbb{R}^d)$. Then, the following holds:*

$$D_\psi^+ E_\varphi(\eta) = \langle \psi, f_\varphi \rangle_{L^2(\mathbb{R}_+^* \times \mathbb{R}^d)} E_\varphi(\eta)$$

Proof. Apply statement 4 of Proposition 3.42 to each summand. □

3.5.4 Commutation Relations for Discrete Gradients on $\mathbb{K}(\mathbb{R}^d)$

Especially in the theories of operator algebras and mathematical physics, the notion of a commutator is of importance. Let us calculate the commutator for some combinations of the discrete gradients introduced above.

Proposition 3.45. *We have the following relations:*

1. *Let $g, h \in C_c(\mathbb{R}^d)$. Then*
$$[D_g^-, D_h^-] = 0.$$

2. *Let $\varphi, \psi \in L^2(\mathbb{R}_+^* \times \mathbb{R}^d, d\varkappa)$. Then*
$$[D_\varphi^+, D_\psi^+] = 0$$

3. *For $h \in C_c(\mathbb{R}^d)$ and $\varphi \in L^2(\mathbb{R}_+^* \times \mathbb{R}^d, d\varkappa)$, we have*
$$[D_h^-, D_\varphi^+] = D_{f_h\varphi}^+$$
where $f_h(s,x) = sh(x)$, as above.

Proof. 1. Let $g, h \in C_c(\mathbb{R}^d)$ and $F \in \mathcal{F}C(\mathbb{K}(\mathbb{R}^d))$ a test function. Using the fact that

$$\sum_{y \in \tau(\eta)} \sum_{x \in \tau(\eta - s_y\delta_y)} = \sum_{x \in \tau(\eta)} \sum_{y \in \tau(\eta - s_y\delta_y)},$$

we see that

$$
\begin{aligned}
\big([D_g^-, D_h^-]F\big)(\eta) &= D_g^-\left[\sum_{x\in\tau(\cdot)} s_x h(x)[F(\cdot - s_x\delta_x) - F(\cdot)]\right](\eta)\\
&\quad - D_h^-\left[\sum_{x\in\tau(\cdot)} s_x g(x)[F(\cdot - s_x\delta_x) - F(\cdot)]\right](\eta)\\
&= \sum_{y\in\tau(\eta)} s_y g(y)\Bigg[\sum_{x\in\tau(\eta - s_y\delta_y)} s_x h(x)[F(\eta - s_y\delta_y - s_x\delta_x) - F(\eta - s_y\delta_y)]\\
&\qquad\qquad - \sum_{x\in\tau(\eta)} s_x h(x)[F(\eta - s_x\delta_x) - F(\eta)]\Bigg]\\
&\quad - \sum_{y\in\tau(\eta)} s_y h(y)\Bigg[\sum_{x\in\tau(\eta - s_y\delta_y)} s_x g(x)[F(\eta - s_y\delta_y - s_x\delta_x) - F(\eta - s_y\delta_y)]\\
&\qquad\qquad - \sum_{x\in\tau(\eta)} s_x g(x)[F(\eta - s_x\delta_x) - F(\eta)]\Bigg]\\
&= \sum_{y\in\tau(\eta)}\sum_{x\in\tau(\eta - s_y\delta_y)} s_y g(y) s_x h(x)[F(\eta - s_y\delta_y - s_x\delta_x) - F(\eta - s_y\delta_y)]\\
&\quad - \sum_{y\in\tau(\eta)}\sum_{x\in\tau(\eta)} s_y g(y) s_x h(x)[F(\eta - s_x\delta_x) - F(\eta)]\\
&\quad - \sum_{y\in\tau(\eta)}\sum_{x\in\tau(\eta - s_y\delta_y)} s_y h(y) s_x g(x)[F(\eta - s_y\delta_y - s_x\delta_x) - F(\eta - s_y\delta_y)]\\
&\quad + \sum_{y\in\tau(\eta)}\sum_{x\in\tau(\eta)} s_y g(y) s_x h(x)[F(\eta - s_x\delta_x) - F(\eta)]\\
&= 0
\end{aligned}
$$

2. Let $\varphi, \psi \in L^2(\mathbb{R}_+^* \times \mathbb{R}^d, d\varkappa)$ and $F \in \mathcal{F}C(\mathbb{K}(\mathbb{R}^d))$. Then

$$
\begin{aligned}
(D_\varphi^+ D_\psi^+ F)(\eta) &= D_\varphi^+\left(\int_{\mathbb{R}_+^*\times\mathbb{R}^d} \psi(s,x)[F(\cdot + s\delta_x) - F(\cdot)]\varkappa(ds,dx)\right)(\eta)\\
&= \int_{\mathbb{R}_+^*\times\mathbb{R}^d} \varphi(t,y)\Bigg(\int_{\mathbb{R}_+^*\times\mathbb{R}^d} \psi(s,x)[F(\eta + t\delta_y + s\delta_x) - F(\eta + t\delta_y)]\varkappa(ds,dx)\\
&\qquad\qquad - \int_{\mathbb{R}_+^*\times\mathbb{R}^d} \psi(s,x)[F(\eta + s\delta_x) - F(\eta)]\varkappa(ds,dx)\Bigg)\varkappa(dt,dy)\\
&= \int_{\mathbb{R}_+^*\times\mathbb{R}^d}\int_{\mathbb{R}_+^*\times\mathbb{R}^d} \varphi(t,y)\psi(s,x)[F(\eta + t\delta_y + s\delta_x) - F(\eta + t\delta_y)\\
&\qquad - F(\eta + s\delta_x) + F(\eta)]\varkappa(ds,dx)\varkappa(dt,dy)\\
&= \int_{\mathbb{R}_+^*\times\mathbb{R}^d}\int_{\mathbb{R}_+^*\times\mathbb{R}^d} \psi(s,x)\varphi(t,y)[F(\eta + t\delta_y + s\delta_x) - F(\eta + s\delta_x)
\end{aligned}
$$

$$- F(\eta + t\delta_y) + F(\eta)]\varkappa(dt,dy)\varkappa(ds,dx)$$

$$= (D^+_\psi D^+_\varphi F)(\eta)$$

Where the last step is done by backtracking the calculation with the variables and functions switched.

3. Let $h \in C_c(\mathbb{R}^d)$ and $\varphi \in L^2(\mathbb{R}^*_+ \times \mathbb{R}^d, d\varkappa)$. Then

$$\left([D^-_h, D^+_\varphi]F\right)(\eta) = (D^-_h D^+_\varphi F)(\eta) - (D^+_\varphi D^-_h F)(\eta)$$

$$= D^-_h \left(\int_{\mathbb{R}^*_+ \times \mathbb{R}^d} \varphi(s,x)[F(\cdot + s\delta_x) - F(\cdot)]\varkappa(ds,dx)\right)(\eta)$$

$$- D^+_\varphi \left(\sum_{x\in\tau(\cdot)} s_x h(x)[F(\cdot - s_x\delta_x) - F(\cdot)]\right)(\eta)$$

$$= \sum_{y\in\tau(\eta)} s_y h(y) \int_{\mathbb{R}^*_+ \times \mathbb{R}^d} \varphi(s,x)[F(\eta - s_y\delta_y + s\delta_x) - F(\eta - s_y\delta_y)]\varkappa(ds,dx)$$

$$- \sum_{y\in\tau(\eta)} s_y h(y) \int_{\mathbb{R}^*_+ \times \mathbb{R}^d} \varphi(s,x)[F(\eta + s\delta_x) - F(\eta)]\varkappa(ds,dx)$$

$$- \int_{\mathbb{R}^*_+ \times \mathbb{R}^d} \varphi(t,y) \sum_{x\in\tau(\eta+t\delta_y)} s_x h(x)[F(\eta + t\delta_y - s_x\delta_x) - F(\eta + t\delta_y)]\varkappa(dt,dy)$$

$$+ \int_{\mathbb{R}^*_+ \times \mathbb{R}^d} \varphi(t,y) \sum_{x\in\tau(\eta)} s_x h(x)[F(\eta - s_x\delta_x) - F(\eta)]\varkappa(dt,dy)$$

$$= \int_{\mathbb{R}^*_+ \times \mathbb{R}^d} \sum_{y\in\tau(\eta)} s_y h(y)\varphi(s,x)[F(\eta - s_y\delta_y + s\delta_x) - F(\eta - s_y\delta_y)]\varkappa(ds,dx)$$

$$- \int_{\mathbb{R}^*_+ \times \mathbb{R}^d} \sum_{y\in\tau(\eta)} s_y h(y)\varphi(s,x)[F(\eta + s\delta_x) - F(\eta)]\varkappa(ds,dx)$$

$$- \int_{\mathbb{R}^*_+ \times \mathbb{R}^d} \sum_{x\in\tau(\eta+t\delta_y)} s_x h(x)\varphi(t,y)[F(\eta + t\delta_y - s_x\delta_x) - F(\eta + t\delta_y)]\varkappa(dt,dy)$$

$$+ \int_{\mathbb{R}^*_+ \times \mathbb{R}^d} \sum_{y\in\tau(\eta)} s_y h(y)\varphi(s,x)[F(\eta - s_y\delta_y) - F(\eta)]\varkappa(ds,dx)$$

$$= \int_{\mathbb{R}^*_+ \times \mathbb{R}^d} \sum_{y\in\tau(\eta)} s_y h(y)\varphi(s,x)[F(\eta - s_y\delta_y + s\delta_x) - F(\eta - s_y\delta_y)]\varkappa(ds,dx)$$

$$- \int_{\mathbb{R}^*_+ \times \mathbb{R}^d} \sum_{y\in\tau(\eta)} s_y h(y)\varphi(s,x)[F(\eta + s\delta_x) - F(\eta)]\varkappa(ds,dx)$$

$$- \int_{\mathbb{R}^*_+ \times \mathbb{R}^d} \sum_{y\in\tau(\eta)} s_y h(y)\varphi(s,x)[F(\eta + s\delta_x - s_y\delta_y) - F(\eta + s\delta_x)]\varkappa(ds,dx)$$

$$+ \int_{\mathbb{R}^*_+ \times \mathbb{R}^d} \sum_{y\in\tau(\eta)} s_y h(y)\varphi(s,x)[F(\eta - s_y\delta_y) - F(\eta)]\varkappa(ds,dx)$$

$$- \int_{\mathbb{R}^*_+ \times \mathbb{R}^d} t h(y)\varphi(t,y)[F(\eta) - F(\eta + t\delta_y)]\varkappa(dt,dy)$$

We see that everything except the last term cancels out. On the other hand,

$$(D^{+}_{f_h\varphi}F)(\eta) = \int_{\mathbb{R}^*_+\times\mathbb{R}^d} sh(x)\varphi(s,x)[F(\eta+s\delta_x)-F(\eta)]\varkappa(ds,dx),$$

which is equal to the last term of the above calculation.

□

4 Dynamics

Now that the necessary theoretical background on $\mathbb{K}(\mathbb{R}^d)$ is established, it is time to consider specific models on the cone. We consider three models which are typically of interest:

1. Glauber dynamics
2. Generalised Contact Model
3. Bolker-Dieckmann-Law-Pacala (BDLP) Model

Since the direct analysis of the Markov-type operator on $\mathbb{K}(\mathbb{R}^d)$ is too difficult, we resort to the scheme proposed in Chapter 2.11.2. This means that we take the following steps:

1. Define the Markov pre-generator L on a class of observables $F\colon \mathbb{K}(\mathbb{R}^d) \to \mathbb{R}$.
2. Use the K-transform to define an associated operator on a class of quasi-observables $G\colon \mathbb{K}_0(\mathbb{R}^d) \to \mathbb{R}$ via the relation $\hat{L} := K^{-1}LK$. This operator is also called the symbol of L.
3. Assuming that the underlying initial state admits a correlation function, we may use the duality on $\mathbb{K}_0(\mathbb{R}^d)$ to construct the statistical dynamics for correlation functions.
4. Depending on the model, we may show the existence of the dynamics using quasi-observables (L^1-techniques) or correlation functions (L^∞-techniques).
5. The different function spaces on $\mathbb{K}_0(\mathbb{R}^d)$ also enable us to prove certain properties of the models such as *a priori* estimates or asymptotic behaviour.

4.1 Glauber Dynamics

The first model we want to investigate are the Glauber dynamics. This model emerges from the analysis of the underlying Gibbs measure based on a corresponding energy functional. There have been various works on the Glauber dynamics under different circumstances, such as consideration of finite volume in [6]. For configuration spaces, this model has been examined in [37] and [19]. While the former is concerned with the construction of the corresponding Gibbs measure, the latter employs semigroup theory to show the existence of dynamics of various models. The analysis of Gibbs measures on the cone of positive measures and preliminary work to this paper were done in [29]. We want to focus on the first step of the process of showing the existence of a semigroup on L^1-type spaces of quasi-observables. Furthermore, we establish the hierarchical structure associated with the Glauber dynamics.

4.1.1 Generator Corresponding to the Dirichlet Form

The generator of the Glauber dynamics is based on a Dirichlet form corresponding to an underlying Gibbs measure. A brief overview of the construction of Gibbs measures can be found in Chapter 2.8.3 of this work. Note that for this chapter, we restrict ourselves to the intensity measure ν_θ from Definition 2.22.

In the case of $\Gamma(\mathbb{R}^d)$, it has been shown in [37] that there exists a Hunt process with the generator given by the Dirichlet form given below. Furthermore, this process has the corresponding Gibbs measure as invariant measure. These statements were shown using semigroup techniques together with the theory of Dirichlet forms. We want to use the same techniques to obtain similar results as in the configuration space case.

Definition 4.1. *Let $\mu \in G(\phi)$ and $F, G \in \mathcal{F}C(\mathbb{K}(\mathbb{R}^d))$. Define the following form:*

$$\begin{aligned}\mathcal{E}(F,G) &:= \frac{1}{2}\int_{\mathbb{K}(\mathbb{R}^d)}\int_{\mathbb{R}^d} D_x^- F(\eta) D_x^- G(\eta)\eta(dx)\mu(d\eta)\\ &= \frac{1}{2}\int_{\mathbb{K}(\mathbb{R}^d)}\sum_{x\in\tau(\eta)} s_x D_x^- F(\eta) D_x^- G(\eta)\mu(d\eta)\end{aligned}$$

Remark 4.2. *For a more general bilinear form, one may replace s_x by some function $m\colon \mathbb{R}_+ \to \mathbb{R}$, $s_x \mapsto m(s_x)$.*

Proposition 4.3. *The operator associated to the above (Dirichlet) form has the form*

$$\begin{aligned}(LF)(\eta) = &\sum_{x\in\tau(\eta)} s_x \left[F(\eta - s_x\delta_x) - F(\eta)\right]\\ &+ \int_{\mathbb{R}_+^*\times\mathbb{R}^d} \left[F(\eta + s_x\delta_x) - F(\eta)\right] e^{-\Phi((s,x);\eta))} s\nu_\theta(ds)\sigma(dx)\end{aligned}$$

where Φ is defined as in Proposition 2.41 and $F \in \mathcal{F}C(\mathbb{K}(\mathbb{R}^d))$.

Proof. we show

$$\mathcal{E}(F,G) = (LF, G)_{L^2(\mathbb{K}(\mathbb{R}^d), d\mathcal{G})}.$$

By using the Georgii-Ngyuen-Zessin identity (13), we can calculate

$$\begin{aligned}\mathcal{E}(F,G) &= \frac{1}{2}\int_{\mathbb{K}(\mathbb{R}^d)}\int_{\mathbb{R}_+^*\times\mathbb{R}^d} D_x^- F(\eta) D_x^- G(\eta)\eta(dx)\mathcal{G}(d\eta)\\ &= \frac{1}{2}\int_{\mathbb{K}(\mathbb{R}^d)}\int_{\mathbb{R}_+^*\times\mathbb{R}^d} D_x^- F(\eta)(G(\eta - s_x\delta_x) - G(\eta))\eta(dx)\mathcal{G}(d\eta)\\ &= \frac{1}{2}\int_{\mathbb{K}(\mathbb{R}^d)}\int_{\mathbb{R}_+^*\times\mathbb{R}^d} D_x^- F(\eta)(G(\eta - s_x\delta_x)\eta(dx)\mathcal{G}(d\eta)\\ &\quad - \frac{1}{2}\int_{\mathbb{K}(\mathbb{R}^d)}\int_{\mathbb{R}_+^*\times\mathbb{R}^d} D_x^- F(\eta)G(\eta)\eta(dx)\mathcal{G}(d\eta)\\ &\overset{(13)}{=} \frac{1}{2}\int_{\mathbb{K}(\mathbb{R}^d)}\int_{\mathbb{R}_+^*\times\mathbb{R}^d} D_x^- F(\eta + s_x\delta_x)G(\eta)e^{-\Phi((s,x);\eta)} s\nu_\theta(ds)\sigma(dx)\mathcal{G}(d\eta)\\ &\quad - \frac{1}{2}\int_{\mathbb{K}(\mathbb{R}^d)}\int_{\mathbb{R}_+^*\times\mathbb{R}^d} D_x^- F(\eta)G(\eta)\eta(dx)\mathcal{G}(d\eta)\\ &= -\frac{1}{2}\int_{\mathbb{K}(\mathbb{R}^d)}\int_{\mathbb{R}_+^*\times\mathbb{R}^d} (F(\eta + s_x\delta_x) - F(\eta))G(\eta)e^{-\Phi((s,x);\eta)} s\nu_\theta(ds)\sigma(dx)\mathcal{G}(d\eta)\\ &\quad - \frac{1}{2}\int_{\mathbb{K}(\mathbb{R}^d)}\sum_{x\in\tau(\eta)} s_x(F(\eta - s_x\delta_x) - F(\eta))G(\eta)\mathcal{G}(d\eta)\\ &= -\frac{1}{2}\int_{\mathbb{K}(\mathbb{R}^d)}\underbrace{\int_{\mathbb{R}_+^*\times\mathbb{R}^d} (F(\eta + s_x\delta_x) - F(\eta))e^{-\Phi((s,x);\eta)} s\nu_\theta(ds)\sigma(dx)}_{=:(L_1F)(\eta)}\, FG(\eta)\mathcal{G}(d\eta)\end{aligned}$$

$$-\frac{1}{2}\int_{\mathbb{K}(\mathbb{R}^d)} \underbrace{\sum_{x\in\tau(\eta)} s_x(F(\eta - s_x\delta_x) - F(\eta))}_{=:(L_0F)(\eta)} G(\eta)\mathcal{G}(d\eta)$$

The definitions of L_0 and L_1 will be used in the next chapter, when we calculate the symbol of L. □

4.1.2 The Symbol for the Glauber Dynamics

As mentioned above, in many cases, it is convenient to consider the symbol operator on the space of quasi-observables. In this chapter, we consider the Markov-type operator calculated above,

$$\begin{aligned}(LF)(\eta) = &\sum_{x\in\tau(\eta)} s_x\left[F(\eta - s_x\delta_x) - F(\eta)\right]\\ &+ \int_{\mathbb{R}_+^*\times\mathbb{R}^d} \left[F(\eta + s_x\delta_x) - F(\eta)\right] e^{-\Phi((s,x);\eta))} s\nu_\theta(ds)\sigma(dx)\end{aligned}$$

and calculate the corresponding symbol on the space of functions $G\colon \mathbb{K}_0(\mathbb{R}^d) \to \mathbb{R}$.

