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THE MESHFREE LOCALIZED PETROV-GALERKIN APPROACH IN SLOPE STABILITY ANALYSIS

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Abstract

The article focuses on the use of the meshfree numerical method in the field of slope stability computations. There are many meshfree implementations of numerical methods. The article shows the results obtained using the meshfree localized Petrov-Galerkin method (MLPG) – localized weak-form of the equilibrium equations with an often used elastoplastic material model based on Mohr-Coulomb (MC) yield criterion. The most important aspect of MLPG is that the discretization process uses a set of nodes instead of elements. Node position within the computational domain is not restricted by any prescribed relationship. The shape functions are constructed using just the set of nodes present in the simple shaped domain of influence. The benchmark slope stability numerical model was performed using the developed meshfree computer code and compared with conventional finite element (FEM) and limit equilibrium (LEM) codes. The results showed the ability of the implemented theoretical preliminaries to solve the geotechnical stability problems.

Keywords:

Slope stability;
Meshfree localized Petrov-Galerkin method;
Meshfree methods;
Numerical simulation.

1 Introduction

The meshfree localized Petrov-Galerkin method showed to be successful numerical simulation method for various physical phenomena. The element-free Galerkin method (EFG) [1] was the first of meshfree variants applied for mechanical simulations of solid materials. This success of EFG becomes attractive to researchers, from different fields, recognizing the potential of the meshfree simulations and adapting it to various phenomena, e.g. fluid and groundwater flow [2], material plasticity problems [3], showing its ability to describe a complex geometry [1] and, more recently, for numerical analysis in geotechnical engineering [4] and [5].

MLPG uses spatial discretization scheme based on a computational area represented by a scattered set of nodes covering the computational area and its boundary. The discretization nodes maintain its original position in space, or they can change their position in space to avoid any improper deformation of the discretization nodes distribution after each loading step. The mentioned approach removes the disadvantage of the finite element method (FEM) for which the improper mesh element shapes causes numerical problems. This paper shows the results obtained using the computer code that implements an MLPG framework, and the results are validated against the FEM and limit equilibrium solutions for the geotechnical problem of slope stability.

The second chapter presents the theoretical aspects and requirements of the meshfree formulation. This follows the standard FEM formulation (MLPG represents a generalization of FEM in some way), thereby describing the similar theoretical aspects between MLPG and FEM. Particular implementation details are then compared for the situations, where a different approach needs to be used for MLPG. The subsequent section focuses on a simple homogeneous slope stability example to compare the presented meshfree framework results for slope stability, with the results obtained from a FEM (in-house code) and LEM (GeoStudio 2007) code.

2 The weak formulation of the equilibrium equations

The MLPG formulation is not based on prescribed elements, as the finite element formulation. The MLPG adopts the more general concept of support domain, geometrically simple element that gathers all the nodes it contains into some form of "virtual" element. The governing equation is used in the weighted residual form rather than the global energy principle to create the global equation system. The compatibility requirements for the shape functions used in MLPG are restricted only to the area of the quadrature domain. Using another point of view, the localized meshfree method only requires the local compatibility of shape functions. The radial basis functions (RBF) are one of the widely used alternatives to constructing the shape function. One of the main advantages of RBF functions is their Kronecker delta function property that makes the imposition of essential boundary condition straightforward.

2.1 MLPG formulation of the governing equations

The general weighted residual form defined over quadrature domain Ω_q bounded by Γ_q has the following matrix form [6]

$$\mathbf{K}_I \mathbf{u} = \mathbf{f}_I \,, \tag{1}$$

where $\mathbf{K}_I$ is the matrix called nodal stiffness matrix for the I-th field node, which is computed using the following formula

$$\mathbf{K}_I = \int_{\Omega_q} \mathbf{V}_I^{\mathbf{T}} \mathbf{D}\mathbf{B} d\Omega - \int_{\Gamma_{qt}} \mathbf{W}_I^{\mathbf{T}} \mathbf{n}\mathbf{D}\mathbf{B} d\Gamma - \int_{\Gamma_{qu}} \mathbf{W}_I^{\mathbf{T}} \mathbf{n}\mathbf{D}\mathbf{B} d\Gamma \,, \tag{2}$$

where, Γ_{qt} and Γ_{qu} represents the part of the local quadrature domain boundary with prescribed natural and essential boundary condition respectively. The $\mathbf{f}_I$ is a nodal force vector with contribution from the body forces applied in the model domain, and the tractions applied on the natural boundary (Γ_{qt})

$$\mathbf{f}_I = \int_{\Omega_q} \mathbf{W}_I^{\mathbf{T}} \mathbf{b} d\Omega + \int_{\Gamma_{qt}} \mathbf{W}_I^{\mathbf{T}} \bar{\mathbf{t}} d\Gamma \,. \tag{3}$$

