

(Super-)Gravities of a different sort

José D. Edelstein^{†1,2,3} and Jorge Zanelli^{*1}

¹ Centro de Estudios Científicos (CECS), Casilla 1469, Valdivia, Chile

² Departamento de Física de Partículas, Universidade de Santiago de Compostela and Instituto Galego de Física de Altas Enerxías, E-15782 Santiago de Compostela, Spain

³ Instituto de Física de La Plata (IFLP), Universidad Nacional de La Plata, Argentina

E-mail: [†][jedels\[at\]usc\[dot\]es](mailto:jedels@usc.es), ^{*}[jz\[at\]cecs\[dot\]cl](mailto:jz[at]cecs[dot]cl)

Abstract. We review the often forgotten fact that gravitation theories invariant under local de Sitter, anti-de Sitter or Poincaré transformations can be constructed in all odd dimensions. These theories belong to the Chern–Simons family and are particular cases of the so-called Lovelock gravities, constructed as the dimensional continuations of the lower dimensional Euler classes. The supersymmetric extensions of these theories exist for the AdS and Poincaré groups, and the fields are components of a single connection for the corresponding Lie algebras. In 11 dimensions these supersymmetric theories are gauge theories for the $osp(1|32)$ and the M algebra, respectively. The relation between these new supergravities and the standard theories, as well as some of their dynamical features are also discussed.

1. Gravity as a gauge theory

In 1915, Einstein invented the first nonabelian gauge theory [1], although at the time nobody –certainly not Einstein himself– had any clue of this fact, and much less, of its importance. It took some forty years and the discovery of Yang and Mills [2] to conceive gravitation as a nonabelian gauge theory [3, 4]. The original observation that led Einstein to General Relativity (**GR**) was that the content of the Equivalence Principle is the possibility of retaining the Lorentz symmetry of Special Relativity in every local neighborhood of a curved spacetime. This turns the global $SO(3, 1)$ symmetry of Special Relativity into a local (gauge) symmetry in GR.

Local Lorentz invariance is an exact gauge symmetry of GR, closely related to the gauge symmetries that characterize the other forces of nature. In spite of this formal similarity between gravity and the other fundamental forces of nature, there exist a number of differences, which may be at the root of the obstructions towards the quantum description of gravitational phenomena.

The principle of equivalence states that spacetime is a differentiable pseudo-Riemannian manifold M , endowed with a tangent bundle of flat Minkowski spaces at each point. Thus, spacetime is the base manifold for a fiber bundle where each fiber is the Lorentz group. Note that the local Lorentz symmetry is unrelated to the freedom to make arbitrary coordinate choices on M –diffeomorphism invariance or general covariance. Coordinates are just auxiliary labels and, as such, any well-posed description of the physical world must be insensitive to the choice of coordinates. General covariance is neither an exclusive feature of GR, nor is it a useful physical symmetry. Proving invariance of a physical system under coordinate transformations is as fundamental as proving invariance of ideas under a change of printer’s font.

A more compelling reason to avoid using general covariance as a symmetry principle is the fact that its first class generators do not form a Lie algebra but an *open* algebra, where the analogues of the structure constants are functions of the phase space variables instead of being invariants under the action of the group [5].

Invariance of gravity under local Lorentz transformations is manifest when one writes the Einstein–Hilbert action in the first order formalism. For example, in four dimensions

$$I_{EH} = \int d^4x \sqrt{-g} R = \int \epsilon_{abcd} R^{ab} \wedge e^c \wedge e^d . \quad (1)$$

Here $R^{ab} = d\omega^{ab} + \omega^a_c \wedge \omega^{cb}$ is the curvature two-form, ω^{ab} is the Lorentz (spin) connection, and $e^a = e^a_\mu dx^\mu$ is the local orthonormal frame (vierbein). Under a local Lorentz transformation $\Lambda^a_b(x)$, e_a and ω^{ab} transform, respectively, as a vector and as a connection

$$\begin{aligned} e^a &\rightarrow e'^a(x) = \Lambda^a_b(x) e^b(x) , \\ \omega^{ab} &\rightarrow \omega'^{ab}(x) = \Lambda^a_c(x) \Lambda^b_d(x) \omega^{cd}(x) + \Lambda^a_c(x) d\Lambda^c_d(x) . \end{aligned} \quad (2)$$

Clearly, the form of the Lagrangian (1) is quite different from the Yang-Mills (YM) one, $(1/4)Tr[\mathbf{F}^2]$. An obvious difference is that (1) is linear rather than quadratic in the curvature. More importantly, gravity requires two fields of different nature: a gauge

connection for the Lorentz group, ω^a_b , and a vector under the same group, e^a . Yang-Mills theory, on the other hand, requires no other dynamical field but the gauge connection \mathbf{A} . In YM theory the spacetime metric represents a non-dynamical background of fixed geometry. On the contrary, for a theory like Gravitation, that dynamically determines the spacetime geometry, a prescribed background geometry would make no sense.

