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A posteriori error estimates for nonconforming approximations of Steklov eigenvalue problems

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ABSTRACT

This paper deals with a posteriori error estimators for the non conforming Crouzeix–Raviart finite element approximations of the Steklov eigenvalue problem. First, we define an error estimator of the residual type which can be computed locally from the approximate eigenpair and we prove the equivalence between this estimator and the broken energy norm of the error with constants independent of the corresponding eigenvalue. Next, we prove that edge residuals dominate the volumetric part of the residual and that the volumetric part of the residual terms dominate the normal component of the jumps of the discrete fluxes across interior edges. Finally, based on these results, we introduce two simpler equivalent error estimators. The analysis shows that these a posteriori error estimates are optimal up to higher order terms and that may be used for the design of adaptive algorithms.

1. Introduction

Eigenvalue problems of the Steklov type occur in many applications. As examples, we can cite the problem of determining the vibration modes of a structure in contact with an incompressible fluid [1], the analysis of the stability of mechanical oscillators immersed in a viscous media [2] and the dynamics of liquids in moving containers [3], the so-called sloshing problem.

In a recent paper [4], we analyzed the first order Crouzeix–Raviart finite element approximation of this spectral problem. We proved convergence and obtained *a priori* error estimates for the eigenvalues and the eigenfunctions. The purpose of this paper is to derive *a posteriori* error estimators for the nonconforming method studied in [4].

A posteriori error analysis for nonconforming finite element approximations experienced a remarkable development during the last ten years.

For second-order elliptic boundary value problems, a posteriori error estimates were first obtained for the Crouzeix–Raviart method by using a Helmholtz type decomposition of the gradient of the error [5]. The same technique has been generalized for a large class of nonconforming finite element methods in [6].

Hierarchic basis type estimators were presented in [7] where two sided bounds on the error were obtained by solving suitable local problems and assuming that a saturation condition is valid.

An alternative approach based on the use of a smoothing procedure of the nonconforming solution is presented in [8]. Similar ideas have been proposed in the previous work [9] in order to derive an error estimator for nonconforming approximations of a nonlinear problem. However, the analysis of the efficiency of the error estimators obtained in this way seems to depend on additional assumptions on the regularity of the true solutions.

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Later, by extending the equilibrated residual method to nonconforming finite element schemes, a robust a posteriori estimator for the Crouzeix–Raviart approximations of Darcy's equation was proposed and analyzed in [10]. This approach was generalized to the Fortin–Soulie element in [11].

Recently, a posteriori error estimates of the residual type were derived within an unifying framework for lowest order conforming, nonconforming and mixed finite element methods [12–14] for the Laplace, Stokes and Navier-Lamé equations. Another interesting result concerning the linear convergence of an appropriated adaptive finite element algorithm for the lowest order Crouzeix–Raviart elements was presented in [15].

Much less attention has been paid to nonconforming methods for eigenvalue problems. This might be due to the fact that eigenvalue problems have a nonlinear character. Therefore, the extension of the techniques originally developed for source problems is neither obvious nor direct.

In particular, for eigenvalue problems of the Steklov type, to the best of the authors' knowledge, no a posteriori error estimates for nonconforming methods have been obtained yet.

The analysis presented in this paper is carried out along the lines of [8]. Roughly speaking, it consists of the following steps.

First, we split the error into two components, usually called the *conforming part* and the *nonconforming part* of the error. This splitting is obtained by introducing a post-processing procedure which is based on an averaging technique applied directly to the nonconforming approximation and requires only explicit local computations.

The derivation of the a posteriori estimates relies on the possibility of estimating each part of the error separately.

The nonconforming part of the error is related directly with the difference between the Crouzeix–Raviart approximation of the eigenfunctions and the smoothed approximation of them given by the post-process. We prove a posteriori estimates for the broken energy norm of this difference and we show that these estimates can be established in terms of the jumps of the discrete solutions. In other words, there is no need of actually computing the post-processed eigenfunctions in order to obtain these error estimations.

To deal with the conforming part of the error, we use the existing techniques for conforming finite element methods [16–18].

The error estimator obtained in this way resembles one of the estimators introduced by Dari et al. in [5] for the approximation by non conforming finite element methods of Poisson type problems. As we mentioned before, the approach considered in that paper is based on the use of a Helmholtz decomposition in combination with some orthogonality relations for the error. No direct extension of these techniques seems to be possible in order to deal with Steklov type problems.

It is well known that edge residuals, i.e., jump terms in the normal derivatives of the approximated solution across interior boundaries, dominate the error in linear conforming finite element approximation of source problems (see [19,20], for instance). This result has been extended to eigenvalue problems and conforming methods in [17,18]. It is also known that in the nonconforming case, edge residuals include the jumps across the element boundaries of the tangential derivatives as well.

Our next step is to obtain edge residuals dominated error estimates for the Crouzeix–Raviart approximation of the Steklov problem. In fact, we prove that

– the volumetric part of the residual terms is dominated exactly by the edge part of the residuals,
– the normal component of the jump of the discrete fluxes across interior edges is dominated by the volumetric part of the residuals up to higher order terms.

In particular, these results allow introducing two simpler error estimators which turn out to be equivalent to the broken energy norm of the error also up to higher order terms. Similar results, but for source problems, were first presented in [15].

We end this paper by proving optimal a priori estimates for the L^2 norm of the error in the restriction to the boundary of the domain of the approximate eigenfunctions. These results improve the previous ones obtained in [4] and are crucial in order to prove the equivalence, up to higher order terms, between the error and the proposed error estimators.

Finally, let us remark that the results presented in this paper are valid for a general simply connected polygonal domain and general meshes satisfying the usual regularity assumptions. The error estimators introduced here are easy to compute locally from the approximated eigenpair and can be used for the design of adaptive algorithms.

2. The Steklov eigenvalue problem and its discretization

Let $\Omega \subset \mathbb{R}^2$ be a simply connected and bounded domain with a polygonal boundary $\partial\Omega = \Gamma$. We consider the following spectral problem:

Find $\lambda \in \mathbb{R}$ and $u \neq 0$ such that

$$\begin{cases} -\operatorname{div}(\alpha \nabla u) + \beta u = 0 & \text{in } \Omega, \\ \alpha \dfrac{\partial u}{\partial \mathbf{n}} = \lambda u & \text{on } \Gamma, \end{cases} \tag{2.1}$$

where the coefficients $\alpha = \alpha(x)$ and $\beta = \beta(x)$ are bounded above and below by positive constants. We assume that $\alpha \in C^1(\bar{\Omega})$.

Here and in the sequel, we shall use standard notation for Sobolev spaces $H^s(\Omega)$, their norms $\|\cdot\|_s$ and seminorms $|\cdot|_s$.

Let $V := H^1(\Omega)$. Let a and b be the symmetric bilinear forms defined by

$$a(u,v) := \int_\Omega \alpha \nabla u \cdot \nabla v + \int_\Omega \beta u v, \quad \forall u, v \in V,$$

$$b(u,v) := \int_\Gamma u v, \quad \forall u, v \in V.$$

Since α and β are bounded in $\bar{\Omega}$, a is continuous and coercive on V. Then, the variational formulation of the spectral problem (2.1) is given by

Find $\lambda \in \mathbb{R}$ and $u \in V$, $u \neq 0$, such that

$$a(u,v) = \lambda b(u,v), \quad \forall v \in V. \tag{2.2}$$

From the classical theory of abstract elliptic eigenvalue problems [21], we can infer that problem (2.2) attains a sequence of finite multiplicity eigenvalues $\lambda_n > 0$, $n \in \mathbb{N}$, diverging to $+\infty$, with corresponding $L^2(\Gamma)$-orthonormal eigenfunctions u_n belonging to V.

We introduce the following spaces:

$$X := L^2(\Omega) \times L^2(\Gamma)$$

$$W := \{(u,\xi) \in H^1(\Omega) \times H^{1/2}(\Gamma) : \xi = u|_\Gamma\},$$

endowed with the norms defined by

$$|(u,\xi)| := (\|u\|_0^2 + \|\xi\|_{0,\Gamma}^2)^{1/2},$$

$$\|(u,\xi)\| := (\|u\|_1^2 + \|\xi\|_{0,\Gamma}^2)^{1/2}.$$

We consider the bounded linear operator $\mathbf{T} : X \to X$ defined by $\mathbf{T}(f,\tau) = (u,\xi) \in W$ and

$$a(u,y) + b(\xi,\mu) = b(\tau,\mu), \quad \forall (y,\mu) \in W. \tag{2.3}$$

By virtue of Lax–Milgram Lemma, we have

$$\|(u,\xi)\| \le C|(f,\tau)|.$$

Since a and b are symmetric, $\mathbf{T}$ is self-adjoint with respect to a and b. Clearly, $(\lambda, (u,\xi))$ is an eigenpair of $\mathbf{T}$ if and only if $\frac{1}{\lambda} - 1$ and (u,ξ) is a solution of problem (2.2). Therefore, the knowledge of the spectrum of $\mathbf{T}$ gives complete information about the solutions of our original problem.

The following proposition states a priori estimates for the solution of problem (2.3) depending on the regularity of the data.

Lemma 2.1. *Let (u,ξ) be the solution of problem (2.3). There exist constants $r \in (1/2, 1]$ and $C > 0$ such that*

- *if $\tau \in L^2(\Gamma)$, $u \in H^{1+r/2}(\Omega)$ and*

$$\|u\|_{1+r/2} \le C\|\tau\|_{0,\Gamma}, \tag{2.4}$$

- *if $\tau \in H^\epsilon(\Gamma)$, with $\epsilon \in (0, r - 1/2)$, $u \in H^{3/2+\epsilon}(\Omega)$ and*

$$\|u\|_{3/2+\epsilon} \le C\|\tau\|_{\epsilon,\Gamma}, \tag{2.5}$$

- *if $\tau \in H^{1/2}(\Gamma)$, $u \in H^{1+r}(\Omega)$ and*

$$\|u\|_{1+r} \le C\|\tau\|_{1/2,\Gamma}. \tag{2.6}$$

Proof. It follows directly from classical regularity results (see [22]). □

In the previous proposition, $r = 1$ if Ω is a convex region and $r < \frac{\pi}{\theta}$, with θ being the largest interior angle of Ω, otherwise (see [23]). As a consequence, the eigenfunctions (u_n, ξ_n) of $\mathbf{T}$ belong to $H^{1+r}(\Omega) \times H^{1/2+r}(\Gamma)$ and satisfy

$$\|u_n\|_{1+r} \le C\|(u_n,\xi_n)\| \le C\|\xi_n\|_{0,\Gamma}. \tag{2.7}$$

Let $\{\mathcal{T}_h\}$ be a family of triangulations of Ω satisfying the following conditions:

- any two triangles in $\mathcal{T}_h$ share at most a vertex or an edge
- the minimal angle of all the triangles in $\mathcal{T}_h$ is bounded below by a positive constant which does not depend on h.

