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A regularity condition and a limit theorem for Harris ergodic Markov chains

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Abstract

Let $(X_n)_{n\geqslant 0}$ be a Harris ergodic Markov chain and f be a real function on its state space. Consider the block sums $\zeta(i)$ for $|f|$, $i \geqslant 1$, between consecutive visits to the atom given by the splitting technique of Nummelin. A regularity condition on the invariant probability measure π and a drift property are introduced and proven to characterize the finiteness of the third moment of $\zeta(i)$. This is applied to obtain versions of an almost sure invariance principle for the partial sums of $(f(X_n))$, which is moreover given in the general case, due to Philipp and Stout for the countable state space case and to Csáki and Csörgő when the chain is strongly aperiodic. Conditions on the strong mixing coefficients are considered. A drift property equivalent to the finiteness of the second moment of $\zeta(i)$ is also given and applied to the functional central limit theorem.

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1. Introduction and preliminaries

Throughout the paper we will consider the canonical Markov chain $(X_n)_{n\geqslant 0}$ taking its values in a countably generated measurable space $(E, \mathscr{E})$ with given transition probability kernel P and initial distribution μ: the X_n's, $n \geqslant 0$, are the coordinate projections on

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the measurable space $(\Omega, \mathscr{F}) := (E^{\infty}, \mathscr{E}^{\infty})$, on which we have the Markovian probability measure $\mathbb{P}_{\mu}$ (with expectation operator $\mathbb{E}_{\mu}$; we write $\mathbb{P}_x$ and $\mathbb{E}_x$ if $\mu = \delta_x$) associated to the stochastic kernel P and the probability measure (p.m.) μ on $(E, \mathscr{E})$. We assume that $(X_n)_{n \geqslant 0}$ is Harris ergodic (i.e. aperiodic and positive Harris recurrent, Nummelin, 1984) and denote by π the invariant p.m. It is well known (Nummelin, 1984; Meyn and Tweedie, 1993) that under these assumptions there exist triples (m_0, s, v) satisfying

$$
\begin{aligned}
&\text{for all } x \in E,\ A \in \mathscr{E} : P^{m_0}(x, A) \geqslant (s \otimes v)(x, A)\ (:= s(x)v(A)),\\
&m_0 \geqslant 1 \text{ is an integer},\\
&s \text{ is an } \mathscr{E}\text{-measurable function with } 0 \leqslant s \leqslant 1 \text{ such that } \pi(s) := \int s \, \mathrm{d}\pi > 0,\\
&v \text{ is a p.m. on } \mathscr{E} \text{ such that } v(s) > 0. \qquad (1.1)
\end{aligned}
$$

If we fix any (m_0, s, v) as in (1.1), then the splitting technique of Nummelin (1978) gives a sequence $(Y_n)_{n \leqslant 0}$ such that $(X_{nm_0}, Y_n)_{n \geqslant 0}$ is a Markov chain with a positive recurrent atom α (the split chain; see below). This allows to extend to this setting the regeneration method used in the countable state space case: the partial sums $S_N = \sum_{n=0}^{N} f(X_n)$ of a functional of the chain can be divided into sums between consecutive visits to α, the sums over the α-blocks, which form a 1-dependent stationary sequence $(\zeta(i))_{i \geqslant 1}$ (see Nummelin, 1984; Meyn and Tweedie, 1993; Chen, 1999a; and Lemma 1.1).

Then, in formulating limit theorems for S_N, one can impose moment conditions on $\zeta'(1)$, where we consider $|f|$ in place of f (see, for example, Meyn and Tweedie 1993, Theorem 17.3.6; Philipp and Stout, 1975, Section 10.1). Since this random variable depends on the particular triple (m_0, s, v) chosen, it is natural to wonder whether those conditions do not depend on (m_0, s, v) and can be expressed in terms of (X_n). For example, the hypothesis of $|f|$-regularity (Nummelin, 1984, Definition 5.4) of the measure $|f|\mathrm{d}\pi$ (condition (R_2) in Proposition 2.1 below) in the central limit theorem (CLT) in Nummelin (1984, Theorem 7.6) is equivalent to the finiteness of the second moment of $\zeta'(1)$ for any (m_0, s, v); on the other hand it is also sufficient for the functional CLT (Niemi and Nummelin, 1982, Proposition 3.1 and Theorem 2; Kaplan and Sil'vestrov, 1979, Theorem 4; see Proposition 2.1 and Remark 2.2 below). Moreover, Meyn and Tweedie (1993, Section 17.5) contains a useful drift property (a Foster–Lyapumov criterion) with an integrability condition on the test function, which implies that moment condition; see also Glynn and Meyn (1996). For a treatment of integrability and tail properties formulated in terms of the original chain, see Chen (1999b).

In this paper we introduce a regularity condition, (R_3) in Proposition 2.3, which is shown to be equivalent to the finiteness of the third moment of $\zeta'(1)$ for any (m_0, s, v) satisfying (1.1) and which leads to an equivalent drift property with integrability, involving tow concatenated test functions; this is condition (D_3) in Proposition 2.3. We observe that the finiteness of the third moment of $\zeta'(1)$ appears as an intermediate condition in Bolthausen (1982) (it is (3.4) there when it is applied to the split chain; see Lemma 1.1 and (1.12) below), which deals with the strongly aperiodic case, that is, when (1.1) is satisfied with $m_0 = 1$.

We also give versions for the general state space case of an almost sure invariance principle due to Philipp and Stout (1975, Theorem 10.1) for the countable state space case and to Csáki and Csörgő (1995, Theorem 2.1) for the strongly aperiodic case (the null recurrent case is also considered there). This theorem requires a finite $2+\delta$ moment of $\zeta'(1)$ and gives an approximation of the order of $t^{1/2-\epsilon}$ (for its consequences, see Philipp and Stout, 1975, Chapter 1). In Proposition 4.2 this is obtained under hypotheses dependent on (m_0, s, v), as in Csáki and Csörgő (1995), by using the methods of Philipp and Stout (1975).

In Corollary 4.3, which deals with ergodic chains of degree 2, the dependence on (m_0, s, v) is dropped. The assumptions in (i), (ii) are made on the strong mixing coefficients $(\alpha(n))$ of the chain; this uses some results about them obtained from Bolthausen (1982) and Rio (2000) which are collected in Section 3. Corollary 3.2 characterizes the convergence of $\sum_{n=1}^{\infty} \alpha(n)$, that is, ergodicity of degree 2, and that of $\sum_{n=1}^{\infty} n\alpha(n)$, by drift conditions. Part (iii) of Corollary 4.3 and its consequences (iv) and (v) are written in similar terms to those of Nummelin (1984, Theorem 7.6 and Corollary 7.3).

We consider as well the second moment of $\zeta'(1)$. Proposition 2.1 shows that its finiteness is equivalent to the drift condition with integrability (D_2) introduced there. Then it can be used in the CLT part of Meyn and Tweedie (1993, Theorem 17.5.3) (and its functional version, by the results quoted above) in place of the CLT moment condition in Meyn and Tweedie (1993, Section 17.5). Likewise, a condition involving (D_2) ensures the expression in that theorem of the asymptotic variance as the sum of the covariances of the functional of the chain in stationary regime; this is shown in Corollary 4.5 to Proposition 4.4, which is proved by using Theorem II-3.1 of Chen (1999a) and results in Nummelin (1984), Meyn and Tweedie (1993) and extends Proposition 2.2 of de Acosta (1997). This allows to show that for the random walk on a half line, the finiteness of the fourth moment of the positive part of the increment variable is sufficient for the functional CLT and the mentioned expression for the limiting variance; this is Proposition 5.3.2 under condition (a-ii) and it appears to be interesting in view of Proposition 17.6.1 and a result suggested on p. 445 in Meyn and Tweedie (1993).

In Section 5, after some remarks about the use of (D_2) and (D_3) when E is a subset of an euclidean space, we deal with three well-known examples and derive almost sure invariance principles from Corollary 4.3(iii) and functional CLTs from Kaplan and Sil'vestrov (1979), Niemi and Nummelin (1982) and Proposition 2.1.

Now we fix any (m_0, s, v) as in (1.1) and describe the construction (the condition $v(s) > 0$ is not needed here) of a version of $(X_n)_{n\geqslant 0}$ and the aforementioned sequence $(Y_n)_{n\geqslant 0}$ on a common probability space (Nummelin, 1984, proof of Theorem 7.6; see also Meyn and Tweedie, 1993, Section 17.3.1; Levental, 1988; Chen, 1999a). Let $\check{\Omega} = \prod_{n\geqslant 0} E_n$ and $\check{\mathscr{F}} = \bigotimes_{n\geqslant 0} \mathscr{F}_n$ where $(E_0, \mathscr{F}_0) := (E \times \{0,1\}, \mathscr{E} \otimes \mathscr{P}(\{0,1\}))$, $(E_n, \mathscr{F}_n) := (E^{m_0} \times \{0,1\}, \mathscr{E}_{m_0} \otimes \mathscr{P}(\{0,1\}))$ for $n \geqslant 1$.

Denote $\check{X}_n$ and Y_n, $n \geqslant 0$, the measurable functions on $(\check{\Omega}, \check{\mathscr{F}})$ such that $(\check{X}_0, Y_0)$ is the projection onto E_0 and $((\check{X}_{(n-1)m_0+1}, \ldots, \check{X}_{nm_0}), Y_n) =: (\check{X}_n, Y_n)$ is the projection onto E_n if $n \geqslant 1$. For $n \geqslant 0$ consider the σ-algebras $\check{\mathscr{F}}_n^{\check{X}} := \sigma(\check{X}_k, 0 \leqslant k \leqslant n)$, $\check{\mathscr{F}}_n^{Y} := \sigma(Y_k, 0 \leqslant k \leqslant n)$, $\check{\mathscr{F}}_n := \check{\mathscr{F}}_{nm_0}^{\check{X}} \vee \check{\mathscr{F}}_n^{Y}$, and $\check{\mathscr{F}}_{-1}^{Y} := \{\emptyset, \check{\Omega}\}$. Define the map $\theta : \check{\Omega} \to \check{\Omega}$ defined by $\theta((x_0, y_0), (x_1, y_1), \ldots) = ((p_{m_0}(x_1), y_1), (x_2, y_2), \ldots)$, $((x_0, y_0), (x_1, y_1), \ldots) \in \check{\Omega}$, where $p_{m_0} : E^{m_0} \to E$ is the projection $p_{m_0}(x_1, \ldots, x_{m_0}) = x_{m_0}$.

Recall that if μ is a measure on a measurable space $(F,\mathscr{F})$ and f is a nonnegative $\mathscr{F}$-measurable function on F we have the kernel (Nummelin, 1984) $I_f(y,B)=f(y)1_B(y)$, $y\in F$, $B\in\mathscr{F}$, and the measure $(\mu I_f)(C)=\int\mu(\mathrm{d}y)I_f(y,C)=\int_C f\,\mathrm{d}\mu$, $C\in\mathscr{F}$; the identity kernel is $I:=I_{1_E}$. Given a p.m. μ on $(E^{m_0},\mathscr{E}^{m_0})$ we define the p.m. $\check{\mu}$ on $(E_1,\mathscr{F}_1)$ by $\check{\mu}=(\mu I_{1-s\circ p_{m_0}})\otimes\delta_0+(\mu I_{s\circ p_{m_0}})\otimes\delta_1$.

Let r be a nonnegative, $\mathscr{E}\otimes\mathscr{E}$-measurable function on $E\times E$ such that $0\leqslant r\leqslant 1$ and for each $x\in E, r(x,\cdot)$ is a version of the Radon–Nikodym derivative $\mathrm{d}(s(x)v)/\mathrm{d}P^{m_0}(x,\cdot)$. Consider Q_0,Q_1 defined on $E\times\mathscr{E}^{m_0}$ by

$$Q_0(x,H)=\begin{cases}\dfrac{1}{1-s(x)}\displaystyle\int P(x,\mathrm{d}t_1)\int\cdots\int P(t_{m_0-1},\mathrm{d}t_{m_0})\{1-r(x,t_{m_0})\}1_H(t_1,\ldots,t_{m_0}),\\ \qquad\qquad\text{if } s(x)<1,\\ \delta_{(x,\ldots,x)}(H), \quad \text{if } s(x)=1,\end{cases}$$

$$Q_1(x,H)=\begin{cases}\dfrac{1}{s(x)}\displaystyle\int P(x,\mathrm{d}t_1)\int\cdots\int P(t_{m_0-1},\mathrm{d}t_{m_0})r(x,t_{m_0})1_H(t_1,\ldots,t_{m_0}),\\ \qquad\qquad\text{if } s(x)>0,\\ (\delta_{(x,\ldots,x)}\otimes v)(H), \quad \text{if } s(x)=0 \ (x\in E,\ H\in\mathscr{E}^{m_0}).\end{cases}$$

Q_0 and Q_1 are stochastic kernels between $(E,\mathscr{E})$ and $(E^{m_0},\mathscr{E}^{m_0})$, that is, for every $x\in E$, $Q_i(x,\cdot)$ is a p.m. on $(E^{m_0},\mathscr{E}^{m_0})$ and for every $H\in\mathscr{E}^{m_0}$, $Q_i(\cdot,H)$ is an $\mathscr{E}$-measurable function $i=0,1$. Define the stochastic kernel $\check{P}$ between the measurable spaces $(E_0,\mathscr{F}_0)$ and $(E_1,\mathscr{F}_1)$ by

$$\check{P}((x,y),\cdot)=1_{\{0\}}(y)Q_0(x,\cdot)^{\vee}+1_{\{1\}}(y)Q_1(x,\cdot)^{\vee}\quad (x,y)\in E_0.$$

Given a p.m. λ on $(E_0,\mathscr{F}_0)$ let $\dot{\mathbb{P}}_\lambda$ be the unique p.m. on $(\dot{\Omega},\dot{\mathscr{F}})$ satisfying

$$\begin{aligned}\dot{\mathbb{P}}_\lambda\left(\prod_{k\geqslant 0}F_k\right)&=\int_{F_0}\lambda(\mathrm{d}(x_0,y_0))\int_{F_1}\check{P}((x_0,y_0),\mathrm{d}(x_1,y_1))\\&\times\int_{F_2}\check{P}((p_{m_0}(x_1),y_1),\mathrm{d}(x_2,y_2))\\&\times\int_{F_3}\cdots\int_{F_n}\check{P}((p_{m_0}(x_{n-1}),y_{n-1}),\mathrm{d}(x_n,y_n))\end{aligned}\tag{1.2}$$

for every measurable rectangle $\prod_{k\geqslant 0}F_k$ such that $F_k=E_k$ if $k>n$, $n\geqslant 2$ (use a theorem of Ionescu Tulcea, Neveu, 1965, Proposition V.1.1 and Corollary 2). Note that if $\eta:\dot{\Omega}\to[0,\infty]$ is $\dot{\mathscr{F}}$-measurable, its $\dot{\mathbb{P}}_\lambda$-expectation satisfies $\dot{\mathbb{E}}_\lambda[\eta]=\int_{E_0}\lambda(d(x_0,y_0))\times\dot{\mathbb{E}}_{(x_0,y_0)}[\eta]$ where $\dot{\mathbb{E}}_{(x_0,y_0)}$ is the expectation with respect to $\dot{\mathbb{P}}_{(x_0,y_0)}:=\dot{\mathbb{P}}_{\delta_{(x_0,y_0)}}$,