The two 1-form fields e^a , ω^a_b , embody two essential aspects of geometry: metricity and parallelism. These are conceptually independent properties, the first related to the notion of distance, area, orthogonality, and the second to the definition of parallel transport of vectors in open neighborhoods. Since these definitions are logically independent, it is fitting to describe them by means of dynamically independent fields. Hence, the equivalence principle can be taken to mean that a D -dimensional spacetime geometry should be described mathematically by an action principle of the form

$$I[e, \omega] = \int_M L_D(e, \omega, de, d\omega) , \quad (3)$$

where the Lagrangian L_D is a D -form constructed out of the fundamental fields and their exterior derivatives. In order to ensure the Lorentz invariance of the dynamics, it would be sufficient to require the Lagrangian to be a Lorentz scalar. This last requirement is not strictly necessary and can be relaxed, requesting instead that under a Lorentz transformation (2), the action changes by a surface term. To construct L_D , the two invariant tensors of the Lorentz group,

$$\eta_{ab}, \quad \text{and} \quad \epsilon_{a_1 a_2 \dots a_D} , \quad (4)$$

should also be used.

Since the metric is not a basic field in this formulation ($g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$), it cannot be assumed a priori to be invertible, as in a theory defined on a prescribed background geometry. In fact, it is conceivable that the ground state (vacuum) of gravity may correspond to a configuration with $e^a_\mu = 0$ [6]. Thus, it would be inconsistent to introduce a structure like the Hodge dual (*) which requires the existence of a metric and its inverse, $g^{\mu\nu}$. The absence of the *-dual does not represent a limitation since all gravity theories which yield second order field equations for the metric can be obtained in this way. Indeed, by taking exterior derivatives of e^a and ω^a_b the only

Lorentz tensors that can be produced are the curvature R^{ab} , and torsion $T^a = de^a + \omega^a_b \wedge e^b$ two forms. Moreover, by virtue of the Bianchi identities,

$$\begin{aligned} DR^{ab} &= dR^{ab} + \omega^a_c \wedge R^{cb} + \omega^b_c \wedge R^{ac} \equiv 0 \\ DT^a &= R^a_b \wedge e^b, \end{aligned} \quad (5)$$

it is clear that no new tensors can be generated. These tensors, together with the invariants (4) are the only ingredients at hand to build up all gravity actions in any dimension.

2. Three series

There are relatively few Lagrangians that can be written in a given spacetime dimension D , with the ingredients listed above that are Lorentz invariant D -forms. These candidate actions fall into three families.

2.1. Lovelock series

General Relativity, viewed as a dynamical theory for the metric without torsion, is generalized for a spacetime dimension $D > 4$, by the so-called Lovelock theories of gravity [7, 8]. Their Lagrangians are the most general D -forms built out of R^{ab} and e^a . They take the form ¹

$$L = \sum_{p=0}^n \alpha_p L_{(p)}, \quad (6)$$

where $n = [(D-1)/2]$, α_p are arbitrary coefficients and

$$L_{(p)} = \epsilon_{a_1 a_2 \dots a_D} R^{a_1 a_2} \dots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \dots e^{a_D}. \quad (7)$$

Included in the series (6) are the cosmological constant term $L_{(0)} = \epsilon_{a_1 a_2 \dots a_D} e^{a_1} \dots e^{a_D}$, the Einstein–Hilbert density $L_{(1)} = \epsilon_{a_1 a_2 \dots a_D} R^{a_1 a_2} e^{a_3} \dots e^{a_D}$, the Gauss-Bonnet density $L_{(2)} = \epsilon_{a_1 a_2 \dots a_D} R^{a_1 a_2} R^{a_3 a_4} e^{a_5} \dots e^{a_D}$, etc. For even D , the last term is the D -dimensional Euler density $L_{(D/2)} = \epsilon_{a_1 a_2 \dots a_D} R^{a_1 a_2} R^{a_3 a_4} \dots R^{a_{D-1} a_D}$. Each $L_{(p)}$ corresponds to the dimensional continuation to D dimensions of the $2p$ -dimensional Euler density.

Varying the action with respect to the vielbein yields a generalization of Einstein equations for arbitrary dimensions known as Lovelock equations. The variation with respect to ω^{ab} yields an equation involving torsion which is always solved by $T^a = 0$, but this is not the most general solution.

2.2. Torsional series

Torsion is not included in the Lagrangian (6), although it is not set identically equal to zero. This means that including torsion in the Lagrangian is as legitimate as including curvature. This means that there is a series of Lorentz invariant polynomials, which are not included in the Lovelock series that can be added in each dimension

$$L_D^{\text{Tor}}(e^a, \omega^{ab}) = \sum_s \beta_s P^s(e^a, R^{ab}, T^a). \quad (8)$$

Notice that these polynomials cannot involve the totally antisymmetric symbol $\epsilon_{abc\dots}$. The explicit form of these terms is not very illuminating and takes a different form in each dimension. The construction of these polynomials, as well as a broad discussion about them, were given

¹ Hereafter, wedge product between forms is always understood.

in [9]. These polynomials include the Pontryagin invariant $4k$ -forms, $\mathcal{P}_{4k}(R) = R_{a_2}^{a_1} \cdots R_{a_1}^{a_{2k}}$, as particular cases.

