To My Father and My Mother

DYNAMICS OF STRUCTURAL VIBRATIONS
BY
FREQUENCY DEPENDENT MATRICES
AND
MODAL ANALYSIS

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SUMMARY:

This report deals with the solution of natural vibration problems in structural machanics by means of frequency dependent mass and stiffness matrices and the solution of structural response problems by classical modal analysis.

The frequency dependent matrices for each structural member are contructed by means of the solution of equations of motion or by means of its normal modes. The overall frequency dependent matrices are formed by conventional assembling process. The nonlinear eigenvalue problem thus obtained is solved by Sturm sequence technique and inverse iteration. All normal modes in a specified frequency range are obtained accurately. Emphasis is on skeletal structures. The computed results of a plane frame were verified by experiment.

Numerical examples are given for the response analysis of three dimensional frame works. Computer programmes for natural vibration problems of space frames, for response analysis with arbitrary excitations, and for numerical linear algebra for matrices of different forms are given in FORTRAN.

ACKNOWLEDGEMENT

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LIST OF SYMBOLS

a, b	length and width of a plate
a_i, b_i	constants
A_o	cross sectional area of a beam
A, B, C, D, A_i	integration constants
$[a], [b]$	relation matrices as defined in eq(2.6.1),(2.6.2)
A_s	effective area of shear
b	bending parameter, eq(3.6.41)
$[B_i]$	boundary differential operator, eq(3.5.3)
$C_{ijkl}, [C]$	elastic constants
c	tapering constant of a straight beam
$[\mathcal{D}(\omega)]$	dynamic stiffness matrix
$[\mathcal{D}_o]$	constant stiffness matrix, independent of frequency
$D, D_{11}, D_{12}, \cdots$ etc.	flexural rigidities of plate
e	base of natural logarithm
$\{e_i\}$	unit vectors or base vectors
E	Young's modulus
E or $\langle \cdots \rangle$	average operator
EI	flexural rigidity
$\{F\}$	force vector
$\{f(t)\}$	forcing function
$F_i(\omega)$	frequency functions, eq(3.2.36)
$\mathcal{F}_i(\omega)$	frequency functions, eq.(3.6.5)
$F(i\omega)$	harmonic response function, receptance
G	shear modulus
GJ	beam torsional rigidity

G_{ij}	constants, eq(3.5.27)
G_i	frequency functions, eq(3.2.28)
h	thichness of plate,shell etc.
$h(t)$	impluse reponse function
i,j,k,L,m,n	integer indice
I_x, I_y, I_z	moment of inertia of area about X axis, etc.
I_o	polar monent of inertia
I_n, J_n, K_n, Y_n	Bessel functions
J	torsional constant
$[K(\omega)]$	stiffness matrix
k_1, k_2, etc.	spring constants
l	length
L	Lagrangian, eq(2.5.12)
$[L]$	linear differential operator, eq(3.5.2),(5.2.2)
$[M(\omega)]$	mass matrix
$[M_.]$	constant mass matrix, independent of frequency
m_1, m_2, etc.	mass points
N_x, N_y, N_{xy}	axial forces and twist of a plate
P_i	the ith modal force
p_i	the ith modal displacement
P	axial force parameter, eq(3.6.41)
q	uniformly distributed load
$q_i, \{q\}$	generalized coordinates of displacements
$Q_i, \{Q\}$	generalized forces
Q	shear force
$\{r\}$	position vector

$\alpha_i, \alpha_{ij}, \beta_i, \beta_{ij}$	constants to be determined
ω_i	the ith natural frequency
ξ, η, ζ	nondimensionalized spatial coordinates
$[\partial]$	differential operator, eq(3.3.2)
θ	angle of rotation
δ_{ij}	Kornecker delta
ν	Poison's Ratio
$[\chi], \{\chi_i\}$	spatial functions, eq(3.5.32)
Ψ, Φ, θ	Euler's angles
Δ, Δ_i	determinants
ζ, ω_b	damping ratio to the critical damping
τ	time variable, time lag
$\Phi_{xx}, \Phi_{x_i x_j}$	spectral densities
$(\bar{\cdots})$	quantity orientated in line with the global coord.
$(\tilde{\cdots})$	quantity coincident with global coord., after the Boolean transformation
$(\dot{})$	derivative with respect to time
$(\,)'$	derivative with respect to argument

Γ	rotatory inertia parameter, eq(3.6.41)
R	radius of a circularly curved beam
$R_{xx}, R_{x_i x_j}$	correlation functions, eq(7.9.3)
s	shearing parameter, eq(3.6.41)
S	surface area
$[S]$	stability matrix
t	time variable
T	kinetic energy
T	torsion
$\{u\}$	displacement vector
u, v, w	displacement components in X,Y,Z direction
U	strain energy
U_0	strain energy density, eq(2.5.3)
V	potential energy
vol	volume, elemental volume
W	work
$\{X\}$	body force vector
X, Y, Z	coordinate axis
x, y, z	spatial coordinates
$X(t), X_i(t)$	random variable
ω	frequency of vibration
$\{\phi_i(x, y, z)\}$	the ith normal mode vector
$\sigma_1, \sigma_2, \lambda, \nu, \gamma, \mu, \nu$	frequency parameters
$\rho, [\rho]$	mass density, the matrix of measure of inertia
σ_{ij}	stress components
ε_{ij}	strain components
$\{\Phi(x, y, z)\}$	surface force vector

CHAPTER ONE

INTRODUCTION

1.1 DYNAMIC EFFECTS OF STRUCTURES

During recent years, the dynamic effects of structures are receiving much more attention than before due to the increased precision called for in design.

Modern civil engineering structures are required to span larger distances than in the past and to stay within the economic limits set down for them. This leads to slender structures which are very sensitive to dynamic effects. It is further aggravated by the fact that the dynamic loads they are expected to support grow continually during their working lives as a consequence of the ever increasing speeds of vehicles moving over them, the intensified soil vibration induced by incessantly heavier traffic, and by the dynamic effects of the machinery they carry,etc.. Static analysis in itself can no longer ensure the safety of such structures.

In ship dynamics, problems of propeller and wave excited vibrations have recently been accentuated by the general growth in ship size and resulting reduction in natural frequencies of primary and secondary structures. The increased power and speed and the decreased structural damping of many modern ships have also contributed to these problems.

Certain types of buildings will increase in height and will be contructed of new lighter materials with less internal damping during the naxt decade. The buildings could suffer from wind generated oscillations as well as earthquake excitations. Such matters should be considered at the design stage.

In agriculture mechanics, the dynamic effects of resonance have been applied to fruit harvesting machines by treating a tree as an assemblage of beams with viscous dampings. In biomechanics, many mechanical organs which take the advantages of the natural vibrations of structures are in their design stages. As far as applications to manufacturing industry arè concerned, the efficiency of conveyors and assembling machines reaches the maximum if they are operated at their natural frequencies. Therefore, it is nessary for research organizations to study accurate and reliable dynamic analysis procedures.

1.2 DYNAMIC ANALYSIS OF STRUCTURES

A structure is considered as a system whose primary function is to support loads, and also an assemblage of discrete structural elements interconnected at a finite number of nodes or generalized coordinates (to be defined in chapter two). The word "analysis" referes to the evaluation of the displacements of the assemblage and of the element forces acting at the nodes. And "dynamic" implies time-varying; hence if the structure is subject to time-varying loads it represents a structural dynamic problem. So far the analytical tools are mainly confined to the study of the dynamic effects of structures which are in states not far away from their static equlibrium configurations.

For the convenience of study, the dynamic problems are classified as the dynamic stability problems and the dynamic response problems. The latter are further divided into asymptotic responses and transient responses. The analysis under the heading of dynamic stability involves the study of the effects of cyclic loads on the structure and intends to answer under what circumstances the structure will fail to hold it static equilibrium configurations. And the dynamic response analysis is the study of the

relationships of displacements and loadings in the time history under the assumption that the structure is in a dynamically stable state. Transient response analysis is the study in short time duration while the asymptotic response is the study in long time duration. This report will consider the dynamic stability problems and the response problems in structural dynamics.

1.3 RELIABILITY AND ACCURACY OF SOLUTIONS

A procedure in dynamic analysis involves four major steps. They are the idealized description of the structural system by a mathematical model, the establishment of a system of governing equations of motion, the solution of this set of equations, and the confirmation of the solution by experiments. By reliability, we mean that every solution obtained by such a procedure is a solution, exactly or approximately, of the original system, and there are no solutions of the original system are missed out by the procedure within any domain of interest. And by accuracy we mean the closeness of a solution from the procedure to the corresponding actual solution of the original system. In other words, the reliability of a dynamic analysis procedure is determined by the "completeness" of its results to the solutions of the original system and the accuracy is determined by their "closeness". The requirements of "how reliable" and "how accurate" of the results within some economic limits of computation will determine the choice of procedures of analysis.

This report is concerned mainly with a computational procedure of vibration analysis in the branch of dynamics of elastic systems by a reliable and accurate method called the dynamic stiffness method.

1.4 METHODS OF STUDY

The analysis of structural response in this report is basically the classical modal analysis. This modal analysis is a special application of the method of Galerkin and uses the normal modes of the structure as basic coordinate functions, and reduces the response problem essentially to the problem of computing natural modes of vibration of the structure. The natual modes of vibration are studied by the method of dynamic stiffness. The method is outlined as below.

We first choose a finite number of nodes of the structure, in such a way that at any instant the coordinate systems associated to these nodes can describe the configurations of the structure sufficiently. If we denote the displacements of these coordinates by the displacement vector $\{q\}$ and the forces at these coordinates by the force vector $\{Q\}$, then the following relationship between $\{q\}$ and $\{Q\}$ is set up by considering the dynamic equilibrium of the structure and its members,

$$[D]\{q\} = \{Q\} \tag{1.4.1}$$

The matrix $[D]$ in equation (1.4.1) is a symmetric matrix and is called the dynamic stiffness matrix. The setting up of this matrix of the structure occupies a very important position in this report.

The requirement for free vibration is that the loadings $\{Q\}$ vanish, therefore,

$$[D]\{q\} = \{0\} \tag{1.4.2}$$

represents the equation of motion of free vibrations. The necessary and sufficient conditions for having nontrivial solutions of $\{q\}$ is that

$$\det [D] = 0 \tag{1.4.3}$$

This equation is to be solved for natural frequencies by a reliable Sturm sequence method which has only fair accuracy. Then the matrix $[\mathcal{D}]$ is separated into two positive definite matrices $[\mathcal{K}]$ and $[\mathcal{M}]$ by a new theorem such that equation (1.4.2) becomes

$$[\mathcal{K}]\{q\} = \omega^2[\mathcal{M}]\{q\} \qquad (1.4.4)$$

where ω is the frequency of vibration, $[\mathcal{K}]$ and $[\mathcal{M}]$ are the stiffness and mass matrices respectively. The nontrivial solutions of equation (1.4.4) for ω and $\{q\}$, when approximated values of ω are known, are obtained by a very stable and accurate method of inverse iteration in conjuction with the Rayleigh's Quotient of the system. Since the method is iterative in nature, the accumulative numerical errors of arithmetic operations are eliminated.

The response of the structure to arbitrary excitations are then studied in terms of the free vibration modes. The structural damping effects are taken into account by considering complex elastic modulus.

1.5 HISTORICAL BACKGROUND

A brief note on the history of the development of vibration theory of continuous systems is presented in this section.

The first mathematician to consider the nature of the resistance of solids to rupture was Galileo (1638). Discussions of the vibrations of solid bodies were first given by Euler (1744) and Bernoulli (1751) who obtained the differential equation of elastic vibrations of beam by variational methods. The formulation of the general analytical tools in vibration theory had their forms during the time of Rayleigh (1842-1919). Since then, more accuracy theories of vibrations which take into account of secondary effects such as rotatory inertia, shear, etc. were formulated and

particular attention was given to the theories of single elements such as beams, plates,and shells. These analytical methods had their limits when solving structural vibration problems due to the resulting highly complicated boundary value problems.

The urge to know the natural frequencies of complex systems to predict the flutter speeds of aircraft in the 1930's forced the scientists to investigate simple solution methods. The methods involving piecewise representation of the continuous system were found reliable and convenient. The three most frequently used methods were known as (a) the finite diffence, (b) the lumped-mass, and (c)the finite element methods. Of these, the finite element method is probably the most popular and useful currently. The finite difference method involves the replacement of the differential operators in the governing equations of motion by their finite difference equivalents. Collatz (ref 101) gives a comprehensive list of finite difference equivalents to differential operators. The resulting equations can be conveniently solved by hand using the method of relaxation (ref 102). The method of lumped-mass assumes that the masses of the original system are lumped at discrete points, each connected by a massless elastic element. The contribution of Argyris is outstanding in this field. The mathematical model of lumped-mass system will give a closer representation of the potential energy than of the kinetic energy since the elastic properties remain distributed while the masses are arbitrarily lumped. The kinetic energy would be represented to the same degree of accuracy as potential energy if the mass were considered distributed in the same manner. To accomplish this without reverting to the exact classical method, it is necessary to assume the displacement (and/or stress) distribution patterns everywhere in the structure before the analysis is carried out. The solution involves using the so-called "finite element" approach. Clough (ref 104 introduced the concept whereby continuous systems were represented by a

connected set of finite elements, the synthesis being possible by Argyris's method. In Clough's original paper the method was applied to the analysis of static plane stress problems, but since then it has been applied to the a wide range of static and dynamic problems.

Complete understanding of the influence of the selected displacement function on the results of most system has not yet been achieved. Consequently, their selection for a particular system is still an art and many workers prefer to make the selection at the shape function stage, where they may also use their physical insight in guiding the selection. For a system represented by a given number of finite elements and a fixed number of degrees of freedom per element, the number of natural frequencies which may be found within a given accuracy increases as the accuracy with which the deflection function satisfies the governing differential equation increases. This was illustrated indepently by J. S. Przemieniecki (ref 8) and Cohen and McCallion (ref 105) for beam structures. Both authors found difficult in solving the resulting eigenvalue problem, because at some frequencies of vibration, the matrices involved become non-definite.

This report will base on this concept to improve the finite element representation of the original system so that higher modes of free vibration can be obtained accurately without subdividing the system. The dynamic matrices resulted are always positive definite. The method has been extented to simple two- and three- dimensional elements.

1.6 THESIS

The construction of the thesis is given in this section. The domain of interest is the frequency parameter instead of time variable in this report

so that the amplitudes of displacements and forces are related to each other by means of the conditions of dynamic equilibrium and compatibility. The formulation is similar to the displacement finite element approach in static analysis of structures. Therefore, the knowledge of the fundamentals of finite element method is assumed.

Although the method is based on the well-known slope-deflection equations and dynamic stiffness, the formulation of the dynamic stiffness by adding frequency terms in the conventional finite element formulae, and the solution of the complete nonlinear eigenvalue problem thus obtained for natural frequencies and the corresponding modes are new. The solution is exact in classical sense for skeletal structures and is accurate and stable for other structures.

Therefore, the report is in four parts. Chapter two contains the basic knowledge required for the remaining part of the report. Chapter three, four, and five present the theory of the method, from a single structural member to the solution of the overall system. Chapter six refers to the development of the relevant computer programmes and the comparison of theorectical and experimental results. And finally, chapter SEVEN demonstrates the applications of the results to various branches of structural engineering.

Chapter two, Configuration, Energy & vibration, has an introduction on general vibration analysis, and is followed by a discussion on generalized coordinates and discretization of the overall structure into elements whose properties and performance are convient to study. Then the various forms of energy are discussed and a special set of generalized coordinates, i.e. normal coordinates, is introduced. The basic equations in elasticity are then summarized, and, finally, the finite element method in vibration is outlined.

The properties and performance of a single element are studied in chapter three. New methods of constructing the dynamic stiffness and separating the dynamic stiffness matrix into two positive definite matrices, which enable a numerically stable and accurate method of inverse iteration for natural frequencies and modes, are discussed in depth. These matrices are listed explicitly for some one-, two- and three- dimensional members.

The standard routine of composing the overall matrices from individual members is carried out in chapter four. New methods of transforming the local coordinates of elements to the same orientations of the overall system by Euler's angles which minimize the storage requirements and additional calculations are also presented. The methods are extended to general structural elements.

Chapter five deals with the solution of the overall nonlinear eigenvalue problem. After a general view of the nonlinear eigenvalue problem, a complete and accurate method of solution which combines the advantages of the classical Sturm sequence technique and the modern inverse iteration method, is studied in detail. An example of a simple structure is given and compared with the classical methods.

The development of relevant computer programmes and the confirmation of results by experiment are presented in chapter six. Most of the programmes are newly designed because of the distinct nature of the nonlinear eigenvalue problem. The experimental results prove the completeness and the closeness of the theory to the behaviour of a typical structure.

Chapter seven is concerned mainly with the applications of the natural frequencies and the vibration modes of a structure. Examples given include the response analysis of a structure due to loadings of stationary and nonstationary, deterministic and random natures.

Finally, chapter eight contains the general discussion, the possibility of further development and the overall conclusion.

CHAPTER TWO

CONFIGURATION, ENERGY AND VIBRATION

When describing the motion of a vibrating body at any instant, a coordinate system to where the states (displacements, forces, etc.) of the body can be refered must be chosen. To describe the vibration of an assemblage of bodies, a single conventional coordinate system, e. g. Cartesian, cylindrical, etc., can be undesirable, because they all result in a system of coupled equations of motion with very complicated regions and boundary conditions. However, we may choose one coordinate frame attached to each body, then the motion of the set of the bodies can be totally described by the motion of it individual members in their own frame of reference, and in addition, the kinematic relations between these frames. The coupling effects of each individual body are given by the conditions of constraints. The motion of a body can be described in many ways. Since the vibration phenomenon is basically a process of transferring energy from one form to the other for every vibrating body, therefore one of the best ways is by reference to its energy.

Therefore, after a review of the basic terms in vibration analysis in the first section of this chapter, we discuss various types of coordinate systems and the forms of energy referred to these coordinate systems for later use inthe remaining chapters. Finally the method of finite element in vibration analysis is briefly outlined.

2.1 FUNDAMENTALS OF VIBRATION (ref 1,2,3)

An assemblage of coupled objects is called a system. A mechanical system possesses inertia and elasticity. When a small disturbance is applied to this system, the propagation of this small disturbances in the medium of the system is called vibration. Most machines and engineering structures experience vibration in differing degrees.

If the medium of a system vibrates sinusoidally in time, the vibration is harmonic. The geometric state (displacements, strains etc.) of the system at any instant is called its configuration. When a harmonic vibration exists, the configuration of the system will repeat itself in equal intervals of time. The time elapsed while the motion repeats itself is called the period; the motion completed during the period is referred to as a cycle; and the number of complete cycles in a unit time is the frequency of vibration; and the peak value of motion is called the amplitude. The set of parameters needed to specify the configuration is called the set of generalised coordinates, and the time rates of change of generalized coordinates are called generalized velocities.

The vibration of a system is generally nonlinear in nature. However, if there exists an equilibrium configuration of the system, i.e. a configuration in which the system can remain permanently at rest or about which the system undergoes a prescribed steady state motion, we can expand all the nonlinearities (geometric, material etc.) in Taylor series about the configuration of equilibrium in terms of the generalized coordinates and their time derivatives. When the vibration of the system is not far away from its equilibrium configuration and when the disturbances are small, we can study the vibration approximately by the first two terms of the Taylor series of the nonlinearities and we say the system is linearized.

Vibrations of linear systems fall into two general classes, free and forced. Free vibration takes place when a system vibrates under the action of forces inherent in the system itself after the starting point of vibration, and in the absence of external applied forces. When a system describable by a finite number of generalized coordinates is subjected to arbitrary initial conditions, the free vibration is periodic with several frequency components. However, among these there will be some simple harmonic motions called principal modes or natural modes of vibration. These are characterized by a certain distribution of amplitude over the body, in which each point in the body undergoes harmonic motion of common frequency (the natural frequency) with all points passing through their equilibrium configuration simultaneously.

Vibration that take place under the excitation of external forces is called forced vibration. When the exciting force is harmonic, the forced vibration take place at the frequency of the excitation (independent of the natural frequencies). When the frequency of the exciting forces is coincident with one of the natural frequencies of the system, a condition of resonance is encountered and dangerously large amplitudes may result. Consequently, the calculation of natural frequencies is of interest in all types of vibrating systems.

Vibration systems are all more or less subject to damping because energy is dissipated by friction and other resistances. Since no external energy is supplied in free vibrations, the motion in free vibration will diminish with time, and is said to be damped. On the other hand, forced vibration may be maintained at constant amplitude with the required energy supplied by an external force. For the convenience of study, we shall assume no damping effect present unless otherwise stated.

The behaviour of an oscillatory system may be examined in terms of the type of excitation to which the system is subjected. These forces of excitation may be divided into the headings of harmonic, periodic, non-periodic, and stochastic, where the applied forces are ststistical. Since a periodic excitation can be expressed as a Fourier series or a trigonometric polynomial in terms of frequency excitation, non-periodic and stochastic excitations can be expressed in terms of Fourier integrals, the time variables are transformed to harmonic frequency spectra, and the free vibration of a system can be considered as harmonic motion with its natural frequencies, Therefore, we will study the harmonic vibration of a mechanical system with great detail; the free vibrations and the other types of forced vibrations are treated as applications of the harmonic vibrations.

2.2 CONTRAINTS AND DEGREES OF FREEDOM (ref 2,4)

The essential characteristics of a system is that the objects are coupled, and hence restricted or constrained in their motion. If we consider a degenerated system of only point, such as a bead, this bead or point is unconstrained or free; whereas if the bead slides on a wire bent to some space curve, the system is the bead and the wire, and the bead is constrained, as shown in fig (2.2.1).

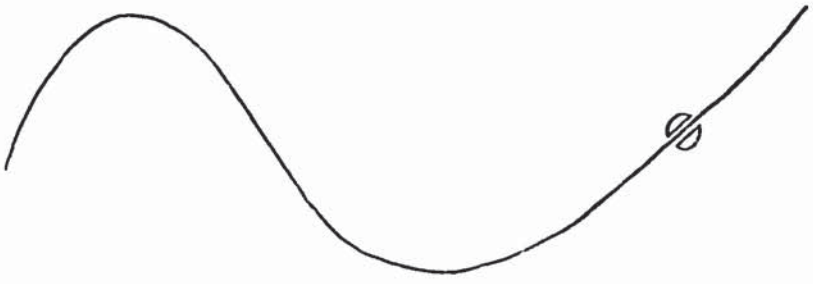


Fig (2.2.1) A system consists of a constrained bead on a wire

The wire is called the constraint and the resultant of the surface traction at the wire-bead interface yields the force of constraint, or reaction. In this particular example, only one generalized coordinate, such as distance along the wire, is needed to specify the configuration if the wire is stationary.

Consider a system whose state is specified by a number of N generalized coordinates. If each one of these generalized coordinates can vary separately without affecting the others and without violating the conditions of constraint, the system is called holonomic with N degrees of freedom. If these generalized coordinates are not capable of varying in this manner, this set of generalized coordinates in nonholonomic. If the constraints are moving (for example, a bead sliding on a moving wire), the system is called rheonomic; otherwise it is called scleronomic.

Contraints are further classified as bilateral if they satisfy an equality and unilateral if they satisfy an inequality. The conditions of constraints which the displacements must satisfy are conditions of compatibility and the conditions of constraints which the forces must satisfy are conditions of equilibrium (dynamic or static). The displacements satisfying all constraint conditions are compatible displacements; and the forces satisfying all constraint conditions are equilibrium forces. Constraints will be assumed scleronomic and generalized coordinates are holonomic in the remaining part of the thesis unless otherwise stated.

2.3 COORDINATES AND COORDINATE SYSTEMS (ref 4)

In order to fix the meanings of coordinates, the coordinate system identifying the coordinates must be specified. If a system consists of two mass points, m_1 and m_2, on the plane of the paper, as shown in fig (2.3.1), the generalised coordinates may be chosen as x_1, y_1, x_2,y_2 referred to the Cartesian coordinate system (OXY) as shown in fig (a); or, alternatively, x_1, x_2, x_3, x_4 referred to the coordinate system (OX_1X_2) as shown in fig(b); or, alternatively, x_1, x_2, x_3, x_4 referred to the coordinate system consisting of two Cartesian coordinate system ($O_1X_1Y_1$) and ($O_2X_2Y_2$) as shown in fig (c).

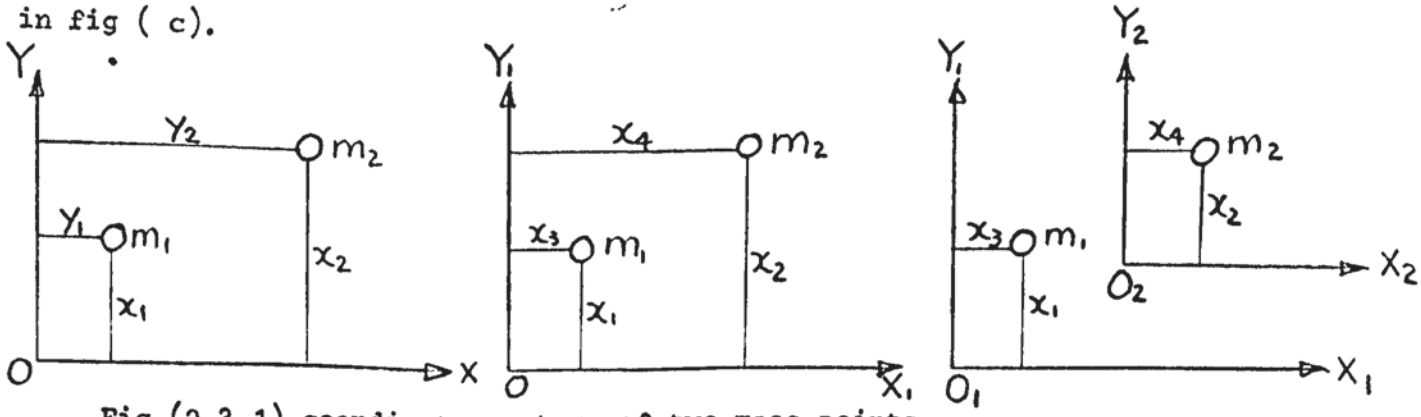


Fig (2.3.1) coordinate systems of two mass points

For the convenience of systematical study, the coordinate systems of figs (b) and (c) are often used because the notation can be unified to x_i, $i = 1,2,3,4$.

If two small disturbing force vectors in the plane are acting on the two mass points and causing them to displace, the projections of these force vectors onto the generalized coordinates are called the generalized coordinates of forces and the projections of these displacements onto the generalized coordinates are called the generalized coordinates of displacements. The projections of the position vectors of these two mass points onto the generalized coordinates are called the generalized coordinates of positions.

And so on. The generalized coordinates of forces are referred to briefly as generalized forces and the generalized coordinates of displacements generalized displacements. If no confus-ion exists, the word generalized coordinates will include generalized forces and generalized displacements.

There are two forms of coordinate systems: discrete coordinate systems and distributed coordinate systems. When the mathematical model of a vibrating system consists of discrete parameters only, discrete coordinate systems are convenient. Otherwise, distributed coordinate systems are employed to obtained a more realistic approximation. We describe these two types of coordinate systems separately.

DISCRETE COORDINATE SYSTEMS

Consider a mass spring system with four mass points which are undergoing deformations in the lateral direction at a, b, c, and d associated with four coordinates as shown in fig (2.3.2). The initial displacements are assumed zero. The vectors $\{u\}$ and $\{F\}$ describe horizontal displacement configurations and force groups associated with the four mass points. The components of $\{u\}$ and $\{F\}$ represent amplitudes of displacement configurations and force groups identified at discrete points. For instance, u_1 is a scalar which amplifies the shape identified by the unit vector $\{e_1\} = [1,0,0,0]^T$, and F_3 is a scalar which amplifies the force group identified by $\{e_3\} = [0,0,1,0]^T$. No information is given for the displacements or forces at points other than at the discrete points a,b,c and d (the dashed lines between mass points in fig (2.3.2) can have an arbitrary shape). Coordinates that identify information at discrete points will be referred to as discrete coordinates.

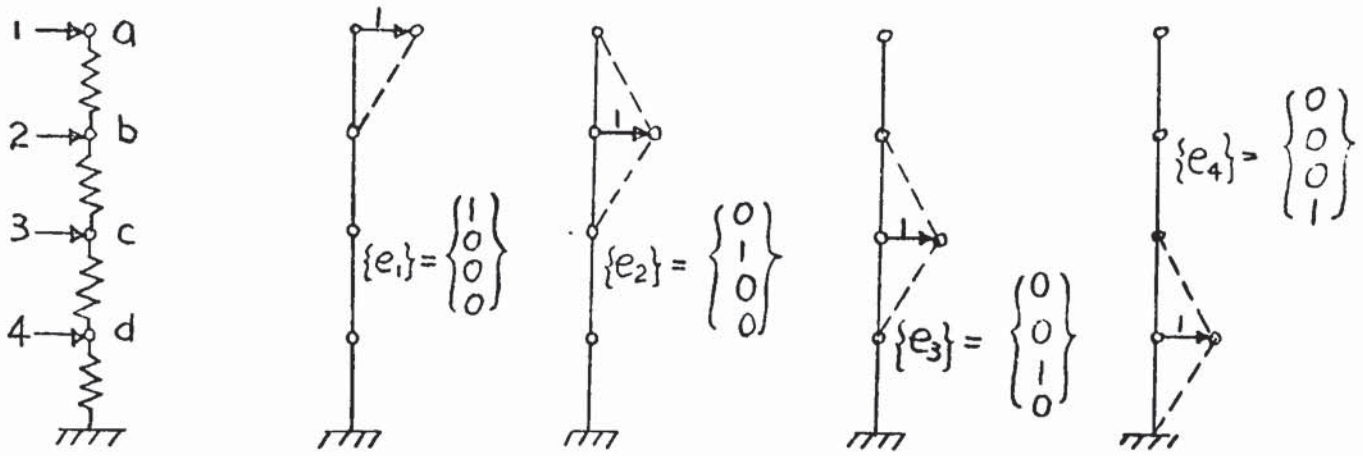


Fig (2.3.2) Discrete coordinate systems

The displacement $\{u\}$ at any mass point is given by

$$\{u\} = \sum_{i=1}^{4} u_i \{e_i\}$$

The discrete coordinate system in the example is the collection of base vectors $\{e_i\}$.

If the points a,b,c, and d are given more degrees of freedom such that displacements along the three dimensions (x_1, x_2, x_3) are possible, then there will be twelve generalized coordinates. Each three of these will describe the displacements of the same point, this point is imagined as the origin of these three generalized coordinates. It will be called the generalized origin of these three generalized coordinates for the convience of the coordinate transformation in chapter 4.

DISTRIBUTED COORDINATES

Consider a cantilever beam with four mass points at a,b,c, and d and four coordinates as shown in fig (2.3.3). Again, the initial displacements are assumed zero and only transverse displacements are considered. Now the displaced configuration associated with $u_1=1$ is a distributed function $\phi_1(x)$, such that

$$\phi_1(x_a) = 1,$$
$$\phi_1(x_b) = 0,$$
$$\phi_1(x_c) = 0,$$
$$\phi_1(x_d) = 0.$$

$\phi_i(x)$ gives information of displacements at every point on the structure. Hence, the displacement u(x) at any point x is

$$u(x)=\sum_{i=1}^{4} u_i \phi_i(x).$$

The displacements u_i represent amplitudes of corresponding shapes $\phi_i(x)$. The collection of $\phi_i(x)$ is a distributed coordinate system. Coordinates that identify distributed displacements (or forces etc.) will be referred to as distributed coordinates.

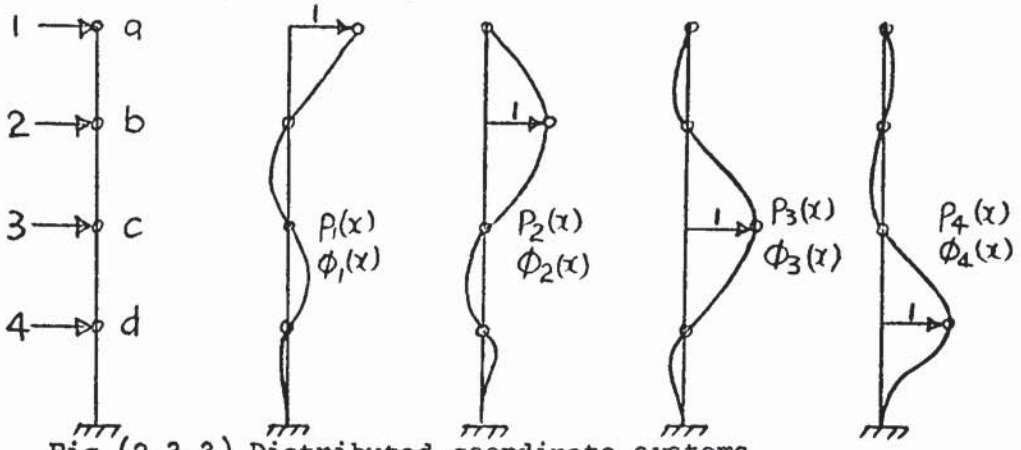


Fig (2.3.3) Distributed coordinate systems

In this example, although the system consists of a continuous member, the knowledge of the four displacement amplitudes u_i, i=1,2,3,4 will define the whole deformed configuration. Therefore the system is said to have four generalized coordinates associated with the distributed coordinates. These distributed coordinates are functions of coordinates and sometimes called coordinate functions or modes (ref 50).

Beside the polynomial coordinate functions, the most widely used ones are the trigonometric coordinate functions. For a three dimensional vibrating body, a possible way to express its displacements u_i, i=1,2,3 is by N terms of a sine series, depends on boundary conditions,

$$u_i=\sum_{r=1}^{N} q_{ir} \sin\frac{r\pi x_i}{\ell_i} \qquad i=1,2,3$$

where $\frac{\ell_i}{r\pi}$ is called the wavelength of the coordinate functions $\sin\frac{r\pi x_i}{l_i}$.

For every r, there are three components q_{ir} corresponding the displacements u_i, i=1,2,3. The point from where these three generalized coordinates are measured will be called the generalized origin of them when we carry out the coordinate transformation in chapter 4.

In the following, we shall discussion some more forms of distributed coordinate system, because our formulation of vibration analysis is based on this concept.

As an example of a two dimensional elastic member, we consider a membrane whose continuous boundary displacements at x=0,a and y=0,b are described by functions $\psi_1(y)$, $\psi_2(y)$, $\psi_3(x)$, $\psi_4(x)$ respectively as shown in fig(2.3.4). The deflected configuration of the membrane may be approximated by

$$w(x,y)=\left[(1-\frac{x}{a})\psi_1(y)+\frac{x}{a}\psi_2(y)\right]\left[(1-\frac{y}{b})\psi_3(x)+\frac{y}{b}\psi_4(x)\right].$$

When the functions ψ_i, i=1,2,3,4 are represented by their Fourier series with finite number of terms, N say, then,

$$\psi_1(y) = \tfrac{1}{2} b_o^{(1)} + \sum_{n=1}^{N} \left(a_n^{(1)} \sin\frac{n\pi y}{b} + b_n^{(1)} \cos\frac{n\pi y}{b}\right)$$

$$\psi_2(y) = \tfrac{1}{2} b_o^{(2)} + \sum_{n=1}^{N} \left(a_n^{(2)} \sin\frac{n\pi y}{b} + b_n^{(2)} \cos\frac{n\pi y}{b}\right)$$

$$\psi_3(x) = \tfrac{1}{2} b_o^{(3)} + \sum_{n=1}^{N} \left(a_n^{(3)} \sin\frac{n\pi x}{a} + b_n^{(3)} \cos\frac{n\pi x}{a}\right)$$

$$\psi_4(x) = \tfrac{1}{2} b_o^{(4)} + \sum_{n=1}^{N} \left(a_n^{(4)} \sin\frac{n\pi x}{a} + b_n^{(4)} \cos\frac{n\pi x}{a}\right)$$

and therefore, there are 8N+4 parameters ($b_o^{(i)}$, $a_n^{(i)}$, $b_n^{(i)}$) which must be specified with the associated coordinate functions ($\sin\frac{n\pi y}{b}$, $\cos\frac{n\pi y}{b}$, etc.) in order to described the deflected configuration of the membrane. Under these conditions, the system consisting of this membrane is said to have 8N+4 generalized coordinates.

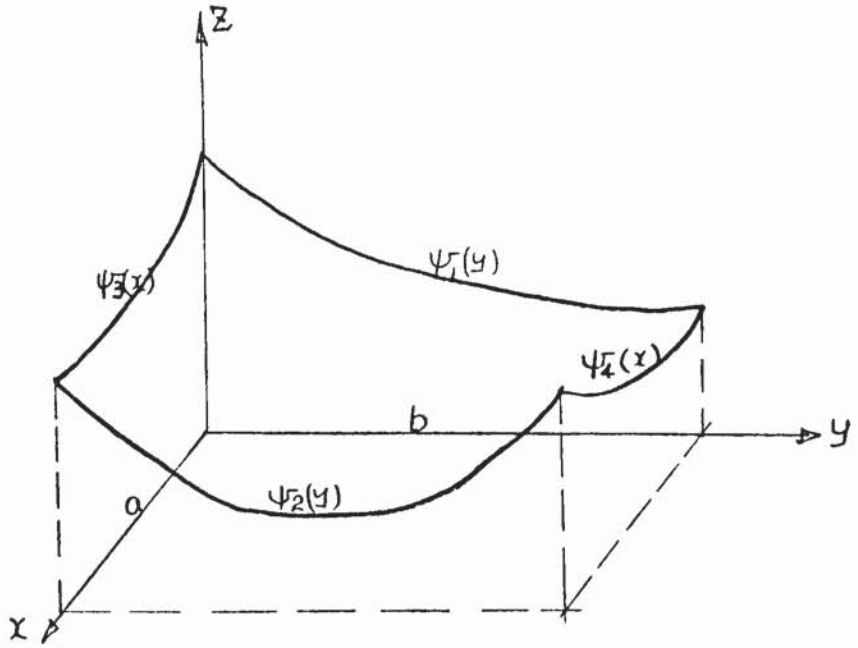


Fig (2.3.4) Continuous boundary displacements of a membrane

When a uniform beam is vibrating with angular frequency ω , the displacement of its transverse vibration can be expressed as

$$u(x,t)=\left(A\cos\frac{\lambda x}{l}+B\sin\frac{\lambda x}{l}+C\cosh\frac{\lambda x}{l}+D\sinh\frac{\lambda x}{l}\right)\sin\omega t$$

where $\lambda^4=\omega^2\rho A_0 l^4/EI$ denotes the frequency parameter

EI =flexural rigidity of the beam

l = the length of the beam

ρA_0 = mass density per unit length

A,B,C,D = integration constants independent of

x = the coordinate of the axis of the beam.

If we choose $\cos\frac{\lambda x}{l}=\phi_1(x)$, $\sin\frac{\lambda x}{l}=\phi_2(x)$, $\cosh\frac{x\lambda}{l}=\phi_3(x)$, and $\sinh\frac{\lambda x}{l}=\phi_4(x)$ as distributed coordinate functions, four parameters A,B,C, and D are needed to define the configuration of the beam and the beam is said to have four generalized coordinates associated with these coordinate functions.

So far, only the linear form of generalized coordinates are considered, e.g. a state vector $\{u\}$ of a system at any instant is expressed in the following linear form,

$$\{u\}=\sum_{i=1}^{N} q_i\{\phi_i(x_1,x_2,x_3)\}$$

where $\{\phi_i(x_1,x_2,x_3)\}$ are either coordinate base vectors or coordinate

functions. When these generalized coordinates are holomonic, we call them linear independent generalized coordinates.

It may be convenient to use nonlinear generalized coordinates in vibration analysis in some cases. For example, in the study of non-stationary vibrations, where the vibrating frequency ω is a function of time, we may express a state vector $\{u\}$ of the system at any instant by

$$\{u\} = a(t)\sin(\psi(t)+\omega t)\{v\}$$

where $\{v\}$ is the shape function of the system in stationary vibrations, $a(t)$ is the amplitude of $\{v\}$, and $\psi(t)$ is the difference of phase angle between the nonstationary vibration and the stationary vibration.

If we choose a and ψ as generalized coordinates, we have a non-linear generalized coordinate system. If a set of non-linear generalized coordinates $(q_1, q_2, \ldots, q_n)$ are chosen, the configuration of the system may be expressed as

$$\{u\} = \{u(x_1, x_2, x_3; q_1, q_2, q_3, \ldots, q_n; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n; t)\}$$

This is the most general form of expressing the state of a vibrating system.

Now, we have discussed all forms of coordinate systems which will be used in the remaining chapters. The next concept to deal with is the discretization of a continuous system. This device is designed to make the numerical calculation and analysis of complicated continuous systems in vibration realistic.

2.4 DISCRETIZATION (ref 6,8)

On a macroscopic scale, every engineering system is continuous. For many vibration problems, it is not possible to obtain analytical solutions which give the values of desired unknown quantities (displacement respon etc.) at any location in acontinuous system. In most of the numerical methods, the solutions yield approximate values of these unknown quantities only at a finite number of coordinates in the system. The process of selecting a finite number of coordinates to represent the configuration of the system can be termed discretization. This discretization process can be classified into discretization by discrete coordinates and discretization by distributed coordinates. Consider two of the ways that a system consisting of a plate may be discretized as shown in fig (2.4.1).

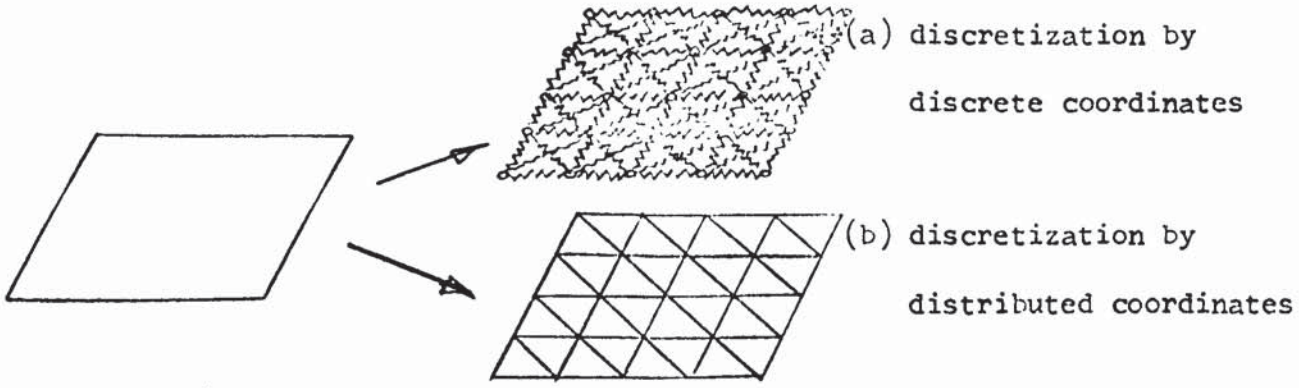


Fig (2.4.1) Two of the ways to discretize a system consisting of a plate

The first way is the discretization of the system by twenty-five discrete masses to which the discrete coordinates are attached, and seventy-two springs (storing potential energy and having negligible masses) connecting the masses as shown in fig (a). The displacement amplitudes at each coordinate are the generalized displacements, the number of which is twenty-five if transverse deformation is considered. There are certain criteria to choose the masses and springs, one of which is by the condition that the kinetic energy and the strain energy of the original

system are being stored by the masses and the springs of the discrete model during the process of deformation. No matter what criteria we have chosen, the displacements are not defined anywhere except at the mass points.

The second way discretized the plate by forty triangles as shown in (b) such that the whole region of the system are occupied by these triangles. These triangles are called the finite elements of the system. These elements are considered interconnected at joints which are called nodes. To express the displacements of each element in terms of the displacements at nodes, we first choose the generalized coordinates to attach to these nodes and determine the associated coordinate functions. If the plate is undergoing flexural deformation, there are three generalized coordinates (one linear and two angular) attaching to each node. If we approximate the displacement of an element by a polynomial in x and y , i.e.

$$w(x,y)=\sum_{i,j=1}^{3} a_{ij}\, x^{i-1}y^{j-1}=\sum_{i,j=1}^{3} a_{ij}\,\phi_{ij}(x,y)=[\chi]\{a\}$$

where $\phi_{ij}(x,y)=x^{i-1}y^{j-1}$, a_{ij} are parameters to be determined by the nodal displacements $\{q\}$, $[\chi]=[\phi_{11},\phi_{12},\cdots,\phi_{33}]$, and $\{a\}=[a_{11},a_{12},\cdots,a_{33}]^T$ If a_{ij} were chosen as generalised coordinates, ϕ_{ij} would have been the coordinate functions. If there exists a relationship $\{a\}=[N]\{q\}$, where $[N]$ is a transformation matrix independent of coordinate variables x,y,z, and $\{q\}$ is the vector of the nodal displacements of the element, then the displacement within a triangle element is given by

$$w(x,y)=[\chi]\{a\}=[\chi][N]\{q\}=[\psi(x,y)]\{q\}$$

where $[\psi(x,y)]=[\chi][N]$ is the matrix of coordinate functions required.

The nodal displacements $\{q\}$ are often taken as generalized coordinates because the condition of compatibility of displacements will be automatically satisfied if all the elements of $\{q\}$ are single-valued.

2.5 WORK AND ENERGY (ref 2,7,9)

Now we have a base, i.e. the set of generalized coordinates, to described the forms of energy. There are two main forms of energy involved in the vibration of a body, which are kinetic energy and strain energy. If the body is rigid, we consider it as a special case where the strain energy vanishes at all time; and when the body is massless, we consider it as a special case where the kinetic energy is zero at all time. These cases do not exist in nature, their introduction is just to simplfy the process of analysis.

Consider a continuous elastic body. We define

$$T = \frac{1}{2}\int_{vol} \{\dot{r}\}^T [\rho] \{\dot{r}\}\, dvol$$
$$= \frac{1}{2}\int_{vol} \{\dot{u}\}^T [\rho] \{\dot{u}\}\, dvol \qquad (2.5.1)$$

as the kinetic energy of a body, where $\{r\}$ is the position vector, which has three components for a three dimensional body, $\{u\}$ is the displacement vector of the body at any instant, $[\rho]$ is a scalar mass density in linear motion or an inertia tensor in angular motion, vol is the total volume of the body. The strain energy of an elastic body is defined as

$$U = \int_{vol} U_o(u_i)\, dvol \qquad (2.5.2)$$

where $U_o(u_i)$ is the strain energy density and is given by

$$dU_o = \sigma_{ij}\, d\varepsilon_{ij} \qquad (2.5.3)$$

where σ_{ij} and ε_{ij} are the components of the stress and strain tensors respectively.

In order to describe the relation between the kinetic energy and strain energy, we shall use the principle of virtual work in dynamics. The principle states (ref 7):

> Assume that the mechanical system is in dynamic equilibrium under applied forces and prescribed geometric constraints. Then the

sum of all the virtual work done over a prescribed time interval by the external and internal forces existing in the system in any arbitrary infinitesimal virtual displacements, which satisfy the prescribed geometrical constraints and which vanish at the ends of the time interval, is zero.

In mathematical expression, the virtual work principle for a dynamic problem states:

$$\int_{t_1}^{t_2} \left[\int_{vol} \{X\}^T \{\delta u\}\, dvol + \int_S \{\Phi\}^T \{\delta u\}\, dS + \delta T - \delta U \right] dt = 0 \qquad (2.5.4)$$

where $t_1 < t < t_2$ is the prescribed time interval,

$\{\delta u\}$ is the compatible virtual displacement vector which vanishes at t_1 and t_2,

$\{X\}$ is the body force vector,

$\{\Phi\}$ is the surface force vector.

If there exist two potentials of the forces, V_1 and V_2 such that

$$\delta V_1 = -\{X\}^T \{\delta u\} \qquad (2.5.5)$$

and

$$\delta V_2 = -\{\Phi\}^T \{\delta u\} \qquad (2.5.6)$$

the above principle reduces to

$$\delta \int_{t_1}^{t_2} \left[T - U - \int_{vol} V_1\, dvol - \int_S V_2\, dS \right] dt = 0 \qquad (2.5.7)$$

This expression represents the Hamilton's principle applied to the dynamic problem of a elastic body, which states,

Among all admissible displacements which satisfy the prescr ibed geometrical constraints and the prescribed conditions at the limits $t=t_1$ and $t=t_2$, the actual solution makes the functional

$$\int_{t_1}^{t_2} \left[T - U - \int_{vol} V_1\, dvol - \int_S V_2\, dS \right] dt \qquad (2.5.8)$$

stationary.

Now, we are going to express the various forms of energy in terms of the generalized coordinates of the vibrating system. We assume the generalized coordinates $(q_1, q_2, \ldots, q_N)$ are of holonomic and non-linear type,

then the state or configuration of the system at any instant is given by

$$u_i = u_i(x_1, x_2, x_3; q_1, q_2, \cdots, q_N, t) \tag{2.5.9}$$

where the time derivatives of q_j are omitted, since they are expressible by the generalized coordinates and the time variable. (Note that a holonomic system satisfies all geometric constraints.) From eq(2.5.9), we obtain

$$\dot{u}_i = \sum_{r=1}^{N} \frac{\partial u_i}{\partial q_r} \dot{q}_r + \frac{\partial u_i}{\partial t} \tag{2.5.10}$$

$$\delta u_i = \sum_{r=1}^{N} \frac{\partial u_i}{\partial q_r} \delta q_r \tag{2.5.11}$$

The Lagrangian is defined as

$$L = T - U \tag{2.5.12}$$

which is a function of generalized coordinates, since T and U are functions of u_i from equations (2.5.1) and (2.5.2). The last two terms of equation (2.5.4) of the principle of virtual work are transformed to

$$\begin{aligned}\int_{t_1}^{t_2} \delta L \, dt &= \int_{t_1}^{t_2} \left[\sum_{r=1}^{N} \left(\frac{\partial L}{\partial q_r} \delta q_r + \frac{\partial L}{\partial \dot{q}_r} \delta \dot{q}_r \right) \right] dt \\ &= \sum_{r=1}^{N} \frac{\partial L}{\partial \dot{q}_r} \delta q_r \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_{r=1}^{N} \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_r} - \frac{\partial L}{\partial q_r} \right) \delta q_r \, dt \\ &= - \int_{t_1}^{t_2} \sum_{r=1}^{N} \left[\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_r} - \frac{\partial L}{\partial q_r} \right] \delta q_r \, dt \end{aligned} \tag{2.5.13}$$

since for $t=t_1$ and $t=t_2$, $\delta q_r = 0$.

The remaining terms of equation (2.5.4) become

$$\int_{vol} \{X\}^T \{\delta u\} \, d\,vol + \int_S \{\Phi\}^T \{\delta u\} \, dS = \sum_{r=1}^{N} Q_r \delta q_r \tag{2.5.14}$$

where

$$Q_r = \int_{vol} \{X\}^T \left\{ \frac{\partial u}{\partial q_r} \right\} d\,vol + \int_S \{\Phi\}^T \left\{ \frac{\partial u}{\partial q_r} \right\} dS \,, \quad r = 1, 2, \cdots, N, \tag{2.5.15}$$

are the generalized force components. Therefore, by equations(2.5.13) and (2.5.14), the equation (2.5.4) of the principle of virtual work gives

$$\int_{t_1}^{t_2} \sum_{r=1}^{N} \left[\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_r} - \frac{\partial L}{\partial q_r} - Q_r \right] \delta q_r \, dt = 0 \tag{2.5.16}$$

Since it holds for any prescribed time interval, therefore the integrand must be zero, i.e.,

$$\sum_{r=1}^{N} \left[\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_r} - \frac{\partial L}{\partial q_r} - Q_r \right] \delta q_r = 0 \tag{2.5.17}$$

If the property of holonomy of q_r is used, i.e., δq_r are independent,

equation (2.5.17) leads to N simultaneous equations describing the motion:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_r} - \frac{\partial L}{\partial q_r} = Q_r \qquad r=1,2,\cdots,N \qquad (2.5.18)$$

These are Lagrange's equations of motion of an elastic body.

If the vibrating system consists of many elastic bodies, which are interconnected with each other, the set of the generalized coordinates (2.5.9) are not independent (i.e. non-holonomic), the system of equations (2.5.17) still holds true, but the system of equations (2.5.18) needs to be modified to take account of inter-body constraints (ref 2). Assume that the δq_r are connected by n<N differential constraint equations of the form

$$\phi_i(q_1, q_2, \cdots, q_N, t) = 0, \quad i=1,2,\cdots,n, \qquad (2.5.19)$$

which, for a virtual displacement, takes the form

$$\sum_{j=1}^{N} B_{ij}\,\delta q_j = 0, \qquad i=1,2,\cdots,n. \qquad (2.5.20)$$

If we denote

$$A_r = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_r} - \frac{\partial L}{\partial q_r} - Q_r, \qquad r=1,2,\cdots,N, \qquad (2.5.21)$$

then a system satisfying both systems (2.5.17) and (2.5.20) is found to be

$$\sum_{r=1}^{N}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_r} - \frac{\partial L}{\partial q_r} - Q_r\right)\delta q_r - \sum_{i=1}^{n}\lambda_i \sum_{r=1}^{N} B_{ir}\,\delta q_r = 0$$

or

$$\sum_{r=1}^{N}\left(A_r - \sum_{i=1}^{n}\lambda_i B_{ir}\right)\delta q_r = 0 \qquad (2.5.22)$$

where λ_i are arbitrary constants with respect to q_r.

Equation (2.5.22) can be written as

$$\sum_{r=1}^{n}\left[A_r - \sum_{i=1}^{n}(\lambda_i B_{ir})\right]\delta q_r + \sum_{r=n+1}^{N}\left(A_r - \sum_{i=1}^{n}\lambda_i B_{ir}\right)\delta q_r = 0 \qquad (2.5.23)$$

Since n<N, we can rearrange the generalized coordinates such that the variations of generalized coordinates in the second part of equation (2.5.23) are independent. Further, we can choose the parameters λ_i, i=1,2,...,n, such that

$$A_r - \sum_{i=1}^{n}\lambda_i B_{ir} = 0, \qquad r=1,2,\cdots,n. \qquad (2.5.24)$$

With condition(2.5.24), equation (2.5.23) becomes

$$\sum_{r=n+1}^{N}\left(A_r - \sum_{i=1}^{n}\lambda_i B_{ir}\right)\delta q_r = 0.$$

But δq_r, r=n+1,n+2,...,N, have been chosen as independent, therefore

$$A_r - \sum_{i=1}^{n}\lambda_i B_{ir} = 0, \qquad r=n+1,n+2,\cdots,N. \qquad (2.5.25)$$

Combining equations (2.5.24),(2.5.25) and (2.5.21), we have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_r} - \frac{\partial L}{\partial q_r} = Q_r + \sum_{i=1}^{n} \lambda_i B_{ir}, \qquad r=1,2,\cdots,N, \qquad (2.5.26)$$

which is Lagrange's equations of motion for a nonholonomic system. The $n+N$ parameters q_r, $r=1,2,\ldots,N$ and λ_i, $i=1,2,\ldots,n$, can be determined by N equations (2.5.26) and n equations of (2.5.29). The λ_i are called Lagrange's multipliers.

From equations(2.5.26), we can see that $\sum_{i=1}^{n} \lambda_i B_{ir}$ may be interpreted as the addition of generalized forces corresponding to any constraints which exist.

In the formulation in this section, no assumptions about the forms of elasticity and displacement patterns have been made. If we say the materials composing the elastic bodies of the systems are linear elastic, the generalized Hooke's law relating the stress components and the strain components holds, i.e.

