

## Stabilizing equilibrium by linear feedback control for controlling chaos in Chen system

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**Abstract.** Stabilization of a chaotic system in one of its unstable equilibrium points by applying small perturbations is studied. A two-stage control strategy based on linear feedback control is applied. Improvement of system performance is addressed by exploiting the ergodicity of the original dynamics and using Lyapunov stability results for control design. Extension to the not complete observability case is also analyzed.

### 1. Introduction

It is well known that one of the Control of Chaos objectives is to suppress the chaotic dynamical behaviour naturally arising in a given system. Since the beginning of the nineties, much work has been developed on this field [1, 2, 3]. An appropriate question was addressed. Given a chaotic system: how a desired time-periodic motion with improved performance can be achieved by applying only small perturbations on some accessible system parameter or system variable? The problem was firstly approached in [4, 5] as follows: a) determination of some of the unstable low-period (or even steady-state) orbit that are embedded in the chaotic attractor, b) choice of one which yields improved system performance and, c) application of small control action to stabilize this already existing orbit. This method, -the OGY method- described in detail in the discrete-time case, applies to the given continuous-time chaotic system by means of Poincaré section.

Not much later, several authors concentrated on controlling chaos in continuous-time systems without using Poincaré discretization and resorted to the well developed machinery of Modern Control Theory. It is known that fundamental features of controlling chaos like taking full account of the special aspects of chaotic motion and applying only small perturbations have been often neglected by them. For example, with the objective of suppressing chaotic behaviour, in references [6, 7, 8, 9] linear feedback control is explored while in reference [10], a controller based on a PI regulator control is addressed. However, these approaches only take care of local stabilization.

This work concentrates on Chen system but the approaches presented here are straightly applicable to other chaotic systems like Lorenz, Chua, Rossler, etc. Our purpose is to stabilize the system in one of its (unstable) equilibrium points by using linear feedback control. Improvement of system

performance is addressed by exploiting the ergodicity of the original dynamics and using Lyapunov stability results for control design [11].

## 2. Purpose and method

Let us assume that we have a chaotic dynamical system described by:

$$\dot{X} = F(X)$$

and that  $E$  is one of its unstable equilibrium points embedded in its chaotic attractor.

Our aim is controlling chaos by applying feedback control. For this purpose, we will construct a linear feedback control, depending on a gain parameter  $k$ , to stabilize the system in the equilibrium point  $E$ . The following facts will be of relevance:

- due to ergodicity of the free system, (almost) every trajectory initialized in the strange attractor reaches a chosen  $E$ -neighbourhood,  $B(E, \delta)$ ;
- fixed  $k$  such that locally asymptotic convergence is guaranteed, the corresponding attraction region of the controlled system may be estimated as a set  $\Omega_k$ .

We look for a  $k$ -depending control law such that  $B(E, \delta) \subset \Omega_k$  and which remains bounded by a desired fixed bound. The control strategy consists in making the free system run till it reaches the neighbourhood  $B(E, \delta)$ . At the time that the trajectory reaches the neighbourhood  $B(E, \delta)$ , the feedback control is activated and it is kept so for all future times. Therefore, once the trajectory enters  $\Omega_k$ , it will stay there for ever while the feedback control will be kept under the desired bound.

Chen dynamical system [12, 13] is described by:

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) \\ \dot{x}_2 &= (c - a)x_1 + cx_2 - x_1x_3 \\ \dot{x}_3 &= -bx_3 + x_1x_2 \end{aligned} \quad (1)$$

being  $x_1$ ,  $x_2$  and  $x_3$ , the state variables and  $a$ ,  $b$  and  $c$ , positive real constants. For  $a=35$ ,  $b=3$  and  $c=28$ , it has a chaotic attractor and its unstable equilibrium points are  $E_1 = (0, 0, 0)$ ,  $E_2 = (\sqrt{63}, \sqrt{63}, 21)$  and  $E_3 = (-\sqrt{63}, -\sqrt{63}, 21)$  (see Figure 1).

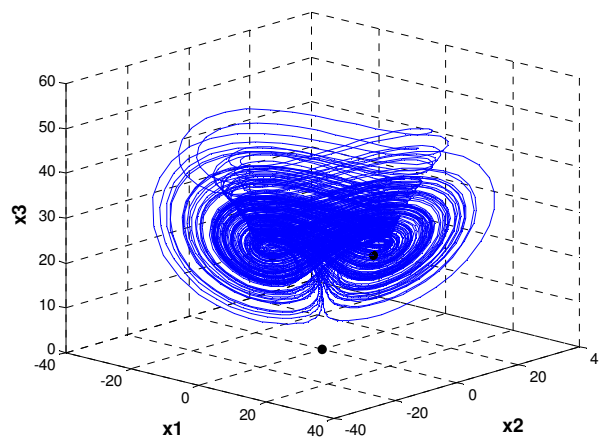


Figure 1. Chen attractor and equilibrium points

As in [6], we assume that all the state variables are observable and that each system equation may be affected by an additive control, i.e.