For convenience, we recall the following notation.

Reminder. *The Lebesgue-Poisson exponent is defined by*

$$e_\lambda(f,\eta) := \prod_{x\in\tau(\eta)} f(s_x, x)$$

for $f\colon \mathbb{R}_+^* \times \mathbb{R}^d \to \mathbb{R}$ and $\eta \in \mathbb{K}_0(\mathbb{R}^d)$ whenever the above expression is defined. Note that we use the notation e_λ for a function on $\mathbb{K}_0(\mathbb{R}^d)$ here, as opposed to $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$.

Also, the following combinatorial identities are needed when calculating $\hat{L}$:

Lemma 4.4. *The Lebesgue-Poisson exponent has the following properties:*

$$\begin{aligned}Ke_\lambda(f,\eta) &= \prod_{y\in\tau(\eta)} (1 + f(s_y, y)),\ \ \eta \in \mathbb{K}_0(\mathbb{R}^d)\\ (G \star e_\lambda(f))(\eta) &= \sum_{\xi\subset\eta} G(\xi)e_\lambda(f+1,\xi)e_\lambda(f,\eta-\xi)\end{aligned}$$

provided, both sides of the equations make sense.

We are now ready to give the form of the symbol on quasi-observables.

Proposition 4.5. *The symbol $\hat{L}$ corresponding to L is given by*

$$\begin{aligned}(\hat{L}G)(\eta) = &-\left(\sum_{x\in\tau(\eta)} s_x\right) G(\eta)\\ &+ \int_{\mathbb{R}_+^*\times\mathbb{R}^d} s\sum_{\xi\subset\eta} G(\xi + s\delta_x)e_\lambda(e^{-2ss_y\phi(x,y)},\xi)e_\lambda(e^{-2ss_y\phi(x,y)} - 1, \eta - \xi)\nu_\theta(ds)\sigma(dx)\end{aligned}$$

for $G \in B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$.

Proof. We set $L = L_0 + L_1$, where L_0 and L_1 are defined in the proof of Proposition 4.3. The same decomposition can be done for $\hat{L}$. We begin by calculating $\hat{L}_0$ using Lemma 2.76.

$$
\begin{aligned}
(\hat{L}_0 G) &= K^{-1}\left[\sum_{x\in\tau(\eta)} s_x\left[KG(\eta - s_x\delta_x) - KG(\eta)\right]\right] \\
&= K^{-1}\left[\sum_{x\in\tau(\eta)} s_x\left[-(KG(\cdot + s_x\delta_x))(\eta - s_x\delta_x)\right]\right] \\
&= K^{-1}\sum_{x\in\tau(\eta)} s_x\left(-\sum_{\xi\Subset\eta - s_x\delta_x} G(\xi + s_x\delta_x)\right) \\
&= -K^{-1}\sum_{x\in\tau(\eta)}\sum_{\xi\Subset\eta - s_x\delta_x} s_x\left(G(\xi + s_x\delta_x)\right) \\
&= \underbrace{-K^{-1}\sum_{\xi\Subset\eta}}_{=\mathrm{id}}\sum_{x\in\tau(\xi)} s_x G(\xi) \\
&= -\left[\sum_{x\in\tau(\xi)} s_x\right] G(\xi)
\end{aligned}
$$

Next, we calculate $\hat{L}_1$ using Lemma 4.4.

$$
\begin{aligned}
(\hat{L}_1 G) &= K^{-1}\int_{\mathbb{R}_+^*\times\mathbb{R}^d}\left[KG(\eta + s_x\delta_x) - KG(\eta)\right]e^{-\Phi((s,x),\eta)} s\nu_\theta(ds)\sigma(dx) \\
&= K^{-1}\int_{\mathbb{R}_+^*\times\mathbb{R}^d} KG(\cdot + s_x\delta_x)(\eta)e^{-\Phi((s,x),\eta)} s\nu_\theta(ds)\sigma(dx) \\
&= K^{-1}\int_{\mathbb{R}_+^*\times\mathbb{R}^d} KG(\cdot + s_x\delta_x)(\eta)K\left[e^{-\Phi((s,x),\cdot)} - 1\right] s\nu_\theta(ds)\sigma(dx) \\
&= K^{-1}\int_{\mathbb{R}_+^*\times\mathbb{R}^d} K\left[G(\cdot + s_x\delta_x)(\eta)\star\left(e^{-\Phi((s,x),\cdot)} - 1\right)\right] s\nu_\theta(ds)\sigma(dx) \\
&= \int_{\mathbb{R}_+^*\times\mathbb{R}^d} s\sum_{\xi\subset\eta} G(\xi + s\delta_x)\prod_{y\in\tau(\xi)}\left(e^{-2ss_y\phi(x,y)}\right)\prod_{y\in\tau(\eta-\xi)}\left(e^{-2ss_y\phi(x,y)} - 1\right)\nu_\theta(ds)\sigma(dx)
\end{aligned}
$$

Note that we used η as a placeholder for legibility. □

4.1.3 Existence of a Semigroup for the Glauber Dynamics

For the Glauber model, the existence of dynamics is shown using semigroup theory on L^1-type spaces of quasi-observables. In the following chapter, we prove the existence of an analytic semigroup associated to the generator $\hat{L}$.

We start by introducing the spaces on which we want to construct our semigroup. These spaces are of L^1-type and have fixed densities depending on the number of particles and the size of the corresponding marks, which is represented by the coefficients C and α.

Definition 4.6. *For $C > 0$ and $\alpha \in \mathbb{R}$, define the family of L^1-type spaces*

$$\mathbf{L}_{\alpha,C} := L^1 \left(\mathbb{K}_0(\mathbb{R}^d), C^{|\tau(\eta)|} e^{\alpha \sum_{x\in\tau(\eta)} s_x} d\lambda_{\mathcal{G}} \right)$$

with the usual L^1-norm defined by the stated measure. We denote this norm by $\| \cdot \|_{\alpha,C}$.

Recall that by Proposition 2.80, the space $B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$ is dense in $\mathbf{L}_{\alpha,C}$ for any $C > 0$ and $\alpha \in \mathbb{R}$.

The construction scheme for the semigroup is as follows: First, we prove the existence of a semigroup for the generator $\hat{L}_0$. Next, we give sufficient conditions on the parameters α, C and the coefficients of $\hat{L}_1$, namely, the potential ϕ, to ensure that this part can be seen as a suitable perturbation of the operator $\hat{L}_0$.

Proposition 4.7. *For any α, C, the operator $\hat{L}_0$ defined above with the domain*

$$\mathcal{D}(\hat{L}_0) := \left\{ G \in \mathbf{L}_{\alpha,C} \middle| \sum_{x\in\tau(\eta)} s_x \cdot G(\eta) \in \mathbf{L}_{\alpha,C} \right\}$$

generates a contraction semigroup on $\mathbf{L}_{\alpha,C}$. Moreover, this semigroup is analytic.

The proof of the statement follows the same outline as in [19] and is as follows: The first part of the statement will be shown using Hille-Yosida. For the second part, recall the following variation. We denote by $R(\zeta, A) := (\zeta \mathbb{1} - A)^{-1}$ the resolvent of A (provided it exists).

Lemma 4.8 ([14, Ex. II.4.12(6)]). *Let $(A, \mathcal{D}(A))$ be a closed, densely defined linear operator on a Banach space X. If there exist $\delta > 0, r > 0$ and $M \geq 1$ s.t.*

$$\Sigma_\delta := \{\zeta \in \mathbb{C} \colon |\zeta| > r \text{ and } |\arg(\zeta)| < \frac{\pi}{2} + \delta\} \subseteq \rho(A)$$

and

$$\|R(\zeta, A)\| \leq \frac{M}{|\zeta|}$$

for all $\zeta \in \Sigma_\delta$, then A generates an analytic semigroup.

Proof of Proposition 4.7. The closedness of $\hat{L}_0$ is clear since it is a multiplication operator.

Dense domain: By Proposition 2.80, we have that $B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d)) \subseteq \mathbf{L}_{\alpha,C}$ is dense. Hence, it is enough to show that $B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d)) \subseteq \mathcal{D}(\hat{L}_0)$. Let $G \in B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$. Then there exists an interval $[a, b] \subset \mathbb{R}_+^*$ such that $G(\eta) = 0$ whenever there exists s_x such that $s_x \notin [a, b]$ as well as $N \in \mathbb{N}$ such that $G(\eta) = 0$ whenever $|\tau(\eta)| > N$. Hence,

$$\begin{aligned}
\|\hat{L}_0 G\|_{\alpha,C} &= \int_{\mathbb{K}_0(\mathbb{R}^d)} \left| \hat{L}_0 G(\eta) \right| C^{|\tau(\eta)|} e^{\alpha \sum_{x\in\tau(\eta)} s_x} \lambda_{\mathcal{G}}(d\eta) \\
&= \int_{\mathbb{K}_0(\mathbb{R}^d)} \sum_{x\in\tau(\eta)} s_x G(\eta) C^{|\tau(\eta)|} e^{\alpha \sum_{x\in\tau(\eta)} s_x} \lambda_{\mathcal{G}}(d\eta) \\
&\leq b \int_{\mathbb{K}_0(\mathbb{R}^d)} |\tau(\eta)| G(\eta) C^{|\tau(\eta)|} e^{\alpha \sum_{x\in\tau(\eta)} s_x} \lambda_{\mathcal{G}}(d\eta) \\
&\leq b \int_{\mathbb{K}_0(\mathbb{R}^d)} N G(\eta) C^{|\tau(\eta)|} e^{\alpha \sum_{x\in\tau(\eta)} s_x} \lambda_{\mathcal{G}}(d\eta) \\
&= bN \|G\|_{\alpha,C} < \infty
\end{aligned}$$

By above arguments, $\hat{L}_0$ is densely defined.

What is left to show is the resolvent bound. Let $\delta \in (0, \frac{\pi}{2})$ and $\zeta \in \Sigma_\delta \cup \mathbb{R}_+^*$. Since $s_x \in \mathbb{R}_+^*$,

$$\left| \sum_{x \in \tau(\eta)} s_x + \zeta \right| > 0$$

and hence, $R(\zeta, \hat{L}_0)$ is well-defined for all $\zeta \in \Sigma_\delta \cup \mathbb{R}_+^*$ as the multiplication operator

$$R(\zeta, \hat{L}_0) G(\eta) = -\frac{1}{\sum_{x \in \tau(\eta)} s_x + \zeta} G(\eta)$$

For $\zeta \in \mathbb{C}$, we show

$$\|R(\zeta, \hat{L}_0)\| \leq \begin{cases} \frac{1}{|\zeta|} & , \text{ if } \Re\zeta \geq 0 \\ \frac{M}{|\zeta|} & , \text{ if } \Re\zeta < 0 \end{cases}$$

which will complete the conditions for Hille-Yosida and Lemma 4.8. Here, $M > 0$ is a constant defined below and $\Re\zeta$ denotes the real part of $\zeta \in \mathbb{C}$.

Case $\Re\zeta \geq 0$: Since $|\sum_{x \in \tau(\eta)} s_x + \zeta| \geq |\zeta|$,

$$\|R(\zeta, \hat{L}_0) G(\eta)\| = \left| \frac{1}{\sum_{x \in \tau(\eta)} s_x + \zeta} \right| \|G(\eta)\| \leq \frac{1}{|\zeta|} \|G(\eta)\|$$

Case $\Re\zeta < 0$: Since $|\arg(\zeta)| \leq \frac{\pi}{2} + \delta$, we have

$$|\Im\zeta| \geq |\zeta| \cdot \left| \sin\left(\frac{\pi}{2} + \delta\right) \right| = |\zeta| \cdot |\cos\zeta|$$

where $\Im\zeta$ denotes the imaginary part of ζ. This yields the estimate

$$\frac{|\zeta|}{\left|\sum_{x \in \tau(\eta)} s_x + \zeta\right|} \leq \frac{|\zeta|}{|\Im\zeta|} \leq \frac{1}{\cos\delta} =: M$$

we can now use this to establish the resolvent bound:

$$\|R(\zeta, \hat{L}_0) G\| \leq \frac{1}{|\zeta|} \left\| \frac{|\zeta|}{\left|\sum_{x \in \tau(\cdot)} s_x + \zeta\right|} G \right\| \leq \frac{1}{|\zeta|} M \|G\| = \frac{M}{|\zeta|} \|G\|$$

by the above considerations, the claim follows. □

Next, we consider $\hat{L}_1$ as a perturbation of $\hat{L}_0$. First, we introduce the notion of a relative bound.

Definition 4.9 ([14], III.2.1)**.** *Let $A \colon \mathcal{D}(A) \subset X \to X$ be a linear operator on the Banach space X. An operator $B \colon \mathcal{D}(B) \subset X \to X$ is called (relatively) A-bounded if $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ and if there exist constants $a, b \in \mathbb{R}_+$ such that*

$$\|Bx\| \leq a\|Ax\| + b\|x\| \tag{39}$$

for all $x \in \mathcal{D}(A)$. The A-bound of B is

$$a_0 := \inf\{a \geq 0 \colon \exists b \in \mathbb{R}_+ \text{ s.t. (39) holds}\}$$

Theorem 4.10 ([14, III.2.10]). *Let the operator $(A, \mathcal{D}(A))$ generate an analytic semigroup $(T(z))_{z\in\Sigma_\delta\cup\{0\}}$ on a Banach space X. Then there exists a constant $\beta > 0$ such that $(A + B, \mathcal{D}(A))$ generates an analytic semigroup for every A-bounded operator B having A-bound $a_0 < \beta$.*

The following identity for the coherent states will be useful in later calculations:

Lemma 4.11. *For the measure $\lambda_{\mathcal{G}}$, the following formula holds:*

$$\int_{\mathbb{K}_0(\mathbb{R}^d)} e_\lambda(f,\eta)\lambda_{\mathcal{G}}(d\eta) = \exp\left(\int_{\mathbb{R}_+^*\times\mathbb{R}^d} f(s,x)\nu_\theta(ds)\sigma(dx)\right)$$

The following Lemma yields the relevant bound for the perturbation $\hat{L}_1$.

Lemma 4.12 (relative bound for $\hat{L}_1$). *Let $\alpha < 1$. For any $C > 0$ with*

$$C \leq \frac{(1-\alpha)\alpha}{2\theta\int_{\mathbb{R}^d}\phi(x,y)\sigma(dy)}, \tag{40}$$

the following estimate holds:

$$\|\hat{L}_1 G\|_{\alpha,C} \leq \frac{1}{C}\|\hat{L}_0 G\|_{\alpha,C}, G \in \mathcal{D}(\hat{L}_0) = \mathcal{D}(\hat{L}_1)$$

Note that the domain of the operators also depend on α and C.