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} \qquad i,j,k,l = 1,2,3 \qquad (2.5.27)$$

where C_{ijkl} are the elastic constants and are independent of the state of stress and strain, i.e. constant repective to σ_{ij} and ε_{kl}. If we say the deformation is small so that the higher terms and products of displacement gradients are negligible, therefore, the following strain-displacement relationship holds,

$$\varepsilon_{ij} = \tfrac{1}{2}\left(u_{i,j} + u_{j,i}\right) \qquad (2.5.28)$$

where $u_{i,j} = \frac{\partial u_i}{\partial x_j}$ etc..

From now on, the word "elastic" implies "linear elastic" and deformations are assumed small, unless otherwise stated. From equation (2.5.28), we can see that deformations are the gradients of displacements. Large displacements do not necessary imply large deformations, although the geometry of the whole system could change considerably as in the stability analysis.

Under the assumption of small deformations and linear elasticity, the energy can be expressed as

$$T = \frac{1}{2}\int_{vol} \{\dot{u}\}^T [\rho] \{\dot{u}\}\, dvol \qquad (2.5.29)$$

$$U = \frac{1}{2}\int_{vol} \{\varepsilon\}^T [C] \{\varepsilon\}\, dvol \qquad (2,5.30)$$

where $\{u\}$ and $\{\varepsilon\}$ are the displacement response vector and the strain vector repectively, $[\rho]$ is the mass density in matrix form, $[C]$ is the matrix of elastic moduli. The dimension of $\{u\}$, denoted by n_1, is six when the rotatory inertia are included. The elements of $[\rho]$ are mass density per volume if linear inertia are concerned, and are moments of inertia per unit volume if rotatory inertia are considered. The dimension of $\{\varepsilon\}$, denoted by n_2, is nine if the rotatory inertia are included. When the rotatory inertia are neglected, the strain tensor is symmetric and $\{\varepsilon\}$ may be reduced to a vector having six components. The dimension of $[C]$ is dependent on the state of strain vector, i.e. whether general three-dimension or plane stress or plane strain.

2.6 MASS AND STIFFNESS MATRICES (ref 8)

It is possible to express the displacements and strains in terms of generalized coordinates in linear form, i.e.

$$\{u\} = [a]\{q\} \qquad (2.6.1)$$

$$\{\varepsilon\} = [b]\{q\} \qquad (2.6.2)$$

where $[a]$ and $[b]$ are independent of $\{u\}$ and $\{q\}$, then, the two forms of energy in equations (2.5.29) and (2.5.30) become,

$$\begin{aligned} T &= \frac{1}{2}\int_{vol} \{\dot{q}\}^T [a]^T [\rho] [a] \{\dot{q}\}\, dvol \\ &= \frac{1}{2}\{\dot{q}\}\int_{vol} [a]^T [\rho] [a]\, dvol\, \{\dot{q}\} \\ &= \frac{1}{2}\{\dot{q}\}[M]\{\dot{q}\} \end{aligned} \qquad (2.6.3)$$

and $$U = \frac{1}{2}\int_{vol} \{q\}^T [b][C][b]\{q\}\, dvol$$
$$= \frac{1}{2}\{q\}^T \int_{vol} [b][C][b]\, dvol\, \{q\}$$
$$= \frac{1}{2}\{q\}^T [K]\{q\} \qquad (2.6.4)$$

where $$[M] = \int_{vol} [a]^T [\rho][a]\, dvol \qquad (2.6.5)$$
$$[K] = \int_{vol} [b]^T [C][b]\, dvol \qquad (2.6.6)$$

are defined as mass matrix and stiffness matrix of the system respectively. They are positive definite symmetrical matrices for the positive definite nature of the energies. Therefore the kinetic energy is expressed as a bilinear form of generalized velocities and strain energy of generalized coordinates.

With equations (2.6.3) and (2.6.4), the Lagrange's equations of motion become

$$[M]\{\ddot{q}\} + [K]\{q\} = \{Q\} \qquad (2.6.7)$$

where $\{Q\}$ is the vector of generalized forces given by equation(2.5.15). If $\{Q\}$ is harmonic in time and represented by $\{Q\}=\{\bar{Q}\}e^{i\omega t}$, then, after introducing $\{q\} = \{\bar{q}\}e^{i\omega t}$, equation (2.6.7) becomes,

$$-\omega^2[M]\{\bar{q}\} + [K]\{\bar{q}\} = \{\bar{Q}\} \qquad (2.6.8)$$

or $$[D]\{\bar{q}\} = \{\bar{Q}\} \qquad (2.6.9)$$

where $[D] = [K] - \omega^2[M]$ is called the dynamic stiffness matrix in harmonic vibrations. If we reduce the amplitude of $\{\bar{Q}\}$ until it reaches zero, equation (2.6.9) becomes

$$[D]\{\bar{q}\} = \{0\} \qquad (2.6.10)$$

which is the equation describing the harmonic free vibration of a system. The necessary and sufficient condition for equation (2.6.10) to have non-trivial solution of $\{\bar{q}\}$ is that

$$\det([D]) = 0 \qquad (2.6.11)$$

which is the frequency equation determining the natural frequencies of the free vibration and the nontrivial solutions of $\{\bar{q}\}$ thus obtained are the

modal shapes corresponding to each natural frequency.

2.7 ENERGY AND NORMAL COORDINATES (ref 10)

By the methods of piecewise representation, the vibrating behaviour of a continuous elastic system can be described approximately by a finite number of generalized coordinates. The energy forms are expressed in terms of these generalized coordinates. Since the choice of these generalized coordinates is not unique, transformations of coordinates are required when the same vibrating system is described by different coordinate systems. As mentioned in section (2.1), there is a special coordinate system which is unique for a specified system and is called the normal coordinate system. In this section we shall discuss the expression of energy in terms of the normal coordinates.

Consider the free vibration of an elastic system whose vibrating configuration is defined by the initial conditions only. Let us assume the initial conditions are such that this system will oscillate freely in two of its normal modes simultaneously while the other modes are not excited. Now, the two normal coordinates, p_1 and p_2 say, associated with these modes are varying harmonically at their cooresponding natural frequencies ω_1 and ω_2. The equations (2.6.3) and (2.6.4) will have the forms

$$\begin{aligned} 2T &= m_{11}\dot{p}_1^2 + m_{22}\dot{p}_2^2 + 2m_{12}\dot{p}_1\dot{p}_2 \\ 2U &= k_{11}p_1^2 + k_{22}p_2^2 + 2k_{12}p_1p_2 \end{aligned} \qquad (2.7.1)$$

where m_{ij}, k_{ij} are components of $[M]$ and $[K]$.

Assume that the free vibration modes are

$$p_i = A_i \sin \omega_i t, \qquad i = 1,2 \qquad (2.7.2)$$

Substitute equation (2.7.2) into equation (2.7.1), and sum together, we obtain the total energy of the system at any time,

$$2(T+U) = A_1^2[m_{11}\omega_1^2\cos^2\omega_1 t + k_{11}\sin^2\omega_1 t] + A_2^2[m_{22}\omega_2^2\cos^2\omega_2 t + k_{22}\sin^2\omega_2 t]$$
$$+ 2A_1A_2[m_{12}\omega_1\omega_2\cos\omega_1 t\cos\omega_2 t + k_{12}\sin\omega_1 t\sin\omega_2 t] \qquad (2.7.3)$$

The right- hand side of equation (2.7.3) must be independent of t because there is no force acting on the system and no means by which energy may be dissipated or created within it. For this to be so, it is necessary that $m_{11}\omega_1^2 = k_{11}$, $m_{22}\omega_2^2 = k_{22}$,

$$m_{12} = m_{21} = k_{12} = k_{21} = 0. \qquad (2.7.4)$$

In this case, equation (2.7.3) has the form

$$2(T+U) = m_{11}A_1^2\omega_1^2 + m_{22}A_2^2\omega_2^2$$
$$= k_{11}A_1^2 + k_{22}A_2^2 \qquad (2.7.5)$$

and equations (2.7.1) become

$$2T = m_{11}\dot{\rho}_1^2 + m_{22}\dot{\rho}_2^2$$
$$2U = k_{11}\rho_1^2 + k_{22}\rho_2^2 \qquad (2.7.6)$$

Similarly, when the number of normal modes excited is more than two, the fact that the coupling terms of principal coordinates in energy expression vanish can be proved. The con-ditions (2.7.4) are called the conditions of orthogonality of normal modes.

If the displacements and strains are expressed in terms of normal modes ϕ_i , $i=1,2,\ldots,n$, i.e.

$$\{u\} = \sum_{i=1}^{n} \{\phi_i\}\rho_i ;$$
$$\{\varepsilon\} = \sum_{i=1}^{n} \{\varepsilon_i\}\rho_i , \qquad (2.7.7)$$

then the kinetic energy and strain energy in equations (2.5.29) and (2.5.30) become

$$T = \frac{1}{2}\sum_{i,j=1}^{n} \dot{\rho}_i\dot{\rho}_j \int_{vol} \{\phi_i\}^T[\rho]\{\phi_j\}\, dvol$$
$$U = \frac{1}{2}\sum_{i,j=1}^{n} \rho_i\rho_j \int_{vol} \{\varepsilon_i\}^T[C]\{\varepsilon_j\}\, dvol$$

or, when the conditions of orthogonality are introduced,

$$T = \frac{1}{2}\sum_{i=1}^{n} \dot{\rho}_i^2 M_i$$
$$U = \frac{1}{2}\sum_{i=1}^{n} \omega_i^2\rho_i^2 M_i \qquad (2.7.8)$$

where the generalized mass corresponding to the ith normal mode M_i, is given by,

$$M_i = \int_{vol} \{\phi_i\}^T [\rho] \{\phi_i\} \, dvol \qquad (2.7.9)$$

The generalized force, P_i, associated with the ith mode is obtained from equation (2.5.15) as

$$P_i = \int_{vol} \{X\}^T \left\{ \frac{\partial u}{\partial p_i} \right\} dvol + \int_S \{\Phi\}^T \left\{ \frac{\partial u}{\partial p_i} \right\} dS$$

$i = 1, 2, \ldots, n.$

But $$\left\{ \frac{\partial u}{\partial p_r} \right\} = \sum_{i=1}^{n} \left(\frac{\partial}{\partial p_r} \{\phi_i\} p_i \right) = \{\phi_r\},$$

for p_r are independent coordinates, then we have,

$$P_r = \int_{vol} \{X\}^T \{\phi_r\} \, dvol + \int_S \{\Phi\}^T \{\phi_r\} \, dS \qquad (2.7.10)$$

$r = 1, 2, \ldots, n.$

By means of equations (2,7,8) and (2.7.10), the Lagrange's equations take the form,

$$\ddot{p}_i + \omega_i^2 p_i = P_i \qquad i = 1, 2, \ldots, n \qquad (2.7.11)$$

with initial conditions

$$\{u(0)\} = \sum_{i=1}^{n} \{\phi_i\} p_i(0)$$

$$\{\dot{u}(0)\} = \sum_{i=1}^{n} \{\phi_i\} \dot{p}_i(0)$$

or

$$p_i(0) = \frac{1}{M_i} \int_{vol} \{\phi_i\}^T [\rho] \{u(0)\} \, dvol$$

$$\dot{p}_i(0) = \frac{1}{M_i} \int_{vol} \{\phi_i\}^T [\rho] \{\dot{u}(0)\} \, dvol \qquad (2.7.12)$$

where $\{u\} = \{u(t)\}$ and $p_i = p_i(t)$.

One of the significant features is that the principal coordinates are uncoupled in these equations.

The solution of a vibrating system by means of its normal modes resulting in the set of differential equations (2.7.11) with the initial conditions (2.7.12) is called the modal analysis. This technique in vibration analysis will be discussed in some details in chapter seven.

2.8 RAYLEIGH'S THEOREM OF ONE CONSTRAINT (ref 3)

The Rayleigh's theorem of one constraint on a vibrating system is of central important in solving the frequency equation in chapter five. We shall study the theorem in this section.

Considering what effect the imposition of an additional constraint has on the natural modes of a dynamical system vibrating about a configuration of stable equilibrium, we suppose that the motion of the system is approximated by N terms of its normal coordinates $(p_1, p_2, \ldots, p_N)$, so that the kinetic and potential energies have the forms

$$T = \frac{1}{2} \sum_{i=1}^{N} \dot{p}_i^2 \tag{2.8.1}$$

$$U = \frac{1}{2} \sum_{i=1}^{N} \lambda_i^2 p_i^2 \tag{2.8.2}$$

where the generalized masses are assumed unity, i.e. , the normal modes are normalized, λ_i are parameters proportional to original natural frequencies. Let an additional constraint be expressed by

$$f(p_1, p_2, \cdots p_N) = 0 . \tag{2.8.3}$$

For small disturbances, we can expand the function f in ascending powers of p_i and retain only the first terms. We can thus express the constraint by

$$\sum_{i=1}^{N} A_i p_i = 0 . \tag{2.8.4}$$

where A_i are constants with respect to p_i. As the equilibrium configuration is supposed to be compatible with the constraint, there will be no constant terms. By means of this equation , we can eliminate, p_N say, in the equations (2.8.1) and (2.8.2), and we have

$$T = \frac{1}{2} \left[\sum_{i=1}^{N-1} \dot{p}_i^2 + \frac{1}{A_N^2} \left(\sum_{i=1}^{N-1} A_i \dot{p}_i \right)^2 \right] \tag{2.8.5}$$

$$U = \frac{1}{2} \left[\sum_{i=1}^{N-1} \lambda_i^2 p_i^2 + \frac{\lambda_N^2}{A_N^2} \left(\sum_{i=1}^{N-1} A_i p_i \right)^2 \right] \tag{2.8.6}$$

The Lagrange's equations of motion (2.5.26) of the modified system are therefore the N-1 equations

$$\ddot{p}_r + \lambda_r^2 p_r + \mu A_r = 0 , \qquad r = 1, 2, \ldots, N-1$$

where $$\mu = \frac{1}{A_N^2}\sum_{i=1}^{N-1} A_i \ddot{\rho}_i + \frac{\lambda_N^2}{A_N^2}\sum_{i=1}^{N-1} A_i \rho_i$$

$$= -\frac{\ddot{\rho}_N}{A_N} - \frac{\lambda_N^2 \rho_N}{A_N}$$

or $$\ddot{\rho}_N + \lambda_N^2 \rho_N + \mu A_N = 0 .$$

So that the equations of motion of the constrained system are the N equations

$$\ddot{\rho}_r + \lambda_r^2 \rho_r + \mu A_r = 0 , \qquad r = 1,2, \ldots , N \qquad (2.8.7)$$

where μ is undetermined.

Now consider a normal mode of vibration of the modified system, we can express $$\rho_i = \alpha_i \cos\lambda t , \qquad i = 1, 2, \cdots, N ,$$

$$\mu = \nu \cos\lambda t .$$

Substituting in the equations (2.8.7), we have

$$\alpha_r(\lambda_r^2 - \lambda^2) + \nu A_r = 0 , \qquad r = 1,2, \ldots ,N. \qquad (2.8.8)$$

With the contraint equation (2.8.4), we sum all equations of (2.8.8), and obtain

$$\sum_{i=1}^{N} \frac{A_i^2}{\lambda_i^2 - \lambda^2} = 0 . \qquad (2.8.9)$$

This equation in λ^2 has N-1 roots, which, from the form of the equation, are evidently interspaced between the quantities λ_1^2, λ_2^2, ... , λ_N^2 ; it follows that the N-1 natural frequencies of the modified system are interspaced between the n natural frequencies of the original system. This is the Rayleigh's theorem of natural frequencies with ane additional contraint.

2.9 DISCRETE VIBRATING SYSTEMS (ref 11)

In order to demonstrate the application of the materials in the previous sections, an example of mass and spring system is given in this section. Fig (2.9.1) shows the schematic of a conservative spring-mass system of n degrees of freedom with one end fixed and the other end free. Let the

magnitudes of all masses be m and let the springs have the same spring constant k. The generalized coordinates of the n masses are denoted by $q_1, q_2, \ldots, q_n$, each of which is measured relative to the static equilibrium positions of each mass.

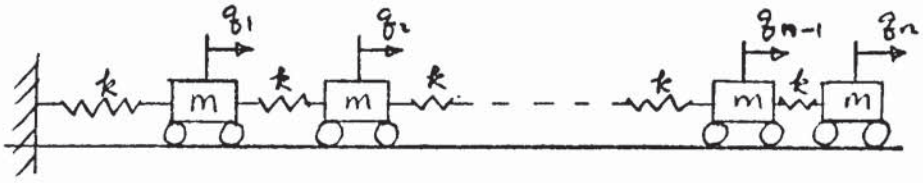


Fig (2.9.1) a conservative spring-mass system of n degrees of freedom

The kinetic and potential energies of the system are given by

$$T = \frac{1}{2} \sum_{i=1}^{n} q_i^2 \, m$$

$$U = \frac{1}{2}\left[\sum_{i=2}^{n} k(q_i - q_{i-1})^2 + k q_1^2\right]$$

For free vibrations, the generalized forces, Q_i , associated with q_i are zero and the Lqgrange's equations (2.5.18) become

$$[m]\{\ddot{q}\} + [K]\{q\} = \{0\} \qquad (2.9.1)$$

where the components of $[m]$ and $[K]$ are given by

$$m_{ij} = \begin{cases} 0 & i \neq j \\ \text{when} & \\ m & i = j \end{cases}$$

and

$$k_{ij} = \begin{cases} 2k & i = j \neq n \\ -k & \text{when } i \neq j \\ k & i = j = n \end{cases} \qquad (2,9.2)$$

repectively. The dynamic stiffness matrix (2.6.9) has the form

$$[D] = \begin{bmatrix} 2k-\omega^2 m & -k & & & \\ -k & 2k-\omega^2 m & -k & & 0 \\ & -k & \ddots & \ddots & \\ & & \ddots & \ddots & -k \\ 0 & & -k & 2k-\omega^2 m & -k \\ & & & -k & k-\omega^2 m \end{bmatrix} \qquad (2.9.3)$$

The vanishing of $\det[D]$ gives the natural frequencies of the system. These natural frequencies are given by

$$\omega_r = 2\sqrt{\frac{k}{m}} \sin\frac{(2r-1)\pi}{2(2n+1)} \qquad r = 1, 2, \ldots, n \qquad (2.9.4)$$

and the corresponding vibrational mode for the j^{th} mass with the r^{th} natural frequency is

$$\phi_{rj} = \sin \frac{r(2j-1)\pi}{(2n+1)} \qquad r,j=1,2, \ldots , n \quad (2.9.5)$$

When the generalized forces, $Q_i(t)$, are not zero, the solution of the vibration problem for $q_i(t)$ can be preceeded by a coordinate transformation transforming the $q_i(t)$ to the normal coordinates $p_i(t)$ according to equation (2.7.7). That is

$$q_i(t) = \sum_{j=1}^{n} \phi_{ij} p_j(t) \qquad i=1,2, \ldots ,n \quad (2.9.6)$$

where ϕ_{ij} is given by equation (2.9.5).

The generalized forces P_i are obtained by using equation (2.7.10), which is

$$\begin{aligned} P_r &= \int_{vol} \{x\}^T \{\phi_r\} \, dvol + \int_S \{\Phi\}^T \{\phi_r\} \, dS \\ &= \{Q\}^T \{\phi_r\} \\ &= \sum_{i=1}^{n} Q_i \phi_{ri} \qquad r=1,2, \ldots ,n \quad (2.9.7) \end{aligned}$$

The Lagrange's equations of motion in terms of normal coordinates are given by equation (2.7.11), i.e.

$$\ddot{p}_i + \omega_i^2 p_i = P_i \qquad i=1,2, \ldots ,n \quad (2.9.8)$$

and ω_i is given by (2.9.4). The set of equations (2.9.8) will be solved in conjunction with the initial conditions

$$\begin{aligned} p_i(0) &= \frac{1}{M_i} \int_{vol} \{\phi_i\}^T [\rho] \{q(0)\} \, dvol \\ &= \sum_{j=1}^{n} \phi_{ij} q_j(0) \\ \dot{p}_i(0) &= \sum_{j=1}^{n} \phi_{ij} \dot{q}_j(0) \end{aligned} \qquad (2.9.9)$$

since $M_i = \int_{vol} \rho \, dvol = m.$

For a system of discrete masses, the product termsof the generalized coordinates in the kinetic energy expression vanish and hence the mass matrix is always diagonal.

2.10 INTRODUCTION TO FINITE ELEMENT METHODS

Since our method of study is very close to the finite element methods, a brief introduction of these methods to elastic systems may be worth to mention.

To the best of our knowledge, the finite element method dates back to a paper by Courant on torsional problems in 1943 (ref 96). It was mathematically based. The term "finite element method" was not introduced until the middle of the fifties. At that time electronic computers were rapidly entering the field of technical computations, and matrix method of structural analysis was proved powerful. An extension of these methods to general structures was natural. P neers in this development were Langefors (ref 97), Argris (ref 98) and Clough (ref 99), and this time the approach was based on simple engineering arguments. The continuous material was regarded as being split physically into finite elements. Each element was analyzed as being aseparated piece of material, making up the complete structure when joined to the other elements. For a thorough study of the finite element methods, textbooks like the one by Zienkiewicz (ref 100) should be recommended. Here, only a brief account of the theory may be included.

Elastic problems are governed by three categories of field equations, viz.

stress equilibrium equations

stress- strain relations (constitutive material laws)

strain- displacement relations (kinematic relations).

In addition, boundary conditions may be given as

specified boundary stress

specified boundary displacements

specified relations between boundary stresses and boundary displacements.

For linear theory of elasticity, these equations are particularly simple. In terms of retangular Cartesian coordinates and by means of standard tensor notation, they may be written

1. stress equilibrium,

$$\sigma_{ij,j}+F_i = 0 , \qquad i,j=1,2,3$$

where σ_{ij} = stress tensor components

F_i = components of body forces

2. stress-strain relations,

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} \qquad i,j,k,l=1,2,3$$

or inversely, $\varepsilon_{ij} = S_{ijkl}\,\sigma_{kl}$

where the new notations are

ε_{ij} = components of strain tensor

C_{ijkl} = elastic stiffness coefficients

S_{ijkl} = elastic flexibility coefficients

3. strain-displacement relations for small displacements,

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j}+ u_{j,i}) \qquad i,j=1,2,3$$

where u_i denotes displacement in the direction i.

For the formulation of stress boundary conditions internal stresses must be related to surface tractions. The surface traction $\overset{\nu}{T}_i$ in direction i at some part of the boundary S may be written

$$\overset{\nu}{T}_i = \sigma_{ij}\nu_j$$

where ν_j is the direction cosine of the outward unit normal vector of the surface S .

All energy principles may be used as a basis for numerical analysis by the finite element method. The finite element discretization implies a division of the total volume V into subvolumes or subdomains denoted finite elements. The functions chosen to represent approximate displacement and stress field are specified within each element, and conditions imposed

on certain parameters at interelement boundaries provide the necessary continuity requirement of field functions.

In the case of the standard displacement method the displacement field is assumed to be $\{u(x,y,z)\} = [\phi(x,y,z)]\{\alpha\}$.

where $\{\phi(x,y,z)\}$ is the vector of chosen modes of displacement

$\{\alpha\}$ is a vector of constants to be determined by the nodal displacements. At any node i, the vector of displacement components is given by

$$\{q_i\} = \{u(x_i,y_i,z_i)\} = [\phi(x_i,y_i,z_i)]\{\alpha\}$$

where (x_i, y_i, z_i) are the coordinates of the node. If all displacement components of the nodes of the element are arranged in a vector $\{q\}$, then $\{q\} = [\Phi]\{\alpha\}$

where the constant matrix $[\Phi]$ is given by

$$[\Phi] = \begin{bmatrix} [\phi(x_1,y_1,z_1)] \\ [\phi(x_2,y_2,z_2)] \\ \vdots \\ [\phi(x_n,y_n,z_n)] \end{bmatrix}, \qquad n = \text{number of nodes},$$

and the displacement field is expressed in terms of nodal displacements,

$$\{u(x,y,z)\} = [\phi(x,y,z)][\Phi]^{-1}\{q\} = [a(x,y,z)]\{q\} \qquad (2.10.1)$$

where $[a(x,y,z)] = [\phi(x,y,z)][\Phi]^{-1}$.

The strain field is obtained from the kinematic relations as

$$\{\varepsilon(x,y,z)\} = [b(x,y,z)]\{q\} \qquad (2.10.2)$$

For vibration analysis, if the external force can be expressed by a potential V, the most convenient energy principle is the Hamilton's principle, which states, see section (2.5):

Among all admissible displacements which satisfy the prescribed geometrical constraints and the prescribed condition at the limits $t=t_1$ and $t=t_2$, the actual condition makes the functional

$$\int_{t_1}^{t_2} [T - U - \int_{vol} V \, dvol] \, dt \qquad (2.10.3)$$

stationary.

Now the kinetic energy and the strain energy are given by

$$T = \frac{1}{2}\int_{vol} \{\dot{u}\}^T [\rho]\{\dot{u}\}\, d\,vol \qquad (2.10.4)$$

and $$U = \frac{1}{2}\int_{vol} \{\varepsilon\}^T [C]\{\varepsilon\}\, d\,vol$$

respectively. From equations (2.10.1) to (2.10.4), we have

$$\delta\left(\frac{1}{2}\{\dot{q}\}^T[M]\{\dot{q}\} + \frac{1}{2}\{q\}^T[K]\{q\} - \{q\}^T\{Q\}\right) = 0 \qquad (2.10.5)$$

where the mass matrix $[M]$, and the stiffness matrix $[K]$ are given by

$$[M] = [a]^T[\rho][a] \qquad (2.10.6)$$

and $$[K] = [b]^T[C][b]$$

respectively, and $\{Q\}$ is the load vector resulting from the volume integral of the expression (2.10.3).

Since the kinetic energy of the system is the summation of the kinetic energies associated with the individual elements, therefore,

$$T = \frac{1}{2}\sum_{\substack{all\\elements}} \int_{\substack{vol\ of\\element}} \{\dot{u}_e\}^T[\rho_e]\{\dot{u}_e\}\, d\,vol \qquad (2.10.7a)$$

and so is the potential energy

$$U = \frac{1}{2}\sum_{\substack{all\\elements}} \int_{\substack{vol\ of\\element}} \{\varepsilon_e\}^T[C_e]\{\varepsilon_e\}\, d\,vol \qquad (2.10.7b)$$

where the subscript e denotes the quanties referred to the individual elements. Applying the requirement of stationary energies (2.10.3) and with reference to equations (2.10.1) and (2.10.2), we have

$$\delta\left(\sum_{\substack{all\\elements}} \frac{1}{2}\{\dot{q}_e\}^T[M_e]\{\dot{q}_e\} + \sum_{\substack{all\\elements}} \frac{1}{2}\{q_e\}^T[K_e]\{q_e\} - \sum_{\substack{all\\elements}} \{q_e\}^T\{Q_e\}\right) = 0 \qquad (2.10.8)$$

Now, if all the coordinate vectors $\{q_e\}$ are transformed to a common cooordinate vector base $\{q\}$ by

$$\{q_e\} = [n_e]\{q\}$$

then we have

$$\delta\left[\frac{1}{2}\{\dot{q}\}^T\left(\sum_{\substack{all\\elements}} [\bar{M}_e]\right)\{\dot{q}\} + \frac{1}{2}\{q\}^T\left(\sum_{\substack{all\\elements}} [\bar{K}_e]\right)\{q\} - \{q\}^T\sum_{\substack{all\\elements}} \{\bar{Q}_e\}\right] = 0 \qquad (2.10.9)$$

where

$$[\bar{K}_e] = [n_e]^T[K_e][n_e]$$
$$[\bar{M}_e] = [n_e]^T[M_e][n_e]$$
$$[\bar{Q}_e] = [n_e]^T[Q_e]$$

Comparing the equations (2.10.6) and (2.10.9), we have, for the system,

$$[M] = \sum_{\substack{\text{all} \\ \text{elements}}} [\bar{m}_e]$$

$$[K] = \sum_{\substack{\text{all} \\ \text{elements}}} [\bar{k}_e]$$

$$[Q] = \sum_{\substack{\text{all} \\ \text{elements}}} [\bar{Q}_e] \qquad (2.10.10)$$

Equations (2.10.10) are used to assemble the system equations of motion.

If we perform the variation of equation (2.10.5), we have

$$[M]\{\ddot{q}\} + [K]\{q\} = \{Q\} \qquad (2.10.11)$$

which is the governing equation of motion in matrix form.

CHAPTER THREE

FORMULATION OF ELEMENT MATRICES IN HARMONIC VIBRATIONS

3.1 INTRODUCTION

As we have mentioned in chapter two, the solution of the free vibration problem, the forced vibration problem and the stochastic vibration problem can be decomposed into the solution of harmonic vibration problems. The investigation of this harmonic vibration problem is therefore very important and forms the main body of this report.

To study the harmonic oscillatory behaviour of a complicated structure, which consists of different types of structural members of different size and geometry, by the method of dynamic stiffness, a common coordinate system for all the structural members is first chosen. This coordinate system is called the global coordinate system. Then the structure is divided into a set of continuous structural members such that the harmonic oscillatory behaviour, e.g. the stiffness properties and the inertia properties etc., of each member can be formulated in a systematic and convenient way suitable for handling by automatic calculating machines. Their behaviour is usually described by the general three dimensional theory of elasticity. In order to simplify the formulation, we may take the advantages of their geometry and approximate their description by one or two dimensional theory of elasticity.

To a first approximation a beam may be represented by its centroidal axis and analyzed as if it were a line element. Because this axis is described be a curve of one spatial parameter, it can be regarded as a one-dimensional member even though the axis itself may be curved as in the case of an arch or helix.

On the other hand, a thin flate plate or a thin curved shell is represented, for the purpose of analysis, by its middle surface. This surface is completely defined by an equation or a set of parametric equations having two independent spatial coordinate parameters. They are accordingly classified as two dimensional.

If a beam is so thick that it cannot be described sufficiently by its axis or a plate is so thick that two spatial parameters are not enough for the purpose of analysis, then they will be classified as three dimensional elements. Although many structural members are three dimensional in reality, few are analyzed as such because of the complexity of formulation.

It is the aim of the chapter to obtain the inertia and stiffness matrices for certain types of structural members. These structural members include one-dimensional structural members such as straight uniform beams, tapered beams, circular beam segments; two- dimensional ones such as square plates retangular plates, cylindrical shells; and three-dimensional members such as cuboids. For more complicated structural members, the present method of dynamic stiffness is inconvenient, although possible, because the mathematical functions involved will be too complicated for practical applications. Fortunately, for the requirements of economics and ease of manufacture, many engineering structures are composed of simple uniform members as mentioned above. When the structural members have only small deviations from their uniform equivalences, Chi, Dame, and Basdekas (ref 61) recommended a variational method based on the natural

frequencies and normal modes of the unmodified structures to study the dynamic response of the modified structures.

The coordinate system chosen for each individual member is called a local coordinate system. The harmonic vibration analysis of structural elements in this chapter will be based on the local coordinate systems of individual members.

3.2 THE FUNDAMENTAL MATRICES

When an elastic body, which has been separated from a structure, is undergoing harmonic vibration with frequency ω, the kinetic energy T and the strain energy U are given by equations (2.5.29) and (2.5.30) respectively,

$$T = \frac{1}{2} \int_{vol} \{\dot{u}\}^T [\rho] \{\dot{u}\} \, dvol \qquad (3.2.1)$$

$$U = \frac{1}{2} \int_{vol} \{\varepsilon\}^T [C] \{\varepsilon\} \, dvol \qquad (3.2.2)$$

where $\{u\} = [u_x, u_y, u_z]^T$ is the displacement vector, $[\rho]$ the inertia tensor, $[C]$ the matrix of elastic moduli and $\{\varepsilon\} = [\varepsilon_x, \varepsilon_y, \varepsilon_z, \varepsilon_{yz}, \varepsilon_{zx}, \varepsilon_{xy}]^T$ is a vector consisting of strain components. When a set of N generalized coordinates q_i, $i=1,2, \ldots, n$ is chosen such that

$$\{u(x,y,z,t)\} = [a(x,y,z,t)] \{q(t)\} \qquad (3.2.3)$$

$$\{\varepsilon(x,y,z,t)\} = [b(x,y,z,t)] \{q(t)\} \qquad (3.2.4)$$

with $[a(x,y,z,t)]$ and $[b(x,y,z,t)]$ being the relation matrices independent of generalized coordinates, then the energy expressions (3.2.1) and (3.2.2) become

$$T = \frac{1}{2} \{\dot{q}\}^T \left(\int_{vol} [a]^T [\rho] [a] \, dvol \right) \{\dot{q}\}$$

$$= \frac{1}{2} \{\dot{q}\}^T [m] \{\dot{q}\} \qquad (3.2.5)$$

and $$U = \frac{1}{2}\{q\}^T\left(\int_{vol}[b]^T[C][b]\,dvol\right)\{q\}$$

$$= \frac{1}{2}\{q\}^T[K]\{q\} \qquad (3.2.6)$$

where $$[m] = \int_{vol}[a]^T[\rho][a]\,dvol, \qquad (3.2.7)$$

$$[K] = \int_{vol}[b]^T[C][b]\,dvol. \qquad (3.2.8)$$

If the body force, excluding the inertia force, is $\{X(x,y,z,t)\}$ and the surface traction is $\{\Phi(x,y,z,t)\}$ then the associated generalized forces Q_i, $i=1,2,\ldots,N$ are given by equation (2.5.15),

$$Q_i(t) = \int_{vol}\{X\}^T\frac{\partial\{u\}}{\partial q_i}\,dvol + \int_{vol}\{\Phi\}^T\frac{\partial\{u\}}{\partial q_i}\,dvol \qquad (3.2.9)$$

Using the Lagrange's equations of motion, equation (2.6.7), we then have

$$[m]\{\ddot{q}\} + [K]\{q\} = \{Q\} \qquad (3.2.10)$$

For harmonic excitation, $\{Q\}$ and $\{q\}$ are represented by

$$\{Q\} = \{\bar{Q}\}e^{i\omega t} \qquad (3.2.11)$$

$$\{q\} = \{\bar{q}\}e^{i\omega t} \qquad (3.2.12)$$

and equation (3.2.10) becomes

$$[D]\{\bar{q}\} = \{\bar{Q}\} \qquad (3.2.13)$$

where $$[D] = [K] - \omega^2[m] \qquad (3.2.14)$$

Note that in forming equation (3.2.13), the strain and displacement variables have been transformed from the time domain to the frequency domain, e.g.

$$\{u(t)\} \rightarrow \{u(\omega)\} = \{ue^{i\omega t}\},$$

$$\{\varepsilon(t)\} \rightarrow \{\varepsilon(\omega)\} = \{\varepsilon e^{i\omega t}\},$$

etc., and therefore equations (3.2.3) and (3.2.4) are rewritten as, when the time variable is replaced by the frequency variable,

$$\{u(x,y,z,\omega)\} = [a(x,y,z,\omega)]\{q(\omega)\} \qquad (3.2.15)$$

$$\{\varepsilon(x,y,z,\omega)\} = [b(x,y,z,\omega)]\{q(\omega)\} \qquad (3.2.16)$$

The matrices $[m]$, $[K]$ and $[D]$ are the fundamental element matrices and are called mass matrix, stiffness matrix and dynamic stiffness matrix respectively. The mass matrix corresponds to the kinetic energy;

the stiffness matrix corresponds to the strain energy; and the dynamic stiffness matrix is relating the amplitude of the displacement vector to that of the force vector. It is our purpose to construct these fundamental matrices for some structural members of practical interest.

The most common method to contruct these matrices is the method of energy (ref 8) which may be outlined as follows:

1. find out the relationship between the displacement vector and the spatial coordinates, i.e. $\{u\} = \{u(x,y,z,\omega)\}$ (3.2.17)

2. choose a set of generalized displacements $\{q\}$ and establish the relation between $\{u\}$ and $\{q\}$, i.e. $\{u\} = [a]\{q\}$ (3.2.18)

3. from the strain-displacement relationships, i.e. $\varepsilon_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i})$ (3.2.19) establish the relation between $\{\varepsilon\}$ and $\{q\}$,

i.e. $\{\varepsilon\} = [b]\{q\}$ (3.2.20)

4. Integrate the matrix products to obtain the mass and stiffness matrices,

i.e. $[m] = \int_{vol} [a]^T[\rho][a]\, dvol$ (3.2.21)

$[K] = \int_{vol} [b]^T[C][b]\, dvol$ (3.2.22)

and obtain $[D] = [K] - \omega^2[m]$. (3.2.23)

Let us consider some elements which are assumed discrete to show the application of the above formulae to construct the fundamental matrices.

For a mass point as shown in fig (3.2.1.a), where q is the generalized coordinate chosen along the direction of oscillation, the kinetic and strain energies are, respectively,

$$T = \tfrac{1}{2} m q^2 \quad \text{and} \quad U = 0 .$$

Therefore $[m] = m$, $[K] = 0$, $[D] = -m\omega^2$.

Similarly, we obtain these fundamental matrices for the other elements shown in fig (3.2.1) and tabulate the results in table (3.2.1).

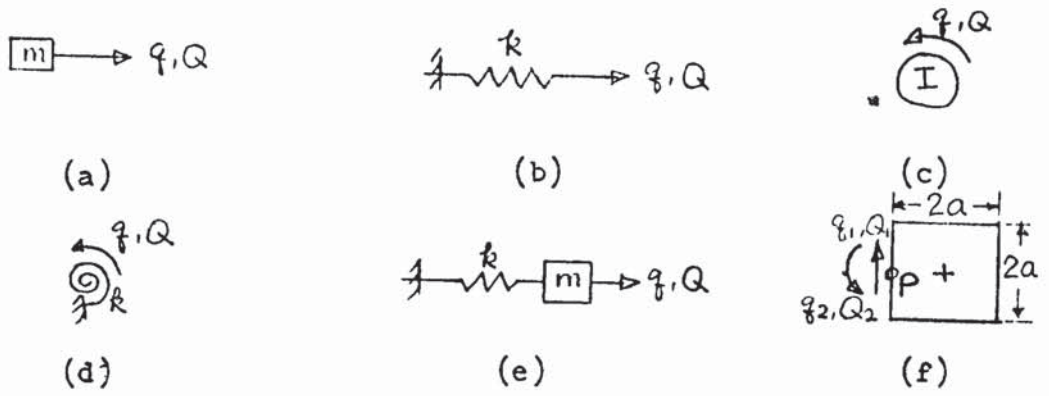


Fig (3.2.1) Some discrete elements

fig (3.2.1)	description	mass matrix	stiffness matrix	dynamic stiffness matrix
(a)	Single mass m	m	0	$-m\omega^2$
(b)	Single spring k	0	k	k
(c)	Rotary inertia I	I	0	$-I\omega^2$
(d)	Rotary spring k	0	k	k
(e)	Mass and spring	m	k	$k-m\omega^2$
(f)	Rigid block with mass m and rotary inertia I	$\begin{bmatrix} m & ma_2 \\ ma & ma^2+I \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$-\omega^2\begin{bmatrix} m & ma \\ ma & ma^2+I \end{bmatrix}$

Table (3.2.1) Fundamental matrices for discrete elements

So far, we have not said any thing about the nature of the fundamental matrices. From equations(3.2.7), (3.2.8) and (3.2.14), these matrices are symmetrical. And also from the positive definite properties of the kinetic and strain energies, the matrices $[\mathcal{M}]$ and $[\mathcal{K}]$ are positive definite. These three matrices depend on the properties of the material of the element and also its geometry etc.. We shall emphasise the dependency of these matrices on frequency of vibration, as described in (ref 8), because we want to study the harmonic vibrations of a system. This frequency

dependendency is not obvious for discrete systems, since the relation matrices of equations (3.2.15) and (3.2.16) are independent of frequency. For continuous or deformable bodies, as we shall see in the next section, this frequency dependency of fundamental matrices becomes explicit.

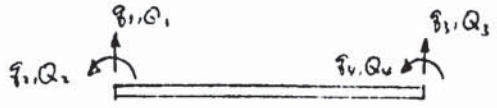


Fig (3.2.2)

3.2.1 BEAM ELEMENTS

Consider the transverse vibration of a uniform beam element as shown in fig (3.2.2), where q_i and Q_i, i=1,2,3,4 , are the end displacements and end forces respectively. The quantities such as u, q, ξ, Q ,etc. are referred to their amplitudes of vibration. A simple displacement pattern of the beam may be assumed to be

$$u(x)= \alpha_0+ \alpha_1 x+ \alpha_2 x^2+ \alpha_3 x^3 \qquad (3.2.24)$$

where α_i, $i = 0,1,2,3$ are constants to be determined. The displacement vector in this case consists of one component, i.e. the transverse displacement, therefore we may drop the brackets for vector notation. Were the functions x^i, i= 0,1,2,3 chosen as coordinate functions, α_i would have been taken as the generalized coordinates. Now since α_i's do not have direct physical interpretation, we prefer to transform them to the q_i's, so that the conditions of compatibility between elements can be applied directly to form the overall system equations. The transformation may proceed as follows.

In order to determine the coefficients α_i, $i=0,1,2,3$ in equation (3.2.24), we use the following boundary conditions

$$u(0)=q_1 \ , \ u'(0)=q_2 \ , \ u(l)=q_3, \ u'(l)=q_4 \ . \qquad (3.2.25)$$

Substituting equation (3.2.24) into equations (3.2.25), we obtain a set of four equations for α_i, $i=0,1,2,3$. After solving these equations for α_i in terms of q_i , i=1,2,3,4,the equation (3.2.24) can be rewritten in the

form $u(x)=[a(x)]\{q\}$ (3.2.26)

where $\{q\} = [q_1 \quad q_2 \quad q_3 \quad q_4]^T$

and $[a(x)] = [1-3(\frac{x}{l})^2+2(\frac{x}{l})^3 \quad (\frac{x}{l}-2(\frac{x}{l})^2+(\frac{x}{l})^3)l$

$3(\frac{x}{l})-2(\frac{x}{l})^3 \quad (-(\frac{x}{l})^2+(\frac{x}{l})^3)l]$ (3.2.27)

The strain and displacement relationship for a beam is

$$\varepsilon_x(x) = \frac{\partial v}{\partial x} = -y\frac{\partial^2 u}{\partial x^2} \quad (3.2.28)$$

where v is the longitudinal displacement and y is the coordinate normal to x and in the plane of vibration. From equations (3.2.28) and (3.2.26) we get the strain and generalized displacement relationship,

$$\varepsilon_x(x) = [b(x)]\{q\} \quad (3.2.29)$$

where $[b(x)] = -\frac{y}{l^2}[-6+12(\frac{x}{l}) \quad (-4+6(\frac{x}{l}))l \quad 6-12(\frac{x}{l}) \quad (-2+6(\frac{x}{l}))l]$ (3.2.30)

The substitution of equations (3.2.30) and (3.2.27) into (3.2.21) and (3.2.22) gives the mass and stiffness matrices

$$[m] = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \quad (3.2.31)$$

and

$$[k] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \quad (3.2.32)$$

respectively.

Instead of assuming the displacement pattern of the form (3.2.24), we can choose the displacement pattern as

$$u(x) = A_1\cos\frac{\lambda x}{l} + A_2\sin\frac{\lambda x}{l} + A_3\cosh\frac{\lambda x}{l} + A_4\sinh\frac{\lambda x}{l} \quad (3.2.33)$$

which satisfy the equation of small oscillation of a beam, i.e.

$$EI\frac{d^4u}{dx^4} - \rho A\omega^2 u = 0 \quad (3.2.34)$$

where $\lambda^4 = \omega^2\rho A l^4/EI$. The boundary conditions represented by eq. (3.2.25) result in four linear equations for the constants A_i, $i=1,2,3,4$.

After solving these equations for A_i, in terms of q_i, i=1,2,3,4, the equation (3.2.33) is rewritten as

$$u(x)=[a(x)]\{q_1\}$$

where

$$[a(x)]=\left[\cos\frac{\lambda x}{l}\ \ \sin\frac{\lambda x}{l}\ \ \cosh\frac{\lambda x}{l}\ \ \sinh\frac{\lambda x}{l}\right]\begin{pmatrix} \frac{1}{2}-\frac{F_4}{2\lambda^2} & \frac{F_2 l}{2\lambda^2} & -\frac{F_3}{2\lambda^2} & \frac{F_1 l}{2\lambda^2} \\ -\frac{F_6}{2\lambda^3} & \frac{l}{2\lambda}+\frac{F_4 l}{2\lambda^3} & -\frac{F_5}{2\lambda^3} & -\frac{F_3 l}{2\lambda^3} \\ \frac{1}{2}+\frac{F_4}{2\lambda^2} & -\frac{F_2 l}{2\lambda^2} & \frac{F_3}{2\lambda^2} & -\frac{F_1 l}{2\lambda^2} \\ \frac{F_6}{2\lambda^3} & \frac{l}{2\lambda}-\frac{F_4 l}{2\lambda^3} & \frac{F_5}{2\lambda^3} & \frac{F_3 l}{2\lambda^3} \end{pmatrix} \qquad (3.2.35)$$

where the frequency functions F_i are given by

$$\begin{aligned} F_1 &= -\lambda(\sinh\lambda-\sin\lambda)/\delta \\ F_2 &= -\lambda(\cosh\lambda\sin\lambda-\sinh\lambda\cos\lambda)/\delta \\ F_3 &= -\lambda^2(\cosh\lambda-\cos\lambda)/\delta \\ F_4 &= \lambda^2(\sinh\lambda\sin\lambda)/\delta \\ F_5 &= \lambda^3(\sinh\lambda+\sin\lambda)/\delta \\ F_6 &= -\lambda^3(\cosh\lambda\sin\lambda+\sinh\lambda\cos\lambda)/\delta \\ \delta &= \cosh\lambda\cos\lambda-1. \end{aligned} \qquad (3.2.36)$$

The mass matrix obtained by means of equations (3.2.21) and (3.2.35) is

$$[m]=\int_{vol}[a]^T[\rho][a]\,d\,vol$$

$$=\rho A l\int_0^1[a]^T[a]\,d\xi$$

$$=\rho A l\begin{pmatrix} 1/2-F_4/2\lambda^2 & -F_6/2\lambda^3 & 1/2+F_4/2\lambda^2 & F_6/2\lambda^3 \\ F_2 l/2\lambda^2 & l/2\lambda+F_4 l/2\lambda^3 & -F_2 l/2\lambda^2 & l/2\lambda-F_4 l/2\lambda^3 \\ -F_3/2\lambda^2 & -F_5/2\lambda^3 & F_3/2\lambda^2 & F_5/2\lambda^3 \\ F_1 l/2\lambda^2 & -F_3 l/2\lambda^3 & -F_1 l/2\lambda^2 & F_3 l/2\lambda^3 \end{pmatrix} \times$$

$$\int_0^1\begin{pmatrix} \cos\lambda\xi\cos\lambda\xi & \cos\lambda\xi\sin\lambda\xi & \cos\lambda\xi\cosh\lambda\xi & \cos\lambda\xi\sinh\lambda\xi \\ \sin\lambda\xi\cos\lambda\xi & \sin\lambda\xi\sin\lambda\xi & \sin\lambda\xi\cosh\lambda\xi & \sin\lambda\xi\sinh\lambda\xi \\ \cosh\lambda\xi\cos\lambda\xi & \cosh\lambda\xi\sin\lambda\xi & \cosh\lambda\xi\cosh\lambda\xi & \cosh\lambda\xi\sinh\lambda\xi \\ \sinh\lambda\xi\cos\lambda\xi & \sinh\lambda\xi\sin\lambda\xi & \sinh\lambda\xi\cosh\lambda\xi & \sinh\lambda\xi\sinh\lambda\xi \end{pmatrix} d\xi \times$$

$$\begin{pmatrix} 1/2-F_4/2\lambda^2 & F_2 l/2\lambda^2 & -F_3/2\lambda^2 & F_1 l/2\lambda^2 \\ -F_6/2\lambda^3 & l/2\lambda+F_4 l/2\lambda^3 & -F_5/2\lambda^3 & -F_3 l/2\lambda^3 \\ 1/2+F_4/2\lambda^2 & -F_2 l/2\lambda^2 & F_3/2\lambda^2 & -F_1 l/2\lambda^2 \\ F_6/2\lambda^3 & l/2\lambda-F_4 l/2\lambda^3 & F_5/2\lambda^3 & F_3 l/2\lambda^3 \end{pmatrix}$$

Or, after simplifying,

$$= \rho A \ell \begin{bmatrix} G_6 & -G_4\ell & G_5 & G_3\ell \\ -G_4\ell & G_2\ell^2 & -G_3\ell & G_1\ell^2 \\ G_5 & -G_3\ell & G_6 & G_4\ell \\ G_3\ell & G_1\ell^2 & G_4\ell & G_2\ell^2 \end{bmatrix} \qquad (3.2.37)$$

where $\xi = x/\ell$, and the frequency functions G_i are given by

$$\begin{aligned} G_1 &= (F_1F_2 - F_3 - F_1)/4\lambda^4 \\ G_2 &= (F_1^2 - F_2)/4\lambda^4 \\ G_3 &= -(F_1F_4 + 2F_3)/4\lambda^4 \\ G_4 &= -(F_1F_3 + 2F_4)/4\lambda^4 \\ G_5 &= (F_3F_4 - 3F_5)/4\lambda^4 \\ G_6 &= (F_3^2 - 3F_6)/4\lambda^4 \ . \end{aligned} \qquad (3.2.38)$$

When carrying out the integrations in equation (3.2.37), we have used the integration tables of section (6.16). Similarly, for the stiffness matrix we differentiate equation (3.2.35) with respect to x twice according to eq. (3.2.28). We have

$$\varepsilon(x) = [b(x)]\{q\} \qquad (3.2.39)$$

where

$$[b(x)] = \frac{y\lambda^2}{\ell^2}\left[\cos\lambda\xi \ \sin\lambda\xi \ -\cosh\lambda\xi \ -\sinh\lambda\xi\right] \begin{pmatrix} \frac{1}{2} - \frac{F_4}{2\lambda^2} & \frac{F_2\ell}{2\lambda^2} & -\frac{F_5}{2\lambda^2} & \frac{F_1\ell}{2\lambda^2} \\ -\frac{F_6}{2\lambda^3} & \frac{1}{2\lambda} + \frac{F_4\ell}{2\lambda^3} & -\frac{F_5}{2\lambda^3} & -\frac{F_3\ell}{2\lambda^3} \\ \frac{1}{2} + \frac{F_4}{2\lambda^2} & -\frac{F_2\ell}{2\lambda^2} & \frac{F_3}{2\lambda^2} & -\frac{F_1\ell}{2\lambda^2} \\ \frac{F_6}{2\lambda^3} & \frac{1}{2\lambda} - \frac{F_4\ell}{2\lambda^3} & -\frac{F_5}{2\lambda^3} & \frac{F_3\ell}{2\lambda^3} \end{pmatrix}$$

The stiffness matrix is calculated by equation (3.2.22) as

$$[K] = \int_{vol} [b]^T [C] [b] \, dvol$$

$$= \frac{EI}{\ell^3}\lambda^4 \begin{pmatrix} \frac{1}{2} - \frac{F_4}{2\lambda^2} & -\frac{F_6}{2\lambda^3} & \frac{1}{2} + \frac{F_4}{2\lambda^2} & \frac{F_6}{2\lambda^3} \\ \frac{F_2\ell}{2\lambda^2} & \frac{1}{2\lambda} + \frac{F_4\ell}{2\lambda^3} & -\frac{F_2\ell}{2\lambda^2} & \frac{1}{2\lambda} - \frac{F_4\ell}{2\lambda^3} \\ -\frac{F_3}{2\lambda^2} & -\frac{F_5}{2\lambda^3} & \frac{F_3}{2\lambda^2} & \frac{F_5}{2\lambda^3} \\ \frac{F_1\ell}{2\lambda^2} & -\frac{F_3\ell}{2\lambda^3} & -\frac{F_1\ell}{2\lambda^2} & \frac{F_3\ell}{2\lambda^3} \end{pmatrix} \times$$

$$\int_0^1 \begin{bmatrix} \cos\lambda\xi\cos\lambda\xi & \cos\lambda\xi\sin\lambda\xi & -\cos\lambda\xi\cosh\lambda\xi & -\cos\lambda\xi\sinh\lambda\xi \\ \sin\lambda\xi\cos\lambda\xi & \sin\lambda\xi\sin\lambda\xi & -\sin\lambda\xi\cosh\lambda\xi & -\sin\lambda\xi\sinh\lambda\xi \\ -\cosh\lambda\xi\cos\lambda\xi & -\cosh\lambda\xi\sin\lambda\xi & \cosh\lambda\xi\cosh\lambda\xi & \cosh\lambda\xi\sinh\lambda\xi \\ -\sinh\lambda\xi\cos\lambda\xi & -\sinh\lambda\xi\sin\lambda\xi & \sinh\lambda\xi\cosh\lambda\xi & \sinh\lambda\xi\sinh\lambda\xi \end{bmatrix} d\xi \times$$

$$\times \begin{bmatrix} 1/2 - F_4/2\lambda^2 & F_2\ell/2\lambda^2 & -F_3/2\lambda^2 & F_1\ell/2\lambda^2 \\ -F_6/2\lambda^3 & \ell/2\lambda + F_4\ell/2\lambda^3 & -F_5/2\lambda^3 & -F_2\ell/2\lambda^3 \\ 1/2 + F_4/2\lambda^2 & -F_2\ell/2\lambda^2 & F_3/2\lambda^2 & -F_1\ell/2\lambda^2 \\ F_6/2\lambda^3 & \ell/2\lambda - F_4\ell/2\lambda^3 & F_5/2\lambda^3 & F_3\ell/2\lambda^3 \end{bmatrix}$$

$$= \omega^2\rho A\ell \begin{bmatrix} G_6 & -G_4\ell & G_5 & G_3\ell \\ -G_4\ell & G_2\ell^2 & -G_3\ell & G_1\ell^2 \\ G_5 & -G_3\ell & G_6 & G_4\ell \\ G_3\ell & G_1\ell^2 & G_4\ell & G_2\ell^2 \end{bmatrix} + \frac{EI}{\ell^3} \begin{bmatrix} F_6 & -F_4\ell & F_5 & F_3\ell \\ -F_4\ell & F_2\ell^2 & -F_3\ell & F_1\ell^2 \\ F_5 & -F_3\ell & F_6 & F_4\ell \\ F_3\ell & F_1\ell^2 & F_4\ell & F_2\ell^2 \end{bmatrix} \qquad (3.2.40)$$

In order to compare the results of equations (3.2.31), (3.2.32) and those of equations (3.2.37) and (3.2.40), we expand the frequency functions in Taylor series with respect to λ in ascending powers about $\lambda = 0$.

$$F_1 = 2 + .007142857\lambda^4 + .000015704\lambda^8 + .000000032\lambda^{12} + \ldots$$

$$F_2 = 4 - .009523810\lambda^4 - .000016262\lambda^8 - .000000032\lambda^{12} - \ldots$$

$$F_3 = 6 + .030952381\lambda^4 + .000072193\lambda^8 + .000000148\lambda^{12} + \ldots$$

$$F_4 = -6 + .052380952\lambda^4 + .000076617\lambda^8 + .000000149\lambda^{12} + \ldots$$

$$F_5 = -12 - .128571429\lambda^4 - .000329571\lambda^8 - .000000684\lambda^{12} - \ldots$$

$$F_6 = 12 - .371428571\lambda^4 - .000364873\lambda^8 - .000000693\lambda^{12} - \ldots \qquad (3.2.41)$$

$$G_1 = -.007142857 - .000031408\lambda^4 - .000000095\lambda^8 - \ldots$$

$$G_2 = .009523810 + .000032525\lambda^4 + .000000096\lambda^8 + \ldots$$

$$G_3 = -.030952381 - .000144386\lambda^4 - .000000443\lambda^8 - \ldots$$

$$G_4 = -.052380952 - .000659142\lambda^4 - .000000447\lambda^8 - \ldots$$

$$G_5 = .128571429 + .000659142\lambda^4 + .000002053\lambda^8 + \ldots$$

$$G_6 = .371428571 + .000729746\lambda^4 + .000002080\lambda^8 + \ldots \qquad (3.2.42)$$

The corresponding functions from equations (3.2.31) and (3.2.32) are

$F_1=2$, $F_2=4$, $F_3=6$, $F_4=-6$, $F_5=-12$, $F_6=12$;

$G_1=-3/420 = -.007142857$, $G_2= 4/420= .009523810$, $G_3=-13/420=-.030952381$,

$G_4=-22/420=-.052380952$, $G_5=54/420= .128571429$, $G_6= .371428571$.