$(x_0, y_0) \in E_0$. We have the following Markovian property:

$$\check{\mathbb{E}}_\lambda[\eta \circ \theta_n | \check{\mathscr{F}}_n] = \check{\mathbb{E}}_{(X_{nm_0}, Y_n)}[\eta], \quad \check{\mathbb{P}}_\lambda\text{-a.s.}, \quad n \geqslant 0 \tag{1.3}$$

for each p.m. λ on $(E_0, \mathscr{F}_0)$ and every $\check{\mathscr{F}}$-measurable function $\eta : \check{\Omega} \to [0, \infty]$ (θ_0 is the identify map in $\check{\Omega}$, θ_n is the nth iterate of θ). For any p.m. μ on $(E, \mathscr{E})$ define the p.m. μ^* on $(E_0, \mathscr{F}_0)$ by $\mu^* = (\mu I_{1-s}) \otimes \delta_0 + (\mu I_s) \otimes \delta_1$. If μ is a p.m. on $(E, \mathscr{E})$ (λ is a p.m. on $(E_0, \mathscr{F}_0)$) and $\eta : \check{\Omega} \to [0, \infty]$ is $\check{\mathscr{F}}$-measurable then

$$\check{\mathbb{E}}_{\mu^*}[\eta \circ \theta_n \mid \check{\mathscr{F}}^X_{nm_0} \vee \check{\mathscr{F}}^Y_{n-1}] = \check{\mathbb{E}}_{\delta^*_{\check{X}_{nm_0}}}[\eta]$$

$$\text{if } n \geqslant 0 \quad (\check{\mathbb{E}}_\lambda \text{ on the left if } n \geqslant 1), \tag{1.4}$$

this follows from

$$\check{\mathbb{P}}_{\mu^*}[Y_n = 1 | \check{\mathscr{F}}^X_{nm_0} \vee \check{\mathscr{F}}^Y_{n-1}] = s(\check{X}_{nm_0}) \quad \text{if } n \geqslant 0 \quad (\check{\mathbb{P}}_\lambda \text{ on the left if } n \geqslant 1). \tag{1.5}$$

Given any p.m. μ on $(E, \mathscr{E})$, another consequence of (1.3) is that, with respect to $(\check{\Omega}, \check{\mathscr{F}}, \check{\mathbb{P}}^*_\mu)$, $(\check{X}_n)_{n \geqslant 0}$ is a Markov chain with transition probability kernel P and initial distribution μ (use (1.4) and the fact that $\check{\mathbb{E}}_{\delta^*_x}[1_H(\check{X}_1)] = (1 - s(x))Q_0(x, H) + s(x)Q_1(x, H) = \int P(x, dt_1) \int \cdots \int P(t_{m_0-1}, dt_{m_0}) 1_H(t_1, \ldots, t_{m_0})$ if $x \in E$ and $H \in \mathscr{E}^{m_0}$). Then we will also write X_n and $\mathscr{F}^Y_n$ in place of $\check{X}_n$ and $\check{\mathscr{F}}^Y_n$, respectively.

Define the stochastic kernel $P^* : E_0 \times \mathscr{F}_0 \to [0, 1]$ by

$$P^*((x, y), \cdot) = 1_{\{0\}}(y) Q(x, \cdot)^* + 1_{\{1\}}(y) v^*, \quad (x, y) \in E_0, \tag{1.6}$$

where $Q(x, A) = (P^{m_0}(x, A) - s(x)v(A))/(1 - s(x))$, if $s(x) < 1$, $\delta_x(A)$ if $s(x) = 1$ ($x \in E$, $A \in \mathscr{E}$). If λ is a p.m. on $(E_0, \mathscr{F}_0)$ we have that with respect to $(\check{\Omega}, \check{\mathscr{F}}, \check{\mathbb{P}}_\lambda)$, $(X_{nm_0}, Y_n)_{n \geqslant 0}$ is a Markov chain with initial distribution λ and transition probability kernel P^* (note that $\check{P}((x, y), E_{m_0-1} \times A) = P^*((x, y), A)$ if $(x, y) \in E_0$, $A \in \mathscr{F}_0$); it is positive Harris recurrent with invariant p.m. π^* (Nummelin, 1978) and the set $\alpha := E \times \{1\}$ is a recurrent atom: $\pi^*(\alpha) > 0$, $P^*((x, 1), \cdot) = v^*$ for all $x \in E$, for every p.m. λ on $(E_0, \mathscr{F}_0)$ we have $\check{\mathbb{P}}_\lambda[(X_{nm_0}, Y_n) \in \alpha \text{ i.o.}] = 1$ and then the λ-a.s. finite $\mathscr{F}^Y_n$-stopping times $T_\alpha = T_\alpha(0) := \inf\{n \geqslant 0 : Y_n = 1\}$ ($= \infty$ if the set is empty), $T_\alpha(i) := \inf\{n > T_\alpha(i-1) : Y_n = 1\}, i \geqslant 1, S_\alpha := \inf\{n \geqslant 1 : Y_n = 1\}$. If μ is a p.m. on $(E, \mathscr{E})$ and $\varphi : E \to [0, \infty]$ is $\mathscr{E}$-measurable, by the definitions of μ^* and P^*,

$$\check{\mathbb{E}}_{\mu^*}[\varphi(X_{km_0}) 1_{\{T_\alpha \geqslant k\}}] = \mu(P^{m_0} - s \otimes v)^k \varphi, \quad k \geqslant 0. \tag{1.7}$$

The following kernels are considered:

$$G_{m_0, s, v} := \sum_{n=0}^{\infty} (P^{m_0} - s \otimes v)^n, \quad \bar{G}_{m_0, s, v} := GV \quad \text{where } V = V_{m_0} := \sum_{m=0}^{m_0-1} P^m \tag{1.8}$$

and we will also write $G = G_{m_0, s, v}$, $\bar{G} = \bar{G}_{m_0, s, v}$.

Assume f is a real $\mathscr{E}$-measurable function on E. The sums over the α-blocks are defined by

$$\zeta(i) = \zeta(f,i) = \sum_{n=T_\alpha(i-1)+1)m_0}^{(T_\alpha(i)+1)m_0-1} f(X_n) = \sum_{n=T_\alpha(i-1)+1}^{T_\alpha(i)} Z_n, \quad i \geqslant 1,$$
$$\text{where } Z_n = Z_n(f) = \sum_{m=0}^{m_0-1} f(X_{nm_0+m}), \quad n \geqslant 1, \tag{1.9}$$

if f is nonnegative, we have for every p.m. μ on $(E,\mathscr{E})$ (by (1.7) and (1.4))

$$\mu G f = \dot{\mathbb{E}}_{\mu^*}\left[\sum_{n=0}^{T_\alpha} f(X_{nm_0})\right], \qquad \mu \bar{G} f = \dot{\mathbb{E}}_{\mu^*}\left[\sum_{n=0}^{T_\alpha} Z_n\right]. \tag{1.10}$$

Using (1.3) (with $\dot{\mathscr{F}}_q$ and $\dot{\mathscr{F}}_1$) and the properties of α and (X_{nm_0}, Y_n) we can prove

$$\dot{\mathbb{E}}_\lambda[\xi 1_{\{Y_q=1\}}(\eta \circ \theta_{q+1})] = \dot{\mathbb{E}}_\lambda[\xi 1_{\{Y_q=1\}}]\dot{\mathbb{E}}_{\nu^*}[\eta]$$
$$\text{if } \xi \geqslant 0 \text{ is } \dot{\mathscr{F}}_q\text{-measurable}, \ \eta \geqslant 0 \text{ is } \dot{\mathscr{F}}\text{-measurable}, \ q \geqslant 0, \tag{1.11}$$

for every p.m. λ on $(E_0, \mathscr{F}_0)$. From this we can obtain

Lemma 1.1. *Let λ be a p.m. on $(E_0, \mathscr{F}_0)$. With respect to $(\dot{\Omega}, \dot{\mathscr{F}}, \dot{\mathbb{P}}_\lambda)$, the sequence $(\zeta(i))_{i\geqslant 1}$ is 1-dependent, strictly stationary and has the same distribution as the $\dot{\mathbb{P}}_{\nu^*}$-distribution of $(\sum_{n=0}^{T_\alpha} Z_n, \sum_{n=T_\alpha(0)+1}^{T_\alpha(1)} Z_n, \ldots, \sum_{n=T_\alpha(i-2)+1}^{T_\alpha(i-1)} Z_n, \ldots)$.*

Finally, we observe that since $\dot{\mathbb{E}}_{(x,1)}[\eta \circ \theta] = \dot{\mathbb{E}}_{\nu^*}[\eta]$ if $\eta \geqslant 0$ is $\dot{\mathscr{F}}$-measurable and $x \in E$ (by (1.3) and the definition of $\dot{P}$), we have (with a standard notation)

$$\mathscr{L}_{\dot{\mathbb{P}}_{\nu^*}}\left(\sum_{n=0}^{T_\alpha} Z_n\right) = \mathscr{L}_{\dot{\mathbb{P}}_{(x,1)}}\left(\sum_{n=1}^{S_\alpha} Z_n\right) = \mathscr{L}_{\dot{\mathbb{P}}_{(x,1)}}\left(\sum_{n=1}^{T_\alpha(1)-T_\alpha(0)} Z_n\right). \tag{1.12}$$

2. A regularity condition and the third moment of block sums

Assume $(X_n)_{n\geqslant 0}$ is as in Section 1. For every (m_0,s,ν) as in (1.1) we have the preceding construction and we are interested in the third moment of $\zeta(i)$ under $\dot{\mathbb{P}}_{\mu^*}$ for any initial distribution μ of $(X_n)_{n\geqslant 0}$. Given $A \in \mathscr{E}$ we have the stopping time $S_A := \inf\{n \geqslant 1: X_n \in A\}$ (defined on Ω) and the kernel $U_A(x,B) := \mathbb{E}_x[\sum_{n=1}^{S_A} 1_B(X_n)]$, $x \in E$, $B \in \mathscr{E}$. Let $\mathscr{E}^+ = \{A \in \mathscr{E}: \pi(A) > 0\}$. First we consider the second moment of $\zeta(i)$.

Proposition 2.1. *Let f be a real-valued function in $\mathscr{L}^1_+(\pi)$ (the set of nonnegative π-integrable functions). Then the following conditions are equivalent:*

(M_2) For some (for every) p.m. λ on $(E_0, \mathscr{F}_0)$ and for some (for every) triple (m_0,s,ν) satisfying (1.1),

$$\dot{\mathbb{E}}_{\nu^*}\left(\sum_{n=0}^{T_\alpha} Z_n\right)^2 < \infty.$$

(R_2) *For every* $A \in \mathscr{E}^+$, $\pi(f(I+U_A)f) < \infty$.

(D_2) *There exist an* $\mathscr{E}$*-measurable function* $V : E \to [0,\infty]$, *a small set* $C \in \mathscr{E}^+$ (*i.e.* $(m_0', \beta 1_C, v')$ *satisfies* (1.1) *for some* $\beta > 0$, m_0' *and* v' (Nummelin, 1984, Definition 2.3) *and a constant* $b \in [0,\infty)$ *such that*

$$PV + f \leqslant V + b1_C \quad \textit{and} \quad \pi(fV) < \infty.$$

We omit the proof, which is similar to that of Proposition 2.3 below.

Remark 2.2. Condition (R_2) means that the measure πI_f is f-regular (Nummelin, 1984, Definition 5.4); it is the second moment assumption on $|f|$ in the CLT in Nummelin (1984, Theorem 7.6) and is also sufficient for the functional CLT (Niemi and Nummelin, 1982, Theorem 2 and Proposition 3.1; Kaplan and Sil'vestrov, 1979, Theorem 4). Its equivalence with (M_2) seems to be known. See Nummelin (1984, Theorem 7.6) and Niemi and Nummelin (1982, Proposition 3.1) where it is shown that (M_2) is equivalent to $\pi(f^2) < \infty$ and the f-regularity of the measure $\int \pi(\mathrm{d}x) f(x) \times P(x,\cdot)$; the equivalence of this last property with (R_2) can be seen using Lemma 2.6 and Nummelin (1984, Proposition 5.13).

Condition (R_2) (or (D_2)) shows that the finiteness of the expectation in (M_2) does not depend on (m_0, s, v) and is the link with (D_2) (we mention that this gives a drift characterization of Meyn and Tweedie, 1993 (17.31); cf. also Meyn and Tweedie, 1993, Lemma 17.5.2(ii)).

In this section we prove the following version of Proposition 2.1.

Proposition 2.3. *Let* f *be a real-valued function in* $\mathscr{L}^1_+(\pi)$. *Then the following conditions are equivalent*:

(M_3) *For some (for every) p.m.* λ *on* $(E_0, \mathscr{F}_0)$ *and for some (for every) triple* (m_0, s, v) *satisfying* (1.1),

$$\check{\mathbb{E}}_{v^*}\left(\sum_{n=0}^{T_\alpha} Z_n\right)^3 < \infty. \tag{2.1}$$

(G_3) *For some (for every) triple* (m_0, s, v) *satisfying* (1.1),

$$\pi(f\bar{G}_{m_0,s,v}f) < \infty \tag{2.2i}$$

and

$$\pi(f\bar{G}_{m_0,s,v}(f\bar{G}_{m_0,s,v}f)) < \infty. \tag{2.2ii}$$

(R_3) *For every* $A \in \mathscr{E}^+$,

$$\pi(f(I+U_A)(f(I+U_A)f)) < \infty. \tag{2.3}$$

(R_3') *For every* A *and* B *in* $\mathscr{E}^+, \pi(f(I+U_A)(f(I+U_B)f)) < \infty$.

(D_3) *There exist* $\mathscr{E}$*-measurable functions* $V_1, V_2 : E \to [0,\infty]$, *small sets* C_1, C_2 *in* $\mathscr{E}^+$ *and constants* $b_1, b_2 \in [0,\infty)$ *such that*

$$PV_1 + f \leqslant V_1 + b_1 1_{C_1}, \quad \pi(fV_1) < \infty,$$

$$PV_2 + fV_1 \leqslant V_2 + b_2 1_{C_2}, \quad \pi(fV_2) < \infty.$$

For the proof, first we observe that given (m_0,s,ν) as in (1.1), if $A\in\mathscr{E}^+$ and $g\in\mathscr{L}^1_+(\pi)$, $(I+U_A)g\leqslant \bar{G}_{m_0,s,\nu}g+M$ for some constant $M=M_{A,g}>0$ (see Nummelin, 1984, p. 80, 82).