Remark 4.13. *It is interesting to note the additional restriction imposed by the introduction of α, which yields an upper bound on the parameter C. Compare to the condition given in [19], Example 1.*

Proof. We write $\boldsymbol{C}(\eta) := C^{|\tau(\eta)|}e^{\alpha\sum_{y\in\tau(\eta)}s_y}$ for brevity.

$$\begin{aligned}
\|\hat{L}_1 G\| &= \int_{\mathbb{K}_0(\mathbb{R}^d)} \left|\hat{L}_1 G(\eta)\right| \boldsymbol{C}(\eta)\lambda_{\mathcal{G}}(d\eta) \\
&\leq \int_{\mathbb{K}_0(\mathbb{R}^d)} \sum_{\xi\subset\eta}\int_{\mathbb{R}_+^*\times\mathbb{R}^d} se^{-\Phi((s,x),\xi)}|G(\xi+s\delta_x)|\times \\
&\qquad\qquad \times e_\lambda\left(\left|e^{-2ss_y\phi(x,y)}-1\right|,\eta-\xi\right)\boldsymbol{C}(\eta)\nu_\theta(ds)\sigma(dx)\lambda_{\mathcal{G}}(d\eta) \\
&= \int_{\mathbb{K}_0(\mathbb{R}^d)}\int_{\mathbb{R}_+^*\times\mathbb{R}^d}\sum_{\xi\subset\eta} se^{-\Phi((s,x),\xi)}|G(\xi+s\delta_x)|\times \\
&\qquad\qquad \times e_\lambda\left(\left|e^{-2ss_y\phi(x,y)}-1\right|,\eta-\xi\right)\boldsymbol{C}(\eta)\nu_\theta(ds)\sigma(dx)\lambda_{\mathcal{G}}(d\eta) \\
&\overset{2.77}{=} \int_{\mathbb{K}_0(\mathbb{R}^d)}\int_{\mathbb{K}_0(\mathbb{R}^d)}\int_{\mathbb{R}_+^*\times\mathbb{R}^d} se^{-\Phi((s,x),\xi_1)}|G(\xi_1+s\delta_x)|\times \\
&\qquad\qquad \times e_\lambda\left(\left|e^{-2ss_y\phi(x,y)}-1\right|,\xi_2\right)\boldsymbol{C}(\xi_1+\xi_2)\nu_\theta(ds)\sigma(dx)\lambda_{\mathcal{G}}(d\xi_1)\lambda_{\mathcal{G}}(d\xi_2) \\
&\overset{2.77}{=} \int_{\mathbb{K}_0(\mathbb{R}^d)}\int_{\mathbb{K}_0(\mathbb{R}^d)}\sum_{x\in\tau(\xi_1)} s_x e^{-\Phi((s,x),\xi_1-s_x\delta_x)}|G(\xi_1)|\times \\
&\qquad\qquad \times e_\lambda\left(\left|e^{-2s_xs_y\phi(x,y)}-1\right|,\xi_2\right)\boldsymbol{C}(\xi_1-s_x\delta_x+\xi_2)\lambda_{\mathcal{G}}(d\xi_1)\lambda_{\mathcal{G}}(d\xi_2)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{K}_0(\mathbb{R}^d)} |G(\xi_1)| \boldsymbol{C}(\xi_1) \sum_{x\in\tau(\xi_1)} \int_{\mathbb{K}_0(\mathbb{R}^d)} s_x \underbrace{e^{-\Phi((s,x),\xi_1 - s_x\delta_x)}}_{\leq 1} \times \\
&\qquad \times e_\lambda \left(\left| e^{-2s_x s_y \phi(x,y)} - 1 \right|, \xi_2 \right) \boldsymbol{C}(\xi_1 - s_x\delta_x + \xi_2) \boldsymbol{C}(\xi_1)^{-1} \lambda_{\mathcal{G}}(d\xi_1)\lambda_{\mathcal{G}}(d\xi_2) \\
&\leq \int_{\mathbb{K}_0(\mathbb{R}^d)} |G(\xi_1)| \boldsymbol{C}(\xi_1) \sum_{x\in\tau(\xi_1)} \int_{\mathbb{K}_0(\mathbb{R}^d)} s_x \times \\
&\qquad \times e_\lambda \left(\left| e^{-2s_x s_y \phi(x,y)} - 1 \right|, \xi_2 \right) \boldsymbol{C}(\xi_2) \boldsymbol{C}(s_x\delta_x)^{-1} \lambda_{\mathcal{G}}(d\xi_1)\lambda_{\mathcal{G}}(d\xi_2) \\
&= C^{-1} \int_{\mathbb{K}_0(\mathbb{R}^d)} |G(\xi_1)| \boldsymbol{C}(\xi_1) \sum_{x\in\tau(\xi_1)} s_x e^{-\alpha s_x} \times \\
&\qquad \times \int_{\mathbb{K}_0(\mathbb{R}^d)} e_\lambda \left(\left| e^{-2s_x s_y \phi(x,y)} - 1 \right| C e^{\alpha s_y}, \xi_2 \right) \lambda_{\mathcal{G}}(d\xi_2)\lambda_{\mathcal{G}}(d\xi_1) \\
&\overset{4.11}{=} C^{-1} \int_{\mathbb{K}_0(\mathbb{R}^d)} |G(\xi_1)| \boldsymbol{C}(\xi_1) \sum_{x\in\tau(\xi_1)} s_x e^{-\alpha s_x} \times \\
&\qquad \times \exp\left(\int_{\mathbb{R}_+^* \times \mathbb{R}^d} \left| e^{-2s_x s \phi(x,y)} - 1 \right| C e^{\alpha s} \nu_\theta(ds)\sigma(dx) \right) \lambda_{\mathcal{G}}(d\xi_1) \\
&\leq C^{-1} \int_{\mathbb{K}_0(\mathbb{R}^d)} |G(\xi_1)| \boldsymbol{C}(\xi_1) \sum_{x\in\tau(\xi_1)} s_x e^{-\alpha s_x} \times \\
&\qquad \times \exp\left(\int_{\mathbb{R}_+^* \times \mathbb{R}^d} 2 s_x s \phi(x,y) C e^{\alpha s} \nu_\theta(ds)\sigma(dx) \right) \lambda_{\mathcal{G}}(d\xi_1) \\
&\overset{2.22}{=} C^{-1} \int_{\mathbb{K}_0(\mathbb{R}^d)} |G(\xi_1)| \sum_{x\in\tau(\xi_1)} s_x \times \\
&\qquad \times \underbrace{\exp\left[\left(2C\theta \int_{\mathbb{R}_+^* \times \mathbb{R}^d} \phi(x,y) e^{(\alpha-1)s} ds\sigma(dx) - \alpha \right) s_x \right]}_{(*)} \boldsymbol{C}(\xi_1)\lambda_{\mathcal{G}}(d\xi_1) \\
&\leq \frac{1}{C} \| \hat{L}_0 G \|
\end{aligned}
$$

The last estimate holds if $(*) \leq 1$, i.e.

$$
\begin{aligned}
0 &\geq 2C\theta \int_{\mathbb{R}_+^* \times \mathbb{R}^d} \phi(x,y) e^{(\alpha-1)s} ds\sigma(dx) - \alpha \\
&= 2C\theta \int_{\mathbb{R}^d} \phi(x,y)\sigma(dx) \int_{\mathbb{R}_+^*} e^{(\alpha-1)s} ds - \alpha \\
&= \frac{2C}{1-\alpha} \theta \int_{\mathbb{R}^d} \phi(x,y)\sigma(dx) - \alpha
\end{aligned}
$$

rewriting yields the claim. □

Combining this estimate with a condition on C emerging from Theorem 4.10 yields the final existence result:

Theorem 4.14. *Let $C \geq 2$ and (40) holds. Then $\hat{L}$ generates an analytic semigroup on the space $\mathbf{L}_{\alpha,C}$.*

Proof. Condition (40) yields the relative boundedness of $\hat{L}_1$ w.r.t. $\hat{L}_0$. The condition $C \geq 2$ ensures that the estimate is sharp enough to guarantee the existence of the perturbed semigroup according to Theorem 4.10. □

4.1.4 Statistical Dynamics of the Glauber Model

For completeness, let us calculate the operator on the space of correlation functions corresponding to $\hat{L}$ according to the scheme proposed in Chapter 2.11.2.

Proposition 4.15. *The operator $L^\triangle$ is given by*

$$\begin{aligned}
\left(L^\triangle k\right)(\eta) &= \left(L_0^\triangle k\right)(\eta) + \left(L_1^\triangle k\right)(\eta) \\
&= -\sum_{x\in\tau(\eta)} s_x k(\eta) \\
&\quad + \int_{\mathbb{K}_0(\mathbb{R}^d)} \sum_{x\in\tau(\eta)} s_x e_\lambda(e^{-2s_xs_y\phi(x,y)}, \eta - s_x\delta_x)\times \\
&\qquad \times e_\lambda(e^{-2s_xs_y\phi(x,y)} - 1, \xi) k(\eta + \xi - s_x\delta_x)\lambda_{\mathcal{G}}(d\xi)
\end{aligned}$$

Proof. We start again with $L_0^\triangle$. Since $\hat{L}$ is a multiplication operator, we directly obtain

$$\left(L_0^\triangle k\right)(\eta) = -\sum_{x\in\tau(\eta)} s_x k(\eta).$$

To calculate $L_1^\triangle$, we fix $\xi \subset \eta$ and look at one summand first. Additionally, we exchange ξ and $\eta - \xi$. We also write

$$\begin{aligned}
f(\zeta) &:= e_\lambda(e^{-2ss_y\phi(x,y)}, \zeta) \\
g(\zeta) &:= e_\lambda(e^{-2ss_y\phi(x,y)} - 1, \zeta)
\end{aligned}$$

for short. Note that the application of Minlos Lemma in the following calculation will change $s \to s_x$ in f and g.

$$\begin{aligned}
&\int_{\mathbb{K}_0(\mathbb{R}^d)} \int_{\mathbb{R}_+^*\times\mathbb{R}^d} sf(\eta-\xi)g(\xi)G(\eta-\xi+s\delta_x)k(\eta)\nu_\theta(ds)\sigma(dx)\lambda_{\mathcal{G}}(d\eta) \\
&\quad = \int_{\mathbb{K}_0(\mathbb{R}^d)} \sum_{x\in\tau(\eta-\xi)} s_x f(\eta-\xi-s_x\delta_x)g(\xi)G(\eta-\xi)k(\eta-s_x\delta_x)\lambda_{\mathcal{G}}(d\eta)
\end{aligned}$$

For the last steps, we need to look at the complete sum to apply the other version of Minlos Lemma.

$$\begin{aligned}
\langle\langle \hat{L}_1 G, k\rangle\rangle &= \int_{\mathbb{K}_0(\mathbb{R}^d)} \sum_{\xi\subset\eta} \sum_{x\in\tau(\eta-\xi)} s_x f(\eta-\xi-s_x\delta_x)g(\xi)G(\eta-\xi)k(\eta-s_x\delta_x)\lambda_{\mathcal{G}}(d\eta) \\
&\overset{2.77}{=} \int_{\mathbb{K}_0(\mathbb{R}^d)} \int_{\mathbb{K}_0(\mathbb{R}^d)} \sum_{x\in\tau(\eta_1)} s_x f(\eta_1 - s_x\delta_x)g(\eta_2)G(\eta_1)k(\eta_1+\eta_2-s_x\delta_x)\lambda_{\mathcal{G}}(d\eta_2)\lambda_{\mathcal{G}}(d\eta_1) \\
&= \int_{\mathbb{K}_0(\mathbb{R}^d)} G(\eta) \underbrace{\int_{\mathbb{K}_0(\mathbb{R}^d)} \sum_{x\in\tau(\eta_1)} s_x f(\eta_1 - s_x\delta_x)g(\eta_2)k(\eta_1+\eta_2-s_x\delta_x)\lambda_{\mathcal{G}}(d\eta_2)}_{=L_1^\triangle k(\eta_1)} \lambda_{\mathcal{G}}(d\eta_1)
\end{aligned}$$

□

4.1.5 Hierarchical Structure for the Glauber Dynamics

Rewriting the equation for the correlation function in terms of the hierarchical structure, we arrive at the following operators:

Proposition 4.16. *For $n \in \mathbb{N}$, the operator $L_n^{\triangle}$ on the hierarchy of correlation functions of the Glauber model is given by*

$$
\begin{aligned}
(L^{\triangle} k^{(n)})(s_1, \dots, x_n) = \\
&- \sum_{i=1}^{n} s_i k^{(n)}(s_1, \dots, x_n) \\
&+ \sum_{m=0}^{\infty} \frac{1}{m!} \int_{(\mathbb{R}_+^* \times \mathbb{R}^d)^m} \sum_{i=1}^{n} s_i \prod_{\substack{j=1 \\ j \neq i}}^{n} e^{-2 s_i s_j \phi(x_i, x_j)} \prod_{k=1}^{m} \left(e^{-2 s_i t_k \phi(x_i, y_k)} - 1 \right) \times \\
&\quad \times k^{(n+m-1)}(s_1, \dots, \check{s}_i, \check{x}_i, \dots, x_n, t_1, \dots, y_m) \nu_\theta(dt_1) \sigma(dy_1) \dots \nu_\theta(dt_m) \sigma(dy_m)
\end{aligned}
$$

where $\check{s}_i, \check{x}_i$ means that these variables are omitted.

Remark 4.17. *In the special case of $n = 1$, this reduces to*

$$
\begin{aligned}
L^{(1)} k^{(1)}(s, x) = -s k^{(1)}(s, x) \\
&+ s \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\mathbb{R}_+^* \times \mathbb{R}^{dn}} \prod_{k=1}^{m} (e^{-2 s t_k \phi(x, y_k)} - 1) \times \\
&\times k^{(m)}(t_1, y_1, \dots, t_m, y_m) \nu_\theta(dt_1) \sigma(dy_1) \dots \nu_\theta(dt_n) \sigma(dy_n)
\end{aligned}
$$

We see that each correlation function depends on all other functions of higher order.

4.2 Continuous Contact Model on the Cone

The next model to consider is the continuous contact model on $\mathbb{K}(\mathbb{R}^d)$. Heuristically, this model is a simple birth-and-death process used to describe e.g. infection spreading, where each agent $x \in \tau(\eta), \eta \in \mathbb{K}(\mathbb{R}^d)$ represents an infected individual. Another application is biological growth in abundance of recources: Each agent of the system may spawn a new agent independent of all other agents of the system. On the other hand, each agent dies independently after some random time. In the discrete case, this model has been thoroughly examined, see e.g. the monographs [44, 45] by Liggett. Lately, a version on continuous state spaces has become established as well. A first description can be found in [42]. Furthermore, the analysis of its correlation functions and invariant measures was done in [36]. In the case of compact marks, the analysis of invariant states was carried out in [41]. Other generalisations of the model include the introduction of fecundity or establishment parameters [20] or an underlying random environment [34, 35].

While the model itself is easy to describe, the analysis of it poses some rather unique problems. Since neither birth nor death is regulated by population size, in the supercritical case, the system may grow exponentially fast. Therefore, standard perturbation techniques used for the Glauber dynamics are not applicable here, unless we extend the function space considered for the dynamics. Furthermore, the analysis of this model is mostly described on the level of correlation fucntions as opposed to quasi-observables as it was the case for the Glauber dynamics.

On the other hand, the model allows a nice description in terms of its hierarchical structure, since the dependence on the order of correlation functions is only downwards. This enables a deep analysis on the side of correlation functions.

We start the chapter by setting up the model via its Markov-type operator. Next, we proceed as proposed in Chapter 2.11.2 to derive the corresponding operator in the space of correlation functions. After stating the hierarchical structure of the system of correlation functions, we may derive *a priori* estimates. On the one hand, we show such kind of estimates for a fixed order $n \in \mathbb{N}_0$. On the other hand, we calculate uniform bounds which also hold for the correlation function $k\colon \mathbb{K}_0(\mathbb{R}^d) \to \mathbb{R}$.

Consider the model

$$\begin{aligned}(LF)(\eta) = &\sum_{x\in\tau(\eta)} m(s_x)[F(\eta - s_x\delta_x) - F(\eta)] \\ &+ \sum_{x\in\tau(\eta)} \int_{\mathbb{R}_+^*\times\mathbb{R}^d} q(s_x,s)a(x-y)[F(\eta+s\delta_y) - F(\eta)]\nu_\theta(ds)\sigma(dy)\end{aligned}$$

for $F \in \mathcal{F}C(\mathbb{K}(\mathbb{R}^d))$. On the example of infection spreading mentioned above, the objects of the above expression mean the following: The agents in $\eta \in \mathbb{K}(\mathbb{R}^d)$ represent infected individuals. An individual recovers from its infection with the rate $m(s)$, while an infected individual $s_x\delta_x$ may infect new agents with rate and distribution given by

$$q(s_x,s)a(x-y)\nu_\theta(ds)\sigma(dy).$$

The form and properties of the rates m, q and a will be prescribed later when discussing the properties of the model.

4.2.1 The Symbol for the Contact Model

To follow the scheme of Markov evolution, we need to calculate the corresponding operator on the space of quasi-observables.

Proposition 4.18. *The symbol $\hat{L}$ corresponding to L of the contact model has the following form for functions $G \in B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$:*

$$\begin{aligned}(\hat{L}G)(\eta) = &-\sum_{x\in\tau(\eta)} m(s_x)G(\eta) \\ &+ \sum_{x\in\tau(\eta)} \int_{\mathbb{R}_+^*\times\mathbb{R}^d} q(s_x,s)a(x-y)[G(\eta - s_x\delta_x + s_y\delta_y) + G(\eta+s\delta_y)]\nu_\theta(ds)\sigma(dy) \\ = &-\sum_{x\in\tau(\eta)} m(s_x)G(\eta) \\ &+ \sum_{x\in\tau(\eta)} \int_{\mathbb{R}_+^*\times\mathbb{R}^d} q(s_x,s)a(x-y)G(\eta - s_x\delta_x + s\delta_y)\nu_\theta(ds)\sigma(dy) \\ &+ \sum_{x\in\tau(\eta)} \int_{\mathbb{R}_+^*\times\mathbb{R}^d} q(s_x,s)a(x-y)G(\eta+s\delta_y)\nu_\theta(ds)\sigma(dy)\end{aligned}$$

Proof. Use the relation $\hat{L} = K^{-1}LK$ and calculate similarly to the Glauber model. □

Note that if we considered the hierarchy $\{G^{(n)}\}_{n=0}^{\infty}$ in terms of quasi-observables, the equation for fixed $n \in \mathbb{N}_0$ would depend on the function of order $n+1$. On the side of correlation functions, this dependence is switched.

4.2.2 Statistical Dynamics of the Contact Model

For the contact model, perturbation methods on typical spaces fail due to a "too strong" birth rate. Instead, it is customary to analyse the system of correlation functions. By using duality (26), we may calculate the dual operator for the statistical dynamics.

Proposition 4.19. *The operator $L^{\triangle}$ is given by*

$$\begin{aligned}(L^{\triangle}k)(\eta) = &- \sum_{x\in\tau(\eta)} m(s_x)k(\eta) \\ &+ \sum_{y\in\tau(\eta)} \int_{\mathbb{R}_+^*\times\mathbb{R}^d} q(s,s_y)a(x-y)k(\eta - s_y\delta_y + s\delta_x)\nu_\theta(ds)\sigma(dx) \\ &+ \sum_{y\in\tau(\eta)} \sum_{x\in\tau(\eta - s_y\delta_y)} q(s_x,s_y)a(x-y)k(\eta - s_y\delta_y)\end{aligned}$$

where $k\colon \mathbb{K}_0(\mathbb{R}^d)\to\mathbb{R}$ is from a class of L^∞-type functions which will be specified later.

Due to the asymmetry of q, it is worth noting that the sum and integral in the second line of the operator switch places.