Therefore, the coefficients in equations (3.2.31) and (3.2.32) are those of equations (3.2.37) and (3.2.40) when λ or ω =0. Actually, the shape functions in equation (3.2.27) are the solution of the differential equation (3.2.34) with the boundary conditions (3.2.25) when $\omega = 0$. Therefore, the fundamental matrices of equations (3.2.31) and (3.2.32) are valid only when the frequency of vibration is small, i.e. about $\omega = 0$.

The coefficients in equations (3.2.41) and (3.2.42) can also be obtained as recommended by (ref 8) by assuming

$$u(x) = \sum_{r=0}^{\infty} \omega^r [a_r(x)] \{q\} \qquad (3.2.43)$$

and substituting into the differential equation (3.2.34). Then,

$$\left(EI \sum_{r=0}^{\infty} \omega^r \frac{d^4}{dx^4} [a_r(x)] - \omega^2 \rho A \sum_{r=0}^{\infty} \omega^r [a_r(x)] \right) \{q\} = \{0\}.$$

For nontrivial solutions of $\{q\}$,

$$EI \sum_{r=0}^{\infty} \omega^r \frac{d^4}{dx^4} [a_r(x)] = \omega^2 \rho A \sum_{r=0}^{\infty} \omega^r [a_r(x)]$$

Comparing the coefficients of the same powers of ω^r, we obtain

$$\frac{d^4}{dx^4}[a_0(x)] = [0]$$

$$\frac{d^4}{dx^4}[a_1(x)] = [0]$$

$$EI \frac{d^4}{dx^4}[a_2(x)] = \rho A [a_0(x)]$$

$$EI \frac{d^4}{dx^4}[a_3(x)] = \rho A [a_1(x)] \qquad \text{etc,.}$$

By solving these equations , we have

$$[a_0(x)] = [(1-3\xi^2+2\xi^3) \quad (\xi-2\xi^2+\xi^3)l \quad (3\xi^2-2\xi^3) \quad (-\xi^2+\xi^3)l]$$

$$[a_1(x)] = [0]$$

$$[a_2(x)] = \frac{\rho A l^4}{2520 EI} [(66\xi^2-156\xi^3+105\xi^4-21\xi^6+6\xi^7)$$
$$(12\xi^2-22\xi^3+21\xi^5-14\xi^6+3\xi^7)l \quad (39\xi^2-54\xi^3+21\xi^6-6\xi^7)$$
$$(-9\xi^2+13\xi^3-7\xi^6+3\xi^7)l]$$

$$[a_3(x)] = [0] \quad \text{etc..}$$

The substitution of these equations in to equation (3.2.21) we obtain the mass matrix in the form

$$[m] = \rho A \ell \begin{bmatrix} .37142857 & & \text{symm.} & \\ .052380952\ell & .009523810\ell^2 & & \\ .12857429 & .030952381\ell & .37142857 & \\ -.030952381\ell & -.007142857\ell^2 & -.052380952\ell & .009523810\ell^2 \end{bmatrix}$$

$$+\rho A \ell \lambda^4 \begin{bmatrix} .729746 & & \text{symm.} & \\ .153233\ell & .0325248\ell^2 & & \\ .659142 & .144386\ell & .729746 & \\ -.144386\ell & -.0314082\ell^2 & -.153233\ell & .0325248\ell^2 \end{bmatrix} \times 10^{-3}$$

$$+ \cdots \qquad (3.2.44)$$

When comparing the coefficients of (3.2.44) and (3.2.42), it shows that the coefficients of (3.2.44) are those of the first two terms of equations (3.2.42). The fundamental matrices resulting from this method do not converge to the true ones when λ is large, and therefore its application is very limited.

3.3 DYNAMIC STIFFNESS & FORCE-DISPLACEMENT RELATIONS

In general dynamic analysis, there are two approaches in obtaining the equations of motion of a mechanical system. One is the energy method which is based on the stationary properties of total energy and the other is the method of Newton which establishes the force and displacement relationship without involving energies. Likewise, these two approaches exist in vibration analysis. The general approach of the energy method in vibration analysis and its application to simple elastic members was discussed in section (3.2). Now we are going to present an approach for obtaining the dynamic stiffness matrix from the force-displacement relationship within the context of the Newtonian method of force.

The internal displacement vector relates to the generalized coordinates by $\{u(x,y,z)\} = [a(x,y,z)]\{q\}$ (3.3.1)

Assuming that the boundary points of the member have the coordinates (x_i, y_i, z_i), $i=1,2, \ldots, n$, where n is the number of the boundary points, then the boundary forces Q_i, at these points may be obtained by the conditions of equilibrium of the member, which is, in symbolic form,

$$\{Q_i\} = [\partial]\{u_i(x_i, y_i, z_i)\} \tag{3.3.2}$$

where $[\partial]$ is an operation matrix denoting the process of differentiating the displacement for strains, transfoming the strains on the boundaries by Hooke's law to boundary forces. From equations (3.3.2) and (3.3.1), we have

$$\{Q\} = [\partial][a(x_i, y_i, z_i)]\{q\} \tag{3.3.3}$$

When comparing the definition of dynamic stiffness matrix , i.e.

$$\{Q\} = [\vartheta]\{q\}$$

the dynamic stiffness matrix of equation (3.3.3) is

$$[\vartheta] = [\partial][a(x_i, y_i, z_i)] \tag{3.3.4}$$

We shall explain the practical meaning of $[\partial]$ by two examples of the following sections.

3.3.1 STRAIGHT BEAM MEMBERS

We derive the dynamic stiffness matrix for a straight beam member by means of force-displacement relationship in this section. Upon choosing the end displacements of the beam as generalized displacements q_i, $i=1,2,3,4$, the displacement pattern satisfying the governing equation of a beam is given by equation (3.2.35) , which is

$$u(x) = [a(x)]\{q\} \tag{3.3.8}$$

where

$$[a(x)] = \left[\cos\frac{\lambda x}{l} \ \sin\frac{\lambda x}{l} \ \cosh\frac{\lambda x}{l} \ \sinh\frac{\lambda x}{l}\right] \begin{pmatrix} \frac{1}{2} - \frac{F_4}{2\lambda^2} & \frac{F_2 l}{2\lambda^2} & -\frac{F_3}{2\lambda^2} & \frac{F_1 l}{2\lambda^2} \\ -\frac{F_6}{2\lambda^3} & \frac{l}{2\lambda} + \frac{F_4 l}{2\lambda^3} & -\frac{F_5}{2\lambda^3} & -\frac{F_3 l}{2\lambda^3} \\ \frac{1}{2} + \frac{F_4}{2\lambda^2} & -\frac{F_2 l}{2\lambda^2} & \frac{F_3}{2\lambda^2} & -\frac{F_1 l}{2\lambda^2} \\ \frac{F_6}{2\lambda^3} & \frac{l}{2\lambda} - \frac{F_4 l}{2\lambda^3} & \frac{F_5}{2\lambda^3} & -\frac{F_3 l}{2\lambda^3} \end{pmatrix} \tag{3.3.9}$$

and the frequency functions F_i, i=1,2,3,4,5,6 are found from eq (3.2.36).

The generalized forces Q_i, i=1,2,3,4 are obtained from elementary beam theory,

Q_1= shear force at (x=0) = EI $d^3u(0)/dx^3$,

Q_2= Bending moment at (x=0) = -EI $d^2u(0)/dx^2$,

Q_3= shear force at (x=ℓ) = -EI $d^3u(\ell)/dx^3$,

Q_4= Bending moment at (x=ℓ) = EI $d^2u(\ell)/dx^2$. (3.3.10)

Carry out the differentiation in equations (3.3.10(, we can relate $\{q\}$ and $\{Q\}$ through the following expression,

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \frac{EI}{\ell^3} \begin{bmatrix} F_1 & -F_4\ell & F_5 & F_3\ell \\ -F_4\ell & F_2\ell^2 & -F_3\ell & F_1\ell^2 \\ F_5 & -F_3\ell & F_6 & F_4\ell \\ F_3\ell & F_1\ell^2 & F_4\ell & F_2\ell^2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} \qquad (3.3.11)$$

or $$\{Q\} = [\mathcal{D}]\{q\} \qquad (3.3.12)$$

where

$$[\mathcal{D}] = \frac{EI}{\ell^3} \begin{bmatrix} F_1 & -F_4\ell & F_5 & F_3\ell \\ -F_4\ell & F_2\ell^2 & -F_3\ell & F_1\ell^2 \\ F_5 & -F_3\ell & F_6 & F_4\ell \\ F_3\ell & F_1\ell^2 & F_4\ell & F_2\ell^2 \end{bmatrix} \qquad (3.3.13)$$

is the dynamic stiffness matrix required.

3.3.2 FOLDED PLATE MEMBERS

A rectangular plate with two opposite edges simply supported and with the other two edges connected to other structures by prescribed displacement patterns will be discussed in this section. Distributed coordinates on the edges will be used in this example. The materials presented in this

section are independent of the rest of the thesis.

To satisfy the boundary conditions of two opposite edges being simply supported, the displacement pattern of the plate may be written as,

$$W(x,y) = \sum_{m=1}^{N} Y_m(y) \sin \frac{m\pi x}{a} \qquad (3.3.14)$$

where N is the number of terms taken, a x b is the dimension of the plate as shown in fig (3.3.1), and $Y_m(y)$ are the functions to be determined to satisfy the governing equation of vibration.

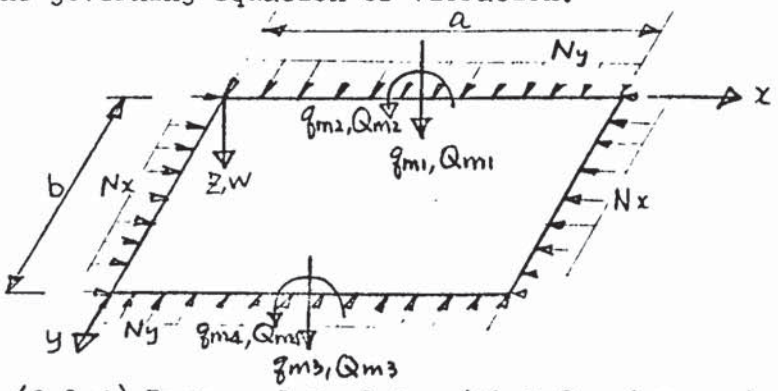


Fig (3.3.1) Rectangular plate with x=0 and x=a simply supported

The generalized displacements q_{mi}, m=1,2, ... , N and i=1,2,3,4 are defined by

$$\begin{aligned}
W(x,o) &= \sum_{m=1}^{N} q_{m1} \sin \frac{m\pi x}{a} \\
W(x,b) &= \sum_{m=1}^{N} q_{m2} \sin \frac{m\pi x}{a} \\
\frac{\partial W(x,0)}{\partial y} &= \sum_{m=1}^{N} q_{m3} \sin \frac{m\pi x}{a} \\
\frac{\partial W(x,b)}{\partial y} &= \sum_{m=1}^{N} q_{m4} \sin \frac{m\pi x}{a}
\end{aligned} \qquad (3.3.15)$$

and the generalized forces Q_{mi} are defined by

$$\begin{aligned}
Q_y(x,0) &= \sum_{m=1}^{N} Q_{m1} \sin \frac{m\pi x}{a}, \quad M_y(x,0) = \sum_{m=1}^{N} Q_{m2} \sin \frac{m\pi x}{a}, \\
-Q_y(x,b) &= \sum_{m=1}^{N} Q_{m3} \sin \frac{m\pi x}{a}, \quad -M_y(x,b) = \sum_{m=1}^{N} Q_{m4} \sin \frac{m\pi x}{a},
\end{aligned} \qquad (3.3.16)$$

where Q_y and M_y are the Kirchoff's shear and the bending monent of the plate along y=constant (ref 7). The generalized forces are related to displacement through the conditions of equilibrium on edges, y=0 and y=b,

$$Q_y(x,y) = -D\left[\frac{\partial^3 W}{\partial y^3} + (2-\nu)\frac{\partial^3 W}{\partial x^2 \partial y}\right]$$
$$M_y(x,y) = -D\left[\frac{\partial^2 W}{\partial y^2} + \nu\frac{\partial^2 W}{\partial x^2}\right] \qquad (3.3.17)$$

where $D = Eh^3/12(1-\nu^2)$ is the flexural rigidity of the plate

h = thickness and ν = the Poisson's ratio.

Before we can apply equations (3.3.15) to (3.3.17) to find the dynamic stiffness matrix, we must find out the functions $Y_m(y)$ in equation (3.3.14). If the loadings are harmonic with time, the governing equation of vibration of the plate with frequency ω is given by

$$D\nabla^4 W - \rho h\omega^2 W + N_x\frac{\partial^2 W}{\partial x^2} + N_y\frac{\partial^2 W}{\partial y^2} = p(x,y) \qquad (3.3.18)$$

where N_x and N_y are the compressive inplane load in x and y directions respectively, ∇^4 is the biharmonic operator in (x,y) coordinate, p(x,y) is the downward distributed load intensity and represented by

$$p(x,y) = \sum_{m=1}^{N} p_m \sin\frac{m\pi x}{a} \qquad \text{per unit area.} \qquad (3.3.19)$$

Substituting equation (3.3.14) into (3.3.18), we have

$$\sum_{m=1}^{N} \sin\frac{m\pi x}{a}\left\{\left(\frac{m\pi}{a}\right)^4 Y_m - 2\left(\frac{m\pi}{a}\right)^2 Y_m'' + Y_m^{IV} - \frac{\rho h\omega^2}{D}Y_m - \frac{N_x}{D}\left(\frac{m\pi}{a}\right)^2 Y_m + \frac{N_y}{D}Y_m'' - \frac{p_m}{D}Y_m\right\}$$
$$= 0 .$$

Multiply by $\sin\frac{n\pi x}{a}$, and integrate over x=0,a, where n is a positive integer, and use the orthogonality of sine functions,

$$Y_m^{IV} - 2\left\{\left(\frac{m\pi}{a}\right)^2 - \frac{N_y}{2D}\right\}Y_m'' + \left\{\left(\frac{m\pi}{a}\right)^4 - \frac{\rho h\omega^2}{D} - \frac{N_x}{D}\left(\frac{m\pi}{a}\right)^2 - \frac{p_m}{D}\right]Y_m = 0,$$
$$m=1,2, \ldots \qquad (3.3.20)$$

The associated boundary conditions for these fourth order differential equations are obtained from equation (3.3.15),

$$Y_m(0) = q_{m1} \ , \ Y_m(b) = q_{m3} \ , \ Y_m'(0) = q_{m2} \ , \ Y_m'(b) = q_{m4} \qquad (3.3.21)$$

The auxiliary roots of equations (3.3.20) are obtained by letting $Y_m = e^{\sigma y}$

$$\sigma^2 = \left[\left(\frac{m\pi}{a}\right)^2 - \frac{N_y}{2D}\right] \pm \left\{\left[\left(\frac{m\pi}{a}\right)^2 - \left(\frac{N_y}{D}\right)\right]^2 - \left(\frac{m\pi}{a}\right)^4 + \frac{\rho h\omega^2}{D} + \frac{N_x}{D}\left(\frac{n\pi}{a}\right)^2 + \frac{P_m}{D}\right\}^{\frac{1}{2}} \qquad (3.3.22)$$

Therefore, $Y_m(y)$ will have four different forms of solutions depending on whether σ^2 is positive, negative or complex. We study these four cases as follows, since, to the knowledge of the author, nowhere in literature has so far considered the complete solutions for all four cases.

Case (1) When all four roots are real, which are $\pm\sigma_1$, $\pm\sigma_2$, then the general solution has the form

$$Y_m(y) = A\cosh\frac{\sigma_2 y}{b} + B\sinh\frac{\sigma_2 y}{b} + C\cosh\frac{\sigma_1 y}{b} + D\sinh\frac{\sigma_1 y}{b} \qquad (3.3.23)$$

where A,B,C,D are integration constants and are determined from the boundary conditions (3.3.21) as

$$A = \left(\frac{\sigma_1^2 - F_4}{\sigma_1^2 - \sigma_2^2}\right)q_{m1} + \left(\frac{-F_2}{\sigma_1^2 - \sigma_2^2}\right)bq_{m2} + \left(\frac{-F_3}{\sigma_1^2 - \sigma_2^2}\right)q_{m3} + \left(\frac{F_1}{\sigma_1^2 - \sigma_2^2}\right)bq_{m4}$$

$$B = \left(\frac{F_6}{\sigma_2^2 - \sigma_1^2}\right)q_{m1} + \left(\frac{\sigma_2^2 - F_4}{\sigma_2^2 - \sigma_1^2}\right)bq_{m2} + \left(\frac{F_5}{\sigma_2^2 - \sigma_1^2}\right)q_{m3} + \left(\frac{F_3}{\sigma_2^2 - \sigma_1^2}\right)bq_{m4}$$

$$C = q_{m1} - A, \quad D = \frac{bq_{m2}}{\sigma_1} - \frac{B\sigma_2}{\sigma_1} \qquad (3.3.24)$$

where
$$F_1 = -(\sigma_2\sinh\sigma_1 - \sigma_1\sinh\sigma_2)(\sigma_1^2 - \sigma_2^2)/\delta$$
$$F_2 = -(\sigma_1\cosh\sigma_1\sinh\sigma_2 - \sigma_2\sinh\sigma_1\cosh\sigma_2)(\sigma_1^2 - \sigma_2^2)/\delta$$
$$F_3 = -\sigma_1\sigma_2(\sigma_1^2 - \sigma_2^2)(\cosh\sigma_1 - \cosh\sigma_2)/\delta$$
$$F_4 = \sigma_1\sigma_2[(\sigma_1^2 + \sigma_2^2)(\cosh\sigma_1\cosh\sigma_2 - 1) - 2\sigma_1\sigma_2\sinh\sigma_1\sinh\sigma_2]/\delta$$
$$F_5 = \sigma_1\sigma_2(\sigma_1^2 - \sigma_2^2)(\sigma_1\sinh\sigma_1 - \sigma_2\sinh\sigma_2)/\delta$$
$$F_6 = -\sigma_1\sigma_2(\sigma_1^2 - \sigma_2^2)(-\sigma_2\cosh\sigma_1\sinh\sigma_2 + \sigma_1\sinh\sigma_1\cosh\sigma_2)/\delta$$
$$\delta = 2\sigma_1\sigma_2(\cosh\sigma_1\cosh\sigma_2 - 1) - (\sigma_1^2 + \sigma_2^2)\sinh\sigma_1\sinh\sigma_2 \qquad (3.3.25)$$

case (2) When there are two real and two imaginary roots, and which are $\pm\sigma_1$, $\pm i\sigma_2$, then the general solution has the form

$$Y_m(y) = A\cos\frac{\sigma_2 y}{b} + B\sin\frac{\sigma_2 y}{b} + C\cosh\frac{\sigma_1 y}{b} + D\sinh\frac{\sigma_1 y}{b} \qquad (3.3.26)$$

where A,B,C,D are integration constants depending on the boundary conditions (3.3.21), and they are found as

$$A=\left(\frac{\sigma_2^2-F_4}{\sigma_1^2+\sigma_2^2}\right)q_{m1} + \left(\frac{F_2}{\sigma_1^2+\sigma_2^2}\right)bq_{m2} + \left(\frac{-F_3}{\sigma_1^2+\sigma_2^2}\right)q_{m3} + \left(\frac{F_1}{\sigma_1^2+\sigma_2^2}\right)bq_{m4}$$

$$B=\left(\frac{-F_6}{\sigma_1^2+\sigma_2^2}\right)\frac{q_{m1}}{\sigma_2} + \left(\frac{\sigma_2^2+F_4}{\sigma_1^2+\sigma_2^2}\right)\frac{bq_{m2}}{\sigma_2} + \left(\frac{-F_5}{\sigma_1^2+\sigma_2^2}\right)\frac{q_{m3}}{\sigma_2} + \left(\frac{-F_3}{\sigma_1^2+\sigma_2^2}\right)\frac{bq_{m4}}{\sigma_2}$$

$$C= q_{m1} - A \quad \text{and} \quad D= \frac{bq_{m2}}{\sigma_1} - \frac{\sigma_2}{\sigma_1}B, \qquad (3.3.27)$$

where the frequency functions are given by

$$\begin{aligned}
F_1 &= -(\sigma_2\sinh\sigma_1 - \sigma_1\sin\sigma_2)(\sigma_1^2+\sigma_2^2)/\delta \\
F_2 &= -(\sigma_1\cosh\sigma_1\sin\sigma_2 - \sigma_2\sinh\sigma_1\sin\sigma_2)(\sigma_1^2+\sigma_2^2)/\delta \\
F_3 &= -\sigma_1\sigma_2(\sigma_1^2+\sigma_2^2)(\cosh\sigma_1-\cos\sigma_2)/\delta \\
F_4 &= \sigma_1\sigma_2[+(\sigma_1^2-\sigma_2^2)(\cosh\sigma_1\cos\sigma_2-1)+2\sigma_1\sigma_2\sinh\sigma_1\sin\sigma_2]/\delta \\
F_5 &= \sigma_1\sigma_2(\sigma_2^2+\sigma_1^2)(\sigma_2\sin\sigma_2+\sigma_1\sinh\sigma_1)/\delta \\
F_6 &= -\sigma_1\sigma_2(\sigma_2^2+\sigma_1^2)(\sigma_2\cosh\sigma_1\sin\sigma_2+\sigma_1\sinh\sigma_1\cos\sigma_2)/\delta \\
\delta &= 2\sigma_1\sigma_2(\cosh\sigma_1\cos\sigma_2-1)+(\sigma_2^2-\sigma_1^2)\sinh\sigma_1\sin\sigma_2
\end{aligned} \qquad (3.3.28)$$

Case (3) when all four roots are imaginary, which are $\pm i\sigma_1$, $\pm i\sigma_2$, then the general solution has the form

$$Y_m(y) = A\cos\frac{\sigma_2 y}{b} + B\sin\frac{\sigma_2 y}{b} + C\cos\frac{\sigma_1 y}{b} + D\sin\frac{\sigma_1 y}{b} \qquad (3.3.29)$$

where $A=\left(\frac{-\sigma_2^2-F_4}{\sigma_2^2-\sigma_1^2}\right)q_{m1} + \left(\frac{F_2}{\sigma_2^2-\sigma_1^2}\right)bq_{m2} + \left(\frac{-F_3}{\sigma_2^2-\sigma_1^2}\right)q_{m3} + \left(\frac{F_1}{\sigma_2^2-\sigma_1^2}\right)bq_{m4}$

$$B=\left(\frac{-F_6}{\sigma_2^2-\sigma_1^2}\right)\frac{q_{m1}}{\sigma_2} + \left(\frac{\sigma_2^2+F_4}{\sigma_2^2-\sigma_1^2}\right)\frac{bq_{m2}}{\sigma_2} + \left(\frac{-F_5}{\sigma_2^2-\sigma_1^2}\right)\frac{q_{m3}}{\sigma_2} + \left(\frac{-F_3}{\sigma_2^2-\sigma_1^2}\right)\frac{bq_{m4}}{\sigma_2}$$

$$C= q_{m1} - A \quad \text{and} \quad D= \frac{bq_{m2}}{\sigma_1} - B\frac{\sigma_2}{\sigma_1}, \qquad (3.3.30)$$

where the frequency functions are given by

$$F_1 = -(\sigma_2 \sin\sigma_1 - \sigma_1 \sin\sigma_2)(\sigma_2^2 - \sigma_1^2)/\delta$$
$$F_2 = -(\sigma_1 \cos\sigma_1 \sin\sigma_2 - \sigma_2 \sin\sigma_1 \cos\sigma_2)(\sigma_2^2 - \sigma_1^2)/\delta$$
$$F_3 = -\sigma_1\sigma_2(\sigma_2^2 - \sigma_1^2)(\cos\sigma_1 - \cos\sigma_2)/\delta$$
$$F_4 = \sigma_1\sigma_2[-(\sigma_1^2 + \sigma_2^2)(\cos\sigma_1\cos\sigma_2 - 1) - 2\sigma_1\sigma_2 \sin\sigma_1 \sin\sigma_2]/\delta$$
$$F_5 = \sigma_1\sigma_2(\sigma_2^2 - \sigma_1^2)(\sigma_2 \sin\sigma_2 - \sigma_1 \sin\sigma_1)/\delta$$
$$F_6 = -\sigma_1\sigma_2(\sigma_2^2 - \sigma_1^2)(\sigma_2 \cos\sigma_1 \sin\sigma_2 - \sigma_1 \sin\sigma_1 \cos\sigma_2)/\delta$$
$$\delta = 2\sigma_1\sigma_2(\cos\sigma_1\cos\sigma_2 - 1) + (\sigma_1^2 + \sigma_2^2)\sin\sigma_1 \sin\sigma_2 \qquad (3..3.31)$$

Case (4) When all four roots are complex and which are $\sigma_2 \pm i\sigma_1$, $-\sigma_2 \pm i\sigma_1$,

then the general solution will have the form

$$Y_m(x) = A \cos\sigma_1\frac{y}{b}\cosh\frac{\sigma_2 y}{b} + B\cos\frac{\sigma_1 y}{b}\sinh\frac{\sigma_2 y}{b} + C\sin\frac{\sigma_1 y}{b}\cosh\frac{\sigma_2 y}{b} + D\sin\frac{\sigma_1 y}{b}\sinh\frac{\sigma_2 y}{b} \qquad (3.3.32)$$

where the integration constants A,B,C,D are found from the boundary conditions (3.3.21) as: $A = q_{m1}$

$$B = \{q_{m1}(\sigma_1\sigma_2 \sin\sigma_1\cos\sigma_1 + \sigma_2^2 \sinh\sigma_2\cosh\sigma_2) + bq_{m2}(\sigma_2 \sin^2\sigma_1) - q_{m3}\sigma_1(\sigma_1\cos\sigma_1\sinh\sigma_2 + \sigma_2\sin\sigma_1\cosh\sigma_2) + bq_{m4}(\sigma_1\sin\sigma_1\sinh\sigma_2)\}/\delta$$

$$C = \{-q_{m1}(\sigma_1\sigma_2\sinh\sigma_2\cosh\sigma_2 + \sigma_2^2\sin\sigma_1\cos\sigma_1) - bq_{m2}\sigma_1\sinh^2\sigma_2 + q_{m3}\sigma_2(\sigma_1\cos\sigma_1\sinh\sigma_2 + \sigma_2\sin\sigma_1\cosh\sigma_2) - bq_{m4}(\sigma_2\sin\sigma_1\sinh\sigma_2)\}/\delta$$

$$D = \{q_{m1}\sigma_1\sigma_2(\sin^2\sigma_1 + \sinh^2\sigma_2) + bq_{m2}(\sigma_1\sinh\sigma_2\cosh\sigma_2 - \sigma_2\sin\sigma_1\cos\sigma_1) - q_{m3}[(\sigma_2^2 + \sigma_1^2)\sin\sigma_1\sinh\sigma_2 + bq_{m4}(\sigma_1\sin\sigma_1\cosh\sigma_2 - \sigma_2\cos\sigma_1\sinh\sigma_2)\}/\delta$$

$$\delta = \sigma_2^2\sin^2\sigma_1 - \sigma_1^2\sinh^2\sigma_2 \qquad (3.3.33)$$

The frequency functions are given by

$$F_1 = -2\sigma_1\sigma_2(\sigma_2 \sin\sigma_1\cosh\sigma_2 - \sigma_1\cos\sigma_1\sinh\sigma_2)/\delta$$
$$F_2 = -2\sigma_1\sigma_2(\sigma_1\sinh\sigma_2\cosh\sigma_2 - \sigma_2\sin\sigma_1\cos\sigma_1)/\delta$$
$$F_3 = 2\sigma_1\sigma_2(\sigma_1^2 + \sigma_2^2)(\sin\sigma_1\sinh\sigma_2)/\delta$$
$$F_4 = (\varkappa_2\sigma_2\sin^2\sigma_1 + \varkappa_1\sigma_1\sinh^2\sigma_2)/\delta$$
$$F_5 = 2\sigma_1\sigma_2(\sigma_1^2 + \sigma_2^2)(\sigma_1\sinh\sigma_2\cos\sigma_1 + \sigma_2\sin\sigma_1\cosh\sigma_2)/\delta$$
$$F_6 = -2\sigma_1\sigma_2(\sigma_1^2 + \sigma_2^2)(\sigma_1\sinh\sigma_2\cosh\sigma_2 + \sigma_2\cos\sigma_1\sin\sigma_1)/\delta$$
$$\delta = \sigma_2^2\sin^2\sigma_1 - \sigma_1^2\sinh^2\sigma_2$$
$$\varkappa_1 = \sigma_1^3 - 3\sigma_1\sigma_2^2 + (2-\nu)\sigma_1$$
$$\varkappa_2 = \sigma_2^3 - 3\sigma_1^2\sigma_2 - (2-\nu)\sigma_2. \qquad (3.3.34)$$

The general solutions of the differential equations (3.3.20) in these forms are not found in the literature. When we study the secondary effects of a beam member, the same type of governing equations are met, and we shall refer to these formulae again. However, we have not studied the physical implication of the various natures of the auxiliary roots and therefore, these formulae are presented here merely for the completeness of the formulation.

Having determined the functions $Y_m(y)$ explicitly in terms of the generalized displacements q_{mi} ,we can carry out the differentiation in equations (3.3.17) and make use of equations (3.3.16) and the orthogonality of sine functions to obtain the relationship between the generalized forces and the generalized displacements.

After some simplification, the dynamic stiffness relations for all these four cases have the form

$$\begin{Bmatrix} Q_{m1} \\ Q_{m2} \\ Q_{m3} \\ Q_{m4} \end{Bmatrix} = \frac{D}{b^3} \begin{bmatrix} F_6 & -F_4 b & F_5 & F_3 b \\ -F_4 b & F_2 b^2 & -F_3 b & F_1 b^2 \\ F_5 & -F_3 b & F_6 & F_4 b \\ F_3 b & F_1 b^2 & F_4 b & F_2 b^2 \end{bmatrix} \begin{Bmatrix} q_{m1} \\ q_{m2} \\ q_{m3} \\ q_{m4} \end{Bmatrix}$$

$$m=1,2, \ \ldots \ ,N \qquad (3.3.35)$$

where the frequency functions F_i have different forms for the four cases and should be calculated under the individual heading, i.e. from expressions (3.3.25), or (3.3.28), or (3.3.31), or (3.3.34) according to the nature of the auxiliary roots. The vibration shape for every m is given by expressions (3.3.23), or (3.3.26), or (3.3.29), or (3.3.32), and the overall shape of vibration at frequency ω , is obtained from equation (3.3.14).

3.3.3 THE INTERACTION BETWEEN BEAMS AND PLATES

It is a common engineering practice to stiffen a plate system by beams. The effect of a stiffening beam is three fold: axial, flexural, and torsional. In the following analysis, we just consider the flexural and torsional effects separately.

The governing equation of a beam in flexural vibration is

$$EI\frac{\partial^4 W}{\partial x^4} + \rho A_0 \frac{\partial^2 W}{\partial t^2} + N_x \frac{\partial^2 W}{\partial x^2} = V \qquad (3.3.36)$$

where N_x is the axial compressive force and V the distributed transverse load per unit length along the beam. For harmonic excitation of a simply supported beam, we write

$$V = \sum_{m=1}^{N} V_m \sin\frac{m\pi x}{a}\, e^{i\omega t}$$

and

$$W = \sum_{m=1}^{N} W_m \sin\frac{m\pi x}{a}\, e^{i\omega t} \qquad (3.3.37)$$

From equations (3.3.36) and (3.3.37), we have

$$\sum_{m=1}^{N} \sin\frac{m\pi x}{a}\left[EI\left(\frac{m\pi}{a}\right)^4 W_m - \rho A_0 \omega^2 W_m - N_x\left(\frac{m\pi}{a}\right)^2 W_m - V_m\right] = 0.$$

Multiply the whole equation by $\sin\frac{n\pi x}{a}$ and integrate over x=0 and x=a, we have

$$V_m = \left[EI\left(\frac{m\pi}{a}\right)^4 - \rho A_0 \omega^2 - N_x\left(\frac{m\pi}{a}\right)^2\right] W_m \qquad (3.3.38)$$

which is the stiffness relation required.

The torsional effect is derived as following.

The differential equation governing the torsional vibration of a beam, when the shear centre coincides with the mass centre of cross sectional area, is

$$GJ\frac{\partial^2 \theta}{\partial x^2} - \rho I_0 \frac{\partial^2 \theta}{\partial t^2} + T = 0 \qquad (3.3.39)$$

where T is the torsional moment acting on the beam per unit length, ρ is the mass density, I_o is the polar moment of inertia, GJ is the torsional rigidity.

For harmonic oscillation of a simply supported beam, we write

$$T = \sum_{m=1}^{N} T_m \sin\frac{m\pi x}{a}\, e^{i\omega t}$$
$$\theta = \sum_{m=1}^{N} \theta_m \sin\frac{m\pi x}{a}\, e^{i\omega t} \qquad (3.3.40)$$

and equation (3.3.39) becomes

$$\sum_{m=1}^{N} \sin\frac{m\pi x}{a}\left[-GJ\left(\frac{m\pi}{a}\right)^2\theta_m + \rho I_o \omega^2 \theta_m + T_m\right] = 0 .$$

Multily the equation by $\sin\frac{n\pi x}{a}$ and integrate over x=0, and x=a then, we have

$$T_m = \left[GJ\left(\frac{m\pi}{a}\right)^2 - \rho I_o \omega^2\right]\theta_m \qquad (3.3.41)$$

which is the stiffness relation required.

When the beam member is on an edge of a folded plate then, the generalized displacements of the beam W_m and θ_m will correspond to the generalized displacements of the plate either q_{m1} and q_{m2} or q_{m3} and q_{m4} respectively, depending on which edge of the plate where the beam is situated.

Due to the limited period of research the author was unable to give any numerical or experimetal example to check the results in these two subsections.

3.4 SEPARATION OF MASS MATRIX FROM DYNAMIC STIFFNESS MATRIX

The dynamic stiffness matrix is ususlly obtained from the conditions of equilibrium without integrating the products of matrices which is unavoidable in forming the mass and stiffness matrices using energy principles. Therefore, it is much simpl-er to obtain the dynamic stiffness matrix directly if the shape functions are complicated.

For a chosen set of generalized coordinates, the generalized displacements $\{q\}$ and the generalized forces $\{Q\}$ are related by the dynamic stiffness matrix $[\mathcal{D}]$, i.e.

$$[\mathcal{D}]\{q\} = \{Q\} . \tag{3.4.1}$$

When free vibration is concerned, $\{Q\} = \{0\}$, and equation (3.4.1) becomes

$$[\mathcal{D}]\{q\} = \{0\} \tag{3.4.2}$$

The necessary and sufficient conditions for non-trivial solution of $\{q\}$ is that

$$\det[\mathcal{D}] = 0 \tag{3.4.3}$$

The solution of (3.4.3) for natural frequencies can be ach ved by usual methods of equation solving. This will be discussed in chapter 5. Because of the fact that at each natural frequency of vibration, $[\mathcal{D}]$ will be singular, any method of solving for $\{q\}$ directly is very unstable numerically (ref 30), other methods must be studied. One of the stable methods recommended by Wilkinson is the inverse iteration method for eigenvectors. The method required that the matrix $[\mathcal{D}]$ be separated into two positive definite matrices $[m]$ and $[K]$, such that

$$[\mathcal{D}] = [K] - \omega^2[m] \tag{3.4.4}$$

The most natural choice is to regard $[\mathcal{K}]$ as the stiffness matrix and $[m]$ the mass matrix of the system. The purpose of this section is to establish the following theorem, so that $[\mathcal{D}]$ can be separated according to equation (3.4.4). To the best the author is aware, this theorem has not appeared in the literature. We shall discuss the advantages of having this theorem in next section

THEOREM (3.4.1):

For an elastic body vibrating harmonically at frequency ω, and for a specific set of generalized coordinates, the mass matrix, $[m]$, and the dynamic stiffness matrix, $[\mathcal{D}(\omega)]$, are related by

$$[m(\omega)] = -\frac{1}{2\omega}\frac{\partial}{\partial\omega}[\mathcal{D}(\omega)] \qquad (3.4.5)$$

In the proof of the theorem, we shall make use the reciprocal theorem (ref 12), which states:

If an elastic body is subjected to two systems of forces, then the work that would be done by the first system in acting through the displacements due to the second system of forces is equal to the work that would be done by the forces of second system in acting through the displacements due to the first system of forces.

For a vibrating body, there are two kinds of forces involved. The first is the external force. If the body is subjected to external body force $\{X\}e^{i\omega t}$ and surface force $\{\Phi\}e^{i\omega t}$, then we define the equivalent

generalized forces $Q_r e^{i\omega t}$ by equation (2.5.15), as

$$Q_r = \int_{vol} \{X\}^T \frac{\partial \{u\}}{\partial q_r} \, dvol + \int_S \{\Phi\}^T \frac{\partial \{u\}}{\partial q_r} \, dS \qquad (3.4.6)$$

where $\{u\} e^{i\omega t}$ is the displacement pattern of the body.

The second is the inertia force, which is

$$\begin{aligned} -[\rho] \frac{\partial^2}{\partial t^2} \{u\} e^{i\omega t} &= \omega^2 [\rho] \{u\} e^{i\omega t} \\ &= \omega^2 [\rho][a] \{q\} e^{i\omega t} \end{aligned} \qquad (3.4.7)$$

where the relation $\{u\} = [a]\{q\}$

has been used. Therefore the total force acting on the body $\{F\} e^{i\omega t}$ is given by

$$\{F\} = \{Q\} + \omega^2 [\rho][a]\{q\} . \qquad (3.4.8)$$

Now, we begin to prove the theorem (3.4.1).

Consider an elastic body subjected to two sets of forces whose equivalent generalized forces are denoted by $\{Q_1\} e^{i\omega_1 t}$ and $\{Q_2\} e^{i\omega_2 t}$. The respective displacement patterns are $\{q_1\} e^{i\omega_1 t}$ and $\{q_2\} e^{i\omega_2 t}$. The total force on the body in state 1 is $\{F_1\} e^{i\omega_1 t}$ and in state 2 $\{F_2\} e^{i\omega_2 t}$, where

$$\begin{aligned} &\{F_1\} = \{Q_1\} + \omega_1^2 [\rho][a_1]\{q_1\} , \\ &\{F_2\} = \{Q_2\} + \omega_2^2 [\rho][a_2]\{q_2\} , \\ &[a_1] = [a(x, y, z, \omega_1)] , \\ &[a_2] = [a(x, y, z, \omega_2)] . \end{aligned} \qquad (3.4.9)$$

Then the reciprocal theorem states,

$$\int_{vol} \{F_1\}^T \{u_2\}\, dvol = \int_{vol} \{F_2\}^T \{u_1\}\, dvol \qquad (3.4.10)$$

Substituting equations (3.4.9) into equation (3.4.10), we have

$$\int_{vol} \{Q_2\}^T \{u_1\}\, dvol + \int_{vol} \omega_2^2 \{q_2\}^T [a_2]^T [\rho] \{u_1\}\, dvol$$
$$= \int_{vol} \{Q_1\}^T \{u_2\}\, dvol + \int_{vol} \omega_1^2 \{q_1\}^T [a_1]^T [\rho] \{u_2\}\, dvol \qquad (3.4.10a)$$

Now, the generalized force at point (x_i, y_i, z_i) is Q_i if (x_i, y_i, z_i) is a generalized coordinate of position where the projection of $u(x_i, y_i, z_i)$ along the direction of the generalized coordinate equals to q_i, and the generalized force at other points is zero. Therefore

$$\int_{vol} \{Q\}^T \{u\}\, dvol = \{Q\}^T \{q\}$$

for states 1 and 2. And equation (3.4.10a) becomes

$$\{Q_2\}^T \{q_1\} + \omega_2^2 \{q_2\}^T \int_{vol} [a_2]^T [\rho] [a_1]\, dvol\, \{q_1\}$$
$$= \{Q_1\}^T \{q_2\} + \omega_1^2 \{q_1\}^T \int_{vol} [a_1]^T [\rho] [a_2]\, dvol\, \{q_2\} \qquad (3.4.11)$$

Substituting the dynamic stiffness relations

$$\{Q_1\} = [D_1]\{q_1\} \qquad \text{and} \qquad \{Q_2\} = [D_2]\{q_2\}$$

into equation (3.4.11) and remembering that the dynamic stiffness matrices are symmetrical, we have

$$\{q_2\}^T [D_2] \{q_1\} + \omega_2^2 \{q_2\}^T \left(\int_{vol} [a_2]^T [\rho] [a_1]\, dvol \right) \{q_1\}$$
$$= \{q_1\}^T [D_1] \{q_2\} + \omega_1^2 \{q_1\}^T \left(\int_{vol} [a_1]^T [\rho] [a_2]\, dvol \right) \{q_2\} \qquad (3.4.12)$$

Now, since $\{q_1\}^T[D_1]\{q_2\}$ is a scalar,

$$\{q_1\}^T[D_1]\{q_2\} = \{q_2\}^T[D_1]\{q_1\}$$

and $\{q_1\}^T\left(\int_{vol}[a_1]^T[\rho][a_2]\,dvol\right)\{q_2\} = \{q_2\}^T\left(\int_{vol}[a_2]^T[\rho][a_1]\,dvol\right)\{q_1\}$

we have, from equation (3.4.12),

$$(\omega_1^2-\omega_2^2)\{q_1\}^T\left(\int_{vol}[a_1]^T[\rho][a_2]\,dvol\right)\{q_2\}$$

$$= \{q_1\}^T([D_2]-[D_1])\{q_2\},$$

or $$\{q_1\}^T\left(\int_{vol}[a_1]^T[\rho][a_2]\,dvol + \frac{[D_2]-[D_1]}{\omega_2^2-\omega_1^2}\right)\{q_2\} = 0.$$

Since $\{q_1\}e^{i\omega_1 t}$ and $\{q_2\}e^{i\omega_2 t}$ arbitrary, then,

$$\int_{vol}[a_1]^T[\rho][a_2]\,dvol + \frac{[D_2]-[D_1]}{\omega_2^2-\omega_1^2} = [0]. \qquad (3.4.13)$$

If we let ω_1 and ω_2 approach to a identical value ω, then $[a_1]$ and $[a_2]$ approach to $[a]$ and we have

$$\int_{vol}[a]^T[\rho][a]\,dvol + \frac{1}{2\omega}\frac{\partial}{\partial\omega}[D] = [0]. \qquad (3.4.14)$$

Now, by definition of a mass matrix

$$[m] = \int_{vol}[a]^T[\rho][a]\,dvol,$$

therefore $$[m] = -\frac{1}{2\omega}\frac{\partial}{\partial\omega}[D],$$

and we have prove the theorem (3.4.1).

In the proof of the theorem, we do not impose any restrictions to the matrix $[a]$, therefore, if the same relation $\{u\} = [a]\{\gamma\}$ has been used for deriving the frequency dependent dynamic stiffness matrix $[\mathcal{D}]$ as shown in last section, than the mass matrix thus obtained is the consistent mass matrix of the displacement pattern represented by $[a]$.

Finally, if one of the forces systems, say system 1, is a static force system, then $\omega_1 = 0$, and from equation (3.4.13) we have

$$[\mathcal{D}_2] = [\mathcal{D}_1] - \omega_2^2 \int_{vol} [a_1]^T [\rho] [a_2] \, dvol. \qquad (3.4.15)$$

If we denote the static quantities in equation (3.4.15) by the subscript o, and the dynamic quantities without subscripts, then

$$[\mathcal{D}] = [\mathcal{D}_o] - \omega^2 \int_{vol} [a_o]^T [\rho] [a] \, dvol \qquad (3.4.16)$$

If the dynamic displacement pattern $[a]$ is approximated by the static one $[a_o]$, then we have

$$\begin{aligned} [\mathcal{D}] &\doteq [\mathcal{D}_o] - \omega^2 \int_{vol} [a_o]^T [\rho] [a_o] \, dvol \\ &= [\mathcal{D}_o] - \omega^2 [m_o] \end{aligned} \qquad (3.4.17)$$

where $[m_o] = \int_{vol} [a_o]^T [\rho] [a_o] \, dvol$,

may be called the consistant mass matrix of static deflection curves because it represents the inertia property of a structural member consistent with the displacement assumed.

Examples of applications of the theorem are given in the following

subsections.

3.4.1 UNIFORM BEAM MEMBERS

We are going to apply theorem (3.4.1) to a uniform beam member. Firstly, we consider the approximated formulation where the displacement pattern is approximated by equations (3.2.26) and (3.2.27). The dynamic stiffness matrix thus obtained is given by

$$[\mathcal{D}] = \frac{EI}{l^3}\begin{pmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l & -6l & 4l^2 \end{pmatrix} - \frac{\omega^2 \rho A l}{420}\begin{pmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{pmatrix}$$

$$= [\mathcal{K}] - \omega^2[m]$$

Since $[\mathcal{K}]$ and $[m]$ are independent of ω in this example, therefore

$$-\frac{1}{2\omega}\frac{\partial}{\partial\omega}[\mathcal{D}] = -\frac{1}{2\omega}\frac{\partial}{\partial\omega}\left([\mathcal{K}] - \omega^2[m]\right) = [m]$$

which is correct.

Now, we consider the exact formulation where the displacement pattern is given by equation (3.2.25). The dynamic stiffness matrix in this case is

$$[\mathcal{D}] = \frac{EI}{l^3}\begin{pmatrix} F_6 & -F_4 l & F_5 & F_3 l \\ -F_4 l & F_2 l^2 & -F_3 l & F_1 l^2 \\ F_5 & -F_3 l & F_6 & F_4 l \\ F_3 l & F_1 l^2 & F_4 l & F_2 l^2 \end{pmatrix}$$

where the frequency functions are given by equation (3.2.36).

Now, $\lambda^4 = \omega^2 \frac{\rho A \ell^4}{EI}$

$$4\lambda^3 d\lambda = 2\omega \frac{\rho A \ell^4}{EI} d\omega$$

Therefore,

$$-\frac{1}{2\omega}\frac{\partial}{\partial\omega}[\mathcal{S}(\omega)] = -\frac{1}{4\lambda^3}\frac{\partial}{\partial\lambda}[\mathcal{D}(\lambda)]\frac{\rho A \ell^4}{EI} \qquad (3.4.18)$$

In order to carry out the differentiation of $[\mathcal{D}(\lambda)]$ with respect to λ, we must differentiate the frequency functions one by one. For F_1, we have

$$F_1 = -\lambda(\sinh\lambda - \sin\lambda)/(\cosh\lambda\cos\lambda - 1),$$

$$\frac{\partial F_1}{\partial\lambda} = -\frac{\sinh\lambda - \sin\lambda}{\cosh\lambda\cos\lambda - 1} - \lambda\frac{\cosh\lambda - \cos\lambda}{\cosh\lambda\cos\lambda - 1} + \frac{\lambda(\sinh\lambda - \sin\lambda)(\sinh\lambda\cos\lambda - \cosh\lambda\sin\lambda)}{(\cosh\lambda\cos\lambda - 1)^2}$$

$$= \frac{1}{\lambda}(-F_1F_2 + F_3 + F_4) = -4\lambda^3 G_1,$$

where G_i, i=1,2, ... , 6 are given by equations (3.2.38).

Similarly, for the other frequency functions,

$$\frac{\partial F_i}{\partial\lambda} = -4\lambda^3 G_i \qquad i=1,2, \ldots ,6$$

Corresponding to equation (3.4.18),

$$-\frac{1}{4\lambda^3}\frac{\partial}{\partial\lambda}F_i\frac{\rho A \ell^4}{EI} \times \frac{EI}{\ell^3} = \rho A \ell G_i$$

Therefore,

$$-\frac{1}{4\lambda^3}\frac{\partial}{\partial\lambda}[\mathcal{D}(\lambda)]\frac{\rho A \ell^4}{EI} = \rho A \ell \begin{bmatrix} G_6 & -G_4\ell & G_5 & G_3\ell \\ -G4\ell & G_2\ell^2 & -G_3\ell & G_1\ell^2 \\ G_5 & -G_3\ell & G_6 & G_4\ell \\ G_3\ell & G_1\ell^2 & G_4\ell & G_2\ell^2 \end{bmatrix}$$

and which is the same mass matrix as given by equation (3.2.37).

For a straight beam element, the effort required to form the mass and stiffness matrices according to equations (3.2.37) and (3.2.40) is more than about ten times the effort required in forming the dynamic stiffness matrix according to equation (3.3.10), and it may be the reason why many authors prefer the dynamic stiffness than the energy approach in the frequency dependent formulation (ref 26).

The advantages of the existence of the theorem (3.4.1) are listed below:

1. In deriving the fundamental matrices, it enables the mass matrix to be separated from the dynamic stiffness matrix. The dynamic stiffness matrices of structural members can be found from classical methods such as mechanical receptance, impedence, admittance and mobility etc., which are well documented in the literature. It is particular useful when the dynamic stiffness can be obtained experimentally only and the governing equations of motion are not known

2. For very complicated dynamic stiffness matrices, where the differentiation with respect to frequency parameter becomes difficult, the theorem provides a way to approximate the mass matrix by replacing the differential by difference, i.e.

$$[m] = -\frac{1}{2\omega}\frac{\partial[\mathcal{D}]}{\partial\omega} \doteq -\frac{1}{2\omega}\frac{\delta[\mathcal{D}]}{\delta\omega}$$

3. For large structural systems having a dynamic stiffness matrix of large order, numerical instability will result as the rounding off errors in the elimination process for evaluating the frequency determinant increases. With the mass matrix, we can apply the Rayleigh's Quotient and inverse iteration to avoid the Guass elimination process. In this way, the modal shapes as well as the natural frequencies are obtained accurately (ref 30, 64, 65), as shown in chapter five.

4. Even if the modal shapes are not required, the fact that the mass matrix is proportional to the first derivative of the dynamic stiffness matrix enables us to calculate the natural frequency very rapidly by the method of root finding of Newton. See chapter five.

5. Moreover, that the positive definite matrix $[m]$ is obtained from the nondefinite matrix $[\mathcal{D}]$ through differentiation make many theories on nonlinear eigenvalue problems applicable to the mechanical vibration problems (ref 66, 67).

We shall use this theorem extensively in deriving the fundamental matrices for various types of structural elenemts in sections (3.6) to (3.9).

3.5 IMPROVING THE FUNDAMENTAL MATRICES

In sections (3.2) to (3.4), we have introduced some methods to construct the fundamental matrices of vibration analysis using the exact displacement patterns or the approximated displacement patterns. It was seen that the formulation is very complicated even for a uniform beam member or a retangular plate member with simple boundary conditions. For other structural elements, especially for three dimensional elements, approximated methods are unaviodable. The usual approximated displacement patterns chosen are frequency independent. These contradict the fact that displacement patterns of a vibrating body do change with frequency of vibration. Therefore, a sufficient number of elements must be taken to ensure convergence. Other methods of improving the results by using the frequency independent functions such as Rayleigh-Ritz, Galerkin, least square, hybrid, and collocation can be found in (ref 68). It was concluded that comparatively large order dynamic matrices are required to give a satisfactory degree of accuracy even in the first few frequencies. Condensation techniques are often used to reduce the size of he matrices which in turn introduce certain rounding off errors so that the results are unreliable beyond certain limits. Therefore, there is a need to modify the frequency independent displacement assumed. This section is devoted to such a modification.

For uniform beam elements, Przemienicki (ref 69) formed the fundamental matrices by using displacement patterns in ascending orders of frequency of excitation. The method was found unsatisfactory by himself because the resulting eigenvalue problem gives negative eigenvalues and the series diverges at high frequency when the frequency parameter $\lambda^4 = \omega^2 \rho A l^4 / EI$ is greater than unity. Cohen and McCallion (ref 70) examined the

possibility of using deformation functions which partially satisfy the continuity required and the governing differential equation. And the conclusion was the closer the shape function assumed to the solution of the governing equation of vibration the more accurate the results to the true ones.

For elements other than uniform beams, it seems in available literature that improvement can only be acheived by employing more generalized coordinates, or equivalently, by increasing the number of elements. However, as a result, large matrices are involved and we go back to the same problem.

In this section, we shall present a convergent method which improves on the fundamental matrices by assumed frequency independent displacement patterns and keeps the order of the matrices fixed at the same time. Although distributed coordinate system will be used in the region of the element, we adopt discrete generalized coordinates on the boundary so that the order of the matrices is equal to the number of generalized coordinates.

Since we can not improve the fundamental matrices formulated from arbitrarily assumed displacement patterns, therefore we are going to improve on the fundamental matrices resulting from the assumed displacement patterns with the dynamic terms neglected as those assumed in the finite element method. By this we can make use of the formulae available in the finite element formulation. The second tool on which our improvement process is based is a set of coordinate functions. They are the normal modes of the individual members with all the generalized coordinates vanished. These normal modes correspond to the partial vibration patterns which occur in individual members independently of the other parts of the structure.

These modes will be discussed in subsection (3.5.2). Then the formulation of the improved matrices is derived in subsection (3.5.3).

Because the method presented in this section serves as an improvement to the normal finite element method in vibration analysis and since it does not seem to have appeared in the literature, we shall use three-dimensional theory throughout and give an overall picture to the subject.

3.5.1 FINITE ELEMENT MODELLING

Finite element models of Rayleigh-Ritz, force, Galerkin, Least square, Hybrid and collocation methods were presented in (ref 68) by Murty et al. for natural vibration problems. A comparative assessment was also made of these methods. If a fixed number of generalized coordinates are chosen, then the method of minimizing the total energy and the method of approximating the governing differential equations while keeping the error a minimium according to some weighting criteria will give more or less the same results. We shall study the latter method here.

If an elastic body is subjected to small perturbation forces with exciting frequency ω , the governing equations of the displacements are given by the Navier equilibrium equations (ref 31) as

$$G\nabla^2 u_r e^{i\omega t} + (\lambda_o + G) u_{j,jr} e^{i\omega t} + X_r = 0, \qquad r, j = 1,2,3,$$

where the comma denotes differentiation with respect to spatial variables, and G and λ_o are the elastic constants. When the body is subjected to harmonic boundary forces only, then

$$X_r = -\rho \frac{\partial^2}{\partial t^2}(u_r e^{i\omega t}) = \rho\omega^2 u_r e^{i\omega t} \qquad r = 1,2,3,$$

and we have

$$G\nabla^2 u_r + (\lambda_0 + G)\, u_{j,jr} + \rho\omega^2 u_r = 0$$

$$r, j = 1,2,3. \qquad (3.5.1)$$

We can write equations (3.5.1) in the following vector form,

$$[L]\{u\} = \lambda[\rho]\{u\} \qquad (3.5.2)$$

where $[L]$ is a matrix of second order linear differential operator of spatial coordinates, and λ is a scalar given by $\lambda = \omega^2$, and $[\rho]$ is a symmetric matrix of density.