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) + u_1 \\ \dot{x}_2 &= (c - a)x_1 + cx_2 - x_1x_3 + u_2 \\ \dot{x}_3 &= -bx_3 + x_1x_2 + u_3 \end{aligned} \quad (2)$$

Let us choose  $E=E_2$  and introduce the following linear controls:

$$u_1 = -k(x_1 - \sqrt{63}), \quad u_2 = -k(x_2 - \sqrt{63}), \quad \text{and} \quad u_3 = -k(x_3 - 21).$$

From the stability analysis on the linearization of the controlled system (2), local convergence is assured if  $k \geq 4.22$ . By means of Lyapunov function construction [11], we estimate the region of attraction of system (2) as being an ellipsoid centered in  $E$ ,  $\Omega_k$ . In Table 1, the lengths of the major and the minor ellipsoid axis,  $s_M(k)$  and  $s_m(k)$  respectively, for some values of  $k$  are displayed.

**Table 1.** Estimated region of attraction.  
 $s_M(k)$ : length of the ellipsoid major axis,  
 $s_m(k)$ : length of the ellipsoid minor axis

$k$	$s_M(k)$	$s_m(k)$
7	0.3196	0.0459
10	0.7344	0.1485
20	2.8445	0.9300
30	6.195	2.65
50	15.387	8.73
100	28.399	20

To state the strategy we must choose  $k$  that verifies:

$$(x_1(t), x_2(t), x_3(t)) \in \Omega_k \Rightarrow \|(u_1(t), u_2(t), u_3(t))\|_2 \leq U \quad (3)$$

being  $U$ , the desired fixed control bound.

As

$$\|(u_1(t), u_2(t), u_3(t))\|_2 \leq k \|(x_1(t), x_2(t), x_3(t)) - E\|_2 \leq k \cdot s_M(k), \quad (4)$$

(3) is verified by  $k$  such that

$$k \cdot s_M(k) \leq U. \quad (5)$$

Due to ergodicity, every sphere  $B(E, \delta)$  with  $\delta > \Delta$  will be visited at any time, by (almost) every trajectory of system (1), initialized in the strange attractor. From numerical experience, we estimate  $\Delta=0.82$ . On the other hand, convergence is guaranteed if  $B(E, \delta) \subset \Omega_k$  which is valid if  $\delta \leq s_m(k)$ . So, we need to choose  $\delta$  such that

$$\Delta \leq \delta \leq s_m(k). \quad (6)$$

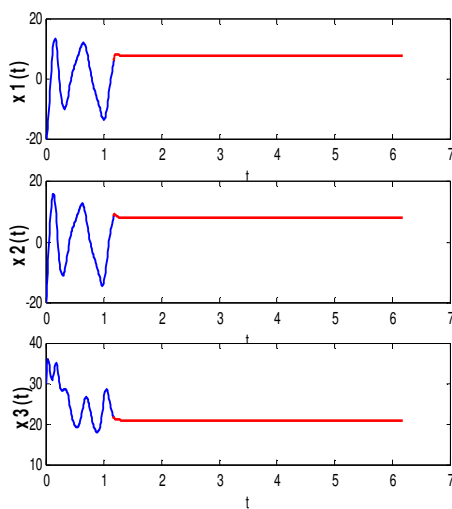
Hence, let us fix  $k$  and  $\delta$  according to (5) and (6). The algorithm consists of two stages. In the first stage, the system runs freely (control not activated). Let  $t_f$  the instant of time at which its trajectory reaches  $B(E, \delta)$ . The second stage begins at time  $t=t_f$  at which the feedback control is activated. Note that, differently from the OGY method [4], the control is kept activated for all  $t > t_f$ .

### 3. Results and extension

We emphasize that under the stated requirements, trajectories convergence and  $U$ -bounded controls are formally proved, for (almost) initial conditions in the strange attractor.