Remark 4.20. *We can rewrite the operator $L^{\triangle}$ as follows:*

$$\begin{aligned}(L^{\triangle}k)(\eta) &= \sum_{y\in\tau(\eta)} \int_{\mathbb{K}_0(\mathbb{R}^d)} q(s,s_y)a(x-y)\left[k(\eta - s_y\delta_y + s\delta_x) - k(\eta)\right]\nu_\theta(ds)\sigma(dx) \\ &\quad + \left[\sum_{y\in\tau(\eta)} \int_{\mathbb{K}_0(\mathbb{R}^d)} q(s,s_y)a(x-y)\nu_\theta(ds)\sigma(dx) - \sum_{y\in\tau(\eta)} m(s_y)\right]k(\eta) \\ &\quad + \sum_{y\in\tau(\eta)} \sum_{x\in\tau(\eta - s_y\delta_y)} q(s_x,s_y)a(x-y)k(\eta - s_y\delta_y) \\ &= \sum_{y\in\tau(\eta)} L_y k(\eta) + \sum_{y\in\tau(\eta)} (\kappa(s_y) - m(s_y))k(\eta) \\ &\quad + \sum_{y\in\tau(\eta)} \sum_{x\in\tau(\eta - s_y\delta_y)} q(s_x,s_y)a(x-y)k(\eta - s_y\delta_y) \\ &= (Mk)(\eta) + (Vk)(\eta) + (Wk)(\eta)\end{aligned}$$

where $\kappa(s_y) = \int_{\mathbb{R}^d} a(x)\sigma(dx)\cdot\int_{\mathbb{R}_+^*} q(s,s_y)\nu_\theta(ds)$ and we set for a function $k\colon \mathbb{K}_0(\mathbb{R}^d)\to\mathbb{R}$

and $\eta \in \mathbb{K}_0(\mathbb{R}^d)$,

$$\begin{aligned}
L_y k(\eta) &= \int_{\mathbb{K}_0(\mathbb{R}^d)} q(s, s_y) a(x-y) \left[k(\eta - s_y\delta_y + s\delta_x) - k(\eta)\right] \nu_\theta(ds)\sigma(dx),\ y \in \tau(\eta)\\
r(s) &= \kappa(s) - m(s),\ s \in \mathbb{R}_+^*\\
Mk(\eta) &= \sum_{y\in\tau(\eta)} L_y k(\eta)\\
Vk(\eta) &= \sum_{y\in\tau(\eta)} r(s_y)k(\eta)\\
Wk(\eta) &= \sum_{y\in\tau(\eta)} \sum_{x\in\tau(\eta - s_y\delta_y)} q(s_x, s_y)a(x-y)k(\eta - s_y\delta_y)
\end{aligned}$$

Remark 4.21. *The operators M and V do not commute, i.e. $MV \neq VM$. Therefore, the approach via Duhamel formula used in [36] is not directly applicable here. Hence, we need to refine the approach by approximating the involved semigroups. As it turns out, one viable approximation is given by Trotter's product formula.*

Proposition 4.22 ([14, Corollary III.5.8])**.** *Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be strongly continuous semigroups on a Banach space X satisfying the stability condition*

$$\left\| \left[T\left(\frac{t}{n}\right) S\left(\frac{t}{n}\right)\right]^n \right\| \leq Me^{wt} \text{ for all } t \geq 0, n \in \mathbb{N} \tag{41}$$

and some constants $M \geq 1$, $w \in \mathbb{R}$. Consider the sum $A + B$ on $D := D(A) \cap D(B)$ of the generators $(A, D(A))$ and $(B, D(B))$ of $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$, respectively. Assume that D and $(\lambda_0 - A - B)D$ are dense in X for some $\lambda_0 > w$. Then $C := \overline{A + B}$ generates a strongly continuous semigroup $(U(t))_{t\geq 0}$ given by the following formula:

$$U(t)x = \lim_{n\to\infty} \left[T\left(\frac{t}{n}\right) S\left(\frac{t}{n}\right)\right]^n,\ x \in X$$

with uniform convergence for t in compact intervals.

Remark 4.23. *If the existence of the semigroup generated by $A + B$ is known* a priori*, we may use formula* (41) *directly. Especially, for bounded operators A, B, we have*

$$e^{t(A+B)} = \lim_{m\to\infty} e^{\frac{t}{m}A} e^{\frac{t}{m}B} \tag{42}$$

4.2.3 Hierarchical structure for the Contact Model

As mentioned above, the growth that may occur in the contact model is too strong to find a uniform bound for the operators M and V defined on $L^\infty(\mathbb{K}_0(\mathbb{R}^d))$. Instead, we consider the equation componentwise to show estimates for each correlation function $k^{(n)}, n \in \mathbb{N}$. Later, we will extend the function space to derive some global estimates as well.

Proposition 4.24. *The operators $L_n^{\triangle}$ are given by*

$$L_n^{\triangle} k^{(n)}(s_1, x_1, \dots, s_n, x_n) = -\sum_{i=1}^{n} m(s_i) k^{(n)}(s_1, x_1, \dots, s_n, x_n)$$
$$+ \sum_{i=1}^{n} \int_{\mathbb{R}_+^* \times \mathbb{R}^d} q(s, s_i) a(x - x_i) k^{(n)}(s_1, x_1, \dots, x_{i-1}, s, x, s_{i+1}, \dots, s_n, x_n) \nu_\theta(ds) \sigma(dx)$$
$$+ \sum_{i=1}^{n} \sum_{j \neq i} q(s_j, s_i) a(x_j - x_i) k^{(n-1)}(s_1, x_1, \dots, \check{s}_i, \check{x}_i, \dots, s_n, x_n)$$

where $\check{x}_i$ means that this variable is omitted.

In the sequel, we may omit the dependence on variables which are left fixed. We can rewrite the above expression in the following way:

$$\begin{aligned}
(L^{(n)} k^{(n)})(s_1, \dots, x_n) &= \sum_{i=1}^{n} \int_{\mathbb{R}_+^* \times \mathbb{R}^d} q(s, s_i) a(x - x_i) \left[k^{(n)}(s, x) - k^{(n)}(s_i, x_i) \right] \nu_\theta(ds) \sigma(dx) \\
&\quad + \left[\sum_{i=1}^{n} \int_{\mathbb{R}_+^* \times \mathbb{R}^d} q(s, s_i) a(x - x_i) \nu_\theta(ds) \sigma(dx) - \sum_{i=1}^{n} m(s_i) \right] k^{(n)} \\
&\quad + \sum_{i=1}^{n} \sum_{j \neq i} q(s_j, s_i) a(x_j - x_i) k^{(n-1)}(\check{s}_i, \check{x}_i) \\
&= (M_n + V_n) k^{(n)} + W_n k^{(n-1)}
\end{aligned}$$

where M_n, V_n and W_n represent the operators in the first, second and third summand, respectively. Furthermore, from now on we assume that r is bounded.

The representation of the correlation functions in this hierarchical fashion enables us to consider the solution for fixed n separately.

Lemma 4.25. *The operators M_n and V_n are bounded on $L^\infty((\mathbb{R}_+^* \times \mathbb{R}^d)^n)$.*

Proof. Let $k^{(n)} \in L^\infty((\mathbb{R}_+^* \times \mathbb{R}^d)^n)$.

$$\begin{aligned}
|M_n k^{(n)}| &= \left| \sum_{i=1}^{n} \int_{\mathbb{R}_+^* \times \mathbb{R}^{d^n}} q(s, s_i) a(x - x_i) \left[k^{(n)}(s, x) - k^{(n)}(s_i, x_i) \right] \nu_\theta(ds) \sigma(dx) \right| \\
&\leq \|q\|_{L^1} \|a\|_{L^1} 2n \|k^{(n)}\|_\infty \\
&= 2n \|q\|_{L^1} \|a\|_{L^1} \|k^{(n)}\|_\infty \\
|V_n k^{(n)}| &\leq \sum_{i=1}^{n} |r(s_i)| |k^{(n)}(s_1, \dots, x_n)| \leq \sup_{s > 0} |r(s)| n \|k^{(n)}\|_\infty
\end{aligned}$$

□

Since M_n and V_n are bounded operators, both are the generators of C_0-semigroups on $L^\infty(\mathbb{R}_+^* \times \mathbb{R}^{d^n})$. In fact, M_n is a Markov generator and therefore, its semigroup e^{tM_n} is contractive. Note that M_n is not a self-adjoint operator, because the kernel q is not symmetric.

Lemma 4.26. *The semigroup e^{tM_n} is positive and conservative and therefore contractive.*

Proof. Denote by $\mathbb{1} \in L^\infty(\mathbb{R}_+^* \times \mathbb{R}^d)$ the function which is equal to 1 everywhere. The conservativity $M_n\mathbb{1} = \mathbb{1}$ is clear from the definition of M_n. For positivity, we split the operator into $M_n = M_n^{(1)} + M_n^{(2)}$, where for $f \in L^\infty((\mathbb{R}_+^* \times \mathbb{R}^d)^n)$, we set

$$M_n^{(1)} f(s_1, x_1, \ldots, s_n, x_n) = \sum_{i=1}^n \int_{\mathbb{R}_+^* \times \mathbb{R}^d} q(s, s_i) a(x - x_i) f(\ldots, \underbrace{s, x}_{i-th}, \ldots) \nu_\theta(ds)\sigma(dx)$$
$$M_n^{(2)} f(s_1, x_1, \ldots, s_n, x_n) = -\sum_{i=1}^n \kappa(s_i) f(s_1, x_1, \ldots, s_n, x_n)$$

Then for $f \geq 0$, we have $M_n^{(1)} f \geq 0$ and therefore also $e^{tM_n^{(1)}} f \geq 0$ for any $t \geq 0$. On the other hand,

$$e^{tM_n^{(2)}} f = e^{-t\sum_{i=1}^n \kappa(s_i)} f,$$

where κ is a real-valued function. Therefore, this semigroup is also positive. The positivity of e^{tM_n} now follows from Trotter's product formula (42).

Using these two properties, we obtain for a function $0 \leq f \in L^\infty(\mathbb{R}_+^* \times \mathbb{R}^d)$ with $\|f\|_\infty \leq 1$,

$$e^{tM_n} f \leq e^{tM_n} \mathbb{1} = \mathbb{1}.$$

By applying the supremum, we see that

$$\|e^{tM_n} f\| \leq \|\mathbb{1}\| = 1.$$

□

Since the sum $M_n + V_n$ is bounded, it generates a C_0-semigroup as well. It is given by the Trotter product formula

$$e^{t(M_n+V_n)} = \lim_{m\to\infty} \left[e^{\frac{t}{m}M_n} e^{\frac{t}{m}V_n} \right]^m. \tag{43}$$

Equation (25) can now be rewritten as a hierarchical sequence of equations, $n \in \mathbb{N}$:

$$\begin{aligned} \frac{\partial}{\partial t} k_t^{(n)}(s_1, \ldots, x_n) &= (M_n + V_n) k_t^{(n)}(s_1, \ldots, x_n) + (W_n k^{(n-1)})(s_1, \ldots, x_n) \\ k_{t|t=0}^{(n)}(s_1, \ldots, x_n) &= k_0^{(n)}(s_1, \ldots, x_n) \end{aligned} \tag{44}$$

The solution to this equation is given by Duhamel's formula:

$$\begin{aligned} k_t^{(n)}(s_1, x_1, \ldots, s_n, x_n) &= e^{t(M_n+V_n)} k_0^{(n)}(s_1, x_1, \ldots, s_n, x_n) \\ &\quad + \int_0^t e^{(t-\tau)(M_n+V_n)} W_n k_\tau^{(n-1)}(s_1, x_1, \ldots, s_n, x_n) d\tau \end{aligned} \tag{45}$$

Remark 4.27. *Compare to the formula used in [36]: The non-commutativity of the operators M_n and V_n forces a slightly more complicated approach for the calculation of* a priori *estimates.*

4.2.4 *A Priori* Estimates

The preparations of the previous chapter are now used to give estimates for the correlation functions $k_t^{(n)}$. Using Trotter's formula (43), we may rewrite the solution (45) as

$$\begin{aligned} k_t^{(n)}(s_1,\dots,x_n) &= \lim_{m\to\infty}\left(e^{\frac{t}{m}M_n}e^{\frac{t}{m}V_n}\right)^m k_0^{(n)}(s_1,\dots,x_n) \\ &\quad + \int_0^t \lim_{m\to\infty}\left(e^{\frac{t-\tau}{m}M_n}e^{\frac{t-\tau}{m}V_n}\right)^m L_n k^{(n-1)}(s_1,\dots,x_n)d\tau \end{aligned}$$

We show that the Trotter approximation can be estimated independently of m.

Lemma 4.28. *For any $m\in\mathbb{N}$ and any $k\in L^\infty((\mathbb{R}^d)^n)$, we have*

$$\left\|\left(e^{\frac{t}{m}M_n}e^{\frac{t}{m}V_n}\right)^m k\right\|_\infty \le e^{tnR}\|k_0^{(n)}\|_\infty,$$

where $R := \sup_{s>0} r(s)$.

Proof. Let $m\in\mathbb{N}$ and $k\in L^\infty((\mathbb{R}^d)^n)$. First, we estimate the action of the semigroup generated by V_n:

$$\begin{aligned} \left|e^{tV_n}k\right| &= \left|e^{t\sum_{i=1}^n r(s_i)}\right| |k(s_1,\dots,x_n)| \le e^{t\sum_{i=1}^n r(s_i)}\|k\|_\infty \\ &\le e^{t\sum_{i=1}^n R}\|k\|_\infty = e^{tnR}\|k\|_\infty \end{aligned}$$

The desired expression can be estimated in the following way:

$$\left\|\left(e^{\frac{t}{m}M_n}e^{\frac{t}{m}V_n}\right)^m k_0^{(n)}\right\| \le \left\|e^{\frac{t}{m}M_n}\right\|^m \left\|e^{\frac{t}{m}V_n}\right\|^m \left\|k_0^{(n)}\right\|_\infty \le e^{tnR}\left\|k_0^{(n)}\right\|_\infty$$

where we used that M_n generates a contraction semigroup. □

Remark 4.29. *The non-commutativity prevents a pointwise estimate in the marks $s_i, i=1,\dots,n$, as can be seen in the proof above.*

The following estimate follows if we apply the Lemma to the solution (45):

Lemma 4.30. *We have the following estimate for the solution* (45)*:*

$$\|k_t^{(n)}(s_1,\dots,x_n)\|_\infty \le e^{tnR}\|k_0^{(n)}\|_\infty + \int_0^t e^{(t-\tau)nR}\|W_n k_\tau^{(n-1)}(s_1,\dots,x_n)\|_\infty d\tau$$

Proof. Use the previous lemma as well as the fact that the Trotter formula provides the strong limit, which commutes with the underlying norm. □

We are now ready to state the main result of this section.

Theorem 4.31. *Assume that the kernels a and q are bounded. Furthermore, assume that the initial condition to the Cauchy problem* (44) *obeys the following bound for all $n\in\mathbb{N}$ and some $C>0$:*

$$\|k_0^{(n)}\|_\infty \le C^n n!$$

Then, the following a priori *estimate holds:*

$$\|k_t^{(n)}\|_\infty \le \alpha(t)^n e^{nRt}(C+t)^n n! K(a,q)^n$$

where

$$\alpha(t) = \max\{1, e^{-tR}\},\ K(a,q) = (1+\|a\|_\infty\|q\|_\infty).$$

Proof. The statement is proven by induction. For $n = 1$, we have by Lemma 4.30,

$$k_t^{(1)}(x_1, s_1) \le e^{tR} C \le (C + t) e^{tR} \alpha(t) K(a, q).$$

For the induction step, we also start with Lemma 4.30:

$$\begin{aligned}
\|k_t^{(n+1)}\|_\infty &\le e^{t(n+1)R} \|k_0^{(n+1)}\|_\infty + \int_0^t e^{(t-\tau)(n+1)R} \|W_{n+1} k_\tau^{(n)}\|_\infty d\tau \\
&\le e^{t(n+1)R} C^{n+1} (n+1)! \\
&\quad + e^{t(n+1)R} \int_0^t e^{-\tau(n+1)R} \sum_{i=1}^{n} \sum_{j \ne i} \|q(s_j, s_i) a(x_j - x_i) k_\tau^{(n)}(\check{s}_i, \check{x}_i)\|_\infty d\tau \\
&\le e^{t(n+1)R} (C + t)^{n+1} (n+1)! \\
&\quad + e^{t(n+1)R} \|q\|_\infty \|a\|_\infty \int_0^t e^{-\tau(n+1)R} \alpha(\tau)^n e^{\tau n r_0} (C + \tau)^n n! K(a, q)^n d\tau \\
&\le e^{t(n+1)R} (C + t)^{n+1} (n+1)! \\
&\quad + e^{t(n+1)R} \|q\|_\infty \|a\|_\infty (n+1)! \alpha(t)^{n+1} K(a, q)^n n \int_0^t (C + \tau)^n d\tau \\
&\le (n+1)! e^{t(n+1)R} (C + t)^{n+1} K(a, q)^{n+1} \alpha(t)^{n+1}
\end{aligned}$$

□

Remark 4.32. *Note that the parameter R is not assumed to be nonnegative. Therefore, if the mortality rate m dominates the birth rate κ globally, we have exponential decay of the correlation functions, which is in accordance with the homogeneous model [36]. On the other hand, even for small "peaks" of the birth rate, we can not guarantee the decay of the solutions. In fact, in similar situations, it was shown that these small flucuations may already lead to a growth of population, see e.g. [35].*

Let us now consider lower bounds for the contact model in a subcritical regime. As it can be seen from Theorem 4.31, the correlation functions of the contact model decay if R is negative. Nevertheless, even in the subcritical regime, the system will admit so-called "clustering". This means that local peaks in the birth rate are still visible, even if the system as a whole is decaying. We concentrate us on the translation invariant case, i.e. the system starts out in a Poissonian state.