There exist two additional terms in this family that can be included in four dimensions, $t = T^a T_a$ and $r = R^{ab} e_a e_b$. It turns out, however, that the combination $N_4 \equiv t - r$ is a total derivative (the Nieh–Yan invariant) and hence the two terms are equivalent Lagrangians. This type of invariants are also related to Chern–Pontryagin classes, and may also contribute to the chiral anomaly in spacetimes with torsion [10, 11, 12].

2.3. Lorentz CS series

There is another class of actions that are not exactly invariant, but *quasi-invariant*, under local Lorentz transformations. These are the Lorentz Chern–Simons (**LCS**) forms that exist for all odd dimensions. The simplest LCS form in three dimensions is $L_3^{CS} = \omega_b^a d\omega_a^b + \frac{2}{3} \omega_b^a \omega_c^b \omega_a^c$, and, in general, in $4k - 1$ dimensions, it takes the form

$$L_{4k-1}^{CS} = [\omega(d\omega)^{2k-1}] + \gamma_1 [\omega^3(d\omega)^{2k-2}] + \gamma_2 [\omega^5(d\omega)^{2k-3}] + \cdots + \gamma_{2k-1} [\omega^{4k-1}], \quad (9)$$

where the coefficients γ_s are fixed rational numbers and the bracket $[\cdots]$ denotes a trace. These forms yield Lorentz invariant equations despite the fact that they involve explicitly the connection and are not truly Lorentz invariants. This "miracle" stems from the fact that the exterior derivative of CS forms are topological invariant densities. In this case, $dL_{4k-1}^{CS} = \mathcal{P}_{4k}(R) = [R^{2k}]$.

3. Miraculous choice

The number of Lagrangians generated in this way increases with dimension, and so does the number of arbitrary dimensionful coupling constants $\alpha_1, \cdots, \alpha_n; \beta_1 \cdots \beta_m$. Fortunately, for $D = 4$ there is complete agreement with GR: in the absence of torsion, only the Einstein–Hilbert and the cosmological constant term can be present. What is the meaning of all those coupling constants in higher dimensions? It can be easily seen that they represent the existence of several scales in the theory, which are related to different radii of curvature in the solutions, or to different cosmological constants $\lambda_1, \lambda_2, \cdots, \lambda_s$. If the cosmological constant is a problem in four dimensions, the problem is a priori much worse for $D > 4$. It can be seen that field equations admit spacetime geometries that jump discontinuously from one with $\lambda = \lambda_1$, to another with $\lambda = \lambda_2$ [13, 14].

The presence of so many dimensionful constants endows the theory with a bad prospect for its quantization. How could the theory be protected from uncontrollable ultraviolet divergences? The ideal situation is closer to the opposite extreme: a Lagrangian with no arbitrary dimensionful constants. That is the case of a Chern–Simons theory, in which all constants are fixed dimensionless rational numbers.

The good news is that in every dimension there exists a choice of coefficients $\alpha_1 \cdots \alpha_n; \beta_1 \cdots \beta_m$ such that all cosmological constants are the same and therefore there is only one scale in the theory. In odd dimensions, this choice is even more miraculous since all dimensionful coefficients in the action can be absorbed by means of a rescaling of the vielbein, $e^a \rightarrow l^{-1} e^a$. Indeed, choosing the Lovelock coefficients in (6) as

$$\alpha_p = \frac{l^{2p-D}}{D-2p} \left(\frac{D-1}{2} \right)^p, \quad (10)$$

produces a Lagrangian that describes a theory of gravity with no built-in scale, being, therefore, scale-invariant. This choice has an additional bonus feature because the gauge symmetry is now enlarged from the Lorentz to the AdS group.

As it is well known, miracles don't exist; they are either hoaxes or surprises from our poor understanding of things. All the miracles that come with the choice (10) are consequence of the fact that with this choice the vielbein and the Lorentz connection are combined into a connection for the AdS group. In other words, the gauge group $SO(D-1, 1)$ has been embedded into $SO(D-1, 2)$, in the form

$$\mathbf{A} = \frac{e^a}{l} J_a + \frac{1}{2} \omega^{ab} J_{ab} , \quad (11)$$

and the action becomes a functional of this connection \mathbf{A} , and not a functional of e^a and ω^{ab} separately. The Lagrangian can now be expressed as

$$L_{2n-1}^{\text{CS}}(\mathbf{A}) = \kappa \langle \mathbf{A}(d\mathbf{A})^{n-1} + \gamma_1 \mathbf{A}^3 (d\mathbf{A})^{n-2} \dots \gamma_{n-1} \mathbf{A}^{2n-1} \rangle , \quad (12)$$

where $\langle \dots \rangle$ denotes a trace on the matrix representation of the AdS generators J_a, J_{ab} [15, 16] (for details and a comprehensive list of references see, e.g., Refs.[17, 18]).