The index h denotes, as usual, the maximal mesh size of $\mathcal{T}_h$, namely, $h := \max_{T\in\mathcal{T}_h} h_T$, with h_T being the diameter of T. Let $\mathcal{E}_h$ denote the set of all the edges of triangles $T \in \mathcal{T}_h$. We split this set as follows: $\mathcal{E}_h = \mathcal{E}_h^I \cup \mathcal{E}_h^\Gamma$, with $\mathcal{E}_h^I := \{\ell \in \mathcal{E}_h : \ell \not\subset \Gamma\}$ and $\mathcal{E}_h^\Gamma := \{\ell \in \mathcal{E}_h : \ell \subset \Gamma\}$ being the sets of inner and boundary edges, respectively. Let $\mathcal{N}_h$ denote the set of vertices of the elements in $\mathcal{T}_h$ and $\mathcal{M}_h$ the set of midpoints of the edges in $\mathcal{E}_h$.

With the triangulation $\mathcal{T}_h$, we consider the lowest-order Crouzeix–Raviart finite element spaces:

$$V_h := \{v_h \in L^2(\Omega) : v_h|_T \in \mathcal{P}_1(T),\ \forall T \in \mathcal{T}_h,\ \text{and } v_h \text{ is continuous at } \mathcal{M}_h \cap \Omega\},$$

and we introduce the nonconforming spaces

$$\mathcal{L}_h := \{\mu_h \in L^2(\Gamma) : \mu_h|_\ell \in \mathcal{P}_1(\ell), \forall \ell \in \mathcal{E}_h^\Gamma\},$$
$$W_h := \{(v_h, \mu_h) \in V_h \times \mathcal{L}_h : \mu_h = v_h|_\Gamma\}.$$

We choose

$$\|(v_h, \mu_h)\|_h = \left(\sum_{T\in\mathcal{T}_h} |v_h|_{1,T}^2 + \|v_h\|_0^2 + \|\mu_h\|_{0,\Gamma}^2\right)^{1/2}$$

as a norm over the space $W + W_h$. Clearly,

$$W + W_h \hookrightarrow X,$$
$$\|v\| = \|v\|_h, \quad \forall v \in W.$$

Let a_h and b_h be the symmetric bilinear forms defined by

$$a_h(u, v) := \sum_{T\in\mathcal{T}_h} \int_T \alpha \nabla u \cdot \nabla v + \int_\Omega \beta u v, \quad \forall u, v \in V + V_h,$$

$$b_h(\xi, \mu) := b(\xi, \mu), \quad \forall \xi, \mu \in L^2(\Gamma).$$

Then, the discretization of the spectral problem (2.2) is given by

Find $\lambda_h \in \mathbb{R}$ and $(u_h, \xi_h) \in W_h$, $(u_h, \xi_h) \neq (0, 0)$, such that

$$a_h(u_h, v_h) = \lambda_h b(\xi_h, \mu_h), \quad \forall (v_h, \mu_h) \in W_h. \tag{2.8}$$

Next we consider the bounded linear operator $\mathbf{T}_h : X \to W + W_h$ defined by $\mathbf{T}_h(f, \tau) = (u_h, \xi_h) \in W_h$ and

$$a_h(u_h, v_h) + b_h(\xi_h, \mu_h) = b_h(\tau, \mu_h), \quad \forall (v_h, \mu_h) \in W_h. \tag{2.9}$$

Once more, the eigenvalue problem for $\mathbf{T}_h$ is equivalent to the spectral problem (2.8) in the sense that $(\lambda_h^*, (u_h, \xi_h))$ is an eigenpair of $\mathbf{T}_h$ if and only if $(\lambda_h, (u_h, \xi_h))$ is a solution of (2.8) with $\lambda_h = \frac{1}{\lambda_h^*} - 1$.

Optimal order error estimates are known for the eigenfunctions normalized with the norm $\|\cdot\|_h$. More precisely, let

$$(\hat{u}, \hat{\xi}) := \frac{(u, \xi)}{\|(u, \xi)\|_h} \quad \text{and} \quad (\hat{u}_h, \hat{\xi}_h) := \frac{(u_h, \xi_h)}{\|(u_h, \xi_h)\|_h}.$$

The following error estimates hold (see [4]).

Theorem 2.2. *There exist strictly positive constants C and h_0 such that, if $h \le h_0$, then*

$$\|(\hat{u}, \hat{\xi}) - (\hat{u}_h, \hat{\xi}_h)\|_h \le Ch^r,$$
$$|\lambda - \lambda_h| \le Ch^{2r},$$

with r being the regularity constants as in Lemma 2.1.

The following lemma shows that similar estimates are valid for the eigenfunctions normalized by

$$\|(u, \xi)\|_{0,\Gamma} = 1 \quad \text{and} \quad \|(u_h, \xi_h)\|_{0,\Gamma} = 1. \tag{2.10}$$

Lemma 2.3. *There exist strictly positive constants C and h_0 such that, if $h \le h_0$, then*

$$\|(u, \xi) - (u_h, \xi_h)\|_h \le Ch^r.$$

Proof. Let us denote

$$\gamma := (u, \xi), \qquad \gamma_h := (u_h, \xi_h), \qquad \hat{\gamma} := (\hat{u}, \hat{\xi}), \quad \text{and} \quad \hat{\gamma}_h := (\hat{u}_h, \hat{\xi}_h).$$

Straightforward computations yield

$$\begin{aligned} \|\gamma - \gamma_h\|_h &\le \|\gamma\|_h \|\hat{\gamma} - \hat{\gamma}_h\|_h + \|\hat{\gamma}_h\|_h | \|\gamma\|_h - \|\gamma_h\|_h| \\ &= \|\gamma\|_h \left(\|\hat{\gamma} - \hat{\gamma}_h\|_h + \left|1 - \frac{\|\gamma_h\|_h}{\|\gamma\|_h}\right|\right). \end{aligned}$$

Because of (2.10), $\|\hat{\gamma}\|_{0,\Gamma} = 1/\|\gamma\|_h$ and $\|\hat{\gamma}_h\|_{0,\Gamma} = 1/\|\gamma_h\|_h$. Then

$$\begin{aligned} \left|1 - \frac{\|\gamma_h\|_h}{\|\gamma\|_h}\right| = \left|\frac{\|\hat{\gamma}_h\|_{0,\Gamma} - \|\hat{\gamma}\|_{0,\Gamma}}{\|\hat{\gamma}_h\|_{0,\Gamma}}\right| &\le \|\gamma_h\|_h \|\hat{\gamma} - \hat{\gamma}_h\|_{0,\Gamma} \\ &\le (\|\gamma - \gamma_h\|_h + \|\gamma\|_h)\|\hat{\gamma} - \hat{\gamma}_h\|_h. \end{aligned}$$

Therefore,

$$\|\gamma - \gamma_h\|_h \le \|\gamma\|_h(1 + \|\gamma\|_h)\|\hat{\gamma} - \hat{\gamma}_h\|_h + \|\gamma\|_h \|\gamma - \gamma_h\|_h \|\hat{\gamma} - \hat{\gamma}_h\|_h.$$

By virtue of Theorem 2.2, there exists a constant C' such that $\|\hat{\gamma} - \hat{\gamma}_h\|_h \leq C'h^r$. Hence, we can choose h_0 such that $C'\|\gamma\|_h h_0^r < 1/2$, and we finally obtain

$$\|\gamma - \gamma_h\|_h \leq 2(1 + \|\gamma\|_h)\|\gamma\|_h\|\hat{\gamma} - \hat{\gamma}_h\|_h \leq Ch^r, \quad \forall h < h_0.$$

Thus, we conclude the proof. □

We end this paragraph by introducing some notation that we will use in the subsequent analysis.

Let ℓ be an interior edge shared by elements T_1 and T_2, i.e., $\ell = \partial T_1 \cap \partial T_2$. We define the jump of a function $v_h \in V_h$ on ℓ by

$$[v_h] := (v_h|_{T_2})|_\ell - (v_h|_{T_1})|_\ell.$$

Given an edge $\ell \in \mathcal{E}_h^I$, we choose a unit normal vector $\mathbf{n}_\ell$, pointing outwards T_2, and we set

$$[\![\nabla v_h]\!]_{\mathbf{n}} := \nabla v_h|_{T_2} \cdot \mathbf{n}_\ell - \nabla v_h|_{T_1} \cdot \mathbf{n}_\ell$$
$$[\![\nabla v_h]\!]_{\mathbf{t}} := \nabla v_h|_{T_2} \times \mathbf{n}_\ell - \nabla v_h|_{T_1} \times \mathbf{n}_\ell,$$

which correspond to the jumps of the normal and tangential derivatives of v_h across ℓ, respectively. Notice that these values are independent of the chosen direction of the normal vector $\mathbf{n}_\ell$. Moreover, if $\mathbf{n}_\ell = \left(n_\ell^1, n_\ell^2\right)$, we define the tangent on ℓ by $\mathbf{t}_\ell = \left(-n_\ell^2, n_\ell^1\right)$ and we write

$$[\![\nabla v_h]\!]_{\mathbf{t}} := \nabla v_h|_{T_2} \cdot \mathbf{t}_\ell - \nabla v_h|_{T_1} \cdot \mathbf{t}_\ell.$$

From now on, C will denote a constant independent of h and u, but not necessarily the same at each occurrence. This constant will also be independent of the particular approximated eigenvalue if we do not mention it.

3. A post-processing operator

In order to construct an a posteriori error estimator, we define a smoothing conforming procedure for the discontinuous approximations obtained by using the Crouzeix–Raviart method. To do this, we introduce a post-processing operator based on the ideas given by Schieweck in [24], which can be directly applied to our problem.