Theorem 4.33. *Assume that there exists a bounded set $B \subset \mathbb{R}_+^* \times \mathbb{R}^d$ such that*

$$\alpha := \inf_{(s_1,x_1),(s_2,x_2) \in B} q(s_1, s_2) a(x_1 - x_2) > 0 \tag{46}$$

Furthermore, assume that the mortality is bounded from above by some $\delta > 0$, i.e. $m \le \delta$ and $\alpha[\nu \otimes \sigma](B) \le \delta$. Consider the Poissonian initial condition $k^{(n)} \equiv C^n$ for some $C > 0$. Then for any $n \in \mathbb{N}, \{(s_1, x_1), \dots, (s_n, x_n)\} \subset B$ and $t \ge t_n := \sum_{j=1}^{n-1} \frac{1}{j}$, the following holds:

$$k_t^{(n)}(s_1, x_1, \dots, s_n, x_n) \ge \beta^n e^{(\alpha[\nu \otimes \sigma](B) - \delta)t}$$

The next few pages will be dedicated to the proof of the theorem. First and foremost, we need to rewrite the operator $L_n^\triangle$ in a different form. Next, we apply Trotter's product formula and show the estimate for the approximated semigroup. In the end, we will put the preliminary results together to show the above statement. We consider the following operator:

$$A_n^i k^{(n)} = \int_{\mathbb{R}^d} \int_{\mathbb{R}_+^*} q(s, s_i) a(x - x_i) k_t^{(n)}(s_1, \ldots, x_{i-1}, s, x, s_{i+1}, \ldots, x_n) \nu(ds) \sigma(dx)$$

This way, we may rewrite the equation in the following way:

$$\frac{\partial}{\partial t} k_t^{(n)}(s_1, \ldots, x_n) = \sum_{i=1}^n A_n^i k_t^{(n)} - \sum_{i=1}^n m(s_i) k_t^{(n)}(s_1, \ldots, x_n) + W_n k^{(n-1)}(s_1, \ldots, x_n)$$

Set

$$A_n := \sum_{i=1}^n A_n^i \text{ and } \vartheta_n(s_1, \ldots, s_n) := \sum_{i=1}^n m(s_i).$$

Then, the above equation can be rewritten as

$$\begin{aligned} \frac{\partial}{\partial t} k_t^{(n)}(s_1, \ldots, x_n) = A_n k_t^{(n)}(s_1, \ldots, x_n) - \vartheta_n(s_1, \ldots, s_n) k_t^{(n)}(s_1, \ldots, x_n) \\ + W_n k_t^{(n-1)}(s_1, \ldots, x_n) \end{aligned}$$

Using Duhamel and Trotter, we may write this equation as

$$\begin{aligned} k_t^{(n)}(s_1, \ldots, x_n) &= e^{t(A_n - \vartheta_n)} k_0^{(n)}(s_1, \ldots, x_n) \\ &\quad + \int_0^t e^{(t-\tau)(A_n - \vartheta_n)} W_n k_\tau^{(n-1)}(s_1, \ldots, x_n) d\tau \\ &= \lim_{m \to \infty} \left[e^{\frac{t}{m} A_n} e^{-\frac{t}{m} \vartheta_n} \right]^m k_0^{(n)}(s_1, \ldots, x_n) \\ &\quad + \int_0^t \lim_{m \to \infty} \left[e^{\frac{t-\tau}{m} A_n} e^{-\frac{t-\tau}{m} \vartheta_n} \right]^m W_n k_\tau^{(n-1)}(s_1, \ldots, x_n) d\tau \end{aligned}$$

Assume that the initial conditions are of the form $k_0^{(n)} = C^n$ for some constant $C > 0$. Furthermore, assume that the function $r(s)$ is bounded from below, i.e.

$$r(s) = \kappa(s) - m(s) \geq \rho$$

for some $\rho \in \mathbb{R}$. We also assume that $m(s) \leq \delta$. For $n = 1$, we use the reaction-diffusion type equation to obtain

$$k_t^{(1)}(s_1, x_1) = e^{t(M_1 + V_1)} k_0^{(1)}(s_1, x_1) = C \lim_{m \to \infty} \left[e^{\frac{t}{m} M_1} e^{\frac{t}{m} V_1} \right]^m \mathbb{1} \geq C e^{t\rho}$$

since M_1 is positive and conservative. For general $n \geq 2$, we drop the first term in Duhamel's formula to obtain the following estimate:

$$\begin{aligned} k_t^{(n)}(s_1, \ldots, x_n) &\geq \int_0^t \lim_{m \to \infty} \left[e^{\frac{t-\tau}{m} A_n} e^{-\frac{t-\tau}{m} \vartheta_n} \right]^m W_n k_\tau^{(n-1)}(s_1, \ldots, x_n) d\tau \\ &= \sum_{i=1}^n \sum_{j \neq i} \int_0^t \lim_{m \to \infty} \left[e^{\frac{t-\tau}{m} A_n} e^{-\frac{t-\tau}{m} \vartheta_n} \right]^m (q(s_i, s_j) a(x_i - x_j)) k^{(n-1)}(\check{s}_i, \check{x}_i) d\tau \end{aligned} \tag{47}$$

For the main calculation, we need to estimate terms of the form

$$e^{uA_n}e^{-u\vartheta_n}[q(s_i,s_j)a(x_i-x_j)].$$

Since we assume that m is bounded from above, ϑ_n is bounded from above by $n\delta$. Since A_n is a positive operator, we see that

$$e^{uA_n}e^{-u\vartheta_n}[q(s_i,s_j)a(x_i-x_j)] \geq e^{-n\delta u}e^{uA_n}[q(s_i,s_j)a(x_i-x_j)]. \tag{48}$$

Let us estimate the term on the right-hand-side.

Lemma 4.34. *Under assumption* (46) *and* $n \geq 2$*, the following holds for* $1 \leq i, j \leq n$*:*

$$e^{uA_n}(q(s_i,s_j)a(x_i-x_j)) \geq \alpha e^{n\alpha[\nu\otimes\sigma](B)u}$$

Proof. Since A_n is the sum of two non-commuting operators, we need to apply Trotter's product formula again to obtain

$$e^{uA_n}\left(q(s_1,s_2)a(x_1-x_2)\right) = \lim_{l\to\infty}\left[e^{\frac{u}{l}A_n^n}e^{\frac{u}{l}\bar{A}_n^{n-1}}\right]^l\left(q(s_1,s_2)a(x_1-x_2)\right)$$

where we set $\bar{A}_n^j = \sum_{i=1}^j A_n^i$. We proceed by double induction. Denote by IH_n and IH_l the induction hypotheses for the induction in n and l, respectively. For the case $n=2$, rewrite the semigroup using Cauchy's product formula:

$$e^{uA_2^1}e^{uA_2^2} = \sum_{l=0}^{\infty}\frac{u^l}{l!}\sum_{i=0}^{l}\binom{l}{i}(A_2^1)^i(A_2^2)^{l-i}$$

Using Lemma 4.35, we obtain

$$\begin{aligned} e^{uA_2^1}e^{uA_2^2}(q(s_i,s_j)a(x_i-x_j)) &\geq \sum_{l=0}^{\infty}\frac{u^l}{l!}\sum_{i=0}^{l}\binom{l}{i}\alpha(\alpha[\nu\otimes\sigma](B))^l \\ &= \alpha e^{n\alpha[\nu\otimes\sigma](B)u} \end{aligned}$$

This proves the inductive base. Assume that the statement of the lemma holds for some $n-1 \geq 2$. Note that by slight abuse of notation, we use that $\bar{A}_n^{n-1} = A_{n-1}$. We want to analyse the term

$$e^{uA_n^n}e^{u\bar{A}_n^{n-1}}(q(s_i,s_j)a(x_i-x_j)) = \sum_{l=0}^{\infty}\frac{u^l}{l!}(A_n^n)^l\left[e^{u\bar{A}_n^{n-1}}(q(s_i,s_j)a(x_i-x_j))\right] \tag{49}$$

Therefore, it suffices to consider

$$(A_n^n)^l\left[e^{u\bar{A}_n^{n-1}}(q(s_i,s_j)a(x_i-x_j))\right]$$

for fixed $l \in \mathbb{N}_0$. We claim that for any $l \in \mathbb{N}_0$,

$$(A_n^n)^l\left[e^{u\bar{A}_n^{n-1}}(q(s_i,s_j)a(x_i-x_j))\right] \geq \alpha e^{\alpha(n-1)[\nu\otimes\sigma](B)u}(\alpha[\nu\otimes\sigma](B)u)^l \tag{50}$$

for $l = 0$, this is clear by IH_n. Assume that (50) holds for some $l-1$. Then

$$\begin{aligned}
&(A_n^n)^l \left[e^{u\bar{A}_n^{n-1}} (q(s_i, s_j) a(x_i - x_j)) \right] \\
&= \int_{\mathbb{R}_+^* \times \mathbb{R}^d} q(s, s_n) a(x - x_n) \left((A_n^n)^{l-1} \left[e^{u\bar{A}_n^{n-1}} (q(s_i, s_j) a(x_i - x_j)) \right] \right) \nu(ds)\sigma(dx) \\
&\overset{(46)}{\geq} \int_B \alpha \left((A_n^n)^{l-1} \left[e^{u\bar{A}_n^{n-1}} (q(s_i, s_j) a(x_i - x_j)) \right] \right) \nu(ds)\sigma(dx) \\
&\overset{\mathrm{IH}_l}{\geq} \int_B \alpha e^{\alpha(n-1)[\nu\otimes\sigma](B)u} (\alpha[\nu \otimes \sigma](B)u)^{l-1} \nu(ds)\sigma(dx) \\
&= \alpha e^{\alpha(n-1)[\nu\otimes\sigma](B)u} (\alpha[\nu \otimes \sigma](B)u)^l
\end{aligned}$$

combining (49) with (50) yields

$$\begin{aligned}
e^{uA_n^n} e^{u\bar{A}_n^{n-1}} (q(s_i, s_j) a(x_i - x_j)) &\geq \alpha e^{\alpha(n-1)[\nu\otimes\sigma](B)u} \sum_{l=0}^{\infty} \frac{u^l}{l!} (\alpha[\nu \otimes \sigma](B)u)^l \\
&= \alpha e^{\alpha n[\nu\otimes\sigma](B)u}
\end{aligned}$$

□

We still need to prove the statement used in the proof of Lemma 4.34.

Lemma 4.35. *Assume condition (46). For all $i, j \in \mathbb{N}_0$, we have*

$$(A_2^1)^i (A_2^2)^j [q(s_1, s_2) a(x_1 - x_2)] \geq \alpha(\alpha[\nu \otimes \sigma](B))^{i+j} \tag{51}$$

the same statement holds for the function $q(s_2, s_1) a(x_2 - x_1)$.

Proof. The statement is shown by double induction. For $i = j = 0$, the statement follows directly from condition (46). Let $i = 0$ and assume that

$$(A_2^2)^j (q(s_1, s_2) a(x_1 - x_2) \geq \alpha(\alpha[\nu \otimes \sigma](B))^j$$

for some $j \in \mathbb{N}_0$. Then

$$\begin{aligned}
(A_2^2)^{j+1} (q(s_1, s_2) a(x_1 - x_2)) &= \int_{\mathbb{R}_+^* \times \mathbb{R}^d} q(s, s_2) a(x - x_2) \times \\
&\quad \times \left[(A_2^2)^j (q(s_1, s_2) a(x_1 - x_2) \right] (s_1, x_1, s, x) \nu(ds)\sigma(dx) \\
&\geq \alpha \int_{\mathbb{R}_+^* \times \mathbb{R}^d} \mathbb{1}_{B^2}(s, x, s_2, x_2) \times \\
&\quad \times \left[(A_2^2)^j (q(s_1, s_2) a(x_1 - x_2) \right] (s_1, x_1, s, x) \nu(ds)\sigma(dx) \\
&\overset{\mathrm{IH}}{\geq} \alpha \int_B \alpha(\alpha[\nu \otimes \sigma](B))^j \nu(ds)\sigma(dx) \\
&= \alpha(\alpha[\nu \otimes \sigma](B))^{j+1}
\end{aligned}$$

For the next step, fix j and assume that (51) holds for some $i \in \mathbb{N}_0$. Then

$$
\begin{aligned}
&(A_2^1)^{i+1}(A_2^2)^j(q(s_1,s_2)a(x_1-x_2)) \\
&\quad = \int_{\mathbb{R}_+^* \times \mathbb{R}^d} q(s,s_1)a(x-x_1)\left[(A_2^1)^i(A_2^2)^j(q(s_1,s_2)a(x_1-x_2))\right](s,x,s_2,x_2)\nu(ds)\sigma(dx) \\
&\quad \geq \alpha \int_{\mathbb{R}_+^* \times \mathbb{R}^d} \mathbb{1}_{B^2}(s,x,s_1,x_1)\left[(A_2^1)^i(A_2^2)^j(q(s_1,s_2)a(x_1-x_2))\right](s,x,s_2,x_2)\nu(ds)\sigma(dx) \\
&\quad = \alpha \int_B \left[(A_2^1)^i(A_2^2)^j(q(s_1,s_2)a(x_1-x_2))\right](s,x,s_2,x_2)\nu(ds)\sigma(dx) \\
&\quad \overset{\text{IH}}{\geq} \alpha \int_B \alpha(\alpha[\nu\otimes\sigma](B))^{i+j}\nu(ds)\sigma(dx) = \alpha(\alpha[\nu\otimes\sigma](B))^{i+j+1}
\end{aligned}
$$

Therefore, the statement of the lemma holds. □

Remark 4.36. *In a similar fashion, one can show that*

$$(A_2^1)^i(A_2^2)^j[\mathbb{1}] \geq (\alpha[\nu\otimes\sigma](B))^{i+j}.$$

Now we can backtrack using Trotter's formula to obtain the desired estimate.

Proposition 4.37. *Assume that $m \leq \delta$ for some $\delta > 0$. For any $u \geq 0$ and $n \in \mathbb{N}$, $\{(s_1,x_1),\dots,(s_n,x_n)\} \subset B$ such that (46) holds for B, we have the following estimate:*

$$e^{u(A_n-\vartheta_n)}[q(s_i,s_j)a(x_i-x_j)] \geq \alpha e^{n(\alpha[\nu\otimes\sigma](B)-\delta)u}$$

Proof. Using Trotter's formula, we have

$$e^{u(A_n-\vartheta_n)} = \lim_{m\to\infty}\left[e^{\frac{u}{m}A_n}e^{-\frac{u}{m}\vartheta_n}\right]^m$$

therefore, the estimate holds if for each $m \in \mathbb{N}$,

$$\left[e^{\frac{u}{m}A_n}e^{-\frac{u}{m}\vartheta_n}\right]^m[q(s_i,s_j)a(x_i-x_j)] \geq \alpha e^{n(\alpha[\nu\otimes\sigma](B)-\delta)u}.$$

but this is clear by Lemma 4.34 and estimate (48). □

Proof of Theorem 4.33. For the case $n = 1$, we obtain

$$k_t^{(1)}(x_1,s_1) = Ce^{t(A_1-\vartheta_1)}\mathbb{1} \geq Ce^{-\delta t}e^{tA_1}\mathbb{1}$$

This can be seen by writing the semigroup as the exponential series and using induction for the summands similar to the proofs of the above lemmas. For general $n \in \mathbb{N}$, assume that the estimate holds for $n-1$. Then by (47) and the above lemmas, we obtain

$$
\begin{aligned}
k_t^{(n)}(s_1,\dots,x_n) &\geq \int_0^t e^{(t-\tau)(A_n-\vartheta_n)} W_n k_\tau^{(n)}(s_1,\dots,x_n)d\tau \\
&\overset{\mathrm{IH}}{\geq} \beta^{n-1}(n-1)! \int_{t_{n-1}}^t e^{(n-1)(\alpha[\nu\otimes\sigma](B)-\delta)\tau} e^{(t-\tau)(A_n-\vartheta_n)} \sum_{i=1}^n \sum_{j\neq i} q(s_i,s_j)a(x_i-x_j)d\tau \\
&\geq \beta^{n-1}(n-1)! \int_{t_{n-1}}^t e^{(n-1)(\alpha[\nu\otimes\sigma](B)-\delta)\tau} e^{-n(t-\tau)\delta} \sum_{i=1}^n \sum_{j\neq i} e^{(t-\tau)A_n}[q(s_i,s_j)a(x_i-x_j)]d\tau \\
&\overset{4.37}{\geq} \beta^{n-1}(n-1)! e^{-n\delta t} \int_{t_{n-1}}^t e^{(n-1)\alpha[\nu\otimes\sigma](B)\tau} e^{\delta\tau} n(n-1)\alpha e^{\alpha n[\nu\otimes\sigma](B)(t-\tau)}d\tau \\
&\geq \beta^n n! e^{-n\delta t} e^{\alpha[\nu\otimes\sigma](B)t} \int_{t_{n-1}}^t (n-1) \underbrace{e^{-(\alpha[\nu\otimes\sigma](B)-\delta)\tau}}_{\geq 1} d\tau \\
&\geq \beta^n n! e^{n(\alpha[\nu\otimes\sigma](B)-\delta)t} \int_{t_{n-1}}^t (n-1)d\tau \\
&\geq \beta^n n! e^{n(\alpha[\nu\otimes\sigma](B)-\delta)t}
\end{aligned}
$$

where the last inequality holds if the integral term is greater or equal one, i.e.

$$\int_{t_{n-1}}^t (n-1)d\tau \geq 1 \Leftrightarrow t - t_{n-1} \geq 1 \Leftrightarrow t \geq t_n$$

This proves the claim. □

Later, we consider a way to prevent clustering in the system. Namely, the so-called BDLP model has a self-regulating mechanism via competition. This model is analysed in Chapter 4.3.

4.2.5 Uniform Estimates With Respect to the Number of Particles

In this part, we show that solutions of (25) which start from a particular class of initial conditions will stay in this class. Namely, if the initial condition is bounded by a term including $(|\tau(\eta)|!)^2$, this bound will be preserved for all time. It should be noted that this bound holds uniformly with respect to the number of particles, while the *a priori* estimates in the previous chapters were done for fixed $|\tau(\eta)| \in \mathbb{N}$.

