Three-point function in perturbed Liouville gravity

Gaston Giribet

Department of Physics, Universidad de Buenos Aires, Ciudad Universitaria, 1428 Buenos Aires, Argentina
Institute of Physics, Universidad Nacional de La Plata, C.C. 67, 1900 La Plata, Argentina

Abstract

Three-point correlation function in perturbed conformal field theory coupled to two-dimensional quantum gravity (perturbed Liouville gravity) is explicitly computed by using the free field approach. The representation considered here is the one recently proposed in [G. Giribet, Nucl. Phys. B 737 (2006) 209] to describe the string theory in AdS_3 space. Consequently, this computation extends previous results which presented free field calculations of particular cases of string amplitudes, and confirms that the free field approach leads to the exact result. Remarkably, this representation allows to compute winding violating three-point functions without making use of the spectral flow operator.
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1. Introduction

In a recent paper [1], a new free field representation of string theory in AdS_3 was introduced in order to realize the explicit identities that, according to what was proven in [2], turn out to connect the correlation functions in both Liouville and $SL(2)_k$ WZNW theories. Such representation corresponds to a perturbed conformal field theory coupled to two-dimensional quantum gravity (perturbed Liouville gravity). Then, this enables to make use of all what we have learned about Liouville field theory and then gain information about the WZNW model. The purpose of this brief note is that of emphasizing the usefulness of such realization by explicitly showing how the free field approach can be used to compute three-point scattering amplitudes in AdS_3. It is known that the free field approach and the Coulomb gas-like prescription were previously employed to this end [3]; however, the computation here regards those cases that were not worked out in previous free field calculations. Our attention will be focussed on the three-point functions that violate the winding number conservations. In fact, even though free field computations of such observables were previously presented in the literature [4], it was done by assuming some kind of kinematic restriction, e.g. the assumption that one of the incoming strings was represented by a highest-weight state of the $SL(2, R)_k$ representations. Moreover, previous free field computations also considered particular relations between left-moving m and right-moving $\bar{m}$ momenta, imposing in such a way certain constraints on the angular momentum of the interacting strings. Here, we relax such assumptions and calculate the generic "winding violating" three-point amplitude in AdS_3 within the framework of the Coulomb gas-like prescription. Besides, we are able to compute correlations involving states of generic winding number, without introducing intricate tricks for the definition of states with winding number grater than one and, remarkably, without resorting to the introduction of the spectral flow operator.

In the following section we briefly review the free field representation that will be used. In Section 3, we compute the three-point amplitude that violates the winding conservation. We do this in detail, by emphasizing the steps through the calculation.

E-mail address: gaston@df.uba.ar (G. Giribet).

2. Perturbed Liouville gravity

2.1. Free field representation

Let us begin by briefly reviewing the free field representation we will employ. The action of the model is that of a matter conformal model S_M coupled to the Liouville action S_L. This takes the form

$$S = \frac{1}{4\pi} \int d^2z \left(-\partial\varphi\bar{\partial}\varphi + QR\varphi + \mu e^{\sqrt{2}b\varphi} \right) + S_M, \tag{1}$$

where $Q = b + b^{-1}$, and we define the convenient notation $b^{-2} = k - 2 \in \mathbb{R}_{>0}$. We will set the value of the Liouville cosmological constant as $\mu = 1$ by properly rescaling the zero mode of $\varphi(z)$ (see [5] for an excellent review on Liouville theory). The specific model representing the "matter sector" corresponds to a $c < 1$ conformal field theory defined by the action

$$S_M = \frac{1}{4\pi} \int d^2z \left(\partial X^0 \bar{\partial}X^0 - \partial X^1 \bar{\partial}X^1 - i\sqrt{k}RX^1 + \Phi_{\text{aux}} \right),$$

where the auxiliary field $\Phi_{\text{aux}}(z)$ is a perturbation, represented by a relevant primary operator of the matter sector and properly dressed with the coupling to the Liouville field in order to turn it into a marginal deformation. This takes the form

$$\Phi_{\text{aux}}(z) = (1/c_k)e^{-\sqrt{\frac{k-2}{2}}\varphi(z)+i\sqrt{\frac{k}{2}}X^1(z)}, \tag{2}$$

where c_k is simply a k-dependent numerical factor, see [2] for details. This is a perturbed CFT coupled to Liouville gravity in the spirit of the models studied in Ref. [6]. The stress tensor of the theory is then given by

$$T(z) = -\frac{1}{2}(\partial\varphi)^2 + \frac{Q}{\sqrt{2}}\partial^2\varphi - \frac{1}{2}(\partial X^1)^2 - i\sqrt{\frac{k}{2}}\partial^2 X^1 + \frac{1}{2}(\partial X^0)^2,$$

and leads to the central charge

$$c = \frac{3k}{k-2}.$$

The fields $X^0(z)$ and $X^1(z)$ have time-like and space-like signatures respectively; namely

$$\left\langle X^0(z_1)X^0(z_2) \right\rangle = -\left\langle X^1(z_1)X^1(z_2) \right\rangle = 2\ln|z_1 - z_2|.$$