It is also possible to construct a de-Sitter invariant action, (with $SO(D, 1)$ as the gauge group) which is obtained by replacing $l^2 \rightarrow -l^2$ in (10). Finally, there is also the possibility of taking the vanishing cosmological constant limit, $l \rightarrow \infty$, which yields a theory invariant under the Poincaré group [19, 20].

It is sometimes argued that the Einstein–Hilbert action with cosmological constant in four dimensions provides a gauge theory for the (A)dS group, because its dynamical fields (e^a and ω^{ab}) are components of the (A)dS connection (11) [21, 22]. The problem with this point of view is that the (A)dS symmetry cannot be respected by the action because there is no Lagrangian for the connection \mathbf{A} invariant under the (A)dS gauge group in four dimensions. Arguing that the symmetry is spontaneously broken is also hard to sustain since there seems to be no regime of the theory in which the symmetry can be restored.

4. Surface terms and transgressions

For mathematicians, Chern–Simons forms are not natural objects. They are not truly invariant, changing by an exact form under a gauge transformation. In physics this is not a serious problem because exact forms in the action are surface terms that, generically, don't affect the field equations or the conservation laws. However, invariances “up to surface terms” have other important physical consequences. The value of the conserved charges, and of the action itself can be renormalized by surface terms. This in turn affects the definition of thermodynamic quantities like the energy and entropy of a black hole.

On the other hand, boundary conditions sufficient to ensure that the action attains an extremum on the classical orbits, require to supplement the action by a surface term of a particular form. In asymptotically locally AdS spaces (**ALADS**) the boundary term takes the form $B_{2n}[\mathbf{A}, \overline{\mathbf{A}}]$, where the field $\overline{\mathbf{A}}$ is only defined at the surface of spacetime, whose rôle is to match the boundary conditions under which the action is to be varied [23]. This addition cures several problems at once: it provides a well-defined variation while, at the same time, it renders the charges and the on-shell value of the action finite, producing well defined thermodynamic quantities, which can also be computed by other means [24]. In particular, the energy of the vacuum turns out to be exactly the Casimir energy for AdS without additional regularizations (counterterms) or ad-hoc background subtractions [25].

As in other cases in physics, the solution to this problem comes from the requirement of gauge invariance. The field $\overline{\mathbf{A}}$ and the boundary term $B_{2n}[\mathbf{A}, \overline{\mathbf{A}}]$ correspond precisely to what is required to turn the Chern–Simons form into a *transgression form* [26, 27, 28]. Unlike CS forms, transgression forms depend on two connections, $\mathcal{T}_{2n-1}(\mathbf{A}, \overline{\mathbf{A}})$. Transgressions are gauge invariants provided \mathbf{A} and $\overline{\mathbf{A}}$ transform as connections for the same gauge group. The defining

property of a transgression is that its exterior derivative is the difference of two invariant classes, for \mathbf{A} and $\overline{\mathbf{A}}$, respectively,

$$d\mathcal{T}_{2n-1}(\mathbf{A}, \overline{\mathbf{A}}) = \langle F^n(\mathbf{A}) \rangle - \langle \overline{F}^n(\overline{\mathbf{A}}) \rangle .$$

Thus, the transgression form can also be written as

$$\mathcal{T}_{2n-1} = L_{2n-1}^{CS}(\mathbf{A}) - L_{2n-1}^{CS}(\overline{\mathbf{A}}) + dB_{2n}(\mathbf{A}, \overline{\mathbf{A}}) ,$$

where $B_{2n}(\mathbf{A}, \overline{\mathbf{A}})$ is defined on a local chart over the boundary of the spacetime manifold M .

The presence of the second connection field $\overline{\mathbf{A}}$ in the transgression form is puzzling if considered as a second dynamical field on the same footing as \mathbf{A} . However, $\overline{\mathbf{A}}$ need only be defined at the boundary of the spacetime manifold M ; it is sufficient to define $\overline{\mathbf{A}}$ on a different manifold \overline{M} that shares a common boundary with (cobordant to) M , $\partial M = \partial \overline{M}$. Then, the action principle based on the transgression form can be written as

$$I_{Trans}[\mathbf{A}, \overline{\mathbf{A}}] = \int_M L_{CS}(\mathbf{A}, d\mathbf{A}) - \int_{\overline{M}} L_{CS}(\overline{\mathbf{A}}, d\overline{\mathbf{A}}) + \int_{\partial M} B_{2n}(\mathbf{A}, \overline{\mathbf{A}}) . \quad (13)$$

The main advantage of this expression is that it allows to compute the conserved charges by direct application of Noether's theorem in covariant language and without subtractions [29, 30].