The matrices $\mathbf{W}$ and $\mathbf{V}$ in the equations (2) and (3) represents the matrix of the weight function and weight function derivatives evaluated over the quadrature domain [6]. In equation (2) $\mathbf{D}$ represents the matrix of elastic constants, $\mathbf{B}$ is the matrix containing the spatial derivatives of displacement field, $\mathbf{n}$ is a vector representing boundary normal, $\mathbf{t}$ is the boundary traction and $\mathbf{b}$ represents the body force. The weight function W used in this study represents the cubic spline and is defined according to [6], as follows

$$W(x - x_i) = W_{ix}(x - x_i) W_{iy}(y - y_i) = W(r_{ix}) W(r_{iy}) \,, \tag{4}$$

$$r_{ix} = \frac{\|x - x_i\|}{d_{sx}} \qquad r_{ix} = \frac{\|y - y_i\|}{d_{sy}} \,, \tag{5}$$

where d_{sx} and d_{sy} is the 2D dimension of the integration area used to evaluate governing equation.

$$W(r_{ix}) = \begin{cases} \frac{2}{3} - 4r_{ix}^2 + 4r_{ix}^3 & r_{ix} \le 0.5 \\ \frac{4}{3} - 4r_{ix} + 4r_{ix}^2 - \frac{4}{3} r_{ix}^3 & 0.5 \prec r_{ix} \le 1 \\ 0 & r_{ix} \succ 1 \end{cases} \qquad W(r_{iy}) = \begin{cases} \frac{2}{3} - 4r_{iy}^2 + 4r_{iy}^3 & r_{iy} \le 0.5 \\ \frac{4}{3} - 4r_{iy} + 4r_{iy}^2 - \frac{4}{3} r_{iy}^3 & 0.5 \prec r_{iy} \le 1 \\ 0 & r_{iy} \succ 1 \end{cases} \,. \tag{6}$$

The global characteristic linear equation system is not symmetric, which makes the computation more expensive compared to standard FEM [6] solvers. Numerical simulations presented, in this article uses the Bi-Conjugate Gradient stabilized (BiCGStab) iterative solver to solve the global equation system [7]. Because the deformation analysis of the Mohr-Coulomb material is nonlinear, the numerical code needs some algorithm to deal with nonlinearities. The generation of the body forces caused by plastic failure combined with the Newton-Raphson method can't be implemented straightforwardly as in FEM. To overcome this complication, the load is divided into small sub-steps which are applied at small increments at each pseudo-time step, having the change of constitutive matrix of each node at failure in mind. If there is any node at which the deformation overcome the prescribed tolerance the whole structure is assumed to be collapsing. There is no iterative procedure as in conventional FEM codes, but the global stiffness matrix has to be altered at node failure for each pseudo-time step, using the elastic-plastic constitutive matrix defined in section 2.2.

2.2 Mohr-Coulomb material equations

The procedures for conventional nonlinear material analysis start using the incremental constitutive relation as follows [8]:

$$d\boldsymbol{\sigma} = \mathbf{D}_{\mathbf{ep}} d\boldsymbol{\varepsilon}, \tag{7}$$

where $\mathbf{D}_{ep}$ is tangential material constants matrix. In this article, the Mohr-Coulomb yield criterion was implemented in the MLPG because it is the first choice for almost every stability or failure simulation in geotechnical engineering.

$$F = \sigma_m \sin\varphi + \bar{\sigma}\left(\frac{\cos\theta}{\sqrt{3}} - \frac{\sin\theta\sin\varphi}{3}\right) - c\cos\varphi, \tag{8}$$

where c and φ are the material strength parameters. In the field of soil strength these two parameters represent the cohesion and internal friction angle respectively. The other values used in equation (8) σ_m, $\bar{\sigma}$, θ are the stress tensor invariants used to decouple the mean and deviatoric part of the stress. Because the simulations presented in this article are focused on the stability evaluation the associated plasticity is adopted in the material model implementation so the plastic potential is the same as yield function.

3 Shape functions construction

The shape functions in standard FEM implementation is based on polynomials defined over a relatively simple element. The number of points used in the element is known in advance and so is the order of the polynomial used for interpolation. Because the number of nodes in the support domain is not guaranteed the using of polynomials in meshfree is complicated and depends not only on the node count but also on the node distribution [6]. To remove the mentioned difficulties, the radial basis functions (RBF) is one of the best solutions [6]. There are various types of radial functions such as multi-quadratic (MQ) or thin-plate spline (TPS) that can be used as a base function. The thin-plate spline is radial basis function [2] defined as a fundamental solution for the biharmonic equation and for 2D it is defined as

$$R_i(\mathbf{X}) = r_i^2 \log(r_i), \tag{9}$$

where r_i is the distance between the response node ($\mathbf{X}$) and the source node $i(\mathbf{X}_i)$ defined simply as the 2d norm