Auxiliary field $\Phi_{\text{aux}}(z)$ enters in the action as an interaction term, involving the Liouville field $\varphi(z)$ and coupling it with the field $X^1(z)$. From the viewpoint of the computation of correlation functions, both the operator $\Phi_{\text{aux}}(z)$ and the cosmological term $\mu e^{\sqrt{2}b\varphi(z)}$ play the role of screening charges in the Coulomb gas-type realization. Actually, these are $(1,1)$-operators of the theory. Then, different amounts of both operators would be required for the correlation functions to be non-vanishing. However, we will focus the attention to those correlators that do not involve insertion of the perturbation field $\Phi_{\text{aux}}(z)$. These cases lead to the violation of winding number conservation. A similar free field realization was independently considered in [7].

The vertex operators in the theory are given by

$$\Phi_{j,m,\bar{m}}^{\omega}(z) = \frac{c_k\Gamma(-m-j)}{\Gamma(j+1+\bar{m})}e^{\sqrt{\frac{2}{k-2}}(j+\frac{k}{2})\varphi(z)+i\sqrt{\frac{2}{k}}(m-\frac{k}{2})X^1(z)-i\sqrt{\frac{2}{k}}(m+\frac{k}{2}\omega)X^0(z)} \times \text{h.c.}, \tag{3}$$

where h.c. stands for the anti-holomorphic part, which also contains the dependence on $\bar{m}$. It is worth pointing out that the normalization $\frac{c_k\Gamma(-m-j)}{\Gamma(j+1+\bar{m})}$ is the precisely the one required in order to reproduce the one-to-one correspondence between correlation functions in WZNW theory and Liouville theory. These are primary operators and have conformal dimension

$$h_{j,m,\omega} = -\frac{j(j+1)}{k-2} - m\omega - \frac{k}{2}\omega^2.$$

This yields the mass spectrum of the theory through the Virasoro constraint $h_{j,m,\omega} = 1$. On the other hand, the energy of the string states is given by the quantity $E = m + \bar{m} + k\omega$, which includes both kinetic and winding contributions. Now, we move to the correlation functions involving these states.

2.2. Particular correlation functions

Here, we are interested in particular N-point correlation functions in the theory. These are denoted as

$$A_{j_1,j_2,\ldots,j_N;m_1,m_2,\ldots,m_N}^{\omega_1,\omega_2,\ldots,\omega_N} = \left\langle \Phi_{j_1,m_1,\bar{m}_1}^{\omega_1}(z_1)\Phi_{j_2,m_2,\bar{m}_2}^{\omega_2}(z_2)\cdots\Phi_{j_N,m_N,\bar{m}_N}^{\omega_N}(z_N) \right\rangle$$

and are those satisfying the particular relation $\omega_1 + \omega_2 + \cdots + \omega_N = 2 - N$. According to the free field realization described in [1], these observables admit an integral representation of the form

$$
A^{\omega_1, \omega_2, \ldots, \omega_N}_{j_1, j_2, \ldots, j_N; m_1, m_2, \ldots, m_N} = \frac{\Gamma(-s)}{(c_k)^{2-N}} \prod_{d=1}^{N} \frac{\Gamma(-m_d - j_d)}{\Gamma(j_d + 1 + \bar{m}_d)} \prod_{r=1}^{s} \int d^2 w_r \left(\prod_{n=1}^{s} \prod_{a=1}^{N} |z_a - w_n|^{\frac{4}{2-k}(j_a + \frac{k}{2})} \right.
$$

$$
\left. \times \prod_{l=1}^{s} \prod_{t=1}^{l-1} |w_l - w_t|^{\frac{4}{2-k}} \prod_{b=1}^{N} \prod_{c=1}^{b-1} |z_b - z_c|^{\frac{4}{2-k}(j_b + \frac{k}{2})(j_c + \frac{k}{2}) - 2(m_b + m_c + \omega_b m_c + \omega_c m_b) + k(1 - \omega_b \omega_c)} \right) \tag{4}
$$