In what follows, we consider the family of meshes $\{\mathcal{T}_h\}$ and the notation introduced above. Let

$$V_h^c := \{v_h \in H^1(\Omega) : v_h|_T \in \mathcal{P}_1(T), \ \forall T \in \mathcal{T}_h\}$$

be the standard linear finite element space. Let $\{\psi_\nu : \nu \in \mathcal{N}_h\}$ be the canonical basis of this space consisting of continuous piecewise linear functions attaining the value 1 at $\mathbf{x}_\nu$ and vanishing at all other vertices in $\mathcal{T}_h$.

For each $\nu \in \mathcal{N}_h$, let

$$\omega_\nu := \{T \in \mathcal{T}_h : \nu \in T\}$$

be the set of all elements in $\mathcal{T}_h$ having a vertex in $\mathbf{x}_\nu$. Clearly, $\omega_\nu = \operatorname{supp} \psi_\nu$. We denote by $|\omega_\nu|$ the cardinality of ω_ν.

For any $v_h \in V_h$, we define a post-processing operator $\mathbf{R}_h : V_h \to V_h^c$ by

$$\mathbf{R}_h v_h := \sum_{\nu \in \mathcal{N}_h} \alpha_\nu \psi_\nu$$

with α_ν being the average of the values of v_h at the node ν given by

$$\alpha_\nu := \frac{1}{|\omega_\nu|} \sum_{T \in \omega_\nu} (v_h|_T)(\mathbf{x}_\nu).$$

Note that the coefficients α_ν can be easily computed by using the basis representation

$$(v_h|_T)(\mathbf{x}_\nu) = \sum_{\ell \subset \partial T} \beta_\ell \varphi_\ell(\mathbf{x}_\nu), \tag{3.1}$$

where $\{\varphi_\ell : \ell \in \mathcal{E}_h\}$ are the edge-oriented basis functions of the Crouzeix–Raviart space, i.e., piecewise linear functions which equal 1 at the midpoint of ℓ and vanishing at the midpoints of all the other edges $\tilde{\ell} \neq \ell$.

The function $\mathbf{R}_h v_h$ is uniquely defined by the values at the nodes of the partition given by Eq. (3.1). It is clear that $\mathbf{R}_h$ is a linear operator. Moreover, we have the following result.

Lemma 3.1. *The post-processing operator $\mathbf{R}_h$ satisfies*

- *for any $v_h \in V_h^c$, $\mathbf{R}_h v_h = v_h$,*
- *for each $T \in \mathcal{T}_h$, there exists a constant C, independent of h, such that*

$$\|\mathbf{R}_h v_h\|_{1,T} \leq C \left(\sum_{T \subset \theta_T} \|v_h\|_{1,T}^2 \right)^{1/2}, \quad \forall v_h \in V_h,$$

where θ_T is the union of T and a few neighboring elements.

Proof. The proof is essentially contained in those of Lemmas 2, 3 and 7 in [24]. □

Lemma 3.2. *For any $u_h \in V_h$, the following estimate holds*

$$\sum_{T\in\mathcal{T}_h} \|u_h - \mathbf{R}_h u_h\|_{1,T}^2 \le C \sum_{\ell\in\mathcal{E}_h^I} |\ell|^{-1} \|[u_h]\|_{0,\ell}^2,$$

where C is a positive constant only depending on the regularity of the mesh.

Proof. The proof is essentially contained in that of Theorem 2.2 in [25]. □

Lemma 3.3. *For each edge $\ell \in \mathcal{E}_h^I$ such that $\ell = T_1 \cap T_2$,*

$$|\ell|^{-1/2} \|[u_h]\|_{0,\ell} \le C \sum_{i=1}^{2} |u - u_h|_{1,T_i}.$$

Proof. First we observe that u is continuous because of Lemma 2.1. Consequently, we can write

$$\int_\ell [u_h]^2 = \int_\ell [u - u_h][u_h] = \int_\ell (u - u_h)|_{T_2}[u_h] - \int_\ell (u - u_h)|_{T_1}[u_h]. \tag{3.2}$$

Let us denote

$$z_h^+ := (u - u_h)|_{T_2} \qquad z_h^- := (u - u_h)|_{T_1}$$

for the sake of notational simplicity. Let P_ℓ denote the $L^2(\ell)$-projection of $H^{1/2}(\ell)$ onto the constants. Since $[u_h]$ is a linear function vanishing at the midpoint of ℓ, we have

$$\int_\ell z_h^+[u_h] = \int_\ell (z_h^+ - P_\ell z_h^+)[u_h] \le C|\ell|^{1/2} \|\nabla(u - u_h)\|_{0,T_2} \|[u_h]\|_{0,\ell}$$

where the last inequality follows from the standard estimates for P_ℓ.

The second term in the right hand side of Eq. (3.2) can be dealt with analogously. We obtain

$$\int_\ell z_h^-[u_h] = \int_\ell (z_h^- - P_\ell z_h^-)[u_h] \le C|\ell|^{1/2} \|\nabla(u - u_h)\|_{0,T_1} \|[u_h]\|_{0,\ell}.$$

Thus, we conclude the proof. □

Lemma 3.4. *Let $u \in V$ and $u_h \in V_h$ be the solutions of problems (2.2) and (2.8), respectively. Then, there exist a constant C, independent of h, such that*

$$\|u - \mathbf{R}_h u_h\|_{1,\Omega} \le Ch^r.$$

Proof. The estimate follows immediately from the triangle inequality, Lemmas 3.2, 3.3 and 2.3. □

Let us remark that there is no need of actually computing $\mathbf{R}_h u_h$ to calculate the error indicators defined below.

4. A posteriori error estimator

From the point of view of applications, it is highly important to be able to design meshes correctly refined as to reduce the approximation errors as much as possible with the lowest computational effort.

The standard approach to attain this goal is to compute an approximation of the eigenpair of interest on an initial coarse mesh $\mathcal{T}_h$ and to use the obtained approximate eigenpair to compute indicators of some local measure of the error for each element $T \in \mathcal{T}_h$ in order to know which of them should be further refined.

We choose the discrete norm

$$\|v_h\|_{1,h,\Omega}^2 := \|v_h\|_{0,\Omega}^2 + \sum_{T\in\mathcal{T}_h} |v_h|_{1,T}^2 \tag{4.1}$$

on $V + V_h$ to measure the error of the computed eigenfunction u_h.

In what follows, we will define error indicators η_T for each element $T \in \mathcal{T}_h$. These indicators are expected to satisfy the following properties:

1. *Reliability*: they should provide an upper estimate of the global error:

$$\|u - u_h\|_{1,h,\Omega} \le C \left(\sum_{T\in\mathcal{T}_h} \eta_T^2 \right)^{1/2} + \text{h.o.t.}, \tag{4.2}$$

where *h.o.t.* denotes higher order terms, i.e., terms which becomes negligible in comparison with the other ones in the estimate, when the mesh size becomes small.

2. *Efficiency*: they should provide lower error estimates, as local as possible, in order to indicate which elements should be effectively refined:
$$\eta_T \leq C\|u - u_h\|_{1,h,\omega_T} + \text{h.o.t.}, \tag{4.3}$$
where ω_T is the union of T and a few neighboring elements.
3. *Low computational cost*: the effective computation of η_T should be inexpensive in comparison with the overall computation of u_h and λ_h.

4.1. Definition of the error indicators. Reliability of the error estimates

To define these error indicators, we begin by providing some error equations which will be the starting point of our analysis.

We consider a particular eigenpair (λ, u) and its corresponding finite discrete approximation (λ_h, u_h). Let $e_h = u - u_h$ denote the error in the approximation. By the triangle inequality, we have
$$\|e_h\|_{1,h,\Omega} \leq \|u - \mathbf{R}_h u_h\|_{1,h,\Omega} + \|u_h - \mathbf{R}_h u_h\|_{1,h,\Omega},$$
where $\mathbf{R}_h u_h \in V_h^c$ is the post-processed finite element approximation associated to the non conforming solution u_h defined in the previous section.

From the definition of the discrete norms $\|\cdot\|_{1,h,\Omega}$, it follows that the bilinear forms a_h are continuous and coercive uniformly on $V + V_h$. Then, there are positive constants c_1 and c_2 such that
$$c_1\|u - \mathbf{R}_h u_h\|_{1,h,\Omega} \leq \sup_{v\in V} \frac{a_h(u - \mathbf{R}_h u_h, v)}{\|v\|_{1,\Omega}} \leq \sup_{v\in V} \frac{a_h(e_h, v)}{\|v\|_{1,\Omega}} + \sup_{v\in V} \frac{a_h(u_h - \mathbf{R}_h u_h, v)}{\|v\|_{1,\Omega}},$$
$$\frac{a_h(u_h - \mathbf{R}_h u_h, v)}{\|v\|_{1,\Omega}} \leq c_2\|u_h - \mathbf{R}_h u_h\|_{1,h,\Omega}.$$
Notice that c_1 and c_2 are actually the coerciveness and the continuity constants and depend only on the variable coefficients α and β (see the definition of problem (2.1)). In what follows, for the sake of simplicity, we shall restrict ourselves to the case $\alpha = \beta = 1$. The general case requires only technical modifications.

Then, with the previous assumption, we obtain
$$\|e_h\|_{1,h,\Omega} \leq \sup_{v\in V} \frac{a_h(e_h, v)}{\|v\|_{1,\Omega}} + 2\,\|u_h - \mathbf{R}_h u_h\|_{1,h,\Omega}. \tag{4.4}$$
For each ℓ of the triangulation, let
$$J_{\ell,\mathbf{t}} := \begin{cases} [\![\nabla u_h]\!]_{\mathbf{t}}, & \text{if } \ell \in \mathcal{E}^I \\ 0, & \text{if } \ell \in \mathcal{E}^\Gamma. \end{cases}$$
We define
$$\eta_{1,T} := \left(\frac{1}{24}\sum_{\ell\subset\partial T} |\ell|\,\|J_{\ell,\mathbf{t}}\|_{0,\ell}^2\right)^{1/2}$$
and
$$\eta_1 = \left(\sum_{T\in\mathcal{T}_h} \eta_{1,T}^2\right)^{1/2}.$$
The arguments of Section 3, in particular Lemma 3.2, yield the following upper bound for the second term in Eq. (4.4).