Lemma 2.4. *Let (m_0,s,ν) satisfy (1.1). If $g\in\mathscr{L}^1_+(\pi)$ then there exist $B=B_g\in\mathscr{E}^+$ and $M=M_g>0$ such that*

$$\check{\mathbb{E}}_{(x,y)}\left[\sum_{n=0}^{T_\alpha} g(\check{X}_{nm_0})\right]\leqslant M \quad \textit{for all } x\in B, \quad y\in\{0,1\}.$$

Proof. Define on E_0, $g^*(x,y)=g(x)$, $(U_\alpha g^*)(x,y)=\check{\mathbb{E}}_{(x,y)}[\sum_{n=1}^{S_\alpha} g^*(X_{nm_0},Y_n)]$ and note that $\check{\mathbb{E}}_{(x,y)}[\sum_{n=0}^{T_\alpha} g(X_{nm_0})]\leqslant g(x)+(U_\alpha g^*)(x,y)$ for all $x\in E, y\in\{0,1\}$. The value of $U_\alpha g^*$ on α is $\check{\mathbb{E}}_{\nu^*}[\sum_{n=0}^{T_\alpha} g^*(X_{nm_0},Y_n)]$ and this constant must be finite because $\pi^*(\{U_\alpha g^* =\infty\})=0$ by Proposition 5.11 of Nummelin (1984). Assume that $\pi(\{s<1\})>0$; since $g\in\mathscr{L}^1_+(\pi)$ and $\pi(\{s<1,(U_\alpha g^*)(\cdot,0)=\infty\})=0$ $((\pi I_{1-s})(\{(U_\alpha g^*)(\cdot,0)=\infty\})=\pi^*((E\times\{0\})\cap\{U_\alpha g^*=\infty\})=0)$, $\pi(\{g<\infty,(U_\alpha g^*)(\cdot,0)<\infty\})>0$ and there exists $k\geqslant 1$ such that $B:=\{g\leqslant k,(U_\alpha g^*)(\cdot,0)\leqslant k\}\in\mathscr{E}^+$ and the conclusion holds with $M:=\max\{2k,k+\check{\mathbb{E}}_{\nu^*}[\sum_{n=0}^{T_\alpha} g^*(X_{nm_0},Y_n)]\}$. When $\pi(\{s<1\})=0$ there exists $k\geqslant 1$ such that $B:=\{s=1,g\leqslant k\}\in\mathscr{E}^+$; in this case, for $x\in B, P^*((x,0),\cdot)=\delta^*_x=\delta_{(x,1)}$ and we have

$$(U_\alpha g^*)(x,0)=\check{\mathbb{E}}_{P^*((x,0),\cdot)}\left[\sum_{n=0}^{T_\alpha} g^*(X_{nm_0},Y_n)\right]=\check{\mathbb{E}}_{(x,1)}\left[\sum_{n=0}^{T_\alpha} g^*(X_{nm_0},Y_n)\right]$$

$$=\check{E}_{(x,1)}[1_{\{T_\alpha=0\}}g^*(X_0,Y_0)]\leqslant g(x).$$

The constant M can be taken as in the other case. □

Lemma 2.5. *Let (m_0,s,ν) satisfy (1.1). If $f_1,f_2\in\mathscr{L}^1_+(\pi)$ then there exist $C=C_{f_1,f_2}\in\mathscr{E}^+$ and $M=M_{f_1,f_2}>0$ such that for $i=1,2$, $\bar{G}f_i\leqslant(I+U_C)f_i+M$.*

Proof. Let $B\in\mathscr{E}^+$ and $M'>0$ be obtained from Lemma 2.4 for $g:=\max\{Vf_1,Vf_2\}$. Define ${}_{m_0}S_B:=\inf\{n\geqslant 1: X_{nm_0}\in B\}$ and ${}_{m_0}U_B(x,A):=\mathbb{E}_x[\sum_{n=1}^{{}_{m_0}S_B} 1_A(X_{nm_0})]$, $x\in E$, $A\in\mathscr{E}$. By the strong Markov property of $(X_{nm_0},Y_n)_{n\geqslant 0}$, if $i=1,2$,

$$\check{\mathbb{E}}_{\delta^*_x}\left[\sum_{n=0}^{T_\alpha}(Vf_i)(X_{nm_0})\right]\leqslant[(I+_{m_0}U_B)Vf_i](x)$$

$$+\check{\mathbb{E}}_{\delta^*_x}\left[\check{\mathbb{E}}_{(X_{m_0 S_B m_0},Y_{m_0 S_B})}\left[\sum_{n=0}^{T_\alpha} g(X_{nm_0})\right]\right]$$

$(x\in E)$. Then by (1.10), $\bar{G}f_i\leqslant(I+_{m_0}U_B)Vf_i+M'$ and it is sufficient to prove that there exist $C\in\mathscr{E}^+$ and $M''>0$ such that $(I+_{m_0}U_B)Vf_i\leqslant(I+U_C)f_i+M''$ for $i=1,2$.

By Proposition 5.12 of Nummelin (1984), there exists $k \geqslant 1$ such that $C_1 := \{\bar{G}f_1 \leqslant k, \bar{G}f_2 \leqslant k\} \in \mathscr{E}^+$. Arguing as in the proof of Proposition 2.6 of Nummelin (1984) we can obtain a subset C of C_1 which is small with the same measure v: $P^{k_0} \geqslant \beta 1_C \otimes v$ for some integer $k_0 \geqslant 1$ and some $\beta > 0$. This implies that (5.10) of Nummelin (1984) holds for an integer $n_0 = n_0(B) \geqslant 2m_0$ and $\gamma = \gamma(B) > 0$. Now argue as in the proof of Lemma 5.3 in Nummelin (1984) with the same $\eta(i), \mathscr{F}_i, \tau$ but with $\xi_i^{(1)}$ and $\xi_i^{(2)}$ defined for f_1 and f_2, respectively, in place of ξ_i defined for f, taking into account that C is both f_1 and f_2-regular, since $C \subset C_1$ (Nummelin, 1984, Proposition 5.13(i)). □

Lemma 2.6. *Let (m_0, s, v) satisfy (1.1). If $g \in \mathscr{L}^1_+(\pi)$ then there exists $M = M_g > 0$ such that $P^m\bar{G}g \leqslant M + 2P^m g + 2P^{m+1}\bar{G}g$ for any integer $m \geqslant 0$.*

Proof. By Lemma 2.5, and the result quoted before Lemma 2.4, it is sufficient to show that for nonnegative g and $C \in \mathscr{E}^+$, $P^m(I + U_C)g \leqslant 2P^m g + 2P^{m+1}(I + U_C)g$. Define $\sigma : E^\infty \to \mathbb{N}^* \cup \{\infty\}$ by $\sigma(x_0, x_1, \ldots) = \min\{n \geqslant 1 : x_n \in C\}$. Then for every $x \in E$

$$
\begin{aligned}
P^m(I + U_C)g &= \int P^m(x, \mathrm{d}y)\mathbb{E}_y\left[\sum_{n=0}^{\sigma(X_0, X_1, \ldots)} g(X_n)\right] \\
&= \mathbb{E}_x\left[1_{\{X_{m+1} \in C\}} \sum_{k=m}^{\sigma(X_m, \ldots)+m} g(X_k)\right] + \mathbb{E}_x\left[1_{\{X_{m+1} \notin C\}} \sum_{k=m}^{\sigma(X_m, \ldots)+m} g(X_k)\right] \\
&= J_1 + J_2, \quad \text{say}.
\end{aligned}
$$

We have $J_1 \leqslant \mathbb{E}_x[g(X_m) + g(X_{m+1})] = (P^m g)(x) + (P^{m+1}g)(x)$ and, since $\sigma(x_m, x_{m+1}, \ldots) = \sigma(x_{m+1}, \ldots) + 1$ if $x_{m+1} \notin C$,

$$
J_2 \leqslant \mathbb{E}_x\left[g(X_m) + \sum_{k=m+1}^{\sigma(X_m+1, \ldots)+m+1} g(X_k)\right] = (P^m g)(x) + P^{m+1}(I + U_C)g. \quad \square
$$

We also need the following version of Theorem 14.2.3(i) in Meyn and Tweedie (1993).

Lemma 2.7. *If $f, V : E \to [0, \infty]$ are $\mathscr{E}$-measurable functions, $C \in \mathscr{E}^+$ is small and $b \in [0, \infty)$ with $PV + f \leqslant V + b1_C$ then for every $B \in \mathscr{E}^+$ there exists $c = c_{C,b,B} \in [0, \infty)$ (depending neither on f nor on V) such that*

$$
(I + U_B)f \leqslant V + c.
$$

Proof of Proposition 2.3. Let λ be a p.m. on $(E_0, \mathscr{F}_0)$ and (m_0, s, v) a triple as in (1.1). First we show that (we use the notation Z_n in Ω and in $\check{\Omega}$)

$$
\check{\mathbb{E}}_\lambda[(\zeta(1))^3] = \frac{1}{\pi(s)}\mathbb{E}_\pi[Z_0^3] + \frac{3}{\pi(s)}\check{\mathbb{E}}_{\check{\pi}^*}\left[Z_0^2 1_{\{Y_0=0\}} \sum_{n=1}^{S_0} Z_n\right]
$$

$$+\frac{3}{\pi(s)}\check{\mathbb{E}}_{\pi^*}\left[Z_0 1_{\{Y_0=0\}}\sum_{n=1}^{S_\alpha} Z_n^2\right]$$

$$+\frac{6}{\pi(s)}\check{\mathbb{E}}_{\pi^*}\left[Z_0 1_{\{Y_0=0\}}\sum_{m=1}^{S_\alpha}\check{\mathbb{E}}_{\delta^*_{X_{mm_0}}}\left[Z_0 1_{\{Y_0=0\}}\sum_{n=1}^{S_\alpha} Z_n\right]\right]. \tag{2.4}$$

The left-hand side is equal to (Lemma 1.1)

$$\check{\mathbb{E}}_{\nu^*}\left[\left(\sum_{n=0}^{T_\alpha} Z_n\right)^3\right] = \check{\mathbb{E}}_{\nu^*}\left[\sum_{n=0}^{T_\alpha} Z_n^3\right] + 3\check{\mathbb{E}}_{\nu^*}\left[\sum_{m=0}^{T_\alpha}\sum_{n=m+1}^{T_\alpha} Z_m^2 Z_n\right]$$

$$+3\check{\mathbb{E}}_{\nu^*}\left[\sum_{m=0}^{T_\alpha}\sum_{n=m+1}^{T_\alpha} Z_m Z_n^2\right] + 6\check{\mathbb{E}}_{\nu^*}\left[\sum_{k=0}^{T_\alpha}\sum_{m=k+1}^{T_\alpha}\sum_{n=m+1}^{T_\alpha} Z_k Z_m Z_n\right]$$

$$= A_1 + 3A_2 + 3A_3 + 6A_4 \quad \text{say.}$$

These terms are equal to the corresponding ones of the right-hand side in (2.4). We prove this for A_3 and A_4. One has $A_3 = \sum_{m=0}^{\infty} a_m$ with, by conditioning with respect to $\check{\mathscr{F}}^X_{mm_0} \vee \check{\mathscr{F}}^Y_{m-1}$ and (1.4),

$$a_m = \check{\mathbb{E}}_{\nu^*}\left[Z_m 1_{\{T_\alpha \geqslant m\}}\sum_{n=m+1}^{\infty} Z_n^2 1_{\{Y_m=\cdots=Y_{n-1}=0\}}\right]$$

$$= \check{\mathbb{E}}_{\nu^*}\left[\check{\mathbb{E}}_{\delta^*_{X_{mm_0}}}\left[Z_0\sum_{k=1}^{\infty} Z_k^2 1_{\{Y_0=\cdots=Y_{k-1}=0\}}\right] 1_{\{T_\alpha \geqslant m\}}\right]$$

$$= \check{\mathbb{E}}_{\nu^*}[\varphi(X_{mm_0})1_{\{T_\alpha \geqslant m\}}] = \nu(P^{m_0} - s\otimes\nu)^m\varphi, \qquad m \geqslant 0,$$

where $\varphi(x) = \check{\mathbb{E}}_{\delta_x^*}[Z_0 1_{\{Y_0=0\}}\sum_{k=1}^{S_\alpha} Z_k^2]$, by (1.7); Corollary 5.2 of Nummelin (1984) then shows that $A_3 = \pi(s)^{-2}\pi(\varphi) = \pi(s)^{-1}\check{\mathbb{E}}_{\pi^*}[Z_0 1_{\{Y_0=0\}}\sum_{k=1}^{S_\alpha} Z_k^2]$. Analogously, $A_4 = \sum_{k=0}^{\infty}\sum_{m=k+1}^{\infty} b_{km}$ with

$$b_{km} = \check{\mathbb{E}}_{\nu^*}\left[Z_k Z_m \sum_{n=m+1}^{\infty} Z_n 1_{\{T_\alpha \geqslant n\}}\right] = \check{\mathbb{E}}_{\nu^*}[Z_k\psi(X_{mm_0})1_{\{T_\alpha \geqslant m\}}]$$

$$= \check{\mathbb{E}}_{\nu^*}[Z_k 1_{\{T_\alpha \geqslant k+1\}}\check{\mathbb{E}}_{\delta^*_{X_{(k+1)m_0}}}[\psi(X_{(m-k-1)m_0})1_{\{T_\alpha \geqslant m-k-1\}}]], \qquad k+1 \leqslant m,$$

where $\psi(x) = \check{\mathbb{E}}_{\delta_x^*}[Z_0 1_{\{Y_0=0\}}\sum_{n=1}^{S_\alpha} Z_n]$; then, for each $k \geqslant 0$,

$$\sum_{m=k+1}^{\infty} b_{km} = \check{\mathbb{E}}_{\nu^*}[Z_k 1_{\{T_\alpha \geqslant k\}}1_{\{Y_k=0\}}(G\psi)(X_{(k+1)m_0})] = \check{\mathbb{E}}_{\nu^*}[\psi'(X_{km_0})1_{\{T_\alpha \geqslant k\}}]$$

with $\psi'(x) = \dot{\mathbb{E}}_{\delta_x^*}[Z_0 1_{\{Y_0=0\}}(G\psi)(X_{m_0})]$. Since, by (1.10) and (91.4), $(G\psi)(X_{m_0}) = \dot{\mathbb{E}}_{\pi^*}[\sum_{m=1}^{S_\alpha} \psi(X_{mm_0})|\check{\mathscr{F}}_{m_0}^X \vee \check{\mathscr{F}}_0^Y]$, we obtain

$$\pi(s)A_4 = \pi(\psi') = \dot{\mathbb{E}}_{\pi^*}[Z_0 1_{\{Y_0=0\}}(G\psi)(X_{m_0})]$$

$$= \dot{\mathbb{E}}_{\pi^*}\left[Z_0 1_{\{Y_0=0\}} \sum_{m=1}^{S_\alpha} \dot{\mathbb{E}}_{\delta^*_{X_{mm_0}}}\left[Z_0 1_{\{Y_0=0\}} \sum_{n=1}^{S_\alpha} Z_n\right]\right].$$

We also have

$$\check{\mathbb{E}}_\lambda[(\zeta(1))^2\zeta(2)] \geqslant \frac{1}{\pi(s)}\dot{\mathbb{E}}_{\pi^*}\left[Z_0^2 1_{\{Y_0=1\}} \sum_{n=1}^{S_\alpha} Z_n\right] + \frac{2}{\pi(s)}\check{\mathbb{E}}_{\pi^*}\left[Z_0 1_{\{Y_0=0\}} \sum_{m=1}^{S_\alpha} \check{\mathbb{E}}_{\delta^*_{X_{mm_0}}}\left[Z_0 1_{\{Y_0=1\}} \sum_{n=1}^{S_\alpha} Z_n\right]\right], \quad (2.5a)$$

$$\check{\mathbb{E}}_\lambda[\zeta(1)(\zeta(2))^2] \geqslant \frac{1}{\pi(s)}\dot{\mathbb{E}}_{\pi^*}\left[Z_0 1_{\{Y_0=1\}} \sum_{n=1}^{S_\alpha} Z_n^2\right] + \frac{2}{\pi(s)}\check{\mathbb{E}}_{\pi^*}\left[Z_0 1_{\{Y_0=1\}} \sum_{m=1}^{S_\alpha} \check{\mathbb{E}}_{\delta^*_{X_{mm_0}}}\left[Z_0 1_{\{Y_0=0\}} \sum_{n=1}^{S_\alpha} Z_n\right]\right], \quad (2.5b)$$

$$\check{\mathbb{E}}_\lambda[\zeta(1)\zeta(2)\zeta(3)] \geqslant \frac{1}{\pi(s)}\check{\mathbb{E}}_{\pi^*}\left[Z_0 1_{\{Y_0=1\}} \sum_{m=1}^{S_\alpha} \check{\mathbb{E}}_{\delta^*_{X_{mm_0}}}\left[Z_0 1_{\{Y_0=1\}} \sum_{n=1}^{S_\alpha} Z_n\right]\right]. \quad (2.5c)$$