and, as it was mentioned above, correspond to those correlators which do not receive perturbations of the form $\int d^2 w\, \Phi_{\text{aux}}(w)$ but merely contributions of the Liouville screening charge $\int d^2 w\, e^{\sqrt{2}b\varphi(w)}$. These represent "maximally violating winding" scattering amplitudes in AdS_3 spacetime. In fact, except for the case of the 2-point function, the total winding number is not conserved in such correlation functions as can be verified from the following conservations laws

$$
\sum_{i=1}^{N} j_i + (N - 2)\frac{k}{2} + s + 1 = 0, \tag{5}
$$

$$
\sum_{i=1}^{N} \omega_i + N - 2 = 0. \tag{6}
$$

These conservation laws are due to $\delta(x)$-functions arising in the integration over the zero modes of the fields $\varphi(z)$, $X^0(z)$ and $X^1(z)$. Besides, correlators that also include "screenings" of the type $\int d^2 w\, \Phi_{\text{aux}}(w)$ do satisfy a different compensation relation, in particular: $\omega_1 + \omega_2 + \cdots + \omega_N = 2 - N + M$, where M is the amount of screening fields $\int d^2 w\, \Phi_{\text{aux}}(w)$ involved in the correlators (see [1] for details). Then, in the case of the three-point function, the only non-trivial result including the perturbation field $\Phi_{\text{aux}}(w)$ would be the conserving winding three-point function which is certainly well known. Let us focus on the non-conservative amplitude.

3. The three-point function

3.1. Integral representation

The intention is to compute the three-point function that describes string scattering amplitudes in AdS_3 for the case where the conservation of the total winding number is violated; and we want to do this by using free fields and without imposing any kinematic restriction on the involved states. We denote such correlation function as

$$
A^{\omega_1, \omega_2, \omega_3}_{j_1, j_2, j_3; m_1, m_2, m_3} = \left\langle \Phi^{\omega_1}_{j_1, m_1, \bar{m}_1}(z_1) \Phi^{-1-\omega_1-\omega_3}_{j_2, \frac{k}{2} - m_1 - m_3, \frac{k}{2} - \bar{m}_1 - \bar{m}_3}(z_2) \Phi^{\omega_3}_{j_3, m_3, \bar{m}_3}(z_3) \right\rangle,
$$

where the quantum numbers are such that satisfy the conservation laws leading to the non-vanishing result. Then, we will compute it by using the approach described in [1]. By means of the standard techniques of the Coulomb gas-like prescription, this leads to the following multiple integral in the whole complex plane

$$
A^{\omega_1, \omega_2, \omega_3}_{j_1, j_2, j_3; m_1, m_2, m_3}
$$

$$
= \Gamma(-s) c_k \prod_{a \neq b}^{3} |z_a - z_b|^{2(h_1 + h_2 + h_3 - 2h_a - 2h_b)}
$$

$$
\times \prod_{c=1}^{3} \frac{\Gamma(-m_c - j_c)}{\Gamma(j_c + 1 + \bar{m}_c)} \prod_{r=1}^{s} \int d^2 w_r \left(\prod_{n=1}^{s} |w_n|^{\frac{4}{2-k}(j_1 + \frac{k}{2})} |1 - w_n|^{\frac{4}{2-k}(j_2 + \frac{k}{2})} \prod_{l=1}^{s} \prod_{t=1}^{l-1} |w_l - w_t|^{\frac{4}{2-k}} \right)
$$

$$
\times \delta(m_1 + m_2 + m_3 - k/2) \delta(\bar{m}_1 + \bar{m}_2 + \bar{m}_3 - k/2) \delta(s + j_1 + j_2 + j_3 + 1 + k/2), \tag{7}
$$

where $\int d^2 w_r = \frac{1}{2\pi i} \int dw_r \int d\bar{w}_r$. The integration over the zero-mode of the fields $\varphi(z)$, $X^0(z)$ and $X^1(z)$ states that the amount of integrals to be performed is given by $s = -j_1 - j_2 - j_3 - \frac{k}{2} - 1$, while the momenta obey the conservation laws $m_1 + m_2 + m_3 = \bar{m}_1 + \bar{m}_2 + \bar{m}_3 = \frac{k}{2}$. Consequently, the conservation of the winding number is violated in one unit, namely $\omega_1 + \omega_2 + \omega_3 = -1$. Notice that the integral (7) is a Dotsenko–Fateev integral (similar to those arising in the minimal models) and can be explicitly solved by using the results of Ref. [8]. It is worth pointing out that, as it is usual within similar contexts, the integral formula of the type (4) has to be understood formally, and a kind of analytic extension of it is required in order to construct generic correlators with non-integer s. The features related to such analytic extension are basically two: First, it is evident that the products of the