5. Supersymmetric extensions

According to the Haag–Lopuszanski–Sohnius theorem [31], supersymmetry is essentially the only nontrivial way to extend a spacetime symmetry, circumventing the well known obstruction pointed out by Coleman and Mandula [32]. The question then naturally arises, whether there exist supersymmetric extensions for the theories described here. The answer is in the affirmative in the odd-dimensional CS-AdS theories. Moreover, these theories admit a supersymmetry which is realized like in any standard non-abelian gauge theory, namely, the dynamical field is a connection for the (super)group and the action turns out to be invariant off-shell (up to surface terms).

In three dimensions, the standard Einstein–Hilbert plus cosmological constant action is a Chern–Simons theory and its supersymmetric extension has been known for almost twenty years [33]. The resulting 2+1 AdS supergravity is a gauge theory for the group $OSp(p|2; R) \otimes OSp(q|2; R)$. In five dimensions, the locally supersymmetric extension of gravity was found by Chamseddine [19], and its purely gravitational sector is the CS-AdS action described above. The generalization to higher dimensions was found in [34, 35], and the supersymmetric extensions of the Poincaré theory was presented in [36].

The idea is to extend the action by introducing all the necessary fields to produce a connection for the gauge supergroup that contains AdS_D as a subgroup in a given dimension. This can be done from first principles, if one has an *a priori* knowledge of what are the semisimple superalgebras containing AdS_D . Alternatively, one can start by adding to (11) the supersymmetry generators Q and \overline{Q} , with the corresponding gauge fields $\overline{\psi}$ and ψ ,

$$\mathbf{A} = \frac{e^a}{l} J_a + \frac{1}{2} \omega^{ab} J_{ab} + \overline{Q} \psi + \overline{\psi} Q + \dots , \quad (14)$$

and subsequently check the closure of the extended algebra. This requires, in general, extra bosonic generators and, in some cases, several copies of the fermions (this is what the dots mean in (14)). The result is quite unique. It is summarized in the next table, where the field content of the resulting theories for $D = 5, 7, 11$, and the corresponding algebras, are confronted with the standard supergravities.

D	CS – AdS supergravity	Algebra	Standard supergravity
5	$e_\mu^a \omega_\mu^{ab} A_\mu \psi_\mu^\alpha \bar{\psi}_{\alpha\mu}$	$usp(2, 2 1)$	$e_\mu^a \psi_\mu^\alpha A_\mu \bar{\psi}_{\alpha\mu}$
7	$e_\mu^a \omega_\mu^{ab} A_{j\mu}^i \psi_\mu^{i\alpha}, i, j = 1, 2$	$osp(2 8)$	$e_\mu^a A_{[3]} A_{j\mu}^i \lambda^\alpha \phi \psi_\mu^{i\alpha}, i, j = 1, 2$
11	$e_\mu^a \omega_\mu^{ab} b_\mu^{abcde} \psi_\mu^\alpha$	$osp(32 1)$	$e_\mu^a A_{[3]} \psi_\mu^\alpha$

Some general comments are in order at this point (for a detailed discussion, see [17, 18]):

Supergravities. The actions obtained in this way are, by construction, invariant under the gauge superalgebra and diffeomorphisms. Since they include gravity, they are supergravities, albeit of a different sort. Some authors would reserve the word *supergravity* for supersymmetric theories whose gravitational sector is described by the Einstein–Hilbert Lagrangian. This narrow definition is correct in three and four dimensions, but seems unwarranted for $D > 4$ in view of the numerous possibilities beyond EH. If one wishes to be precise, the supergravities described here seem to belong to a separate class and the connection with the standard ones is still an open problem.

Local supersymmetry. The supersymmetry transformations are those of a connection, namely, $\delta\mathbf{A} = -\nabla\mathbf{\Lambda} = -(d\mathbf{\Lambda} + [\mathbf{A}, \mathbf{\Lambda}])$, where $\mathbf{\Lambda}$ is a zero-form with values in the Lie algebra, and ∇ is the exterior covariant derivative in the representation of \mathbf{A} . In particular, under a supersymmetry transformation, $\mathbf{\Lambda} = \bar{\epsilon}^i Q_i - \bar{Q}^i \epsilon_i$. For instance, in terms of the component fields of the five dimensional $usp(2, 2|1)$ theory, this means

$$\begin{aligned}
\delta e^a &= \frac{1}{2} (\bar{\epsilon}^r \Gamma^a \psi_r - \bar{\psi}^r \Gamma^a \epsilon_r) , \\
\delta \omega^{ab} &= -\frac{1}{4} (\bar{\epsilon}^r \Gamma^{ab} \psi_r - \bar{\psi}^r \Gamma^{ab} \epsilon_r) , \\
\delta A_s^r &= -i (\bar{\epsilon}^r \psi_s - \bar{\psi}^r \epsilon_s) , \\
\delta \psi_r &= -\nabla \epsilon_r , \quad \delta \bar{\psi}^r = -\nabla \bar{\epsilon}^r , \\
\delta A &= -i (\bar{\epsilon}^r \psi_r - \bar{\psi}^r \epsilon_r) .
\end{aligned}$$

where ∇ is the covariant derivative on the bosonic connection,

$$\nabla \epsilon_r = \left(d + \frac{1}{2} \omega^{ab} \Gamma_{ab} + \frac{1}{2l} e^a \Gamma_a \right) \epsilon_r - A_r^s \epsilon_s - \frac{3i}{4} A \epsilon_r .$$