Lemma 4.1. *The following estimate holds:*
$$\|u_h - \mathbf{R}_h u_h\|_{1,h,\Omega} \leq C\eta_1,$$
where C only depends on the regularity of the mesh.

Proof. Let us denote by T^- and T^+ two adjacent triangles and by $\hat{\mathbf{x}}$ the midpoint of the common side ℓ. Let $P_i = (x_i, y_i)$, $i = 1, 2$, denote the endpoints of ℓ. Then,
$$x(\eta) = \frac{x_1 + x_2}{2} + \frac{x_2 - x_1}{2}\eta, \qquad y(\eta) = \frac{y_1 + y_2}{2} + \frac{y_2 - y_1}{2}\eta, \quad -1 \leq \eta \leq 1$$
is a parametric representation of ℓ.

Let us consider the natural extensions of the linear functions $u_h|_{T^-}$ and $u_h|_{T^+}$ to the larger set $T^- \cup T^+$. For notational convenience, we will denote these extended functions again by $u_h|_{T^-}$ and $u_h|_{T^+}$. In this situation,
$$u_h|_{T^\pm}(\eta) = u_h|_{T^\pm}(\hat{\mathbf{x}}_\ell) + \frac{|\ell|}{2}\,(\nabla u_h|_{T^\pm} \cdot \mathbf{t}_\ell)\,\eta, \quad -1 \leq \eta \leq 1. \tag{4.5}$$

Then, since $u_h|_{T^-}$ and $u_h|_{T^+}$ coincide at $\hat{\mathbf{x}}$,

$$[u_h](\eta) = \frac{|\ell|}{2}(\nabla u_h|_{T^+} \cdot \mathbf{t}_\ell - \nabla u_h|_{T^-} \cdot \mathbf{t}_\ell)\eta, \quad -1 \le \eta \le 1,$$

from which we have

$$|\ell|^{-1}\|[u_h]\|_{0,\ell}^2 = \frac{|\ell|}{12}\|[\![\nabla u_h]\!]_{\mathbf{t}}\|_\ell^2. \tag{4.6}$$

Then, summing up on all the edges $\ell \in \mathcal{E}_h^I$ and using Lemma 3.2, we conclude the proof. □

Regarding the first term in the right hand side of Eq. (4.4), the error indicator we are going to use is quite similar to the one derived for the standard linear elasticity equations (see [16]).

For each $T \in \mathcal{T}_h$, let

$$\eta_{2,T} := \left(|T|\,\|u_h\|_{0,T}^2 + \frac{1}{2}\sum_{\ell\subset\partial T}|\ell|\,\|J_{\ell,\mathbf{n}}\|_{0,\ell}^2\right)^{1/2},$$

where

$$J_{\ell,\mathbf{n}} := \begin{cases} [\![\nabla u_h]\!]_{\mathbf{n}}, & \text{if } \ell \in \mathcal{E}^I \\ 2(\nabla u_h \cdot \mathbf{n}_\ell - \lambda_h u_h)|_\ell, & \text{if } \ell \in \mathcal{E}^\Gamma. \end{cases}$$

Let

$$\eta_2 = \left(\sum_{T\in\mathcal{T}_h}\eta_{2,T}^2\right)^{1/2}.$$

Lemma 4.2. *The following estimate holds:*

$$\sup_{v\in V}\frac{a_h(e_h, v)}{\|v\|_{1,\Omega}} \le C\left(\eta_2 + \|\lambda u - \lambda_h u_h\|_{0,\Gamma}\right),$$

where C only depends on the regularity of the mesh.

Proof. Integrating by parts, we obtain from (2.2) and any $v \in V$

$$\begin{aligned} a_h(e_h, v) &= \lambda\int_\Gamma uv - \sum_{T\in\mathcal{T}_h}\left(\int_{\partial T}\nabla u_h\cdot\mathbf{n}v + \int_T u_h v\right) \\ &= \lambda\int_\Gamma uv - \sum_{T\in\mathcal{T}_h}\int_T u_h v - \sum_{\ell\in\mathcal{E}_h^I}\int_\ell [\![\nabla u_h]\!]_{\mathbf{n}}\, v - \sum_{\ell\in\mathcal{E}_h^\Gamma}\int_\ell(\nabla u_h\cdot\mathbf{n} - \lambda_h u_h)v - \lambda_h\int_\Gamma u_h v \\ &= \int_\Gamma(\lambda u - \lambda_h u_h)v - \sum_{T\in\mathcal{T}_h}\left(\int_T u_h v - \frac{1}{2}\sum_{\ell\subset\partial T}\int_\ell J_{\ell,\mathbf{n}}v\right). \end{aligned} \tag{4.7}$$

Let v^I be a continuous piecewise linear approximation of v such that

$$\|v - v^I\|_{0,T} \le C|v|_{1,\theta_T}|T|^{1/2}, \quad \forall T \in \mathcal{T}_h, \tag{4.8}$$

$$\|v - v^I\|_{0,\ell} \le C|v|_{1,\theta_T}|\ell|^{1/2}, \quad \forall \ell \subset \partial T, \tag{4.9}$$

where θ_T is the union of T and a few neighboring elements (see [26], also [27]). Since $v^I \in V \cap V_h$, we can use (2.2) and (2.8) to obtain

$$a_h(u - u_h, v^I) = \lambda\int_\Gamma uv^I - \lambda_h\int_\Gamma u_h v^I.$$

Then, straightforward computations yield

$$\begin{aligned} a_h(e_h, v) &= \int_\Gamma(\lambda u - \lambda_h u_h)v^I + a_h(e_h, v - v^I) \\ &= \int_\Gamma(\lambda u - \lambda_h u_h)v - \int_\Omega u_h(v - v^I) - \frac{1}{2}\sum_{T\in\mathcal{T}_h}\sum_{\ell\subset\partial T}\int_\ell J_{\ell,\mathbf{n}}(v - v^I). \end{aligned}$$

Therefore, the Cauchy–Schwarz inequality, estimates for the interpolation error (4.8) and (4.9), and the definition of the error estimator η lead to

$$a_h(e_h, v) \leq \|\lambda u - \lambda_h u_h\|_{0,\Gamma} \|v\|_{0,\Gamma} + C\eta_2 |v|_{1,\Omega},$$

from which we can conclude the proof. □

An estimator for the nonconforming error on element T is given by

$$\eta_T^2 := \eta_{1,T}^2 + \eta_{2,T}^2.$$

Let

$$\eta = \left(\sum_{T \in \mathcal{T}_h} \eta_T^2 \right)^{1/2}.$$

The following theorem states an upper estimate for the error in terms of η plus one more term which is proved to be of higher order.

Theorem 4.3. *The following estimate holds:*

$$\|u - u_h\|_{1,h,\Omega} \leq C \left(\eta + \|\lambda u - \lambda_h u_h\|_{0,\Gamma} \right).$$

Proof. It is an immediate consequence of Lemmas 4.1 and 4.2 and the definition of the error estimator η. □

Remark 4.4. The previous theorem shows that the global estimator provides an upper bound of the error in the discrete energy norm up to a term. As we will show below, this term is of higher order than the error estimate given by Theorem 2.2. See Section 6.

4.2. Efficiency of the error indicators

Our next goal is to show that the local error estimators η_T provide local lower bounds for the error on a neighborhood of T. Notice that the efficiency of the error indicator $\eta_{1,T}$ is an immediate consequence of (4.6) and Lemma 3.3. The following theorem yields this result.

Theorem 4.5. *For all $T \in \mathcal{T}_h$, let $\omega_T := \bigcup\{\widetilde{T} \in \mathcal{T}_h : \widetilde{T}$ shares an edge with $T\}$. There exists a positive constant C, depending only on the regularity of the elements of ω_T, such that*

$$\eta_{1,T} \leq C \|u - u_h\|_{1,h,\omega_T}.$$

The following lemmas provide an upper estimate for each term in the definition of $\eta_{2,T}$.

Lemma 4.6. *For each element $T \in \mathcal{T}_h$,*

$$|T|^{1/2} \|u_h\|_{0,T} \leq C \left(|T|^{1/2} \|u - u_h\|_{0,T} + |u - u_h|_{1,T} \right).$$

Proof. Let $\varphi_T := u_h b_T$, with b_T being a cubic bubble scaled as to satisfy

$$\int_T u_h \varphi_T = \int_T (u_h)^2 b_T = \|u_h\|_{0,T}^2 |T|.$$

Then, standard homogeneity arguments yield

$$\|\varphi_T\|_{0,T} \leq C \|u_h\|_{0,T} |T|,$$
$$|\varphi_T|_{1,T} \leq C \|u_h\|_{0,T} |T|^{1/2}.$$

Since φ_T vanishes on ∂T, we have

$$\int_T \nabla u_h \cdot \nabla \varphi_T = \int_{\partial T} \nabla u_h \cdot \mathbf{n} \varphi_T = 0,$$

whereas, extending φ_T by zero outside of T and using (2.2) with $\varphi_T \in V$, we obtain

$$\int_T \nabla u \cdot \nabla \varphi_T + \int_T u \varphi_T = 0.$$

So, as a consequence of all this, we have

$$\begin{aligned}\|u_h\|_{0,T}^2|T| &= \int_T u_h\varphi_T - \int_T u\varphi_T - \int_T \nabla u\cdot\nabla\varphi_T\\ &= -\int_T (u-u_h)\varphi_T - \int_T \nabla(u-u_h)\cdot\nabla\varphi_T\\ &\le \|u-u_h\|_{0,T}\|\varphi_T\|_{0,T} + \|\nabla(u-u_h)\|_{0,T}\|\nabla\varphi_T\|_{0,T}\\ &\le C\left(\|u-u_h\|_{0,T}|T| + \|\nabla(u-u_h)\|_{0,T}|T|^{1/2}\right)\|u_h\|_{0,T},\end{aligned}$$

which allows us to conclude the lemma. □

Lemma 4.7. *For each edge ℓ such that $\ell = T_1\cap T_2$, with $T_1, T_2\in\mathcal{T}_h$,*

$$|\ell|^{1/2}\,\|J_{\ell,\mathbf{n}}\|_{0,\ell} \le C\sum_{i=1}^{2}\left(|T_i|^{1/2}\,\|u-u_h\|_{0,T_i} + |u-u_h|_{1,T_i}\right).$$