The prescribed boundary displacement conditions of the body can be written symbolically,

$$[B_i]\{u(x_i, y_i, z_i)\} = q_i \qquad i = 1,2, \ldots, N \qquad (3.5.3)$$

where N= number of generalized coordinates on boundaries and $[B_i]$ is a row of differential operators. Equations (3.5.3) define the generalized displacements.

According to finite element modelling, we specify the "approximate displacement patterns" as those functions which satisfy approximately the equations (3.5.2) with $\lambda = 0$ and the boundary conditions (3.5.3). If we denote these functions by

$$\{u(x,y,z)\} = [a_0(x,y,z)]\{q\} \qquad (3.5.4)$$

where $[a_0(x,y,z)] = [\{a_1(x,y,z)\}, \{a_2(x,y,z)\}, \cdots, \{a_N(x,y,z)\}]$, (3.5.5)

and $\{q\} = [q_1, q_2, \cdots, q_N]^T$

then $\{a_i(x,y,z)\}$ will satisfy

$$[L]\{a_i\} = \{0\} \qquad \text{in the region of the body} \qquad (3.5.6)$$

and $$[B_i]\{a_j\} = \delta_{ij} \qquad \text{on the boundaries} \qquad (3.5.7)$$

where δ_{ij} is the Kornecker delta.

Before we go any further from here, it is appropriate for us to make clear the physical meanings of these equations. As a first example, let us consider a uniform straight beam member whose governing equation of vibration is

$$\frac{d^4u}{dx^4} - \frac{\rho A l^4 \omega^2}{EI} u = 0,$$

with the boundary conditions $u(0)=q_1$, $u'(0)=q_2$, $u(1)=q_3$, $u'(1)=q_4$.

Therfore, in equation (3.5.2), the operator $[L] = d^4/dx^4$, the displacement vector contains one element u, and the frequency parameter $\lambda = \rho A l^4 \omega^2 / EI$.

The boundary operators are scalars in this case,

$[B_1]\{u(x_1)\} = 1 \times u(0) = q_1$; $[B_2]\{u(x_2)\} = \frac{d}{dx} u(0) = q_2$; $[B_3]\{u(x_3)\} = q_3$; $[B_4]\{u(x_4)\} = q_4$

$\therefore [B_1] = [B_3] = 1$, $[B_2] = [B_4] = \frac{d}{dx}$, $x_1 = x_2 = 0$, $x_3 = x_4 = l$.

Equation (3.5.4) represents the solution of

$$\frac{d^4v}{dx^4} = 0$$

with the boundary conditions

$$v(0)=q_1,\ v'(0)=q_2,\ v(1)=q_3,\ v'(1)=q_4,$$

which gives $v(x) = [1-3x^2+2x^3 \quad (x-2x^2+x^3)l \quad 3x^2-2x^3 \quad (-x^2+x^3)l]\{q\} = [a_0]\{q\}$

where $[a_0] = [1-3x^2+2x^3 \quad (x-2x^2+x^3)l \quad 3x^2-2x^3 \quad (-x^2+x^3)l]$

$\{q\} = [q_1\ q_2\ q_3\ q_4]^T$

and $\{a_i\}$, $i=1,2,3,4$ are scalar given by

$$a_1 = 1-3x^2+2x^3$$
$$a_2 = (x-2x^2+x^3)l$$
$$a_3 = 3x^2-2x^3$$
$$a_4 = (-x^2+x^3)l$$

As a second example, consider a more complicated element, say a cylindrical shell element as shown in fig (3.5.1). The non-dimensional long-itudinal coordinate $\alpha = x / R$ is used throughout, where R is the radius of the cylinder. We denote the displacement along the generatrix by $ue^{i\omega t}$, the peripheral displacement by $ve^{i\omega t}$, and the radial displacement by $we^{i\omega t}$.

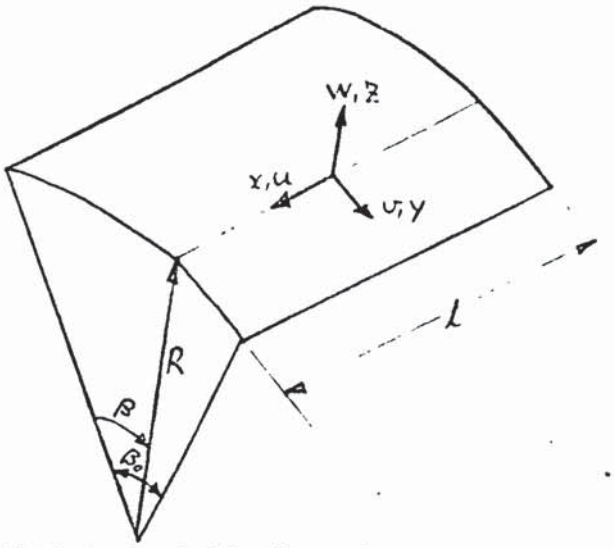


Fig (3.5.1) A cylindrical shell element

The the governing equations of motion in harmonic vibration with frequency ω according to Donnell's theory (ref 55) are given by

$$\begin{bmatrix} \frac{\partial^2}{\partial\alpha^2}+\frac{1-\nu}{2}\frac{\partial^2}{\partial\beta^2} & \frac{1+\nu}{2}\frac{\partial^2}{\partial\alpha\partial\beta} & \nu\frac{\partial}{\partial\alpha} \\ \frac{1+\nu}{2}\frac{\partial^2}{\partial\alpha\partial\beta} & \frac{\partial^2}{\partial\beta^2}+\frac{1-\nu}{2}\frac{\partial^2}{\partial\alpha^2} & \frac{\partial}{\partial\beta} \\ \nu\frac{\partial}{\partial\alpha} & \frac{\partial}{\partial\beta} & k\nabla^2\nabla^2 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \lambda \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \qquad (3.5.8)$$

where $k = h^2/12R^2$,

h = thickness of the shell,

$\nabla^2 = \frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}$,

and $\lambda = \rho R^2(1-\nu^2)\omega^2/E$.

Therefore, the operator

$$[L] \equiv \begin{bmatrix} \frac{\partial^2}{\partial\alpha^2}+\frac{1-\nu}{2}\frac{\partial^2}{\partial\beta^2} & \frac{1+\nu}{2}\frac{\partial^2}{\partial\alpha\partial\beta} & \nu\frac{\partial}{\partial\alpha} \\ \frac{1+\nu}{2}\frac{\partial^2}{\partial\alpha\partial\beta} & \frac{\partial^2}{\partial\beta^2}+\frac{1-\nu}{2}\frac{\partial^2}{\partial\alpha^2} & \frac{\partial}{\partial\beta} \\ \nu\frac{\partial}{\partial\alpha} & \frac{\partial}{\partial\beta} & k\nabla^2\nabla^2 \end{bmatrix},$$

the vector $\{u\} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$

K.S.Rao et al. employed seven generalized coordinates $\partial w/\partial\alpha$, $\partial w/\partial\beta$, w, $\partial u/\partial\beta$, u, $\partial v/\partial\beta$ and v for each corner of the shell (ref 72) and totally 28 generalized coordinates per member. Now, the boundary operators are 3 x 1 row matrices and examplified by

$$q_1 = \frac{\partial w(0,0)}{\partial\alpha} = [0 \quad 0 \quad \frac{\partial}{\partial\alpha}] \begin{Bmatrix} u(0,0) \\ v(0,0) \\ w(0,0) \end{Bmatrix} = [B_1]\{u(0,0)\}$$

where $[B_1] = [\,0 \quad 0 \quad \frac{\partial}{\partial \alpha}\,]$.

The matrix $[a(\alpha,\beta)]$ in equation (3.5.4) is a 3x28 matrix and $\{a_i(\alpha,\beta)\}$, i=1,2, ... ,28 are 3x1 matrix functions of spatial variables α and β. The formulae for these matrices are found in (ref 72) and will not be repeated here.

3.5.2 PARTIAL VIBRATIONS OF STRUCTURAL MEMBERS

A structure would vibrate unstably when one of the natural frequencies of any one of its composite members with all edges clamped is excited. This is called a partial vibration and the corresponding frequency is called a partial frequency. Some details of the physical significance can be found from (ref 28). In this subsection we just give some numerical consideration of the partial vibration modes because we shall consider this again in the calculation of the overall problem in chapter five.

Consider the general elastic body we have discussed in subsection (3.5.1). The governing equations of vibration are

$$[L]\{u\} = \lambda[\rho]\{u\} \qquad \text{in the region}$$

$$[B_i]\{u(x_i,y_i,z_i)\} = q_i \qquad i = 1,2, \ldots ,N \text{ on the boundaries.}$$

When partial vibration occurs, i.e. when all the boundary displacements are held rigid, we we denote the modal patterns by $\{\phi_j\}$, then, they satisfy

$$[L]\{\phi_j\} = \lambda_j[\rho]\{\phi_j\} \qquad \text{in the region of the body (3.5.9)}$$

$$[B_i]\{\phi_j(x_i,y_i,z_i)\} = 0 \qquad \text{on the boundaries (3.5.10)}$$

where λ_j is the partial frequency parameter. Because there are an infinite number sets of λ and $\{\phi\}$ satisfying the partial vibration conditions if the body is a continuum, j runs from 1,2, ... , ∞. We arrange the values of j such that

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots .$$

For general three-dimensional solids, numerical solutions of the eigenvalue

problem (3.5.9) with homogeneous boundary conditions (3.5.10) can be done by the method of shooting as shown in (ref 73). Computer programmes in ALGOL and FORTRAN languages can be found in " Nottingham Algorithm Group ", I.C.L 1900 system, Document No. 505, algorithm D02AGA and D02 AGF. For simple elements such as uniform beams, the eigenfunctions can be obtained by solving the equations of the eigenvalue problem directly. The eigenfunctions of a uniform beam member are given by

$$\phi_r(\tfrac{x}{l}) = \cosh\tfrac{\lambda_r x}{l} - \cos\tfrac{\lambda_r x}{l} - \sigma_r(\sinh\tfrac{\lambda_r x}{l} - \sin\tfrac{\lambda_r x}{l}) \qquad (3.5.11)$$

where $\lambda_1 = 4.73004075$
$\lambda_2 = 7.85320462$
$\lambda_3 = 10.99560784$
$\lambda_4 = 14.13716549$
$\lambda_5 = 17.27875966$
$\lambda_r = (r+1/2)*3.14159265 \qquad r>5$

$\sigma_r = (\cosh\lambda_r - \cos\lambda_r)/(\sinh\lambda_r - \sin\lambda_r)$

The eigenfunctions of a semicircular beam member are listed in fig(3.6.8) and those of a tapered beam can be found in subsection (3.6.2). For two-dimensional members such as plates and shells, series form of solutions only are possible. One of the most effective ways of obtaining the eigenfunctions of a clamped rectangular plate is the Rayleigh-Ritz method wherein products of clamped beam functions as coordinate functions are used as shown by Young (ref 74). Ashton (ref 45) generalized the method to anistropic plates. We describe the method briefly for a vibrating orthotropic rectangular plate as follows. If the transverse displacement is expressed in the form

$$W(x,y) = \sum_{m=1}^{p}\sum_{n=1}^{q} a_{mn} X_m(x) Y_n(y)$$

where $X_m(x)$ and $Y_n(y)$ are clamped beam functions, p and q are numbers of terms used in each direction, a_{mn} is to be determined, then, the Hamilton's principle

$U+V-T$= stationary in one cycle of vibration,

will give

$$\sum_{m=1}^{p}\sum_{n=1}^{q}(K_{ijmn}-S_{ijmn}-\omega^2 M_{ijmn})\,a_{mn}=0 \qquad (3.5.12)$$

for i=1,2, ... ,p; j=1,2, ... ,q

where

$$K_{ijmn}=\frac{b}{a}D_{11}\int_0^1 X_i''X_m''dx\int_0^1 Y_jY_n dy+\frac{D_{12}}{ab}\left(\int_0^1 X_i''X_m dx\int_0^1 Y_jY_n''dy+\int_0^1 X_m''X_i dx\int_0^1 Y_j''Y_n dy\right)$$
$$+\frac{aD_{22}}{b}\int_0^1 X_iX_m dx\int_0^1 Y_j''Y_n''dy+\frac{2D_{16}}{a^2}\left(\int_0^1 X_m''X_i'dx\int_0^1 Y_j'Y_n dy+\int_0^1 X_i''X_m'dx\int_0^1 Y_n'Y_j dy\right)$$
$$+\frac{2D_{26}}{b^2}\left(\int_0^1 X_i'X_m dx\int_0^1 Y_n''Y_j'dy+\int_0^1 X_m'X_i dx\int_0^1 Y_j''Y_n'dy\right)$$
$$+\frac{4D_{66}}{ab}\int_0^1 X_i'X_m'dx\int_0^1 Y_j'Y_n'dy\ ,$$

$$S_{ijmn}=\frac{bN_x}{a}\int_0^1 X_i'X_m'dx\int_0^1 Y_jY_n dy+\frac{aN_y}{b}\int_0^1 X_iX_m dx\int_0^1 Y_j'Y_n'dy$$
$$+N_{xy}\left(\int_0^1 X_i'X_m dx\int_0^1 Y_n'Y_j dy+\int_0^1 X_i'X_m dx\int_0^1 Y_n'Y_j dy\right)\ ,\ \text{and}$$

$$M_{ijmn}=\rho hab\int_0^1 X_iX_m dx\ Y_jY_n dy\ ,$$

where a,b the dimensions of the retangular plate

h is the thickness,

D_{ij} i,j=1,2,..6 are the flexural rigidities

N_x, N_y are the compressive inplane load per unit length

N_{xy} is the shear inplane load per unit lehgth .

As part of the present investigation, a computer program has been designed for calculating the eigenvalues ω^2 and eigenvectors a_{mn} of equation (3.5.12). The details are given in chapter six. If we take p=q=6, $N_x=N_y=N_{xy}=0$, $a=b=h=D_{11}=D_{22}=D_{12}=1$, $D_{16}=D_{26}=D_{66}=0$, we obtain the natural modes of a isotropic square plate free of axial effects. The calculated frequencies are given in fig (3.5.2) .

35.9914838	73.4132961	73.4132961	108.270813
131.636539	132.244894	165.153267	165.153267
210.602226	210.602226	220.501774	242.406359
243.338633	297.035658	297.035658	309.043483
309.273939	340.963862	340.963862	372.282395
394.120028	395.007265	427.518884	427.518884
458.747070	459.242959	469.043961	469.043961
512.111766	512.111766	565.532410	585.297458
586.183215	681.683240	681.683240	796.936053

Fig (3.5.2) The first 36 natural frequencies of a square plate

$[L]$

In all cases, if the operator is self-adjoint, i.e.

$$\int_{vol} \{\phi_i\}^T [L] \{\phi_j\} \, dvol = \int_{vol} \{\phi_j\}^T [L] \{\phi_i\} \, dvol \qquad (3.5.13)$$

than, the eigenfunctions $\{\phi_i\}$ will satisfy the condition of orthogonality, such that

$$\int_{vol} \{\phi_j\}^T [\rho] \{\phi_k\} \, dvol = 0 \qquad (3.5.14)$$

when $j \neq k$. We prove it as follows.

Premultiply equation(3.5.9) by $\{\phi_i\}^T$ and integrate over the whole volume,

$$\int_{vol} \{\phi_i\}^T [L] \{\phi_j\} \, dvol = \lambda_j \int_{vol} \{\phi_i\}^T [\rho] \{\phi_j\} dvol . \qquad (3.5.15)$$

Interchange the subscripts i and j ,

$$\int_{vol} \{\phi_j\}^T [L] \{\phi_i\} dvol = \lambda_i \int_{vol} \{\phi_j\}^T [\rho] \{\phi_i\} dvol . \qquad (3.5.16)$$

Substract equation (3.5.16) from equation (3.5.15) and make use of equation (3.5.13)

$$0 = \lambda_j \int_{vol} \{\phi_i\}^T [\rho] \{\phi_j\} dvol - \lambda_i \int_{vol} \{\phi_j\}^T [\rho] \{\phi_i\} dvol .$$

But $\{\phi_i\}^T [\rho] \{\phi_j\} = \{\phi_j\}^T [\rho] \{\phi_i\}$, since both sides are scalar,

therefore, $0 = (\lambda_j - \lambda_i) \int_{vol} \{\phi_i\}^T [\rho] \{\phi_j\} dvol$

If $i \neq j$, then, $\int_{vol} \{\phi_i\}^T [\rho] \{\phi_j\} dvol = 0$.

If $i = j$, then $\int_{vol} \{\phi_i\}^T [\rho] \{\phi_i\} dvol = C$,

where C is a constant.

We can normalize the eigenfunction by dividing $\{\phi_i\}$ by $\sqrt{C}$ and we have,

$$\int_{vol} \{\phi_i\}^T [\rho] \{\phi_i\} dvol = 1 ,$$

The normal modes satisfy the governing equations of motion completely, and therefore, if they are chosen as generalized coordinate functions, the governing equations will not be violated.

3.5.3 SERIES FORM OF DYNAMIC MATRICES

Having discussed the background materials, we proceed on to the improvement of the fundamental matrices.
Assume the solutions $[a_i(x,y,z)]$ of

$$[L]\{a_i\} = \{0\} \qquad \text{in the region of the body} \qquad (3.5.17)$$

and $$[B_i]\{a_j\} = \delta_{ij} \qquad \text{on the boundaries} \qquad (3.5.18)$$

$i=1,2,\ldots,N$, the number of gen. coord.,

are known and the eigenfunctions $\{\phi_j(x,y,z)\}$ of

$$[L]\{\phi_j\} = \lambda_j[\rho]\{\phi_j\} \qquad \text{in the region of the body} \qquad (3.5.19)$$

$$[B_i]\{\phi_j\} = 0 \qquad \text{on the boundaries} \qquad (3.5.20)$$

are given. We are going to solve for $\{u(x,y,z)\}$ of the equations

$$[L]\{u\} = \lambda[\rho]\{u\} \qquad \text{in the region of the body} \qquad (3.5.21)$$

$$[B_i]\{u\} = q_i \qquad \text{on the boundaries} \qquad (3.5.22)$$

$i=1,2,\ldots,N$

in terms of $\{a_i\}$ and $\{\phi_j\}$.

We first look for some conditions for the constants α_j such that we can express the solution $\{u\}$ in the form

$$\{u\} = \sum_{j=1}^{\infty} \alpha_j\{\phi_j\} + \sum_{i=1}^{N} \{a_i\} q_i \qquad (3.5.23)$$

Substituting equation (3.5.23) into equation (3.5.22), we find that all the boundary conditions are satisfied. Therefore we substitute equation (3.5.23) into equation(3.5.21) to see whether the differential equation is satisfied.

We have $$\sum_{j=1}^{\infty} \alpha_j[L]\{\phi_j\} + \sum_{i=1}^{N} q_i[L]\{a_i\} = \lambda\sum_{j=1}^{\infty} \alpha_j[\rho]\{\phi_j\} + \lambda\sum_{i=1}^{N} q_i[\rho]\{a_i\}$$

or $$\sum_{j=1}^{\infty} \alpha_j\lambda_j[\rho]\{\phi_j\} = \lambda\sum_{j=1}^{\infty} \alpha_j[\rho]\{\phi_j\} + \lambda\sum_{i=1}^{N} q_i[\rho]\{a_i\} \qquad (3.5.24)$$

To apply the condition of orthogonality, we pre-multiply equation (3.5.24) by $\{\phi_k\}^T$ and integrate over the whole volume, we have

$$\alpha_j\lambda_j = \lambda\alpha_j + \lambda\sum_{i=1}^{N} q_i G_{ij} \qquad (3.5.25)$$

where we have assumed that the eigenfunctions are normalized and that

$$\{a_i\} = \sum_{j=1}^{\infty} G_{ij}\{\phi_j\} \qquad (3.5.26)$$

where $$G_{ij} = \int_{vol} \{\phi_j\}^T[\rho]\{a_i\}\, d\,vol \qquad (3.5.27)$$

Then, from equation (3.5.25), the constants α_j are given by

$$\alpha_j = \frac{\lambda}{\lambda_j - \lambda} \sum_{i=1}^{N} G_{ij} q_i \qquad (3.5.28)$$

Substituting equations(3.5.28) and (3.5.26) into (3.5.23), we have

$$\{u\} = \sum_{j=1}^{\infty} \left(\frac{\lambda}{\lambda_j - \lambda} \{\phi_j\} \sum_{i=1}^{N} G_{ij} q_i \right) + \sum_{i=1}^{N} q_i \{a_i\}$$

$$= \sum_{i=1}^{N} \left(\{x_i\} + \{a_i\} \right) q_i = [a]\{q\}, \qquad (3.5.29)$$

where $$\{x_i\} = \sum_{j=1}^{\infty} \frac{\lambda G_{ij}}{\lambda_j - \lambda} \{\phi_j\}, \qquad (3.5.30)$$

$$[a] = [\{x_1\} + \{a_1\} \quad \{x_2\} + \{a_2\} \cdots \{x_N\} + \{a_N\}].$$

Therefore, we have expressed the displacement pattern in terms of generalized coordinates. In the formulae (3.5.29), $\{a_i\}$ are assumed shape functions compatible with the boundary conditions of the body and satifying the governing equations of vibration when $\lambda = 0$. One of the choices are those functions used in finite element methods. The expansion of $\{a_i\}$ in terms of the eigenfunctions $\{\phi_j\}$ is convergent. The shape functions are adjusted to suit the governing equation (3.5.21) through equation (3.5.29). The solution vector represented by equation (3.5.29) satisfies both the governing equations and the boundary conditions. Finally, we can take as many terms of correction as we wish in $\{x_i\}$ without affecting the size of the dynamic matrices.

For the dynamic stiffness matrix, we proceed as follows. Assume the dynamic matrix $[D_0]$ and the consistent mass matrix $[m_0]$ corresponding to the assumed "static" displacement patterns (3.5.4) are known. We apply the reciprocal theorem to two boundary force systems $\{F_0\}$ and $\{F\}e^{i\omega t}$. $\{F_0\}$ denotes static force system and $\{F\}e^{i\omega t}$ harmonic force system with frequency ω. Then, we have, from equation (3.4.16),

$$[D] = [D_0] - \omega^2 \int_{vol} [a_0]^T [\rho] [a] \, dvol \qquad (3.5.31)$$

where

$[\mathcal{D}]$ = the dynamic stiffness matrix to be determined,

$[\mathcal{D}_0]$ = the static stiffness matrix corresponding to the "static" displacement pattern,

$$[a] = [\{x_1\}+\{a_1\} \quad \{x_2\}+\{a_2\} \quad \cdots \quad \{x_N\}+\{a_N\}]$$
$$= [\{x_1\} \quad \{x_2\} \quad \cdots \quad \{x_N\}] + [\{a_1\} \quad \{a_2\} \quad \cdots \quad \{a_N\}]$$
$$= [x] + [a_0], \tag{3.5.32}$$

$$[x] = [\{x_1\} \quad \{x_2\} \quad \cdots \quad \{x_N\}] \tag{3.5.33}$$

Substituting equation (3.5.32) into (3.5.31), we have

$$[\mathcal{D}] = [\mathcal{D}_0] - \omega^2[m_0] - \omega^2 \int_{vol} [a_0]^T [\rho] [x] \, dvol \tag{3.5.34}$$

where $[m_0] = \int_{vol} [a_0]^T [\rho] [a_0] \, dvol$

By means of equations (3.5.33) the last term in equation (3.5.34) can be evaluated as follows. Let

$$[J] = \int_{vol} [a_0]^T [\rho] [x] \, dvol$$
$$= \int_{vol} [\{a_1\} \quad \{a_2\} \cdots \{a_N\}]^T [\rho] [\{x_1\} \quad \{x_2\} \cdots \{x_N\}] \, dvol$$
$$= \left[\int_{vol} \{a_i\}^T [\rho] \{x_j\} \, dvol \right]$$

Because of equations (3.5.30) and (3.5.26),

$$[J] = \left[\int_{vol} \{a_i\}^T [\rho] \left(\sum_{k=1}^{\infty} \frac{\lambda G_{jk}}{\lambda_k - \lambda} \{\phi_k\} \right) dvol \right]$$
$$= \left[\sum_{k=1}^{\infty} \frac{\lambda G_{jk}}{\lambda_k - \lambda} \int_{vol} \{a_i\}^T [\rho] \{\phi_k\} \, dvol \right]$$
$$= \left[\sum_{k=1}^{\infty} \frac{\lambda G_{jk} G_{ik}}{\lambda_k - \lambda} \right]$$

Therefore, from equation (3.5.34)

$$[\mathcal{D}] = [\mathcal{D}_0] - \omega^2 [m_0] - \omega^4 \sum_{k=1}^{\infty} \left[\frac{G_{jk} G_{ik}}{\omega_k^2 - \omega^2} \right] \tag{3.5.35}$$

For mass matrix, we apply theorem (3.4.1)

$$[m] = -\frac{1}{2\omega} \frac{\partial}{\partial \omega} [\mathcal{D}]$$
$$= [m_0] + \sum_{k=1}^{\infty} \left(\frac{\omega^4}{(\omega_k^2 - \omega^2)^2} + \frac{2\omega^2}{\omega_k^2 - \omega^2} \right) [G_{ik} G_{jk}] \tag{3.5.35a}$$
$$= [m_0] + \sum_{k=1}^{\infty} \left(\left(\frac{\omega_k}{\omega}\right)^2 + 1 \right) \Big/ \left(\left(\frac{\omega_k}{\omega}\right)^2 - 1 \right)^2 [G_{ik} G_{jk}].$$

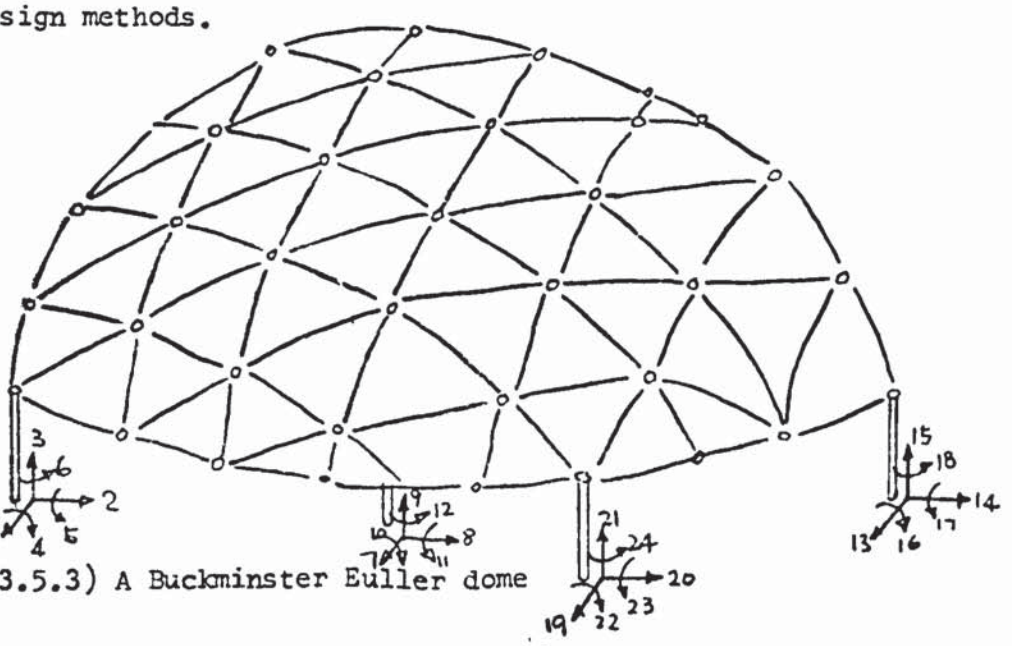
In the following subsections, we shall compare the performance of the finite element method and the present method for beam members in vibration analysis. A general consideration of applying these formulae is given below.

In the development of equations(3.5.35), we have made no assumption about the size and the shape of an elemental member as long as we are able to determine the normal modes of the member. For skeletal members, the number of generalized coordinates on the boundar-ies is finite, the equations (3.5.35) will give exact dynamic matrices because the eigenfunctions are always obtainable by the computer programme in chapter six on the assumption that the computer is big enough to handle the matrices of such an element. For example, we can apply the formula(3.5.36) to a space frame as shown in Fig (3.5.3), where the four free ends are going to be connected to the other part of a structure, say the earth in earthquake response problems, then an exact dynamic stiffness matrix of order 24 may be constructed because it is possible to determine the normal modes of such a structure with the four free ends clamped by means of the computer programme in chapter six and to calculate the static displacement patterns when each of the 24 generalized coordinates is displaced by a unit, one at a time. With these displacement patterns and eigenfunctions, the constants G_{ij} in equation (3.5.27) can be obtained once and for all for such a particular giant structural member. The internal degrees of freedom, which are actually not free to move but depend on the 24 generalized coordinates and the frequency of vibration, are taken by the use of the eigenfunctions already.

The major advantages of forming the dynamic stiffness matrix of a substructure in this way over the method of condensation (ref8), which is

one of the most effective methods for substructural analysis, are in the following three three phases. (i) The method of condensation requires elimination processes to reduce the order of the frequency determinant. These processes introduce rounding off error and are numerically unstable about the natural frequencies and the frequencies of anti-resonance, but the present method required no elimination of equations and thus saves computing time. (ii) The method of condensation destroys the Sturm Sequence property of a vibrating system (see chapter five) and the present method does not. Therefore the numerical methods of the vibration problem presented in chapter five are valid for the resulting dynamic stiffness matrix. (iii) Finally, although the method of condensation could make use of backing storage for its large storage requirement, yet it is a time consuming process. The only storage requirement for the present method is for the constants ω_k and G_{ik}. Experience (as discussed in next subsection) show that six terms of normal modes is enough to give eight digit accuracy of the dynamic stiffness matrix and therefore the storage used is very limited.

The penalty of the present method is the computation of normal modes and the integration for the constants G_{ij}. This is very well balanced if the same structure is going to be used more than one time and it is often the case in design methods.



Fig(3.5.3) A Buckminster Euller dome

This approach is particularly useful when the gorvening equations are not known or these equations are very difficult to solve but the eigenfunctions can be obtained by experiment.

3.5.4 STRAIGHT BEAM ELEMENTS

We now give some numerical examples of the application of formulae (3.5.35) in forming the dynamic stiffness matrices. We shall begin with a simple straight beam element where we can compare our results with the exact one. Then examples of two- and three-dimensional elements will also be given.

Consider a conventional beam element as shown in fig (3.5.4).

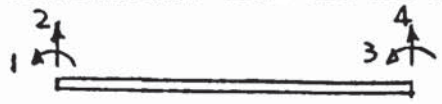


Fig (3.5.4) A uniform beam element

The row matrix of static deflection patterns corresponding to unit generalized coordinates is

$$[a(\xi)] = [a_1(\xi) \quad a_2(\xi) \quad a_3(\xi) \quad a_4(\xi)]$$
$$= [1-3\xi^2+2\xi^3 \quad (\xi-2\xi^2+\xi^3)l \quad 3\xi^2-2\xi^3 \quad (-\xi^2+\xi^3)l] \qquad (3.5.36)$$

where $\xi = x/l$, and a_i, i=1,2,3,4 are scalar functions of ξ.

The eigenfunctions of a clamped-clamped beam are given by

$$\phi_r(\xi) = \cosh\lambda_r\xi - \cos\lambda_r\xi - \sigma_r(\sinh\lambda_r\xi - \sin\lambda_r\xi) \qquad (3.5.37)$$

where the values of λ_r and σ_r are given in equation (3.5.11).

The constants G_{ij}, i=1,2,3,4; j=1,2,3, ... are calculated from equations (3.5.27), i.e.

$$G_{ij} = \rho A l \int_0^1 a_i(\xi)\,\phi_j(\xi)\,d\xi$$

The integrals $I_{ij} = \int_0^1 a_i(\xi)\,\phi_j(\xi)\,d\xi$

have been evaluated up to 50 modes by using a computer programme described in chapter six and listed in appendix 16. Since by symmetry,

$$I_{3j} = (-1)^{j+1} I_{1j}$$
$$I_{4j} = (-1)^{j} I_{2j},$$

we just list the integrals I_{1j} and I_{2j} in fig (3.5.5).

j	I_{1j}	I_{2j}
1	0.415430762	0.039309305
2	0.254571965	0.032409195
3	0.181992796	0.015542153
4	0.141471231	0.010057033
5	0.115749042	0.006099941
6	0.097941504	0.004796069
7	0.084882636	0.003602531
8	[illegible]	[illegible]
9	0.067012693	0.002245345
10	0.060630455	0.001830006
11	0.055358241	0.001530037
12	0.050992592	0.001296011
13	0.047157200	0.001111093
14	0.043704212	0.000963316
15	0.041079243	0.000843065
16	0.038533917	0.000744205
17	0.036379273	0.000661603
18	0.034411790	0.000592029
19	0.032647163	0.000532019
20	0.031054695	0.000480195
21	0.029610922	0.000435373
22	0.028294212	0.000400921
23	0.027090203	0.000362040
24	0.025984481	0.000337597
25	0.024965431	0.000311639
26	0.024023338	0.000288562
27	0.023149810	0.000267257
28	0.022337536	0.000249483
29	0.021580331	0.000232[illegible]
30	0.020872772	0.000217736
31	0.020210152	0.000204025
32	[illegible]	0.000191[illegible]
33	0.019003575	0.000180562
34	0.018452747	0.000170239
35	0.017930951	0.000160795
36	0.017441639	0.000152165
37	0.016976507	0.000144191
38	0.016535579	0.000136713
39	0.016116956	0.000129773
40	0.015719097	0.000123544
41	0.015340035	0.000117601
42	0.014979070	0.000111914
43	0.014634937	0.000107421
44	0.014306862	0.000102292
45	0.013991643	0.000097[illegible]3
46	0.013690748	0.000093318
47	0.013405569	[illegible]
48	0.013126171	0.000086147
49	0.012861006	0.000082[illegible]
50	0.012606302	0.000079169

Fig (3.3.5) The projection of the static curves $a_i(\xi)$ onto the normal modes $\phi_j(\xi)$ of the same beam

Therefore the dynamic stiffness matrix of such a beam is given by eq(3.5.35) as,

$$[\mathcal{D}] = \frac{EI}{l^3}\begin{pmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{pmatrix} - \frac{\omega^2 \rho A l}{420}\begin{pmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{pmatrix}$$

$$- \omega^4 \rho^2 A^2 l^2 \sum_{k=1}^{\infty}\left(\frac{I_{ik} I_{jk}}{\omega_k^2 - \omega^2}\right) \qquad (3.5.38)$$

The lowest natural frequencies of a cantilever beam were calculated by the method using only one element. The percentage error vis the number of terms taken is plotted in fig (3.5.6). We see that the error reduces as the number of terms taken increases and that six terms in equation (3.5.8) will give satisfactory results.

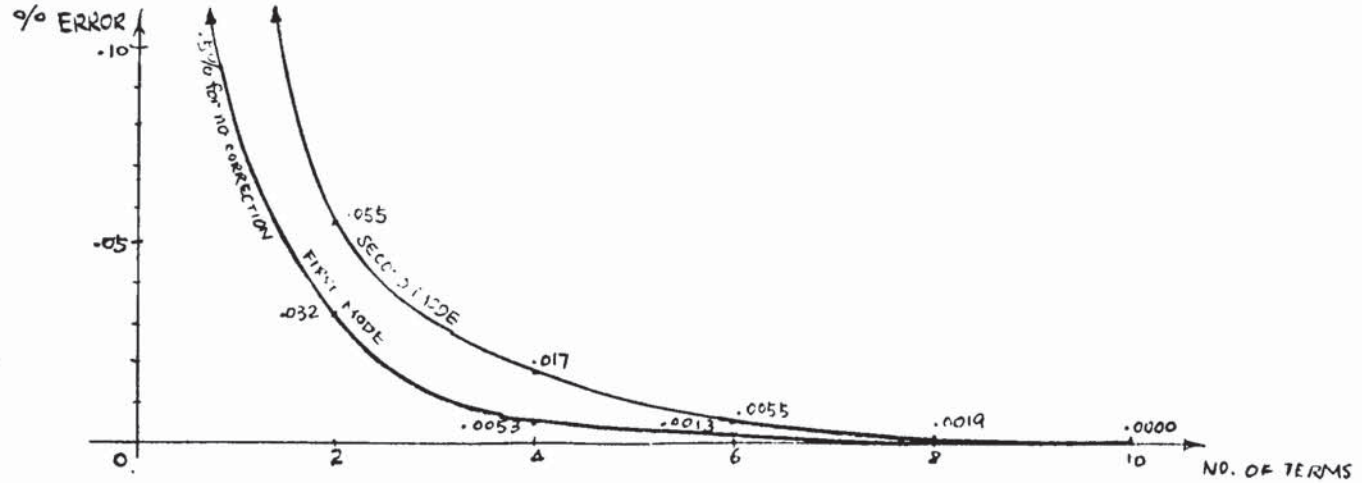


Fig (3.5.6) Percentage error vs number of terms taken to calculate the natural frequencies of a cantilever beam.

One of the most popular ways of finding the approximate eigenfunctions of a complicated member is the method of Rayleigh-Ritz (ref 16). In the following, the dynamic stiffness matrix of a tapered beam whose flexural rigidity EI and density ρA are varying according to

$$EI(x) = EI_0 \sum_{k=0}^{n_1} \alpha_k x^k \qquad (3.5.39)$$

and

$$\rho A(x) = \rho A_0 \sum_{k=0}^{n_2} \beta_k x^k \qquad (3.5.40)$$

where EI_0, ρA_0, $\alpha_k < 1$, $k=1,2,\ldots,n_1$, $\beta_k < 1$, $k=1,2,\ldots,n_2$, are known constants, and $\alpha_0 = \beta_0 = 1$, will be calculated with the help of the Rayleigh-

Ritz method. If we express a typical normal mode of the tapered beam with both ends clamped in the following form

$$\psi(x) = \sum_{i=1}^{n} A_i \phi_i(x) \tag{3.5.41}$$

where A_i are constants to be determined and $\phi_i(x)$ are the normal modes of a uniform beam of rigidity EI_o and density ρA_o. The maximum strain energy and kinetic energy are given by

$$U = \int_o^l EI(x) \sum_{i,j=1}^{n} A_i \phi_i''(x) A_j \phi_j''(x) dx$$
$$= \sum_{i,j=1}^{n} A_i A_j \int_o^l EI_o \sum_{k=0}^{n_1} x^k \alpha_k \phi_i'' \phi_j'' dx$$
$$T = \int_o^l \rho A(x) \sum_{i,j=1}^{n} A_i \phi_i(x) A_j \phi_j(x) dx$$
$$= \sum_{i,j=1}^{n} A_i A_j \int_o^l \rho A_o \sum_{k=0}^{n_2} x^k \beta_k \phi_i \phi_j dx$$

where l is the overall length of the beam. Rayleigh's theorem states that the following quotient is stationary and equals the square of the corresponding natural frequency if $\psi(x)$ in equation (3.5.41) is a normal mode of vibration,

$$\omega^2 = \frac{\sum_{i,j=1}^{n} A_i A_j \alpha_{ij}}{\sum_{i,j=1}^{n} A_i A_j \beta_{ij}} \tag{3.5.42}$$

where
$$\alpha_{ij} = \int_o^l EI_o \sum_{k=0}^{n_1} x^k \alpha_k \phi_i'' \phi_j'' dx \tag{3.5.43}$$
$$\beta_{ij} = \int_o^l \rho A_o \sum_{k=0}^{n_2} x^k \beta_k \phi_i \phi_j dx \tag{3.5.44}$$

A computer program was designed and presented in section (6.10) to evaluate the integrals (3.5.43) and (3.5.44). For k=0,1,2,3,4,5,6, these integrals are explicitly calculated and stored in a data file, the details can be found in chapter six.

The necessary condition for ω^2 in equation(3.5.42) to be stationary is
$$[\alpha_{ij} - \omega^2 \beta_{ij}]\{A_j\} = \{0\}. \tag{3.5.45}$$
The solution of the algebraic eigenvalue problem will give the natural frequencies ω_k and the corresponding eigenvectors $\{A_j^{(k)}\}$.

For the constants G_{ij} in the equation (3.5.35) for dynamic matrices we assume the static deflections of the tapered beam are approximately

those of the equivalent uniform beam. Then, G_{ij} is given by equation (3.5.27) as

$$G_{ik} = \frac{\int_0^l \rho A(x)\, a_i(x)\, \bar{\psi}_k(x)\, dx}{\int_0^l \bar{\psi}_k^2(x)\, dx} \qquad (3.5.46)$$

where $\bar{\psi}_k(x) = \sum_{j=1}^{n} A_j^{(k)} \phi_j(x)$ is the approximation to k^{th} mode of the tapered beam.

The expression (3.5.46) can be written as follows,

$$G_{ik} = \int_0^l \rho A_0 \Big(\sum_{m=0}^{n_2} \beta_m x^m\Big)\, a_i(x) \Big(\sum_{j=1}^{n} A_j^{(k)} \phi_j(x)\Big) dx \Big/ \int_0^l \Big(\sum_{j=1}^{n} A_j^{(k)} \phi_j\Big)^2 dx$$

$$= \rho A_0 \sum_{m=0}^{n_2} \sum_{j=1}^{n} \beta_m A_j^{(k)} \int_0^l x^m a_i(x)\, \phi_j(x) dx \Big/ \sum_{i=1}^{n} (A_i^{(k)})^2 \qquad (3.5.47)$$

And the computation is reduced to the eigenvalue problem of equation (3.5.45) for the constants $A_j^{(k)}$, if the integrals $\int_0^l x^m \phi_j(x)\, dx$ are available. We are going to study some common structural members more detailly in the following sections.

3.6 ONE DIMENSIONAL ELEMENTS

One dimensional elements are examplified by beam elements. A beam element is usually represented by its centroidal axis and analyzed as if it were a line element. It may be straight or curved, and have uniform or non-uniform properties over its length. After the time of Euler and Bernoulli the literature concerning the vibrations of a uniform straight beam is plentiful. However, the study of the vibrations of complex structural systems consisting of beams were not found until the second world war. Bishop and Johnson (ref 10) gave explicit formulae to analyze the vibrations of uniform straight beam structures by the method of receptances. This method has been extended to slightly internally damped structures of beams by Snowdon (ref 83) using complex elastic modulus. The method of receptance

expresses displacements in terms of boundary forces. A dual method to the receptance is called the dynamic stiffness method which expresses forces in terms of displacements. A comparison of the methods was given by Simpson (ref 84).

As far as tapered beam elements are concerned, Sanger (ref 86) gave explicit formulae of the receptances for some special forms of tapering. Leung (ref 85) showed the formulation of the dynamic stiffness matrices for the same types of beams. Kolousek (ref 87) expressed the dynamic stiffness matrix in determinantal forms. Explicit formulae for a general tapered beam element has not been found so far.

For curved beams, explicit formulae for deflections are obtained for curved uniform beam members only. These can be found from (ref 37) and (ref 38). The governing equations of motion for a generally curved beam in space was given by Massoud (ref 88)

When cosidering the secondary effects, Howson and Williams gave the dynamic stiffness matrix for a beam member when the effects of axial loads, shear deformation and rotatory inertia are taken into account. Rao and Rao (ref 89) solved the longitudinal vibration of a beam including the effects of shear and lateral inertia. Carr (ref 90) obtained the torsional modes when the effect of warping is considered.

Since the frequency dependent mass matrices for structural members are new in literature, therefore we list the mass and dynamic stiffness matrices for some common one-dimensional elements is this section. These will include uniform straight beams, tapered beams, circularly curved beams and straight beams with secondary effect considered.

3.6.1 UNIFORM BEAM ELEMENTS

The simplest one dimensional element is a uniform straight beam element. As shown in fig (3.6.1), for the convenience of computation we choose the local coordinate axes to be coincident with the principal axes of the beam, such that the controid of the cross sectional area is on the x axis.

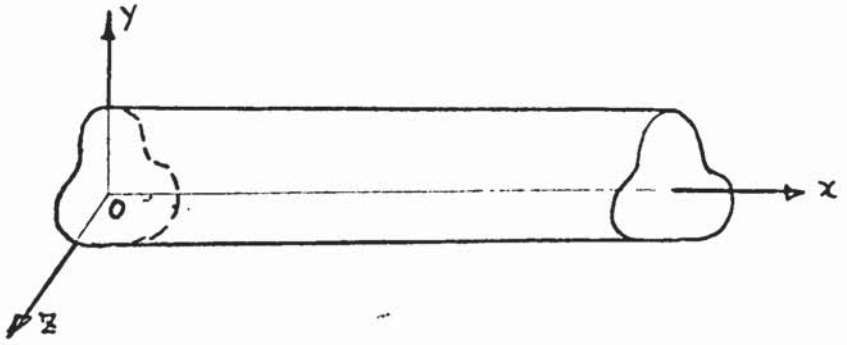


Fig (3.6.1) A uniform beam member

There are four types of vibrations for this simple element, namely, flexural vibration in xy plane, flexural vibration in xz plane, longitudinal vibration in x direction and torsional vibration along the beam. These vibrations are uncoupled when the vibration amplitudes are small and when the shear centre of the cross sectional area is coincident with its controid.

When this beam is vibrating at a frequency ω , then it is approximated by a straight member as shown in the fig (3.6.2). The symbols used in the subsection are explained below:

ω = vibrating frequency

l = length of the beam

A_o = cross section of the beam

E = Young's modulus

ρ = density per unit volume

I_y = moment of inertia of A_o about y axis

I_z = moment of inertia of A_o about z axis

J = torsional constant of A_o

kA_o = effective shear cross sectional area

G = shear modulus

$\{u\}e^{i\omega t} = [u, v, w]^T e^{i\omega t}$ = displacement vector

$\{q\}e^{i\omega t}$ = generalized displacement vector

$\{Q\}e^{i\omega t}$ = generalized force vector

I_o = polar moment of inertia of area A_o

ν = Poisson's ratio

$\theta e^{i\omega t}$ = angle of twist

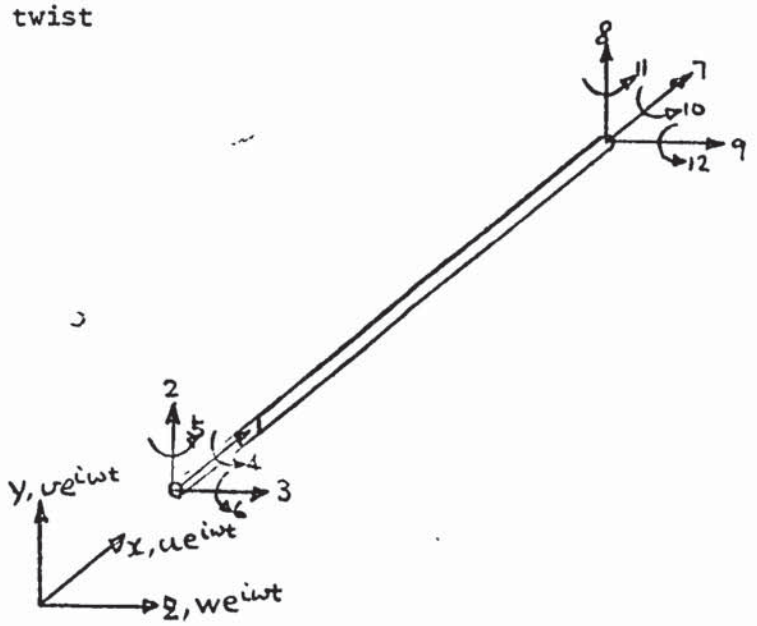


Fig (3.6.2) Generalized coordinates of a straight beam member

The governing equations of motion, for vibration with boundary forces only, have been obtained from Hamilton's principle as below:

(1) Flexural vibration in xy plane,

$$EI_z \frac{d^4v}{dx^4} - \rho A_o \omega^2 v = 0$$

with boundary conditions

$v(0)=q_2$, $v(\ell)=q_8$, $v'(0)=q_6$, $v'(\ell)=q_{12}$;

(2) flexural vibration in xz plane,

$$EI_y \frac{d^4w}{dx^4} - \rho A_0 \omega^2 w = 0$$

with boundary conditions $w(0)=q_3$, $w(l)=q_9$, $w'(0)=q_5$, $w'(l)=q_{11}$;

(3) longitudinal vibration,

$$EA_0 \frac{d^2u}{dx^2} + \rho A_0 \omega^2 u = 0$$

with boundary conditions $u(0)=q_1$, $u(l)=q_7$;

(4) torsional vibration

$$GJ \frac{d^2\theta}{dx^2} + \rho I_0 \omega^2 \theta = 0$$

with boundary conditions $\theta(0)=q_4$, $\theta(l)=q_{10}$;

The solutions of these differential equations in terms of generalized coordinates $q_1, q_2, \ldots q_{12}$ are given by

(1) $v(x) = A_1 \cos\lambda_z\xi + B_1 \sin\lambda_z\xi + C_1 \cosh\lambda_z\xi + D_1 \sinh\lambda_z\xi$

where $\xi = \frac{x}{l}$, $\lambda_z^4 = \rho A_0 \omega^2 l^4 / EI_z$

and $A_1 = (-F_4(\lambda_z)q_2/l + F_2(\lambda_z)q_6 - F_3(\lambda_z)q_8/l + F_1(\lambda_z)q_{12})l/2\lambda_z^2 + q_2/2$

$B_1 = (-F_6(\lambda_z)q_2/l + F_4(\lambda_z)q_6 - F_5(\lambda_z)q_8/l - F_3(\lambda_z)q_{12})l/2\lambda_z^3 + lq_6/2\lambda_z$

$C_1 = -A_1 + q_2$, $D_1 = -B_1 + q_6 l/\lambda_z$

and the frequency functions are shown in fig(3.5.3);

(2) $w(x) = A_2 \cos\lambda_y\xi + B_2 \sin\lambda_y\xi + C_2 \cosh\lambda_y\xi + D_2 \sinh\lambda_y\xi$

where $\xi = x/l$, $\lambda_y^4 = \rho A_0 \omega^2 l^4 / EI_y$

and $A_2 = [-F_4(\lambda_y)q_3/l + F_2(\lambda_y)q_5 - F_3(\lambda_y)q_9/l + F_1(\lambda_y)q_{11}]l/2\lambda_y^2 + q_3/2$

$B_2 = [-F_6(\lambda_y)q_3/l + F_4(\lambda_y)q_5 - F_5(\lambda_y)q_9/l - F_3(\lambda_y)q_{11}]l/2\lambda_y^3 + lq_5/2\lambda_y$

$C_2 = -A_2 + q_3$, $D_2 = -B_2 + q_5 l/\lambda_y$

and the frequency functions are shown in fig(3.6.3);

(3) $u(x) = A_3 \cos\psi\xi + B_3 \sin\psi\xi$

where $\xi = x/l$, $\psi^2 = l^2 \rho A_0 \omega^2 / EA_0$,

$A_3 = q_1$, $B_3 = -q_1 \cot\psi + q_7 \csc\psi$;

(4) $\theta(x) = A_4 \cos\gamma\xi + B_4 \sin\gamma\xi$

where $\xi = x/\ell$, $\gamma^2 = \ell^2 \rho I_o \omega^2 / GJ$,

$A_4 = q_4$, $B_4 = -q_4 \cot\gamma + q_{10} \csc\gamma$.

The generalized forces and generalized displacements are related according to

$$\begin{Bmatrix} Q_2 \\ Q_6 \\ Q_8 \\ Q_{10} \end{Bmatrix} = \frac{EI_y}{\ell^3} \begin{bmatrix} F_6(\lambda_y) & -F_4(\lambda_y)\ell & F_5(\lambda_y) & F_3(\lambda_y)\ell \\ -F_4(\lambda_y)\ell & F_2(\lambda_y)\ell^2 & -F_3(\lambda_y)\ell & F_1(\lambda_y)\ell^2 \\ F_5(\lambda_y) & -F_3(\lambda_y)\ell & F_6(\lambda_y) & F_4(\lambda_y)\ell \\ F_3(\lambda_y)\ell & F_1(\lambda_y)\ell^2 & F_4(\lambda_y)\ell & F_2(\lambda_y)\ell^2 \end{bmatrix} \begin{Bmatrix} q_2 \\ q_6 \\ q_8 \\ q_{12} \end{Bmatrix}$$

$$\begin{Bmatrix} Q_3 \\ Q_5 \\ Q_9 \\ Q_{11} \end{Bmatrix} = \frac{EI_z}{\ell^3} \begin{bmatrix} F_6(\lambda_z) & -F_4(\lambda_z)\ell & F_5(\lambda_z) & F_3(\lambda_z)\ell \\ -F_4(\lambda_z)\ell & F_2(\lambda_z)\ell^2 & -F_3(\lambda_z)\ell & F_1(\lambda_z)\ell^2 \\ F_5(\lambda_z) & -F_3(\lambda_z)\ell & F_6(\lambda_z) & F_4(\lambda_z)\ell \\ F_3(\lambda_z)\ell & F_1(\lambda_z)\ell^2 & F_4(\lambda_z)\ell & F_2(\lambda_z)\ell^2 \end{bmatrix} \begin{Bmatrix} q_3 \\ q_5 \\ q_9 \\ q_{11} \end{Bmatrix}$$

$$\begin{Bmatrix} Q_1 \\ Q_7 \end{Bmatrix} = \frac{EA_o}{\ell} \begin{bmatrix} \psi\cot\psi & -\psi\csc\psi \\ -\psi\csc\psi & \psi\cot\psi \end{bmatrix} \begin{Bmatrix} q_1 \\ q_7 \end{Bmatrix}$$

$$\begin{Bmatrix} Q_4 \\ Q_{10} \end{Bmatrix} = \frac{GJ}{\ell} \begin{bmatrix} \gamma\cot\gamma & -\gamma\csc\gamma \\ -\gamma\csc\gamma & \gamma\cot\gamma \end{bmatrix} \begin{Bmatrix} q_4 \\ q_{10} \end{Bmatrix}$$

The complete dynamic stiffness matrix $[\mathcal{D}]$ which relates the generalized forces and displacements, i.e.

$$\{Q\} = [\mathcal{D}]\{q\}$$

where $\{Q\} = [Q_1\ Q_2 \cdots Q_{12}]^T$

and $\{q\} = [q_1\ q_2 \cdots q_{12}]^T$

is given by equation (3.6.1) and the mass matrix $[m]$ by equation (3.6.2). The frequency functions F_i, G_i and $\psi_1, \psi_2, \gamma_1, \gamma_2$ are listed in fig (3.6.3). The torsion constants J for several shapes of cross are listed in fig (3.6.3a).

Torsional Constants J

If a circular bar of constant section and of length l is subjected to a constant torque T, the angle of twist between the two bar ends is

$$\theta = \frac{Tl}{GJ}$$

where G is the shear modulus and J the polar moment of inertia.

When the cross section of the bar is noncircular, plane cross sections do not remain plane after deformation and warping will occur caused by longitudinal displacements of points in the cross section. Nevertheless, the above equation can be used with good accuracy for noncircular cross sections, but J should be taken as the appropriate torsion constant. The torsion constants for several shapes of cross sections are listed below.