Off-shell symmetry. These actions are invariant (up to surface terms) under these transformations, and neither on-shell conditions nor auxiliary fields are necessary to realize the symmetry. This is in contrast with the standard cases, which often require torsional on-shell conditions in order to close the symmetry algebra. Symmetries requiring on-shell conditions are likely to be troublesome, since they are not necessarily respected in the quantum theory.

Extended susy. These algebras allow for extensions with $\mathcal{N} > 1$, and the field multiplets for these algebras can be easily constructed in all cases. These algebras possess a periodic nature which is inherited from the well known periodicity *mod* 8 of the Clifford algebras. Thus, the relevant groups are $OSp(\mathcal{N}|2^{4k+1})$ for $D = 8k + 3$, $OSp(2^{4k-1}|\mathcal{N})$ for $D = 8k - 1$, and $SU(2^k, 2^k|\mathcal{N})$ for $D = 4k + 1$.

Odd D , $\Lambda < 0$. No similar construction is known for positive cosmological constant. The reason is that the de Sitter group does not admit supersymmetric extensions [37, 38]. Chern–Simons actions exist only in odd dimensions; therefore, a similar construction does not exist in even dimensions. It is possible, however that in even dimensions a construction similar to the standard supergravity in four dimensions could be carried out, where some on-shell conditions are assumed in order to close the algebra including diffeomorphisms.

Matching of degrees of freedom. In these theories there is no matching between bosonic and fermionic degrees of freedom. The matching present in standard supersymmetric theories results from two assumptions which do not hold in the present case: the first assumption is that the spacetime symmetry group is Poincaré, while in our case it is the AdS group. The second is that the fields form a vector multiplet under supersymmetry, which is not the case here since all the fields are parts of a connection and therefore belong to the adjoint representation. It is worth mentioning that global issues –like the presence of a deficit angle– may also spoil the boson–fermion degeneracy in standard supergravity [39, 40, 41, 42].

Polarization states. All component fields in these theories carry only one spacetime index (they are 1-forms), and they are antisymmetric tensors of arbitrary rank under the Lorentz group (*i.e.*, $b_{\mu}^{ab\dots}$). Thus, they belong to representations of the rotation group whose Young tableaux have an arbitrarily long single column and one row with two squares. This means that the fundamental fields in these theories describe states of spin 2 or less, which goes in the opposite direction of the recent interest on higher spin fields [43, 44].

Degeneracy. Chern–Simons systems in dimensions $D \geq 5$ possess remarkable dynamical features, unexpected in a field theory but often found in fluid dynamics. One of these features stems from the fact that the symplectic form is a function of the connection, and its rank depends on the configuration [45, 46]. There are regions in phase space where the symplectic form has maximal rank (*generic* configurations), where the counting of degrees of freedom is the usual one [47]. Other regions, where the rank is smaller (*degenerate* configurations), possess fewer propagating degrees of freedom. There are even *maximally degenerate* configurations, around which the theory is topological and has no local degrees of freedom. An example of such maximally degenerate configuration is the standard vacuum of Yang-Mills theory, $\mathbf{A} = 0$.

Another unexpected feature is that degenerate systems may loose degrees of freedom in their time evolution. A simple mechanical model shows that a degenerate system may start from a nondegenerate configuration reaching a state where the degeneracy occurs in a finite time. There, some degrees of freedom cease to be dynamical and become gauge coordinates. After that, those degrees of freedom, as well as their initial data, are irreversible lost [48].

Irregularity. An independent issue, also present in CS theories is the fact that the functional independence of the gauge generators (first class constraints) may break down for certain configurations [49], and a careful analysis is required in order to have a well defined canonical formalism [50, 51].

6. Manifest M -covariant theory

An obvious advantage of the CS construction is the economy of assumptions. The only information required to define the Lagrangian is the gauge (super)group and the dimensionality of the manifold. The field content, the coupling constants, the dynamics of the spacetime manifold, the vacuum structure, are all outputs of the theory.