The result will be shown as follows: We consider the homogeneous problem on the space of quasi-observables

$$
\begin{aligned}
\frac{\partial}{\partial t} G_t(\eta) &= (\tilde{M} + \tilde{V}) G_t(\eta) \\
G_t(\eta)_{|t=0} &= G_0(\eta)
\end{aligned}
\tag{52}
$$

where $\tilde{M}$ and $\tilde{V}$ are the pre-duals of M and V. We show the existence of a C_0-semigroup on an L^1-space. This implies the existence of a weak*-semigroup on the corresponding L^∞-type space of correlation functions. This semigroup is then perturbed by the inhomogeneity W. Since we construct a semigroup on this L^∞-space, the global estimate holds.

We start by considering a modified solution to (52) to obtain a contraction semigroup.

Proposition 4.38. *Let $\tilde{M}_n$ and $\tilde{V}_n$ the pre-duals of M_n and V_n as above and R_n the multiplication operator with $R_n := nR$, where R is the bound given in Lemma 4.28. Then the family of operators $\{T(t)\}_{t\geq 0}$ defined by*

$$T(t)G(\eta) = \left(e^{t(\tilde{M}_n+\tilde{V}_n-R_n)}G^{(n)}\right)(\eta)$$

is a C_0-contraction semigroup on the space $\mathcal{L}^1 := L^1(\mathbb{K}_0(\mathbb{R}^d), (|\tau(\cdot)|!)^2 d\lambda)$. Its generator is given by $A := \tilde{M} + \tilde{V} - R$ with domain

$$\mathcal{D}(A) = \{G \in \mathcal{L}^1 \colon |\tau(\cdot)|^2 G \in \mathcal{L}^1\}$$

Note that $\tilde{V} = V$ and $\tilde{V}_n = V_n$ as functions, therefore, we omit the tilde from now on.

Remark 4.39. *[14, Lemma II.1.3.(iv)] Let us recite the following classic result from semigroup theory which is used in the upcoming proof. Let $(A, D(A))$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$. Then*

$$T(t)x - x = A\int_0^t T(s)x ds \tag{53}$$

for $x \in X$.

Proof of Proposition 4.38. The semigroup properties, i.e. $T(0)G = G$ and $T(t+s)G = T(t)T(s)G$, are directly inherited from the semigroups on the fixed-particle spaces. Let us show that $T(t)$ is a contraction on $\mathcal{L}^1$. First, for fixed $n \in \mathbb{N}_0$, the following estimate holds for any $G \in \mathcal{L}^1$:

$$\|e^{t(\tilde{M}_n+V_n-R_n)}G^{(n)}\|_{L^1} = e^{-tR_n}\|e^{t(\tilde{M}_n+V_n)}G^{(n)}\|_{L^1} \leq e^{-tR_n}e^{tR_n}\|G^{(n)}\|_{L^1} = \|G^{(n)}\|_{L^1}$$

where we used that $\tilde{M}_n$ generates a contraction semigroup and $V_n \leq nR$. Therefore, the estimate $\|T(t)G\|_{\mathcal{L}^1} \leq \|G\|_{\mathcal{L}^1}$ holds and $(T(t))_{t\geq 0}$ is a contraction semigroup.

Next, we show the continuity of the semigroup at $t = 0$. First, we show the property for $G \in \mathcal{D}(A)$.

$$\begin{aligned}
&\|T(t)G - G\|_{\mathcal{L}^1} = \\
&\quad= \sum_{n=0}^{\infty} \frac{1}{n!}(n!)^2 \int_{\mathbb{R}_+^* \times \mathbb{R}^{dn}} \left| e^{t(\tilde{M}_n+V_n-R_n)}G - G \right| \nu(ds_1)\ldots\nu(ds_n)\sigma(dx_1)\ldots\sigma(dx_n) \\
&\quad= \sum_{n=0}^{\infty} n! \left\| e^{t(\tilde{M}_n+V_n-R_n)}G^{(n)} - G^{(n)} \right\|_{L^1(\mathbb{R}_+^* \times \mathbb{R}^d)} \\
&\quad= \sum_{n=0}^{\infty} n! \left\| (\tilde{M}_n + V_n - R_n) \int_0^t e^{s(\tilde{M}_n+V_n-R_n)}G^{(n)} ds \right\| \\
&\quad\leq \sum_{n=0}^{\infty} n! \left\| \tilde{M}_n + V_n - R_n \right\| \int_0^t \left\| e^{s(\tilde{M}_n+V_n-R_n)}G^{(n)} ds \right\| \\
&\quad\leq \sum_{n=0}^{\infty} n! \left\| \tilde{M}_n + V_n - R_n \right\| \left\| G^{(n)} \right\| t \\
&\quad\leq \text{const} \cdot t \sum_{n=0}^{\infty} n! n \|G^{(n)}\|
\end{aligned}$$

which converges to zero if $G \in \mathcal{D}(A)$.

Next, we need to show that the continuity holds for general $G \in \mathcal{L}^1$. But since $B_{\mathrm{cm}}(\mathbb{K}_0) \subset \mathcal{D}(A)$ is dense in $\mathcal{L}^1$, there exists a sequence $(G_m)_{m\in\mathbb{N}} \subset \mathcal{D}(A)$ such that $\|G - G_m\|_{\mathcal{L}^1} \to 0$ if $m \to \infty$. Let $\varepsilon > 0$. By above considerations, we can choose $m \in \mathbb{N}$ and $t > 0$ such that

$$\|G - G_m\| < \frac{\varepsilon}{3} \text{ and } \|T(t)G_m - G_m\| < \frac{\varepsilon}{3}$$

Therefore, it follows that

$$\begin{aligned}
\|T(t)G - G\| &\leq \|T(t)G - T(t)G_m\| + \|T(t)G_m - G_m\| + \|G_m - G\| \\
&\leq \|T(t)\| \cdot \|G - G_m\| + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&< \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} \\
&= \varepsilon
\end{aligned}$$

which shows continuity for general $G \in \mathcal{L}^1$.

As a last step, we need to show that A is in fact the generator of $(T(t))_{t\geq 0}$. Let $G \in \mathcal{D}(A)$ as defined above. We use identity (53) twice and obtain

$$\begin{aligned}
&\left\| \frac{T(t)G - G}{t} - (\tilde{M} + V - R)G \right\|_{\mathcal{L}^1} \\
&= \sum_{n=0}^{\infty} n! \frac{1}{t} \left\| e^{t(\tilde{M}_n + V_n - R_n)} G^{(n)} - G^{(n)} - t(\tilde{M}_n + V_n - R_n) G^{(n)} \right\| \\
&= \sum_{n=0}^{\infty} n! \frac{1}{t} \left\| (\tilde{M}_n + V_n - R_n) \int_0^t e^{s(\tilde{M}_n + V_n - R_n)} G^{(n)} ds - (\tilde{M}_n + V_n - R_n) \int_0^t G^{(n)} ds \right\| \\
&= \sum_{n=0}^{\infty} n! \frac{1}{t} \left\| (\tilde{M}_n + V_n - R_n) \int_0^t e^{s(\tilde{M}_n + V_n - R_n)} G^{(n)} - G^{(n)} ds \right\| \\
&= \sum_{n=0}^{\infty} n! \frac{1}{t} \left\| (\tilde{M}_n + V_n - R_n)^2 \int_0^t \int_0^s e^{\tau(\tilde{M}_n + V_n - R_n)} G^{(n)} d\tau ds \right\| \\
&\leq \text{const} \cdot \sum_{n=0}^{\infty} n! n^2 \frac{1}{t} \int_0^t \int_0^t \left\| e^{\tau(\tilde{M}_n + V_n - R_n)} G^{(n)} \right\| d\tau ds \\
&\leq \text{const} \cdot \sum_{n=0}^{\infty} n! n^2 \frac{1}{t} t^2 \|G^{(n)}\| \\
&= \text{const} \cdot t \sum_{n=0}^{\infty} n! n^2 \|G^{(n)}\|
\end{aligned}$$

The series is finite if $G \in \mathcal{D}(A)$ and therefore, the term converges to zero as $t \to 0$. Therefore, A with domain $\mathcal{D}(A)$ is the generator of the semigroup $(T(t))_{t\geq 0}$. □

Next, we may view the pre-dual $\tilde{W}$ of W as bounded perturbation of A. Recall that for a function $G \in \mathcal{L}^1$, the operator is defined as

$$(\tilde{W}G)(\eta) = \sum_{x\in\tau(\eta)} \int_{\mathbb{R}_+^* \times \mathbb{R}^d} q(s_x, s) a(x - y) G(\eta + s\delta_y) \nu(ds) \sigma(dy)$$

provided the right-hand-side makes sense. Let us show that $\tilde{W}$ is in fact a bounded operator.

Proposition 4.40. *Assume that the kernel q fulfills the following boundedness condition:*

$$\sup_{t\in[0,\infty)} \int_0^\infty q(t,s)ds =: \|q\|_{L^\infty(L^1)} < \infty.$$

Then, $\tilde{W}$ is a bounded operator on the space $\mathcal{L}^1$.

Proof. Let $G \in \mathcal{L}^1$. Using combinatorial arguments and Cauchy-Schwarz, we obtain

$$\begin{aligned}
\|\tilde{W}G\|_{\mathcal{L}^1} &= \sum_{n=0}^\infty n! \int_{\mathbb{R}_+^*\times\mathbb{R}^d} |(\tilde{W}G)^{(n)}(s_1,\dots,x_n)|\nu(ds_1)\dots\sigma(dx_n) \\
&\le \sum_{n=0}^\infty n! \int_{\mathbb{R}_+^*\times\mathbb{R}^d} \sum_{i=1}^n \int_{\mathbb{R}_+^*\times\mathbb{R}^d} |q(s_i,s)a(x_i-y)|\times \\
&\quad \times |G^{(n+1)}(s_1,\dots,x_n,s,y)|\nu(ds)\sigma(dy)\nu(ds_1)\dots\sigma(dx_n) \\
&\le \sum_{n=0}^\infty n! \sum_{i=1}^n \int_{\mathbb{R}_+^*\times\mathbb{R}^d} |q(s_i,s)a(x_i-y)|\nu(ds)\sigma(dy)\times \\
&\quad \times \int_{\mathbb{R}_+^*\times\mathbb{R}^{d^{n+1}}} |G^{(n+1)}|d\nu^{\otimes n+1}d\sigma^{\otimes n+1} \\
&\le \|a\|_{L^1}\|q\|_{L^\infty(L^1)} \sum_{n=0}^\infty n! \sum_{i=1}^n \|G^{(n+1)}\|_{L^1} \\
&= \|a\|_{L^1}\|q\|_{L^\infty(L^1)} \sum_{n=1}^\infty (n-1)!(n-1)\|G^{(n)}\|_{L^1} \\
&\le \|a\|_{L^1}\|q\|_{L^\infty(L^1)} \sum_{n=1}^\infty n!\|G^{(n)}\|_{L^1} \\
&\le \|a\|_{L^1}\|q\|_{L^\infty(L^1)}\|G\|_{\mathcal{L}^1}
\end{aligned}$$

Therefore, the operator $\tilde{W}$ is bounded on $\mathcal{L}^1$. □

By the bounded perturbation theorem (e.g. [14, Thm. III.1.3]), the operator $(A+\tilde{W},\mathcal{D}(A))$ generates a C_0-semigroup as well. Since A generates a contractive semigroup, the perturbed semigroup obeys the following bound:

$$\|e^{t(A+\tilde{W})}\| \le e^{t\|a\|_{L^1}\|q\|_{L^\infty(L^1)}}.$$

We have shown that $(A+\tilde{W},\mathcal{D}(A))$ is the generator of a C_0-semigroup on $\mathcal{L}^1$. Therefore, the dual semigroup $T^*(t)$ exists with generator $A^*+W = (M+V-R)+W$ and is weak*-continuous on the space

$$\mathcal{K} := \left\{ k \in L^\infty(\mathbb{K}_0(\mathbb{R}^d)) \colon \frac{1}{(|\tau(\cdot)|!)^2}k \in L^\infty(\mathbb{K}_0(\mathbb{R}^d)) \right\}$$

Furthermore, the semigroup $(T^*(t))_{t\ge 0}$ yields the solution to the equation

$$\begin{aligned}
\frac{\partial}{\partial t}k_t(\eta) &= (A^*+W)k(\eta) \\
k_{|t=0}(\eta) &= k_0(\eta)
\end{aligned} \tag{54}$$

Note that the equation is understood in the weak*-sense. Recall that for the contact model, equation (25) can be rewritten as

$$\begin{aligned}&\frac{\partial}{\partial t}k_t(\eta) = (A^* + W - R)k_t(\eta)\\&k_{|t=0}(\eta) = k_0(\eta)\end{aligned} \tag{55}$$

where $k_0 \in \mathcal{K}$. Denote by $\tilde{k}_t$ the solution to (54). Then the solution to (55) is given by

$$k_t(\eta) = \left(e^{tR_n}\tilde{k}_t^{(n)}\right)(\eta).$$

For $\tilde{k}_t$ and $\eta \in \mathbb{K}_0(\mathbb{R}^d)$, we have the following estimate:

$$|\tilde{k}_t(\eta)| = (|\tau(\eta)!|^2)\left[\frac{k_t(\eta)}{(|\tau(\eta)|!)^2}\right] \leq (|\tau(\eta)|!)^2\|\tilde{k}_t\|_{\mathcal{K}} \leq (|\tau(\eta)|!)^2 e^{t\|W\|}\|k_0\|_{\mathcal{K}}$$

And therefore, the solution k_t to (55) satisfies

$$|k_t(\eta)| \leq e^{tR|\tau(\eta)|}e^{t\|W\|}(|\tau(\eta)|!)^2\|k_0\|_{\mathcal{K}} = e^{t(R|\tau(\eta)|-\|W\|)}(|\tau(\eta)|!)^2\|k_0\|_{\mathcal{K}}.$$

Assuming $R \leq 0$, this gives us the following estimate on $\mathcal{K}$:

Proposition 4.41. *Let $k_0 \in \mathcal{K}$ and $R := \sup_{s>0} r(s) \leq 0$. Then the solution to* (55) *satisfies the following norm estimate for all $t \geq 0$:*

$$\|k_t\|_{\mathcal{K}} \leq e^{-t\|W\|}\|k_0\|_{\mathcal{K}}$$

4.2.6 Invariant Measures for the Contact Model

The hierarchical dependence of the correlation functions in the contact model enables us to show the existence of invariant measures. One approach which may be used in the case of compact marks uses the theorem of Krein-Rutman to rewrite equation (58) as an eigenvalue equation, see [41]. Note that this approach relies on the fact that the integral operator on marks is compact, which is not given in our case. Instead, we use the approach from the homogeneous contact model [36]. In this case, we analyse the marks in the same way as the position variables by applying harmonic analysis on $\mathbb{R}_+^*$ as explained in Chapter 2.2. Note that this approach imposes certain symmetry and integrability conditions on the kernels.

In general, we want to find a measure for which the evolution stays constant, i.e.

$$\frac{\partial}{\partial t}\mu_t = L^*\mu_t = 0.$$

Where L^* is again the dual operator to L defined for the contact model. By standard arguments from harmonic analysis above, this implies that for all $n \in \mathbb{N}_0$, the corresponding n-point correlation functions fulfill the following equation:

$$L_n^{\triangle}k^{(n)} + f^{(n)} = 0 \tag{56}$$

where $f^{(1)} \equiv 0$ and

$$f^{(n)} = \sum_{i=1}^{n} k^{(n-1)}(\check{s}_i, \check{x}_i)\sum_{j\neq i} q(s_i, s_j)a(x_i - x_j)$$

Our goal is to express $k^{(n)}$ via $f^{(n)}$, which means that we need to invert the operator $L_n^{\triangle}$, i.e.

$$k^{(n)} = -(L_n^{\triangle})^{-1} f^{(n)}.$$

since $f^{(n)}$ is defined via $k^{(n-1)}$, this gives us a recursive description of the sequence of correlation functions.

The following theorem is an adaption of a similar theorem from the theory of configuration spaces $\Gamma(\mathbb{R}^d)$, see [36]. Set dh to be the Haar measure on $\mathbb{R}_+^*$, i.e.

$$h(ds) = \frac{1}{s} ds.$$

Theorem 4.42. *Let $d \geq 3$ and assume the following conditions on Q and a:*

1. $\|a\|_{L^1(\mathbb{R}^d)} = \|Q\|_{L^1(\mathbb{R}_+^*, dh)} = 1$
2. *All second moments exist, i.e.*

$$\int_{\mathbb{R}^d} x_k x_j a(x) dx < \infty \text{ for all } 1 \leq k, j \leq d$$
$$\int_{\mathbb{R}_+^*} s^2 Q(s) ds < \infty$$

3. $\mathcal{F}_{\mathbb{R}^d} a \in L^1(\mathbb{R}^d)$, $\mathcal{F}_{\mathbb{R}_+^*} Q \in L^1(\mathbb{R}_+^*, dh)$

Then, for any $\rho \geq 0$, there exists a unique measure $\mu^\rho \in \mathcal{M}^1(\mathbb{K}(\mathbb{R}^d))$ such that the corresponding sequence of correlation functions satisfies (56), *is translation invariant in all variables with respect to the corresponding group action and fulfills the following estimate:*

$$k^{(n)} \leq C_\rho^n (n!)^2. \tag{57}$$

Furthermore, the density of the system is $k^{(1)} \equiv \rho$.

The rest of this chapter is devoted to the proof of the theorem. Define the jump operator with respect to marks as

$$\mathcal{Q} k(s) = \int_{\mathbb{R}_+^*} Q\left(\frac{t}{s}\right) k(s) h(ds)$$

For $n = 1$, the equation has the form

$$\int_{\mathbb{R}_+^* \times \mathbb{R}^d} Q\left(\frac{s}{s_1}\right) a(x - x_1) k^{(1)}(s, x) h(ds) \sigma(dx) = k^{(1)}(s_1, x_1) \tag{58}$$

Remark 4.43. 1. *The operator $\mathcal{Q}$ needs to be normalised in two ways simultaneously: First, the principal eigenvalue should be $\lambda = 1$. On the other hand, we also need $\|Q\|_{L^1} = 1$ to obtain equation* (58).