In order to verify (2.5a), observe that its left-hand side equals

$$\check{\mathbb{E}}_{\nu^*}\left[\left(\sum_{m=0}^{T_\alpha} Z_m\right)^2 \sum_{n=T_\alpha+1}^{T_\alpha(1)} Z_n\right] = \check{\mathbb{E}}_{\nu^*}\left[\left(\sum_{m=0}^{T_\alpha} Z_m^2\right) \sum_{n=T_\alpha+1}^{T_\alpha(1)} Z_n\right]$$

$$+ 2\check{\mathbb{E}}_{\nu^*}\left[\left(\sum_{m=0}^{T_\alpha}\sum_{n=m+1}^{T_\alpha} Z_m Z_n\right) \sum_{k=T_\alpha+1}^{T_\alpha(1)} Z_k\right]$$

$$= A + 2B \quad \text{say},$$

moreover, $A = \sum_{m=0}^{\infty} a_m$ with

$$a_m = \check{\mathbb{E}}_{\nu^*}\left[Z_m^2 1_{\{T_\alpha \geqslant m\}} \sum_{n=m+1}^{\infty} Z_n 1_{\{T_\alpha+1 \leqslant n \leqslant T_\alpha(1)\}}\right]$$

$$\geqslant \check{\mathbb{E}}_{\nu^*}\left[Z_m^2 1_{\{Y_m=1\}} 1_{\{T_\alpha \geqslant m\}} \sum_{n=m+1}^{\infty} Z_n 1_{\{Y_j=0 \text{ if } m+1 \leqslant j \leqslant n-1\}}\right]$$

$$= \check{\mathbb{E}}_{\nu^*}\left[\check{\mathbb{E}}_{\delta^*_{X_{mm_0}}}\left[Z_0^2 1_{\{Y_0=1\}} \sum_{k=1}^{S_\alpha} Z_k\right] 1_{\{T_\alpha \geqslant m\}}\right]$$

and $B = \sum_{m=0}^{\infty}\sum_{n=m+1}^{\infty} b_{m,n}$ with

$$b_{m,n} = \check{\mathbb{E}}_{\nu^*}\left[Z_m Z_n 1_{\{T_\alpha \geqslant n\}} \sum_{k=n+1}^{\infty} Z_k 1_{\{T_\alpha+1 \leqslant k \leqslant T_\alpha(1)\}}\right]$$

$$\geqslant \check{\mathbb{E}}_{\nu^*}\Big[Z_m 1_{\{Y_m=0\}} 1_{\{T_\alpha \geqslant m\}} 1_{\{Y_j=0 \text{ if } m+1 \leqslant j \leqslant n-1\}} Z_n 1_{\{Y_n=1\}}$$

$$\times \sum_{k=n+1}^{\infty} Z_k 1_{\{Y_j=0 \text{ if } n+1 \leqslant j \leqslant k-1\}}\Big]$$

$$= \check{\mathbb{E}}_{\nu^*}\Big[1_{\{T_\alpha \geqslant m\}} \check{\mathbb{E}}_{\delta^*_{X_{mm_0}}}\Big[Z_0 1_{\{Y_0=0\}} 1_{\{S_\alpha \geqslant n-m\}}$$

$$\times \check{\mathbb{E}}_{\delta^*_{X_{(n-m)m_0}}}\left[Z_0 1_{\{Y_0=1\}} \sum_{r=1}^{S_\alpha} Z_r\right]\Big]\Big].$$

These facts imply (2.5a). For the proof of (2.5b) one can argue as for A above to obtain $\check{\mathbb{E}}_{\nu^*}[(\sum_{m=0}^{T_\alpha} Z_m)(\sum_{n=T_\alpha+1}^{T_\alpha(1)} Z_n)^2] \geqslant (1/\pi(s))\check{\mathbb{E}}_{\pi^*}[Z_0 1_{\{Y_0=1\}}(\sum_{n=1}^{S_\alpha} Z_n)^2]$ which is equal to the right-hand side of (2.5b) (for $m \geqslant 1$, write $Z_m \sum_{n=m+1}^{\infty} Z_n \times 1_{\{S_\alpha \geqslant n\}} = 1_{\{S_\alpha \geqslant m\}} Z_m 1_{\{Y_m=0\}} \sum_{n=m+1}^{\infty} Z_n 1_{\{Y_j=0 \text{ if } m<j<n\}}$). For (2.5c) use that if $0 \leqslant m < n$,

$$Z_m Z_n 1_{\{T_\alpha \geqslant m, T_\alpha+1 \leqslant n \leqslant T_\alpha(1)\}} \sum_{k=n+1}^{\infty} Z_k 1_{\{T_\alpha(1)+1 \leqslant k \leqslant T_\alpha(2)\}}$$

$$\geqslant Z_m Z_n 1_{\{T_\alpha \geqslant m, Y_m=1, Y_j=0 \text{ if } m<j<n, Y_n=1\}} \sum_{k=n+1}^{\infty} Z_k 1_{\{Y_j=0 \text{ if } n<j<k\}}.$$

From (2.4) and (2.5) we have that (2.1) is equivalent to

$$\pi(f^3) < \infty, \quad M_1 := \check{\mathbb{E}}_{\pi^*}\left[Z_0^2 \sum_{n=1}^{S_\alpha} Z_n\right] < \infty, \quad M_2 := \check{\mathbb{E}}_{\pi^*}\left[Z_0 \sum_{n=1}^{S_\alpha} Z_n^2\right] < \infty,$$
$$M_3 := \check{\mathbb{E}}_{\pi^*}\left[Z_0 \sum_{m=1}^{S_\alpha} \check{\mathbb{E}}_{\delta^*_{X_{mm_0}}}\left[Z_0 \sum_{n=1}^{S_\alpha} Z_n\right]\right] < \infty. \tag{2.6}$$

Noting that $\check{\mathbb{E}}_\gamma[\sum_{n=1}^{S_\alpha} Z_n | \check{\mathscr{F}}^X_{m_0} \vee \mathscr{F}^Y_0] = (\bar{G}f)(X_{m_0})$ and that $\check{\mathbb{E}}_\gamma[\sum_{n=1}^{S_\alpha} Z_n^2 | \check{\mathscr{F}}^X_{m_0} \vee \mathscr{F}^Y_0] = \sum_{m=0}^{m_0-1}(GP^m f^2)(X_{m_0}) + 2\sum_{\ell=0}^{m_0-2}\sum_{m=\ell+1}^{m_0-1}(GP^\ell[fP^{m-\ell}f])(X_{m_0})$ for any p.m. γ on

$(E_0, \mathscr{F}_0)$, we can show

$$M_1 = \sum_{m=0}^{m_0-1} \pi(f^2 P^{m_0-m} \bar{G} f) + 2 \sum_{\ell=0}^{m_0-2} \sum_{m=\ell+1}^{m_0-1} \pi(f P^{m-\ell} [f P^{m_0-m} \bar{G} f]), \tag{2.7a}$$

$$M_2 = \sum_{m=0}^{m_0-1} \sum_{k=0}^{m_0-1} \pi(f P^{m_0-k} G P^m f^2) + 2 \sum_{\ell=0}^{m_0-2} \sum_{m=\ell+1}^{m_0-1} \sum_{k=0}^{m_0-1} \pi(f P^{m_0-k} G P^{\ell} [f P^{m-\ell} f]), \tag{2.7b}$$

$$M_3 = \sum_{h=1}^{m_0} \sum_{m=0}^{m_0-1} \pi(f P^h G P^m [f P^{m_0-m} \bar{G} f]). \tag{2.7c}$$

Assume that (2.1) holds. Then $\bar{\mathbb{E}}_\gamma[(\zeta(1))^2] < \infty$ and $\infty > \bar{\mathbb{E}}_{\pi^*}[Z_0 \sum_{n=1}^{S_\alpha} Z_n] = \sum_{m=1}^{m_0} \pi(f P^m \bar{G} f)$ (see Nummelin, 1984, pp. 138–139; or argue as above); then Lemma 2.6 proves (2.2i) and $\alpha_m := \pi(f P^m \bar{G} f) < \infty$ if $0 \leqslant m \leqslant m_0$. On the other hand, we claim that $\gamma_{mk} := \pi(f G P^m [f P^m [f P^k G f]) < \infty$ if $0 \leqslant m \leqslant m_0 - 1$ and $0 \leqslant k \leqslant m_0 - m$. That $\gamma_{m,m_0-m} < \infty$ if $0 \leqslant m \leqslant m_0 - 1$ follows from (2.6), (2.7c) (consider the term corresponding to $h = m_0$ and m), the equality

$$g + P^{m_0} G g = G g + \frac{\pi(g)}{\pi(s)} s, \quad g : E \to [0, \infty] \ \mathscr{E}\text{-measurable} \tag{2.8}$$

and $\pi(f P^m [f P^{m_0-m} \bar{G} f]) < \infty$ (by (2.6) and (2.7a)). Hence it is sufficient to prove that $\gamma_{m,k+1} < \infty$ for some $0 \leqslant m \leqslant m_0 - 1$ and $0 \leqslant k \leqslant m_0 - m - 1$ implies $\gamma_{mk} < \infty$. Assume $\gamma_{m,k+1} < \infty$, $0 \leqslant m \leqslant m_0 - 1$, $0 \leqslant k \leqslant m_0 - m - 1$; by Lemma 2.6 and using that $\alpha_0 < \infty$ we only need to show that $\pi(f G P^m [f P^k f]) < \infty$ and we have two cases: (i) $0 \leqslant m \leqslant m_0 - 2$ and $1 \leqslant k \leqslant m_0 - m - 1$; (ii) $0 \leqslant m \leqslant m_0 - 1$ and $k = 0$. In case (i), $\pi(f P^{m_0} G P^m [f P^k f]) < \infty$ ((2.6) and (2.7b)) and $\pi(f G P^m [f P^k f]) < \infty$ by (2.8) applied to $g = P^m [f P^k f]$ since $\pi(fg) \leqslant \pi(f^3)$. In case (ii), apply (2.8) with $g = P^m f^2$ noting that $\pi(f P^{m_0} G P^m f^2) < \infty$ by (2.6), (2.7b) and that $\pi(fg) \leqslant \pi(f^3) < \infty$. Having proved our claim, we conclude that (2.1) implies (2.2i) and that $\pi(f \bar{G} [f \bar{G} f]) = \sum_{m=0}^{m_0-1} \gamma_{m,0} < \infty$, that is (2.2ii).

For the proof that (2.2) implies (2.1) consider, besides α_m, $\beta_{m,n} := \pi(f P^m \bar{G}(f P^n \bar{G} f))$ for $0 \leqslant m \leqslant m_0$, $0 \leqslant n \leqslant m_0$. Starting from $\alpha_0 < \infty$ and $\beta_{0,0} < \infty$ one can show that $\alpha_{m_0} < \infty$ and $\beta_{m_0,m_0} < \infty$ using (2.8) and then that $\alpha_m < \infty$ and $\beta_{m,n} < \infty$ if $0 \leqslant m \leqslant m_0$, $0 \leqslant n \leqslant m_0$, using Lemma 2.6. Thus $M_i < \infty$, $i = 1, 2, 3$, in (2.7) and we obtain (2.1) from its equivalence to (2.6).

Hence (2.1) and (2.2) are equivalent for every (m_0, s, v) satisfying (1.1). That the validity of (2.2) for some (m_0, s, v) as in (1.1) implies (R_3') and consequently (R_3), follows from the inequality quoted before Lemma 2.4. Conversely, assume (R_3) holds and that (m_0, s, v) is any triple satisfying (1.1). We will prove that (2.2) is verified.

First we fix $A \in \mathscr{E}^+$ and show that $\pi(f(I+U_A)f) < \infty$. Observe that

$$\int_A \pi(\mathrm{d}x)\mathbb{E}_x\left[\left(\sum_{n=1}^{S_A} f(X_n)\right)^3\right]$$

$$= \mathbb{E}_\pi[f(X_0)^3] + 3\mathbb{E}_\pi\left[f(X_0)^2 1_{A^c}(X_0)\mathbb{E}_{X_0}\left[\sum_{n=1}^{S_A} f(X_n)\right]\right]$$

$$+ 3\mathbb{E}_\pi\left[f(X_0)1_{A^c}(X_0)\mathbb{E}_{X_0}\left[\sum_{n=1}^{S_A} f(X_n)^2\right]\right]$$

$$+ 6\mathbb{E}_\pi\left[f(X_0)1_{A^c}(X_0)\mathbb{E}_{X_0}\left[\sum_{m=1}^{S_A} f(X_m)1_{A^c}(X_m)\mathbb{E}_{X_m}\left[\sum_{n=1}^{S_A} f(X_n)\right]\right]\right]. \tag{2.9}$$

For example, the last term comes from $\varphi(x) := \mathbb{E}_x[\sum_{m=1}^{S_A} f(X_m)\sum_{n=m+1}^{S_A} f(X_n) \times \sum_{k=n+1}^{S_A} f(X_k)] = \sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty} a_{mn}$ where

$$a_{m,n} = \mathbb{E}_x\left[f(X_m)f(X_n)\sum_{k=n+1}^{\infty} f(X_k)1_{\{S_A \geqslant k\}}\right]$$

$$= \mathbb{E}_x\left[f(X_m)f(X_n)1_{\{S_A \geqslant n\}}1_{A^c}(X_n)\mathbb{E}_{X_n}\left[\sum_{k=1}^{S_A} f(X_k)\right]\right].$$

Now $\varphi(x)$ equals

$$\sum_{m=1}^{\infty}\mathbb{E}_x\left[f(X_m)1_{\{S_A \geqslant m\}}1_{A^c}(X_m)\mathbb{E}_{X_m}\left[\sum_{r=1}^{S_A} f(X_r)1_{A^c}(X_r)\mathbb{E}_{X_r}\left[\sum_{k=1}^{S_A} f(X_k)\right]\right]\right]$$

$$= \mathbb{E}_x\left[\sum_{m=1}^{S_A} f(X_m)1_{A^c}(X_m)\mathbb{E}_{X_m}\left[\sum_{r=1}^{S_A} f(X_r)1_{A^c}(X_r)\mathbb{E}_{X_r}\left[\sum_{k=1}^{S_A} f(X_k)\right]\right]\right]$$

and then $\int_A \pi(\mathrm{d}x)\varphi(x)$ gives the last term in (2.9) by Proposition 5.9 of Nummelin (1984). Since

$$\pi(f(I+U_A)(f(I+U_A)f)) = \mathbb{E}_\pi\left[f(X_0)\mathbb{E}_{X_0}\left[\sum_{m=0}^{S_A} f(X_m)\mathbb{E}_{X_m}\left[\sum_{n=0}^{S_A} f(X_n)\right]\right]\right],$$

we obtain

$$\int_A \pi(\mathrm{d}x)\mathbb{E}_x\left[\left(\sum_{n=1}^{S_A} f(X_n)\right)^3\right] \leqslant 13\pi(f(I+U_A)(f(I+U_A)f)) < \infty.$$

Using Proposition 5.9 of Nummelin (1984), the Markov property and $\pi(f^3)<\infty$,

$$\pi(f(I+U_A)f)=\int_A \pi(\mathrm{d}x)\mathbb{E}_x\left[\sum_{j=0}^{S_A-1} f(X_j)\sum_{k=j}^{S_A} f(X_k)\right]$$

$$\leqslant \int_A \pi(\mathrm{d}x)\mathbb{E}_x\left[\left(\sum_{n=0}^{S_A} f(X_n)\right)^2\right]$$

$$\leqslant \pi(A)+\int_A \pi(\mathrm{d}x)\mathbb{E}_x\left[\left(\sum_{n=0}^{S_A} f(X_n)\right)^3\right]<\infty.$$

Since this holds for every $A\in\mathscr{E}^+$, Lemma 2.5 ensures that $f_1:=f\bar{G}f\in\mathscr{L}^1_+(\pi)$, that is, (2.2i) is verified; let $C\in\mathscr{E}^+$ and $M>0$ be obtained from that lemma for f_1 and $f_2:=f$; then

$$\pi(f\bar{G}[f\bar{G}f])\leqslant\pi(f\{(I+U_C)f_1+M\})=\pi(f(I+U_C)f_1)+M\pi(f)$$

and

$$\pi(f(I+U_C)f_1)\leqslant\pi(f(I+U_C)[f\{(I+U_C)f+M\}])$$

$$=\pi(f(I+U_C)[f(I+U_C)f])+M\pi(f(I+U_C)f),$$

which proves (2.2ii).