Proof. For $J_{\ell,\mathbf{n}} = [\![\nabla u_h]\!]_{\mathbf{n}}$, let $\varphi_\ell\in H_0^1(T_1\cup T_2)$ be such that

$$\int_\ell J_{\ell,\mathbf{n}}\varphi_\ell = |\ell|\,\|J_{\ell,\mathbf{n}}\|_{0,\ell}^2,$$
$$\int_{T_i}\psi\varphi_\ell = 0\quad\forall\psi\in\mathcal{P}_1(T_i),\ i=1,2.$$

The function φ_ℓ can be taken as a continuous piecewise quadratic polynomial augmented with local bubbles of degree four. Standard homogeneity arguments yield

$$\|\varphi_\ell\|_{0,T_i}\le C|\ell|^{1/2}\,\|J_{\ell,\mathbf{n}}\|_{0,\ell}|T_i|^{1/2},\quad i=1,2$$
$$|\varphi_\ell|_{1,T_i}\le C|\ell|^{1/2}\,\|J_{\ell,\mathbf{n}}\|_{0,\ell},\quad i=1,2.$$

Extending φ_ℓ by zero outside of $T_1\cup T_2$ and using (2.2) with $\varphi_\ell\in V$, we obtain

$$\int_{T_1\cup T_2}\nabla u\cdot\nabla\varphi_\ell + \int_{T_1\cup T_2}u\varphi_\ell = 0$$

whereas, by integrating by parts on each triangle, we have

$$\sum_{i=1}^{2}\left(\int_{T_i}\nabla u_h\cdot\nabla\varphi_\ell + \int_{T_i}u_h\varphi_\ell\right) = \int_\ell \nabla u_h\cdot\mathbf{n}\varphi_\ell = \int_\ell J_{\ell,\mathbf{n}}\varphi_\ell.$$

Hence,

$$\begin{aligned}|\ell|\,\|J_{\ell,\mathbf{n}}\|_{0,\ell}^2 &= -\sum_{i=1}^{2}\left(\int_{T_i}\nabla(u-u_h)\cdot\nabla\varphi_\ell + \int_{T_i}(u-u_h)\varphi_\ell\right)\\ &\le \sum_{i=1}^{2}\left(|u-u_h|_{1,T_i}|\varphi_\ell|_{1,T_i} + \|u-u_h\|_{0,T_i}\|\varphi_\ell\|_{0,T_i}\right)\\ &\le C\sum_{i=1}^{2}\left(|u-u_h|_{1,T_i} + |T_i|^{1/2}\,\|u-u_h\|_{0,T_i}\right)|\ell|^{1/2}\,\|J_{\ell,\mathbf{n}}\|_{0,\ell}\end{aligned}$$

and the proof is concluded. □

Lemma 4.8. *For each ℓ of a triangle $T\in\mathcal{T}_h$ such that $\ell\subset\Gamma$,*

$$|\ell|^{1/2}\,\|J_{\ell,\mathbf{n}}\|_{0,\ell}\le C\left(\|u-u_h\|_{1,T} + |\ell|^{1/2}\,\|\lambda u - \lambda_h u_h\|_{0,\ell}\right).$$

Proof. It is simple to show that there is a unique $\varphi_T\in\mathcal{P}_3(T)$ vanishing on the two edges $\ell'\neq\ell$ of T and satisfying

$$\int_\ell\varphi_T\psi = -2\int_\ell J_{\ell,\mathbf{n}}\psi\quad\forall\psi\in\mathcal{P}_1(\ell),$$
$$\int_T\varphi_T = 0.$$

Furthermore, standard homogeneity arguments yield

$$\|\varphi_T\|_{0,T} \le C|\ell|^{1/2}\, \|J_{\ell,\mathbf{n}}\|_{0,\ell},$$
$$|\varphi_T|_{1,T} \le C|\ell|^{-1/2}\, \|J_{\ell,\mathbf{n}}\|_{0,\ell}.$$

Let us take $\psi = J_{\ell,\mathbf{n}} \in \mathcal{P}_1(\ell)$. Extending φ_T by zero outside of T and using the residual equation (4.7), we obtain

$$\begin{aligned}\|J_{\ell,\mathbf{n}}\|_{0,\ell}^2 &= \int_\ell (\lambda_h u_h - \nabla u_h \cdot \mathbf{n})\varphi_T \\ &= \int_T \nabla(u-u_h)\cdot\nabla\varphi_T + \int_T (u-u_h)\varphi_T - \int_\ell (\lambda u - \lambda_h u_h)\varphi_T + \int_T u_h\varphi_T \\ &\le |u-u_h|_{1,T}|\varphi_T|_{1,T} + \|u-u_h\|_{0,T}\|\varphi_T\|_{0,T} + \|\lambda u - \lambda_h u_h\|_{0,\ell}\|\varphi_T\|_{0,\ell} + \|u_h\|_{0,T}\|\varphi_T\|_{0,T} \\ &\le C\left(|\ell|^{-1/2}|u-u_h|_{1,T} + |\ell|^{1/2}\|u-u_h\|_{0,T} + \|\lambda u - \lambda_h u_h\|_{0,\ell} + |\ell|^{1/2}\|u_h\|_{0,T}\right)\|J_{\ell,\mathbf{n}}\|_{0,\ell},\end{aligned}$$

where we have used the estimates above for $\|\varphi_T\|_{0,T}$ and $|\varphi_T|_{1,T}$, and a standard local trace inequality to estimate $\|\varphi_T\|_{0,\ell}$. Notice that, since we are assuming regularity of the family of meshes $\{\mathcal{T}_h\}$, we have $\ell \sim |T|^{1/2}$. Therefore, we can write

$$|\ell|^{1/2}\|J_{\ell,\mathbf{n}}\|_{0,\ell} \le C\left(\|u-u_h\|_{1,T} + |\ell|^{1/2}\|\lambda u - \lambda_h u_h\|_{0,\ell} + |T|^{1/2}\|u_h\|_{0,T}\right)$$

and we conclude the proof by applying Lemma 4.6. □

The following lemma shows that the term $\|\lambda\, u - \lambda_h u_h\|_{0,\ell}|\ell|^{1/2}$ can be split into two parts: one is bounded by the local error and the other is of higher order than the local estimator.

Lemma 4.9. *For each ℓ of a triangle $T \in \mathcal{T}_h$ such that $\ell \subset \Gamma$,*

$$|\ell|^{1/2}\|\lambda u - \lambda_h u_h\|_{0,\ell} \le C\left(\lambda\|u-u_h\|_{1,T} + h^\sigma \eta_{2,T}\right),$$

with $\sigma > 0$.

Proof. By using the triangle inequality, we can write

$$\|\lambda u - \lambda_h u_h\|_{0,\ell}|\ell|^{1/2} \le \lambda\|u-u_h\|_{0,\ell}|\ell|^{1/2} + |\lambda-\lambda_h|\,\|u_h\|_{0,\ell}|\ell|^{1/2}. \tag{4.10}$$

Since $u - u_h \in H^1(T)$, a local trace inequality leads to

$$\|u-u_h\|_{0,\ell} \le C\left(|\ell|^{-1/2}\|u-u_h\|_{0,T} + |\ell|^{1/2}|u-u_h|_{1,T}\right).$$

Then, the first term in the right hand side of inequality (4.10) is bounded by

$$\|u-u_h\|_{0,\ell}|\ell|^{1/2} \le C\left(\|u-u_h\|_{0,T} + |\ell|\,|u-u_h|_{1,T}\right) \le C\|u-u_h\|_{1,T}.$$

In order to bound the second term, we can use again a local trace inequality and an inverse estimate to obtain

$$\|u_h\|_{0,\ell} \le C\left(|\ell|^{-1/2}\|u_h\|_{0,T} + |\ell|^{1/2}|u_h|_{1,T}\right) \le C|\ell|^{-1/2}\|u_h\|_{0,T}.$$

Therefore,

$$|\lambda-\lambda_h|\,\|u_h\|_{0,\ell}|\ell|^{1/2} \le C|\lambda-\lambda_h|\,\|u_h\|_{0,T} \le Ch^{2r-1}\eta_{2,T},$$

the last inequality because of Theorem 2.2 and the definition of $\eta_{2,T}$. Thus, the result follows from the fact that $r \in (1/2, 1]$. □

As a direct consequence of all the previous lemmas we have the following theorem.

Theorem 4.10. *For all $T \in \mathcal{T}_h$, let $\omega_T := \bigcup\{\widetilde{T} \in \mathcal{T}_h : \widetilde{T}$ shares an edge with $T\}$. There exists a positive constant C, depending only on the regularity of the elements of ω_T, such that*

1. *If T has only inner edges, then*

$$\eta_{2,T} \le C\|u-u_h\|_{1,h,\omega_T},$$

2. *If T has an edge lying on Γ, then*

$$\eta_{2,T} \le C(1+\lambda)\|u-u_h\|_{1,h,\omega_T} + \mathcal{O}(h^\sigma)\eta_{2,T}.$$

5. Another two a posteriori error estimators

The goal of this section is to define simpler estimators which also yield global upper and local lower bounds on the error of the approximations of the Steklov eigenvalue problem.

Let $\{\varphi_\ell : \ell \in \mathcal{E}_h\}$ be the natural basis of the Crouzeix–Raviart space associated with ℓ, i.e., the piecewise linear function attaining the value 1 at the midpoint of ℓ and vanishing at any other midpoint.

Lemma 5.1. *Let $T \in \mathcal{T}_h$ such that $\partial T \cap \Gamma = \emptyset$. There holds*

$$\frac{1}{3}|T|\,\|u_h\|^2_{0,T} \le \sum_{\ell\subset\partial T} |\ell|\,\|J_{\ell,\mathbf{n}}\|^2_{0,\ell} \le \frac{4}{3}|T|\,\|u_h\|^2_{0,T} + \frac{2}{3}\sum_{\widetilde{T}\subset\omega_T} |\widetilde{T}|\,\|u_h\|^2_{0,\widetilde{T}},$$

where $\omega_T := \bigcup\{\widetilde{T} \in \mathcal{T}_h : \widetilde{T}$ shares an edge with $T\}$.