form $\prod_{n=1}^{s}$ in (7) only make sense for positive integers s. Then, the analytic continuation of the formulas containing such products (after integration) is needed in order to consider generic values of the momenta (see Ref. [10] for more details). For instance, this is similar to what occurs in the computation of correlation functions in 2D minimal gravity, [9]. The second issue is the presence of the overall factor $\Gamma(-s)$, which arises after integrating over the zero mode of the Liouville field $\varphi(z)$. This factor diverges for positive integers s, and such a divergence is associated to the non-compactness of the theory, [11]. Here, we follow standard paths in this kind of computation and proceed by assuming an analytic continuation of the formulas obtained after the integration. Then, we can integrate out (7) by using the following identity (see Ref. [4])

$$
\begin{aligned}
I_s(J_1, J_2; k) &= \prod_{r=1}^{s} \int d^2 w_r \left(\prod_{n=1}^{n} |w_n|^{\frac{4}{k-2}J_1-2}|1-w_n|^{\frac{4}{k-2}J_2} \prod_{l=1}^{s}\prod_{t=1}^{l-1} |w_l - w_t|^{\frac{4}{2-k}} \right) \\
&= \frac{(k-2)}{\Gamma(-s)}\left(\frac{\pi\Gamma(\frac{1}{k-2})}{\Gamma(1-\frac{1}{k-2})} \right)^s \frac{\Gamma(-1-J_1-J_2-J_3)\Gamma(2J_2+1)\Gamma(J_1-J_2-J_3)\Gamma(-J_1-J_2+J_3)}{\Gamma(2+J_1+J_2+J_3)\Gamma(-2J_2)\Gamma(1-J_1+J_2+J_3)\Gamma(1+J_1+J_2-J_3)} \\
&\quad \times \frac{G_k(-2-J_1-J_2-J_3)G_k(-1-J_1+J_2-J_3)G_k(-1+J_1-J_2-J_3)G_k(-1-J_1-J_2+J_3)}{G_k(-1)G_k(-2J_1-1)G_k(-2J_1-1)G_k(-2J_3-1)},
\end{aligned} \tag{8}
$$

where J_3 has been defined by $J_3 = s - J_1 - J_2 - 1$, and where the special function $G_k(x)$ is defined through

$$G_k(x) = (k-2)^{\frac{x(k-1-x)}{2(k-2)}} \Gamma_2(-x|1, k-2)\Gamma_2(k-1+x|1, k-2),$$

where the Barnes function $\Gamma_2(x|1, y)$ is given by

$$\ln \Gamma_2(x|1, y) = \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \left((x+n+my)^{-\varepsilon} - (1-\delta_{n,0}\delta_{m,0})(n+my)^{-\varepsilon} \right),$$

where the presence of the factor $(1-\delta_{n,0}\delta_{m,0})$ in the right hand side means that the sum in the second term does not take into account the step $m = n = 0$.

Some useful functional relations of these functions are the following

$$G_k(x) = G_k(-x-k+1), \tag{9}$$

$$G_k(x) = G_k(x+1)\gamma\left(1 + \frac{1+x}{k-2} \right), \tag{10}$$

$$G_k(x) = G_k(x-k+2)(k-2)^{2x+1}\gamma(-x), \tag{11}$$

where we have made use of the standard notation

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}.$$

The $G_k(x)$ function develops simple poles at $x = p + q(k-2)$ and $x = -1 - p - (1+q)(k-2)$, for $p, q \in \mathbb{Z}_{\geqslant 0}$. The functional properties (9)–(11), due to the fact that these involve the $\gamma(x)$ function as well, can be used to prove the above integral formula for $I_s(J_1, J_2; k)$ in the case one prefers starting with the Dotsenko–Fateev expression in terms of product of $\Gamma(x)$ functions (see Appendix of Ref. [8]). The important point here is that the integral $I_s(J_1, J_2; k)$ precisely agrees with the one we have to compute through the identification $J_1 = -1 - j_1$, $J_2 = -\frac{k}{2} - j_2$ and $J_3 = -1 - j_3$. Then, we are ready to evaluate the three-point function. First, notice that the relations (9)–(11) help us in writing