That (D_3) implies (R_3') follows by two applications of Lemma 2.7. It remains to prove that (R_3') implies (D_3). Assume (R_3') and take a set $C_1\in\mathscr{E}^+$ which is both f-regular and regular (Nummelin, 1984, Definition 5.4) (consider $\bar{f}:=\max\{1_E,f\}\in\mathscr{L}^1_+(\pi)$ in Proposition 5.13(ii) of Nummelin (1984). Then C_1 is small (Meyn and Tweedie, 1993, proof of Proposition 11.3.8) and if $V_1:E\to[0,\infty]$ is defined by $V_1(x):=\mathbb{E}_x[\sum_{n=0}^{T_{C_1}} f(X_n)]$, $x\in E$ ($T_A:=\inf\{n\geqslant 0\colon X_n\in A\}$ if $A\in\mathscr{E}$), then $PV_1+f\leqslant V_1+b_1 1_{C_1}$ (Meyn and Tweedie, 1993, Theorem 14.2.3(ii)) where $b_1:=\sup_{x\in C_1}[(I+U_{C_1})f](x)\in[0,\infty)$. Moreover $\pi(fV_1)\leqslant\pi(f(I+U_{C_1})f)\leqslant\pi(C_1)+13\pi(f(I+U_{C_1})(f(I+U_{C_1})f))<\infty$ by (R_3') (the second inequality was proved above). Therefore we can take again $C_2\in\mathscr{E}^+$ which is fV_1-regular and regular, then small. Defining $V_2(x):=\mathbb{E}_x[\sum_{n=0}^{T_{C_2}}(fV_1)(X_n)]$, $x\in E$, we have $PV_2+fV_1\leqslant V_2+b_2 1_{C_2}$ where $b_2:=\sup_{x\in C_2}[(I+U_{C_2})f](x)\in[0,\infty)$. Finally, $\pi(fV_2)\leqslant\pi(f(I+U_{C_2})(fV_1))\leqslant\pi(f(I+U_{C_2})(f(I+U_{C_1})f))<\infty$ by (R_3'). □

3. Strong mixing conditions

Denoting by α and β the strong mixing and absolute regularity coefficients, respectively, between σ-algebras in $(\Omega,\mathscr{F},\mathbb{P}_\pi)$ (see, for example, Bradley, 1986 or Rio, 2000), for $(X_n)_{n\geqslant 0}$ we have $\alpha(n):=\sup_{k\geqslant 0}\alpha(\sigma(X_j,j\leqslant k),\sigma(X_j,j\geqslant k+n))$ and $\beta(n)$ defined similarly. By Bradley (1986, Theorem 4.1), Bolthausen (1982, Lemma 1) and

Davydov (1973, Proposition 1) we have

$$\alpha(n) = \alpha(\sigma(X_0), \alpha(X_n))$$
$$= \tfrac{1}{2} \sup \left\{ \int \pi(\mathrm{d}x) |(P^n f)(x) - \pi(f)| \colon 0 \leqslant f \leqslant 1,\ \mathscr{E}\text{-measurable} \right\}$$

and

$$\beta(n) = \beta(\sigma(X_0), \sigma(X_n)) = \int \pi(\mathrm{d}x) \| P^n(x, \cdot) - \pi \|, \tag{3.1}$$

where $\| \cdot \|$ is the total variation norm.

Recall that by Theorem 1 of Athreya and Pantula (1986), $\alpha(n) \to 0$ as $n \to \infty$. The following version of Theorem 2 of Bolthausen (1980) is obtained from Bolthausen (1982) and Rio (2000); see also Remark 3.3. Here n^p can be replaced by the functions of n in the class Λ_0 considered in Rio (2000, Sections 9.5, 9.6).

Proposition 3.1. *Let $p \geqslant 0$, $p \in \mathbb{R}$. Assume (m_0, s, v) satisfies (1.1). The following conditions are equivalent*:

(i) $\sum_{n=1}^{\infty} n^p \alpha(n) < \infty$.
(ii) $\sum_{n=1}^{\infty} n^p \beta(n) < \infty$.
(iii) $\check{\mathbb{E}}_{\pi^*}[S_\alpha^{p+1}] < \infty$.
(iv) $\check{\mathbb{E}}_{(x,1)}[S_\alpha^{p+2}] < \infty$ *for some (for every)* $x \in E$.

Proof (*Sketch*). (iii) $\Leftrightarrow$ (iv) is shown by the equality $\check{\mathbb{E}}_{\pi^*}[S_\alpha^r] = \pi(s) \sum_{k=1}^{\infty} k^r \check{\mathbb{P}}_{(x,1)}[S_\alpha \geqslant k+1] + \pi(s) \check{\mathbb{E}}_{(x,1)}[S_\alpha^r]$ for any real $r \geqslant 0$ (we observe that (5.7) in Nummelin, 1984 has a version for π^*). (i) $\Rightarrow$ (iv) follows from Lemmas 5 and 3 in Bolthausen (1982) (or Rio, 2000, Proposition 9.7) applied to $(X_{nm_0})_{n \geqslant 1}$. By the first equality in (3.1), (iii) $\Rightarrow$ (ii) is a consequence of Corollaire 9.1 in Rio (2000) applied to $(X_{nm_0})_{n \geqslant 1}$. □

Using (1.12), Lemma 1.1, Proposition 2.1 and Proposition 5.16 in Nummelin (1984), we see that when $p = 0$, (iv) is equivalent to the ergodicity of degree 2 (Nummelin, 1984, Section 6.4) of $(X_n)_{n \geqslant 0}$. Hence we obtain (a) of the following result; for (b) use Proposition 2.3.

Corollary 3.2. (a) *$(X_n)_{n \geqslant 0}$ is ergodic of degree 2 if and only if $\sum_{n=1}^{\infty} \alpha(n) < \infty$, if and only if $\sum_{n=1}^{\infty} \beta(n) < \infty$, if and only if (D_2) holds for $f = 1_E$.*

(b) $\sum_{n=1}^{\infty} n\alpha(n) < \infty$ if and only if (D_3) holds for $f = 1_E$.

Remark 3.3. That ergodicity of degree 2 implies $\sum_{n=1}^{\infty} \beta(n) < \infty$ (in the expression given by (3.1)) was proved before in Proposition 2.1 of de Acosta (1997) and the converse in Theorem II-4.1 of Chen (1999a) through the remarkable equivalence to the property that the CLT holds for every bounded mean zero functional of the chain.

Remark 3.4. As a technical aside, we mention the following version (for the general case $m_0 \geqslant 1$) of Lemma 5 in Bolthausen (1982) and Lemma 9.4 in Rio (2000). *For any triple* (m_0, s, v) *as in* (1.1), *define*

$$\check{\alpha}_{(m_0,s,v)}(n) := \sup_{m \geqslant 0} \sup\{|\check{\mathbb{P}}_{\pi^*}(A \cap B) - \check{\mathbb{P}}_{\pi^*}(A)\check{\mathbb{P}}_{\pi^*}(B)| : A \in \check{\mathscr{F}}^X_{mm_0} \vee \check{\mathscr{F}}^Y_{m-1}, B \in \check{\mathscr{F}}^{m+n}\},$$

$$n \geqslant 1,$$

where $\check{\mathscr{F}}^k := \sigma((\check{X}_{km_0}, Y_k), (\check{X}_{k+1}, Y_{k+1}), \ldots)$. *Then* $\check{\alpha}_{m_0,s,v}(n) \leqslant 4\alpha(nm_0)$, $n \geqslant 1$, *and* $\alpha(n) \leqslant \check{\alpha}_{m_0,s,v}([n/m_0] - 1)$ *for* $n \geqslant 2m_0$. We prove the first assertion. Fix $n \geqslant 1, m \geqslant 0$, $A \in \check{\mathscr{F}}^X_{mm_0} \vee \check{\mathscr{F}}^Y_{m-1}$ and $B \in \check{\mathscr{F}}^{m+n}$; there exists $B' \in \check{\mathscr{F}}$ such that $B = \{((\check{X}_{(m+n)m_0}, Y_{m+n}), (\bar{X}_{m+n+1}, Y_{m+n+1}), \ldots) \in B'\}$. By (1.4), (1.3) and (1.5),

$$\check{\mathbb{E}}_{\pi^*}[1_B | \check{\mathscr{F}}^X_{mm_0} \vee \check{\mathscr{F}}^Y_{m-1}] = \check{\mathbb{E}}_{\delta^*_{\check{X}_{mm_0}}}[1_{B'} \circ \theta_n] = \check{\mathbb{E}}_{\delta^*_{\check{X}_{mm_0}}}[\check{\mathbb{P}}_{(\check{X}_{nm_0}, Y_n)}(B')]$$

$$= \check{\mathbb{E}}_{\delta^*_{\check{X}_{mm_0}}}[f_0(\check{X}_{nm_0})] + \check{\mathbb{E}}_{\delta^*_{\check{X}_{mm_0}}}[f_1(\check{X}_{nm_0})]$$

with $f_0(x) = (1 - s(x))\check{\mathbb{P}}_{(x,0)}(B')$ and $f_1(x) = s(x)\check{\mathbb{P}}_{(x,1)}(B')$, $x \in E$. By (1.3), $\check{\mathbb{P}}_{\pi^*}(B) = \check{\mathbb{E}}_{\pi^*}[\check{\mathbb{P}}_{(\check{X}_{(m+n)m_0}, Y_{m+n})}(B')] = \pi(f_0) + \pi(f_1)$. Then $|\check{\mathbb{P}}_{\pi^*}(A \cap B) - \check{\mathbb{P}}_{\pi^*}(A)\check{\mathbb{P}}_{\pi^*}(B)| \leqslant I_0 + I_1$ where $I_i = \int |\check{\mathbb{E}}_{\delta^*_{\check{X}_{mm_0}}}[f_i(\check{X}_{nm_0})] - \pi(f_i)| \mathrm{d}\check{\mathbb{P}}_{\pi^*}$, $i = 1, 2$.

Using that the $\check{\mathbb{P}}_{\mu^*}$-distribution of $(\check{X}_n)$ equals the $\mathbb{P}_\mu$-distribution of (X_n), we get (take $\mu = \delta_x$ and π) for $i = 1, 2$,

$$I_i = \int |\mathbb{E}_{\delta_{X_{mm_0}}}[f_i(X_{nm_0})] - \pi(f_i)| \,\mathrm{d}\mathbb{P}_\pi = \int \pi(\mathrm{d}x)|(P^{nm_0} f_i)(x) - \pi(f_0)| \leqslant 2\alpha(nm_0)$$

by Lemma 1 in Bolthausen (1982).

Finally, we show that the argument in the remark preceding Corollary 3 in Bolthausen (1980) gives the following extension of Lemma 4 in Bolthausen (1982).

Corollary 3.5. *Let* f *be a real* $\mathscr{E}$*-measurable function on* E. *Assume there exist* $p > 2$ *and* $\rho > 2/(p-2)$ *such that* $\int |f|^p \,\mathrm{d}\pi < \infty$ *and* $\sum_{n=1}^{\infty} n^\rho \alpha(n) < \infty$. *Then for any triple* (m_0, s, v) *satisfying* (1.1), $\check{\mathbb{E}}_\lambda[(\zeta(|f|, 1))^{p_1}] < \infty$ *for every p.m.* λ *on* $(E_0, \mathscr{F}_0)$, *where* $p_1 := p(2 + \rho)/(p + \rho + 1) \in (2, p)$.

Proof. Writing $p' = p/p_1$, q' for its conjugate exponent, $r = p_1 - p_1/p$, $s = -r$, we have $rq' = 2 + \rho$, $sp' + p_1 p' - 1 = 0$ and, for $x \in E$, $\check{\mathbb{E}}_{(x,1)}[(\sum_{n=1}^{S_\alpha} Z_n(|f|))^{p_1}] \leqslant A^{1/q'} B^{1/p'}$ with $A = \check{\mathbb{E}}_{(x,1)}[S_\alpha^{2+\rho}]$,

$$B = \check{\mathbb{E}}_{(x,1)}\left[S_\alpha^{sp'}\left(\sum_{n=1}^{S_\alpha} Z_n\right)^{p_1 p'}\right] \leqslant \check{\mathbb{E}}_{(x,1)}\left[\sum_{n=1}^{S_\alpha} Z_n^p\right] = \frac{1}{\pi(s)}\mathbb{E}_\pi\left[\left(\sum_{m=0}^{m_0-1} |f(X_n)|\right)^p\right]$$

by (1.4) and (5.7) in Nummelin (1984). □

4. An almost sure invariance principle

From Theorem 7.6 in Nummelin (1984) and its proof we have

Lemma 4.1. *Let f be a real-valued function in $\mathscr{L}_0^1(\pi)$ (the set of π-integrable functions with $\pi(f)=0$) and let (m_0,s,v) be a triple satisfying (1.1). Assume $\check{\mathbb{E}}_{v^*}[(\sum_{n=0}^{T_\alpha} Z_n(|f|))^2]<\infty$. Then the constant*

$$\sigma^2=\sigma^2(f):=\pi(f^2)+\frac{2}{m_0}\sum_{n=1}^{m_0}(m_0-n)\pi(fP^nf)+\frac{2}{m_0}\sum_{n=1}^{m_0}\pi(fP^n\bar{G}_{m_0,s,v}f)$$

$$=\frac{\pi(s)}{m_0}\left(\check{\mathbb{E}}_{v^*}\left[\left(\sum_{n=0}^{T_\alpha}Z_n(f)\right)^2\right]+2\check{\mathbb{E}}_{v^*}\left[\left(\sum_{n=0}^{T_\alpha}Z_n(f)\right)\left(\sum_{n=T_\alpha(0)+1}^{T_\alpha(1)}Z_n(f)\right)\right]\right)$$

$$=\frac{\pi(s)}{m_0}(\check{\mathbb{E}}_{\mu^*}[\zeta(f,1)^2]+2\mathbb{E}_{\mu^*}[\zeta(f,1)\zeta(f,2)]), \tag{4.1}$$

μ being any p.m. μ on $(E,\mathscr{E})$, is finite and nonnegative and does not depend on (m_0,s,v) nor on μ.

We give a version of Theorem 10.1 in Philipp and Stout (1975) and Theorem 2.1 in Csáki and Csörgő (1995), in similar terms to those of Csáki and Csörgő (1995) and Theorem 17.3.6 in Meyn and Tweedie (1993).

Proposition 4.2. *Let f be a real-valued function in $\mathscr{L}_0^1(\pi)$ and (m_0,s,v) be a triple satisfying (1.1).*

Assume there exists $\delta>0$ such that

$$\check{\mathbb{E}}_{v^*}[T_\alpha^{1+\delta/2}]<\infty \tag{4.2}$$

and

$$\check{\mathbb{E}}_{v^*}\left[\left(\sum_{n=0}^{T_\alpha}Z_n(|f|)\right)^{2+\delta}\right]<\infty. \tag{4.3}$$

Then for every p.m. μ on $(E,\mathscr{E})$ there exist a probability space $(\Omega',\mathscr{F}',\mathbb{P}')$ and a sequence of r.v.'s $(X_n')_{n\geqslant 0}$ together with a continuous standard Brownian motion $\{B'(t):\ t\in[0,\infty)\}$ defined on it such that

(i) *the distribution of $(X_n')_{n\geqslant 0}$ equals the $\mathbb{P}_\mu$-distribution of $(f(X_n))_{n\geqslant 0}$,*
(ii) *$|\sum_{0\leqslant n\leqslant t}X_n'-B'(\sigma^2 t)|=\mathrm{O}(t^{1/2-\epsilon})$ as $t\to\infty$, $\mathbb{P}'$-almost surely, for some $\epsilon\in(0,\frac{1}{2})$ (the constant implied by O being random),*

where σ^2 is the constant defined in (4.1).