2. *We consider invariant measures with respect to the Haar measure h and not with respect to the Lévy measure ν_θ.*

3. *We show estimate* (57), *which implies the moment growth condition of Theorem 2.97. The rest of the existence proof follows the same scheme as in [36].*

Assume that we are looking for a solution which is translation invariant in all variables with respect to the corresponding group operation. Then, the first correlation function is constant:

$$k^{(1)}(s_1, x_1) = \rho > 0.$$

Note that if we choose to have a solution of this form, we need to assume that $\kappa = m$. Otherwise, the solution will decay or explode, depending on which coefficient is larger. This situation is also known as the critical case. We also assume that a and Q are even in the sense that

$$a(x) = a(-x), \; Q(s) = Q(s^{-1})$$

Remark 4.44. *For the position kernel, evenness corresponds to the case of a homogeneous environment, which is natural if we do not want to make additional assumptions. This interpretation can also be used to explain the above property of Q: The natural distance on $(\mathbb{R}_+^*, \cdot)$ is given by*

$$d(s,t) = \left|\log \frac{s}{t}\right| = d(t,s).$$

If we assume that Q also only depends on the distance of the two marks, i.e. there exists a function $\Phi\colon \mathbb{R}_+^* \to \mathbb{R}$ such that $Q := \Phi \circ d\colon (\mathbb{R}_+^*)^2 \to \mathbb{R}$, it makes sense to assume the above symmetry.

Let us consider the case $n = 2$. In the translation invariant case, this means that

$$v^{(2)}(s_1, x_1, s_2, x_2) = v^{(2)}\left(\frac{s_1}{s_2}, x_1 - x_2\right).$$

Then (56) becomes

$$\begin{aligned}
&Q *_1 (a * v^{(2)})\left(\frac{s_1}{s_2}, x_1 - x_2\right) + Q *_2 (a * v^{(2)})\left(\frac{s_1}{s_2}, x_1 - x_2\right) - 2v^{(2)}\left(\frac{s_1}{s_2}, x_1 - x_2\right)\\
&= \rho a(x_1 - x_2)\left[Q\left(\frac{s_1}{s_2}\right) + Q\left(\frac{s_2}{s_1}\right)\right]\\
&= 2\rho a(x_1 - x_2) Q\left(\frac{s_1}{s_2}\right)
\end{aligned}$$

Replacing $\frac{s_1}{s_2} \to t$ and $x_1 - x_2 \to \xi$, the equation becomes

$$Q *_1 (a * v^{(2)})(t,\xi) + Q *_2 (a * v^{(2)})(t,\xi) - 2v^{(2)}(t,\xi) = -2\rho a(\xi) Q(t)$$

Remark 4.45. *The convolution in s_1 works in the following sense:*

$$\begin{aligned}
L_2^1 v^{(2)}\left(\frac{s_1}{s_2}, x_1 - x_2\right) &= \int_{\mathbb{R}_+^* \times \mathbb{R}^d} Q\left(\frac{s_1}{s}\right) a(x_1 - x) v^{(2)}\left(\frac{s}{s_2}, x - x_2\right) h(ds)\sigma(dx)\\
&= \int_{\mathbb{R}_+^* \times \mathbb{R}^d} Q\left(\frac{s_1}{s_2 t}\right) a(x_1 - x) v^{(2)}(t, x - x_2) \frac{s_2}{s_2 t} dt \sigma(dx)\\
&= Q * (a * v^{(2)})\left(\frac{s_1}{s_2}, x_1 - x_2\right)
\end{aligned}$$

If we dropped the assumption of evenness on Q, we would need to be careful at the convolution $*_2$ in s_2, since Q is not even in the sense that

$$Q(s) \neq Q(s^{-1}).$$

Instead, we would consider the function

$$\widetilde{Q}(s) := Q(s^{-1})$$

which means that

$$\begin{aligned} L_2^2 v^{(2)}\left(\frac{s_1}{s_2}, x_1 - x_2\right) &= \int_{\mathbb{R}_+^* \times \mathbb{R}^d} Q\left(\frac{s_2}{s}\right) a(x_1 - x) v^{(2)}\left(\frac{s_1}{s}, x - x_2\right) h(ds)\sigma(dx) \\ &= \int_{\mathbb{R}_+^* \times \mathbb{R}^d} \widetilde{Q}\left(\frac{s}{s_2}\right) a(x_1 - x) v^{(2)}\left(\frac{s_1}{s}, x - x_2\right) h(ds)\sigma(dx) \\ &= \int_{\mathbb{R}_+^* \times \mathbb{R}^d} \widetilde{Q}\left(\frac{s_1}{s_2} t\right) a(x_1 - x) v^{(2)}\left(\frac{1}{t}, x - x_2\right) h(dt)\sigma(dx) \\ &= \widetilde{Q} * (a * v^{(2)})\left(\frac{s_1}{s_2}, x_1 - x_2\right) \end{aligned}$$

Rewritten, the equation looks as follows:

$$Q * (a * v^{(2)})(t, \xi) + \widetilde{Q} * (a * v^{(2)})(t, \xi) - 2v^{(2)}(t, \xi) = -\rho a(\xi) \left(Q(t) + \tilde{Q}(t)\right)$$

Let us return to the case where Q is even. In this case, the equation simplifies to

$$Q * (a * v^{(2)})(t, \xi) + Q * (a * v^{(2)})(t, \xi) - 2v^{(2)}(t, \xi) = -2\rho a(\xi) Q(t)$$

After taking the Fourier transform, the equation becomes

$$\begin{aligned} 2\hat{Q}(z)\hat{a}(p)\hat{v}^{(2)}(z, p) - 2\hat{v}^{(2)}(z, p) &= -2\rho\hat{a}(p)\hat{Q}(z) \\ \Leftrightarrow 2\left[\hat{Q}(z)\hat{a}(p) - 1\right]\hat{v}^{(2)}(z, p) &= -2\rho\hat{a}(p)\hat{Q}(z) \\ \Leftrightarrow \hat{v}^{(2)}(z, p) &= \frac{\rho\hat{a}(p)\hat{Q}(z)}{\left[1 - \hat{Q}(z)\hat{a}(p)\right]}, \end{aligned}$$

assuming that the right hand side is well-defined.

Under the conditions of Theorem 4.42, the function $\hat{v}^{(2)}(z, p)$ is integrable and we can take

$$v^{(2)}(t, \xi) = \mathcal{F}^{-1}\left(\hat{v}^{(2)}(z, p)\right).$$

Note that there might be problems with the integrability of $k^{(2)}$ in the original equation, which we can to circumvent by considering the second Ursell function instead. Therefore, the actual solution to (56) for $n = 2$ has the form

$$k^{(2)}(t, \xi) = v^{(2)}(t, \xi) + \rho^2.$$

This consideration is only a technical step to apply the Fourier transform properly. If we set

$$A = \frac{1}{(2\pi)^{d+1}} \int_0^\infty \int_{\mathbb{R}^d} \frac{|\hat{a}(p)\hat{Q}(z)|}{1 - \hat{a}(p)\hat{Q}(z)} dp h(dz)$$

we get the estimate

$$k^{(2)}(s_1, x_1, s_2, x_2) \leq \rho A + \rho^2 \leq C^2 (2!)^2$$

where

$$C := \max\left\{A, \frac{\sqrt{\rho(A + \rho)}}{2}\right\}$$

Remark 4.46. *The choice of C will become clear after the general induction step.*

For general $n \geq 3$, we need to solve

$$L_n^{\triangle} k^{(n)}(s_1, x_1, \dots, s_n, x_n) = -\sum_{i=1}^{n} k^{(n-1)}(s_1, x_1, \dots, \check{s}_i, \check{x}_i, \dots, s_n, x_n) \times$$
$$\times \sum_{j \neq i} a(x_i - x_j) Q\left(\frac{s_i}{s_j}\right) =: -f^{(n)}(s_1, \dots, x_n)$$

Then, the following expression is a solution to (56):

$$k^{(n)}(s_1, x_1, \dots, s_n, x_n) = \int_0^\infty e^{tL_n^{\triangle}} f^{(n)}(s_1, x_1, \dots, s_n, x_n) dt$$

Note that this is just the resolvent formula for $\lambda = 0$, i.e. the inverse of the operator $L_n^{\triangle}$. This expression makes sense if the following two conditions are satifsied:

$$\int_0^\infty e^{tL_n^{\triangle}} f^{(n)}(s_1, x_1, \dots, s_n, x_n) dt < \infty$$
$$e^{tL_n^{\triangle}} f^{(n)} \to 0 \text{ for almost all } (s_1, \dots, x_n)$$

Let us check the first condition by induction. We assume that

$$k^{(n-1)} \leq C_\rho^{n-1}((n-1)!)^2$$

This yields

$$f^{(n)}(s_1, \dots, x_n) \leq C_\rho^{n-1}((n-1)!)^2 \sum_{i=1}^{n} \sum_{j \neq i} a(x_i - x_j) Q\left(\frac{s_i}{s_j}\right)$$

By the Markov property of the semigroup $e^{tL_n^{\triangle}}$, we see that the only relevant summands of $L_n^{\triangle}$ are L_n^i and L_n^j. Using this fact and the positivity of the semigroup, we obtain

$$\begin{aligned}
&\int_0^\infty \left[e^{tL_n^{\triangle}} f^{(n)}\right](s_1, \dots, x_n) dt \\
&\quad \leq C_\rho^{n-1}((n-1)!)^2 \sum_{i=1}^{n} \sum_{j \neq i} \int_0^\infty \left[e^{tL_n^{\triangle}} a(x_i - x_j) Q\left(\frac{s_i}{s_j}\right)\right](s_i, x_i, s_j, x_j) dt \\
&\quad = C_\rho^{n-1}((n-1)!)^2 \sum_{i=1}^{n} \sum_{j \neq i} \int_0^\infty \left[e^{t(L_n^i + L_n^j)} a(x_i - x_j) Q\left(\frac{s_i}{s_j}\right)\right](s_i, x_i, s_j, x_j) dt
\end{aligned}$$

By the contraction property of $e^{tL_n^{\triangle}}$ on L^∞, we obtain for the remaining integral term

$$
\begin{aligned}
&\int_0^\infty \left[e^{t(L_n^i+L_n^j)} a(x_i-x_j) Q\left(\frac{s_i}{s_j}\right)\right](s_i,x_i,s_j,x_j)dt \\
&\leq \int_0^\infty \operatorname*{ess\,sup}_{s_j\in\mathbb{R}_+^*, x_j\in\mathbb{R}^d} \left| e^{tL_n^i} a(x_i-x_j) Q\left(\frac{s_i}{s_j}\right)\right|(s_i,x_i)dt \\
&= \int_0^\infty \operatorname*{ess\,sup}_{s_j,x_j} \left|\mathcal{F}^{-1}\mathcal{F}\left[e^{tL_n^i} a(\cdot_i-x_j) Q\left(\frac{\cdot_i}{s_j}\right)\right]\right|(s_i,x_i)dt \\
&= \int_0^\infty \operatorname*{ess\,sup}_{s_j,x_j} \left|\mathcal{F}^{-1}\left[e^{t(\hat{a}(p)\hat{Q}(z)-1)}\mathcal{F}\left(a(\cdot_i-x_j) Q\left(\frac{\cdot_i}{s_j}\right)\right)\right]\right|(s_i,x_i)dt \\
&= \int_0^\infty \operatorname*{ess\,sup}_{s_j,x_j} \left|\mathcal{F}^{-1}\left[e^{t(\hat{a}(p)\hat{Q}(z)-1)} e^{-iz\log s_j} e^{-i(p,x_j)} \hat{a}(p)\hat{Q}(z)\right]\right|(s_i,x_i)dt \\
&\leq \frac{1}{(2\pi)^{d+1}}\int_0^\infty \operatorname*{ess\,sup}_{s_j,x_j} \int_0^\infty\int_{\mathbb{R}^d} \left|e^{t(\hat{a}(p)\hat{Q}(z)-1)} e^{-i(z_i\log s_j-(p_i,x_j))}\hat{a}(p)\hat{Q}(z)\right|(s_i,x_i)dph(dz)dt \\
&= \frac{1}{(2\pi)^{d+1}}\int_0^\infty\int_0^\infty\int_{\mathbb{R}^d} e^{t(\hat{a}(p)\hat{Q}(z)-1)}\left|\hat{a}(p)\hat{Q}(z)\right| dph(dz)dt
\end{aligned}
$$

By using Fubini's theorem, we may exchange the integrals and use

$$
\int_0^\infty e^{t(\hat{a}(p)\hat{Q}(z)-1)}dt = \frac{1}{1-\hat{a}(p)\hat{Q}(p)}
$$

to obtain that the above expression equals

$$
\frac{1}{(2\pi)^{d+1}}\int_0^\infty\int_{\mathbb{R}^d}\frac{|\hat{a}(p)\hat{Q}(z)|}{1-\hat{a}(p)\hat{Q}(z)}dph(dz)
$$

Note that we do not need the absolute value, since the functions a and Q are real and even. By some standard assumptions on the kernels a and Q as above, this expression is finite.

Putting everything together, we obtain

$$
\begin{aligned}
k^{(n)}(s_1,x_1,\dots,s_n,x_n) &= \int_0^\infty e^{tL_n^{\triangle}} f^{(n)}(s_1,\dots,x_n)dt \\
&\leq C_\rho^{n-1}((n-1)!)^2 n(n-1)A \leq C_\rho^n (n!)^2
\end{aligned}
$$

the remainder of the proof follows the same scheme as in [36, Chapter 4.3].

4.3 Bolker-Dieckmann-Law-Pacala Model

As we have seen in Theorem 4.33, particles in the contact model admit clustering, if there exists some area where the birth kernel is strong enough. To counter this effect, we add competition to the contact model to suppress clustering. While semigroup techniques do not work for the dynamics of n-point correlation functions in the contact model, the addition of the competition mechanism enables us to use semigroup theory again to show the existence of the dynamics this way. Heuristically, this is due to the fact that by introducing competition, the mortality rate of the system increases with the population size. Therefore, this mechanism is also called self-regulation. The model is known as the

Bolker-Dieckmann-Law-Pacala model. It was first studied by the aforementioned authors [8, 9, 13]. For the infinite-dimensional case on $\Gamma(X)$, we refer to [17, 18] for more details.

From a modelling perspective, the additional term adds another natural component. Namely, in nature, agents such as animals or plants have to compete for resources. In areas with high population, the mortality will be higher, since more agents have to compete for the same amount of limited resources.

As stated above, we modify the contact model by adding a competition term. The model is given by the following operator for $F \in \mathcal{F}C(\mathbb{K}(\mathbb{R}^d))$:

$$\begin{aligned}(LF)(\eta) = &\sum_{x\in\tau(\eta)} m(s_x)[F(\eta - s_x\delta_x) - F(\eta)]\\ &+ \kappa^- \sum_{x\in\tau(\eta)} \sum_{y\in\tau(\eta - s_x\delta_x)} q^-(s_x, s_y)a^-(x-y)\,[F(\eta - s_x\delta_x) - F(\eta)]\\ &+ \kappa^+ \sum_{x\in\tau(\eta)} \int_{\mathbb{R}_+^*\times\mathbb{R}^d} q^+(s_x, s)a^+(x-y)[F(\eta + s\delta_y) - F(\eta)]\nu(ds)\sigma(dy)\end{aligned}$$

where the kernels are normalised, i.e.

$$\int_{\mathbb{R}^d} a^\pm(x)\sigma(dx) = 1, \ \int_{\mathbb{R}_+^*} q^\pm(s)\nu(ds) = 1$$

and $\kappa^+, \kappa^- > 0$ are birth and competition rates, respectively.

Remark 4.47. *While the model can be considered in a general setting for a mortality function function $m\colon \mathbb{R} \to \mathbb{R}$, we only consider $m(s) \equiv m$ constant for the proof of existence.*

4.3.1 The Symbol for the BDLP Model

As usual in the scheme, we proceed by considering the operator $\hat{L}$ on the space of quasi-observables.