$$G_k(-1-j_1+j_2+j_3+k/2) = \frac{G_k(j_1-j_2-j_3-k/2)}{\gamma(-j_1+j_2+j_3+k/2)}(k-2)^{k-1-2(j_1-j_2-j_3)},$$

$$G_k(-1+j_1+j_2-j_3+k/2) = \frac{G_k(-j_1-j_2+j_3-k/2)}{\gamma(j_1+j_2-j_3+k/2)}(k-2)^{k-1+2(j_1+j_2-j_3)}$$

and also

$$G_k(2j_2+k-1) = G_k(1-2j_2-k)(k-2)^{2(2j_2+k-1)}\gamma(1-2j_2-k)\gamma\left(1 - \frac{2j_2+k-1}{k-2} \right).$$

Since we are mainly interested in the string theory applications (and consequently in "correlation numbers" instead of correlation functions) we can make use of the projective invariance and set the worldsheet inserting points as usual: $z_1 = 0$, $z_2 = 1$ and $z_3 = \infty$. By integrating out and using the functional relations (9)–(11) we find the following expression for the violating three-point correlation number

$$A^{\omega_1,\omega_2,\omega_3}_{j_1,j_2,j_3;m_1,m_2,m_3}$$
$$= (k-2)\left(\pi\gamma\left(\frac{1}{k-2}\right)\right)^{-j_1-j_2-j_3-\frac{k}{2}-1} \frac{c_k\Gamma(-m_1-j_1)\Gamma(-m_2-j_2)\Gamma(-m_3-j_3)}{\Gamma(j_1+1+\bar{m}_1)\Gamma(j_2+1+\bar{m}_2)\Gamma(j_3+1+\bar{m}_3)}$$
$$\times \frac{G_k(j_1+j_2+j_3+\frac{k}{2})G_k(-j_1-j_2+j_3-\frac{k}{2})G_k(j_1-j_2-j_3-\frac{k}{2})G_k(1+j_1-j_2+j_3-\frac{k}{2})}{\gamma(-j_1-j_2-j_3-\frac{k}{2})\gamma(-\frac{2j_2+1}{k-2})G_k(-1)G_k(2j_1+1)G_k(1-k-2j_2)G_k(2j_3+1)}$$
$$\times \delta(m_1+m_2+m_3-k/2)\delta(\bar{m}_1+\bar{m}_2+\bar{m}_3-k/2)\delta(s+j_1+j_2+j_3+1+k/2), \tag{12}$$

where we preferred writing this in such a way because it permits to compare with the results in the literature (although the replacement $j_i \to -j_i$ is still necessary to compare with Ref. [12]). In fact, this formula exactly agrees with the one found in the literature by using rather different approaches (see also [13]). Besides, this extends previous computations which were done by using free field techniques because it does represent the "generic" three-point violating winding amplitude in AdS_3. Notice that this computation does not require the insertion of the spectral flow operator (conjugate representations of the identity operator) and seems to be valid for states with generic winding number (spectral flow parameter ω). In particular, the fact that this computation did not make use of the spectral flow operator is actually interesting. The inclusion of such additional vertex in the correlators with the purpose of realizing the violation of the winding number is actually one of the most ingenious tricks; however, from the viewpoint of the standard prescription for computing correlation functions, the introduction of such an operator could appear as a little heterodox; then, having found an alternative way of calculating seems to be a good point. Besides, it is worth mentioning that the formula above is consistent with the FZZ conjecture (cf. Ref. [14]).

3.2. Remarks on the pole structure

Some remarks are in order: First, besides the usefulness of expression (12) in order to compare with the results of [12] and [13], the result can be also written in a way such that the symmetry under interchanges $j_i \leftrightarrow j_j$ for $i, j \in \{1, 2, 3\}$ turns out to be explicit. By using the relations (9) and (10) we can write (12) in the following form, where such symmetry manifestly appears,

$$A^{\omega_1,\omega_2,\omega_3}_{j_1,j_2,j_3;m_1m_2m_3} = (k-2)\left(\pi\gamma\left(\frac{1}{k-2}\right)\right)^{-\sum_{a=1}^3 j_a-\frac{k}{2}-1}\prod_{b=1}^3\frac{\Gamma(-m_b-j_b)G_k(2j_b-\sum_{a=1}^3 j_a-k/2)}{\Gamma(j_b+1+\bar{m}_b)G_k(2j_b+1)}$$
$$\times \frac{c_k\gamma(1+\sum_{a=1}^3 j_a+k/2)G_k(\sum_{a=1}^3 j_a+k/2)}{G_k(-1)}\delta\left(\sum_{a=1}^3 m_a-k/2\right)\delta\left(\sum_{a=1}^3 \bar{m}_a-k/2\right)$$
$$\times \delta\left(s+\sum_{a=1}^3 j_a+1+k/2\right).$$