Proof. Given an interior element $T \in \mathcal{T}_h$, we denote by T_i, $i = 1, 2, 3$ its neighbor triangles. Let ℓ be such that $\ell = T \cap T_i$. Then, from (2.8) we have

$$\begin{aligned}-\int_{T\cup T_i} u_h\varphi_\ell &= \int_T \nabla u_h\cdot\nabla\varphi_\ell + \int_{T_i}\nabla u_h\cdot\nabla\varphi_\ell = \int_{\partial T}\nabla u_h\cdot\mathbf{n}_\ell\varphi_\ell + \int_{\partial T_i}\nabla u_h\cdot\mathbf{n}_\ell\varphi_\ell\\ &= [\![\nabla u_h]\!]_{\mathbf{n}}\,|\ell| = J_{\ell,\mathbf{n}}|\ell|.\end{aligned}$$

Now, since $\|\varphi_\ell\|^2_{0,T_i} = \frac{|T_i|}{3}$,

$$\left|\int_{T\cup T_i} u_h\varphi_\ell\right| \le \frac{1}{\sqrt{3}}\left(\|u_h\|_{0,T}|T|^{1/2} + \|u_h\|_{0,T_i}|T_i|^{1/2}\right)$$

and then

$$\sum_{\ell\subset\partial T}|\ell|\,\|J_{\ell,\mathbf{n}}\|^2_{0,\ell} \le 2|T|\,\|u_h\|^2_{0,T} + \frac{2}{3}\sum_{i=1}^{3}|T_i|\,\|u_h\|^2_{0,T_i}.$$

On the other hand, a straightforward computation yields

$$\int_{T\cup T_i} u_h\varphi_\ell = \frac{|T|+|T_i|}{3}u_h(\hat{\mathbf{x}}_\ell),$$
$$\int_T u_h^2 = \frac{|T|}{3}\sum_{\ell\subset\partial T}u_h^2(\hat{\mathbf{x}}_\ell),$$

where $\hat{\mathbf{x}}_\ell$ denotes the midpoint of the edge ℓ. Therefore,

$$\sum_{\ell\subset\partial T}|\ell|\,\|J_{\ell,\mathbf{n}}\|^2_{0,\ell} = \sum_{i=1}^{3}\frac{(|T|+|T_i|)^2}{9}u_h^2(\hat{\mathbf{x}}_{\ell_i}) \ge \frac{|T|^2}{9}\sum_{\ell\subset\partial T}u_h^2(\hat{\mathbf{x}}_\ell) = \frac{|T|}{3}\|u_h\|^2_{0,T}.$$

So, combining the inequalities above, we conclude the proof. □

Lemma 5.2. *Let $T \in \mathcal{T}_h$ such that $\partial T \cap \Gamma = \ell$. There hold*

$$\frac{4}{3}|T|\,\|u_h\|^2_{0,T} \le \sum_{\tilde{\ell}\subset\partial T}|\tilde{\ell}|\,\|J_{\tilde{\ell},\mathbf{n}}\|^2_{0,\tilde{\ell}},$$

$$|\ell|\,\|J_{\ell,\mathbf{n}}\|^2_{0,\ell} \le \frac{4}{3}|T|\,\|u_h\|^2_{0,T} + \frac{\lambda_h^2}{12}|\ell|^2\|\nabla u_h\cdot\mathbf{t}_\ell\|^2_{0,\ell}.$$

Proof. Let ℓ be such that $\ell = \partial T \cap \Gamma$. Then, from (2.8), we have

$$\begin{aligned}\int_T u_h\varphi_\ell &= \int_\ell \lambda_h u_h\varphi_\ell - \int_T\nabla u_h\cdot\nabla\varphi_\ell = \int_\ell(\lambda_h u_h - \nabla u_h\cdot\mathbf{n}_\ell)\varphi_\ell\\ &\le \|\lambda_h u_h - \nabla u_h\cdot\mathbf{n}_\ell\|_{0,\ell}\|\varphi_\ell\|_{0,\ell} = \frac{1}{2}\|J_{\ell,\mathbf{n}}\|_{0,\ell}\,\|\varphi_\ell\|_{0,\ell}.\end{aligned}$$

Then, since $\|\varphi_\ell\|^2_{0,\ell} = |\ell|$ and $\int_T u_h\varphi_\ell = \frac{|T|}{3}u_h(\hat{\mathbf{x}}_\ell)$,

$$\frac{|T|}{3}u_h(\hat{\mathbf{x}}_\ell) \le \|J_{\ell,\mathbf{n}}\|_{0,\ell}|\ell|^{1/2}.$$

Let us now denote by T_i, $i = 1, 2$, the two triangles sharing an edge with T. Let ℓ_i, $i = 1, 2$, denote the edge in common. Proceeding as in the proof of the previous lemma, we can write

$$|\ell_i|\,\|J_{\ell,\mathbf{n}}\|^2_{0,\ell_i} = \frac{(|T|+|T_i|)^2}{9}u_h^2(\hat{\mathbf{x}}_{\ell_i}) \ge \frac{|T|^2}{9}u_h^2(\mathbf{x}_{\ell_i}),$$

where $\hat{\mathbf{x}}_{\ell_i}$ denotes the midpoint of the edge ℓ_i. Consequently

$$\frac{|T|}{3}\|u_h\|_{0,T}^2 = \frac{|T|^2}{9}\sum_{\hat{\ell}\subset\partial T} u_h^2(\hat{\mathbf{x}}_{\hat{\ell}}) \le \sum_{\hat{\ell}\subset\partial T} |\hat{\ell}|\,\|J_{\hat{\ell},\mathbf{n}}\|_{0,\hat{\ell}}^2.$$

This established the first estimate of the lemma. In order to prove the second one, we need to compute $\|J_{\ell,\mathbf{n}}\|_{0,\ell}$ when $\ell\subset\Gamma$. Let $\ell\in\mathcal{E}_h^{\Gamma}$, we have

$$\int_\ell (\lambda_h u_h - \nabla u_h\cdot\mathbf{n}_\ell)^2 = \lambda_h^2\int_\ell u_h^2 - \left(2\lambda_h u_h(\hat{\mathbf{x}}_\ell)\nabla u_h\cdot\mathbf{n}_\ell - (\nabla u_h\cdot\mathbf{n}_\ell)^2\right)|\ell|.$$

Now, some simple calculations show that for any $u_h\in V_h$,

$$\int_\ell u_h^2 = u_h^2(\hat{\mathbf{x}}_\ell)|\ell| + \frac{1}{12}(\nabla u_h\cdot\mathbf{t}_\ell)^2|\ell|^3.$$

Hence,

$$\|J_{\ell,\mathbf{n}}\|_{0,\ell}^2 = 4\left(|\lambda_h u_h(\hat{\mathbf{x}}_\ell) - \nabla u_h\cdot\mathbf{n}_\ell|^2|\ell| + \frac{1}{12}\|\nabla u_h\cdot\mathbf{t}_\ell\|_{0,\ell}^2|\ell|^2\right).$$

On the other hand, since $\varphi_\ell = 1$ on ℓ and $\|\varphi_\ell\|_{0,T}^2 = \frac{|T|}{3}$,

$$\int_T u_h\varphi_\ell = \int_\ell (\lambda_h u_h - \nabla u_h\cdot\mathbf{n}_\ell)\varphi_\ell = \left(\lambda_h u_h(\hat{\mathbf{x}}_\ell) - \nabla u_h\cdot\mathbf{n}_\ell\right)|\ell|,$$

$$\left|\int_T u_h\varphi_\ell\right| \le \sqrt{\frac{|T|}{3}}\|u_h\|_{0,T}.$$

Therefore, combining the estimates above we obtain

$$\frac{|T|}{3}\|u_h\|_{0,T}^2 \ge \left|\lambda_h u_h(\hat{\mathbf{x}}_\ell) - \nabla u_h\cdot\mathbf{n}_\ell\right|^2|\ell| = \frac{1}{4}\|J_{\ell,\mathbf{n}}\|_{0,\ell}^2 - \frac{1}{12}(\nabla u_h\cdot\mathbf{t}_\ell)^2|\ell|^3$$

which allows us to conclude the proof. □

Lemmas 5.1 and 5.2 above imply that we may omit the volumetric contribution in the expression of $\eta_{2,T}$ and define a simpler error estimator based only on the jumps of the normal and tangential discrete derivatives of the approximate solution. As we show below, this new estimator is equivalent to the approximate eigenfunction error up to higher order terms.

Let

$$\widehat{\eta}_{2,T}^2 := \frac{1}{2}\sum_{\ell\subset\partial T}|\ell|\,\|J_{\ell,\mathbf{n}}\|_{0,\ell}^2$$

and the corresponding global error estimator

$$\widehat{\eta} := \left(\sum_{T\in\mathcal{T}_h}\eta_{1,T}^2 + \widehat{\eta}_{2,T}^2\right)^{1/2}.$$

The following theorems show that this estimator is globally reliable and locally efficient up to higher order terms.

Theorem 5.3. *There exists a positive constant C, depending only on the regularity of the mesh, such that*

$$\|u-u_h\|_{1,h,\Omega} \le C\left(\widehat{\eta} + \|\lambda u - \lambda_h u_h\|_{0,\Gamma}\right).$$

Proof. Lemmas 5.1 and 5.2 yield $\eta_T \le C\widehat{\eta}_T$ for all $T\in\mathcal{T}_h$, with C being a constant depending only of the regularity of the mesh. Then, the result follows directly from Theorem 4.3. □

Theorem 5.4. *For all $T\in\mathcal{T}_h$, let $\omega_T := \bigcup\{\widetilde{T}\in\mathcal{T}_h : \widetilde{T}$ shares an edge with $T\}$. There exists a positive constant C, depending only on the regularity of the elements of ω_T, such that*

1. *If T has only inner edges, then*

$$\widehat{\eta}_T \le C\|u-u_h\|_{1,h,\omega_T},$$

2. *If T has an edge lying on Γ, then*

$$\widehat{\eta}_T \le C(1+\lambda)\|u-u_h\|_{1,h,\omega_T} + \mathcal{O}(h^\sigma)\widehat{\eta}_{2,T}.$$

Proof. It is obvious that $\widehat{\eta}_T \le \eta_T$ for all $T\in\mathcal{T}_h$. Then, the results follow directly from Theorems 4.5 and 4.10 and Lemma 5.2. □

Another error estimator can be defined by observing that the terms corresponding to the jumps of the discrete flux across element boundaries are dominated by the volumetric ones up to higher order terms. In fact, due to Lemmas 5.1 and 5.2, we can define the following local estimator

$$\widetilde{\eta}_{2,T}^2 := |T|\,\|u_h\|_{0,T}^2$$

and the corresponding global error estimator

$$\widetilde{\eta} := \left(\sum_{T\in\mathcal{T}_h} \eta_{1,T}^2 + \widetilde{\eta}_{2,T}^2\right)^{1/2}.$$

The following theorem shows that this estimator yields a global upper bound on the error measured in the V_h-norm.