As an example, consider a CS theory for the supersymmetric extension of the Poincaré group in eleven dimensions. Following the steps outlined here, one arrives almost uniquely at a gauge invariant action for the M -algebra [52, 53]. The connection,

$$\mathbf{A} = \frac{e^a}{l} P_a + \frac{1}{2} \omega^{ab} J_{ab} + \bar{Q} \psi + b^{ab} Z_{ab} + b^{abcde} Z_{abcde} , \quad (15)$$

includes, apart from the vielbein, the Lorentz connection, and the gravitino, a second-rank and a fifth-rank antisymmetric Lorentz tensor one-forms, $b^{[2]}$ and $b^{[5]}$. The superalgebra includes the Poincaré generators (P_a, J_{ab}) , one (Majorana) supersymmetry generator Q and the “central extensions” of the M -algebra, $Z_{[2]}$ and $Z_{[5]}$,

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^a)_{\alpha\beta} P_a + (C\Gamma^{ab})_{\alpha\beta} Z_{ab} + (C\Gamma^{abcde})_{\alpha\beta} Z_{abcde} . \quad (16)$$

The generators $Z_{[2]}$, $Z_{[5]}$ commute with all but the Lorentz generators. The supersymmetric action is found to be [52, 53]

$$\begin{aligned} L_\alpha^M &= \epsilon_{a_1 \dots a_{11}} R^{a_1 a_2} \dots R^{a_9 a_{10}} e^{a_{11}} - \frac{1}{3} R_{abc} \bar{\psi} \Gamma^{abc} D\psi - \frac{1}{12} R_{abc} R_{de} b^{abcde} \\ &\quad + 8 [R^2 R_{ab} - 6(R^3)_{ab}] R_{cd} \left(\bar{\psi} \Gamma^{abcd} D\psi - 6R^{[ab} b^{cd]} \right) , \end{aligned} \quad (17)$$

where $R_{abc} = \epsilon_{abca_1 \dots a_8} R^{a_1 a_2} \dots R^{a_7 a_8}$, $R^2 := R^{ab} R_{ba}$ and $(R^3)^{ab} := R^{ac} R_{cd} R^{db}$.

6.1. Tentative vacuum states

We now turn to the dynamical contents of this system. In particular, one would like to identify a true vacuum of the theory. The field equations take the form

$$\langle F^5 G_A \rangle = 0 , \quad (18)$$

where the curvature $F = dA + A^2 = \frac{1}{2} R^{ab} J_{ab} + \tilde{T}^a P_a + D\psi^\alpha Q_\alpha + \tilde{F}^{[2]} Z_{[2]} + \tilde{F}^{[5]} Z_{[5]}$, with $\tilde{T}^a = De^a - (1/2) \bar{\psi} \Gamma^a \psi$ and $\tilde{F}^{[k]} = Db^{[k]} - (1/2) \bar{\psi} \Gamma^{[k]} \psi$ for $k = 2$ and 5 . The bracket $\langle \dots \rangle$ is a multilinear form of the M -algebra generators G_A .

Obviously, a configuration with a locally flat connection, $F = 0$, solves the field equations (18) and would be a natural candidate for the vacuum in a standard field theory. Moreover, such state is invariant under all gauge transformations being, therefore, maximally supersymmetric, which makes it likely to be a stable (BPS) configuration. Identifying this solution with a vacuum state would seem even more compelling in view of the fact that it has no charge of any kind and is therefore invariant under all spacetime and supersymmetry transformations.

Matter-free eleven-dimensional Minkowski spacetime is an example of such a state. However, no local degrees of freedom propagate on such background: all perturbations around it are zero modes. In fact, for the configuration, $\psi = 0$, $b^{[2]} = 0$, $b^{[5]} = 0$, Eq. (18) is a set of polynomial equations of fifth degree in the curvature two-forms. In particular, the equations obtained varying with respect to the vielbein and the spin connection take the form

$$\epsilon_{aa_1 \dots a_{10}} R^{a_1 a_2} \dots R^{a_9 a_{10}} = 0 , \quad (19)$$

$$\epsilon_{aba_1 \dots a_9} R^{a_1 a_2} \dots R^{a_7 a_8} T^{a_9} = 0 . \quad (20)$$

Thus, in order to have a propagating connection, the spatial components R_{ij}^{ab} cannot be small and must therefore be non-perturbative. Since the derivatives of the field cannot be small either, the deviations are necessarily non-local. In order to have well-defined linearized perturbations, a background solution must be a simple zero of one of the set of equations. In particular, this requires the curvature to be nonvanishing on a submanifold of sufficiently high dimensionality.

6.2. Nontrivial vacuum geometry

Let us consider a torsionless spacetime with a product geometry of the form $X_{d+1} \times S^{10-d}$, where X_{d+1} is a domain wall whose worldsheet is a d -dimensional spacetime M_d . The line element is

$$ds^2 = e^{-2\xi|z|} \left(dz^2 + \tilde{g}_{\mu\nu}^{(d)}(x) dx^\mu dx^\nu \right) + d\Omega_{10-d}^2, \quad (21)$$

where $\tilde{g}_{\mu\nu}^{(d)}$ stands for the worldsheet metric, $d\Omega_{10-d}^2$ is the metric of S^{10-d} with radius r_0 , and ξ is a constant. This ansatz solves (20) identically, and, as it is shown in [52, 53], it also solves (19) and possesses propagating degrees of freedom only if $d = 4$ and $\tilde{g}_{\mu\nu}^{(d)}(x)$ describes a de Sitter geometry of cosmological constant $\Lambda_4 = 3\xi^2$.