On the other hand, notice that we can obtain the two-point function by properly performing the limit $j_2 \to -k/2$ in the expression for the three-point function we just obtained. This is because the 2-point function does conserve the winding number. In fact, by taking into account the functional relation

$$\lim_{\varepsilon\to 0}\frac{G_k(\varepsilon-x)G_k(\varepsilon+x)}{G_k(2\varepsilon+1)} = -2\pi i(k-2)G_k(-1)\gamma\left(1+\frac{1}{k-2}\right)\delta(x)$$

and using (10) we find that in the limit $\varepsilon = -j_2 - k/2 \to 0$ the expression (11) reduces to

$$A^{\omega_1,\omega_3}_{j_1,,j_3;m_1,m_3} = -2\pi i(k-2)^2\left(\pi\gamma\left(\frac{1}{k-2}\right)\right)^{-2j_1-1}\frac{c_k\gamma(2j_1+1)\Gamma(-m_1-j_1)\Gamma(m_1-j_1)}{\gamma(-\frac{2j_1+1}{k-2})\Gamma(j_1+1+\bar{m}_1)\Gamma(j_1+1-\bar{m}_1)}$$
$$\times \delta(m_1+m_3-k/2)\delta(\bar{m}_1+\bar{m}_3-k/2)\delta(j_1-j_3). \tag{13}$$

This is, up to a k-dependent factor, the reflection coefficient, and is non vanishing only for the cases fulfilling the conditions $m_1 + m_3 = \bar{m}_1 + \bar{m}_3 = \omega_1 + \omega_3 = 0$.

Other comment regards the operator product expansion. The OPE and, consequently, the fusion rules of the theory are codified in the pole structure of the three-point function. The OPE for the $\omega = 0$ sector of the Hilbert space was studied in detail in Ref. [15] and was analyzed in relation with the four-point function in Ref. [12]. Here, we want to make a few remarks on the mixing between sectors $\omega = 0$ and $\omega = 1$. Let us consider the short distance behavior

$$\Phi^{\omega_1=0}_{j_1,m_1}(z_1)\Phi^{\omega_2=0}_{j_2,m_2}(z_1) \simeq \sum_\omega \int_C dj\,dm\,d\bar{m}\,|z_1-z_2|^{2(h_{j,m,\omega}-h_{j_1,m_1,\omega_1}-h_{j_2,m_2,\omega_2})}$$
$$\times Q_k(j_1,j_2,j;m_1,m_2,m;\omega)\{\Phi^\omega_{j,m}(z_1)\} + \cdots, \tag{14}$$