Proof. Fix a p.m. λ on $(E_0,\mathscr{F}_0)$. We will show that the conclusion holds in fact for $(f(\check{X}_n))$ and $\check{\mathbb{P}}_\lambda$ in place of $(f(X_n))$ and $\mathbb{P}_\mu$. We follow Philipp and Stout (1975, Section 10.2). In place of (10.2.1) in Philipp and Stout (1975) define, for $N \geqslant 1$, as in Nummelin (1984, proof of Theorem 7.6), $i(N) = \max\{i \geqslant 0\colon (T_\alpha(i)+1)m_0 \leqslant N\}$ if $(T_\alpha(0)+1)m_0 \leqslant N$, and 0 otherwise. For $N \geqslant 1$ define $S_N = \sum_{n=0}^{N} f(X_n)$, $S_N' = \sum_{n=0}^{((T_\alpha+1)m_0-1)\wedge N} f(X_n)$, $\tilde{S}_N = \sum_{i=1}^{i(N)} \zeta(i)$, $S_N'' = \sum_{n=(T_\alpha(i(N))+1)m_0}^{N} f(X_n)$; then $S_N = S_N' + \tilde{S}_N + S_N''$. We have $|S_N'| = \mathrm{O}(1)$ as $N \to \infty$ $\check{\mathbb{P}}_\lambda$-a.s. because $\check{\mathbb{P}}_\lambda[T_\alpha < \infty] = 1$. On the other hand $|S_N''| \leqslant \zeta(|f|, i(N)+1)$ for all sufficiently large N $\check{\mathbb{P}}_\lambda$-a.s. By (4.3), $\check{\mathbb{E}}_\lambda[\zeta(|f|,1)^{2+\delta}] < \infty$ which implies, by the Borel–Cantelli lemma (or as in Philipp and Stout, 1975, Section 10.2), that $|S_N''| = \mathrm{O}(N^{(1-\delta/(2+\delta))/2})$, as $N \to \infty$, $\check{\mathbb{P}}_\lambda$-a.s. ($i(N)+1 \leqslant N/m_0$ if $(T_\alpha+1)m_0 \leqslant N$). Then

$$|S_N - \tilde{S}_N| = \mathrm{O}(N^{(1-\delta/(2+\delta))/2}) \quad \text{as } N \to \infty, \quad \check{\mathbb{P}}_\lambda\text{-a.s.} \tag{4.4}$$

By using Theorem 4.1 in Philipp and Stout (1975), we obtain a probability space $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{\mathbb{P}})$ and a sequence of r.v.'s $(\bar{\zeta}(i))_{i\geqslant 1}$ together with a continuous standard Brownian motion $\{\bar{B}(t)\colon t \in [0,\infty)\}$ defined on it such that $(\bar{\zeta}(i))_{i\geqslant 1}$ has the same distribution as the $\check{\mathbb{P}}_\lambda$-distribution of $(\zeta(i))_{i\geqslant 1}$ and

$$\left|\sum_{0\leqslant i\leqslant t} \bar{\zeta}(i) - \bar{\sigma}\bar{B}(t)\right| = \mathrm{O}(t^{1/2-\bar{\epsilon}}) \quad \text{as } t \to \infty, \quad \bar{\mathbb{P}}\text{-a.s.} \tag{4.5}$$

for some $\bar{\epsilon} \in (0, \frac{1}{2})$, where $\bar{\sigma}^2 := \check{\mathbb{E}}_\lambda[\zeta(1)^2] + 2\check{\mathbb{E}}_\lambda[\zeta(1)\zeta(2)]$. By proof of Theorem 7.6 in Nummelin (1984), $(\pi(s)/m_0)\bar{\sigma}^2 = \sigma^2$.

We need versions of these r.v.'s, even of $i(N)$, defined on a common probability space. Consider the Polish spaces (with their natural topologies) $\mathscr{X} = \mathbb{R}^\infty\{0,1\}^\infty$, $\mathscr{Y} = \mathbb{R}^\infty$, $\mathscr{Z} = C[0,\infty)$ and the p.m.'s $\beta = \mathscr{L}_{\check{\mathbb{P}}_\lambda}(((f(X_k))_{k\geqslant 0},(Y_n)_{n\geqslant 0}),(\zeta(i))_{i\geqslant 1})$ and $\gamma = \mathscr{L}_{\bar{\mathbb{P}}}((\bar{\zeta}(i))_{i\geqslant 1},\bar{B})$ on the product spaces $\mathscr{X}\times\mathscr{Y}$ and $\mathscr{Y}\times\mathscr{Z}$, respectively. Let $\Omega' = \mathscr{X}\times\mathscr{Y}\times\mathscr{Z}$, $\mathscr{F}'$ be the product σ-algebra and denote by $((X_k')_{k\geqslant 0},(Y_n')_{n\geqslant 0})$, $(\zeta'(i))_{i\geqslant 1}$ and B'' the projections onto $\mathscr{X}$, $\mathscr{Y}$ and $\mathscr{Z}$, respectively. Then (Berkes and Philipp, 1979, Lemma A1; de Acosta, 1982, Corollary A.2) there exists a p.m. $\mathbb{P}'$ on $(\Omega',\mathscr{F}')$ such that $\mathscr{L}(((X_k')_{k\geqslant 0},(Y_n'')_{n\geqslant 0}),(\zeta'(i))_{i\geqslant 1}) = \beta$ and $\mathscr{L}((\zeta'(i))_{i\geqslant 1},B'') = \gamma$. Now define on Ω' the r.v.'s $T_\alpha'(i)$, $i \geqslant 0$, $i'(N)$, $N \geqslant 1$, in terms of (Y_n') as $T_\alpha(i)$, $i(N)$ are defined with respect to (Y_n). From (4.4) and (4.5) we get

$$\left|\sum_{0\leqslant n\leqslant t} X_n' - \bar{\sigma}B''(i'([t]))\right| = \mathrm{O}(t^{1/2-\epsilon'}) \quad \text{as } t \to \infty, \quad \mathbb{P}'\text{-a.s.},$$

for some $\epsilon' \in (0,\frac{1}{2})$ (note that $i'(N) < N/m_0$ and $i'(N) \to \infty$ $\mathbb{P}'$-a.s.).

This implies the conclusion with $B'(\mu) := \sigma B''(\sigma^{-2}\mu)$, $\mu \geqslant 0$, if we show that

$$|B''(i'([t])) - B''\left(\frac{\pi(s)}{m_0}t\right)| = \mathrm{O}(t^{1/2-\epsilon''}) \quad \text{as } t \to \infty, \quad \mathbb{P}'\text{-a.s.}, \tag{4.6}$$

for some $\epsilon'' \in (0,\frac{1}{2})$ ($[\,\cdot\,]=$ integer part). First, observe that

$$\left|i'(N) - \frac{\pi(s)}{m_0}N\right| = \mathrm{O}(N^{1-\delta/(2+\delta)}) \quad \text{as the integer } N \to \infty, \quad \mathbb{P}'\text{-a.s.} \tag{4.7}$$

In fact, $i(N) - (\pi(s)/m_0)N = o(N^{2/(2+\delta)})$ as $N \to \infty$, $\mathbb{P}_\lambda$-a.s.; this can be obtained from $T_\alpha(i) - i/\pi(s) = o(i^{2/(2+\delta)})$ as $i \to \infty$, $\mathbb{P}_\lambda$-a.s., which follows from the strong law of large numbers of Marcinkiewicz–Zygmund applied to the 1-dependent sequence $\rho_j = T_\alpha(j) - T_\alpha(j-1)$, $j \geqslant 1$, since $\check{\mathbb{E}}_\lambda[\rho_1^{1+\delta/2}] = \check{\mathbb{E}}_{\nu*}[(1+T_\alpha)^{1+\delta/2}] < \infty$ by (4.2) and $\check{\mathbb{E}}_\lambda \rho_1 = \check{\mathbb{E}}_{\nu*}[1+T_\alpha] = 1/\pi(s)$ (use (1.7)).

Moreover, let $\beta \in (0, \delta/(2+\delta))$ and $\gamma > 0$, to be determined later, and $p_n := n^{\gamma+1/\beta}$. Arguing as in proof of Lemma 3.5.3 in Philipp and Stout (1975) (using (4.7)) we can show that if $M_n = \max_{p_n \geqslant t \geqslant p_{n+1}} |B''(I'([t])) - B''((\pi(s)/m_0)t|$, $c = \pi(s)/m_0$, and $R(a,b) = \max_{a \leqslant s,t \leqslant b} |B''(s) - B''(t)|$ $(0 \leqslant a \leqslant b)$ then $M_n \leqslant R(cp_{n-1}, cp_{n+2})$ for all sufficiently large n, $\mathbb{P}'$-a.s. If we take $\beta \leqslant -1+\sqrt{2}$, $\mathbb{P}'(R(cp_{n-1}, cp_{n+2}) \geqslant p_n^{1/2(1-\beta+\gamma)}) \leqslant \mathbb{P}'(R(0,1) \geqslant Kn^\gamma) \leqslant 2\mathbb{P}'(|B''(1)| \geqslant \frac{1}{2}Kn^\gamma)$ where K is a constant depending on β and γ. The Borel–Cantelli lemma then gives that $M_n = O(p_n^{(1-\beta+\gamma)/2})$ as $n \to \infty$, $\mathbb{P}'$-a.s., which proves (4.6) choosing $0 < \gamma < \beta \leqslant \delta/(2+\delta)$ and $\beta \leqslant -1+\sqrt{2}$. □

Let f be a real-valued function in $\mathscr{L}_0^1(\pi)$. We will say that f *satisfies the almost sure invariance principle* (*ASIP*), if the conclusion of Proposition 4.2 is verified. If the random elements $\{(1/\sqrt{n})\sum_{0 \leqslant k \leqslant nt} X_k \colon 0 \leqslant t \leqslant 1\}$ of the Skorohod space $D[0,1]$ converge in distribution as $n \to \infty$ to $\{B(\sigma^2 t)\colon 0 \leqslant t \leqslant 1\}$ where $\{B(t)\colon 0 \leqslant t \leqslant 1\}$ is a standard Brownian motion and σ^2 is defined by (4.1), we will say that f *satisfies the functional CLT* (*FCLT*).

Now we deal with ergodic chains of degree 2 (recall Corollary 3.2).

Corollary 4.3. *A function $f \in \mathscr{L}_0^1(\pi)$ satisfies the ASIP if either one of the following sets of conditions is verified*:

(i) *There exist $p > 2$ and $\rho > 2/(p-2)$ such that $\pi(|f|^p) < \infty$ and $\sum_{n=1}^\infty n^\rho \alpha(n) < \infty$.*
(ii) *f is bounded and $\sum_{n=1}^\infty n^\delta \alpha(n) < \infty$ for some $\delta > 0$.*
(iii) *$(X_n)_{n \geqslant 0}$ is ergodic of degree 2 and $\pi(|f|(I+U_A)(|f|(I+U_A)|f|)) < \infty$ for every $A \in \mathscr{E}^+$.*
(iv) *$(X_n)_{n \geqslant 0}$ is ergodic of degree 2 and f is bounded and special* (Nummelin, 1984, Definition 5.4).
(v) *$(X_n)_{n \geqslant 0}$ is ergodic of degree 2 and f is bounded and vanishes on the complement of some regular set* (Nummelin, 1984, Definition 5.4).

Proof. For the proof of (i) use Proposition 3.1 with (1.12) and Corollary 3.5 with Lemma 1.1. (ii) is a consequence of (i) (or use Proposition 3.1).

(iii) follows from the fact that ergodicity of degree 2 is equivalent to $\check{\mathbb{E}}_{(x,1)}[S_\alpha^2] < \infty$ for some (for every) $x \in E$ (by the results quoted before Corollary 3.2) and Proposition 2.1. (iv) and (v) follow from (iii). □

By Proposition 2.2 in de Acosta (1997), under the conditions in (iv), (v) above the constant σ^2 in (4.1) verifies (4.8) below. Now we extend that result. In what follows, $f: E \to \mathbb{R}$ is an $\mathscr{E}$-measurable function and $\bar{f} := f - \pi(f)$ when $\pi(f) < \infty$.

Proposition 4.4. *Assume $(X_n)_{n\geqslant 0}$ is ergodic of degree 2. Let g be a real-valued function in $\mathscr{L}^1_+(\pi)$ such that*

(i) *πI_g is g-regular,*
(ii) *π is g-regular, and*
(iii) *πI_g is regular.*

Then, if $|f| \leqslant g$, the constant σ^2 defined in (4.1) satisfies

$$\sigma^2 = \mathbb{E}_\pi[\bar{f}(X_0)^2] + 2\sum_{k=1}^{\infty} \mathbb{E}_\pi[\bar{f}(X_0)\bar{f}(X_k)]$$

$$= \lim_{n\to\infty} \frac{1}{n}\mathbb{E}_\pi\left[\left(\sum_{k=0}^{n} \bar{f}(X_k)\right)^2\right], \tag{4.8}$$

the series being absolutely convergent.

Proof. First we define a version of $P^n h$ for any real $\mathscr{E}$-measurable function h such that $|h| \leqslant Vg$ and any $n \geqslant 0$. Since $\pi(g) < \infty$ there exists $N \in \mathscr{E}$ with $\pi(N) = 0$ such that for each $x \in N^c$ and each $n \geqslant 0$, $(P^n(Vg))(x) = P^n(x,\cdot)(V_g) < \infty$. Then $M := \{y\colon\ P^n(y,N) > 0 \text{ for some } n \geqslant 0\} \in \mathscr{E}$ satisfies $\pi(M) = 0$ (Nummelin, 1984, Proposition 2.4(iii)). Given h as above and $n \geqslant 0$ we define $h^{(n)}(y) = P^n(y,\cdot)(h)$ if $y \notin N$, 0 if $y \in N$. Then, for each $n \geqslant 0$, $h^{(n)}$ is $\mathscr{E}$-measurable, $\pi(h^{(n)}) = \pi(h)$ and $P^n(x,\cdot)(h) = P^m(x,\cdot)h^{(i)}$ for every $x \notin M(\supset N)$ if $n = m + i$, $m, i \geqslant 0$ (by dominated convergence applied to approximating simple functions, noting that $P^m(x,N) = 0$ if $x \notin M$).

Fix f as in the statement. Note that

$$|\mathbb{E}_\pi[\bar{f}(X_0)\bar{f}(X_n)]| = \left|\int_{M^c} \pi(\mathrm{d}x)\bar{f}(x)(f^{(n)}(x) - \pi(f)\right|$$

$$\leqslant \|\bar{f}P^n\bar{f}\|_{\mathscr{L}^1(\pi)} = \int_{M^c} \pi(\mathrm{d}x)|f(x) - \pi(f)||f^{(n)}(x) - \pi(f)|.$$

Then by using Theorem II-3.1 in Chen (1999a) and an argument in proof of Theorem 17.5.3 in Meyn and Tweedie (1993), the result will follow if we show that $\sum_{n=0}^{\infty} \|\bar{f}P^n\bar{f}\|_{\mathscr{L}^1(\pi)} < \infty$. It is sufficient to prove that

$$\sum_{n=0}^{\infty} \int \pi(\mathrm{d}x)|f(x)|\,|f^{(n)}(x) - \pi(f)| < \infty \tag{4.9}$$

and

$$\sum_{n=0}^{\infty} \int \pi(\mathrm{d}x)|f^{(n)}(x) - \pi(f)| < \infty.$$

When $x \in M^c$, $\|P^n(x,\cdot) - \pi\|_g \leqslant \|P^{km_0}(x,\cdot) - \pi\|_{V_a} < \infty$ ($\|\cdot\|_g$ is defined in Nummelin (1984, Section 6.2) if $n = km_0 + i$, $n, m \geqslant 0$, $0 \leqslant i \leqslant m_0 - 1$ (given any real $\mathscr{E}$-measurable function h such that $|h| \leqslant g$, consider $h^{(i)}$; on the other hand, $|P^n(x,\cdot)-\pi|(Vg) \leqslant P^n(x,\cdot)(Vg)+\pi(Vg)<\infty$).