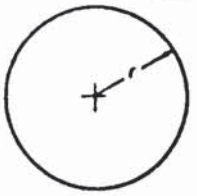
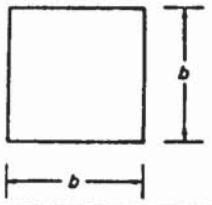
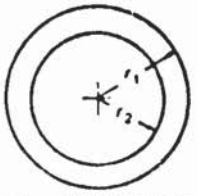
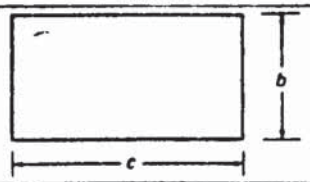
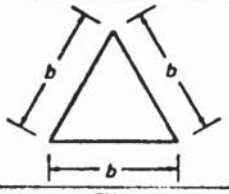
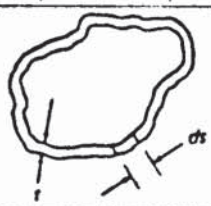
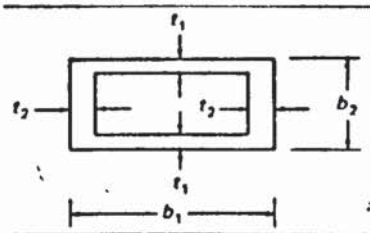
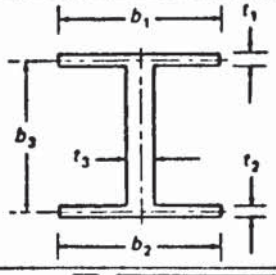
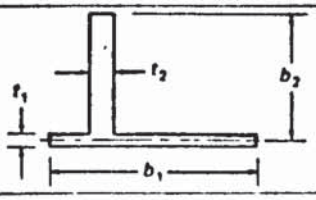
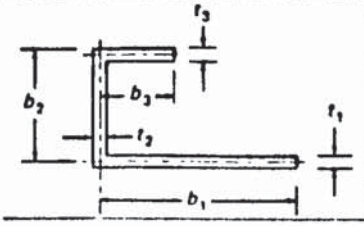
Section	Torsional Constant J
	$J = \frac{\pi r^4}{2}$
	$J = 0.1406b^4$
	$J = \frac{\pi(r_2^4 - r_1^4)}{2}$

Fig (3.6.2a) Torsional constants

Section	Torsional Constant J
	$J = cb^3\left[\frac{1}{3} - 0.21\frac{b}{c}\left(1 - \frac{b^4}{12c^4}\right)\right]$
	$J = \frac{b^4\sqrt{3}}{80}$
	Closed section $J = \frac{4a^2}{\int \frac{ds}{t}}$ where a = area enclosed by a line through the center of the thickness and the integral is carried out over the circumference.
	$J = \frac{2t_1t_2(b_1 - t_2)^2(b_2 - t_1)^2}{b_1t_2 + b_2t_1 - t_2^2 - t_1^2}$
	$J = \frac{b_1t_1^3 + b_2t_2^3 + b_3t_3^3}{3}$
	$J = \frac{b_1t_1^3 + b_2t_2^3}{3}$
	Open section composed of rectangles $J = \frac{1}{3}\Sigma b_it_i^3$

$$[\vartheta] =$$

$$\begin{bmatrix}
\frac{EA}{l}\psi\cot\psi & 0 & 0 & 0 & 0 & 0 & -\frac{EA}{l}\psi\csc\psi & 0 & 0 & 0 & 0 & 0 \\
 & \frac{EI_3}{l^3}F_6(\lambda_2) & 0 & 0 & 0 & -\frac{EI_2}{l^2}F_4(\lambda_2) & 0 & \frac{EI_2}{l^3}F_5(\lambda_2) & 0 & 0 & 0 & \frac{EI_3}{l^2}F_3(\lambda_2) \\
 & & \frac{EI_y}{l^3}F_6(\lambda_y) & 0 & \frac{EI_y}{l^2}F_4(\lambda_y) & 0 & 0 & 0 & \frac{EI_y}{l^3}F_5(\lambda_y) & 0 & -\frac{EI_y}{l^2}F_3(\lambda_y) & 0 \\
 & & & \frac{GJ}{l}\gamma\cot\gamma & 0 & 0 & 0 & 0 & 0 & -\frac{GJ}{l}\gamma\csc\gamma & 0 & 0 \\
 & & & & \frac{EI_y}{l}F_2(\lambda_y) & 0 & 0 & 0 & \frac{EI_y}{l^2}F_3(\lambda_y) & 0 & \frac{EI_y}{l}F_1(\lambda_y) & 0 \\
 & & & & & \frac{EI_2}{l}F_2(\lambda_2) & 0 & -\frac{EI_2}{l^2}F_3(\lambda_2) & 0 & 0 & 0 & \frac{EI_2}{l^2}F_1(\lambda_2) \\
 & & & & & & \frac{EA}{l}\psi\cot\psi & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & \frac{EI_2}{l^3}F_6(\lambda_2) & 0 & 0 & 0 & \frac{EI_2}{l^2}F_4(\lambda_2) \\
 & & & & & & & & \frac{EI_y}{l^3}F_6(\lambda_y) & 0 & -\frac{EI_y}{l^2}F_4(\lambda_y) & 0 \\
 & & & & & & & & & \frac{GJ}{l}\gamma\cot\gamma & 0 & 0 \\
 & & & & & & & & & & \frac{EI_y}{l}F_2(\lambda_y) & 0 \\
 & & & & & & & & & & & \frac{EI_2}{l}F_2(\lambda_2)
\end{bmatrix}$$

SYMMETRICAL

(3.6.1)

$$[m] =$$

$$\begin{bmatrix}
\psi_1 & 0 & 0 & 0 & 0 & 0 & \psi_2 & 0 & 0 & 0 & 0 & 0 \\
 & \frac{G_6(\lambda_2)}{l^3} & 0 & 0 & 0 & -\frac{G_4(\lambda_2)}{l^2} & 0 & \frac{G_5(\lambda_2)}{l^3} & 0 & 0 & 0 & \frac{G_3(\lambda_2)}{l^2} \\
 & & \frac{G_6(\lambda_y)}{l^3} & 0 & \frac{G_4(\lambda_y)}{l^2} & 0 & 0 & 0 & \frac{G_5(\lambda_y)}{l^3} & 0 & -\frac{G_3(\lambda_y)}{l^2} & 0 \\
 & & & \gamma_1 & 0 & 0 & 0 & 0 & 0 & \gamma_2 & 0 & 0 \\
 & & & & -\frac{G_2(\lambda_y)}{l} & 0 & 0 & 0 & \frac{G_3(\lambda_y)}{l^2} & 0 & \frac{G_1(\lambda_y)}{l} & 0 \\
 & & & & & \frac{G_2(\lambda_2)}{l} & 0 & -\frac{G_3(\lambda_2)}{l^2} & 0 & 0 & 0 & \frac{G_1(\lambda_2)}{l} \\
 & & & & & & \psi_1 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & \frac{G_6(\lambda_2)}{l^3} & 0 & 0 & 0 & \frac{G_4(\lambda_2)}{l^2} \\
 & & & & & & & & \frac{G_6(\lambda_y)}{l^3} & 0 & -\frac{G_4(\lambda_y)}{l^2} & 0 \\
 & & & & & & & & & \gamma_1 & 0 & 0 \\
 & & & & & & & & & & \frac{G_2(\lambda_y)}{l} & 0 \\
 & & & & & & & & & & & \frac{G_2\lambda_2}{l}
\end{bmatrix}$$

SYMMETRICAL

(3.6.2)

$$F_1(\lambda) = -\lambda(\sinh\lambda - \sin\lambda)/\delta$$
$$F_2(\lambda) = -\lambda(\cosh\lambda\sin\lambda - \sinh\lambda\cos\lambda)/\delta$$
$$F_3(\lambda) = -\lambda^2(\cosh\lambda - \cos\lambda)/\delta$$
$$F_4(\lambda) = \lambda^2(\sinh\lambda\sin\lambda)/\delta$$
$$F_5(\lambda) = \lambda^3(\sinh\lambda + \sin\lambda)/\delta$$
$$F_6(\lambda) = -\lambda^3(\cosh\lambda\sin\lambda + \sinh\lambda\cos\lambda)/\delta$$
$$\delta = \cosh\lambda\cos\lambda - 1$$

$$G_1(\lambda) = (F_1(\lambda)F_2(\lambda) - F_3(\lambda) - F_1(\lambda))/4\lambda^4$$
$$G_2(\lambda) = (F_1(\lambda)F_1(\lambda) - F_2(\lambda))/4\lambda^4$$
$$G_3(\lambda) = -(F_1(\lambda)F_4(\lambda) + 2F_3(\lambda))/4\lambda^4$$
$$G_4(\lambda) = -(F_1(\lambda)F_3(\lambda) + 2F_4(\lambda))/4\lambda^4$$
$$G_5(\lambda) = (F_3(\lambda)F_4(\lambda) - 3F_5(\lambda))/4\lambda^4$$
$$G_6(\lambda) = (F_3(\lambda)F_3(\lambda) - 3F_6(\lambda))/4\lambda^4$$

These functions have a removable singularity at $\lambda=0$, i.e. when $\lambda=0$, they are in the forms of 0/0. Therefore when λ is small, the functions are expanded in polynomial forms to avoid the numerical instability when using computer to calculate these functions.

$$F_1(\lambda) = 2 + 0.007142857\lambda^4 + 0.000015704\lambda^8 + 0.000000032\lambda^{12}$$

$$F_2(\lambda) = 4 - 0.009523810\lambda^4 - 0.000016262\lambda^8 - 0.000000032\lambda^{12}$$

$$F_3(\lambda) = 3 + 0.030952381\lambda^4 + 0.000072193\lambda^8 + 0.000000148\lambda^{12}$$

$$F_4(\lambda) = -6 + 0.052380952\lambda^4 + 0.000076617\lambda^8 + 0.000000149\lambda^{12}$$

$$F_5(\lambda) = -12 - 0.128571429\lambda^4 - 0.000329571\lambda^8 - 0.000000684\lambda^{12}$$

$$F_6(\lambda) = 12 - 0.371428571\lambda^4 - 0.000364873\lambda^8 - 0.000000793\lambda^{12}$$

$$G_1(\lambda) = -0.007142857 - 0.000031408\lambda^4 - 0.000000095\lambda^8$$

$$G_2(\lambda) = 0.009523810 + 0.000032525\lambda^4 + 0.000000095\lambda^8$$

$$G_3(\lambda) = -0.030952381 - 0.000144386\lambda^4 - 0.000000443\lambda^8$$

$$G_4(\lambda) = -0.052380952 - 0.000153234\lambda^4 - 0.000000447\lambda^8$$

$$G_5(\lambda) = 0.128571429 + 0.000659142\lambda^4 + 0.000002053\lambda^8$$

$$G_6(\lambda) = 0.371428571 + 0.000729745\lambda^4 + 0.000002080\lambda^8$$

Fig(3.3.3) Frequency functions for a straight uniform beam

When λ is large, it has been found that the subtraction of two exponential functions involved in computing these functions will give rise to numerical instability, i.e., the number of useful significant figure is reduced. In this case, the following approximations where the exponential functions have been eliminated should be used.

$$F_1(\lambda) \doteq -\lambda/\cos\lambda$$
$$F_2(\lambda) \doteq \lambda/(1-\tan\lambda)$$
$$F_3(\lambda) \doteq -\lambda^2/\cos\lambda$$
$$F_4(\lambda) \doteq \lambda^2\tan\lambda$$
$$F_5(\lambda) \doteq \lambda^3/\cos\lambda$$
$$F_6(\lambda) \doteq -\lambda^3/(1+\tan\lambda)$$

For longitudinal and torsional vibrations, the functions of frequency used in the element mass matrix are given below,

$$\psi_1 = (\psi^2\csc^2\psi - \psi\cot\psi)/2\psi^2$$
$$\psi_2 = (\psi\csc\psi - \psi^2\csc\psi\cot\psi)/2\psi^2$$
$$\gamma_1 = (\gamma^2\csc^2\gamma - \gamma\cot\gamma)/2\gamma^2$$
$$\gamma_2 = (\gamma\csc\gamma - \gamma^2\csc\gamma\cot\gamma)/2\gamma^2$$

Fig (3.6.3) cont.

The above formulae are valid only when the loadings are presented on the boundaries. These can be modified to include loadings in the region of the beam as well. The idea is borrowed from that of receptance and, as far as the author is aware, is new in the stiffness analysis of structures.

Consider the flexural vibration in xy plane alone, as shown in fig (3.6.4). The loading $f(x)e^{i\omega t}$ may be distributed or concentrated bending moments or forces. The beam element in fig(3.6.4a) is separated in-to an equivalent system consisting of two identical beams, fig(b) and fig(c). The first one is subjected to boundary forces, $\bar{Q}_1$, $\bar{Q}_2$, $\bar{Q}_3$, $\bar{Q}_4$, and boundary displacements q_1, q_2, q_3, q_4 and the second one subjected to interior forces $f(x)$ as well as boundary forces P_1, P_2, P_3, P_4 but no boundary displacements. The displace-ment pattern and the fundamental matrices of the member in fig (b) are found as before as

$$\bar{u}(x) = \bar{A}\cos\lambda\xi + \bar{B}\sin\lambda\xi + \bar{C}\cosh\lambda\xi + \bar{D}\sinh\lambda\xi \qquad (3.6.3)$$

where $\bar{A} = [-F_4(\lambda)q_1/l + F_2(\lambda)q_2 - F_3(\lambda)q_3/l + F_1(\lambda)q_4]\, l/2\lambda^2 + q_1/2$

$\bar{B} = [-F_6(\lambda)q_1/l + F_4(\lambda)q_2 - F_4(\lambda)q_3/l - F_3(\lambda)q_4]\, l/2\lambda^3 + q_2 l/2\lambda$

$\bar{C} = -\bar{A} + q_1$, $\bar{D} = -\bar{B} + q_2 l/\lambda$,

and

$$[\mathcal{D}] = \frac{EI}{l^3}\begin{pmatrix} F_6(\lambda) & -F_4(\lambda)l & F_5(\lambda) & F_3(\lambda)l \\ -F_4(\lambda)l & F_2(\lambda)l^2 & -F_3(\lambda)l & F_1(\lambda)l^2 \\ F_5(\lambda) & -F_3(\lambda)l & F_6(\lambda) & F_4(\lambda)l \\ F_3(\lambda)l & F_1(\lambda)l^2 & F_4(\lambda)l & F_2(\lambda)l^2 \end{pmatrix} \qquad (3.6.4)$$

The displacement $ve^{i\omega t}$ of the clamped beam in fig(c) is governed by the differential equation

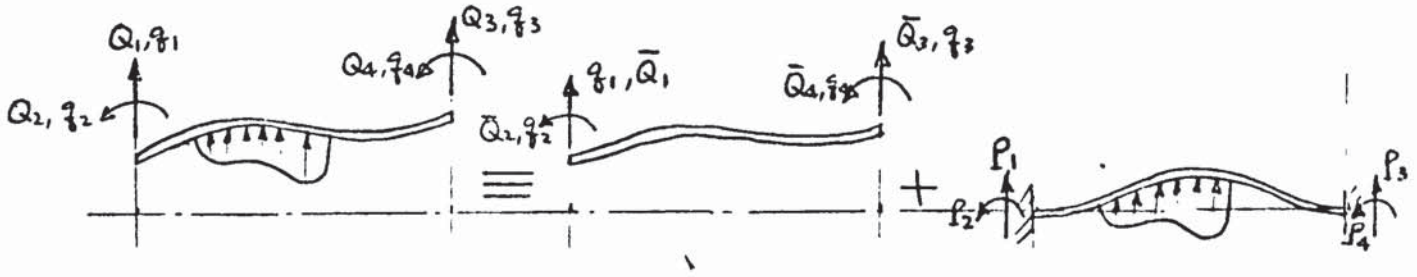
$$EI\frac{d^4v}{dx^4} - \omega^2\rho A_0 v = f(x) \qquad (3.6.5)$$

with boundary conditions $v(0)=v(l)=v'(0)=v'(l)=0$.

The solution is straight forward and has the form

$$v(x)=A\cos\lambda\xi + B\sin\lambda\xi + C\cosh\lambda\xi + D\sinh\lambda\xi + \Phi(x) \qquad (3.6.6)$$

where A,B,C,D are integration constants and $\Phi(x)$ the particular integral.



(a) Generalized coord. of a forced beam

(b) Generalized coord. of a free beam

(c) Generalized coord. of a clamped beam

Fig (3.6.4) Beam element with interior loadings

The function $\Phi(x)$ is different for differnt loadings and some are listed in fig (3.6.5). H(x) is a step function, i.e. when x greater than 0 then H(x)=1 otherwise H(x)=0. After determining the displacement function, we have

$$v(x)= (\Phi F_3 - \Phi' F_1)\mathcal{F}_{10}(\lambda\xi)/2\lambda^2 + (\Phi F_5 + \Phi' F_3)\mathcal{F}_8(\lambda\xi)/2\lambda^3 + \Phi(x) \tag{3.6.7}$$

where $F_i=F_i(\lambda)$ and $\Phi=\Phi(\ell)$, $\Phi'=\Phi'(\ell)$, $\xi=x/\ell$.

$\mathcal{F}_i$, are the frequency functions as shown in fig(3.6.5).

The generalized forces P_1, P_2, P_3, P_4 are obtained by

$$P_1 = EIv'''(0)$$

$$P_2 = -EIv''(0)$$

$$P_3 = -EIv'''(\ell)$$

$$P_4 = EIv''(\ell)$$

Therefore the dynamic stiffness relation of the original system, fig(a), is given by

$$\{Q\} = \{P\} + [\bar{D}]\{q\} \tag{3.6.8}$$

where $\{P\} = [P_1 \ P_2 \ P_3 \ P_4]^T$.

The particular integral $\Phi(x)$ can be extended to additional mass and springs on a beam at point x=c by putting $F=-kv(c)$ or $F=-m\omega^2 v(c)$ in fig(3.6.5a and b) where k is the spring constant and m the mass.

In the case of uniformly distributed loads over the entire span, the displacement and fixing forces are given explicitly as below.

$$v(x)= A\cos\lambda\xi + B\sin\lambda\xi + C\cosh\lambda\xi + D\sinh\lambda\xi \qquad (3.6.9)$$

where $A = \frac{f l^4}{2EI\lambda^6}\left[\lambda^2 - F_3(\lambda) - F_4(\lambda)\right]$

$B=-D= -\frac{f l^4}{2EI\lambda^5}\left[F_5(\lambda) + F_6(\lambda)\right]$

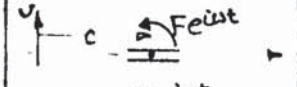
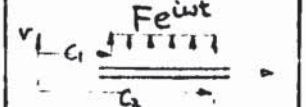
$C = \frac{f l^4}{2EI\lambda^6}\left[\lambda^2 + F_3(\lambda) + F_4(\lambda)\right]$

and $P_1=P_3= \frac{f l}{\lambda^4}\left[F_5(\lambda) + F_6(\lambda)\right]$

$-P_2=P_4= \frac{f l^2}{\lambda^4}\left[F_3(\lambda) + F_4(\lambda)\right]$

where f is the intensity of the U.D.L..

In the main programme, we have not included this device due to lack of time. A useful peice of further work would be to code this into a programme. However, examples of use will be given in chapter seven.

LOADINGS		$\Phi(x)$
(a) Concentrated force F		$-\frac{F l^3}{2EI\lambda^3} H(x-c)\, \mathcal{F}_8\left(\frac{\lambda}{l}(x-c)\right)$
(b) Concentrated moment F		$\frac{F l^2}{2EI\lambda^2} H(x-c)\, \mathcal{F}_{10}\left(\frac{\lambda}{l}(x-c)\right)$
(c) U.D.L. F per unit length		$\frac{F l^4}{2EI\lambda^4}\left[H(x-c_1)\left(\mathcal{F}_9\left(\frac{\lambda}{l}(x-c_1)\right)-2\right) - H(x-c_2)\left(\mathcal{F}_9\left(\frac{\lambda}{l}(x-c_2)\right)-2\right)\right]$

$\mathcal{F}_7(x) = \sin x + \sinh x$
$\mathcal{F}_8(x) = \sin x - \sinh x$
$\mathcal{F}_9(x) = \cos x + \cosh x$
$\mathcal{F}_{10}(x) = \cos x - \cosh x$
$\mathcal{F}_3(x) = \cos x \cosh x - 1$

$$H(x)= \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

Fig (3.6.5) Particular integrals for beam vibrations

3.6.2 TAPERED STRAIGHT BEAM MEMBERS

A tapered beam is defined as a beam whose cross sectional area is a function of the coordinate parameter which defines the axis. We shall study the vibration of a tapered beam member whose axis is a straight line. The governing equation of such an element is given by

$$\frac{d^2}{dx^2}\left(EI(x)\frac{d^2v}{dx^2}\right) - \rho A(x)\omega^2 v = 0 \qquad (3.6.10)$$

where EI(x) = the flexural rigidity about the y axis,

$\rho A(x)$ = the density per unit length,

v(x)= the amplitude of flexural vibration.

This equation may be solved analytically only for certain distributions of A(x) and EI(x). One of them is,(ref 87),

$$\rho A(x) = \rho A_o (1+ cx/l)^n$$
$$EI(x) = EI_o (1+ cx/l)^{n+2} \qquad (3.6.11)$$

where ρA_o and EI_o are the density and rigidity at x=0 respectively, C and n are constants. With this assumption, the solution of equation (3.6.10) is

$$v(\beta) = [C_1 J_n(\beta) + C_2 Y_n(\beta) + C_3 I_n(\beta) + C_4 K_n(\beta)]/\beta^n, \qquad (3.6.12)$$

where $\beta = 2\lambda_o \xi^{\frac{1}{2}}/c$,

$\xi = (1 + cx/l)$,

and $\lambda_o = l(\omega^2 \rho A_o/EI_o)^{\frac{1}{4}}$,

and J_n, Y_n, I_n, K_n are the Bessel's function of the n^{th} order.

For prescribed boundary displacements,

$$v(\beta_o) = q_1, \quad v'(\beta_o) = q_2, \quad v(\beta_1) = q_3, \quad v'(\beta_1) = q_4,$$

where $\beta_o = \beta(x=0)$, $\beta_1 = \beta(x=l)$,

the integration constants of equation (3.6.12) are obtained from

$$\begin{bmatrix} J_n(\beta_o) & Y_n(\beta_o) & I_n(\beta_o) & K_n(\beta_o) \\ J_{n+1}(\beta_o) & Y_{n+1}(\beta_o) & -I_{n+1}(\beta_o) & K_{n+1}(\beta_o) \\ J_n(\beta_1) & Y_n(\beta_1) & I_n(\beta_1) & K_n(\beta_1) \\ J_{n+1}(\beta_1) & Y_{n+1}(\beta_1) & -I_{n+1}(\beta_1) & K_{n+1}(\beta_1) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} \beta_o^n q_1 \\ -l\beta_o^n q_2/\lambda_o \\ \beta_1^n q_3 \\ -l\beta_1^n q_4/\lambda_1 \end{Bmatrix}$$

where $\lambda_1 = l(\omega^2 \rho A(l)/EI(l))^{\frac{1}{4}}$. (3.6.13)

After solving $\{c\} = [\,c_1 \;\; c_2 \;\; c_3 \;\; c_4\,]^T$ from equation (3.6.13) in terms of $\{q\} = [\,q_1 \;\; q_2 \;\; q_3 \;\; q_4\,]^T$, we write

$$\{c\} = [\eta]\{q\} \tag{3.6.14}$$

Therefore, equation (3.6.12) may be written as

$$v(\beta(x)) = [\mathcal{X}]\{c\} = [\mathcal{X}][\eta]\{q\} \tag{3.6.15}$$

where $[\mathcal{X}] = \frac{1}{\beta^n}[\,J_n(\beta) \;\; Y_n(\beta) \;\; I_n(\beta) \;\; K_n(\beta)\,]$

The generalized forces are defined by

$$\begin{aligned} Q_1 &= \frac{d}{dx}[EI(0)v''(\beta_0)] \\ Q_2 &= -EI(0)v''(\beta_0) \\ Q_3 &= -\frac{d}{dx}[EI(l)v''(\beta_1)] \\ Q_4 &= EI(l)v''(\beta_1) \end{aligned} \tag{3.6.16}$$

And , therefore, after substituting equation (3.6.12) or (3.6.15) into equations (3.6.16), we have the dynamic stiffness relation

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \frac{EI_0}{l^3} \begin{bmatrix} \frac{\lambda_0^3}{\beta_0^n} J_{n+1}(\beta_0) & \frac{\lambda_0^3}{\beta_0^n} Y_{n+1}(\beta_0) & \frac{\lambda_0^3}{\beta_0^n} I_{n+1}(\beta_0) & -\frac{\lambda_0^3}{\beta_0^n} K_{n+1}(\beta_0) \\ -\frac{l\lambda_0^2}{\beta_0^n} J_{n+2}(\beta_0) & -\frac{l\lambda_0^2}{\beta_0^n} Y_{n+2}(\beta_0) & -\frac{l\lambda_0^2}{\beta_0^n} I_{n+2}(\beta_0) & -\frac{l\lambda_0^2}{\beta_0^n} K_{n+2}(\beta_0) \\ -\frac{\lambda_1^3}{\beta_1^n} J_{n+1}(\beta_1) & \frac{\lambda_1^3}{\beta_1^n} Y_{n+1}(\beta_1) & \frac{\lambda_1^3}{\beta_1^n} I_{n+1}(\beta_1) & \frac{\lambda_1^3}{\beta_1^n} K_{n+1}(\beta_1) \\ \frac{l\lambda_1^2}{\beta_1^n} J_{n+2}(\beta_1) & \frac{l\lambda_1^2}{\beta_1^n} Y_{n+2}(\beta_1) & \frac{\lambda_1^3}{\beta_1^n} I_{n+2}(\beta_1) & \frac{l\lambda_1^2}{\beta_1^n} K_{n+2}(\beta_1) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} \tag{3.6.17}$$

or in matrix form

$$\{Q\} = [B]\{c\} = [B][\eta]\{q\}$$

By the definition of the dynamic stiffness matrix, we have

$$\{Q\} = [\mathcal{D}]\{q\}$$

where

$$[\mathcal{D}] = [B][\eta] \tag{3.6.18}$$

The calculation of the tapered beam element is not easy because of the Bessel's functions involved. Moreover, if the geometric quantities $\rho A(x)$ and $EI(x)$ can not be expressed in the form (3.6.11), e.g. lincarly varying width and depth, the method will fail. We shall use our new method discussed in section (3.5) to obtain the displacement function and the related dynamic stiffness matrix and mass matrix for a generally tapered

beam element.

Assume that the rigidity and density are expressed in the forms

$$EI(x)=EI_0\sum_{k=0}^{n_1}\alpha_k x^k$$
$$\rho A(x)=\rho A_0\sum_{k=0}^{n_2}\beta_k x^k \qquad (3.6.19)$$

where EI_0, ρA_0, $\alpha_k<1$, $k=1,2,\cdots,n_1$, $\beta_k<1$, $k=1,2,\cdots,n_2$, are known constants and $\alpha_0=\beta_0=1$. We obtain firstly the modal shapes of a single clamped-clamped tapered beam according to Rayleigh-Ritz method by solving the eigenvalue problem represented by equation (3.5.45),

$$[\alpha_{ij}-\omega^2\beta_{ij}]\{A_j\}=\{0\} \qquad (3.6.20)$$

where $\alpha_{ij}=\int_0^l EI_0\sum_{k=0}^{n_1}\alpha_k\phi_i''\phi_j''x^k dx$

$\beta_{ij}=\int_0^l \rho A_0\sum_{k=0}^{n_2}\beta_k\phi_i''\phi_j''x^k dx$

$\phi_i(x)$= the i th mode of a clamped uniform beam having rigidity EI_0 and density ρA_0, i,j=1,2, ... ,n. The integrals $\int_0^l x^k\phi_i\phi_j dx$ and $\int_0^l x^k\phi_i''\phi_j'' dx$ have been calculated for the first twelve modes for k=0 to 6. A general computer programme has been designed in chapter six for this purpose. If the solution of equation (3.6.20) for the k th mode of vibration is ω_k and $\{A^{(k)}\}$, then

$$\psi_k(x)=\sum_{j=1}^{n}A_j^{(k)}\phi_j(x) \qquad (3.6.21)$$

is the k th mode of the tapered beam with clamped ends. The constants G_{ik} are calculated from equation (3.5.47) as

$$G_{i,k}=\frac{\rho A_0\sum_{m=0}^{n_2}\sum_{j=1}^{n}\beta_m A_j^{(k)}\int_0^l x^m a_i(x)\phi_j(x)dx}{\sum_{j=1}^{n}[A_j^{(k)}]^2} \qquad (3.6.22)$$

i,k= 1,2,3,4,

where $a_i(x)$ are given by equation (3.5.36). The integrals $\int_0^l x^m a_i(x)\phi_j(x)dx$ are readily calculated for the first 14 modes, as listed in fig(3.6.6).

A tapered clamped-clamped beam was studied using this method. Two elements were used in the calculation of the natural frequency and six terms approximation of the series form of the dynamic matrix were taken. The rigidity and density were assumed in the forms

$$EI(x) = (1+cx)^3$$

and $\rho A(x) = 1+cx$. The frequency parameter λ, where $\lambda^4 = \omega^2 \rho A_o l^4 / EI_o$, was listed in fig(3.6.6a) for the first four modes and for c=.1(.1)1.

c=	first mode	second mode	third mode	fourth mode
0.1	4.89811	8.13204	11.38589	14.63889
0.2	5.04791	8.38012	11.73286	15.08477
0.3	5.18017	8.59870	12.03831	15.47715
0.4	5.29584	8.78957	12.30484	15.81943
0.5	5.396160	8.95491	12.53561	16.11573
0.6	5.48246	9.09716	12.73417	16.37068
0.7	5.55618	9.21887	12.90415	16.58901
0.8	5.61874	9.32251	13.04913	16.77538
0.9	5.67155	9.41048	13.17248	16.93414
1.0	5.71591	9.48495	13.27726	17.06923

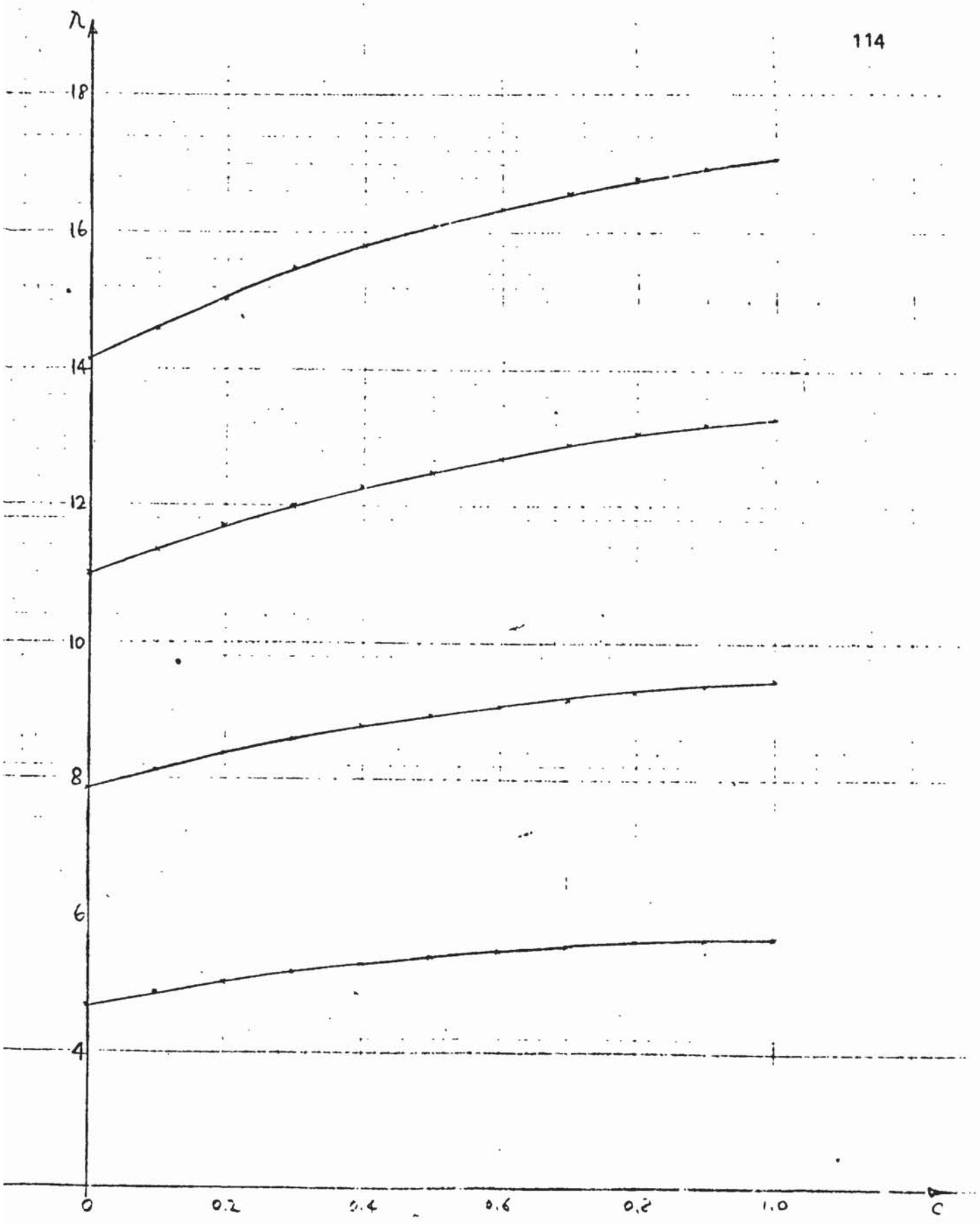
Fig (3.6.6a) The first four natural frequency parameters of a tapered beam with both ends clamped

The results when c=1 were compared with those calculated from the exact formulation and found that these figures are accurate up to 4 significent figures.

$\gamma=0$	$\gamma=1$	$\gamma=2$	$\gamma=3$	$\gamma=4$	$\gamma=5$	$\gamma=6$
K= 1						
0.415431	0.169065	0.079673	0.041613	0.023672	0.014058	0.008837
0.089392	0.039336	0.019916	0.010848	0.006517	0.003873	0.002462
0.415431	0.240366	0.156973	0.105640	0.074226	0.054062	0.043473
-0.089392	-0.047556	-0.029033	-0.016790	-0.012470	-0.00[illegible]5	-0.006123
K= 2						
0.254871	0.032429	-0.011900	-0.017934	-0.015650	-0.012149	-0.009119
0.032429	-0.000000	-0.005995	-0.005995	-0.004028	-0.003601	-0.002729
-0.254871	-0.222442	-0.170023	-0.139599	-0.109504	-0.086571	-0.069161
0.032429	0.032429	0.020434	0.020440	0.015612	0.011951	0.009222
K= 3						
0.181885	0.017737	0.001105	0.002850	0.004404	0.005428	0.005212
0.016542	0.000597	0.000200	0.001270	0.001764	0.001132	0.001500
0.181885	0.164148	0.147600	0.129408	0.111510	0.095239	0.081065
-0.016542	-0.015945	-0.015040	-0.014371	-0.012807	-0.010774	-0.009065
K= 4						
0.141471	0.010007	-0.000730	-0.001095	-0.001494	-0.001307	-0.002125
0.010007	0.000000	-0.000305	-0.000365	-0.000510	-0.000674	-0.000760
-0.141471	-0.131464	-0.120727	-0.110350	-0.100040	-0.089934	-0.080329
0.010007	0.010007	0.009642	0.009277	0.008756	0.008084	0.007324
K= 5						
0.115749	0.006824	0.000125	0.000359	0.000625	0.000807	0.000972
0.006699	0.000062	0.000021	0.000164	0.000230	0.000288	0.000362
0.115749	0.108925	0.102227	0.095295	0.088391	0.081063	0.074727
-0.006699	-0.006637	-0.006605	-0.006441	-0.006211	-0.005923	-0.005561
K= 6						
0.097942	0.004796	-0.000122	-0.000183	-0.000250	-0.000372	-0.000475
0.004796	-0.000000	-0.000061	-0.000061	-0.000099	-0.000137	-0.000169
-0.097942	-0.093145	-0.088227	-0.083370	-0.078498	-0.073649	-0.068872
0.004796	0.004796	0.004735	0.004674	0.004576	0.004439	0.004270
K= 7						
0.084283	0.003629	0.000026	0.000082	0.000145	0.000196	0.000251
0.003603	0.000013	0.000007	0.000038	0.000053	0.000071	0.0000[illegible]
0.084283	0.081254	0.077651	0.073993	0.070341	0.066727	0.063[illegible]
-0.003603	-0.003589	-0.003583	-0.003545	-0.003491	-0.003420	-0.003330
K= 8						
0.074896	0.002805	-0.000033	-0.000049	-0.000072	-0.000108	-0.000142
0.002805	-0.000000	-0.000016	-0.000016	-0.000028	-0.000040	-0.000051
-0.074896	-0.072092	-0.069254	-0.066433	-0.063605	-0.060761	-0.057775
0.002805	0.002805	0.002788	0.002772	0.002744	0.002704	0.002653
K= 9						
0.067113	0.002253	0.000008	0.000026	0.000047	0.000064	0.000085
0.002245	0.000004	0.000002	0.000012	0.000017	0.000024	0.000031
0.067113	0.064759	0.062514	0.060250	0.057909	0.055732	0.053464
-0.002245	-0.002241	-0.002239	-0.002227	-0.002210	-0.002166	-0.002156
K= 10						
0.060630	0.001838	-0.000012	-0.000017	-0.000026	-0.000040	-0.000053
0.001833	-0.000000	-0.000006	-0.000006	-0.000010	-0.000015	-0.000019
-0.060630	-0.058792	-0.056923	-0.055020	-0.053252	-0.051408	-0.049500
0.001836	0.001838	0.001832	0.001826	0.001816	0.001802	0.001762
K= 11						
0.055356	0.001535	0.000003	0.000010	0.000018	0.000026	0.000035
0.001532	0.000002	0.000001	0.000005	0.000007	0.000010	0.000013
0.055356	0.053823	0.052201	0.050751	0.049212	0.047675	0.046141
-0.001532	-0.001531	-0.001530	-0.001525	-0.001516	-0.001509	-0.001490
K= 12						
0.050930	0.001297	-0.000005	-0.000007	-0.000011	-0.000017	-0.000023
0.001297	-0.000000	-0.000002	-0.000002	-0.000004	-0.000006	-0.000009
-0.050930	-0.049633	-0.048331	-0.047032	-0.045731	-0.044435	-0.043142
0.001297	0.001297	0.001294	0.001292	0.001288	0.001281	0.001273
K= 13						
0.047157	0.001113	0.000001	0.000005	0.000008	0.000012	0.000016
0.001112	0.000001	0.000000	0.000002	0.000003	0.000004	0.000006
0.047157	0.046044	0.044937	0.043817	0.042702	0.041567	0.040474
-0.001112	-0.001111	-0.001111	-0.001109	-0.001106	-0.001101	-0.001095
K= 14						
0.043905	0.000964	-0.000002	-0.000004	-0.000005	-0.000008	-0.000012
0.000964	-0.000000	-0.000001	-0.000001	-0.000002	-0.000003	-0.000004
-0.043905	-0.042941	-0.041977	-0.041011	-0.040044	-0.039076	-0.038114
0.000964	0.000964	0.000963	0.000961	0.000959	0.000956	0.000952

Fig(3.6.6b) The values of the integral $\int_0^1 x^{\gamma} a_i(x) \phi_k(x) dx$

for the first 14 modes, the programme is found in section (6.5)



Fig(3.6.6c) The variation of the frequence parameters with the taper const. C in natural vibration of a clamp-clamp tapered beam

3.6.3 CIRCULARLY CURVED BEAMS

Analytical solutions are not possible for generally curved beam members. Even for circularly curved beams, the analytical solutions are very complicated. As for the tapered beam elements, the circularly curved beam element as shown in fig (3.6.7) is studied by the analytical method as well as the new method described in section (3.5). We begin with the analytical method.

The following notations are adopted:

ρA = mass per unit length

ω = frequency of vibration (rad/sec)

$M e^{i\omega t}$ = bending moment

$Q e^{i\omega t}$ = shearing force

$N e^{i\omega t}$ = axial force

$u e^{i\omega t}$ = inward radial displacement

$v e^{i\omega t}$ = out of plane displacement

$w e^{i\omega t}$ = tangential displacement

$\phi e^{i\omega t}$ = torsional displacement

$\zeta e^{i\omega t}$ = angular displacement of beam axis

r = radius

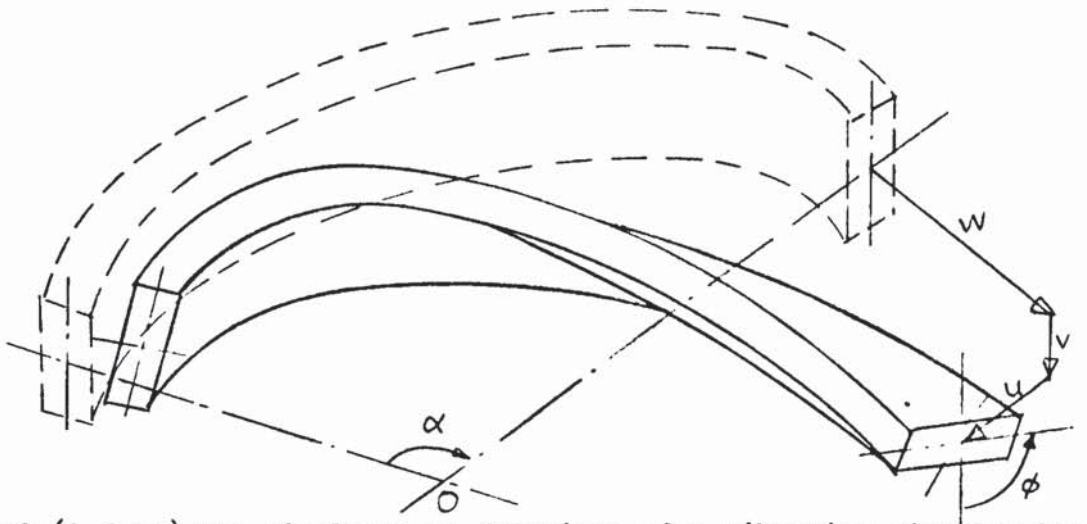
α = angle of opening

$\alpha+\beta+\gamma = 180^{0}$

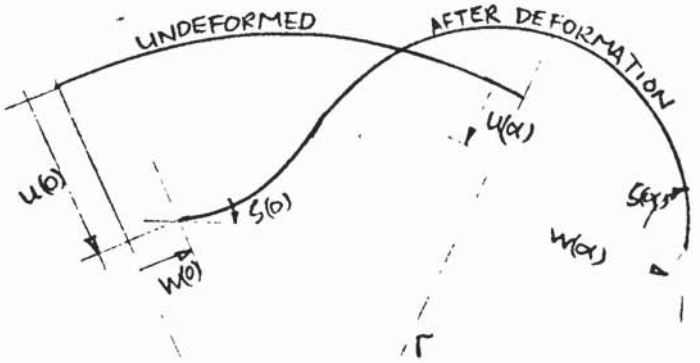
θ = angular coordinate (from 0 to α)

q_i, Q_i, i=1,2,3,4,5,6 are generalized coordinates of displacement and force respectively for inplane vibration.

The equations of motion in vibration, (ref 37,38) are obtained from Hamilton's principle as below:



Fig(3.6.7a) The displacement functions of a vibrating circular beam in space



Fig(3.6.7b) Forces and displacements for in-plane vibration

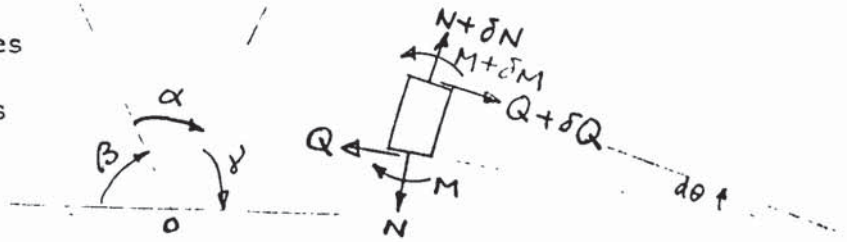


Fig (3.6.7b) forces and displacements for in-plane vibration

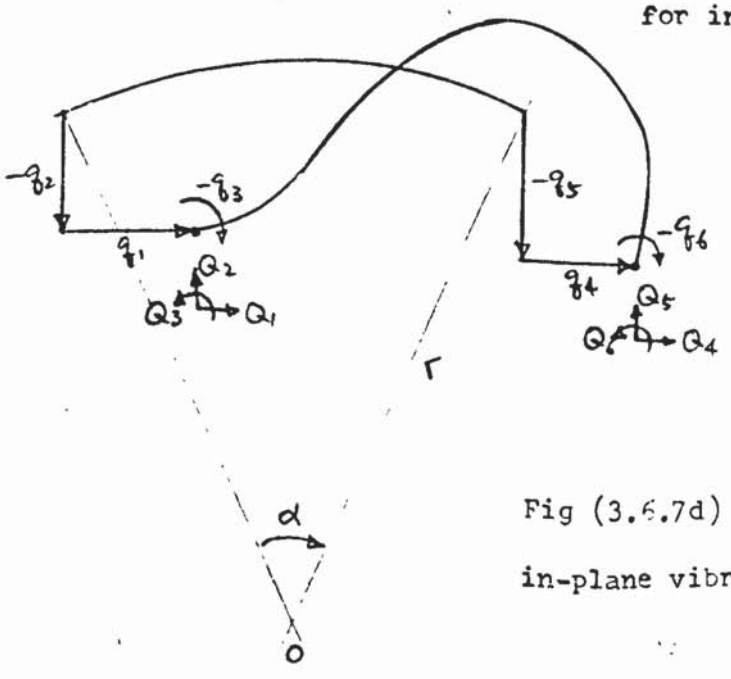


Fig (3.6.7d) Generalized coordinates for in-plane vibration

(i) for inplane vibration

$$u^{IV} + w''' = \rho A r^4 \omega^2 u / EI_y ,$$
$$u''' + w'' = -\rho A r^4 \omega^2 w / EI_y ; \qquad (3.6.23)$$

(ii) for out of plane vibration

$$v^{IV} - r\phi'' - k(v'' + r\phi'') = -\rho A r^4 \omega^2 v / EI_x ,$$
$$v'' - r\phi'' + k(v'' + r\phi'') = 0 , \qquad (3.6.24)$$

where $k = GJ / EI_x$

and GJ is the torsional rigidity.

The uncoupled equations for the inplane vibration are

$$w^{VI} + 2w^{IV} + (1-\lambda)w'' + \lambda w = 0 ,$$
$$u^{VI} + 2u^{IV} + (1-\lambda)u'' + \lambda u = 0 , \qquad (3.6.25)$$

where Roman superscrits denote the differentiations with respect to θ and $\lambda = \rho A r^4 \omega^2 / EI_y$, and inextensional vibration was assumed. The uncoupled equations for the out of plane vibration are

$$\phi^{VI} + 2\phi^{IV} + (1 - k\bar{\lambda})\phi'' + \bar{\lambda}\phi = 0$$
$$v'' = r(\phi - k\phi'') / (1+k) \qquad (3.6.26)$$

where $\bar{\lambda} = \rho A r^4 \omega^2 / EI_x$

From equations (3.6.25) and (3.6.26), the governing equations for w, u, ϕ, have the same form and v may be determined from the second equation of eqs. (3.6.26). Therefore, we shall give solutions for the inplane vibration only.

The auxiliary equation for the differential equations (3.6.25) is

$$\sigma^6 + 2\sigma^4 + (1-\lambda)\sigma^2 + \lambda = 0 \qquad (3.6.27)$$

The roots are of three types depending on the value of λ , and the solutions are given separately below:

Case 1: When $0 < \lambda < 0.113400546$, the roots are of the form $\sigma = \pm\sigma_1 i, \pm\sigma_2 i$, $\pm\sigma_3 i$, where σ_j are real numbers. The displacement has the solution,

$$w(\theta) = [\psi(\theta)]\{C\} \qquad (3.6.28)$$

where $\{C\} = [C_1 \;\; C_2 \;\; C_3 \;\; C_4 \;\; C_5 \;\; C_6]^T$

is a vector of integration constants and

$$[\psi(\theta)] = [\cos\sigma_1\theta \;\; \cos\sigma_2\theta \;\; \cos\sigma_3\theta \;\; \sin\sigma_1\theta \;\; \sin\sigma_2\theta \;\; \sin\sigma_3\theta]$$

Case 2: When $0.113400546 < \lambda < 17.636599455$, the roots are of the form, $\pm\sigma_1 i, \pm(\nu+\mu i), \pm(\nu-\mu i)$, where ν, μ are real. The row vector, $[\psi(\theta)]$ in equation (3.6.28) has the form

$$[\psi(\theta)] = [\cos\sigma_1\theta \;\; \cos\mu\theta\cosh\nu\theta \;\; \cos\mu\theta\sinh\nu\theta \;\; \sin\sigma_1\theta \;\; \sin\mu\theta\sinh\nu\theta \;\; \sin\mu\theta\cosh\nu\theta].$$

Case 3: When $17.636599455 < \lambda$, the roots are of the form $\pm\sigma_1 i, \pm\sigma_2, \pm\sigma_3$ and the row vector, $[\psi(\theta)]$, in equation (3.6.28) is

$$[\psi(\theta)] = [\cos\sigma_1\theta \;\; \cosh\sigma_2\theta \;\; \cosh\sigma_3\theta \;\; \sin\sigma_1\theta \;\; \sinh\sigma_2\theta \;\; \sinh\sigma_3\theta].$$

For the prescribed boundary displacement conditions we have,

$$\begin{aligned}
&r\zeta(0) = u'(0) + w(0) = -q_3 r \\
&u(0)\cos\beta + w(0)\sin\beta = q_1 \\
&u(0)\sin\beta - w(0)\cos\beta = -q_2 \\
&r\zeta(\alpha) = u'(\alpha) + w(\alpha) = -q_6 r \\
&-u(\alpha)\cos\gamma + w(\alpha)\sin\gamma = q_4 \\
&u(\alpha)\sin\gamma + w(\alpha)\cos\gamma = -q_5
\end{aligned} \tag{3.6.29}$$

From equations (3.6.28) and (3.6.29) and the fact of inextensional vibration that $u = \frac{dw}{d\theta}$, we have a set of six linear equations for $\{C\}$,

$$\begin{bmatrix}
[\psi'(0)]\cos\beta + [\psi(0)]\sin\beta \\
-[\psi'(0)]\sin\beta + [\psi(0)]\cos\beta \\
-([\psi''(0)] + [\psi(0)])/r \\
-[\psi'(\alpha)]\cos\gamma + [\psi(\alpha)]\sin\gamma \\
-[\psi'(\alpha)]\sin\gamma - [\psi(\alpha)]\cos\gamma \\
-([\psi''(\alpha)] + [\psi(\alpha)])/r
\end{bmatrix}
\begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{Bmatrix} =
\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} \tag{3.6.30}$$

or in brief, $[\Psi]\{C\} = \{q\}$

and upon solving, $\{C\} = [\Psi]^{-1}\{q\}$ (3.6.31)

The tangential displacement (3.6.28) can be express as

$$w(\theta) = [\psi(\theta)][\Psi]^{-1}\{q\} \tag{3.6.32}$$

The mass and stiffness matrices are given by formulae (2.6.5) and (2.6.6) as

$$[\mathcal{M}] = \rho A r \int_0^{\alpha} [\Psi]^{-T} [\psi(\theta)]^T [\psi(\theta)] [\Psi]^{-1} \, d\theta$$

$$[\mathcal{K}] = \frac{EI}{r^3} \int_0^{\alpha} [\Psi]^{-T} [\psi''(\theta)]^T [\psi''(\theta)] [\Psi]^{-1} \, d\theta \qquad (3.6.33)$$

Obtaining explicit formulae would be a large effort that would be justified only if the circular curved elements were used as common structural members as beams as presently used. Therefore the evaluation of these matrices are based on numerical integrations.

In the following, we list the matrix $[\Psi]$ for all three cases.

(1) When $0 < \lambda < 0.113400546$, then $[\Psi]^T =$

$$\begin{bmatrix}
1 & 0 & -\sigma_1^2 & \cos\sigma_1\alpha & -\sigma_1\sin\sigma_1\alpha & -\sigma_1^2\cos\sigma_1\alpha \\
1 & 0 & -\sigma_2^2 & \cos\sigma_2\alpha & -\sigma_2\sin\sigma_2\alpha & -\sigma_2^2\cos\sigma_2\alpha \\
1 & 0 & -\sigma_3^2 & \cos\sigma_3\alpha & -\sigma_3\sin\sigma_3\alpha & -\sigma_3^2\cos\sigma_3\alpha \\
0 & \sigma_1 & 0 & \sin\sigma_1\alpha & \sigma_1\cos\sigma_1\alpha & -\sigma_1^2\sin\sigma_1\alpha \\
0 & \sigma_2 & 0 & \sin\sigma_2\alpha & \sigma_2\cos\sigma_2\alpha & -\sigma_2^2\sin\sigma_2\alpha \\
0 & \sigma_3 & 0 & \sin\sigma_3\alpha & \sigma_3\cos\sigma_3\alpha & -\sigma_3^2\sin\sigma_3\alpha
\end{bmatrix}$$

(2) When $0.113400546 < \lambda < 17.636599455$, then $[\Psi]^T =$

$$\begin{bmatrix}
1 & 0 & -\sigma_1^2 & \cos\sigma_1\alpha & -\sigma_1\sin\sigma_1\alpha & -\sigma_1^2\cos\sigma_1\alpha \\
1 & 0 & \nu^2-\mu^2 & \cos\mu\alpha\cosh\nu\alpha & -\mu\sin\mu\alpha\cosh\nu\alpha \\ & & & & +\nu\cos\mu\alpha\sinh\nu\alpha & (\nu^2-\mu^2)\cos\mu\alpha\cosh\nu\alpha \\ & & & & & +2\mu\nu\sin\mu\alpha\sinh\nu\alpha \\
0 & \nu & 0 & \cos\mu\alpha\sinh\nu\alpha & -\sin\ \sinh & (\nu^2-\mu^2)\cos\mu\alpha\sinh\nu\alpha \\ & & & & +\cos\ \cosh & -2\mu\nu\sin\mu\alpha\cosh\nu\alpha \\
0 & \sigma_1 & 0 & \sin\sigma_1\alpha & \sigma_1\cos\sigma_1\alpha & -\sigma_1^2\sin\sigma_1\alpha \\
0 & 0 & 2\mu\nu & \sin\mu\alpha\sinh\nu\alpha & \mu\cos\mu\alpha\sinh\nu\alpha & (\nu^2-\mu^2)\sin\mu\alpha\sinh\nu\alpha \\ & & & & +\nu\sin\mu\alpha\cosh\nu\alpha & +2\mu\nu\cos\mu\alpha\cosh\nu\alpha \\
0 & \mu & 0 & \sin\mu\alpha\cosh\nu\alpha & \mu\cos\mu\alpha\cosh\nu\alpha & (\nu^2-\mu^2)\sin\mu\alpha\cosh\nu\alpha \\ & & & & +\nu\sin\mu\alpha\sinh\nu\alpha & +2\mu\nu\cos\mu\alpha\sinh\nu\alpha
\end{bmatrix}$$

(3) When $17.636599455 < \lambda$, then $[\Psi]^T =$

$$\begin{bmatrix}
1 & 0 & -\sigma_1^2 & \cos\sigma_1\alpha & -\sigma_1\sin\sigma_1\alpha & -\sigma_1^2\cos\sigma_1\alpha \\
1 & 0 & \sigma_2^2 & \cosh\sigma_2\alpha & \sigma_2\sinh\sigma_2\alpha & \sigma_2^2\cosh\sigma_2\alpha \\
1 & 0 & \sigma_3^2 & \cosh\sigma_3\alpha & \sigma_3\sinh\sigma_3\alpha & \sigma_3^2\cosh\sigma_3\alpha \\
0 & \sigma_1 & 0 & \sin\sigma_1\alpha & \sigma_1\cos\sigma_1\alpha & -\sigma_1^2\sin\sigma_1\alpha \\
0 & \sigma_2 & 0 & \sinh\sigma_2\alpha & \sigma_2\cosh\sigma_2\alpha & \sigma_2^2\sinh\sigma_2\alpha \\
0 & \sigma_3 & 0 & \sinh\sigma_3\alpha & \sigma_3\cosh\sigma_3\alpha & \sigma_3^2\sinh\sigma_3\alpha
\end{bmatrix}$$

Particularly, when $\alpha = 180^\circ$, the natural frequency parameters of a clamped -clamped end circular ring are all in range (3). In such a case, the natural frequencies and the corresponding modes are calculated and tabulated in fig(3.6.8) for the first five modes.