$$r_i = \sqrt{(x - x_i)^2 + (y - y_i)^2}\ . \tag{10}$$

The TPS kernel function has a weak singularity as the response node approaches the source node [6], the singularity can be avoided by the simple limit procedure. The approximation of the solution using the RBF kernels augmented with polynomials can be written as

$$u(\mathbf{X}) = \sum_{i=1}^{n} R_i(\mathbf{X}) a_i + \sum_{j=1}^{m} p_j(\mathbf{X}) b_j = \mathbf{R}^{\mathrm{T}}(\mathbf{X})\mathbf{a} + \mathbf{p}^{\mathrm{T}}(\mathbf{X})\mathbf{b}, \tag{11}$$

where $R_i(\mathbf{X})$ is the radial basis function (RBF), n is the number of RBFs which depends on the number of nodes in the support domain, $p_j(\mathbf{X})$ is polynomial, m is an order of augmenting polynomial, a_i and b_j are interpolation coefficients. For more details, please see [2] and [6].

4 Meshfree numerical solution of the slope stability

The analyzed problem is a homogeneous slope of soil material, described using nonlinear MC model, loaded by self-weight. The stability of the slope is described by the scalar quantity describing the ratio of active and passive forces - a factor of safety (FS). The FS evaluation from the results of numerical simulation is not so straight forward as in the calculations performed using limit equilibrium methods. To find the factor of safety many calculations have to be performed with change (or reduction) of the strength parameters by the scalar valued strength reduction factor (SRF). The SRF factor equals to FS for the value at which there is a transition from stable to unstable state.

Gravity loads are generated using the global RBF shape functions and applied to the slope in small (difference) increments. The loading is applied in small steps while no failure occurs. At failure, the overall loading is compared to the model (slope) gravity loading and its ratio represents FS.

Table1: Factors of safety (FS) obtained using compared methods.

Computational method	The factor of safety (FS)
FEM – In-house code	1.57
LEM – Jambu (GeoStudio 2007)	1.62
Meshfree – MLPG with RBF	1.52

The boundary conditions on the left and right vertical boundaries are represented by fixed zero movements in the horizontal dimension and no movement at the model basement. Fig.1 shows the nodal distribution and displacement field vectors numerical stability simulation of a homogeneous 2:1 slope with the Mohr-Coulomb material model strength parameters φ = 20° and c = 15 kN/m^2. The gravitational load of slope is driven by the volumetric weight γ = 20 kN/m^3. The elastic parameters are given nominal values of E = 5×10^4 kPa and ν = 0.3 since they have little influence on the computed factor of safety.

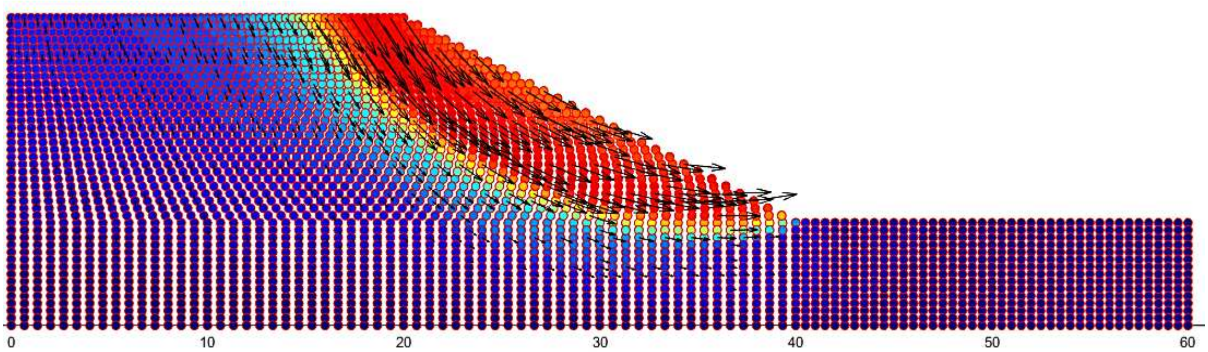

Fig.1: Nodal distribution for the meshfree model with vector and color-coded displacements.