We use concepts, notations and results developed in Nummelin (1984, Chapters 4 and 6). Consider the sequences $u=(u_n)_{n\geqslant 0}$ and, for $x\in E$, $a(x)=(a_n(x))_{n\geqslant 0}$ defined by $u_0=1$, $u_n=v(P^{m_0})^{n-1}s$ if $n\geqslant 1$, $a_n(x)=[(P^{m_0}-s\otimes v)^n s](x)=\check{\mathbb{P}}_{\delta_x^*}(T_\alpha=n)$, $n\geqslant 0$; for any $\mathscr{E}$-measurable function h such that $|h|\leqslant Vg$, consider $\sigma(h)=(\sigma_n(h))_{n\geqslant 0}$ with $\sigma_n(h)=[v(P^{m_0}-s\otimes v)^n](h)$, $n\geqslant 0$. By (4.23) in Nummelin (1984) $v(P^{m_0})^{n-1}(h)=(u\bigstar\sigma(h))_{n-1}$, $n\geqslant 1$. Note that $\sum_{n=0}^{\infty}\sigma_n(Vg)=\pi(s)^{-1}\pi(Vg)<\infty$ (by positive recurrence) and then that $v(P^{m_0})^n(Vg)=(u\bigstar\sigma(Vg))_n<\infty$ for each $n\geqslant 0$.

Using the first-entrance-last-exit decomposition (Nummelin, 1984, (4.24)) it can be shown that if $n=km_0+i$, $n,m\geqslant 0$, $0\leqslant i\leqslant m_0-1$, for $x\notin M$ we have

$$
\begin{aligned}
&|f^{(n)}(x)-\pi(f)|\\
&\quad\leqslant [(P^{m_0}-s\otimes v)^k(Vg)](x)+(|a(x)\bigstar u-u|\bigstar\sigma(Vg))_{k-1}\\
&\qquad+\int_{N^c}\pi(\mathrm{d}z)\left\{\left[(P^{m_0}-s\otimes v)^k(Vg)\right](z)+\left(|a(z)\bigstar u-u|\bigstar\sigma(Vg)\right)_{k-1}\right\}.
\end{aligned}
\tag{4.10}
$$

($|c|:=(|c_n|)_{n\geqslant 0}$ for any sequence c).

For the convergence of the first series in (4.9) it is sufficient to have the convergence of the series whose kth term is the π-integral of the right-hand member of (4.10) multiplied by $g(x)$. This requires the finiteness of the following four quantities: $A_1=\pi(g\bar{G}g)$,

$$
\begin{aligned}
A_2&=\int\pi(\mathrm{d}x)g(x)\left(\sum_{k=0}^{\infty}|a(x)\bigstar u-u|_k\right)\left(\sum_{k=0}^{\infty}\sigma_k(Vg)\right)\\
&\leqslant\frac{m_0}{\pi(s)}\pi(g)Var(u)\left(\int\pi(\mathrm{d}x)g(x)\check{\mathbb{E}}_{\delta_x^*}[T_\alpha]\right)
\end{aligned}
$$

where $Var(u)=1+\sum_{n=1}^{\infty}|u_n-u_{n-1}|<\infty$ by Theorem 6.4(i) in Nummelin (1984) (the hypotheses there are satisfied; for example, the increment sequence of u, $b_0:=0$, $b_n:=v(P^{m_0}-s\otimes v)^{n-1}s$, $n\geqslant 1$, verifies $M_b<\infty$—see Nummelin (1984, p. 74)—and $M_{a(x)}=\check{\mathbb{E}}_{\delta_x^*}[T_\alpha]$), $A_3=\pi(g)\pi(\bar{G}g)$ and

$$
\begin{aligned}
A_4&=\int\pi(\mathrm{d}x)g(x)\int_{N^c}\pi(\mathrm{d}z)\left(\sum_{k=0}^{\infty}|a(z)\bigstar u-u|_k\right)\left(\sum_{k=0}^{\infty}\sigma_k(Vg)\right)\\
&\leqslant\frac{m_0}{\pi(s)}\pi(g)^2Var(u)\left(\int\pi(\mathrm{d}x)\check{\mathbb{E}}_{\delta_x^*}[T_\alpha]\right).
\end{aligned}
$$

We are led to similar quantities when the second series in (4.9) is considered. Both sets of bounds show that (4.9) is verified if the following five quantities are finite: $\pi(g)$, $\pi(g\bar{G}g)$, $\pi(\bar{G}g)$, $\int \pi(\mathrm{d}x)g(x)\mathbb{E}_{\delta^*}[T_\alpha]$ and $\int \pi(\mathrm{d}x)\mathbb{E}_{\delta^*}[T_\alpha]$. These conditions as a whole are equivalent to the hypotheses of the proposition (use Nummelin, 1984, Proposition 5.13(iv)) and (1.10) with $\mu = \delta_x, f = 1_E$). $\square$

The next result shows that in part of Theorem 17.5.3 in Meyn and Tweedie (1993), we can replace $\pi(V^2) < \infty$ by $\pi(gV) < \infty$. For its proof use the preceding proposition and Lemma 2.7.

Corollary 4.5. *Let g be a real-valued function in $\mathscr{L}^1_+(\pi)$ that satisfies (D_2) and such that $g \geqslant 1$. Then $(X_n)_{n\geqslant 0}$ is ergodic of degree 2 and the conclusion of Proposition* 4.4 *holds.*

5. Examples

5.1. Some remarks for the case $E \subseteq \mathbb{R}^d$

Assume that $(X_n)_{n\geqslant 0}$ is a Markov chain on $(E, \mathscr{E})$ with transition probability kernel P, where E is a Borel subset of $\mathbb{R}^d$ and $\mathscr{E}$ is its Borel σ-algebra. Let $\|\cdot\|$ be the Euclidean norm. Given $p \geqslant 0, p \in \mathbb{R}$, we define $g_p : E \to [0,\infty)$ by $g_p(x) = 1 + \|x\|^p$, $x \in E$. For $p \geqslant 1$ consider the property

there exist a constant $b \in [0,\infty)$, a small set $C \in \mathscr{E}^+$
and an $\mathscr{E}$-measurable function $V = E \to [0,\infty)$ verifying
$c_1\|x\|^p \leqslant V(x) \leqslant c_2\|x\|^p$, $x \in E$, for some positive constants c_1, c_2,
such that $PV + g_{p-1} \leqslant V + b1_C$. $(\mathscr{V}_p)$

For $p > 0$, $p \in \mathbb{R}$, let $(\mathscr{V}'_p)$ be the condition obtained from this by replacing g_{p-1} by g_p.

For the rest of this subsection we assume that $(X_n)_{n\geqslant 0}$ is Harris ergodic with invariant probability measure π. The following results will be used in the examples (the part involving $(\mathscr{V}_p)$ when $E = [0,\infty)$ in Sections 5.2 and 5.3; $(\mathscr{V}'_p)$ when $E = \mathbb{R}^d$ in Section 5.4). We omit the proofs concerning $(\mathscr{V}'_p)$.

Lemma 5.1.1. (a) *If $(\mathscr{V}_p)$ is verified with $p \in [1,\infty)$ then $\int_E \|x\|^{p-1}\pi(\mathrm{d}x) < \infty$. If $(\mathscr{V}'_p)$ is verified with $p \in (0,\infty)$ then $\int_E \|x\|^p\pi(\mathrm{d}x) < \infty$.*

(b) *If $(\mathscr{V}_p)$ is verified for $p = 1$ and 2 (or if $(\mathscr{V}'_p)$ is verified for some $p > 0$) then $(X_n)_{n\geqslant 0}$ is ergodic of degree 2.*

Proof. (a) Use Theorem 14.3.7 in Meyn and Tweedie (1993). (b) By $(\mathscr{V}_2)$ and (a), $\int_E \|x\|\pi(\mathrm{d}x) < \infty$; then $(\mathscr{V}_1)$ and Theorem 11.3.12(i) in Meyn and Tweedie (1993) show that π is 1-regular. Use Proposition 5.16(i) in Nummelin (1984). $\square$

In what follows, $f : E \to \mathbb{R}$ is a Borel measurable function and $\bar{f} := f - \pi(f)$ when $\pi(f) < \infty$.

Proposition 5.1.2. *Let $p \in [1,\infty)$ ($p \in (0,\infty)$). Assume that ($\mathscr{V}_r$) (respectively, ($\mathscr{V}'_r$)) is verified for $r = p$ and $2p$. Then g_{p-1} (respectively, g_p) satisfies (R_2). If $|f| \leqslant a g_{p-1}$ (respectively, $|f| \leqslant a g_p$) for some $a \in (0,\infty)$, we have that $\pi(|f|) < \infty$, (R_2) holds for $|\bar{f}|$ and, therefore, $\bar{f}$ satisfies the FCLT and the constant σ^2 in (4.1) verifies (4.8).*

Proof. g_{p-1} satisfies the first inequality in (D_2) with the function V given by ($\mathscr{V}_p$); ($\mathscr{V}_{2p}$) and Lemma 5.1.1(a) show that $\pi(g_{p-1}V) < \infty$. Then g_{p-1} satisfies (R_2) by Proposition 2.1. Assume $|f| \leqslant a g_{p-1}$ for some constant a. If $p = 1$, f is bounded and if $p > 1$, $\pi(|f|^{1+p/(p-1)}) < \infty$ by ($\mathscr{V}_{2p}$) and Lemma 5.1.1(a); moreover $|\bar{f}| \leqslant a' g_{p-1}$ for some a' which now implies that $|\bar{f}|$ verifies (R_2) and then that $\bar{f}$ satisfies the FCLT (see Remark 2.2). Use Corollary 4.5. □

Corollary 5.1.3. (a) *Assume $d = 1$. If ($\mathscr{V}_r$) is verified for $r = 2$ and 4 (or if ($\mathscr{V}'_r$) is verified for $r = 1$ and 2) and $f(x) = x$ we have that $\pi(|f|) < \infty$, (R_2) holds for $|\bar{f}|$ and, therefore, $\bar{f}$ satisfies the FCLT and the constant σ^2 in (4.1) verifies (4.8).*

(b) *The same conclusions are valid if ($\mathscr{V}_r$) is verified for $r = 1$ and 2 (or if ($\mathscr{V}'_r$) is verified for some $r > 0$) and f is bounded.*

Proposition 5.1.4. *Let $p \in [1,\infty)$ ($p \in (0,\infty)$). Assume that ($\mathscr{V}_r$) is verified for $r = p, p+1, 2p$ and $3p$ (respectively, ($\mathscr{V}'_r$) is verified for $r = p$, $2p$ and $3p$). Then g_{p-1} (respectively, g_p) satisfies (R_3). Suppose that $|f| \leqslant a g_{p-1}$ (respectively, $|f| \leqslant a g_p$) for some $a \in (0,\infty)$. We have that $\pi(|f|) < \infty$ and $|\bar{f}|$ satisfies (R_3); if moreover ($\mathscr{V}_1$) and ($\mathscr{V}_2$) are verified (nothing else about ($\mathscr{V}'_r$)), then $(X_n)_{n\geqslant 0}$ is ergodic of degree 2, $\bar{f}$ satisfies the ASIP and the constant σ^2 in (4.1) verifies (4.8).*

Proof. By ($\mathscr{V}_p$) and ($\mathscr{V}_{2p}$), g_{p-1} verifies the first two inequalities in (D_3) with b_1 the constant, C_1 the small set and V_1 the function, with associated constants c_1, c_2, given by ($\mathscr{V}_p$). Let b', C', V' and b'', C'', V'' analogously obtained from ($\mathscr{V}_{p+1}$) and ($\mathscr{V}_{2p}$), respectively. Then g_{p-1} also verifies the third inequality with $b_2 := c_2(b' + b'')$, $C_2 := C' \cup C''$, which is small by Corollary 2.1(iii) in Nummelin (1984) and $V_2 := c_2(V' + V'')$. From ($\mathscr{V}_{3p}$) we obtain that $\pi(g_{p-1}V_2) < \infty$. Hence g_{p-1} verifies (R_3) by Proposition 2.3.

Then, as in the proof of Proposition 5.1.2, the assertions about f ($\pi(|f|^{1+2p/(p-1)}) < \infty$ if $p > 1$) and $\bar{f}$ follow (by ($\mathscr{V}_{3p}$)) using Lemma 5.1.1 and Corollary 4.3(iii). □

Corollary 5.1.5. *If ($\mathscr{V}_r$) is verified for $r = 1,2,3$ (($\mathscr{V}'_r$) is verified for some $r > 0$) then $(X_n)_{n\geqslant 0}$ is ergodic of degree 2 and if f is bounded, we have that (R_3) holds for $|\bar{f}|$, $\bar{f}$ satisfies that ASIP and the constant σ^2 in (4.1) verifies (4.8).*

Corollary 5.1.6. *Assume $d = 1$. If ($\mathscr{V}_r$) is verified for $r = 2,3,4,6$ (($\mathscr{V}'_r$) is verified for $r = 1,2,3$) and $f(x) = x$ we have that $\pi(|f|) < \infty$ and $|\bar{f}|$ satisfies (R_3). If moreover*

$(\mathscr{V}_1)$ *is verified (nothing else about* $(\mathscr{V}'_r)$*), then* $(X_n)_{n\geqslant 0}$ *is ergodic of degree* 2, $\bar{f}$ *satisfies the ASIP and the constant* σ^2 *in* (4.1) *verifies* (4.8).

5.2. Example. δ-skeleton of the forward process

Let $Z_0, Y_1, Y_2, \ldots$ be independent real r.v.'s taking its values in $E=[0,\infty)$, $\mathscr{E}$ being the class of its Borel subsets, $\Gamma_0 := \mathscr{L}(Z_0)$, with $Y_1, Y_2, \ldots$ identically distributed and $\Gamma := \mathscr{L}(Y_i)$, $i \geqslant 1$. Define $Z_n = Z_0 + \sum_{i=1}^n Y_i$ for $n \geqslant 1$. We assume that Γ is spread-out (Nummelin, 1984, Example 2.1(c)), not concentrated at 0, and that $\mathbb{E}Y_1 < \infty$. Fix $\delta > 0$ such that $\Gamma([0,\delta)) < 1$. Then, if $X_n = V^+(n\delta)) := \inf\{Z_k - n\delta\colon Z_k \geqslant n\delta, k \geqslant 0\}$, $n \geqslant 0$, $(X_n)_{n\geqslant 0}$ is a Harris ergodic Markov chain on $(E,\mathscr{E})$ (Meyn and Tweedie, 1993; Nummelin, 1984) with $\pi(\mathrm{d}t) = c^{-1}\Gamma((t,\infty))\,\mathrm{d}t$, $c := \mathbb{E}Y_1$ and $[0,\delta)$ is a small set in $\mathscr{E}^+$.

Lemma 5.2.1. *Assume* $\mathbb{E}[(Y_1)^p] < \infty$ *with* $p \geqslant 1$, $p \in \mathbb{R}$. *Then* $(X_n)_{n\geqslant 0}$ *satisfies* $(\mathscr{V}'_p)$.