NATURAL FREQUENCY PARAMETERS

mode n	λ_n	σ_{1n}	σ_{2n}	σ_{3n}
1	19.22	2.398	1.514	1.208
2	93.16	3.328	2.834	1.023
3	321.2	4.402	4.045	1.006
4	757.6	5.385	5.098	1.003
5	1584.0	6.425	6.186	1.001

NORMAL MODES

	$\cos\sigma_1\theta$	$\cosh\sigma_2\theta$	$\cosh\sigma_3\theta$	$\sin\sigma_1\theta$	$\sinh\sigma_2\theta$	$\sinh\sigma_3\theta$
$W_1(\theta)$	0.4319	3.738	-4.170	0.3317	-3.675	3.987
$W_2(\theta)$	0.3613	0.6272	-0.9885	0.2049	-0.6274	1.071
$W_3(\theta)$	0.2545	0.3380	-0.5925	0.1862	-0.3380	0.5444
$W_4(\theta)$	0.2138	0.2568	-0.4706	0.1476	-0.2568	0.5127
$W_5(\theta)$	0.1697	0.1925	-0.3622	0.1336	-0.1925	0.3323
	$\sin\sigma_1\theta$	$\sinh\sigma_2\theta$	$\sinh\sigma_3\theta$	$\cos\sigma_1\theta$	$\cosh\sigma_2\theta$	$\cosh\sigma_3\theta$
$U_1(\theta)$	-1.036	5.659	-5.037	0.7475	-5.563	4.815
$U_2(\theta)$	-1.202	1.778	-1.012	0.6818	-1.778	1.096
$U_3(\theta)$	-1.120	1.368	-0.5963	0.8196	-1.367	0.5479
$U_4(\theta)$	-1.151	1.309	-0.4719	0.7949	-1.309	0.5141
$U_5(\theta)$	-1.090	1.191	-0.3627	0.8582	-1.191	0.3324

Fig (3.6.8) The natural frequencies and modes of a semicircular ring with clamped ends

The conditions of orthogonality are given by, (ref 39),

$$\int_0^{\alpha} \rho A r (U_i U_j + W_i W_j)\, d\theta = \begin{cases} 0 & \text{when } i \neq j \\ \rho A r \alpha & \text{when } i = j \end{cases}.$$

Next, we establish the dynamic stiffness matrix in series form according to the method described in section (3.5). To begin, the static tangential displacement function,

$$w_o(\theta) = C_1 + C_2 + C_3\cos\theta + C_4\sin\theta + C_5\cos\theta + C_6\sin\theta , \qquad (3.6.34)$$

with satisfies, $w_o^{VI} + 2w_o^{IV} + w_o'' = 0$ (3.6.35)

must be defined. For a particular case where the angle of opening $\alpha = \pi$, and $\beta = \gamma = 0$, i.e., for a semicircular beam, the boundary conditions are,

$$\begin{aligned} w_o''(0) + w_o(0) &= -q_3 r \\ w_o'(0) &= q_1 \\ w_o(0) &= q_2 \\ w_o''(\pi) + w_o(\pi) &= -q_6 r \\ -w_o'(\pi) &= q_4 \\ w_o(\pi) &= q_5 \end{aligned} \qquad (3.6.36)$$

Differentiating expression (3.6.34) with respect to θ, we have,

$$w_o'(\theta) = C_2 - C_3\sin\theta + C_4\cos\theta + C_5(\cos\theta - \theta\sin\theta) + C_6(\sin\theta + \theta\cos\theta)$$

$$w_o''(\theta) = -C_3\cos\theta - C_4\sin\theta + C_5(-2\sin\theta - \theta\cos\theta) + C_6(2\cos\theta - \theta\sin\theta)$$

Therefore, from conditions (3.6.36) we have the equations for the determination of the constants C_i, i=1,2,3,4,5,6,

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -2 \\ 0 & -1 & 0 & 1 & 1 & \pi \\ 1 & \pi & -1 & 0 & -\pi & 0 \\ -1 & -\pi & 0 & 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 r \\ q_4 \\ q_5 \\ q_6 r \end{Bmatrix}$$

and the solution is

$$\begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{Bmatrix} = \begin{bmatrix} -.63662 & 0.00000 & -.59472 & .63662 & .00000 & -.40529 \\ .00000 & .00000, & .31831 & .00000 & .00000 & -.31831 \\ .63662 & 1.00000 & .59472 & -.63662 & .00000 & .40529 \\ 1.40528 & .31831 & -.25801 & -.40528 & .31831 & .89403 \\ -.40529 & -.31831 & -.06030 & .40529 & -.31831 & -.57632 \\ -.31831 & .00000 & .20264 & ,31831 & .00000 & -.20264 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 r \\ q_4 \\ q_5 \\ q_6 r \end{Bmatrix}$$

or $\{c\} = [N]\{q\}$

For inextensional deformation, $u_0 = \frac{dw_0}{d\theta}$,

and from equation (3,6.34), we have

$$w_0 = [1 \quad \theta \quad \cos\theta \quad \sin\theta \quad \cos\theta \quad \sin\theta][N]\{q\}$$

and $u_0 = [0 \quad 1 \quad -\sin\theta \quad \cos\theta \quad (\cos\theta - \sin\theta) \quad (\sin\theta + \theta\cos\theta)][N]\{q\}$

Therefore, the static displacement pattern is given by

$$\begin{Bmatrix} u_0 \\ w_0 \end{Bmatrix} = [a_0]\{q\}$$

where $$[a_0] = \begin{bmatrix} 1 & \theta & \cos\theta & \sin\theta & \cos\theta & \sin\theta \\ 0 & 1 & -\sin\theta & \cos\theta & \cos\theta - \theta\sin\theta & \sin\theta + \theta\cos\theta \end{bmatrix}[N]$$

$$= [\{a_1\} \quad \{a_2\} \quad \{a_3\} \quad \{a_4\} \quad \{a_5\} \quad \{a_6\}], \qquad (3.6.37)$$

And the constants for the dynamic stiffness matrix are given by

$$G_{ik} = \rho A r \int_0^{\pi} \{\phi_k\}^T \{a_i\} d\theta,$$

where $\{\phi_k\} = [u_k \quad w_k]^T$,

is the vector of the kth normal mode of the semicircular beam, the first five of them can be found in fig (3.6.8). The constants G_{ij} may be obtained by numerical integration.

As the curved beam element is not commonly used as the straight beam member, no computer programme was designed for this element.

3.6.4 SECONDARY EFFECTS

In vibration analysis, the wave length of vibration is inversely proportional to the frequency of vibration. Therefore, the higher the vibration frequency of a beam with finite length, the smaller the ratio of wavelength to its thickness. Since the simple beam theory we have discussed in the subsections (3.6.1) to (3.6.3) is based on the assumption that the wave length of the beam is much longer than its thickness, when the ratio is smaller than about ten, the theory needs to be modified. If the wavelength thickness ratio is greater than five, corrections can be made by considering the deformations orthogonal to the beam axis, otherwise, three-dimensional models should be used.

If the longitudinal vibration is concerned, the axial displacement will cause lateral displacements because of the existence of poisson's ratio, and these effects of lateral inertia and shear are regarded as secondary effects. For torsional vibrations, the warping effect will cause axial displacement, and the effects of the inertia and shear resulting from this axial displacement are considered as secondary. The secondary effects of lateral vibrations will be the inertia of rotation of the crossectional area and its shearing deflection. We assume these three different types of vibrations are independent here and their coupling effects are considered in next subsection.

The governing equations of harmonic vibrations for these caes are given by:

(i) flexural vibration (ref 33)

$$EI\frac{d^4v}{dx^4} + \rho A_0\omega^2\left(r^2+\frac{EI}{kGA_0}\right)\frac{d^2v}{dx^2} - \rho A_0\omega^2\left(1-\frac{\rho A\omega^2 r}{kGA_0}\right)v = 0 ;$$

(ii) torsional vibration (ref 32)

$$EF\frac{d^4\theta}{dx^4} - GJ\frac{d^2\theta}{dx^2} - \rho I_0\omega^2\theta = 0 ;$$

(iii) longitudinal vibration (ref 89)

$$\nu^2 G I_o \frac{d^4 u}{dx^4} - (EA_o - \nu^2 \rho I_o \omega^2) \frac{d^2 u}{dx^2} - \rho A_o \omega^2 u = 0 \ ;$$

where ℓ =length of the beam,

ν =Poisson's ratio,

I_o =Polar moment of the cross sectional area,

GJ =torsional rigidity,

EA_o=axial rigidity,

EI =flexural rigidity,

k =shear coefficient,

r =radius of gyration,

F =warping constant,

u =axial displacement,

v =lateral displacement,

θ =torsional displacement.

For the convenience of study, we represent all three equations by

$$\psi^{IV} + \rho \psi'' + q \psi = 0 \ , \qquad (3.6.38)$$

where $\psi = \psi(x)$, $\rho = \rho(\omega)$, $q = q(\omega)$,

and the associated boundary conditions are

$$\psi(0) = q_1 \ , \ \psi'(0) = q_2 \ , \ \psi(\ell) = q_3 \ , \ \psi'(\ell) = q_4 \ . \qquad (3.6.39)$$

The auxiliary equation to equation (3.6.38) has four types of roots, namely (i) four real roots, (ii) four imaginary roots, (iii) four complex roots and (iv) two real and two imaginary roots. In the case of flexural vibrations, when all the physical properties such as density and rigidity are positive, we can conclude from the theory of equation that the solutions will fall into cases (ii) or (iv). Usually, in the ordinary range of frequency when $\omega < \omega_c$, where $\omega_c = kGA/\rho I$, the auxiliary equation has two real and two imaginary roots. For frequencies greater than this critical frequency, the transverse displacement of the beam is very small (ref 33). In fact, (ref 33) shows that the two components of the slope of a simply

supported beam due to bending and to shear are essentially equal in magnitude but are in anti-phase at frequencies above ω_c.

Since the governing equations and boundary conditions are similar to those discussed in section (3.3.2) for folded plate structures, we are not going to repeat here. For all the three types of vibrations and four classes of solutions, the dynamic stiffness relation has the form:

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = C \begin{bmatrix} F_6 & -F_4\ell & F_5 & F_3\ell \\ -F_4\ell & F_2\ell^2 & -F_3\ell & F_1\ell^2 \\ F_5 & -F_3\ell & F_6 & F_4\ell \\ F_3\ell & F_1\ell^2 & F_4\ell & F_2\ell^2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} \qquad (3.6.40)$$

where

$C = EI/\ell^3$ for flexural vibration

$= GJ/\ell$ for torsional vibration

$= EA_o/\ell$ for axial vibration,

and Q_i, i=1,2,3,4 are the generalized forces associated with q_i. F_i, i=1,2,3,4,5,6, must be consistent with the four classes of solution.

In this example, we have shown that the complicating secondary effects can be taken into account without complicating the form of the dynamic stiffness matrices. All these formulae are based on classical solutions and have been arranged in such a form that the main programme of chapter six for framed structures can include these elements without particular difficulty when required.

3.6.5 COUPLING EFFECTS

Since there are three types of vibrations for a traight beam element, three coupled vibrations are possible, which are flexural-axial, flexural-torsional and tosional-axial couplings. There are two clases of coupling effects, one is the static coupling and the other dynamic. When the axial effects on flexural vibration is considered, if the axial forces are constant with respect to time, then the coupling is static, otherwise dynamic. Two examples are considered, i.e. static coupling of axial forces on flexural vibrations and dynamic coupling of flexural-torsional vibration.

It is not our aim here to provide a general computer programme to include all these elements,but rather, to demonstrate how these elements can be developed to be incoperated to the general computer programme. However, computer procedures have been designed for flexural-axial vibrations. The details are found in chapter six.

The mass, stability and dynamic stiffness matrices of a straight beam member taking accounts of the coupling effects of constant axial force P, rotatory inertia, shearing forces on flexural vibrations are studied first. The governing equations of vibration are given by (ref 33,34),

$$\begin{aligned} v^{IV} + \frac{b(q+s)}{1-sp} v'' - \frac{b(1-brs)}{1-sp} v = 0 \\ \psi^{IV} + \frac{b(q+s)}{1-sp} \psi'' - \frac{b(1-brs)}{1-sp} \psi = 0 \end{aligned} \qquad (3.6.41)$$

where $ve^{i\omega t}$ =lateral deflection

$\psi e^{i\omega t}$ =bending slope

$b = \rho A_0 l^4 \omega^2 / EI$ the bending parameter

$r = I / A_0 l^2$ the rotatory inertia parameter

$S = EI / GA_s l^2$ the shearing parameter

$p = Pl^2 / EI$ the axial force parameter

$q = p/b + r(1-sp)$

$\delta = [(q-s)^2 + 4/b]^{\frac{1}{2}}$

ρ = mass density per unit volume

A_0 = cross sectional area

l = length of the beam

ω = vibration frequency

E = Young's modulus

I = moment of inertia of cross sectional area

G = shear modulus

A_s = effective area of shear

P = compressive axial force

$\xi = x/l$

Roman superscripts denote the differentiations with respect to ξ.

The general solutions of equations (3.6.41) have the forms:

(i) when $rsb < 1$

$$v(\xi) = A\cos\beta\xi + B\sin\beta\xi + C\cosh\alpha\xi + D\sinh\alpha\xi$$

$$\psi(\xi) = (-A\bar{a}\sin\beta\xi + B\bar{a}\cos\beta\xi + C\bar{d}\sinh\alpha\xi + D\bar{d}\cosh\alpha\xi)/$$

where $\alpha = (b(\delta-q-s)/2(1-sp))^{\frac{1}{2}}$ $\qquad \beta = (b(\delta+q+s)/2(1-sp))^{\frac{1}{2}}$

$\bar{d} = (1-sp)\alpha + bs/\alpha$ $\qquad \bar{a} = (1-sp)\beta - bs/\beta$

(ii) when $rsb > 1$

$$v(\xi) = A\cos\beta\xi + B\sin\beta\xi + C\cos\alpha\xi + D\sin\alpha\xi$$

$$\psi(\xi) = (-A\bar{a}\sin\beta\xi + B\bar{a}\cos\beta\xi + C\bar{d}\sin\alpha\xi + D\bar{d}\cos\alpha\xi)/$$

where

$\alpha = -(b(q+s-\delta)/2(1-sp))^{\frac{1}{2}}$ $\qquad \beta = (b(\delta+q+s)/2(1-sp))^{\frac{1}{2}}$

$\bar{d} = (1-sp)\alpha - bs/\alpha$ $\qquad \bar{a} = (1-sp)\beta - bs/\beta$.

If the displacement boundary conditions,

$$v(0) = q_1, \quad v(l) = q_3, \quad \psi(0) = q_2, \quad \psi(l) = q_4 \qquad (3.6.42)$$

are introduced, then the constants of integration are determined by

(i) When rbs < 1

$$A = [(\alpha\bar{d} - F_4)q_1 + (F_2 l)q_2 - F_3 q_3 + F_1 l q_4] / (\beta\bar{a} + \alpha\bar{d}),$$

$$B = [-F_6 q_1 + l(\beta\bar{a} + F_4)q_2 - F_5 q_3 - F_3 L] / \bar{a}(\beta\bar{a} + \alpha\bar{d}),$$

$$C = q_1 - A, \quad D = \frac{l q_2}{\bar{d}} - \frac{B\bar{a}}{\bar{d}},$$

where the frequency functions are defined by

$$F_0 = 2\bar{a}\bar{d}(\cosh\beta\cos\alpha - 1) + (\bar{a}^2 - \bar{d}^2)\sinh\alpha\sin\beta$$

$$F_1 = \frac{-1}{F_0}(\bar{a}\sinh\alpha - \bar{d}\sin\beta)(\beta\bar{a} + \alpha\bar{d})$$

$$F_2 = \frac{-1}{F_0}(\bar{d}\cosh\alpha\sin\beta - \bar{a}\sinh\alpha\cos\beta)(\beta\bar{a} + \alpha\bar{d})$$

$$F_3 = \frac{-1}{F_0}\bar{a}\bar{d}(\alpha\bar{d} + \beta\bar{a})(\cosh\alpha - \cos\beta)$$

$$F_4 = \frac{\bar{a}\bar{d}}{F_0}[(\alpha\bar{d} - \beta\bar{a})(\cosh\alpha\cos\beta - 1) + (\bar{a}\alpha + \bar{d}\beta)\sin\beta\sinh\alpha]$$

$$F_5 = \frac{\bar{a}\bar{d}}{F_0}(\alpha\bar{d} + \beta\bar{a})(\bar{d}\sinh\alpha + \bar{a}\sin\beta)$$

$$F_6 = \frac{-\bar{a}\bar{d}}{F_0}(\alpha\bar{d} + \beta\bar{a})(\bar{a}\cosh\alpha\sin\beta + \bar{d}\sinh\alpha\cos\beta);$$

and (ii) When rbs > 1

$$A = \frac{1}{\beta\bar{a} - \alpha\bar{d}}[-(\alpha\bar{d} + F_4)q_1 + F_2 l q_2 - F_3 q_3 + F_1 l q_4]$$

$$B = \frac{1}{\bar{a}(\beta\bar{a} - \alpha\bar{d})}[-F_6 q_1 + (\beta\bar{a} + F_4)l q_2 - F_5 q_3 - F_3 l q_4]$$

$$C = q_1 - A, \quad D = \frac{l q_2}{\bar{d}} - \frac{B\bar{a}}{\bar{d}},$$

where the frequency functions are defined by

$$F_0 = 2\bar{a}\bar{d}(\cos\alpha\cos\beta - 1) + (\bar{a}^2 + \bar{d}^2)\sin\alpha\sin\beta$$

$$F_1 = -\frac{1}{F_0}(\bar{a}\sin\alpha - \bar{d}\sin\beta)(\beta\bar{a} - \alpha\bar{d})$$

$$F_2 = -\frac{1}{F_0}(\bar{d}\cos\alpha\sin\beta - \bar{a}\sin\alpha\cos\beta)(\bar{a}\beta - \alpha\bar{d})$$

$$F_3 = -\frac{\bar{a}\bar{d}}{F_0}(\alpha\bar{d} + \beta\bar{a})(\cos\alpha - \cos\beta)$$

$$F_4 = \frac{\bar{a}\bar{d}}{F_0}[(\alpha\bar{d} + \beta\bar{a})(\cos\alpha\cos\beta - 1) - (\bar{a}\alpha + \bar{d}\beta)\sin\beta\sin\alpha]$$

$$F_5 = \frac{\bar{a}\bar{d}}{F_0}[(\alpha\bar{d} + \beta\bar{a})(\bar{a}\sin\beta - \bar{d}\sin\alpha)]$$

$$F_6 = \frac{-\bar{a}\bar{d}}{F_0}(\alpha\bar{d} + \beta\bar{a})(\bar{a}\cos\alpha\sin\beta - \bar{d}\sin\alpha\cos\beta)$$

There is a removable singularity for all the frequency functions at b,r,s,p =0, and when this is removed, the following approximations apply:

$$F_1=2-36s/(1+12s) + b(1/140 + r/30) + p/30$$

$$F_2=4-36s/(12s+1) - b(1/105 +2r/15) - 2p/15$$

$$F_3=6 - 72s/(12s+1) + b(13/420 - r/10) - p/10$$

$$F_4=-6 + 72s/(1+12s) + b(11/210 +r/10) +p/10$$

$$F_5=-12 + 144s/(1+12s) - b(9/70 -12r/10) +12p/10$$

$$F_6= 12 - 144s/(1+12s) - b(13/35 + 12r/10) - 12p/10.$$

The dynamic stiffness matrix in all cases are determined through the force and displacement relationship as

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \frac{EI}{\ell^3} \begin{bmatrix} F_6 & -F_4\ell & F_5 & F_3\ell \\ -F_4\ell & F_2\ell^2 & -F_3\ell & F_1\ell^2 \\ F_5 & -F_3\ell & F_6 & F_4\ell \\ F_3\ell & F_1\ell^2 & F_4\ell & F_2\ell^2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} \qquad (3.6.43)$$

Computer procedures have been designed to calculate the dynamic stiffness matrix and the derivatives of the matrix with respect to s,b,r,p. They are ready to incorporate in the main programme, see chapter six.

As a second example, we consider the dynamic coupling of torsion and flexure. The coupling effect is due to the eccentricity of the centroidal axis with respect to the shear axis, and this makes it necessary to consider simutaneously the translational and the rotational displacements of the cross section. The governing equations of vibration of a beam member having cross sectional area as shown in fig (3.6.8) are given by (ref 106):

$$EA\, u'' + \lambda u = 0$$

$$EI_z\, v^{IV} - \lambda v + \lambda e_z \theta = 0$$

$$EI_y\, w^{IV} - \lambda w - \lambda e_y \theta = 0$$

$$\lambda e_z v - \lambda e_y w - GJ\theta'' - \lambda(r^2 + e_y^2 + e_z^2)\theta = 0 \qquad (3.6.44)$$

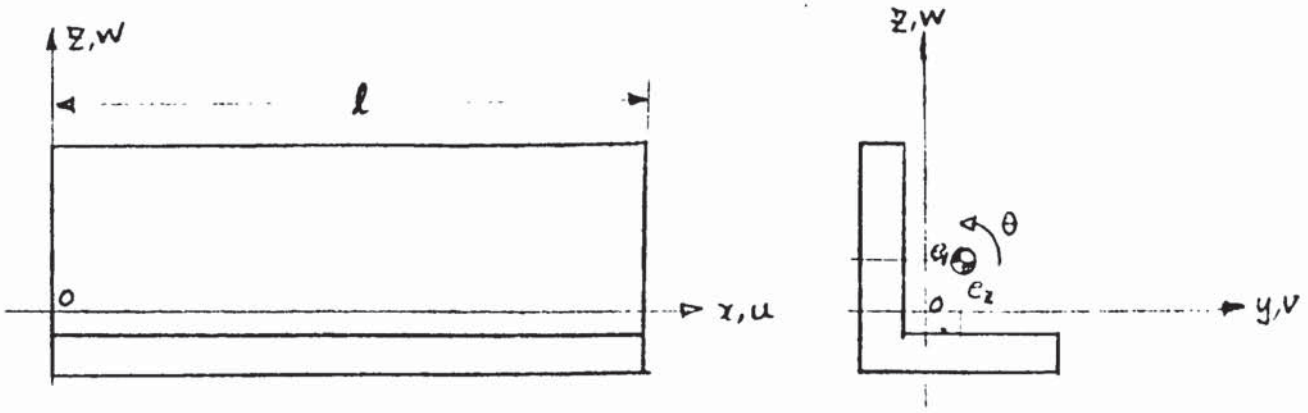


Fig (3.6.8) Coupled torsional-flexural vibration of a beam member

where I_z, I_y, E, w, v, θ, G, J have the usual meanings,

e_z, e_y are distances of centroidal axial relative to the shear axis

$\lambda = \omega^2 \rho A$

r = radius of gyration of cross sectional area about the shear centre.

The general solutions of equations (3.6.44) involve twelve generalized coordinates and it is not advisable to solve them directly because of their complexity. Therefore the new method of section (3.5) is used. The natural vibration of the beam member with both ends clamped can be found by an application of Galerkin's method.

The method begins with the choice of a set of 4N normalized eigen-functions u_i, v_i, w_i, θ_i, $i=1,2, \ldots, N$, where N= number of terms of approximation, they satisfy

$$EAu_i'' = -\zeta_i u_i$$
$$GJ\theta_i'' = -\mu_i \theta_i$$
$$EI_z v_i^{IV} = \nu_i v_i$$
$$EI_y w_i^{IV} = \eta_i w_i \qquad (3.6.45)$$

where ζ_i, μ_i, ν_i and η_i are known frequency parameters. Then the solutions are expressed as

$$u(x) = \sum_i \delta_i u_i(x)$$
$$\theta(x) = \sum_i \alpha_i \theta_i(x)$$
$$v(x) = \sum_i \beta_i v_i(x)$$
$$w(x) = \sum_i \gamma_i w_i(x) \qquad (3.6.46)$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i, i=1,2, \ldots, N$ are to be determined in the following manners. Substituting equations (3.6.46) into (3.6.44) and applying equations (3.6.45), we have

$$\begin{aligned}
&\sum_i (\lambda-\zeta_i)\delta_i u_i = 0 \\
&\sum_i (\nu_i-\lambda)\beta_i v_i + \lambda e_z \sum_i \alpha_i\theta_i = 0 \\
&\sum_i (\eta_i-\lambda)\gamma_i w_i - \lambda e_y \sum_i \alpha_i\theta_i = 0 \\
&\lambda e_z \sum_i \beta_i v_i - \lambda e_y \sum_i \gamma_i w_i + \sum_i [\mu_i - \lambda(r^2+e_y^2+e_z^2)]\alpha_i\theta_i = 0
\end{aligned} \tag{3.6.47}$$

Mutiplying the first, second, third, and fourth equations of (3.6.47) by u_j, v_j, w_j, and θ_j respectively and integrating over the length of the beam member, we have

$$\begin{bmatrix}
\left[\frac{1}{\zeta_i}\right] & [0] & [0] & [0] \\
[0] & \left[\frac{1}{\mu_i}(r^2+e_y^2+e_z^2)\right] & \left[\frac{-e_z}{\sqrt{\mu_i\nu_i}}\int_0^1 v_j\theta_i dx\right] & \left[\frac{e_y}{\sqrt{\mu_i\eta_i}}\int_0^1 w_j\theta_i dx\right] \\
[0] & \left[\frac{-e_z}{\sqrt{\mu_i\nu_i}}\int_0^1 \theta_j v_i dx\right] & \left[\frac{1}{\nu_i}\right] & [0] \\
[0] & \left[\frac{e_y}{\sqrt{\mu_i\eta_i}}\int_0^1 \theta_j w_i dx\right] & [0] & \left[\frac{1}{\eta_i}\right]
\end{bmatrix}
\begin{Bmatrix} \{\delta_i\} \\ \{\sqrt{\mu_i}\alpha_i\} \\ \{\sqrt{\nu_i}\beta_i\} \\ \{\sqrt{\eta_i}\gamma_i\} \end{Bmatrix}
= \frac{1}{\lambda}
\begin{Bmatrix} \{\delta_i\} \\ \{\sqrt{\mu_i}\alpha_i\} \\ \{\sqrt{\nu_i}\beta_i\} \\ \{\sqrt{\eta_i}\gamma_i\} \end{Bmatrix} \tag{3.6.48}$$

Upon solving this eigenvalue problem for a symmetric positive definite matrix of the left hand side of equation (3.6.48), we have, for the k th mode of coupled vibrations, the frequency parameter $\lambda_k = \omega_k^2 \rho A$ and the normal mode

$$\{\phi_k(x)\} = \sum_{i=1}^{N} \{\psi_i^k(x)\} \tag{3.6.49}$$

where $\{\phi_k(x)\} = [u(x),\ \theta(x),\ v(x),\ w(x)]_k^T$

$$\{\psi_i^k(x)\} = [\delta_i^k u_i(x) \quad \alpha_i^k\theta_i(x) \quad \beta_i^k v_i(x) \quad \gamma_i^k w_i(x)]^T$$

and the constants $\delta_i^k, \alpha_i^k, \beta_i^k, \gamma_i^k$, are the eigenvector components of eq. (3.6.48) for the k th mode, such that

$$\int_0^1 \{\phi_i(x)\}^T \{\phi_j(x)\}\, dx = \begin{cases} 0 & \text{when } i \neq j \\ 1 & \text{when } i = j \end{cases}$$

The static deflection patterns for unit end displacements can be found from solving equations (3.6.44) with $\lambda=0$, i.e.,

$$\bar{u}''=0, \quad \bar{v}^{IV}=0, \quad \bar{w}^{IV}=0, \quad \bar{\theta}''=0 \tag{3.6.50a}$$

subjected to the displacement boundary conditions

$$\begin{aligned}
&\bar{u}(0)=q_1, \quad \bar{u}(\ell)=q_7 ,\\
&\bar{v}(0)=q_2, \quad \bar{v}(\ell)=q_8 ,\\
&\bar{v}'(0)=q_3, \quad \bar{v}'(\ell)=q_9 ,\\
&\bar{w}(0)=q_4, \quad \bar{w}(\ell)=q_{10},\\
&\bar{w}'(0)=q_5, \quad \bar{w}'(\ell)=q_{11},\\
&\bar{\theta}(0)=q_6, \quad \bar{\theta}(\ell)=q_{12}.
\end{aligned} \tag{3.6.50b}$$

And the solution will have the form

$$\{\bar{u}(x)\} = [a_0(x)]\{q\}$$

where $\{\bar{u}(x)\} = [\bar{u}(x) \quad \bar{v}(x) \quad \bar{w}(x) \quad \bar{\theta}(x)]^T$

$$\{q\} = [q_1, \ q_2 \ \cdots \ q_{12}]^T$$

and

$$[a_0(x)]^T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccc} \bar{u}(x) & \bar{\theta}(x) & \bar{v}(x) & \bar{w}(x) \end{array} \\ \left[\begin{array}{cccc} 1-\xi & 0 & 0 & 0 \\ 0 & 0 & 1-3\xi^2+2\xi^3 & 0 \\ 0 & 0 & 0 & 1-3\xi^2+2\xi^3 \\ 0 & 1-\xi & 0 & 0 \\ 0 & 0 & 0 & (-\xi+2\xi^2-\xi^3)\ell \\ 0 & 0 & (\xi-2\xi^2+\xi^3)\ell & 0 \\ \xi & 0 & 0 & 0 \\ 0 & 0 & 3\xi^2-2\xi^3 & 0 \\ 0 & 0 & 0 & 3\xi^2-2\xi^3 \\ 0 & \xi & 0 & 0 \\ 0 & 0 & 0 & (\xi^2-\xi^3)\ell \\ 0 & 0 & (-\xi^2+\xi^3)\ell & 0 \end{array}\right] \end{array}$$

$$= [\{a_1\}^T \ \{a_2\}^T \cdots \{a_{12}\}^T], \ \xi = x/\ell . \tag{3.6.51}$$

As we have seen from equations (3.6.50a) that all the deflections are uncoupled when $\lambda = 0$, therefore, the static stiffness matrix $[\mathcal{D}_0]$ and the mass matrix consistent with the static displacement $[m_0]$ will be the same

as in the uncoupled static cases. And the dynamic corrections can be obtained by applying equation (3.5.35), where G_{ik} is given by equation (3.5.27) as

$$G_{ik} = \int_{vol} \{\phi_k\}^T [\rho] \{a_i\} dvol$$
$$= \rho A \ell \int_0^1 \sum_{j=1}^{N} [\delta_j^k u_j(\xi) \ \alpha_j^k \theta_j(\xi) \ \beta_j^k v(\xi) \ \gamma_j^k w_j(\xi)] \{a_i\} d\xi$$

Since we have a general computer programme, in chapter six, to compute the integrals of the type

$$\int_0^1 \phi_j(\xi) \xi^k d\xi$$

where $\phi_j(\xi)$ is either a clamped beam function or trigonometric function, the work reduces to the solution of the eigenvalue problem represented by equation (3.6.48) for λ_k and constants δ_j^k, α_j^k, β_j^k, γ_j^k. Any standard computer subroutine for eigenvalue problem can solve this eigen-problem numerically.

Due to lack of time, we have not designed a computer procedure for this element. However, when it is needed,we should not find any particular difficulty in foming the dynamic mstrices in the manner described above.

3.7 TWO DIMENSIONAL ELEMENTS

A two dimensional element is one whose displacement at any point is described by two spatial parameters. The smallest geometric dimension,i.e. the thickness, of such an element is much smaller than the other two dimensions so that the configuration is usually represented by the configuration of its middle surface. If this element is a flat one, then it is a plate element otherwise it is a shell element.

The dynamic stiffness of a general two dimensional element has not received very much attention in the literature because of its continuous contour of boundaries and the coupling effects between the two dimensions. The only model which may be found is that of a plate element whose two opposite

edges are simply supported (ref 47). In this case, the governing equation of vibration is degenerated into that of a beam by choosing a set of distributed generalized coordinates on the two other edges which are not simply supported. An approximation method of finite strip (ref 91) was introduced by Cheung. Again, this method is limited to plate elements with the boundary conditions of two opp ite edges prescribed and therefore the treatment of the plate element is degenerated to that of one dimensional elements. Dill and Pister (ref 92) introduced a retangular plate element where the displacements of the four edges are expressible by Fourier series. However, the coupling effects between the two spatial coordinates give rise to a large matrix, and the numerical convergency is very poor.

In section (3.3.2), we have discussed a retangular plate element whose two opposite edges are simply supported. In the following, we shall use the new method of section (3.5) to derive a dynamic stiffness matrix for the plate where all the edges are subjected to prescribed boundary displacements.

We need two sets of information about the vibrating plate element to form the dynamic matrices. One is the modal information when all edges of the plate are clamped and the other is the static deflection patterns when the plate is subjected to unit boundary displacements. We study the free vibration of a clamped plate first. To this end, Leissa (ref 93) summarized the natural frequencies and the corresponding modes for various types of plates with triangular, rectangular, polygonal and circular shapes. Therefore, we could construct the dynamic matrices for these plate elements as well. However, we shall demonstrate the method by rectangular members.

So far, the most popular method of calculating the natural frequencies and modes for an individual member is that of Rayleigh-Ritz. Although polynomial coordinate functions have been used by many authors (ref 107), Mikhlin has proved that the Ritz system for polynomial coordinate functions is numerically unstable (ref 16). To eliminate the effect, we use beam

H= 0,0000% 0

	EIGENVALUE 1,295386900 3	5,389512041 3	5,389512041 3
1	-0,999788652	0,000000000	-0,014204368
2	0,000000000	0,871634363	0,000000000
3	-0,000000000	0,488306868	0,000000000
4	-0,000000000	-0,000000000	0,000000000
5	-0,000000000	0,000000000	-0,707011251
6	-0,019735023	-0,000000000	0,704090840
7	-0,000000000	0,028751367	0,000000000
8	-0,000000000	-0,028431466	0,000000000
9	-0,000000000	0,009186979	-0,000000000
10	0,000000000	-0,004283518	0,000000000
11	0,004936516	0,000000000	-0,065355356
12	-0,000000000	0,000000000	0,000000000
13	-0,000000000	-0,000000000	0,000000000
14	-0,000000000	-0,004080334	0,000000000
15	0,000000000	-0,003631408	0,000000000
16	0,000000000	-0,000000000	-0,005970877
17	0,002533115	-0,000000000	0,004796278
18	0,000000000	0,004961832	0,000000000
19	0,000000000	0,003117282	0,000000000
20	-0,000000000	0,000000000	-0,000000000
21	0,000000000	0,000000000	0,010459385
22	0,001509606	0,000000000	-0,003858548
23	-0,000000000	-0,001529929	0,000000000
24	-0,000000000	0,001129718	-0,000000000
25	-0,000000000	0,000000000	-0,000000000
26	0,000000000	-0,000000000	0,000000000
27	0,000000000	-0,001040922	0,000000000
28	-0,000000000	-0,001014994	0,000000000
29	0,000000000	-0,001006082	0,000000000
30	0,000000000	-0,000918902	-0,000000000
31	0,000310000	-0,000000000	0,001466308
32	-0,000000000	0,000000000	-0,000000000
33	0,000000000	-0,000000000	0,000000000
34	-0,000000000	0,000332232	-0,000000000
35	0,000000000	0,000305031	0,000000000
36	0,000000000	-0,000000000	0,000000000

	EIGENVALUE 1.172256892 4	1.732817833 4	1.743871199 4
1	0.000000000	-0.001991802	0.000000000
2	0.008837478	-0.000000000	0.001768280
3	0.004950927	-0.000000000	0,000990625
4	-0.000000000	0.000000000	0,000000000
5	0.000000000	-0.006018745	0.000000000
6	-0.000000000	0.003886341	-0,000000000
7	0.007416740	-0.000000000	0,001178120
8	-0.007334218	0.000000000	-0,001165011
9	-0.903259560	0.000000000	-0,004535635
10	0.421144245	-0.000000000	0,002116649
11	-0.000000000	-0.006147602	-0,000000000
12	-0.000000000	-0.000000000	0,000000000
13	-0.000000000	0.000000000	-0,000000000
14	0.059565842	-0.000000000	0,003321524
15	0.035017294	0.000000000	0,002957064
16	0.000000000	0.704059688	-0,000000000
17	-0.000000000	-0.704125566	-0,000000000
18	-0.000025045	-0.000000000	0,001742015

Fig (3.7.1)

19	-0.000015250	0.000000000	0.001[illegible]1136
20	-0.000000000	0.000000000	-0.000000000
21	-0.000000000	-0.065512457	-0.000000000
22	-0.000000000	-0.005472993	-0.000000000
23	-0.004984909	-0.000000000	0.800557547
24	0.002942505	-0.000000000	-0.590272[illegible]26
25	-0.000000000	-0.000000000	0.000000000
26	-0.000000000	-0.000000000	-0.000000000
27	0.009749909	0.000000000	-0.003114990
28	0.009507046	-0.000000000	-0.003057598
29	-0.002772429	-0.000000000	0.07311[illegible]835
30	-0.002552175	-0.000000000	0.066777936
31	0.000000000	-0.01618[illegible]64	-0.000000000
32	-0.000000000	0.000000000	0.000000000
33	0.000000000	-0.000000000	0.000000000
34	0.001015555	0.000000000	-0.013207729
35	0.000952588	-0.000000000	-0.012751454
36	-0.000000000	-0.000000000	0.000000000

EIGENVALUE 2.72756015[illegible] 4 2.7275601576 4 4.4355297698 4

1	-0.000000000	0.000000000	-0.000000000
2	-0.465506646	-0.000000000	-0.019657140
3	0.871634803	-0.000000000	0.035409504
4	-0.000000000	-0.998881062	-0.000000000
5	-0.000000000	-0.000000000	0.000000000
6	-0.000000000	0.000000000	-0.000000000
7	-0.026431406	-0.000000000	0.761156688
8	-0.028751507	-0.000000000	0.709047874
9	0.004283516	-0.000000000	-0.002591590
10	0.009186979	-0.000000000	-0.00517[illegible]800
11	-0.000000000	-0.000000000	0.000000000
12	0.000000000	-0.000000000	-0.000000000
13	-0.000000000	0.045746562	-0.000000000
14	0.003651408	-0.000000000	-0.037414059
15	-0.004080534	-0.000000000	0.042[illegible]504
16	-0.000000000	0.000000000	-0.000000000
17	-0.000000000	-0.000000000	-0.000000000
18	0.003917262	-0.000000000	0.012562245
19	-0.004961857	0.000000000	-0.020658250
20	0.000000000	-0.007100449	0.000000000
21	-0.000000000	0.000000000	-0.000000000
22	-0.000000000	-0.000000000	0.000000000
23	0.001129718	0.000000000	-0.000058354
24	0.001529929	-0.000000000	-0.000051914
25	0.000000000	0.000000000	-0.000000000
26	-0.000000000	0.009195286	-0.000000000
27	0.001014994	-0.000000000	0.004024355
28	-0.001060922	-0.000000000	-0.004247575
29	0.000918902	-0.000000000	-0.006088851
30	-0.001096088	-0.000000000	0.00615[illegible]522
31	-0.000000000	-0.000000000	-0.000000000
32	0.000000000	0.000000000	-0.000000000
33	-0.000000000	0.002005644	-0.000000000
34	0.000305031	-0.000000000	0.001500592
35	-0.000332252	0.000000000	-0.001036407
36	-0.000000000	0.000708581	-0.000000000

EIGENVALUE 4.435529760[illegible] 4 4.862105240[illegible] 4 5.870842755[illegible] 4

1	-0.000000000	-0.000000000	0.000000000
2	0.000000000	-0.003526017	0.000000000
3	0.000000000	0.006361940	0.000000000
4	-0.052600513	0.000000000	-0.017248540
5	0.000000000	0.000000000	0.000000000
6	-0.000000000	-0.000000000	0.000000000
7	-0.000000000	0.016202832	-0.000000000
8	-0.000000000	0.016890668	-0.000000000
9	-0.000000000	0.000928962	0.000000000
10	-0.000000000	0.001992569	-0.000000000
11	-0.000000000	-0.000000000	0.000000000
12	-0.706801310	0.000000000	-0.014341650
13	-0.702008689	0.000000000	-0.010302946
14	0.000000000	-0.002227417	0.000000000
15	0.000000000	0.010424321	-0.000000000
16	-0.000000000	0.000000000	-0.000000000
17	-0.000000000	0.000000000	-0.000000000
18	-0.000000000	-0.516987258	0.000000000
19	0.000000000	0.850170469	-0.000000000
20	0.076763357	-0.000000000	0.008537709
21	0.000000000	0.000000000	-0.000000000
22	0.000000000	-0.000000000	0.000000000
23	0.000000000	-0.001183627	0.000000000
24	-0.000000000	-0.001603206	0.000000000
25	0.012665234	-0.000000000	-0.702988616
26	0.012673039	0.000000000	-0.702564011
27	-0.000000000	-0.064069162	0.000000000
28	0.000000000	0.066628837	-0.000000000
29	0.000000000	-0.000293665	-0.000000000
30	-0.000000000	0.000321528	0.000000000
31	0.000000000	0.000000000	-0.000000000
32	0.016477233	0.000000000	-0.074841981
33	-0.007726361	0.000000000	-0.075004715
34	-0.000000000	-0.014468211	0.000000000
35	-0.000000000	0.015758406	-0.000000000
36	0.002241754	0.000000000	-0.024016593

EIGENVALUE 5.92136905.. 4   6.025018194.. 4   3.32601619.. 4

1	-0.014204368	0.000000000	0.003051387
2	-0.000000000	0.035409534	-0.000000000
3	0.000000000	0.019337140	0.000000000
4	-0.000000000	-0.000000000	-0.000000000
5	0.707011241	0.000000000	0.000000001
6	0.704090637	0.000000000	0.069177032
7	0.000000000	-0.707047874	-0.000000000
8	0.000000000	0.701158688	0.000000000
9	-0.000000000	-0.095128630	-0.000000000
10	-0.000000000	0.002391390	0.000000000
11	-0.063358348	-0.000000000	0.994051307
12	0.000000000	0.000000000	-0.000000000
13	0.000000000	-0.000000000	0.000000000
14	0.000000000	0.042039314	-0.000000000
15	0.000000000	0.037414039	-0.000000000
16	0.005070879	0.000000000	-0.000000000
17	0.004796278	0.000000000	-0.002301197
18	-0.000000000	0.020658250	0.000000000
19	-0.000000000	0.012562248	0.000000000
20	-0.000000000	0.000000000	-0.000000000
21	-0.010459385	0.000000000	-0.000000000

22	-0.003858548	0.000000000	-0.061222220
23	-0.000000000	0.000151914	-0.000000000
24	-0.000000000	-0.000148534	-0.000000000
25	-0.000000000	-0.000000000	0.000000000
26	0.000000000	-0.000000000	0.000000000
27	0.000000000	-0.004647575	-0.000000000
28	-0.000000000	-0.004324355	0.000000000
29	-0.000000000	0.008856322	-0.000000000
30	-0.000000000	0.008388851	-0.000000000
31	0.001466504	-0.000000000	0.017351439
32	-0.000000000	-0.000000000	-0.000000000
33	0.000000000	-0.000000000	0.000000000
34	-0.000000000	0.001634407	-0.000000000
35	-0.000000000	0.001506542	-0.000000000
36	0.000000000	-0.000000000	0.000000000

EIGENVALUE 9.550787421E 4 9.565036222E 4 1.162543551E 5

1	0.000000000	0.000924018	-0.000000000
2	-0.001906144	-0.000000000	-0.000646741
3	-0.001067657	-0.000000000	-0.000362317
4	-0.000000000	-0.000000000	0.000000000
5	-0.000000000	0.002644237	-0.000000000
6	0.000000000	0.008317160	-0.000000000
7	-0.039696604	0.000000000	-0.010007908
8	0.039255001	-0.000000000	0.009946535
9	-0.071883279	0.000000000	0.018408238
10	0.053516279	0.000000000	-0.013190345
11	-0.000000000	0.043201236	-0.000000000
12	-0.000000000	0.000000000	-0.000000000
13	-0.000000000	-0.000000000	0.000000000
14	-0.741539677	0.000000000	-0.025262404
15	-0.659954057	0.000000000	-0.022483034
16	-0.000000000	0.065592672	-0.000000000
17	-0.000000000	-0.001526476	-0.000000000
18	-0.006075568	-0.000000000	0.002452134
19	-0.003696545	-0.000000000	0.001478977
20	-0.000000000	0.000000000	-0.000000000
21	0.000000000	0.793987724	0.000000000
22	0.000000000	0.678678539	0.000000000
23	0.000854287	0.000000000	0.078167763
24	-0.000632293	0.000000000	-0.057719941
25	-0.000000000	-0.000000000	0.000000000
26	-0.000000000	-0.000000000	-0.000000000
27	0.044597294	0.000000000	0.004341037
28	0.043291393	-0.000000000	0.009595904
29	0.025822686	0.000000000	-0.751274567
30	0.023582943	-0.000000000	-0.667403744
31	-0.000000000	-0.078271417	0.000000000
32	-0.000000000	0.000000000	0.000000000
33	0.000000000	-0.000000000	0.000000000
34	0.005072974	0.000000000	-0.056543762
35	0.004557632	-0.000000000	-0.051745594
36	-0.000000000	-0.000000000	0.000000000

EIGENVALUE 1.162543551E 5 1.583941216E 5 1.755305967E 5

1	0.000000000	-0.000000000	-0.000000000
2	-0.009450927	-0.000000000	0.010657839
3	0.006857478	-0.000000000	-0.001906144

4	-0.000000000	-0.032606513	-0.000000000
5	-0.000000000	-0.000000000	0.000000000
6	-0.000000000	0.000000000	-0.000000000
7	-0.007534216	0.000000000	0.039255001
8	-0.007416740	-0.000000000	0.039096654
9	-0.421144245	0.000000000	-0.055516279
10	-0.903239560	0.000000000	-0.071083279
11	0.000000000	0.000000000	-0.010000000
12	0.000000000	0.796801311	0.010100000
13	-0.000000000	-0.702008687	-0.000000000
14	-0.053012294	-0.000000000	0.659954000
15	0.059565842	0.000000000	-0.741559677
16	-0.000000000	0.000000000	0.000000000
17	-0.000000000	0.000000000	-0.000000000
18	-0.000015230	-0.000000000	-0.093694543
19	0.000025045	-0.000000000	0.096075508
20	-0.000000000	0.076783537	-0.000000000
21	0.000000000	-0.000000000	-0.000000000
22	-0.000000000	-0.000000000	0.000000000
23	0.002942505	-0.000000000	-0.000652293
24	0.003984909	0.000000000	-0.000656297
25	-0.000000000	-0.012665234	0.000000000
26	0.000000000	0.012473039	-0.000000000
27	-0.009507046	0.000000000	-0.043291595
28	0.009749909	-0.000000000	0.044377294
29	0.002532175	-0.000000000	-0.023584643
30	-0.002772479	0.000000000	0.025322606
31	0.000000000	-0.000000000	-0.000000000
32	0.000000000	-0.016477233	0.000000000
33	0.000000000	-0.007726560	0.000000000
34	0.000432366	-0.000000000	0.004557452
35	-0.001015553	0.000000000	-0.005072974
36	-0.000000000	0.002241754	-0.000000000

EIGENVALUE 1.560307397[illegible] 5 1.8277239622 5 1.8277239624 5

1	-0.000000000	0.000000000	-0.000000000
2	0.000000000	0.000528007	-0.000000000
3	-0.000000000	-0.000042500	-0.000000000
4	0.001873456	-0.000000000	0.001001433
5	0.000000000	0.000000000	0.000000000
6	-0.000000000	-0.000000000	0.000000000
7	0.000000000	-0.000740237	0.000000000
8	-0.000000000	-0.000748566	-0.000000000
9	0.000000000	-0.007790424	0.000000000
10	0.000000000	-0.011708536	0.000000000
11	0.000000000	-0.000000000	-0.000000000
12	0.000000000	-0.000000000	0.015040744
13	-0.197771947	0.000000000	-0.014575651
14	0.000000000	0.040253162	-0.000000000
15	0.000000000	-0.045229319	0.000000000
16	-0.000000000	0.000000000	-0.000000000
17	0.000000000	0.000000000	-0.000000000
18	-0.000000000	-0.047315069	-0.000000000
19	0.000000000	0.078794934	-0.000000000
20	-0.997150001	0.000000000	-0.044617773
21	-0.000000000	-0.000000000	-0.000000000
22	0.000000000	-0.000000000	0.000000000
23	0.000000000	0.027741433	-0.000000000
24	0.000000000	0.005662132	0.000000000

25	0.000000000	0.000000000	-0.075145843
26	-0.003103739	0.000000000	-0.070018208
27	-0.000000000	0.692152580	0.000000000
28	-0.000000000	-0.709433771	-0.000000000
29	-0.000000000	0.004259528	0.000000000
30	-0.000000000	-0.004563454	0.000000000
31	-0.000000000	0.000000000	0.000000000
32	-0.000000000	-0.000000000	0.702241805
33	-0.072552562	-0.000000000	0.697111975
34	0.000000000	-0.045306905	0.000000000
35	-0.000000000	0.049347125	0.000000000
36	0.007206064	0.000000000	-0.079783028

EIGENVALUE 2,104488743& 5 2.109040958& 5 2.200022569& 5

1	-0.001991802	0.000000000	0.000924018
2	-0.000000000	0.006061940	0.000000000
3	-0.000000000	0.003396017	-0.000000000
4	0.000000000	0.000000000	0.000000000
5	0.006018745	-0.000000000	-0.009744227
6	0.003886341	0.000000000	0.038517151
7	-0.000000000	-0.016890868	-0.000000000
8	0.000000000	0.016702932	-0.000000000
9	0.000000000	0.001992369	0.000000000
10	0.000000000	-0.000928962	0.000000000
11	-0.006169603	-0.000000000	0.043201256
12	0.000000000	-0.000000000	-0.000000000
13	0.000000000	0.000000000	0.000000000
14	0.000000000	0.010424321	-0.000000000
15	0.000000000	0.009277417	0.000000000
16	-0.704039695	0.000000000	-0.065592675
17	-0.704025559	0.000000000	-0.061526475
18	-0.000000000	-0.850170419	-0.000000000
19	0.000000000	-0.516987258	0.000000000
20	0.000000000	-0.000000000	0.000000000
21	0.065512496	-0.000000000	-0.703987722
22	-0.0[illegible]	-0.[illegible]	0.[illegible]
23	-0.000000000	0.001603266	0.000000000
24	-0.000000000	-0.001183827	0.000000000
25	0.000000000	-0.000000000	0.000000000
26	-0.000000000	-0.000000000	-0.000000000
27	0.000000000	0.056628832	-0.000000000
28	-0.000000000	0.064969162	-0.000000000
29	-0.000000000	0.000321528	-0.000000000
30	-0.000000000	0.000293665	0.000000000
31	-0.016188004	0.000000000	-0.078271417
32	0.000000000	-0.000000000	0.000000000
33	-0.000000000	0.000000000	-0.000000000
34	0.000000000	-0.015758406	0.000000000
35	0.000000000	-0.014468211	0.000000000
36	-0.000000000	0.000000000	-0.000000000

EIGENVALUE 2.200022569& 5 2.[illegible]& 5 2.[illegible]& 5

1	0.000000000	0.000417891	-0.000000000
2	-0.000942500	0.000000000	-0.000455717
3	-0.000528007	0.000000000	-0.000242977
4	-0.000000000	-0.000000000	0.000000000
5	-0.000000000	-0.000000000	0.000000000
6	0.000000000	-0.001267951	0.000000000

7	0.000743560	0.000000000	0.000332655
8	-0.000746257	-0.000000000	-0.000624692
9	-0.016705350	0.000000000	0.000657214
10	0.007790424	-0.000000000	-0.000295845
11	-0.000000000	-0.000264569	0.000000000
12	-0.000000000	0.000000000	-0.000000000
13	0.000000000	-0.000000000	-0.000000000
14	-0.045229319	-0.000000000	-0.000007614
15	-0.040253102	0.000000000	-0.000045988
16	-0.000000000	-0.000000000	0.000000000
17	-0.000000000	-0.032530179	0.000000000
18	-0.078794934	0.000000000	-0.023577455
19	-0.047915069	-0.000000000	-0.014214096
20	0.000000000	-0.000000000	-0.000000000
21	0.000000000	0.000000000	0.000000000
22	-0.000000000	0.108464200	-0.000000000
23	-0.003662132	0.000000000	0.021410855
24	0.002704152	0.000000000	-0.015610019
25	0.000000000	0.000000000	0.000000000
26	0.000000000	0.000000000	0.000000000
27	-0.709853770	-0.000000000	-0.045912850
28	-0.692152560	0.000000000	-0.044769227
29	-0.004563454	-0.000000000	-0.065087276
30	-0.004259328	0.000000000	-0.059994945
31	0.000000000	0.993563758	0.000000000
32	0.000000000	-0.000000000	-0.000000000
33	-0.000000000	0.000000000	-0.000000000
34	-0.049547125	-0.000000000	0.731454696
35	-0.045504905	-0.000000000	0.671753255
36	0.000000000	0.000000000	-0.000000000

EIGENVALUE 3.198269067& 5 3.425731145& 5 3.436107621& 5

1	0.000000000	-0.000000000	0.000000000
2	-0.000998625	-0.000000000	0.001562517
3	0.001768280	-0.000000000	-0.000546741
4	-0.000000000	-0.000274554	0.000000000
5	-0.000000000	-0.000000000	0.000000000
6	-0.000000000	-0.000000000	-0.000000000
7	-0.001165011	0.000000000	0.003496555
8	-0.001174160	-0.000000000	0.001107908
9	-0.002116649	-0.000000000	0.000190345
10	-0.004539635	0.000000000	0.000408238
11	0.000000000	-0.000000000	-0.000000000
12	-0.000000000	0.014541649	0.000000000
13	-0.000000000	-0.010682946	0.000000000
14	-0.002954064	-0.000000000	0.002483039
15	0.003521524	0.000000000	-0.002262464
16	0.000000000	0.000000000	0.000000000
17	0.000000000	0.000000000	-0.000000000
18	0.001061081	-0.000000000	0.001478977
19	-0.001744914	0.000000000	-0.002452134
20	-0.000001030	0.000457709	-0.000000000
21	-0.000000000	0.000000000	-0.000000000
22	0.000000000	-0.000000000	0.000000000
23	-0.579977826	-0.000000000	-0.057713997
24	-0.890537547	0.000000000	-0.076167743
25	-0.000001000	0.722284617	-0.000000000
26	-0.000000000	-0.702564018	-0.000000000
27	0.003037394	-0.000000000	-0.000957904

28	-0.003114990	0.000000000	0.009641057
29	-0.066777956	0.000000000	0.667503744
30	0.073113636	0.000000000	-0.731274567
31	-0.000000000	0.000000000	0.000000000
32	0.000000000	0.074541980	-0.000000000
33	0.000000000	-0.073604716	0.000000000
34	-0.012181454	0.000000000	-0.061095594
35	0.013267729	0.000000000	0.066543702
36	0.000000000	-0.024618573	0.000000000

EIGENVALUE 4.64692040[illegible]8 5 4.6469204048 5 6.3510707278 5

1	-0.000000000	0.000000000	0.000000000
2	0.000000000	0.000242977	0.000000000
3	0.000000000	-0.000433717	0.000000000
4	0.001001905	-0.000000000	0.000647741
5	-0.000000000	-0.000000000	-0.000000000
6	-0.000000000	0.000000000	0.000000000
7	-0.000000000	-0.000626502	0.000000000
8	-0.000000000	-0.000633653	-0.000000000
9	0.000000000	0.000293643	-0.000000000
10	-0.000000000	0.000430212	-0.000000000
11	0.000000000	-0.000000000	0.000000000
12	-0.015940749	0.000000000	-0.000000000
13	-0.014673661	0.000000000	0.001051161
14	0.000000000	0.000085988	0.000000000
15	0.000000000	-0.000096619	0.000000000
16	0.000000000	-0.000000000	0.000000000
17	0.000000000	0.000000000	-0.000000000
18	-0.000000000	-0.014216996	0.000000000
19	0.000000000	0.023373436	-0.000000000
20	-0.049617723	0.000000000	-0.000694103
21	0.000000000	-0.000000000	0.000000000
22	-0.000000000	-0.000000000	-0.000000000
23	0.000000000	-0.015310919	-0.000000000
24	-0.000000000	-0.021410833	0.000000000
25	0.075143843	0.000000000	0.000000000
26	-0.070618268	0.000000000	-0.046251527
27	-0.000000000	0.044769227	-0.000000000
28	-0.000000000	-0.045912880	-0.000000000
29	-0.000000000	0.059924945	0.000000000
30	0.000000000	-0.065687276	0.000000000
31	-0.000000000	0.000000000	0.000000000
32	-0.702941861	0.000000000	0.000000000
33	0.697115060	-0.000000000	0.108951128
34	0.000000000	0.671733235	0.000000000
35	-0.000000000	-0.731634698	0.000000000
36	-0.079783026	-0.000000000	0.942977661

functions as coordinate functions. If we denote the k th natural frequency by ω_k and the corresponding mode by ϕ_k for the plate member, then,

$$\phi_k(x,y) = \sum_{i=1}^{n_i} \sum_{j=1}^{n_j} A_{ij}^k X_i(x) Y_j(y)$$

where X_i and Y_j are beam functions of clamped ends in x and y directions respectively, and n_i, n_j are the number of terms takes in the corresponding directions. The quantities ω_k and A_{ij}^k may be automatically generated by a programme described in chapter six when the plate dimensions are given. We list these quantities for the first 36 modes of a square plate when six modes of beam functions in each direction are used in fig (3.7.1). These quantities are computed by the programme of section (6.5.3)

For the static analysis of a rectangular element by the method of energy we can refer to Przemieniecki (ref 8), Zienkiewicz (ref 63) and Szilard (ref 62). The materials of the static analysis can be found from (ref 8) and summarized here for the completeness of deriving the dynamic matrices.