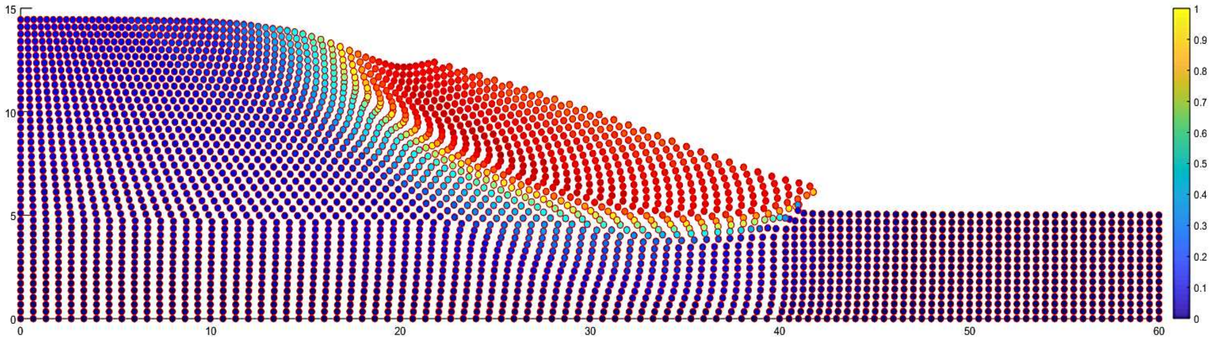

Fig. 2: The deformed model with color-coded normalized displacements, FS = 1.52.

The final output in Fig. 2 gives the "toe" type mechanism of failure. The results of safety factor (FS) where the meshfree model has been cross-validated using conventional finite element model (Fig. 3) and limit equilibrium Janbu method (Fig.4) are summarized in Tab.1.

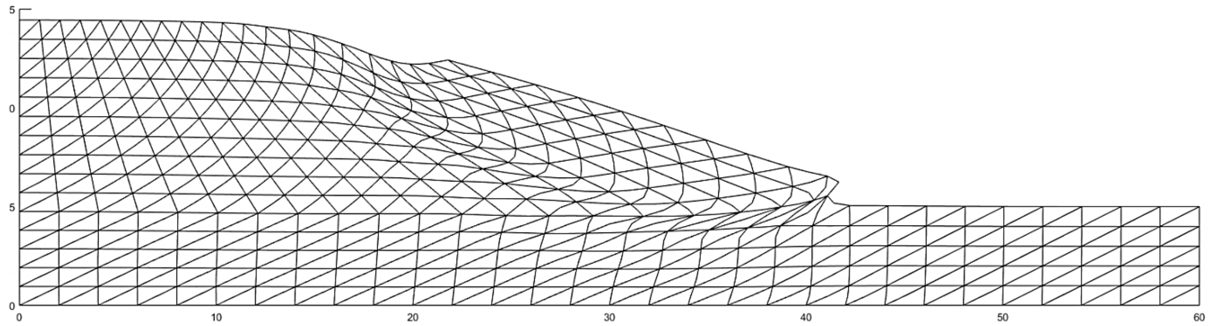

Fig. 3: The deformed FEM mesh at failure, FS = 1.57.

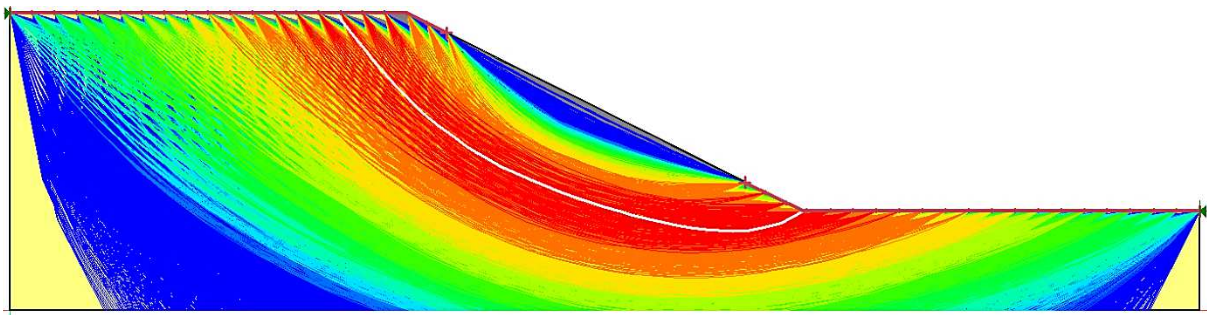

Fig. 4: Trial slips surfaces and critical slip surface LEM - Janbu, FS = 1.62.

The failure mechanisms, as well as FS values, resolved using mentioned computational methods shows excellent agreement with the meshfree model. The results showed the ability of the implemented meshfree method to evaluate the slope stability tasks.

5 Conclusions

A meshfree localized Petrov-Galerkin method (MLPG) code has been developed for nonlinear material analyses. The results of the used method in solving nonlinear problems are compared to the finite element method (FEM) and limit equilibrium (LEM) results of a similar problem.

Slope stability numerical example is provided to illustrate the implementation, performance, and behavior of the MLPG method. The simulations to evaluate and analyze slope stability problems performed using the meshfree showed no stress discontinuities and the evaluation of stability better quantified. This important aspect of meshfree simulations should be focused on future work.

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