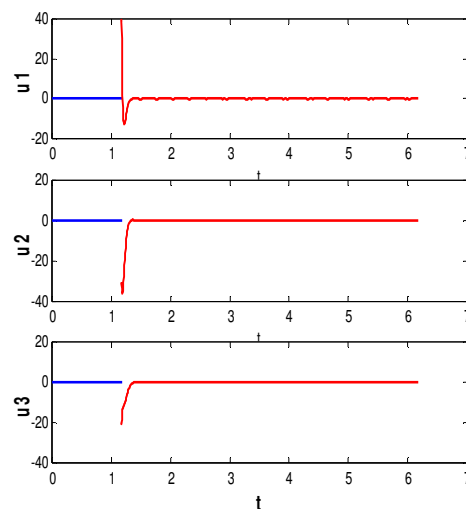
Note that the smaller is  $\delta$ , the smaller  $k$  value may be set what yields to control effort reduction. However, a great reduction on  $\delta$  will probably result on a drastic increase of the waiting time (first stage time). Besides, in general, a too small  $k$  delays too much convergence in the second stage. Therefore, control parameters values must be chosen respecting a compromise between control effort, and total convergence time (that is, the sum of the two stages times).

Conservative feature of our estimations must be pointed out. This is put in evidence by simulations. Let us see two examples. In both of them, the system is initialized in  $(-20, -20, 30)$ . Choosing  $\delta=2$  and  $k=30$ , convergence is guaranteed and  $\|(u_1(t), u_2(t), u_3(t))\|_2 \leq 185.7$  is predicted. In Figure 2, the corresponding states and controls signals are displayed. Note that the control bound is meaningfully less than the predicted one. By instead, setting  $k=7$ , convergence is verified although it is not theoretically proven (see Figure 3).



**Figure 2.1.** States.

— control not activated  
 — control is activated ( $k=30$ )

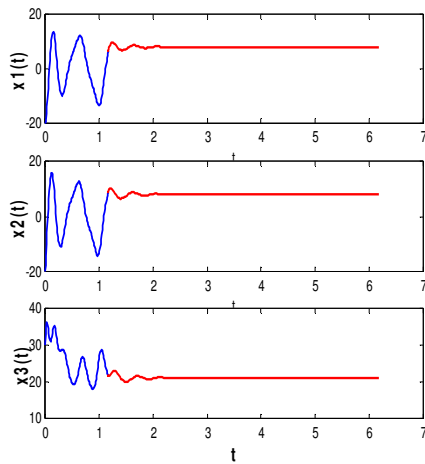


**Figure 2.2.** Controls.

— control not activated  
 — control is activated ( $k=30$ )

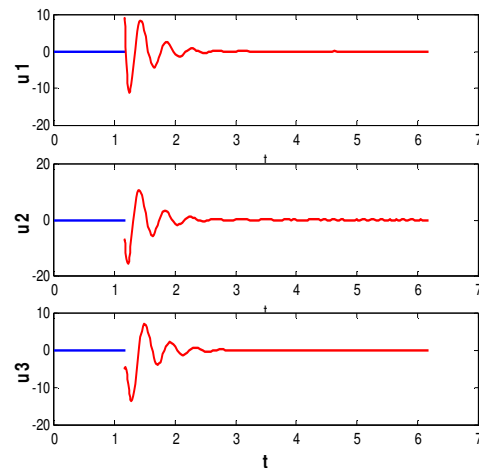
We wonder if this methodology applies when not all states are observable. Suppose that we have system  $\dot{X} = F(X)$  but only the output  $y$  is at our disposal, being  $y=CX$ . The objective is to make the output converge to  $y_E=CE$ . We implement a two stages-algorithm as in the complete observability case, save that: *i*) the criteria for control activation is  $|y - y_E| < \delta$ , *ii*) the condition for control activation must be verified at every time. This is because in this case, we do not have estimated region of attraction.

In spite of this limitation, we obtain experimental evidence of the algorithm success. Let us show it by applying the algorithm to Chen system. The output of system (1) is described by  $C=(0,1,0)$ . Setting in (2) the linear feedback control:  $u_1=0$ ,  $u_2=-k(y-y_E)$  and  $u_3=0$ , the system output locally converges to  $y_E$  if  $k \geq 6.059$ . The parameters values are chosen as  $\delta=2$  and  $k=10$  so we predict  $\|(u_1(t), u_2(t), u_3(t))\|_2 \leq 20$ . Output convergence and control bound are verified through simulation. For example, with initial condition given by  $(5, -15, 40)$ , the resulting output and control are shown in Figure 4.