Consider the rectangular plate element as shown in fig (3.7.2), q_i $i=1,2,\ldots,12$ are chosen as generalized displacements. A deflection function that ensures both the deflection and slope compatibility on adjacent plate elements is given by

$$u(\xi,\eta) = [a_0]\{q\} \qquad (3.7.1)$$

where $u(\xi,\eta)$ is the transverse deflection, $\xi = x/a$, $\eta = y/b$, $\{q\} = [q_1, q_2, \ldots, q_{12}]^T$ and $[a_0]$ is a 1 x 12 matrix given by

$$[a_0] = [\xi_1\eta_1 \quad \xi_1\eta_2 b \quad -\xi_2\eta_1 a \quad \xi_1\eta_3 \quad -\xi_1\eta_4 b \quad -\xi_2\eta_3 a \quad \xi_3\eta_3 a \quad -\xi_3\eta_4 b \quad \xi_4\eta_3 \quad \xi_3\eta_1 \quad \xi_3\eta_2 b \quad \xi_4\eta_1 a] \qquad (3.7.2)$$

where $\xi_1 = (1+2\xi)(1-\xi)^2$ $\quad \eta_1 = (1+2\eta)(1-\eta)^2$

$\xi_2 = \xi(1-\xi)^2$ $\quad \eta_2 = \eta(1-\eta)^2$

$\xi_3 = (3-2\xi)\xi^2$ $\quad \eta_3 = (3-2\eta)\eta^2$

$\xi_4 = \xi^2(1-\xi)$ $\quad \eta_4 = \eta^2(1-\eta)$ $\qquad (3.7.3)$

By substituting equation (3.7.2) into equation (3.2.21), we obtain the consistence mass matrix

$$[m_o] = \frac{\rho ab}{176,400}$$

	1	2	3	4	5	6	7	8	9	10	11	12
1	24,336											
2	3,432 b	$624b^2$										
3	−3,432 a	$-484ab$	$624a^2$									
4	8,424	2,028 b	−1,188 a	24,336								
5	−2,028 b	$-468b^2$	$286ab$	−3,432 b	$624b^2$							
6	−1,188 a	$-286ab$	$216a^2$	−3,432 a	$484ab$	$624a^2$						
7	2,916	702 b	−702 a	8,424	−1,188 b	−2,028 a	24,336					
8	−702 b	$-162b^2$	$156ab$	−1,188 b	$216b^2$	$286ab$	−3,432 b	$624b^2$				
9	702 a	$169ab$	$-162a^2$	2,028 a	$-286ab$	$-468a^2$	3,432 a	$-484ab$	$624a^2$			
10	8,424	1,188 b	−2,028 a	2,916	−702 b	−702 a	8,424	−2,028 b	1,188 a	24,336		
11	1,188 b	$216b^2$	$-286ab$	702 b	$-162b^2$	$-169ab$	2,028 b	$-468b^2$	$286ab$	3,432 b	$624b^2$	
12	2,928 a	$286ab$	$-468a^2$	702 a	$-169ab$	$-162a^2$	1,188 a	$-286ab$	$216a^2$	3,432 a	$484ab$	$624a^2$

SYMMETRICAL

(3.7.4)

For the stiffness matrix, if we denote,

$$[D_o] = \frac{E h^3}{12(1-\nu^2)ab}\begin{bmatrix} K_{11} & K_{21}^T \\ K_{21} & K_{22} \end{bmatrix} \text{ and } \beta = b/a, \quad \text{then,}$$

(3.7.5)

$$[K_{11}] =$$

	1	2	3	4	5	6
1	$\frac{150}{35}(\beta^2+\beta^{-2}) + \frac{72}{25}$					
2	$\left[\frac{22}{35}\beta^2+\frac{78}{35}\beta^{-2} + \frac{6}{25}(1+5\nu)\right]a$	$\left(\frac{4}{35}\beta^2+\frac{52}{35}\beta^{-2} + \frac{8}{25}\right)b$				
3	$-\left[\frac{78}{35}\beta^2+\frac{22}{35}\beta^{-2} + \frac{6}{25}(1+5\nu)\right]a$	$-\left[\frac{11}{35}(\beta^2+\beta^{-2}) + \frac{1}{50}(1+60\nu)\right]ab$	$\left(\frac{52}{35}\beta^2+\frac{4}{35}\beta^{-2} + \frac{8}{25}\right)a^2$			
4	$\frac{54}{35}\beta^2-\frac{156}{35}\beta^{-2} - \frac{72}{25}$	$\left(\frac{13}{35}\beta^2-\frac{78}{35}\beta^{-2} - \frac{6}{25}\right)b$	$\left[-\frac{27}{35}\beta^2+\frac{22}{35}\beta^{-2} + \frac{6}{25}(1+5\nu)\right]a$	$\frac{150}{35}(\beta^2+\beta^{-2}) + \frac{72}{25}$		
5	$\left(-\frac{13}{35}\beta^2 + \frac{78}{35}\beta^{-2}+\frac{6}{25}\right)b$	$\left(-\frac{3}{35}\beta^2 + \frac{26}{35}\beta^{-2}-\frac{2}{25}\right)b^2$	$\left[-\frac{11}{70}\beta^2+\frac{11}{35}\beta^{-2} - \frac{1}{50}(1+5\nu)\right]ab$	$-\left[\frac{22}{35}\beta^2+\frac{78}{35}\beta^{-2} + \frac{6}{25}(1+5\nu)\right]b$	$\left(\frac{4}{35}\beta^2+\frac{52}{35}\beta^{-2} + \frac{8}{25}\right)b^2$	
6	$\left[-\frac{27}{35}\beta^2+\frac{22}{35}\beta^{-2} + \frac{6}{25}(1+5\nu)\right]a$	$\left[-\frac{22}{70}\beta^2+\frac{11}{35}\beta^{-2} + \frac{1}{50}(1+5\nu)\right]ab$	$\left(\frac{13}{35}\beta^2-\frac{4}{35}\beta^{-2} - \frac{8}{25}\right)a^2$	$-\left[\frac{78}{35}\beta^2+\frac{22}{35}\beta^{-2} + \frac{6}{25}(1+5\nu)\right]a$	$\left[-\frac{11}{35}(\beta^2+\beta^{-2}) + \frac{1}{50}(1+60\nu)\right]ab$	$\left(\frac{52}{35}\beta^2+\frac{4}{35}\beta^{-2} - \frac{8}{25}\right)a^2$

SYMMETRICAL

$[K_{21}] =$

	1	2	3	4	5	6
7	$-\frac{54}{35}(\beta^2+\beta^{-2}) + \frac{72}{25}$	$\left(-\frac{13}{35}\beta^2-\frac{27}{35}\beta^{-2} + \frac{6}{25}\right) l$	$\left(\frac{27}{35}\beta^2+\frac{13}{35}\beta^{-2} - \frac{6}{25}\right) a$	$-\frac{156}{35}\beta^2+\frac{54}{35}\beta^{-2} - \frac{72}{25}$	$\left[\frac{22}{35}\beta^2-\frac{27}{35}\beta^{-2} + \frac{6}{25}(1+5\nu)\right] b$	$\left(\frac{78}{35}\beta^2-\frac{13}{35}\beta^{-2} + \frac{6}{25}\right) a$
8	$\left(\frac{13}{35}\beta^2+\frac{27}{35}\beta^{-2} - \frac{6}{25}\right) b$	$\left(\frac{3}{35}\beta^2+\frac{9}{35}\beta^{-2} + \frac{2}{25}\right) b^2$	$\left[\frac{13}{70}(\beta^2+\beta^{-2}) - \frac{1}{50}\right] ab$	$\left[\frac{22}{35}\beta^2-\frac{27}{35}\beta^{-2} + \frac{6}{25}(1+5\nu)\right] b$	$\left(-\frac{4}{35}\beta^2+\frac{18}{35}\beta^{-2} - \frac{8}{25}\right) b^2$	$\left[-\frac{11}{35}\beta^2+\frac{13}{70}\beta^{-2} - \frac{1}{50}(1+5\nu)\right] ab$
9	$\left(-\frac{27}{35}\beta^2-\frac{13}{35}\beta^{-2} + \frac{6}{25}\right) a$	$\left[-\frac{13}{70}(\beta^2+\beta^{-2}) + \frac{1}{50}\right] ab$	$\left(\frac{2}{35}\beta^2+\frac{3}{35}\beta^{-2} + \frac{2}{25}\right) a^2$	$\left(-\frac{28}{35}\beta^2+\frac{13}{35}\beta^{-2} - \frac{6}{25}\right) a$	$\left[\frac{11}{35}\beta^2-\frac{13}{70}\beta^{-2} + \frac{1}{50}(1+5\nu)\right] ab$	$\left(\frac{26}{35}\beta^2-\frac{3}{35}\beta^{-2} - \frac{2}{25}\right) a^2$
10	$-\frac{156}{35}\beta^2 + \frac{54}{35}\beta^{-2}-\frac{72}{25}$	$\left[-\frac{22}{35}\beta^2+\frac{27}{35}\beta^{-2} - \frac{6}{25}(1+5\nu)\right] b$	$\left(\frac{78}{35}\beta^2-\frac{13}{35}\beta^{-2} + \frac{6}{25}\right) a$	$-\frac{54}{35}(\beta^2+\beta^{-2}) + \frac{72}{25}$	$\left(\frac{13}{35}\beta^2+\frac{27}{35}\beta^{-2} - \frac{6}{25}\right) b$	$\left(\frac{27}{35}\beta^2+\frac{13}{35}\beta^{-2} - \frac{6}{25}\right) a$
11	$\left[-\frac{22}{35}\beta^2+\frac{27}{35}\beta^{-2} - \frac{6}{25}(1+5\nu)\right] b$	$\left(-\frac{4}{35}\beta^2+\frac{18}{35}\beta^{-2} - \frac{8}{25}\right) b^2$	$\left[\frac{11}{35}\beta^2-\frac{13}{70}\beta^{-2} + \frac{1}{50}(1+5\nu)\right] ab$	$\left(-\frac{13}{35}\beta^2-\frac{27}{35}\beta^{-2} + \frac{6}{25}\right) b$	$\left(\frac{3}{35}\beta^2+\frac{9}{35}\beta^{-2} + \frac{2}{25}\right) b^2$	$\left[\frac{13}{70}(\beta^2+\beta^{-2}) - \frac{1}{50}\right] ab$
12	$\left(-\frac{78}{35}\beta^2+\frac{13}{35}\beta^{-2} - \frac{6}{25}\right) a$	$\left[-\frac{11}{35}\beta^2+\frac{13}{70}\beta^{-2} - \frac{1}{50}(1+5\nu)\right] ab$	$\left(\frac{26}{35}\beta^2-\frac{3}{35}\beta^{-2} - \frac{2}{25}\right) a^2$	$\left(-\frac{27}{35}\beta^2-\frac{13}{35}\beta^{-2} + \frac{6}{25}\right) a$	$\left[\frac{13}{70}(\beta^2+\beta^{-2}) - \frac{1}{50}\right] ab$	$\left(\frac{9}{35}\beta^2+\frac{3}{35}\beta^{-2} + \frac{2}{25}\right) a^2$

$[K_{22}] =$

	7	8	9	10	11	12
7	$\frac{156}{35}(\beta^2+\beta^{-2}) + \frac{72}{25}$					
8	$-\left[\frac{22}{35}\beta^2+\frac{78}{35}\beta^{-2} + \frac{6}{25}(1+5\nu)\right] b$	$\left(\frac{4}{35}\beta^2+\frac{52}{35}\beta^{-2} + \frac{8}{25}\right) b^2$	SYMMETRICAL			
9	$\left[\frac{78}{35}\beta^2+\frac{22}{35}\beta^{-2} + \frac{6}{25}(1+5\nu)\right] a$	$-\left[\frac{11}{35}(\beta^2+\beta^{-2}) + \frac{1}{50}(1+5\nu)\right] ab$	$\left(\frac{52}{35}\beta^2+\frac{4}{35}\beta^{-2} + \frac{8}{25}\right) a^2$			
10	$\frac{54}{35}\beta^2-\frac{156}{35}\beta^{-2} - \frac{72}{25}$	$\left(-\frac{13}{35}\beta^2+\frac{78}{35}\beta^{-2} - \frac{6}{25}\right) b$	$\left[\frac{27}{35}\beta^2-\frac{22}{35}\beta^{-2} - \frac{6}{25}(1+5\nu)\right] a$	$\frac{156}{35}(\beta^2+\beta^{-2}) + \frac{72}{25}$		
11	$\left(\frac{13}{35}\beta^2-\frac{78}{35}\beta^{-2} - \frac{6}{25}\right) b$	$\left(-\frac{3}{35}\beta^2+\frac{27}{35}\beta^{-2} - \frac{2}{25}\right) b^2$	$\left[\frac{13}{70}\beta^2-\frac{11}{35}\beta^{-2} - \frac{1}{50}(1+5\nu)\right] ab$	$\left[\frac{22}{35}\beta^2+\frac{78}{35}\beta^{-2} + \frac{6}{25}(1+5\nu)\right] a$	$\left(\frac{4}{35}\beta^2+\frac{52}{35}\beta^{-2} + \frac{8}{25}\right) b^2$	
12	$\left[\frac{27}{35}\beta^2-\frac{22}{35}\beta^{-2} - \frac{6}{25}(1+5\nu)\right] a$	$\left[-\frac{13}{70}\beta^2+\frac{11}{35}\beta^{-2} + \frac{1}{50}(1+5\nu)\right] ab$	$\left(\frac{78}{35}\beta^2-\frac{4}{35}\beta^{-2} - \frac{8}{25}\right) a^2$	$\left[\frac{78}{35}\beta^2+\frac{22}{35}\beta^{-2} + \frac{6}{25}(1+5\nu)\right] a$	$\left[\frac{11}{35}(\beta^2+\beta^{-2}) + \frac{1}{50}(1+5\nu)\right] ab$	$\left(\frac{52}{35}\beta^2+\frac{4}{35}\beta^{-2} + \frac{8}{25}\right) a^2$

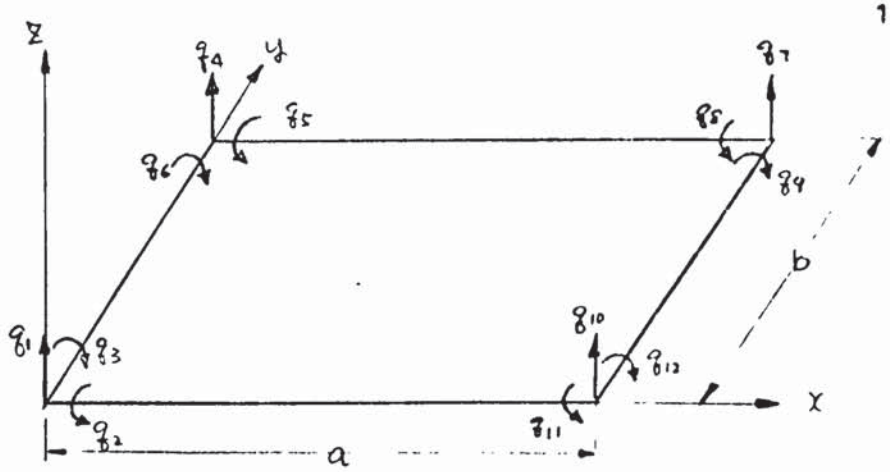


Fig (3.7.2) retangular plate element

The constants G_{lk} can be found from equation (3.5.27), which is

$$G_{lk} = \int_{vol} \{\phi_k\}^T [\rho] \{a_l\} d\,vol$$

$$= \rho h \sum_{i=1}^{n_i} \sum_{j=1}^{n_j} A_{ij}^{k} \int_0^b \int_0^a X_i(x) Y_j(y) a_l(x,y)\, dx\, dy \qquad (3.7.6)$$

where ϕ_k and a_l are scalars and a_l is the l th element of the matrix $[a_0]$ of equation (3.7.2). In order to evaluate the integrals of equation (3.7.6), we must calculate integrals of the form

$$\int_0^1 \psi_i(\xi)\, \xi_m(\xi)\, d\xi \qquad m=1,2,3,4 \qquad (3.7.7)$$

where ψ_i represents either $X_i(x)$ or $Y_i(y)$ and $\xi_m(\xi)$ are polynomials of ξ as given by equation (3.7.3). The values of the integrals (3.7.7) are calculated and listed in fig (3.6.6) for the first 50 modes. And from equation (3.5.35), we have the dynamic stiffness matrix of the retangular plate,

$$[D] = [D_0] - \omega^2 [m_0] - \omega^4 \sum_{k=1}^{n} \left[\frac{G_{ik} G_{jk}}{\omega_k^2 - \omega^2} \right]$$

where n=number of terms taken.

Note that since the expression (3.7.1) is not exact and the modal coorections are made in the interior of plate, to satisfy the differential equation, but not on the boundaries, we can not expect exact results for the dynamic matrices. The accuracy may be increase by increasing the number of generalized coordinates on the edges. This element could be a possibility for future work.

3.8 THREE DIMENSIONAL ELEMENTS

The dynamic stiffness method can be extended to three dimensional elements by the method of section (3.5). We consider a typical element of a parallelapiped as shown in fig (3.8.1).

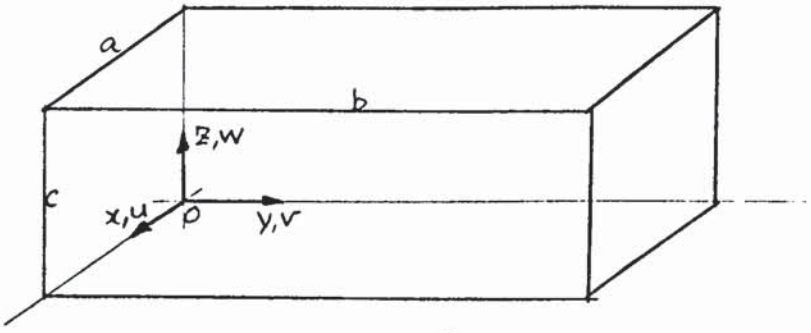


Fig (3.8.1) A three dimensional element

Two sets of information are needed. One is the normal modes of vibration of the element with the displacements on all six boundary surfaces reduced to zero and the other is the mass and stiffness matrices by finite element method of static displacement functions.

For the normal vibrations, when the higher order terms of derivatives are neglected, the governing equations are given by equations (3.5.1), which are, in full forms,

$$
\begin{aligned}
&(\lambda_0+G)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+G\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u+\rho\omega^2 u=0\\
&(\lambda_0+G)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+G\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)v+\rho\omega^2 v=0\\
&(\lambda_0+G)\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+G\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)w+\rho\omega^2 w=0
\end{aligned}
\qquad (3.8.1)
$$

where λ_0 and G are Lame's constants.

To solve the system of equations, we apply the Galerkin's method. Although the method is very popular in two dimensional cases, it does not seem to appear in the literature for three dimensional solids. Assume the displacement vector function $\{u\}=[u,v,w]^T$ has the form

$$\{u\} = \sum_{m,n,p} \{h\} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sin\frac{p\pi z}{c} \qquad (3.8.2)$$

where $\{h\} = \{h(m,n,p)\} = [h_1, h_2, h_3]^T$, and m,n,p are integers. The series of equation (3.8.2) are to be summed over a prescribed number of terms and $\{h\}$ are the constant vectors to be determined for different m,n,p. The function $\{u\}$ is satisfying the boundary conditions on the surfaces of the cuboid. We substitute the equation (3.8.2) into the equations (3.8.1) and multiply all the equations by $\sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \sin\frac{k\pi z}{c}$, where i,j,k are integers. Integrating over the whole volume of the body, by the orthogonality of the trigonometric functions, we have, after simplfications,

$$\begin{bmatrix} (\frac{m\pi}{a})^2(\lambda_0+G)+\Omega^2 & 0 & 0 \\ 0 & (\frac{n\pi}{b})^2(\lambda_0+G)+\Omega^2 & 0 \\ 0 & 0 & (\frac{p\pi}{c})^2(\lambda_0+G)+\Omega^2 \end{bmatrix} \begin{Bmatrix} h_1 \\ h_2 \\ h_3 \end{Bmatrix} = \omega^2 \rho \begin{Bmatrix} h_1 \\ h_2 \\ h_3 \end{Bmatrix} \qquad (3.8.3)$$

where $\Omega^2 = G\left[(\frac{m\pi}{a})^2+(\frac{n\pi}{b})^2+(\frac{p\pi}{c})^2\right]$.

Therefore, for every specific value of m,n, and p, there are three values for the natural frequencies ω^2, which are

$$\begin{aligned} \omega_1^2(m,n,p) &= (\tfrac{m\pi}{a})^2(\lambda_0+G)+\Omega^2 \\ \omega_2^2(m,n,p) &= (\tfrac{n\pi}{b})^2(\lambda_0+G)+\Omega^2 \\ \omega_3^2(m,n,p) &= (\tfrac{p\pi}{c})^2(\lambda_0+G)+\Omega^2 \end{aligned} \qquad (3.8.4)$$

and the corresponding eigenvectors are given by

$$\begin{aligned} \{h(1,m,n,p)\} &= [1 \quad 0 \quad 0]^T \\ \{h(2,m,n,p)\} &= [0 \quad 1 \quad 0]^T \\ \{h(3,m,n,p)\} &= [0 \quad 0 \quad 1]^T \end{aligned} \qquad (3.8.5)$$

independent of m,n,p. For the static deflection patterns, we work with local dimensionless coordinates (ξ,η,ζ) referred to the centroid:

$$\xi = (x-x_c)/a, \quad \eta = (y-y_c)/b, \quad \zeta = (z-z_c)/c \qquad (3.8.6)$$

where (x_c,y_c,z_c) is the coordinate of the centroid of the cuboid. The coordinates ξ,η,ζ range from -1 to +1 and only one coordinate varies along

one edge. A first order model using eight nodes and a second order model using twenty nodes are shown in fig (3.8.2). Note that there are three generalized coordinates associated with each node. The displacement patterns are given by

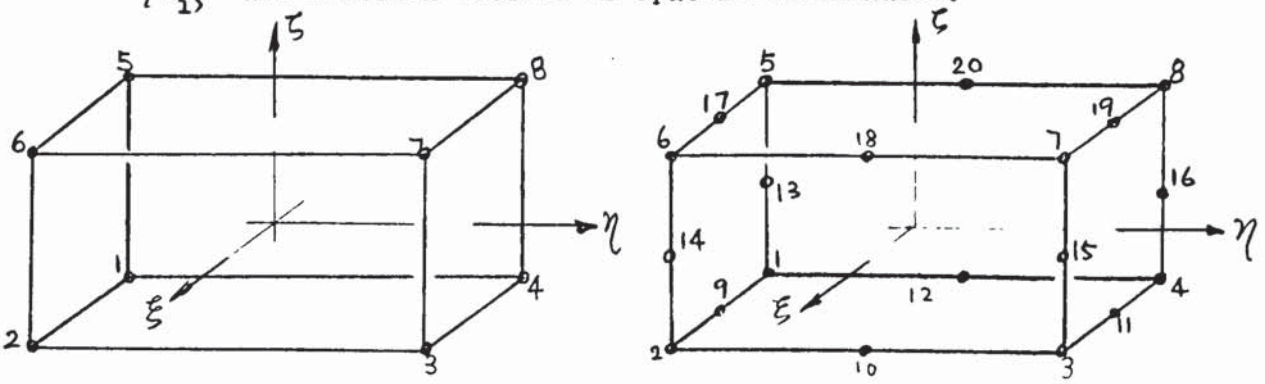
$$\{\bar{u}\} = [\{a_1\}, \{a_2\}, \cdots, \{a_N\}][\{q_1\}, \{q_2\}, \cdots, \{q_N\}]^T \qquad (3.8.7)$$

where $\{\bar{u}\} = [\bar{u}, \bar{v}, \bar{w}]^T$ is the static displacement vector

$\{q_i\}$, i=1,2, ... ,N are 3 x 1 generalized displacement vectors,

N=8 for linear model and N=20 for quadratic model,

$\{a_i\}$ are function vectors of spatial coordinates.



(a) first order model (b) second order model

Fig (3.8.2) Generalized coordinates of a cuboid element

The functions $\{a_i\}$ are given by

$$\{a_i\} = \frac{1}{8}(1+\xi_i\xi)(1+\eta_i\eta)(1+\zeta_i\zeta)\{e\} \qquad i=1,2,\ldots,8$$

for a linear model, where ξ_i, η_i, ζ_i are position coordinates of node i, either +1 or -1, and $\{e\} = [1 \quad 1 \quad 1]^T$ is a vector of unity. For a second order model,

$$\{a_i\} = \frac{1}{8}(1+\xi_i\xi)(1+\eta_i\eta)(1+\zeta_i\zeta)(\xi_i\xi+\eta_i\eta+\zeta_i\zeta-2)\{e\}, \quad i=1,2,\ldots,8$$

$$\{a_i\} = \frac{1}{4}(1-\zeta^2)(1+\xi_i\xi)(1+\eta_i\eta)\{e\}, \quad i=13,14,15,16$$

$$\{a_i\} = \frac{1}{4}(1-\xi^2)(1+\eta_i\eta)(1+\zeta_i\zeta)\{e\}, \quad i=9,11,17,19$$

$$\{a_i\} = \frac{1}{4}(1-\eta^2)(1+\xi_i\xi)(1+\zeta_i\zeta)\{e\}, \quad i=10,12,18,20.$$

The dynamic stiffness matrix is given by

$$[\mathcal{D}] = [\mathcal{D}_o] - \omega^2 [m_o] \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \sum_{l=1}^{3} \left[\frac{G_{jmnpl}\, G_{imnpl}}{\omega_l^2(m,n,p) - \omega^2} \right] \qquad i,j=1,2, \ldots,N$$

where $[\mathcal{D}_o]$ and $[m_o]$ are the stiffness and mass matrices consistence with the displacement pattern (3.8.7) respectively, and

$$G_{jmnpl} = \{h(l,m,n,p)\}^T \int_0^a \int_0^b \int_0^c \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sin\frac{p\pi z}{c} \{a_j\}\, dx\, dy\, dz$$

Implementing the technique for three dimensional continua would involve a considerable effort and no doubt, a number of difficulties would arise. We have not pursued this further from here.

3.9 STRUCTURAL DAMPING EFFECTS

Internal energy dissipation within the structural material is called the structural damping or hysteretic damping. It is essentially nonlinear. Therefore, an equivalent linearized form is required if the effect is to be included while keeping the computational efforts to a minimum. It has been found experimentally (ref 83) that the energy dissipated per cycle by internal damping is independent of frequency for most material and that periodic vibrations are chacterized by a shift of phase between stress and strain, as shown in fig (3.9.1). The energy dissipated per cycle of harmonic vibration is proportional to the area of the loop in the figure. For the convenience of study, as recommended by Snowdon (ref 83), internal damping may be modelled using a complex stiffness term, i.e.

$$\{\sigma\} = ([C] + i[\gamma])\{\varepsilon\} \qquad (3.9.1)$$

where $[\gamma]$ is a diagonal matrix. $[C]$ and $[\gamma]$ are real matrices while $\{\sigma\}$ and $\{\varepsilon\}$ are complex vectors whose real parts are representing the stress and the strain components respectively. This idealization represents the stress- strain relationship as shown in fig (3.9.2). Since only real quantities have physical meanings, both figs(3.9.1) and (3.9.2) are plots of the real parts of stress against the corresponding real part of strain. If viscoelastic material is concerned, both $[C]$ and $[\gamma]$ are frequency dependent matrices.

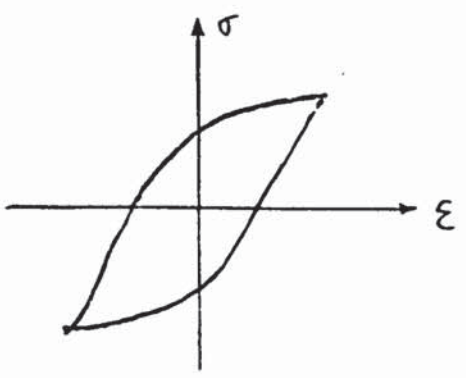


Fig (3.9.1) hysteretic damping

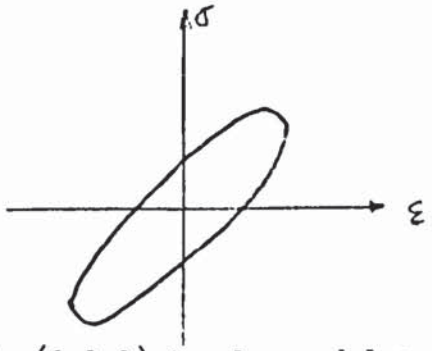


Fig (3.9.2) Complex modulus approximation for hysteretic damping

Under the assumption of complex modulus approximation of internal damping, the governing equations of motion of a vibrating body may be obtained by replacing the elastic constants in the governing equations of the corresponding undamped vibration by complex quantities. These complex quantities are chosen such that the energy dissipated per cycle of vibration is equivalent to the area of the loop of fig (3.9.2). Therefore the governing equation of lateral vibration of a beam has the form

$$(E+i\eta)I\frac{d^4W}{dx^4} + \rho A\frac{\partial^2 W}{\partial t^2} = f(x,t)$$

where η is the damping coefficient. The displacement patterns and fundamental matrices have the same forms as those for undamped elastic beam when replacing E by$(E+i\eta)$. Computer procedure to take account of internal damping for beam member was designed in a convenient form to be included in the main programme. This will be discussed in chapter six. Numerical examples will be given in chapter seven.

For more complicated structural elements, analytical solutions may not be possible even for this linear model. We shall discuss the possibility of applying the method of modal analysis to the damped vibration for complicated structures in chapter seven.

CHAPTER FOUR

FROM ELEMENTS TO SYSTEM

4.1 INTRODUCTION

When a structure is regarded as an assemblage of elements, it is preferable, from a computational point of view, to generate the dynamic matrices, or the governing equations of motion of the system with reference to global or system coordinates by :

(i) generating the element matrices with respect to coordinates local to , and convenient for, the element;

(ii) prior to assembly of the elements, the element properties are related to the global coordinates by a coordinate transformation; and

(iii) the assembly is then effected by the compatibility and equilibrium required among the elements to give the governing equations of motion of the system.

In chapter three we discussed in detail the various aspects of the analysis of an element by the dynamic stiffness method. This chapter is concerned with generating the governing equations of motion of the system in step (ii) and (iii) for harmonic vibration. The solution of these equations will be dealed with in chapter five.

The transformation of the local coordinates to the corresponding global coordinates is done in two stages. Firstly, the orientations of the local coordinates are transformed into alignment with those of global coordinates, and secondly, the alined local coordinates are related to the corresponding global coordinates by a Boolean matrix of transformation. In section(4.2) we discuss the requirements of compatibilty of displacements, equilibrium of forces, the invariarcy of energy, and their relation to the transfomation

of coordinates. Then a straight beam member in space is taken as example to give explicit formulae of transformation in section (4.3), and the generalization to the other structural elements is discussed in section (4.4).

4.2 TRANSFORMATION OF COORDINATES

These conditions in elasto-mechanics will be used to derive the formulae relating to coordinate transformation in the section. These conditions are equilibrium of forces, compatibility of displacements and invariance of energies with respect to transfomation of coordinates. Suppose there are N elements in the structural system considered. The kinetic energy of the j th element is

$$T^{(j)} = \frac{1}{2}\{\dot{q}^{(j)}\}^T [m^{(j)}] \{\dot{q}^{(j)}\} \qquad (4.2.1)$$

where $\{q^{(j)}\}$ is the vector of generalized displacements in the directions of the local generalized coordinates of the body and $[m^{(j)}]$ is the corresponding mass matrix. Before a Boolean matrix relating the global and local coordinates can be formed, the vector $\{q^{(j)}\}$ is transformed to $\{\bar{q}^{(j)}\}$ by a square matrix of direction cosines $[n^{(j)}]$, where $\{\bar{q}^{(j)}\}$ is the vector of local generalized displacements projected onto the corresponding global coordinates. The law of transformation of vector gives

$$\{q^{(j)}\} = [n^{(j)}] \{\bar{q}^{(j)}\} \qquad (4.2.2)$$

Substituting equation (4.2.2) into equation (4.2.1), we have

$$T^{(j)} = \frac{1}{2}\{\dot{\bar{q}}^{(j)}\}^T [n^{(j)}]^T [m^{(j)}] [n^{(j)}] \{\dot{\bar{q}}^{(j)}\} \qquad (4.2.3)$$

However, the kinetic energy of the same body with respect to the coordinates aligned to the global system is

$$T^{(j)} = \frac{1}{2}\{\dot{\bar{q}}^{(j)}\}^T [\bar{m}^{(j)}] \{\dot{\bar{q}}^{(j)}\} \qquad (4.2.4)$$

where $[\bar{m}^{(j)}]$ is the mass matrix corresponding to $\{\bar{q}^{(j)}\}$.

Since the kinetic energy is invariant under coordinate transformation, the comparison of equations (4.2.3) and (4.2.4) shows

$$[\bar{m}^{(i)}] = [n^{(i)}]^T [m^{(i)}] [n^{(i)}] \tag{4.2.5}$$

Similarly, for the stiffness matrix and the dynamic stiffness matrix,

$$[\bar{K}^{(i)}] = [n^{(i)}]^T [K^{(i)}] [n^{(i)}] \tag{4.2.6}$$

$$[\bar{D}^{(i)}] = [n^{(i)}]^T [D^{(i)}] [n^{(i)}] \tag{4.2.7}$$

After the alignment of coordinates, the vector $\{\bar{q}^{(i)}\}$ is further transformed to coincide with the generalized coordinates of the system $\{\tilde{q}^{(i)}\}$ by a Boolean matrix $[B^{(i)}]$, i.e.

$$\{\tilde{q}^{(i)}\} = [B^{(i)}] \{\bar{q}^{(i)}\} \tag{4.2.8}$$

where $[B^{(i)}]$ is a rectangular matrix consisting of zeros and ones such that when the i'th element of $\{\bar{q}^{(i)}\}$ is coincident with the k'th element of then $B^{(i)}_{ki} = 1$ otherwise zero. The actual values of the components of the dynamic matrices under such a transformation are unchanged, but their relative positions in the local and global matrices are changed. The transformed matrices according to equation (4.2.8) are

$$[\tilde{m}^{(i)}] = [B^{(i)}]^T [\bar{m}^{(i)}] [B^{(i)}] \tag{4.2.9}$$

$$[\tilde{K}^{(i)}] = [B^{(i)}]^T [\bar{K}^{(i)}] [B^{(i)}] \tag{4.2.10}$$

$$[\tilde{D}^{(i)}] = [B^{(i)}]^T [\bar{D}^{(i)}] [B^{(i)}] \tag{4.2.11}$$

where $[\tilde{m}^{(i)}]$, $[\tilde{K}^{(i)}]$ and $[\tilde{D}^{(i)}]$ are the dynamic matrices of the element with respect to the generalized displacements of the whole system $\{\tilde{q}^{(i)}\}$.

From the definition of dynamic stiffness matrix,

$$\{Q^{(i)}\} = [D^{(i)}] \{q^{(i)}\} \tag{4.2.12}$$

where $\{Q^{(i)}\}$ is the vector of generalized forces corresponding to $\{q^{(i)}\}$.

For the transformation corresponding to equation (4.2.2),

$$\{\bar{Q}^{(i)}\} = [\bar{D}^{(i)}] \{\bar{q}^{(i)}\} \tag{4.2.13}$$

where $\{\bar{Q}^{(i)}\}$ is the vector of generalized forces corresponding to $\{\bar{q}^{(i)}\}$.

From equations (4.2.7) and (4.2.2), we have

$$\{\bar{Q}^{(i)}\} = [n^{(i)}]^T [D^{(i)}] [n^{(i)}] \{\bar{q}^{(i)}\}$$

$$= [n^{(i)}]^T [D^{(i)}] \{q^{(i)}\},$$

and from equation (4.2.12),

$$\{\bar{Q}^{(j)}\} = [n^{(j)}]^T \{Q^{(j)}\} \tag{4.2.14}$$

which is the law of transformation of forces. Similarly,

$$\{\tilde{Q}^{(j)}\} = [B^{(j)}]^T \{\bar{Q}^{(j)}\} \tag{4.2.15}$$

where $\{\tilde{Q}^{(j)}\}$ is the vector of generalized forces corresponding to $\{\tilde{q}^{(j)}\}$.

Within the theory of Newtonian mechanics, the summation of all forces, including inertia forces at every point of a body must be zero in order to reach the state of dynamic equilibrium and the displacement vector at every point is single valued at all times. These are the requirements of equilibrium of forces and compatibility of displacements respectively. In mathematical terms, we have

$$\sum_{j=1}^{N} \{\tilde{Q}^{(j)}\} = \{\tilde{Q}\} = \{0\} , \tag{4.2.16}$$

and

$$\{\tilde{q}^{(1)}\} = \{\tilde{q}^{(2)}\} = \cdots = \{\tilde{q}^{(N)}\} = \{\tilde{q}\} \text{ , say ,} \tag{4.2.17}$$

where the superscripts denote the element numbers and N is the total number of elements in the system.

Now we have the dynamic stiffness equation,

$$\{\tilde{Q}^{(j)}\} = [\tilde{D}^{(j)}] \{\tilde{q}^{(j)}\} = [\tilde{D}^{(j)}] \{\tilde{q}\} .$$

Equation (4.2.16) gives

$$\sum_{j=1}^{N} \{\tilde{Q}^{(j)}\} = \left(\sum_{j=1}^{N} [\tilde{D}^{(j)}] \right) \{\tilde{q}\} = \{0\}$$

or

$$[\tilde{D}] \{\tilde{q}\} = \{0\} \tag{4.2.18}$$

where

$$[\tilde{D}] = \sum_{j=1}^{N} [\tilde{D}^{(j)}] \tag{4.2.19}$$

is the assembled dynamic stiffness matrix. Equation (4.2.18) is the governing equation of free vibration of the assembled system required.

Note that in generating $[\tilde{D}^{(j)}]$ according to equation (4.2.11), we do not literally carry out the triple matrix product. Because $[B^{(j)}]$ is large and sparse, this performance is very wasteful in time and storage. In fact, we form an algorithm which is an assembly code emerging from the Boolean nature of the connection matrix $[B^{(j)}]$ in global codes. More

precisely, we record only the non-zero element of the connection matrix in a vector $\{C^{(j)}\}$. The dimension of $\{C^{(j)}\}$ is equal to the number of generalized coordinates of the j th member of the system. If $B^{(j)}_{kl} = 1$ then the ith component of $\{C^{(j)}\}$ takes the value k otherwise zero. Suppose there are r generalized coordinates with each element and s generalized coordinates with the overall system. We have reduced the storage requirement of N Boolean matrices of order r x s to N code vectors of order r. The saving is N x s x (r -1). We can reduce the storage requirement even further by the fact that for every point on the boundaries, three (for translation), or six (for translation and rotation) generalized coordinates will come together in groups. Therefore the storage requirement is three or six times less if we give one code only to each group. For example, a straight beam member in space requires 12 generalized coordinates to specify its configuration, but six of them are associated with the same point and there only two (12/6) components are needed for the code vector. This saving is even more obvious for distributed generalized coordinate system on boundaries. If we express the boundaries of a membrane in terms of Fourier series, as in section (2.3), all the Fourier coefficients on edge will come together in a group and the saving of storage in designing such an algorithm is considerable. The computer procedure designed for such a purpose is presented in chapter six.

4.3 STRUCTURE OF STRAIGHT BEAMS

We obtain the explicit formulae of the transformation matrices presented in section(4.2) for a straight beam member in space.

We begin with the transformation matrix of rotation $[\eta^{(\phi)}]$. For the straight beam element as shown in fig (4.3.1), the complete dynamic matrices for the twelve generalized coordinates $q_i, i=1,2, \ldots, n$ are given in section (3.6). XYZ is the local coordinate orientation and $\overline{XYZ}$ is the global coordinate orientation. To fixed the relative orientation of XYZ with respect to $\overline{XYZ}$, at least the coordinates of three points on XYZ relative to $\overline{XYZ}$ must be specified. For the reasons of ease of measurement, the coordinates of A,B, and P are chosen; X is coincident with the axis of the beam and Y,Z are the two principal axes of the cross sectional area prependicular to X. If we denote the coordinates of points A,B, and P by $A(\bar{x}_a, \bar{y}_a, \bar{z}_a)$, $B(\bar{x}_b, \bar{y}_b, \bar{z}_b)$ and $P(\bar{x}_p, \bar{y}_p, \bar{z}_p)$, then after a translation movement such that PAB is translated to P'A'B' as shown in fig (4.3.2), the coordinates of $B'(x_b, y_b, z_b)$ and $P'(x_p, y_p, z_p)$ are given by

$$x_b = \bar{x}_b - \bar{x}_a \; ; \quad y_b = \bar{y}_b - \bar{y}_a ; \quad z_b = \bar{z}_b - \bar{z}_a$$

$$x_p = \bar{x}_p - \bar{x}_a \; ; \quad y_p = \bar{y}_p - \bar{y}_a ; \quad z_p = \bar{z}_p - \bar{z}_a . \qquad (4.3.1)$$

Now we are in a position to decide which quantities shall we retain to specify the transformation matrix $\{\eta^{(\phi)}\}$. Although the matrix $\{\eta^{(\phi)}\}$ is of order 12 x 12, the generalized coordinates can be separated into four groups (1,2,3), (4,5,6),(7,8,9),(10,11,12) such that the rotation matrices for each group are the same. This grouping is possible whenever the local generalized coordinates are in the directions of a Cartesian coordinate system. In this way, we write

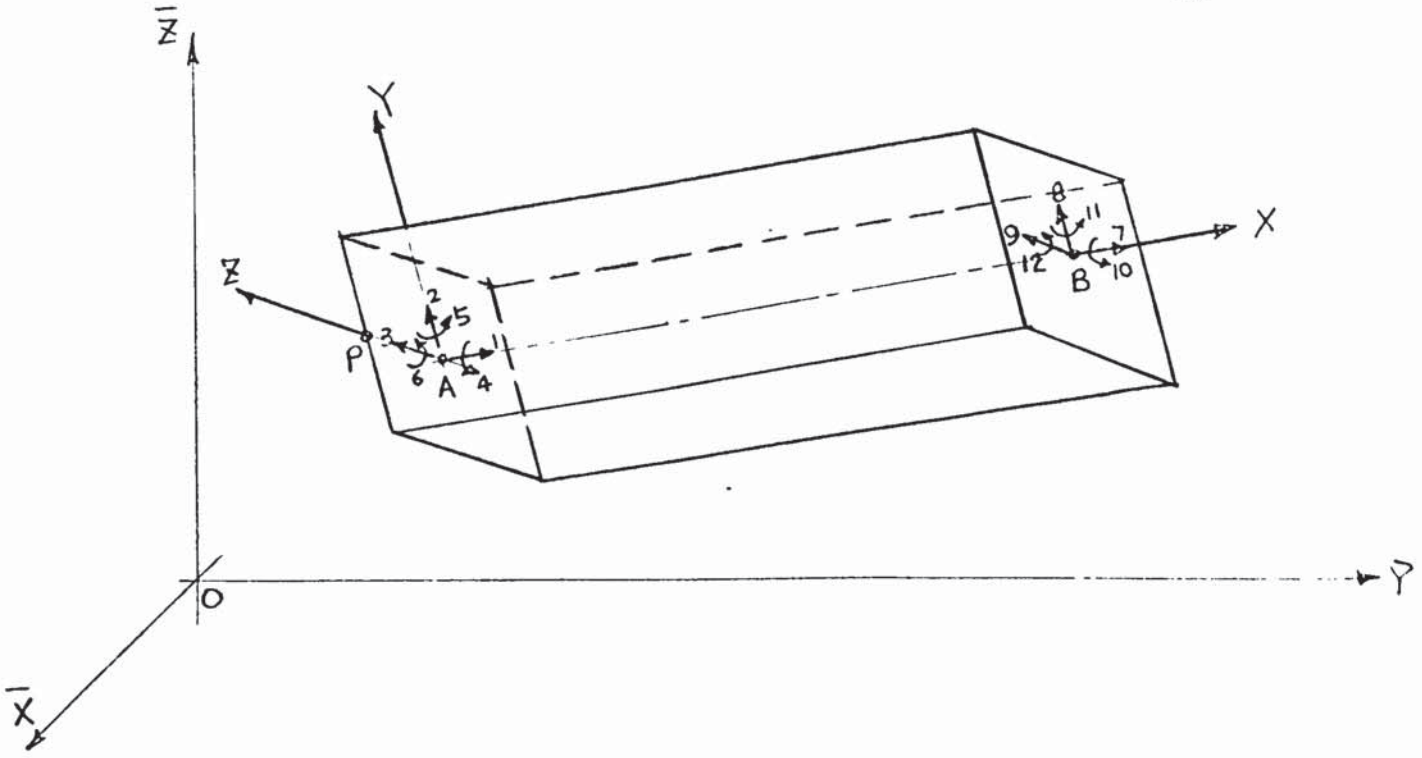


Fig (4.3.1) The coordination of a space beam member

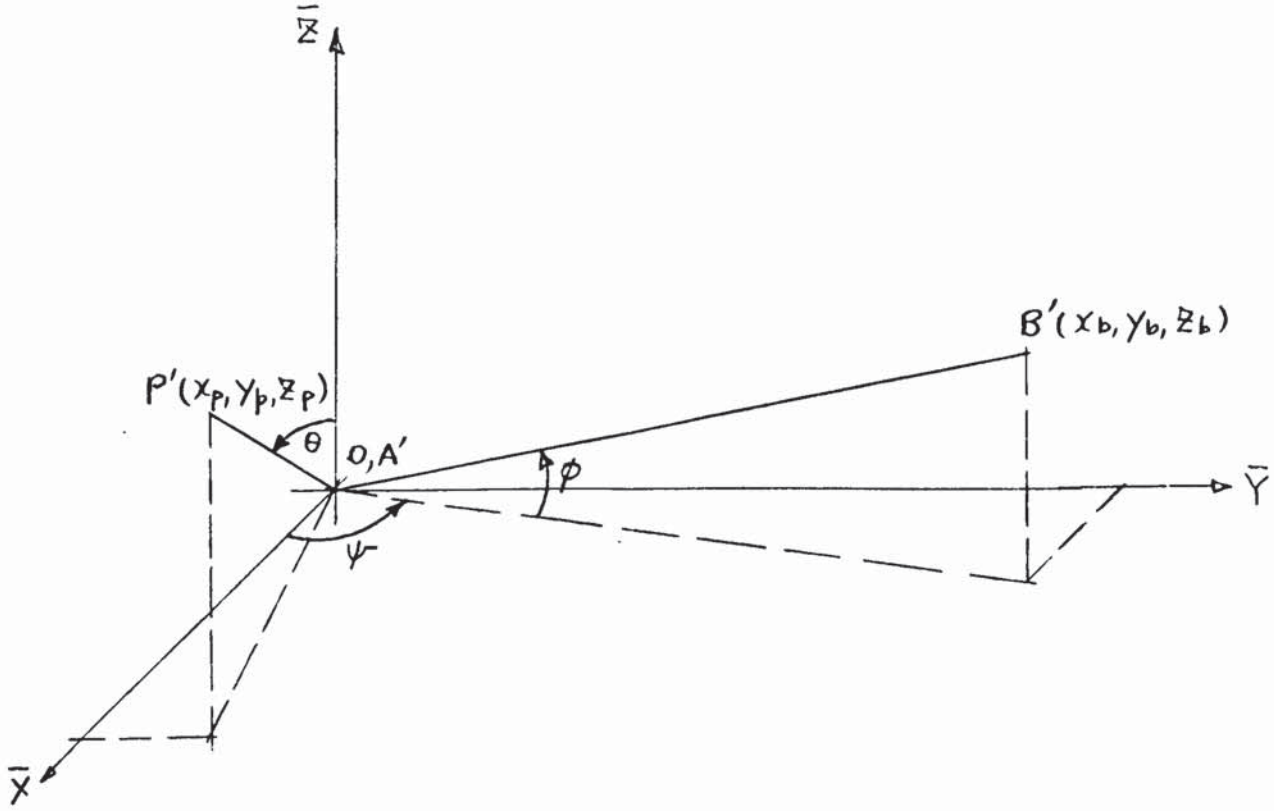


Fig (4.3.2) The Euler's angles and a space beam member

$$[n^{(j)}] = \begin{bmatrix} [n] & & 0 \\ & [n] & \\ & & [n] \\ 0 & & \end{bmatrix} \qquad [n] \qquad (4.3.2)$$

where $[n]$ is a 3 x 3 matrix of rotation. In analytical geometry, the components of $[n]$ are called the direction cosines. There is no need to store all nine components since they satisfy three normalizing conditions and three orthogonal conditions. However, it is not economical to store three components and then solve a set of six second order equations each time for the other six components. We choose a set of angles, the so-called Euler angles to specify the transformation for the following reasons:

(i) only three real quantities needed be retained

(ii) these are transformed to direction cosines by a simple calculation

(iii) these are extensively used in the other branches of mechanices, especially in aerodynamics, and no major changes are needed when modifying the present method to the vibration about steady motion

and (iv) the formulae are already available.

Euler angles are the angles ψ , ϕ , and θ as shown in fig (4.3.2) and they are calculated from the coordinates of B' and P' as

$$\begin{aligned} \psi &= \arctan\left(\frac{Y_b}{X_b}\right) \\ \phi &= \arctan \frac{Z_b}{(X_b^2 + Y_b^2)^{\frac{1}{2}}} \\ \theta &= \arctan \frac{(X_p^2 + Y_p^2)^{\frac{1}{2}}}{Z_p} \end{aligned} \qquad (4.3.3)$$

And the matrix of direction cosines is given by, (ref 19),

$$[n]^T = \begin{bmatrix} \cos\phi\cos\psi - \sin\phi\sin\psi\cos\theta & -\sin\phi\cos\psi - \cos\phi\sin\psi\cos\theta & \sin\theta\sin\psi \\ \cos\phi\sin\psi + \sin\phi\cos\psi\cos\theta & -\sin\phi\sin\psi + \cos\phi\cos\psi\cos\theta & -\sin\theta\cos\psi \\ \sin\theta\sin\phi & \sin\theta\cos\phi & \cos\theta \end{bmatrix} \qquad (4.3.4)$$

Therefore, the rotational transformation of coordinates of a beam is obtained in the following manner:

(i) calculate the transformation matrix $[n]$ according to equation(4.3.4);

(ii) Partition the 12 x12 matrix $[\mathcal{D}]$ into 16 submatrices of order 3 x 3,i.e.

$$[\mathcal{D}] = \begin{bmatrix} [\mathcal{D}_{11}] & [\mathcal{D}_{12}] & [\mathcal{D}_{13}] & [\mathcal{D}_{14}] \\ [\mathcal{D}_{21}] & [\mathcal{D}_{22}] & [\mathcal{D}_{23}] & [\mathcal{D}_{24}] \\ [\mathcal{D}_{31}] & [\mathcal{D}_{32}] & [\mathcal{D}_{33}] & [\mathcal{D}_{34}] \\ [\mathcal{D}_{41}] & [\mathcal{D}_{42}] & [\mathcal{D}_{43}] & [\mathcal{D}_{44}] \end{bmatrix}$$

(iii) the transformed matrix $[\overline{\mathcal{D}}]$ is given by

$$[\overline{\mathcal{D}}] = [[n]^T \times [\mathcal{D}_{ij}] \times [n]]$$

$$i,j=1,2,3,4 \qquad (4.3.5)$$

After aligning the local coordinates in the directions of the global system, we transform the matrix $[\overline{\mathcal{D}}^{(j)}]$ of the j th element of the space frame to coincide with the global coordinates represented by $\{\mathcal{Y}\}$. The code vector $\{c^{(j)}\}$ has two components since every 6 generalized coordinates of $\{\overline{q}^{(j)}\}$ come together. We partition the matrix $[\overline{\mathcal{D}}^{(j)}]$ obtained from equation (4.3.5) into four submatrices of order 6 x 6 such that the first six generalized coordinates are associated with point A and the others associated with B,

$$[\overline{\mathcal{D}}^{(j)}] = \begin{bmatrix} [\overline{\mathcal{D}}_{AA}^{(j)}] & [\overline{\mathcal{D}}_{AB}^{(j)}] \\ [\overline{\mathcal{D}}_{BA}^{(j)}] & [\overline{\mathcal{D}}_{BB}^{(j)}] \end{bmatrix}$$

Now every node of the overall structure is numbered from 1 to p, where p is the total number of nodes of the structure. The overall stiffness matrix will have the form

$$[\widetilde{\mathcal{D}}] = [[\mathcal{D}_{rs}]] \qquad r,s=1,2, \ldots ,p$$

where $[\widetilde{\mathcal{D}}_{rs}]$ are six by six submatrices. Before the assembly is effected, we initialize all the components of $[\widetilde{\mathcal{D}}]$ by setting them zeros.