Theorem 5.5. *There exists a positive constant C, depending only on the regularity of the mesh, such that*

$$\|u-u_h\|_{1,h,\Omega} \le C\left(\widetilde{\eta} + \|\lambda u - \lambda_h u_h\|_{0,\Gamma} + \lambda_h\left(\sum_{\ell\in\mathcal{E}_h^\Gamma} |\ell|\,\|u_h - u_h(\mathbf{x}_\ell)\|_{0,\ell}^2\right)^{1/2}\right)$$

where $\mathbf{x}_\ell$ denotes the midpoint of the boundary edge ℓ.

Proof. For $T\in\mathcal{T}_h$ such that $\partial T\cap\Gamma=\emptyset$, Lemma 5.1 yields directly

$$\eta_T \le C\widetilde{\eta}_T.$$

For $\ell\in\mathcal{E}_h^\Gamma$, let T be the triangle in $\mathcal{T}_h$ such that $\ell\subset\partial T$. Thanks to Lemma 5.2 and expression (4.5), we can write

$$\eta_T^2 \le C\widetilde{\eta}_T^2 + \frac{\lambda_h^2}{12}|\ell|^2\,\|\nabla u_h\cdot\mathbf{t}_\ell\|_{0,\ell}^2 = C\widetilde{\eta}_T^2 + \lambda_h^2|\ell|\,\|u_h - u_h(\mathbf{x}_\ell)\|_{0,\ell}^2.$$

Thus, summing up on all the triangles $T\in\mathcal{T}_h$ and using Theorem 4.3, we conclude the proof. □

The analogue of Theorem 5.4 is the following theorem.

Theorem 5.6. *For all $T\in\mathcal{T}_h$, let $\omega_T := \bigcup\{\widetilde{T}\in\mathcal{T}_h : \widetilde{T}$ shares an edge with $T\}$. There exists a positive constant C, depending only on the regularity of the elements of ω_T, such that*

$$\widetilde{\eta}_T \le C\|u-u_h\|_{1,h,\omega_T}.$$

Proof. The proof follows immediately from Theorem 4.5 and Lemma 4.6. □

Remark 5.7. The final form of estimator $\widetilde{\eta}$ resembles those derived in [5,15] for the Crouzeix–Raviart approximation of source problems.

6. Optimal a priori error estimate for $\|u-u_h\|_{0,\Gamma}$ and terms of higher order

The first goal of this section is to prove an estimate of higher order for the error $\|u-u_h\|_{0,\Gamma}$. We do this by using the abstract spectral approximation theory given in [4]. We preserve the notation of Section 2.

We begin by defining the Steklov–Poincaré operator associated to problem (2.3), i.e., given $\tau\in L^2(\Gamma)$, let u be the unique solution in V of the following problem

$$\int_\Omega \alpha\nabla u\cdot\nabla v + \int_\Omega \beta uv + \int_\Gamma uv = \int_\Gamma \tau v, \quad \forall v\in V. \tag{6.1}$$

Then, we define $\mathbf{B}\tau := u|_\Gamma$ and we note that $\mathbf{B}$ is a bounded linear operator from $L^2(\Gamma)$ into itself. From the definitions of $\mathbf{B}$ and $\mathbf{T}$, (2.3) and (6.1), we can establish

$$\mathbf{B}\tau = \mathbf{C}\circ\mathbf{T}(f,\tau)$$

with $\mathbf{C}$ being the operator defined by

$$\mathbf{C} : X \to L^2(\Gamma)$$
$$(u,\xi)\mapsto \xi.$$

In a similar way, we can define the approximate operator $\mathbf{B}_h : L^2(\Gamma)\to L^2(\Gamma)$ by $\mathbf{B}_h\tau = u_h|_\Gamma$, where u_h is the unique solution in V_h of the discrete problem

$$\int_\Omega \alpha\nabla u_h\cdot\nabla v_h + \int_\Omega \beta u_h v_h + \int_\Gamma u_h v_h = \int_\Gamma \tau v_h, \quad \forall v_h\in V_h. \tag{6.2}$$

Then, we obtain

$$\mathbf{B}_h\tau = \mathbf{C}\circ\mathbf{T}_h(f,\tau).$$

Two properties have to be proved to apply the theory in [4] in order to conclude convergence of the spectral approximation and non existence of spurious modes. The first one means that the operators $\mathbf{B}_h$ provide good approximations of $\mathbf{B}$ when applied to sources (f, τ) in the discrete space. The second one means that the used finite element spaces provide good approximations of the eigenfunctions of $\mathbf{B}$.

Lemma 6.1. *For $(f, \tau) \in W_h$, there exists a positive constant C such that*

$$\|(\mathbf{B} - \mathbf{B}_h)\tau\|_{0,\Gamma} \le C\, h^{r/2}\|\tau\|_{0,\Gamma},$$

with r being the regularity constants as in Lemma 2.1.

Proof. It is a direct consequence of Theorem 4.3 in [4] and the relations defining $\mathbf{B}$ and $\mathbf{B}_h$. $\square$

Lemma 6.2. *For each eigenfunction u of $\mathbf{B}$ associated with λ there exists a strictly positive constant C such that*

$$\inf_{u_h \in V_h} \|u - u_h\|_{0,\Gamma} \le C h^{r+1/2}\|u\|_{1+r}.$$

Proof. It is a direct consequence of Theorem 4.2 in [4] and the relations defining $\mathbf{B}$ and $\mathbf{B}_h$. $\square$

Let λ be a fixed eigenvalue of the operator $\mathbf{T}$ and $\mathcal{S}_\lambda$ its corresponding associated eigenspace. When the source term belongs to $\mathcal{S}_\lambda$, the order of the approximation is larger. In fact, we have the following result.

Lemma 6.3. *For $(f, \tau) \in \mathcal{S}_\lambda$, the following estimate holds*

$$\|(\mathbf{B} - \mathbf{B}_h)\tau\|_{0,\Gamma} \le C h^{3r/2}\|\tau\|_{0,\Gamma},$$

with r as in Lemma 2.1.

Proof. Given $(f, \tau) \in \mathcal{S}_\lambda$, let $(u, \xi) = \mathbf{T}(f, \tau)$, $(u_h, \xi_h) = \mathbf{T}_h(f, \tau)$ and $e_h = u - u_h$. Since $u_h \in V_h$, the error function e_h is discontinuous. We denote by $[e_h]$ the jump of this function across an edge $\ell \in \mathcal{E}_h^I$.

We use a duality argument based on the following auxiliary problem:

$$\begin{cases} -\operatorname{div}(\alpha\nabla\varphi) + \beta\varphi = 0 & \text{in } \Omega, \\ \alpha\dfrac{\partial\varphi}{\partial\mathbf{n}} + \varphi = e_h & \text{on } \Gamma. \end{cases} \tag{6.3}$$

Since $e_h|_\Gamma \in H^\epsilon(\Gamma)$, with $\epsilon \in (0, 1/2)$, the results of Lemma 2.1 yields $\varphi \in H^{3/2+\epsilon}(\Omega) \subset H^{1+r/2}(\Omega)$ and

$$\|\varphi\|_{1+r/2} \le C\|e_h\|_{0,\Gamma}. \tag{6.4}$$

By using Eqs. (6.3), we have

$$\begin{aligned} \int_\Gamma e_h^2 &= \int_\Gamma \left(\alpha\frac{\partial\varphi}{\partial\mathbf{n}} + \varphi\right) e_h = \sum_{\ell\in\mathcal{E}_h^\Gamma} \int_\ell \left(\alpha\frac{\partial\varphi}{\partial\mathbf{n}} + \varphi\right) e_h \\ &= -\sum_{\ell\in\mathcal{E}_h^I} \int_\ell \alpha\frac{\partial\varphi}{\partial\mathbf{n}}[e_h] + \sum_{T\in\mathcal{T}_h}\left(\int_T \operatorname{div}(\alpha\nabla\varphi)e_h + \int_T \alpha\nabla\varphi\cdot\nabla e_h\right) + \int_\Gamma \varphi e_h \\ &= -\sum_{\ell\in\mathcal{E}_h^I} \int_\ell \alpha\frac{\partial\varphi}{\partial\mathbf{n}}[e_h] + \int_\Omega \beta\varphi e_h + \sum_{T\in\mathcal{T}_h}\int_T \alpha\nabla\varphi\cdot\nabla e_h + \int_\Gamma \varphi e_h, \end{aligned} \tag{6.5}$$

where we have used the equality

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T} \alpha\frac{\partial\varphi}{\partial\mathbf{n}} e_h = \sum_{\ell\in\mathcal{E}_h^\Gamma}\int_\ell \alpha\frac{\partial\varphi}{\partial\mathbf{n}} e_h + \sum_{\ell\in\mathcal{E}_h^I}\int_\ell \alpha\frac{\partial\varphi}{\partial\mathbf{n}}[e_h]$$

and integration by parts. Notice that the regularity of φ implies that $\frac{\partial\varphi}{\partial\mathbf{n}}$ is well defined as an $L^2(\Gamma)$-function.