The geometry defined by (21) is an example of a configuration with less than the maximal number of degrees of freedom. In this case, the geometry has the degrees of freedom of $(3+1)$ -dimensional gravity with positive cosmological constant, which is certainly less than those of the full $(10+1)$ theory. This is a generic situation among the ansätze of the form (21): for large enough starting dimension $D > 4$, the d -dimensional spacetime would have propagating degrees of freedom only if $d = 4$.

7. Gauge action for EH theory

As mentioned above, it is still unclear how are CS supergravities related to the standard theories. As a step towards understanding this point, one might look for the minimal deformation or extension of the (super) Poincaré group Chern–Simons theory where the pure gravity sector is described by the Einstein–Hilbert term. It is possible to address this problem by means of an expansion method that allows to deform consistently a Chern–Simons theory into another one but for a Lie (super) algebra of larger dimension [54, 55, 56]. In correspondence to the appearance of extra generators, this requires the introduction of additional fields.

Consider the simplest case of a bosonic deformation of the Poincaré symmetry in five dimensions [57]. In order to cancel the variation of the Einstein–Hilbert action, two additional bosonic fields are included, a vector one-form h^a and an antisymmetric tensorial one-form field κ^{ab} . Consequently, the Poincaré connection is extended by means of two additional generators, Z_a and Z_{ab} ,

$$A = e^a P_a + \frac{1}{2} \omega^{ab} J_{ab} + h^a Z_a + \frac{1}{2} \kappa^{ab} Z_{ab}. \quad (22)$$

The resulting algebra turns out to be an extension of the Poincaré algebra by an Abelian ideal spanned by these generators. The Lagrangian describing the dynamics of the system reads

$$L_{\text{CS}}^{\text{EH}} = \epsilon_{abcdef} \left(\frac{2}{3} R^{ab} e^c e^d e^f + R^{ab} R^{cd} h^f + 2R^{ab} \kappa^{cd} T^f \right). \quad (23)$$

In a torsionless, matter-free configuration (*i.e.*, $h^a = \kappa^{ab} = 0$), the field equations become the Einstein equations with an additional Gauss–Bonnet constraint. This system has as a non-trivial solution a pp-wave. Interestingly enough, allowing for $\kappa^{ab} \neq 0$, the field equations become

$$\epsilon_{abcdef} R^{ab} e^c e^d = -\epsilon_{abcdef} R^{ab} D\kappa^{cd}, \quad (24)$$

$$\epsilon_{[a|cdfg} R^{cd} \kappa^{fg} e_{|b]} = 0, \quad (25)$$

$$\epsilon_{abcdef} R^{ab} R^{cd} = 0. \quad (26)$$

It is not hard to check that a four dimensional de Sitter domain wall, analogous to (21), exists if the extra bosonic field takes the form

$$\kappa^{\mu\nu} = 0, \quad \kappa^{\mu z} = -\frac{1}{2\xi} \text{sgn}(z) e^{-2\xi|z|} \tilde{e}^\mu. \quad (27)$$

This is nothing but the five-dimensional version of the metric solution of the Poincaré invariant gravity theory displayed above.

8. Discussion

We have argued that GR represents a way to implement the Lorentz invariance at a local level. This calls for a first order formalism, where the basic fields are two 1-forms, e^a and ω^a_b . If we further demand an enlargement of the gauge symmetry from the Lorentz to the Poincaré, dS or AdS groups, we are pushed towards quite a unique answer: Chern–Simons (super) gravity. These theories exist for any odd dimensional space-time. Interestingly enough, it is possible to write down an action in eleven dimensions with the symmetries dictated by the M algebra. This algebra, which corresponds to the maximal extension of the $\mathcal{N} = 1$ super Poincaré algebra, plays an important rôle in M-theory [58]. This is very suggestive and the question is unavoidable: is Chern–Simons supergravity for the M-algebra related to M-theory? Does this theory play a rôle in the M-theory diagram? There have been attempts to relate these theories. It was already suggested that M theory could be non-perturbatively equivalent to a Chern–Simons theory, though with a different symmetry group; namely, $OSp(32|1) \times OSp(32|1)$ [59] (see also [60, 61]). This claim was mainly supported on arguments dealing with holography. However, the connection to eleven dimensional supergravity at low energies, to the best of our knowledge, has not been understood yet. We have seen that it is possible to extend the Poincaré algebra in such a way that the Einstein–Hilbert action comes out. However, several bosonic fields need to be introduced and their equations of motion severely constrain the system. On the other hand, standard (super) gravity is not a gauge theory of the (super) Poincaré group. Thus, it seems clear that the connection between these theories possibly demand the existence of a spontaneous symmetry breaking mechanism. One thing seems clear: a lot of interesting results are still to be uncovered.

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