where the dots "$\cdots$" stand for "other contributions", and where the coefficient $Q_k(j_1, j_2, j; m_1, m_2, m; \omega)$ is given by a quotient between the structure constant (12) and the reflection coefficient (13) of two states with winding number $\omega = 1$. To be precise, a change of sign in such expression also appears because of a replacing $m_3 \to -m$. Because the pole structure of the structure constants determines the OPE, the arising of the factor $\gamma^{-1}(-j_1 - j_2 - j_3 - k/2)$ in (12) turns out to be important since it cancels a simple pole coming from the function $G_k(j_1 + j_2 + j_3 + k/2)$. The sum $\sum_\omega$ over the quantum number ω stands for making explicit that the fusion rules can lead to the mixing of sectors due to the spectral flow symmetry and eventually yield the violation of the winding conservation up to one in the three-point function; accordingly, $\omega \in \{0, \pm 1\}$. On the other hand, the region of integration C, schematically represented in the formal sum $\int_C dj \, dm \, d\bar{m}$, is defined in such a way that the integration over the indices $j \in -\frac{1}{2} + i\mathbb{R}$ and $\alpha \in [0, 1)$ of the continuous series $\mathcal{C}_j^{\alpha, \omega}$ is performed, and so for the contributions due to the poles corresponding to states of the discrete series $\mathcal{D}_j^{\pm, \omega=1}$. I.e. the definition of C is understood as running over the sets $\mathcal{C}_j^{\alpha, \omega} = \{j, m \mid j \in -\frac{1}{2} + i\mathbb{R}, \alpha \in [0, 1), m \in \alpha + \mathbb{Z}_{\geq 0}\}$ and encloses the poles belonging to the sets $\mathcal{D}_j^{\pm, \omega} = \{j, m \mid j \in \mathbb{R}_{<-\frac{1}{2}}, m = \pm(j - n), n \in \mathbb{Z}_{\geq 0}\}$. These sets parameterize the (universal covering of the) unitary representations of $SL(2, \mathbb{R})$ that are relevant for the string theory applications. For "picking up" the poles corresponding to the discrete states contributions, the contours included in C have to be properly chosen and a regularization procedure is required in those cases where different poles turn out to coincide, [15]. Besides, the sum over the quantum numbers j, m, $\bar{m}$ and ω in the OPE (14) has to take into account the fact that certain states of discrete representations of both sectors $\omega = 0$ and $\omega = 1$ are related one each other through the identification $\mathcal{D}_j^{\pm, \omega=1} \sim \mathcal{D}_{-k/2-j}^{\mp, \omega=0}$, similarly as what occurs in the compact $SU(2)_k$ case. Besides, a lower bound on the sum over j is required in order to guarantee the unitarity of the spectrum; namely $2j > 1 - k$. In the case on which we were interested here, unlike the case when the OPE is considered as being closed among the states of sector $\omega = 0$, it is not necessary to distinguish between discrete $\mathcal{D}_j^{\pm, \omega}$ and continuous series $\mathcal{C}_j^{\alpha, \omega}$ in order to analyze the m-dependent pole structure of $Q_k(j_1, j_2, j; m_1, m_2, m; 1)$. This is due to the fact that, remarkably, the dependence of the violating winding amplitude (12) on the parameters m and $\bar{m}$ turns out to be substantially simpler than the one that corresponds to the winding conserving case. This is explained by the fact that the field $\Phi_{\text{aux}}(z)$ depends on $X^1(z)$ as well. Hence, the whole pole structure of $Q_k(j_1, j_2, j; m_1, m_2, m; 1)$ is basically given by the poles of (12) and by the poles of the $\Gamma(x)$-functions (occurring at $x \in \mathbb{Z}_{<0}$) arising in the denominator of (13). Within this framework, it would be certainly interesting to extend the study made in [15] and [16] for the case of violating amplitudes. This could help in understanding the factorization properties of the four-point function in the $SL(2, R)_k$ WZNW model. As mentioned before, the OPE was studied in connection to the four-point function in Ref. [12], where it was proven that two incoming states belongings to the sector $\omega = 0$ can produce intermediate states with both $\omega = 0$ and $\omega = 1$. However, further study is necessary to fully understand the factorization of the four-point function and our hope is that the free field representation can help in doing this.

3.3. Remarks on the $sl(2)_k$ invariance

Now, let us make some remarks about the $sl(2)_k$ symmetry of the action (1). Such symmetry should to be present in the theory since what one is actually doing is asserting the identity between the free field realization Liouville $\times U(1) \times U(1)$ and the $SL(2, R)_k$ WZNW model. In fact, ab initio, we know that this construction actually presents such $sl(2)_k$ symmetry since it turns out to reproduce those solutions of the Knizhnik–Zamolodchikov equations that Ribault has found in Ref. [2]. However, even though the solutions we obtain have the appropriate symmetry, the question arises as to why does it happen if the Liouville interaction term $e^{\sqrt{2}b\varphi(z)}$ does not seem to have regular OPE with the $SL(2, R)_k$ currents though. To be precise, even though one knows that the free field representation presented in Ref. [1] turns out to transform properly by construction (it reproduces solutions of the KZ equation), it is also true that it is not obvious that the Liouville interaction term regarded as a screening charge commutes with the free field representation of the $sl(2)_k$ current algebra as one could naively expect. Again, why does it happen? The answer to this question yields from noticing that also the vertex operators $\Phi_{j,m,\bar{m}}^\omega(z)$ do not satisfy the usual OPE that the vectors of the $SL(2, R)$ representations satisfy according to the usual picture. In particular, it is worth noticing that the m-dependent overall factor of such vertex operators plays a crucial role for this condition to hold. To be precise, let me make the following observation: The stress-tensor of the free field theory presented here can be thought of as the Sugawara construction starting from the following generators of the $sl(2)_k$ affine algebra

$$J^\pm(z) = -i\sqrt{\frac{k}{2}}\partial Y^1(z) e^{\mp i\sqrt{\frac{2}{k}}(Y^0(z)+Y^1(z))} \pm \sqrt{\frac{k-2}{2}}\partial\rho(z) e^{\mp i\sqrt{\frac{2}{k}}(Y^0(z)+Y^1(z))}, \qquad J^3(z) = i\sqrt{\frac{k}{2}}\partial Y^0(z)$$

which follow from the free field redefinition [1]

$$\rho(z) = (1-k)\varphi(z) + i\sqrt{k(k-2)}\, X^1(z), \qquad Y^1(z) = (k-1)X^1(z) + i\sqrt{k(k-2)}\, \varphi(z), \qquad Y^0(z) = -X^0(z).$$