Proposition 4.48. *The symbol $\hat{L}$ on $B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$ corresponding to L is given by*

$$\begin{aligned}(\hat{L}G)(\eta) = &-\sum_{x\in\tau(\eta)} m(s_x)G(\eta)\\ &- \kappa^- \sum_{x\in\tau(\eta)} \sum_{y\in\tau(\eta - s_x\delta_x)} q^-(s_x, s_y)a^-(x-y)G(\eta - s_y\delta_y)\\ &- \kappa^- \sum_{x\in\tau(\eta)} \sum_{y\in\tau(\eta - s_x\delta_x)} q^-(s_x, s_y)a^-(x-y)G(\eta)\\ &+ \kappa^+ \sum_{x\in\tau(\eta)} \int_{\mathbb{R}_+^*\times\mathbb{R}^d} q^+(s_x, s)a^+(x-y)G(\eta - s_x\delta_x + s\delta_y)\nu(ds)\sigma(dy)\\ &+ \kappa^+ \sum_{x\in\tau(\eta)} \int_{\mathbb{R}_+^*\times\mathbb{R}^d} q^+(s_x, s)a^+(x-y)G(\eta + s\delta_y)\nu(ds)\sigma(dy)\end{aligned}$$

Proof. Combinatorial arguments similar to the calculations done for the Glauber dynamics and contact model. ∎

4.3.2 Existence of a Semigroup for the BDLP Model

As we have calculated the operator $\hat{L}$, we may use perturbation theory to show the existence of the dynamics as it was the case for the Glauber dynamics. By adding the competition term, the mortality rate is "strong" enough to dominate the birth intensity, at least for a high population. We consider the following decomposition of $\hat{L}$:

$$\begin{aligned}
\hat{L}_0^1 G(\eta) &= -\sum_{x\in\tau(\eta)} m(s_x)G(\eta) \\
\hat{L}_0^2 G(\eta) &= -\kappa^- \sum_{x\in\tau(\eta)} \sum_{y\in\tau(\eta-s_x\delta_x)} q^-(s_x,s_y)a^-(x-y)G(\eta) \\
\hat{L}_1 G(\eta) &= -\kappa^- \sum_{x\in\tau(\eta)} \sum_{y\in\tau(\eta-s_x\delta_x)} q^-(s_x,s_y)a^-(x-y)G(\eta-s_y\delta_y) \\
\hat{L}_2 G(\eta) &= \kappa^+ \sum_{x\in\tau(\eta)} \int_{\mathbb{R}_+^*\times\mathbb{R}^d} q^+(s_x,s)a^+(x-y)G(\eta-s_x\delta_x+s\delta_y)\nu(ds)\sigma(dy) \\
\hat{L}_3 G(\eta) &= \kappa^+ \sum_{x\in\tau(\eta)} \int_{\mathbb{R}_+^*\times\mathbb{R}^d} q^+(s_x,s)a^+(x-y)G(\eta+s\delta_y)\nu(ds)\sigma(dy)
\end{aligned} \tag{59}$$

For the proof of existence, we restrict ourselves to the case where the intrinsic mortality is constant, i.e. $m(s)\equiv m$. We consider the evolution on the spaces $\mathbf{L}_{\alpha,C}$ as defined in Definition 4.6 with the usual L^1-norm denoted by $\|\cdot\|_{\alpha,C}$. Assume the following integrability condition on q^+ and q^-:

There exist constants $\Pi^\pm>0$ such that for all $s_x\in\mathbb{R}_+^*$,

$$\int_{\mathbb{R}_+^*} q^\pm(s_x,s)e^{\alpha s}\nu(ds)\le \Pi^\pm. \tag{60}$$

Let us now state the conditions for relative boundedness of $L_i, i=1,2,3$ w.r.t. L_0. To improve legibility, we write $\boldsymbol{C}(\eta)=C^{|\tau(\eta)|}e^{\alpha\sum_{z\in\tau(\eta)}s_z}$ for short.

Let us first state the result for the main part of the generator.

Proposition 4.49. *The operator $\hat{L}_0=\hat{L}_0^1+\hat{L}_0^2$ generates an analytic semigroup with*

$$\mathcal{D}(\hat{L}_0)=\left\{G\in\mathbf{L}_{\alpha,C}\colon \hat{L}_0G\in\mathbf{L}_{\alpha,C}\right\}$$

Proof. Hille-Yosida, using that $B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$ is dense in $\mathbf{L}_{\alpha,C}$, analogously to the proof of Proposition 4.7. □

We now consider the decomposition (59) and consider each perturbation separately. For convenience, we write $\lambda=\lambda_{\mathcal{G}}$ in the following proofs.

Lemma 4.50. *Assume* (60). *Then*

$$\|\hat{L}_1G\|_{\alpha,C}\le\frac{\kappa^-C\Pi^-}{m}\|\hat{L}_0^1G\|_{\alpha,C}$$

Proof. We have

$$
\begin{aligned}
\|\hat{L}_1 G\|_{\alpha,C} &= \int_{\mathbb{K}_0(\mathbb{R}^d)} \left|\hat{L}_1 G(\eta)\right| C^{|\tau(\eta)|} e^{\alpha \sum_{x\in\tau(\eta)} s_x} \lambda(d\eta) \\
&\leq \int_{\mathbb{K}_0(\mathbb{R}^d)} \kappa^- \sum_{x\in\tau(\eta)} \sum_{y\in\tau(\eta - s_x\delta_x)} q^-(s_x, s_y) a^-(x-y) |G(\eta - s_y\delta_y)| \boldsymbol{C}(\eta)\lambda(d\eta) \\
&= \kappa^- \int_{\mathbb{K}_0(\mathbb{R}^d)} \int_{\mathbb{R}_+^* \times \mathbb{R}^d} \sum_{x\in\tau(\eta)} q^-(s_x, s) a^-(x-y) |G(\eta)| \boldsymbol{C}(\eta + s\delta_y) \nu(ds)\sigma(dy)\lambda(d\eta) \\
&= \kappa^- C \int_{\mathbb{K}_0(\mathbb{R}^d)} |G(\eta)| \boldsymbol{C}(\eta) \sum_{x\in\tau(\eta)} \int_{\mathbb{R}_+^* \times \mathbb{R}^d} q^-(s_x, s) e^{\alpha s} a^-(x-y) \nu(ds)\sigma(dy)\lambda(d\eta) \\
&\overset{(60)}{\leq} \frac{\kappa^- C \Pi^-}{m} \int_{\mathbb{K}_0(\mathbb{R}^d)} m|\tau(\eta)||G(\eta)|\boldsymbol{C}(\eta)\lambda(d\eta) \\
&\leq \frac{\kappa^- C \Pi^-}{m} \|\hat{L}_0^1 G\|_{\alpha,C}
\end{aligned}
$$

□

Lemma 4.51. *For $\alpha \geq 0$ and (60), we have the following estimate:*

$$
\|\hat{L}_2 G\|_{\alpha,C} \leq \frac{\Pi^+ \kappa^+}{m} \|\hat{L}_0^1 G\|_{\alpha,C}
$$

Proof.

$$
\begin{aligned}
&\|\hat{L}_2 G\|_{\alpha,C} \leq \\
&\leq \kappa^+ \int_{\mathbb{K}_0(\mathbb{R}^d)} \sum_{x\in\tau(\eta)} \int_{\mathbb{R}_+^* \times \mathbb{R}^d} q^+(s_x, s) a^+(x-y) |G(\eta - s_x\delta_x + s\delta_y)| \nu(ds)\sigma(dy) \boldsymbol{C}(\eta)\lambda(d\eta) \\
&\overset{2.77}{=} \kappa^+ \int_{\mathbb{K}_0(\mathbb{R}^d)} \sum_{y\in\tau(\eta)} \int_{\mathbb{R}_+^* \times \mathbb{R}^d} q^+(s, s_y) a^+(x-y) |G(\eta)| \boldsymbol{C}(\eta - s_y\delta_y + s\delta_x) \nu(ds)\sigma(dx)\lambda(d\eta) \\
&= \kappa^+ \int_{\mathbb{K}_0(\mathbb{R}^d)} |G(\eta)| \boldsymbol{C}(\eta) \sum_{y\in\tau(\eta)} \int_{\mathbb{R}_+^* \times \mathbb{R}^d} q^+(s, s_y) e^{-\alpha s_y} e^{\alpha s_x} a^+(x-y) \nu(ds)\sigma(dx)\lambda(d\eta) \\
&\leq \kappa^+ \int_{\mathbb{K}_0(\mathbb{R}^d)} |G(\eta)| \boldsymbol{C}(\eta) \sum_{y\in\tau(\eta)} \Pi^+ \lambda(d\eta) \\
&\leq \frac{\Pi^+ \kappa^+}{m} \|\hat{L}_0^1 G\|_{\alpha,C}
\end{aligned}
$$

□

Lemma 4.52. *Assume the following pointwise estimate:*

$$
2e^{-\alpha s} \kappa^+ a^+(x) q^+(s) \leq C\kappa^- a^-(x) q^-(s) \ a.s.
$$

Then, the following estimate holds:

$$
\|\hat{L}_3 G\|_{\alpha,C} \leq \frac{1}{2} \|\hat{L}_0^2\|_{\alpha,C}
$$

Proof.

$$
\begin{aligned}
&\|\hat{L}_3 G\|_{\alpha,C} \le \\
&\le \kappa^+ \int_{\mathbb{K}_0(\mathbb{R}^d)} \sum_{x\in\tau(\eta)} \int_{\mathbb{R}_+^*\times\mathbb{R}^d} q^+(s_x,s)a^+(x-y)|G(\eta+s_y\delta_y)|\boldsymbol{C}(\eta)\nu(ds)\sigma(dy)\lambda(d\eta) \\
&\stackrel{2.77}{=} \int_{\mathbb{K}_0(\mathbb{R}^d)} \sum_{y\in\tau(\eta)} \sum_{x\in\tau(\eta-s_y\delta_y)} \kappa^+ q^+(s_x,s_y)a^+(x-y)|G(\eta)|\boldsymbol{C}(\eta-s_y\delta_y)\lambda(d\eta) \\
&= \int_{\mathbb{K}_0(\mathbb{R}^d)} \sum_{y\in\tau(\eta)} \sum_{x\in\tau(\eta-s_y\delta_y)} \underbrace{\kappa^+ q^+(s_x,s_y)a^+(x-y)C^{-1}e^{-s_y}}_{\le\kappa^-a^-q^-} |G(\eta)|C^{|\tau(\eta)|}e^{\alpha\sum_{z\in\tau(\eta)}s_z}\lambda(d\eta) \\
&\le \int_{\mathbb{K}_0(\mathbb{R}^d)} \sum_{y\in\tau(\eta)} \sum_{x\in\tau(\eta-s_y\delta_y)} \kappa^- q^-(s_x,s_y)a^-(x-y)|G(\eta)|C^{|\tau(\eta)|}e^{\alpha\sum_{z\in\tau(\eta)}s_z}\lambda(d\eta) \\
&= \|\hat{L}_0^2 G\|_{\alpha,C}
\end{aligned}
$$

□

Putting this all together, we obtain the following result:

Theorem 4.53. *Assume integrability condition* (60) *as well as*

$$
\begin{aligned}
&2(\Pi^-\kappa^- C + \Pi^+\kappa^+) < m \\
&2e^{-\alpha s}\kappa^+ q^+(s)a^+(x) \le C\kappa^- q^-(s)a^-(x)\ \nu\otimes\sigma - a.e.
\end{aligned}
$$

Then the operator $\hat{L}$ generates a strongly continuous semigroup on $\mathbf{L}_{\alpha,C}$.

Proof. Take the Lemmas above together with Theorem 4.10. The factor of 2 stems from the proof of the perturbation theorem. □

4.3.3 Statistical Dynamics of the BDLP Model

Let us close the considerations of the BDLP-model by taking into account the evolution of correlation functions. The correlation functions let us show that the competition introduced in this model suppresses clustering of the solution.

Let us start by giving the form of the operator $L^\triangle$ on the space of correlation functions.

Proposition 4.54. *The operator $L^\triangle$ is given by*

$$
\begin{aligned}
(L^\triangle k)(\eta) = &- \sum_{x\in\tau(\eta)} m(s_x)k(\eta) \\
&- \kappa^- \int_{\mathbb{R}_+^*\times\mathbb{R}^d} \sum_{x\in\tau(\eta)} q^-(s_x,s)a^-(x-y)k(\eta+s\delta_y)\nu(ds)\sigma(dy) \\
&- \kappa^- \sum_{x\in\tau(\eta)} \sum_{y\in\tau(\eta-s_x\delta_x)} q^-(s_x,s_y)a^-(x-y)k(\eta) \\
&+ \kappa^+ \sum_{y\in\tau(\eta)} \int_{\mathbb{R}_+^*\times\mathbb{R}^d} q^+(s,s_y)a^+(x-y)k(\eta-s_y\delta_y+s\delta_x)\nu(ds)\sigma(dx) \\
&+ \kappa^+ \sum_{y\in\tau(\eta)} \sum_{x\in\tau(\eta-s_y\delta_y)} q^+(s_x,s_y)a^+(x-y)k(\eta-s_y\delta_y)
\end{aligned}
$$

where the considered class of $k\colon \mathbb{K}_0(\mathbb{R}^d)\to\mathbb{R}$ is specified below.

Proof. The proof works analogously to the calculations done before by using duality (26). □

The fact that the evolution on quasi-observables conserves integrability of the solution can be used to show that the correlation functions obey a sub-Poissonian bound by considering the dual semigroup, similar to [33]. It should be noted, though, that we do not make any statements about the continuity of the dual semigroup. For the evolution in mind, consider the following space for some $C > 0$ and $\alpha \in \mathbb{R}$:

$$\mathcal{K}_{\alpha,C} = \left\{k\colon \mathbb{K}_0(\mathbb{R}^d) \to \mathbb{R} \mid k \cdot C^{-|\tau(\cdot)|} e^{-\alpha \sum_{x\in\tau(\cdot)} s_x} \in L^\infty(\mathbb{K}_0(\mathbb{R}^d), \lambda_{\mathcal{G}})\right\}.$$

which is the dual to the space $\mathcal{L}_{\alpha,C}$ from the previous chapter with respect to duality (26). This means the following: Denote by $(\hat{U}_t)_{t\geq 0}$ the semigroup generated by $\hat{L}$ on $\mathcal{L}_{\alpha,C}$, cf. Theorem 4.53. Then we can construct the evolution of correlation functions using the following relation:

$$\langle\langle G, k_t\rangle\rangle = \langle\langle \hat{U}_t G, k_0\rangle\rangle$$

for an initial condition $k_0 \in \mathcal{K}_{\alpha,C}$. Denote $\hat{U}_t^* k_0 := k_t$. Since the evolution on $\mathcal{K}_{\alpha,C}$ is also given by a semigroup, the bound is preseved. In other words, for an initial condition $k_0 \in \mathcal{K}_{\alpha,C}$, we have

$$|k_t(\eta)| \leq \mathrm{const} \cdot C^{|\tau(\eta)|} e^{\alpha \sum_{x\in\tau(\eta)} s_x}.$$

Note that in comparison to Theorem 4.33, there is no factor $|\tau(\eta)|!$ included, which indicates that the BDLP model suppresses the clustering which is present in the contact model.

Appendix

A The Final Topology on $\mathbb{K}(\mathbb{R}^d)$ and the Relation to Measurable Structures

We want to check that the final topology induced by the mapping $\mathcal{R}\colon \Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \to \mathbb{K}(\mathbb{R}^d)$ fulfills the statements used in this work. In what follows, there are some statements from topology and measure theory.

First, let us show that a homeomorphism preserves the measurable structure.

Lemma A.1. *Let $f\colon X \to Y$ be a homeomorphism on topological spaces X and Y. Then the image-σ-algebra of X under f and the Borel-σ-algebra of Y coincide, i.e. $\mathcal{B}_f(X) = \mathcal{B}(Y)$, where*

$$\mathcal{B}_f(X) := \{f(A) \mid A \in \mathcal{B}(X)\}$$

Proof. Let τ_X and τ_Y be the topologies of the corresponding spaces. Since $\mathcal{B}(X) = \sigma(\tau_X)$ and $\mathcal{B}(Y) = \sigma(\tau_Y)$, it suffices to show $f(\tau_X) = \tau_Y$. Note that we abuse the notation slightly, as we may use the same notation for the pre-image mapping and the inverse mapping of f.

"$\subseteq$": Let $B \in f(\tau_X)$, i.e. there exists $A \in \tau_X$ such that $B = f(A)$. Since f^{-1} is continuous, we have $B \in \tau_Y$.

"$\supseteq$": Let $B \in \tau_Y$. We want to show that there exists $A \in \tau_X$ such that $B = f(A)$. Since f is continuous, we may set $A := f^{-1}(B) \in \tau_X$. Since f is bijective, we see that $f(f^{-1}(B)) = B$.

□

Next, let us define the notion of the final topology, which is the topology induced by a family of mappings. Since we only concider the reflection mapping $\mathcal{R}$, we define the final topology with respect to one mapping.

Definition A.2. *Let (X, τ_X) be a topological space, Y a set and $f\colon X \to Y$ some mapping. A topology τ_Y on Y is called final topology with respect to (X, τ_X, f) if one of the following equivalent properties holds:*

1. *τ_Y is the finest topology such that f is continuous.*
2. *A subset $O \subset Y$ is open if and only if $f^{-1}(O) \subset X$ is open.*
3. *A mapping $g\colon Y \to Z$ is continuous if and only if $g \circ f$ is continuous, where Z topological space.*

Let us show some topological results stemming from this definition.

Lemma A.3. *Let (X, τ_X) be a topological space, Y a set and $f\colon X \to Y$ a bijection. Then $f(\tau_X)$ is the final toplogy on Y, where*

$$f(\tau_X) = \{f(U) \mid U \in \tau_X\}.$$

Proof. Let us first show that $f\colon (X, \tau_X) \to (Y, f(\tau_X))$ is continuous. To this end, let $V \in f(\tau_X)$. Then there exists $U \in \tau_X$ such that $f(U) = V$ by definition of $f(\tau_X)$. Therefore, we also have

$$f^{-1}(V) = U \in \tau_X$$

and f is continuous.

Next, we need to show that $f(\tau_X)$ is the finest topology such that f is continuous. Let $W \subset Y$ with $W \notin f(\tau_X)$ and assume that there exists $U \in \tau_X$ such that $f^{-1}(W) = U$. But since f is a bijection, we have

$$W = f(f^{-1}(W)) = f(U) \in f(\tau_X)$$

which is a contradiction. Therefore, the claim holds. □

Furthermore, the following result is immediate by part 2 of the definition:

Lemma A.4. *Let $f\colon (X, \tau_X) \to (Y, f(\tau_X))$ be a bijection. Then f is a homeomorphism.*

Proof. f is continuous by definition. We need to check that the inverse mapping $f^{-1}\colon Y \to X$ is continuous. To this end, let $U \in \tau_X$. Since f is a bijection, we have

$$\left(f^{-1}\right)^{-1}(U) = f(U) \in f(\tau_X).$$

In fact, this also holds directly by applying part 2 of the definition. □

Remark A.5. *In our case, these results are applied to the bijective reflection mapping $\mathcal{R}\colon \Pi(\mathbb{R}_+^* \times \mathbb{R}^d) \to \mathbb{K}(\mathbb{R}^d)$. It induces the topology on $\mathbb{K}(\mathbb{R}^d)$ and becomes a homeomorphism. Furthermore, the measurable structures on $\Pi(\mathbb{R}_+^* \times \mathbb{R}^d)$ and $\mathbb{K}(\mathbb{R}^d)$ coincide.*

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