Let φ^I be the piecewise linear Lagrange interpolation of φ. Since $\varphi^I \in V \cap V_h$, we can use problems (6.1) and (6.2) to obtain the following residual equation

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T} \alpha\nabla e_h\cdot\nabla\varphi^I + \int_\Omega \beta e_h\varphi^I + \int_\Gamma e_h\varphi^I = 0. \tag{6.6}$$

Therefore, subtracting Eqs. (6.5) and (6.6) we obtain

$$\|e_h\|_{0,\Gamma}^2 = \sum_{T\in\mathcal{T}_h}\int_T \alpha\nabla(\varphi-\varphi^I)\cdot\nabla e_h + \int_\Omega \beta(\varphi-\varphi^I)e_h + \int_\Gamma(\varphi-\varphi^I)e_h - \sum_{\ell\in\mathcal{E}_h^I}\int_\ell \alpha\frac{\partial\varphi}{\partial\mathbf{n}}[e_h].$$

We are going to estimate the terms appearing in the right hand side of the equation above separately. Let us recall here that the coefficients α and β are assumed to be bounded above and below by positive constants.

• First term:

By using the Cauchy–Schwarz inequality and standard error estimates for the Lagrange interpolant φ^I, we have

$$\left|\int_T \alpha\nabla(\varphi-\varphi^I)\cdot\nabla e_h\right| \le \|\alpha\nabla(\varphi-\varphi^I)\|_{0,T}\|\nabla e_h\|_{0,T} \le Ch^{r/2}\|\varphi\|_{1+r/2,T}\|e_h\|_{1,T}.$$

Then, summing up on all the triangles $T\in\mathcal{T}_h$ and using estimate (6.4) and Theorem 4.4 in [4], we conclude

$$\left|\sum_{T\in\mathcal{T}_h}\int_T \alpha\nabla(\varphi-\varphi^I)\cdot\nabla e_h\right| \le Ch^{3r/2}\|e_h\|_{0,\Gamma}\|\tau\|_{0,\Gamma}.$$

• Second term:

Proceeding exactly as in the proof of the previous estimate, we obtain

$$\left|\sum_{T\in\mathcal{T}_h}\int_T \beta(\varphi-\varphi^I)e_h\right| \le Ch^{1+3r/2}\|e_h\|_{0,\Gamma}\|\tau\|_{0,\Gamma}.$$

• Third term:

By using a trace theorem and standard error estimates for the Lagrange interpolant φ^I, we have

$$\|\varphi-\varphi^I\|_{0,\ell} \le C\left(h^{-1/2}\|\varphi-\varphi^I\|_{0,T} + h^{1/2}|\varphi-\varphi^I|_{1,T}\right) \le Ch^{(1+r)/2}\|\varphi\|_{1+r/2,T}.$$

Then, summing up on all the edges $\ell\in\mathcal{E}_h^\Gamma$ and using estimate (6.4) and Theorem 4.4 in [4], we can write

$$\left|\int_\Gamma(\varphi-\varphi^I)e_h\right| \le \|\varphi-\varphi^I\|_{0,\Gamma}\|e_h\|_{0,\Gamma} \le Ch^{(1+3r)/2}\|e_h\|_{0,\Gamma}\|\tau\|_{0,\Gamma}.$$

• Fourth term:

Let P_ℓ denote the $L^2(\ell)$-projection of $H^\epsilon(\ell)$ onto the constants. For an edge $\ell\in\mathcal{E}_h^I$, let $T_1,T_2\in\mathcal{T}_h$ be such that $T_1\cap T_2=\ell$. Since u is continuous and $[u_h]$ is a linear function vanishing at the midpoint of ℓ, we have

$$\begin{aligned}\left|\int_\ell \alpha\frac{\partial\varphi}{\partial\mathbf{n}}[e_h]\right| &= \left|\int_\ell\left(\alpha\frac{\partial\varphi}{\partial\mathbf{n}} - P_\ell\left(\alpha\frac{\partial u}{\partial\mathbf{n}}\right)\right)[e_h]\right|\\ &= \left|\int_\ell\left(\alpha\frac{\partial\varphi}{\partial\mathbf{n}} - P_\ell\left(\alpha\frac{\partial\varphi}{\partial\mathbf{n}}\right)\right)(e_h|_{T_1}) - \int_\ell\alpha\left(\frac{\partial\varphi}{\partial\mathbf{n}} - P_\ell\left(\alpha\frac{\partial\varphi}{\partial\mathbf{n}}\right)\right)(e_h|_{T_2})\right|\\ &\le \sum_{i=1,2}\left|\int_\ell\left(\alpha\frac{\partial\varphi}{\partial\mathbf{n}} - P_\ell\left(\alpha\frac{\partial\varphi}{\partial\mathbf{n}}\right)\right)\left((e_h|_{T_i}) - P_\ell(e_h|_{T_i})\right)\right|.\end{aligned}$$

Let P_T denote the $L^2(T)$-projection of $H^{\epsilon+1/2}(T)$ onto the constants. By using a trace theorem and standard error estimates for P_T, we can write

$$\begin{aligned}\left|\int_\ell \alpha\frac{\partial\varphi}{\partial\mathbf{n}}[e_h]\right| &\le \sum_{i=1,2}\|\alpha\nabla\varphi\cdot\mathbf{n} - P_T(\alpha\nabla\varphi\cdot\mathbf{n})\|_{0,\ell}\,\|(e_h|_{T_i}) - P_T(e_h|_{T_i})\|_{0,\ell}\\ &\le C\sum_{i=1,2}\left(h^{r/2-1/2}\|\nabla\varphi\|_{r/2,T_i}\right)\left(h^{1/2}\|e_h\|_{1,T_i}\right).\end{aligned}$$

Thus, summing up on all the edges $\ell\in\mathcal{E}_h^I$ and using estimate (6.4) and Theorem 4.4 in [4], we obtain

$$\left|\sum_{\ell\in\mathcal{E}_h^I}\int_\ell\alpha\frac{\partial\varphi}{\partial\mathbf{n}}[e_h]\right| \le Ch^{3r/2}\|e_h\|_{0,\Gamma}\|\tau\|_{0,\Gamma}.$$

Then, by combining all these estimates, we conclude the proof. □

Let m denote the multiplicity of the eigenvalue λ of $\mathbf{B}$ and $\mathcal{S}_\lambda$ the corresponding eigenspace as above. Since $\|\mathbf{B}-\mathbf{B}_h\|_{0,\Gamma}\to 0$ as $h\to 0$, there exists m eigenvalues of $\mathbf{B}_h$, $\lambda_{1h},\lambda_{2h},\dots,\lambda_{mh}$, repeated according to their respective multiplicity, converging to λ (see [4]). Let $\mathcal{S}_{\lambda_h}$ be the direct sum of the corresponding associated eigenspaces. The following theorem is a consequence of Theorem 3.12 in [4] and Lemmas 6.2 and 6.3.

Theorem 6.4. *There exist strictly positive constants C and h_0 such that, if $h < h_0$, then*

1. *For each $u_h \in \mathcal{S}_{\lambda_h}$, with $\|u_h\|_{0,\Gamma} = 1$, $\operatorname{dist}(u_h, \mathcal{S}_\lambda) \le Ch^{3r/2}$,*
2. *For each $u \in \mathcal{S}_\lambda$, with $\|u\|_{0,\Gamma} = 1$, $\operatorname{dist}(u, \mathcal{S}_{\lambda_h}) \le Ch^{3r/2}$,*

with r as in Lemma 2.1.

We further conclude that if u is a unit eigenfunction of $\mathbf{B}$ corresponding to λ then there is a unit eigenfunction u_h of $\mathbf{B}_h$ such that

$$\|u - u_h\|_{0,\Gamma} \le Ch^{3r/2}. \tag{6.7}$$

The error estimate established in Theorems 4.3 and 5.3 or 5.5 shows that the proposed global estimator provides an upper bound of the error in the broken energy norm up to a multiplicative constant and some additional terms. We are now in position to prove that these terms are of higher order, i.e., they are asymptotically negligible with respect to $\|u - u_h\|_{1,h,\Omega}$. In fact, we have the following result.

Lemma 6.5. *There exists a constant C independent of h such that*

$$\|\lambda u - \lambda_h u_h\|_{0,\Gamma} \le Ch^{3r/2}, \tag{6.8}$$

$$\lambda_h^2 \sum_{\ell \in \mathcal{E}_h^\Gamma} |\ell| \, \|u_h - u_h(\mathbf{x}_\ell)\|_{0,\ell}^2 \le Ch^{3r+1}, \tag{6.9}$$

with r as in Lemma 2.1.

Proof. We begin by observing that the first inequality follows directly from Theorem 2.2 and estimate (6.7).

Now, let ℓ be a boundary edge. Let P_ℓ denote the $L^2(\ell)$-projection of $H^{r+1/2}(\ell)$ onto the constants. Since $u_h|_\ell$ is a linear function, $P_\ell u_h|_\ell = u_h(\mathbf{x}_\ell)$, with $\mathbf{x}_\ell$ being the midpoint of ℓ. We immediately have

$$\lambda_h(u_h - u_h(\mathbf{x}_\ell)) = (\lambda_h u_h - \lambda u) + \lambda(u - P_\ell u) + P_\ell(\lambda u - \lambda_h u_h)$$

from which we obtain

$$\lambda_h \|u_h - u_h(\mathbf{x}_\ell)\|_{0,\ell} \le C \left(\|\lambda_h u_h - \lambda u\|_{0,\ell} + \lambda \|u - P_\ell u\|_{0,\ell} \right).$$

Let T be the triangle in $\mathcal{T}_h$ such that $\ell \subset \partial T$. Let P_T denote the $L^2(T)$-projection of $H^{r+1}(T)$ onto the constants. By using a suitable trace theorem and standard error estimates for P_T, we have

$$\|u - P_\ell u\|_{0,\ell} \le C \left(h^{-1/2} \|u - P_T u\|_{0,T} + h^{1/2} \|\nabla(u - P_T u)\|_{0,T} \right) \le Ch^{r+1/2} \|u\|_{1+r,T},$$

the last inequality being true because Lemma 2.1 and the fact that u is an eigenfunction of problem (2.2). Then, summing up on all the edges $\ell \in \mathcal{E}_h^\Gamma$, inequality (6.9) follows from estimate (2.7) and inequality (6.8). $\square$

Remark 6.6. The generic constant C appearing in the estimates of the theorem above depends on λ. Although it is not difficult to trace this dependence, we prefer not state it explicitly.

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