Proof. For this chain, its transition operator P_δ, say, applied to any nonnegative $\mathscr{E}$-measurable function h, gives that $(P_\delta h)(x)$ equals

$$\begin{cases} \displaystyle\int_{[0,\infty)} 1_{[0,\delta-x)}(t)\left\{\int_{[\delta-x-t,\infty)} h(s+t-(\delta-x))\Gamma(\mathrm{d}s)\right\} U(\mathrm{d}t) & \text{if } 0 \leqslant x < \delta, \\ h(x-\delta) & \text{if } x \geqslant \delta, \end{cases}$$

where U is the renewal measure $U(A) = \sum_{r=0}^\infty \Gamma^{r*}(A)$, $A \in \mathscr{E}$. Let $c := (\inf_{x\in[\delta,\infty)}(x^p - (x-\delta)^p)/(1+x^{p-1}))^{-1} \in (0,\infty)$ and $b := cU([0,\delta))\mathbb{E}[(Y_1)^p] + 1 + \delta^{p-1} \in (0,\infty)$ (as is well known, $U([0,\delta)) < \infty$). If $V(x) := cx^p$ then $(P_\delta V)(x) + g_{p-1}(x) \leqslant V(x)$ if $x \geqslant \delta$, and $(P_\delta V)(x) \leqslant cU([0,\delta))\mathbb{E}[(Y_1)^p]$ if $0 \leqslant x < \delta$. □

Proposition 5.2.2.

(a) *If any one of the following conditions is verified then* $(X_n)_{n\geqslant 0}$ *is ergodic of degree* 2, $\pi(|f|) < \infty$, $\bar{f}$ *satisfies the FCLT and the constant* σ^2 *in* (4.1) *verifies* (4.8):

(a-i) $\mathbb{E}[(Y_1)^{2p}] < \infty$ *with* $p \in [1,\infty)$ *and* $|f| \leqslant a g_{p-1}$ *for some* $a \in [0,\infty)$.

(a-ii) $\mathbb{E}[(Y_1)^4] < \infty$ *and* $f(x) = x$.

(b) *If any one of the following conditions is verified then* $(X_n)_{n\geqslant 0}$ *is ergodic of degree* 2, $\pi(|f|) < \infty$, $\bar{f}$ *satisfies the ASIP and the constant* σ^2 *in* (4.1) *verifies* (4.8):

(b-i) $\mathbb{E}[(Y_1)^{3p}] < \infty$ *with* $p \in [1,\infty)$ *and* $|f| \leqslant a g_{p-1}$ *for some* $a \in [0,\infty)$.

(b-ii) $\mathbb{E}[(Y_1)^3] < \infty$ *and* f *is bounded.*

(b-iii) $\mathbb{E}[(Y_1)^6] < \infty$ *and* $f(x) = x$.

(b-iv) $\mathbb{E}[(Y_1)^2] < \infty$, D_+ *and* D_- *are two disjoint bounded subsets of* E *such that* $\pi(D_+) = \pi(D_-)$ *and* f *is defined for* $t \in E$ *by* $f(t) = +1$ *if* $t \in D_+$, -1 *if* $t \in D_-$ *and* 0 *otherwise.*

(c) *If* $\mathbb{E}[(Y_1)^3] < \infty$ *and* $\pi(|f|^p) < \infty$ *with* $p \in (4,\infty)$ *then* $(X_n)_{n\geqslant 0}$ *is ergodic of degree* 2 *and* $\bar{f}$ *satisfies the ASIP.*

Proof. (a-i)–(b-iii) allow to apply the results of Section 5.1 and (b-iv) that of Corollary 4.2(v) (by Nummelin, 1984, Example 5.3(e)). For (c), use Corollary 4.3(i) and Corollary 3.2(b). □

5.3. Example. Random walk on a half line (reflected random walk)

Let $Z_0, Y_1, Y_2, \dots$ be independent real r.v.'s, Z_0 taking its values in $E=[0,\infty)$, $\mathscr{E}$ being the class of its Borel subsets, $\Gamma_0 := \mathscr{L}(Z_0)$, with $Y_1, Y_2, \dots$ identically distributed and $\Gamma := \mathscr{L}(Y_i)$, $i \geqslant 1$. Define $Z_n = Z_0 + \sum_{i=1}^n Y_i$ for $n \geqslant 1$. We assume that $\mathbb{E}|Y_1| < \infty$ and $\beta := \mathbb{E}Y_1 < 0$. Then, if $W_0 := Z_0$, $W_n := (W_{n-1} + Y_n)_+$, $n \geqslant 1$, (where $x_+ := x$ if $x \geqslant 0$, 0 if $x < 0$, $x \in \mathbb{R}$), $(W_n)_{n\geqslant 0}$ is a Harris ergodic Markov chain on $(E, \mathscr{E})$; the unit point mass at 0 is an irreducibility measure and $[0,c]$ is a small set in $\mathscr{E}^+$ for any $c \in [0,\infty)$ (Meyn and Tweedie, 1993; Nummelin, 1984).

Lemma 5.3.1. *Assume* $\mathbb{E}[(Y_1^+)^p] = \int_{[0,\infty)} s^p \Gamma(\mathrm{d}s) < \infty$ *with* $p \geqslant 1$, $p \in \mathbb{R}$. *Then* $(W_n)_{n\geqslant 0}$ *satisfies* $(\mathscr{V}_p)$.

Proof. For this chain, its transition operator P, say, applied to any nonnegative $\mathscr{E}$-measurable function h, gives

$$(Ph)(x) = \Gamma((-\infty, -x))h(0) + \int_{[-x,\infty)} h(x+y)\Gamma(\mathrm{d}y).$$

If $V_0(x) := x^p$, $x \geqslant 0$, we have

$$(PV_0)(x) = x^p\Gamma([-x,\infty)) + p\int_{[-x,\infty)} \left\{ \left(\int_0^1 (x+\zeta y)^{p-1}\,\mathrm{d}\zeta \right) y \right\} \Gamma(\mathrm{d}y), \quad x \geqslant 0$$

(note that $\int_{[-x,\infty)} (\int_0^1 (x+\zeta y)^{p-1} \mathrm{d}\zeta)|y|\Gamma(\mathrm{d}y) = \int_{[-x,x]} \cdots + \int_{(x,\infty)} \cdots \leqslant 2^{p-1}x^{p-1}\mathbb{E}|Y_1| + 2^{p-1}E[(Y_1^+)^p] < \infty$). Writing $I(x)$ for the last integral, we claim that $\lim_{x\to\infty}(1/x^{p-1}) \times I(x) = \beta$. In order to prove this, fix any sequence (x_n) in $[1,\infty)$ which tends to infinity. Then $h_n(y) := 1_{[-x_n,\infty)}(\int_0^1 (1+\zeta y/x_n)^{p-1} d\zeta) y \to y$, for every $y \in \mathbb{R}$ and $|h_n(y)| \leqslant 1_{(-\infty,0)}|y| + 1_{[0,\infty)}(1+y)^p$, $y \in \mathbb{R}$, $n \geqslant 1$. The dominated convergence theorem gives our claim. Hence there exists $x_0 \geqslant 0$ such that $(PV_0)(x) \leqslant x^p + \frac{1}{2}\beta p x^{p-1}$ for every $x \geqslant x_0$; taking $c \in (0,\infty)$ such that $-1/c = \frac{1}{3}\beta p$ and defining $V(x) = cx^p$, $x \geqslant 0$, we have some $x_1 \geqslant x_0$ for which $(PV)(x) \leqslant V(x) - g_{p-1}(x)$ for every $x \geqslant x_1$. Observe that $0 \leqslant x \leqslant x_1$ implies $(PV)(x) = c\int_{[-x,\infty)} (x+y)^p \Gamma(\mathrm{d}y) = c\left(\int_{[-x,x]} \cdots + \int_{(x,\infty)} \cdots\right) \leqslant c(2^p x_1^p + 2^p \mathbb{E}[(Y_1^+)^p])$. Now it suffices to take $b = c2^p(x_1^p + \mathbb{E}[(Y_1^+)^p]) + g_{p-1}(x_1) \in (0,\infty)$. □

As in the preceding example we obtain

Proposition 5.3.2.

(a) *If any one of the following conditions is verified then* $(W_n)_{n\geqslant 0}$ *is ergodic of degree* 2, $\pi(|f|) < \infty$, $\bar{f}$ *satisfies the FCLT and the constant* σ^2 *in* (4.1) *verifies* (4.8):

(a-i) $\mathbb{E}[(Y_1^+)^{2p}] < \infty$ *with* $p \in [1,\infty)$ *and* $|f| \leqslant a g_{p-1}$ *for some* $a \in [0,\infty)$. (a-ii) $\mathbb{E}[(Y_1^+)^4] < \infty$ *and* $f(x) = x$.

(b) *If any one of the following conditions is verified then* $(W_n)_{n\geqslant 0}$ *is ergodic of degree* 2, $\pi(|f|) < \infty$, $\bar{f}$ *satisfies the ASIP and the constant* σ^2 *in* (4.1) *verifies* (4.8):

(b-i) $\mathbb{E}[(Y_1^+)^{3p}] < \infty$ *with* $p \in [1,\infty)$ *and* $|f| \leqslant a g_{p-1}$ *for some* $a \in [0,\infty)$.

(b-ii) $\mathbb{E}[(Y_1^+)^3] < \infty$ *and* f *is bounded.* (b-iii) $\mathbb{E}[(Y_1^+)^6] < \infty$ *and* $f(x) = x$.

(c) *If* $\mathbb{E}[(Y_1^+)^3] < \infty$ *and* $\pi(|f|^p) < \infty$ *with* $p \in (4,\infty)$ *then* $(X_n)_{n\geqslant 0}$ *is ergodic of degree* 2 *and* $\bar{f}$ *satisfies the ASIP.*

5.4. Example. Linear state space models

Let $X_0, W_1, W_2, \ldots$ be independent random vectors, X_0 taking its values in $E = \mathbb{R}^d$, $\mathscr{E}$ being the class of its Borel subsets, $W_1, W_2, \ldots$ taking its values in $\mathbb{R}^p$ and identically distributed with $\Gamma := \mathscr{L}(W_i)$, $i \geqslant 1$. Let F be a $d \times d$ matrix and G be a $d \times p$ matrix. We assume that Γ is nonsingular with respect to Lebesgue measure, the eigenvalues of F lie in the open unit disk in $\mathbb{C}$ and the (controllability) matrix $[F^{n-1}G|\cdots|FG|G]$ has rank d. Then, if $X_n := FX_{n-1} + GW_n, n \geqslant 1, (X_n)_{n\geqslant 0}$ is an aperiodic irreducible Markov chain on $(E, \mathscr{E})$, and every compact subset of E is small (Glynn and Meyn, 1996; Meyn and Tweedie, 1993).

Lemma 5.4.1. *Assume* $\mathbb{E}[\|W_1\|^p] = \int_E \|x\|^p \Gamma(\mathrm{d}x) < \infty$ with $p > 0$, $p \in \mathbb{R}$. *Then* $(X_n)_{n\geqslant 0}$ *satisfies* $(\mathscr{V}_p')$ *with a compact set* C.

Proof. As in Meyn and Tweedie (1993, proof of Proposition 12.5.1) consider the positive definite matrix $M := I + \sum_{i=1}^{\infty} (F^\top)^i F^i$ ($F^\top$ is the transpose of F) and the norm $|x|_M := \sqrt{x^\top M x}$ which satisfies $|Fx|_M \leqslant \alpha |x|_M$ and $\|x\| \leqslant |x|_M \leqslant \beta \|x\|$, $x \in E$, for certain positive constants $\alpha < 1$ and β. Define $V_0(x) := |x|_M$, $x \in E$. First we show that

$$P(V_0^p) \leqslant \lambda V_0^p + L \text{ for some constants } \lambda \text{ and } L,$$
$$0 < \lambda < 1, \quad 0 \leqslant L < \infty. \tag{5.1}$$

We have $P(V_0^p)(x) = \mathbb{E}[V_0^p(Fx + GW_1)] \leqslant \mathbb{E}[\{V_0(Fx) + V_0(GW_1)\}^p]$. If $0 < p \leqslant 1$, $P(V_0^p)(x) \leqslant \alpha^p (V_0(x))^p + \mathbb{E}[|GW_1|_M^p]$ and (5.1) holds with $\lambda = \alpha^p$, $L = \mathbb{E}[|GW_1|_M^p]$. Suppose $p > 1$. Put $c_p = \max\{1, 2^{p-2}\}$. In this case $P(V_0^p)(x) \leqslant \alpha^p (V_0(x))^p + p c_p \alpha^{p-1} (V_0(x))^{p-1} \mathbb{E}[|GW_1|_M] + p c_p \mathbb{E}[|GW_1|_M^p] = \alpha^p (V_0(x))^p + \varphi(x)$, say, with $\varphi(x)/(V_0(x))^p \to 0$, as $\|x\| \to \infty$. Take $\delta > 0$, $\delta < 1 - \alpha^p$ and then $t > 0$ such that $\varphi(x) \leqslant \delta (V_0(x))^p$ whenever $\|x\| > t$. Now (5.1) is verified with $\lambda := \alpha^p + \delta$ and $L := \sup\{\varphi(x) \colon \|x\| \leqslant t\}$. Hence (5.1) is proved.

Take now $r > 0$ large enough such that $\lambda + L/r < 1$ and the compact set (then small) $C := \{(V_0)^p \leqslant r\} \in \mathscr{E}^+$. If we choose $\gamma > 0$ with $\lambda + \gamma + (L + \gamma)/r < 1$, then $(\mathscr{V}_p')$ is verified with $V := (1/\gamma)(V_0)^p$, $b := L + \gamma$. □

Proposition 5.4.2. *Assume* $\mathbb{E}[\|W_1\|^p] < \infty$ *for some* $p > 0, p \in \mathbb{R}$. *Then* $(X_n)_{n\geqslant 0}$ *is Harris ergodic of degree* 2.

Proof. We know that (X_n) is irreducible and aperiodic. By Lemma 5.4.1 there exist a compact set C and a function V on E such that $(PV)(x) - V(x) \leqslant -g_p(x) < 0$ for every $x \notin C$; Theorems 9.4.1, 9.2.2 and Proposition 6.3.5 in Meyn and Tweedie (1993) show that (X_n) is Harris recurrent. Positiveness follows from $(\mathscr{V}'_p)$ and Theorem 11.0.1 in Meyn and Tweedie (1993). Now apply Lemma 5.1.1(b). □

The FCLT under (a-iii) in the following result was proved in Glynn and Meyn (1996). From Section 5.1 we obtain

Proposition 5.4.3.

(a) *If any one of the following conditions is verified then* $\pi(|f|) < \infty$, $\bar{f}$ *satisfies the FCLT and the constant* σ^2 *in* (4.1) *verifies* (4.8):

(a-i) $\mathbb{E}[\|W_1\|^{2p}] < \infty$ *with* $p \in (0, \infty)$ *and* $|f| \leqslant a g_p$ *for some* $a \in (0, \infty)$.

(a-ii) $\mathbb{E}[\|W_1\|^{p}] < \infty$ *for some* $p \in (0, \infty)$, *and* f *is bounded*.

(a-iii) $\mathbb{E}[\|W_1\|^{2}] < \infty$ *and* $|f| \leqslant a(1 + \|x\|)$ *for some* $a \in (0, \infty)$.

(b) *If any one of the following conditions is verified then* $\pi(|f|) < \infty$, $\bar{f}$ *satisfies the ASIP and the constant* σ^2 *in* (4.1) *verifies* (4.8):

(b-i) $\mathbb{E}[\|W_1\|^{3p}] < \infty$ *with* $p \in (0, \infty)$ *and* $|f| \leqslant a g_p$ *for some* $a \in (0, \infty)$.

(b-ii) $\mathbb{E}[\|W_1\|^{p}] < \infty$ *for some* $p \in (0, \infty)$, *and* f *is bounded*.

(b-iii) $\mathbb{E}[\|W_1\|^{3}] < \infty$ *and* $|f| \leqslant a(1 + \|x\|)$ *for some* $a \in (0, \infty)$.

(c) If $\mathbb{E}[\|W_1\|^{p}] < \infty$ *for some* $p \in (0, \infty)$ *and* $\pi(|f|^q) < \infty$ *with* $q \in (4, \infty)$ *then* $\bar{f}$ *satisfies the ASIP*.

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