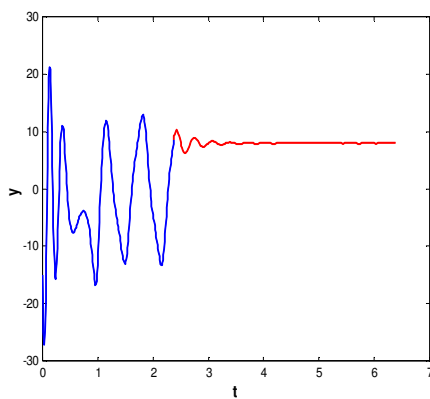
**Figure 3.1.** States.

— control not activated  
 — control is activated ( $k=7$ )



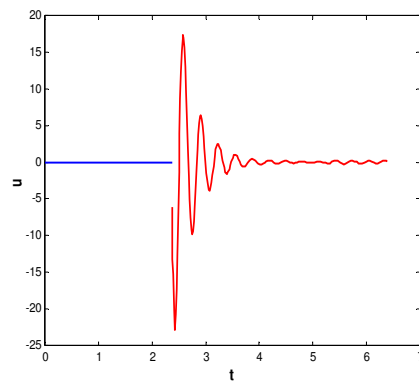
**Figure 3.2.** Controls.

— control not activated  
 — control is activated ( $k=7$ )



**Figure 4.1.** Output.

— control not activated  
 — control is activated ( $k=10$ )



**Figure 4.2.** Controls.

— control not activated  
 — control is activated ( $k=10$ )

#### 4. Discussion and conclusions

Chen system has been stabilized while considering fundamental features on controlling chaos. Hence, some progress with respect to previous works [6, 7, 8, 9] has been made.

As in OGY method, the on-line implementation only requires data on system linearization. But, for control design, extra system information is needed (for estimation of the region of attraction) to choose control parameters which guarantee convergence. Then, fixing these values, not only convergence but also no “kicking” of the trajectory out of the neighbourhood of the equilibrium point is assured. On the other hand, simulated results show us that our theoretical estimation may be too conservative. This

drawback will be object of future investigation as well as the extension of these ideas to other plausible situations like stabilization of periodic orbits, controllability restrictions, system affected by noise, etc.

We have also considered the case of uncompleted observability as in [10]. Bounded controls (under the desired bound) have been achieved by our approach. Of course, issues about compromise between convergence times and control bound restrictions also arise here. The theoretical proof of convergence or any other property of the control strategy promise interesting research.

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### References

- [1] Boccaletti S, Grebogi C, Lai Y C, Mancini H and Maza D 2000 The control of chaos: theory and applications *Physics Reports* **329** 103-197.
- [2] Chen G, Dong X 1998 From Chaos to Order: Perspectives, Methodologies and Applications, Singapore: World Scientific.
- [3] Fradkov A L and Evans R J 2005 Control of chaos: Methods and applications in engineering. *Annual Reviews in Control* **29** 33-56.
- [4] Ott E, Grebogi C and Yorke J 1990 Controlling Chaos *Phys. Rev. Lett.* **64** 1196.
- [5] Romeiras F, Grebogi C, Ott E and Dayawansa W P 1992 Controlling Chaotic Dynamical Systems *Physica D: Nonlinear Phenomena* **58** 165-192
- [6] Yassen M T 2003 Chaos control of Chen chaotic dynamical system, *Chaos, Solitons & Fractals* **15** 271-283.
- [7] Wang X, Tian L and Yu L 2006 Linear Feedback Controlling and Synchronization of the Chen's Chaotic System, *International Journal of Nonlinear Science* **2** no.1 43-49.
- [8] Cui-Xiang L and Jia-Shi T 2008 Equilibrium points and bifurcation control of a chaotic system, *Chinese Physics B* **17** No.1 135-38.
- [9] Plienpanich T, Niamsup P and Lenbury T 2005 Controllability and stability of the perturbed Chen chaotic dynamical system *Applied Mathematics and Computation* **171** 927-947.
- [10] Jiang G P and Wang S P 2003 Stabilizing Unstable Equilibrium of Chaotic Systems using PI regulator *Proceedings of IEEE International Symposium on Circuits and Systems* **III** 64-67.
- [11] Khalil H 2001 Nonlinear Systems *Prentice Hall* 750.
- [12] Chen G and Ueta T 1999 Yet another chaotic attractor *International Journal of Bifurcation and Chaos* **9** 7 1465-1466.
- [13] Zhou T, Tang Y and Chen G 2004 Chen's attractor exists *International Journal of Bifurcation and Chaos* **14** 9 3167-3177 .