Then, as it can be verified, these currents do not have regular OPE with the Liouville cosmological constant term as one could naively expect. However, the non trivial point is that this is precisely what makes the $SL(2, R)_k$ to be recovered. Namely, these currents do not presents regular OPE with the Liouville cosmological term $e^{\sqrt{2}b\varphi(z)}$, but these do not satisfy the usual OPE with the vertex operators $\Phi_{j,m,\bar{m}}^\omega(z)$ either; and both facts seem to combine in such a way that explain why the formulas obtained for the correlators

by using this free field representation turn out to be $SL(2, R)_k$ invariant. Let me emphasize that the proof of such $SL(2, R)_k$ invariance of the correlators simply follows from the fact that these exactly solve the KZ equation, since lead to the solutions of [2] with the appropriate normalization factor, as by means of the Coulomb gas-like prescription in [1]. Furthermore, let us notice that this is precisely one of the two aspects that make of this free field construction in terms of the product Liouville $\times U(1) \times U(1)$ a non trivial one. Namely, the first non trivial point is the fact that this construction does not seem to follow from a simple field redefinitions (i.e. there is no clear way for obtaining this tachyonic interaction term through bosonization, for instance), and the second non trivial point is precisely the use of this non standard representations $\Phi^\omega_{j,m,\bar{m}}$ which, once combined with the Liouville cosmological term, restores the $SL(2, R)_k$ invariance that the correlators one computes manifest.

4. Conclusion

By using the free field representation introduced in [1], we have computed the three-point winding violating amplitude in AdS_3 for the generic case, i.e. without imposing the highest-weight state condition $m_a \pm j_a = 0$ on any vertex and without making assumptions on the angular momenta $m_a - \bar{m}_a$. Besides, this computation seems to involve vertex operators of generic winding number ω_a, without resorting to subtle tricks for defining the vertex of sectors $\omega > 1$. Then, it shows that the free field method turns out to be powerful enough to reproduce the three-point winding violating amplitude on the sphere in complete agreement with other calculations. Notice that even the factor $\gamma^{-1}(-j_1 - j_2 - j_3 - \frac{k}{2})$ has been reproduced here and the correct m-dependent factor has been also obtained. This result represents a consistency check for the realization proposed in [1], which now has shown to be useful to compute string scattering amplitudes. We emphasize that our result is based on the free field representation of Ref. [1], which was defined to exactly realize the solutions of the Knizhnik–Zamolodchikov equation given in Ref. [2].

Acknowledgements

References

[1] G. Giribet, Nucl. Phys. B 737 (2006) 209.
[2] S. Ribault, JHEP 0509 (2005) 045.
[3] K. Becker, M. Becker, Nucl. Phys. B 418 (1994) 206.
[4] G. Giribet, C. Núñez, JHEP 0106 (2001) 010.
[5] Y. Nakayama, Int. J. Mod. Phys. A 19 (2004) 2771.
[6] Al. Zamolodchikov, hep-th/0508044.
[7] V. Fateev, Relation between sine-Liouville and Liouville correlation functions, unpublished.
[8] V. Dotsenko, V. Fateev, Nucl. Phys. B 251 (1985) 691.
[9] V. Dotsenko, Nucl. Phys. B 338 (1990) 747;
     V. Dotsenko, Nucl. Phys. B 358 (1991) 541.
[10] M. Goulian, M. Li, Phys. Rev. Lett. 66 (1991) 2051.
[11] P. Di Francesco, D. Kutasov, Nucl. Phys. B 375 (1992) 119.
[12] J. Maldacena, H. Ooguri, Phys. Rev. D 65 (2002) 106006.
[13] G. Giribet, Phys. Lett. B 628 (2005) 148.
[14] T. Fukuda, K. Hosomichi, JHEP 0109 (2001) 003.
[15] Y. Satoh, Nucl. Phys. B 629 (2002) 188.
[16] K. Hosomichi, Y. Satoh, Mod. Phys. Lett. A 17 (2002) 683.