The vector $\{c^{(j)}\}$ retains the information that if the node A of the element corresponds to the rth node of the system then the first element of $\{c^{(j)}\}$ is equal to r and if the node corresponds to the sth node of the system then the second element of $\{c^{(j)}\}$ is s, otherwise 0. When the superscript j is running from 1 to N, the number of elements, the submatrices

$[\widetilde{\mathcal{D}}_{rr}]$, $[\widetilde{\mathcal{D}}_{rs}]$, $[\widetilde{\mathcal{D}}_{sr}]$, and $[\widetilde{\mathcal{D}}_{ss}]$ are being added by the submatrices $[\bar{\mathcal{D}}^{(j)}_{AA}]$, $[\bar{\mathcal{D}}^{(j)}_{AB}]$, $[\bar{\mathcal{D}}^{(j)}_{BA}]$, and $[\bar{\mathcal{D}}^{(j)}_{BB}]$ respectively. The overall dynamic stiffness matrix is formed. All the other dynamic matrices can be formed in the same way. The related computer programming techniques are discussed in chapter six.

A simpler formulation is possible if the structure is of two dimensions. This should be used to save effort whenever possible. We discuss this as follows. For a beam member in a plane, the dynamic matrices are of order six by six. The generalized coordinates are as shown in fig(4.3.3).

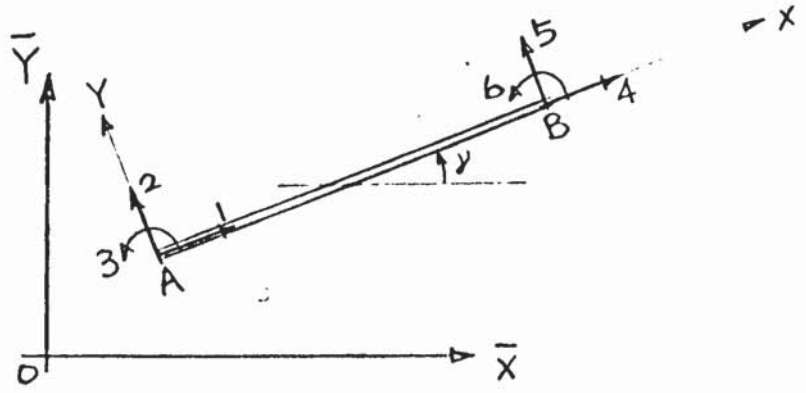


Fig (4.3.3) A beam member in a plane

The dynamic stiffness is first partitioned into four submatrices, each of order three by three,

$$[\mathcal{D}] = \begin{bmatrix} [\mathcal{D}_{11}] & [\mathcal{D}_{12}] \\ [\mathcal{D}_{21}] & [\mathcal{D}_{22}] \end{bmatrix}$$

The transformation matrix is

$$[n]^T = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where γ is the angle of orientation of X relative to $\bar{X}$. In this case, the code vector $\{C\}$ will have two components and its function is exactly as that of space beam except that three coordinates go together instead of six.

CHAPTER FIVE

NATURAL SOLUTION OF THE OVERALL VIBRATION PROBLEM

5.1 INTRODUCTION

The vibration analysis of elastic structural systems involves essentially two distinct procedures, the setting up of governing equations of motion of the structure and the solution of these equations. The first phase is achieved by discretizing the overall structure into structural elements, whose dynamic properties can be studied conveniently and whose motion can be described sufficiently by a set of generalized coordinates; and by forming the equations of motion of the overall structure according to the requirement of equilibrium of forces and compatibility of displacements among the adjacent members. This yields sets of simultaneous algebraic equations for equilibrium and stability (eigenvalue) problems, or sets of simultaneous differential equations with prescribed initial conditions for reponse problems. For the equilibrium problem, we have discussed the formulation of the equations for individual members in chapter three and for the overall system in chapter four. This chapter is devoted to the solution of these equations for the eigenvalue problem.

The solution of the eigenvalue problem is as important as its formulation since solution efficiency in terms of economy and reliability plays a vital part in the overall analysis of practical engineering structures. The eigenvalue problem to be solved has the form

$$[\mathcal{D}(\lambda)]\{q\} = \{0\} \qquad (5.1.1)$$

where λ is the eigenvalue and $\{q\}$ is the eigenvector. Both λ and $\{q\}$ are to be determined. The matrix $[\mathcal{D}]$ is a non-definite matrix. Eq(5.1.1)

may be written in an equivalent form,

$$[\mathcal{K}(\lambda)]\{q\} = \lambda[\mathcal{M}(\lambda)]\{q\} \qquad (5.1.2)$$

where $[\mathcal{K}(\lambda)]$ and $[\mathcal{M}(\lambda)]$ are positive definite matrices, and

$$[\mathcal{D}(\lambda)] = [\mathcal{K}(\lambda)] - \lambda[\mathcal{M}(\lambda)] .$$

The necessary and sufficient conditions for having non-trivial $\{q\}$ is

$$\det [\mathcal{D}(\lambda)] = 0 . \qquad (5.1.3)$$

We solve equation (5.1.3) instead of equation (5.1.2) for eigenvalue, because equation (5.1.3) represents a nonlinear algebraic equation which is much easier to visualize than the unfamiliar nonlinear eigenvalue problem (5.1.2), when no approximation is at hand. The conventional method of solving equation (5.1.3) is by regarding it as an algebraic equation in the unknown λ. Since $\det [\mathcal{D}(\lambda)]$ is highly irregular, the main difficulty of the equation by trial-and-error methods is that some roots may easily missed. If the matrices $[\mathcal{K}]$ and $[\mathcal{M}]$ in equation (5.1.2) were independent of λ, then a linear eigenvalue problem resulted. For linear eigenvalue problems, Wilkinson (ref 30) suggested a method of Sturm sequence which ensures a complete solution of equation (5.1.3). Note that, although for linear eigenvalue problems, $[\mathcal{M}]$ and $[\mathcal{K}]$ in equation (5.1.2) are eigenvalue independent, $[\mathcal{D}]$ in equation (5.1.3) is eigenvalue dependent. Williams and Wittrick (ref 26) extended the method to the eigenvalue problem when $[\mathcal{M}]$ and $[\mathcal{K}]$ are functions of eigenvalue. The success of the algorithm was reported in (ref 75,76). However, the difficulty remains in obtaining the eigenvector $\{q\}$ from equation (5.1.1) when $\det [\mathcal{D}] = 0$, which was also reported (ref 77). A new and stable algorithm which combines the inverse iteration technique, recommended by (ref 30) when approximate eigenvalue is available, and the theorem (3.4.1), for the calculation of the eigenvector $\{q\}$, will be presented in section (5.9).

5.2 THE NATURE OF NONLINEAR EIGENVALUE PROBLEMS

Eigenvalue problems arise when the question of stability of an equilibrium, dynamic or static, configuration of a structural system is concerned. Strictly speaking, every eigenvalue problem is nonlinear in nature, although within a certain limits, a linearized treatment gives satisfactory answers. A brief review of a general nonlinear eigenvalue problem (ref 21) is given in this section because the solution of equation (5.1.1) is a particular case of such a general problem.

A nonlinear eigenvalue problem is defined as the problem of finding appropriate solutions $\{u\}$ of a nonlinear equation of the form

$$\{F_i(u_1, u_2, \cdots, u_N, \lambda)\} = \{0\} \quad i=1,2,\ldots,N \qquad (5.2.1)$$

where F_i are nonlinear operators, depending on the parameter λ, operating on the unknown function $\{u\} = [u_1, u_2, \ldots, u_N]^T$. One of the first questions to be answered is whether or not equation (5.2.1) has any solution $\{u\}$ for a given value λ. If it does, the question of how many solutions arises, and then how this number varies with λ. Of particular interest is the so-called bifurcation process whereby a solution of equation (5.2.1) splits into two or more solutions as λ passes through a critical value λ_o, called an eigenvalue. The main problem is to determine the properties of the solutions $\{u\}$ and how they depend upon λ.

To illustrate the idea of bifurcation, let us consider the linear eigenvalue problem

$$[L]\{u\} = \lambda\{u\} \qquad (5.2.2)$$

where $[L]$ is a constant matrix and λ is a real number. The equation represents a natural vibration problem of a discrete system of mass and springs. λ corresponds to the excitation frequency and $\{u\}$ the vibration amplitude. For every value of λ, a trivial solution of equation (5.2.2) is

$$\{u\} = \{0\} \qquad (5.2.3)$$

For some particular values of λ , $\{u\}$ has nontrivial solutions as well. These particular values are the eigenvalues and the corresponding solutions of $\{u\}$ are the eigenvectors. Because of the homogeneous nature of equation (5.2.3), the eigenvectors are only determined to within a constant factor. An eigenvector is said to be normalized if the constant factor is so chosen that the norm of the vector $\|u\|$, i.e. the sum of the squares of its components, is equal to unity.

Suppose now there is a sequence of eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$ corresponding to a sequence of normalized eigenvectors $\{u_1\}$, $\{u_2\}$, ... , $\{u_N\}$ such that

$$[L]\{u_j\} = \lambda_j \{u_j\} \qquad (5.2.4)$$

where $$\|u_j\| = 1, \qquad j = 1, 2, \ldots, N \qquad (5.2.5)$$

The norm of $\{u\}$ of equation (5.2.3) is zero while that of equation (5.2.5) is unity. A graph of the norms of the solutions versus the parameter λ is called the reponse diagram. One of the possible plots of such diagrams is shown in fig (5.2.1).

This shows that the norm $\|u\|$ splits into two branches at each of the eigenvalues λ_j. The points $\lambda = \lambda_j$ are sometimes called the bifurcation points.

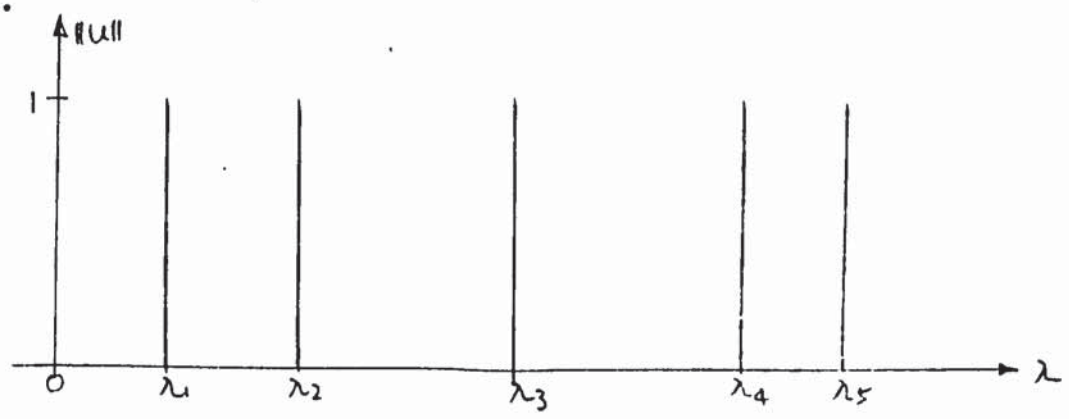


Fig (5.2.1) A respond diagram corresponds to linear eigenvalue problems

Consider now the nonlinear eigenvalue problem represented by equation (5.2.1) which has equation (5.2.2) as its linear approximation. One of the

possible plots of the response diagrams for the nonlinear eigenvalue problem is shown in fig (5.2.2). We give some examples in mechanical vibrations which are represented by the branches of fig(5.2.2).

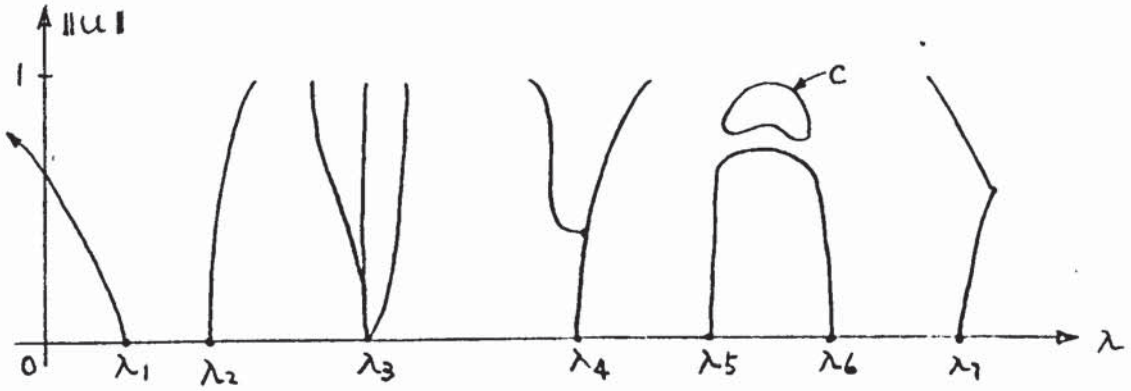


Fig (5.2.2) A response diagram corresponding to Nonlinear Eigenvalue Problems

The branch emanating from λ_1 when $\|u\|=0$ corresponds to the vibration problems of beam columns with the influence of inplane loads. The natural frequency parameter λ decreases with the increase of the loading and becomes zero when the respond curve cuts the axis $\lambda=0$. This signifies that even when the exciting frequency is zero, i.e. static, the structure will be unstable. This is an example of the so-called divergence instability. For the bifurcation at point λ_2, a typical example is the free vibration of a structure with large amplitudes. Since the internal forces increase with the amplitude of vibration, the natural frequency will increase as the amplitude of vibration increase according to the Rayleigh's theorem of constraining forces. At point λ_3, a phenomenon of dynamic instability occurs when a non-conservative structural system reaches its bifurcation point where several branches may be emanating from an eigenvalue of its linearly approximated system. For the problem of post bulking behaviour, the branches sprouting from λ_4 is an exemplifying plot where a secondary bifurcation may occur. As shown by the curve connecting λ_5 and λ_6, the phenomenon is called the flutter instability.

There may be branches that do not emanate from the eigenvalues of the linear approximation, such as the branch C. This is a typical example of variable damping. Finally, there may be a discontinuity as shown in the branch associated with λ_7. This corresponds to the snap-through instability of the vibration of shells.

Let us come back to the eigenvalue problem which represents the natural vibration problem dealt with in this chapter, i.e. equation (5.1.2).

$$[\mathcal{K}(\lambda)]\{q\} = \lambda[\mathcal{M}(\lambda)]\{q\} \qquad (5.2.6)$$

where the matrices $[\mathcal{K}(\lambda)]$ and $[\mathcal{M}(\lambda)]$ are symmetric and positive definite for all positive values of λ. In this case, a typical reponse diagram is given in fig (5.2.3), where λ_i, i=1,2, ... are the natural frequency parameters corresponding to the linearized approximation of the nonlinear eigenvalue problem, and λ'_i, i=1,2, ... are the respective exact values. Because in the linearized model artificial constraining forces are presented, the respective natural frequencies are higher than the exact ones according to Rayleigh's theorem on constraints. In the following sections, we shall restrict ourselves to the eigenvalue problem (5.2.6) and describe a new method for solving the problem and contrast it with existing methods.

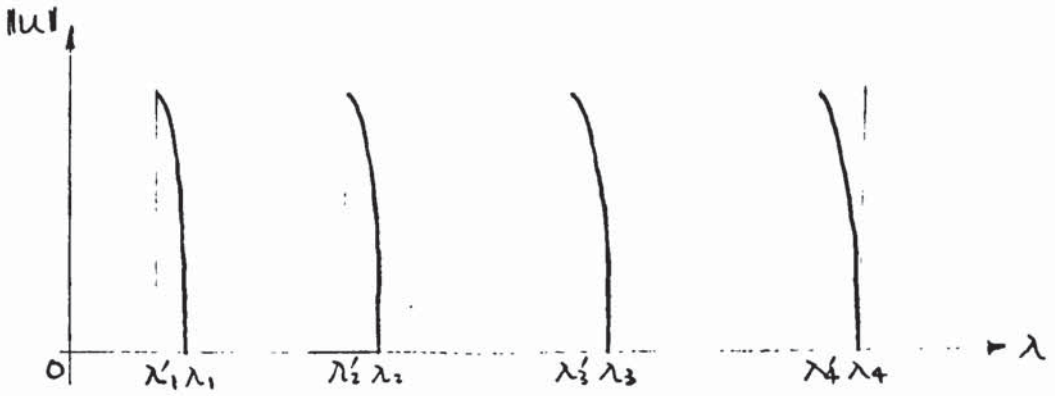


Fig (5.2.3) Response diagram for free vibration of continuous elastic system

5.3 STURM SEQUENCE FOR A POLYNOMIAL (ref 22)

Determining the natural frequencies (eigenvalues) is essentially a problem of solving a nonlinear frequency equation. There is a long list of available methods for such a purpose. Which method to use depends upon a number of factors, such as whether one needs all the roots or a few, whether the roots are complex or real, simple or multiple, whether a first approximation is available, and so on. In the case of vibration analysis, the only information about the roots is that all the natural frequencies are real and positive. The first problem is to locate the roots and find the first approximation. this is usually done by trial- and -error methods and the question of whether all the required roots have been isolated is difficult to answer and is a source of uncertainty. A procedure ultilizing the so-called Sturm sequence is a significant improvement. Before discussing the Sturm sequence more fully for our purpose, we illustrate it by determining the roots of a polynomial equation.

A way of solving the polynomial equation

$$f_o(x)= x^4 - 2.4\, x^3 + 1.03\, x^2 + 0.6\, x - 0.32 = 0 \qquad (5.3.1)$$

is as follows. We first construct a series of polynomial functions $f_i(x)$, i=1,2,3,4 according to

$$f_1(x) = d\, f_o(x)/\, dx$$

$$f_i(x) = (a_i x + b_i)\, f_{i-1}(x) - f_{i-2}(x) \qquad i=2,3,4,$$

where $f_i(x)$ is a polynomial of degree $4-i$ in x , and $a_i x + b_i$ are the quotients of $f_{i-2}/\, f_{i-1}$ and f_i is the remainder. Therefore, we have for equation (5.3.1),

$$f_o(x) = x^4 - 2.4x^3 + 1.03x^2 + 0.6x - 0.32$$

$$f_1(x) = x^3 - 1.8x^2 + 0.515x + 0.15$$

$$f_2(x) = x^2 - 1.3434x + 0.4071$$

$$f_3(x) = x - 0.6645 \,, \quad f_4(x) = 1.$$

Since we are only interested in the signs of these functions, the highest terms in x of these functions have been normalized to unity. The signs of these functions at various points of x are tabulated in fig (5.3.1).

x	f_o	f_1	f_2	f_3	f_4	sign changes
$-\infty$	+	-	+	-	+	4
-1	+	-	+	-	+	4
0	-	+	+	-	+	3
1	-	-	+	+	+	1
2	+	+	+	+	+	0
∞	+	+	+	+	+	0

Fig (5.3.1) Changes of sign in a sequence of polynomial

The last column are the number of sign changes in the sequence $\{f_1, f_2, f_3, f_4\}$ for various values of x. Such a sequence is called the Sturm sequence of f_o. A property of this sequence is that the number of real roots of f_o in the interval (a,b) is precisely the difference between the number of sign changes in the sequence $\{f_o(a), f_1(a), \ldots, f_4(a)\}$ and the corresponding number in $\{f_o(b), f_1(b), \ldots, f_4(b)\}$. Therefore from fig (5.3.1), we know that there is one root in (-1,0), one in (1,2) and two roots in (0,1), etc.. In this example, the roots turn out to be -0.5, 0.5, 0.8 and 1.6. One of the disadvantages of the Sturm sequence technique for polynomial equations is that, the number of polynomials to be evaluated at each value of x is equal to the order of the original polynomial equation. This is very uneconomical as far as computing time is concerned. It may be the reason why this technique is not widely used although it guarrantees the completeness of solution.

In the case of the eigenvalue problem of free vibration analysis, the extra calculations involved in determining the Sturm sequence is very small

when compared with the total so that its use is then worth while.

A Sturm sequence is defined as follows:

A sequence of functions $\{f_o(x), f_1(x), f_2(x), \ldots, f_n(x)\}$ which satisfy on an interval (a,b) of the real line the conditions,

1. each $f_i(x)$ is continuous;
2. the sign of $f_n(x)$ is constant for all x;
3. if $f_i(x)=0$ then $f_{i-1}(x)$ and $f_{i+1}(x)$ not equal to zero;
4. if $f_i(x)=0$ then $f_{i-1}(x)$ and $f_{i+1}(x)$ have opposite signs;
5. if $f_o(x)=0$ then for h sufficiently small

$$\text{sign } f_o(x-h) \,/\, f_1(x-h) = -1,$$

$$\text{sign } f_o(x+h) \,/\, f_1(x+h) = +1$$

is called a Sturm sequence.

A sturm sequence has the property that:

The number of roots of the function $f_o(x)$ in the interval (a,b), is the difference between the number of changes of sign in the sequences $\{f_o(a), f_1(a), \ldots, f_n(a)\}$ and $\{f_o(b), f_1(b), \ldots, f_n(b)\}$.

This called the Sturm's theorem (ref 22).

5.4 STURM SEQUENCE FOR THE LINEAR EIGENVALUE PROBLEM

We shall study the solution of the eigenvalue problem represented by

$$[\mathcal{K}]\{x\} = \omega^2[\mathcal{M}]\{x\} \qquad (5.4.1)$$

where $[\mathcal{K}]$ and $[\mathcal{M}]$ are symmetric positive definite constant matrices with respect to the frequency ω. Equation (5.4.1) represents the free vibration of a system of n generalized coordinates where the coordinate functions are linearized to the first approximation with respect to frequency. The frequency equation of such a system takes the form

$$\det[\mathcal{D}] = \Delta_0(\omega) = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{vmatrix} = 0 \qquad (5.4.2)$$

where $[\mathcal{D}] = [\mathcal{K}] - \omega^2[\mathcal{M}]$, $\alpha_{rs}(\omega)$ are real continuous functions of ω and symmetrical with respect to r,s. In a particular case of equation(5.4.1) when$[\mathcal{K}]$is a tridiagonal matrix and$[\mathcal{M}]$a diagonal matrix, Givens (ref 78) proved that the principal minors of$[\mathcal{D}]$forms a Sturm sequence with respect to ω of equation (5.4.2). Wilkinson (ref 80) and Gupta (ref 79) gave a proof for a more general form of$[\mathcal{K}]$and$[\mathcal{M}]$based on a method of matrix transformation. All these proofs involved advanced linear matrix theory and are difficult to extend to the nonlinear eigenvalue problem. For the nonlinear eigenvalue problem in vibration analysis,Williams and Wittrick (ref 26) gave a proof based on physical arguments.

In readiness for the construction of Sturm sequence of a nonlinear eigenvalue problem in free vibration of continuous system in next section, we design a new proof of the Sturm sequence property of the principal minors of $[\mathcal{D}]$ for general symmetric positive definite matrices$[\mathcal{M}(\omega)]$and $[\mathcal{K}(\omega)]$ without the assumption of the independency of frequence. The proof is based on a formula for the symmetric determinant (5.4.3)$_{\text{below}}$ and the Rayleigh's theorem of one constraint. This is extended to matrices with discontinuity, as in the case of free vibration of continuous elastic system, in next section.

We begin with the equation (5.4.2) where the components $\alpha_{rs}(\omega)$ are assumed continuous. For vibration frequency equation of a continuous system discontinuity of $\alpha_{rs}(\omega)$ may occur. We shall discuss it in next section. Let Δ_1 denote the determinant obtained from Δ_0 by striking out the first row and first column; let Δ_2 denote the determinant obtained from Δ_0 by striking

the first two rows and the first two columns; and so on. Finally, since Δ_n is obtained from by striking all rows and all columns, which is meaningless, we assume that $\Delta_n=1$ to fulfill the second condition of a Sturm sequence. The first condition of a Sturm sequence is automatically satisfied by the continuity of $\alpha_{rs}(\omega)$.

The physical meaning of Δ_i is that, $\Delta_i=0$ is the frequency equation of the original system with the first i generalized displacements constrained to be zeros. For example, if we consider the vibration of a discrete system in section (2.7), which is shown in fig (5.4.1a) with n=4.

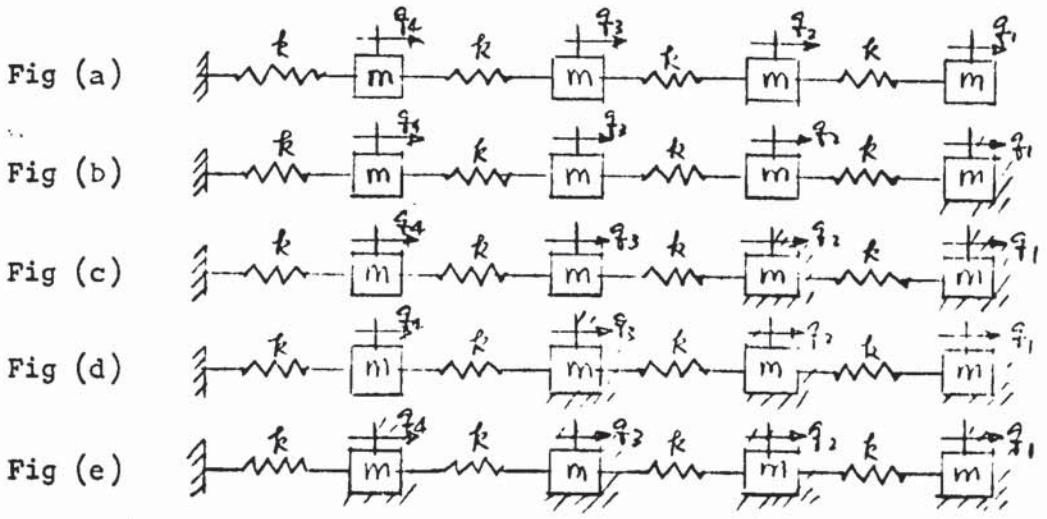


Fig (5.4.1) Sturm sequence and mass-spring system

The the frequency equation for the original system as shown in fig(a) is

$$\Delta_0=\begin{vmatrix} k-\omega^2m & -k & 0 & \\ -k & 2k-\omega^2m & -k & \\ 0 & -k & 2k-\omega^2m & -k \\ & & -k & 2k-\omega^2m \end{vmatrix}=0.$$

When the first row and the first column are struck out, we have the frequency equation to the system shown in fig(b) where the first generalized displacement q_1 is constrained to be zero,

$$\Delta_1 = \begin{vmatrix} 2k-m & -k & \\ -k & 2k-m & -k \\ & -k & 2k-m \end{vmatrix} = 0.$$

Similarly, we have the frequency equations for the system shown in fig (c) to fig (e) as $\Delta_2 = \begin{vmatrix} 2k-m & -k \\ -k & 2k-m \end{vmatrix} = 0,$

$$\Delta_3 = |(2k-m)| = 0,$$

and $\Delta_4 = 1,$ which is chosen deliberately to fulfil the requirement of constant sign.

The rest of this section is devoted to the construction of a Sturm sequence by means of $\Delta_0, \Delta_1, \ldots, \Delta_n$ and the proof the sequence $\{(-1)^i \Delta_i\}$, i=0,1,2, ... ,n is a Sturm sequence.

From the theory fo determinants (ref 24), it is known that

$$\frac{\partial \Delta_i}{\partial \alpha_{ii}} \frac{\partial \Delta_i}{\partial \alpha_{i+1,i+1}} - \left(\frac{\partial \Delta_i}{\partial \alpha_{i,i+1}}\right)^2 = \Delta_i \frac{\partial^2 \Delta_i}{\partial \alpha_{ii} \partial \alpha_{i+1,i+1}} \qquad (5.4.3)$$

$$i=0,1, \ldots, n-1.$$

This ensures the third and the fourth conditions of a Sturm sequence, i.e., if $\Delta_{i-1} = \partial \Delta_i / \partial \alpha_{ii}$ vanishes, the quantities Δ_i and $\Delta_{i-2} = \partial^2 \Delta_i / \partial \alpha_{ii} \partial \alpha_{i+1,i+1}$ must have opposite signs. Now only the condition 5 is to be fulfilled. The Rayleigh's theorem for one constraint upon a finite number, say n, of the generalized coordinates system states, see section (2.8), that, if one extra constraint is imposed on a vibrating system with natural frequencies $\omega_1 \leq \omega_2 \leq \ldots \leq \omega_n$, then the new natural frequencies $\tilde{\omega}_1 \leq \tilde{\omega}_2 \leq \ldots \ldots \leq \tilde{\omega}_{n-1}$ separate the old ones in the sense that $\omega_k \leq \tilde{\omega}_k \leq \omega_{k+1}$, k=1,2, ... , n-1.

This is possible only if the plots of $\Delta_0(\omega)$ and $\Delta_1(\omega)$ against ω have the forms shown in fig (5.4.2a) and (5.4.2b) respectively.

Fig (5.4.2a)

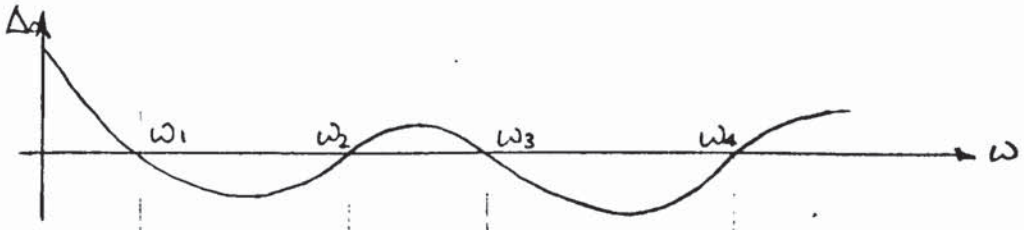
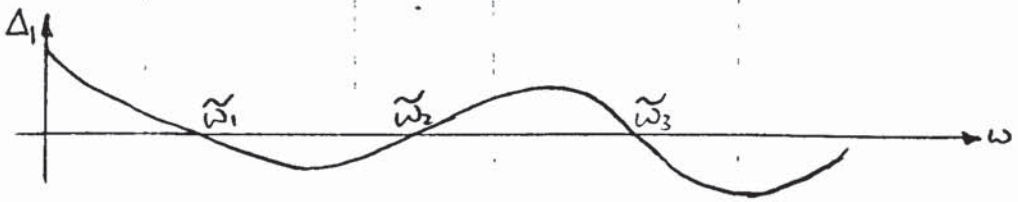


Fig (5.4.2b)



Therefore, for sufficiently small positive number h,

$$\text{sign}\,\frac{\Delta_0(\omega_i-h)}{\Delta_1(\omega_i-h)}=+1, \text{ and sign } \frac{\Delta_0(\omega_i+h)}{\Delta_1(\omega_i-h)}=-1, \quad i=1,2, \ldots ,n$$

which violates condition five. However, introducing a small modification, the following sequence, $\{ (-1)^n\Delta_0, (-1)^{n-1}\Delta_1, \ldots , (-1)\Delta_{n-1}, \Delta_n \}$, satisfies all five conditions required to be a Sturm sequence. It is well known that the principal minors Δ_r, r=0,1,2, ... ,n can be obtained from triangularizing the determinant Δ_0 using the Gauss elimination method without interchanges. This will be discussed in section (5.6). Actually, if the diagonal elements of the triangularized determinant are $d_1, d_2, \ldots , d_n$, then the principal minors are given by

$$(-1)^r\Delta_r = d_1 d_2 \cdots d_{n-r} .$$

The Sturm number of $\{(-1)^r\Delta_r\}$ is equal to the number of negative elements of $\left\{\frac{\Delta_0}{-\Delta_1}, \frac{-\Delta_1}{\Delta_2}, \ldots , \frac{(-1)^{n-1}\Delta_{n-1}}{(-1)^n\Delta_n}\right\}$, which is the number of negative elements of $\{ d_1, d_2, \ldots , d_n \}$.

Therefore, we come to the conclusion that:

The Sturm number of the frequency determinant of the form (5.4.2) is equal to the number of negative elements on the diagonal of the triangularized form of the frequency determinant by Gauss elimination without interchanges.

There is a minor disadvantage of the method that the Gauss elimination without interchanges is unstable numerically when approaching a root (ref 30). A new method similar to the Newton's iteration process for scalar algebraic equation which is not subjected to the restriction and converges very fast when approaching a root will be discussed later in section (5.7). We proceed on to construct a Sturm sequence of the frequency determinant of a vibrating continuous system.

5.5 STEP-BY-STEP METHOD FOR THE NATURAL VIBRATION PROBLEM

When analysing the harmonic vibration of a frame structure as represented in fig (5.5.1) by finite element method, the beam members such as AB, BC, etc. are being subdivided into smaller elements by the internal points a_1, a_2,..., b_1, b_2, ... etc. The harmonic forces are presented at the nodal points, such as B,C, and are arranged in a vector $\{P\}e^{i\omega t}$. If we arrange the generalized coordinates of displacements at the internal points to a vector $\{q\}e^{i\omega t}$ and those at nodal points to $\{Q\}e^{i\omega t}$, then the following equation of motion is obtained:

$$[D_1]\begin{Bmatrix}\{q\}\\ \{Q\}\end{Bmatrix} = \begin{bmatrix}[D_{qq}] & [D_{qQ}]\\ [D_{Qq}] & [D_{QQ}]\end{bmatrix}\begin{Bmatrix}\{q\}\\ \{Q\}\end{Bmatrix} = \begin{Bmatrix}\{0\}\\ \{P\}\end{Bmatrix} \qquad (5.5.1)$$

where $[D_{qq}] = [K_{qq}] - \omega^2[M_{qq}]$ etc.,

and $[K_{qq}]$ and $[M_{qq}]$ are the submatrices of the mass and the stiffness matrices repectively. These are of constant coefficients for the finite element method. Since the coefficient matrices in equation (5.5.1) are linear functions of ω^2, therefore there is no discontinuity over the whole frequency range and the natural frequency counting rule obtained in the last section applies. For a real structure, the dynamic stiffness equation can be set up by the dynamic stiffness method as:

$$[D_2(\omega)]\{Q\} = \{P\} \qquad (5.5.2)$$

where $[D_2]$ is a matrix of coefficients depending on ω ; Since discontinuities may happen in $[D_2]$ for some values of frequency, the natural frequency counting rule fails to apply.

Now, let us return to the equation (5.5.1). If we apply a Gauss elirination process to eliminate $[D_{Qq}]$, we have

$$\begin{bmatrix}[D_{qq}] & [D_{qQ}]\\ [0] & [D_a]\end{bmatrix}\begin{Bmatrix}\{q\}\\ \{Q\}\end{Bmatrix} = \begin{Bmatrix}\{0\}\\ \{P\}\end{Bmatrix} \qquad (5.5.3)$$

where $[D_2] = [D_{22}] - [D_{21}][D_{11}]^{-1}[D_{12}]$, which is a matrix containing coefficients which are complicated functions of ω, due to the present of $[D_{11}]$

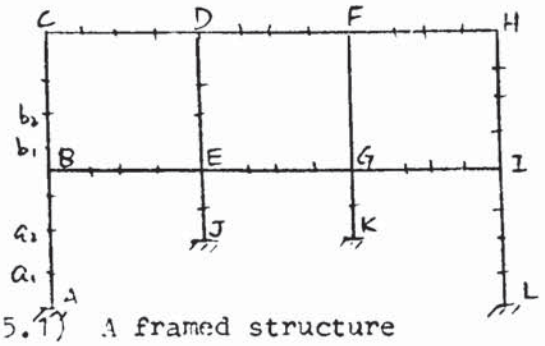


Fig (5.5.1) A framed structure

We try to analysis the results from equation (5.5.3), which is rewritten as :

$$[D_{11}]\{q\} + [D_{12}]\{Q\} = \{0\} \tag{5.5.4}$$

$$[D_2]\{Q\} = \{P\} \tag{5.5.5}$$

Firstly, if we denote the Sturm number of $[D]$ by $s[D]$ then

$$S[D_1] = S[D_{11}] + S[D_2] \tag{5.5.6}$$

Secondly, if the number of internal subdivision increases to approach to the real structure, then the equations (5.5.3) and (5.5.5) will be marched up eventually. Therefore,

$$S[D_1] = S[D_{11}] + S[D_2] \tag{5.5.7}$$

Thirdly, consider the case of partial vibration, i.e., when $\{Q\} = \{0\}$ and $\{q\} \neq \{0\}$. Then from equation (5.5.4), a free vibration problem to be solved is

$$[D_{11}]\{q\} = \{0\} \tag{5.5.8}$$

The physical problem represented by the equation is the natural vibration problem of the structure with all nodal points, such as points B and C, in fig(5.5.1) rigidly clamped. Therefore, the Sturm number of $[D_{11}]$

is equal to the total number of partial frequencies of the individual members of the structure. Therefore, we can conclude from equation (5.5.7) that:

> the Sturm number of a real structure in natural vibration analysis is equal to the sum of all the number of partial frequencies of the individual members of the structure and the Sturm number of the overall dynamic stiffness matrix associated with the generalized coordinates at all the nodal points.

5.6 ISOLATION OF NATURAL FREQUENCIES BY GAUSS ELIMINATION

The solution of the eigenvalue problem, linear or nonlinear, $[\mathcal{D}]\{q\} = \{0\}$, for natural frequencies, by the Sturm sequence technique needs the information of the values of $\det[\mathcal{D}]$ and the Sturm numbers at various values of ω. The information is obtained through the Gauss elimination process of triangularizing the matrix $[\mathcal{D}]$ into its upper triangular form. If the elements on the diagonal of this form are denoted ny $d_i, i=1,2,\ldots,n$, where n is the order of the matrix, then

$$\det[\mathcal{D}] = \prod_{i=1}^{n} d_i \qquad (5.6.1)$$

and the Sturm number is equal to the number of negative elements of the sequence $\{d_i\}$.

The Gauss elimination is equivalent to a triangular decomposition which reduces $[\mathcal{D}]$ to the form

$$[\mathcal{D}] = [L][U] \qquad (5.6.2)$$

where $[L] = l_{ij}$ is a lower triangular matrix with unit diagonal elements and $[U] = u_{ij}$ is an upper triangular matrix. The algorithm for both methods is

$$\begin{aligned} l_{ij} &= \frac{1}{u_{jj}}\left(\mathcal{D}_{ij} - \sum_{k=1}^{j-1} l_{ik}u_{kj}\right) && \text{when } j<i \\ u_{ij} &= \mathcal{D}_{ij} - \sum_{k=1}^{i-1} l_{ik}u_{kj} && \text{when } j \geq i \end{aligned} \qquad (5.6.3)$$

where $i,j=1,2,\ldots,n$ and $[\mathcal{D}]$ is symmetric, and

$$d_i = u_{ii} \qquad i=1,2,\ldots,n\,. \qquad (5.6.4)$$

To illustrate, suppose we are going to decompose the matrix

$$[\mathcal{D}(\omega)] = \begin{pmatrix} \omega^2 & \omega & 1 \\ \omega & \omega & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \qquad (5.6.5)$$

by Gauss elimination method at $\omega = 2$ and evaluate $\det[\mathcal{D}(2)]$ and its Sturm number. Now,

$$[\mathcal{D}(2)] = \begin{bmatrix} 4 & \cdot & 1 \\ 2 & \cdot & 2 \\ 1 & \cdot & 4 \end{bmatrix}$$

Applying formulae (5.6.3), we have

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & 3/2 & 1 \end{bmatrix} \quad \text{and} \quad [U] = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 1 & 3/2 \\ 0 & 0 & 3/2 \end{bmatrix}$$

Then, $\det[\mathcal{D}(2)] = \prod_{i=1}^{3} u_{ii} =$ x 3/2 =6 and the Sturm number =0, since all u_{ii}, i=1,2,3 are positive. Alternatively, we can develop the determinant as $\det[\mathcal{D}(\omega)] = -2\omega^4+2\omega^2-\omega$, when ω =2, then, $\det[\mathcal{D}(2)]$ = which provides a check.

This is a part of the complete algorithm for the solution of the eigenvalue problem which will be disscused in section (5.11). A computer programme has been designed for this purpose and this can be found in chapter six.

5.7 SOLUTION OF FREQUENCY EQUATION BY MODIFIED NEWTON'S METHOD

By the Sturm sequence technique, the eigenvalues (natural frequencies) of a linear or nonlinear eigenvalue problem in vibration analysis are isolated. This was discussed in the sections (5.4) to (5.6). We are in a position to solve the frequence equation

$$\Delta(\omega) = \det[\mathcal{D}(\omega)] = 0 \qquad (5.7.1)$$

for the natural frequencies. The classical method of Newton which solves an algebraic equation at a rate of quadratic convergence (ref 81) is modified to solve the determinantal equation (5.7.1). If an approximation to the root ω is denoted , then the Newton's method give successive

approximations according to

$$\omega^{(n)} = \omega^{(n-1)} - \frac{\Delta(\omega^{(n-1)})}{\Delta'(\omega^{(n-1)})} \qquad n=1,2,3,\ldots \quad (5.7.2)$$

where $\Delta' = \frac{d\Delta}{d\omega}$ (5.7.3)

Now, from the Gauss elimination, we have decomposed the matrix $[\mathcal{D}]$ into

$$[\mathcal{D}(\omega)] = [L(\omega)][U(\omega)] \quad (5.7.4)$$

therefore, $$\Delta(\omega) = \det[\mathcal{D}(\omega)] = \prod_{i=1}^{N} u_{ii}(\omega) \quad (5.7.5)$$

where N is the order of $[\mathcal{D}]$. Differentiating both sides of equation (5.7.4) with respect to ω,

$$\frac{\partial}{\partial\omega}[\mathcal{D}(\omega)] = [L(\omega)]\left(\frac{\partial}{\partial\omega}[U(\omega)]\right) + \left(\frac{\partial}{\partial\omega}[L(\omega)]\right)([U(\omega)])$$

or, in view of equation (3.4.5)

$$-2\omega[M(\omega)] = [L][U'] + [L'][U] \quad (5.7.6)$$

where $[M(\omega)] = [m_{ij}]$ is the mass matrix and

$$[U'] = \frac{\partial}{\partial\omega}[U], \quad [L'] = \frac{\partial}{\partial\omega}[L]$$

In the matrix equation (5.7.6), the unknowns $[L'] = [l'_{ij}]$ and $[U'] = [u'_{ij}]$ are solved as

$$l'_{ij} = \frac{1}{u_{jj}}\left(-2\omega m_{ij} - \sum_{k=1}^{j} l_{ik}u'_{kj} - \sum_{k=1}^{j-1} l'_{ik}u_{kj}\right), \qquad j < i$$

$$u'_{ij} = -2\omega m_{ij} - \sum_{k=1}^{i-1} l_{ik}u'_{kj} - \sum_{k=1}^{i-1} l'_{ik}u_{kj}, \qquad j \geq i$$

$$i,j=1,2,\ldots,n \quad (5.7.7)$$

where l_{ik} and u_{kj} are elements of $[L]$ and $[U]$ respectively. Differentiating both sides of equation (5.7.5) with respect to ω, we have

$$\Delta' = \frac{d\Delta}{d\omega} = \sum_{i=1}^{N} u'_{ii} \prod_{j\neq i} u_{jj} \quad (5.7.8)$$

where u'_{ii} are obtained from equations (5.7.7). Therefore, the Newton's successive approximations are given by

$$\omega^{(n)} = \omega^{(n-1)} - \prod_{i=1}^{N} u_{ii} \Big/ \left(\sum_{i=1}^{N} u'_{ii} \prod_{j\neq i} u_{jj}\right)$$

$$= \omega^{(n-1)} - 1 \Big/ \sum_{i=1}^{N} \left(\frac{u'_{ii}}{u_{ii}}\right) \qquad n=1,2,\ldots \quad (5.7.9)$$

This technique for vibration problem is new and effective and is recommended whenever the eigenvector is not wanted. The analysis of a cantilever beam is given in section (5.11) and compared with other methods.

5.8 FREQUENCY EXTRACTION BY INTERPOLATION METHODS

In the case where the mass matrix is difficult to obtain, or the frequency determinant is not behaving well in the neighbourhood of a natural frequency e.g. multiple roots or discontinuity, Newton's method is not convenient. Interpolation methods of extracting roots are alternatives. The interpolation methods discussed in this section will include the method of bisection, linear interpolation and quadratic interpolation. Then, we shall described a mixed combination of these methods to achieve the best convergency.

The bisection method is characterized by obtaining a better approximation c to a root of $f(x)=0$ within two bounds a and b using the formula $c=0.5*(a+b)$. Although the algorithm converges very slowly to a root, it is able to take care of the numerical instability induced by any badly behaving characteristics of the function.

The linear interpolation acheives a better root c by

$$c = a - (a-b)f(a)/(f(a)-f(b))$$

Ostrowski (ref 108) has shown that if $f(x)$ is continuous up to the second derivative in the neighbourhood of a simple root, then successive linear interpolation from a sufficiently good approximation give superlinear convergence to the root. The convergence is better than the bisection, but it suffers from the numerical numerical instability of the irregularity of the function.

The direct quadratic interpolation (ref 108) is based on fitting a quadratic, $y=Ax^2+Bx+C$, through three previous approximations x_0, x_1, x_2, uses the formula $x_3 = x_2 - 2C/(B \pm \sqrt{B^2 - 4AC})$

where x_3 is a better approximation,

$$A = ((x_1-x_0)y_2 + (x_0-x_2)y_1 + (x_2-x_1)y_0) / ((x_2-x_1)(x_1-x_0)^2),$$

$$B = [(x_1 - x_0)(2x_2 - x_1 - x_0)y_2 - (x_2 - x_0)^2 y_1 + (x_2 - x_1)^2 y_0] / [(x_2 - x_1)(x_1 - x_0)^2],$$

$$C = (x_2 - x_0)y_1 / (x_1 - x_0),$$

$$y_i = f(x_i), \quad i = 0, 1, 2,$$

and the sign which makes the denominater larger in absolute value should be chosen. For well-behaved functions this device saves about half the number of function evaluations per root on the average when compared with the linear method. The condition for applying this method is that all three values of x_0, x_1, x_2 are to be distinct and the function is single-valued within the interval. A computer programme has been designed for this purpose of extracting roots in the main programme. The programme is to be discussed in chapter six.

5.9 DETERMINATION OF EIGENVECTORS

Upon locating the natural frequencies (eigenvalues) by Sturm sequence method in section (5.5), reducing their bounds by Newton's method or interpolation methods in sections (5.7) and (5.8), we are in a position to find the normal modes (eigenvectors) and to improve the approximated natural frequencies.

Suppose ω_o is an approximation which satisfies

$$\det [\mathcal{D}(\omega)] = 0 \ . \qquad (5.9.1)$$

If $\omega^2 = p + \omega_o^2$ is a better approximation, where p is a small correction to be determined, and $\{q\}$ is the corresponding eigenvector, then the free vibration problem is governed by

$$[\mathcal{K}]\{q\} = \omega^2 [\mathcal{M}]\{q\},$$

or
$$[\mathcal{K}]\{q\} = (p+\omega_o^2)[\mathcal{M}]\{q\},$$

or
$$[\mathcal{D}]\{q\} = p[\mathcal{M}]\{q\} \qquad (5.9.2)$$

Here, $[\mathcal{D}]$ is an indefinite symmetric matrix, i.e. its principal minors are either positive or negative, and $[\mathcal{M}]$ is a positive definite symmetric matrix, i.e. its principal minors are positive. We have a reduced problem of solving the linear eigenvalue problem represented by equation (5.9.2) now. Here $[\mathcal{D}]$ and $[\mathcal{M}]$ are evaluated at $\omega = \omega_o$, and it is needed to solve for the smallest eigenvalue p and the corresponding eigenvector $\{q\}$ only.

If the complete solution of equation (5.9.2) were wanted, the Householder transformation method recommended by Wilkinson (ref 30) would be the most economical method in computing operations. Since we need the smallest p in absolute value, this method becomes uneconomoic. One method of solving the largest eigenvalue λ in absolute value for the eigenvalue problem

$$[A]\{x\} = \lambda\{x\} \qquad (5.9.3)$$

is the matrix iteration method characterized by

$$[A]\{x^{(i)}\} = \lambda\{x^{(i+1)}\},$$

$$\{X^{(0)}\} = [1, 1, \cdots, 1]^T \qquad i=0,1,2, \ldots$$

To apply the scheme to our problem, we premultiply equation (5.9.2) by $[m]^{-1}$

$$[m]^{-1}[D]\{q\} = p\{q\} \tag{5.9.4}$$

When comparing equations (5.9.3) and (5.9.4), we have the following iterative algorithm for the largest eigenvalue p in absolute value,

$$[m]^{-1}[D]\{q^{(i)}\} = p\{q^{(i+1)}\}$$

$$\{q^{(0)}\} = [1, 1, \cdots, 1]^T \qquad i=0,1,2, \ldots$$

If the smallest eigenvalue p in absolute value is required, the iteration is performed backward according to

$$[m]^{-1}[D]\{q^{(i+1)}\} = p\{q^{(i)}\}$$

$$\{q^{(0)}\} = [1, 1, \cdots, 1]^T \qquad i=0,1,2, \ldots \tag{5.9.5}$$

The numerical stability of the algorithm was proved by Wilkinson (ref 30). An improved eigenvalue for the original system of equations (5.9.1) may then be obtained from

$$\omega^2 = p + \omega_0^2 \ . \tag{5.9.6}$$

Unfortunately, when the order of the dynamic matrices are large, which is the case in practical engineering structural analysis, this small value p will be affected substantially by rounding off errors. Therefore, the improvement of the eigenvalue is more reliable when applying Rayleigh's Quotient with the improved eigenvector,

$$\text{i.e.} \qquad p = \frac{\{q\}^T[D]\{q\}}{\{q\}^T[m]\{q\}} \tag{5.9.7}$$

For the economical use of computing time and storage, however, the above algorithm is not directly applicable to actual computation. It is because of three reasons. Firstly, the inversion of the mass matrix is undesirable; secondly, the matrix product on the left hand side of equation (5.9.5) is unsymmetrical; and finally, the process will involve a large amount of equation solving routine for each cycle. The following algorithm is recommended by Wilson (ref 64) to improve the efficiency of the process.

The procedure begins with the solution of an initial sarting iteration value ω_0 for the required frequency ω from the root isolation and extraction algorithm, as described in the last few sections. It follows by the evaluation of $[\mathcal{D}]$ and $[m]$ at $\omega = \omega_0$, and the decomposition $[\mathcal{D}] = [L][D][L]^T$ where $[D]$ is the diagonal matrix and $[L]$ lower triagular matrix as described in section (5.6). An initial vector $\{y_0\} = [1, 1, \ldots, 1]^T$ is chosen as the starting iteration vector and the following steps are performed during a typical (k+1)'th cycle for the determination of $\{y\}$.

(i) Solve the following set equations by backward substitution

$$[L][D][L]^T \{x_{k+1}\} = \{y_k\}$$

$$\{\bar{y}_{k+1}\} = [m]\{x_{k+1}\}$$

(ii) Compute a new estimate of p from

$$p_{k+1} = \frac{\{x_{k+1}\}^T \{y_k\}}{\{x_{k+1}\}^T \{\bar{y}_{k+1}\}}$$

(iii) Test the relative error

$$\varepsilon_1 = \left| \frac{p_{k+1}}{\omega_0^2} \right|$$

If it is found less than the accuracy required for ω then $(\omega_0^2 + p_{k+1})^{\frac{1}{2}}$ is accepted as the required natural frequency else if $|p_{k+1}| < |p_k|$ then go to step (i) else decompose $[\mathcal{D}] - p_{k+1}[m] = [L][D][L]^T$ and goto step (i).

This technique is called the inverse iteration and is able to take advantage of the band form of the dynamic matrices. A computer programme was designed for this method and is built into the main programme in chapter six. In case where a number of n close roots are discovered during the root isolation routine, the numerical convergence of inverse iteration is very poor for each eigenvector. Wilson and Bath (ref 64), recommended a block iteration method by which all these n eigenvectors are found simultaneously with the corrections to the natural frequencies. This is also incorperated in the main programme for repeated natural frequencies.

5.10 COMPARISON OF METHODS OF SOLUTION

As mentioned in the previous sections, there are two numerical methods to solve the complete vibration problem of a structure for natural frequencies and the corresponding modes by dynamic stiffness. With the Sturm sequence technique, one locates the natural frequencies by Gauss elimination and determines the eigenvectors by solving the displacements from the sets of resulting singular linear equations, and the o er employs the inverse iteration technique. For the comparison of the performance of these methods, a cantilever beam of unit geometric and elastic parameters, i.e. $l=A=EI=\rho=1$, is used. Suppose the cantilever is clamped at the left hand end and the displacement and slope at the other end are u_1 and u_2 respectively. Then from (ref 10), the natural frequencies and the corresponding modes for the first three modes are shown in Fig (5.10.1).

For the first mode of vibration, we round the frequency parameter λ to 1.88. Since there are two homogeneous linear equations in two unknowns for the eigenvector, the direct solutions of u_2 from these equations when u_1 is set to unity are 1.36894 and 1.37514 repectively, as shown in fig (5.10.2). The percentage error of u_2 is about the same as that of the eigenvalue. For the second mode, the percentage error of u_2 thus obtained is about three times that of the eigenvalue and for the third mode, the percentage error of u_2 is about seven times that of the eigenvalue. Therefore, we see that the accuracy of the eigenvectors are uncertained relative to the eigenvalue by this method. Alternatively, the inverse iteration method gives very accurate results of u_2 for all modes, regardless the accuracy of the eigenvalue assumed. The improved frequency parameters by Rayleigh's quotient are very close to the true ones.

mode no.	natural freq.	end displacement $u_1 e^{i\omega t}$	end slope $u_2 e^{i\omega t}$
1	1.87510	1	1.37650
2	4.69409	1	4.78077
3	7.85476	1	7.84867

Fig (5.10.1) Normal modes of a cantilever

mode number		1	2	3
exact values	freq. param.	1.87510	4.69409	7.85476
	u_1	1	1	1
	u_2	1.37650	4.78077	7.84867
approx. values	λ	1.88	4.70	8.00
	% error	(0.260)	(0.126)	(1.85)
Gauss elimination	u_2 from 1st eq.	1.36894	4.75899	6.82347
	% error	(0.584)	(0.456)	(13.062)
	u_2 from 2nd eq.	1.37514	4.75820	6.97429
	% error	(0.205)	(0.472)	(11.140)
inverse iteration	u_2	1.37650	4.78286	7.83332
	% error	(0.000)	(0.087)	(0.020)
	improved λ	1.87511	4.69409	7.85476
	% error	(0.000)	(0.000)	(0.000)

Fig (5.10.2) Comparison of results by Gauss elimination and inverse iteration methods

In engineering practi e, the order of the dynamic matrices are very large . The solution of the frequency determinant for natural frequencies by Gauss elimination is subjected to rounding off error and very close estimations of natural frequencies may not be obtained in many cases. The direct solution for eigenvector will involve the solution of a set of homogeneous equations with their determinant singular. The solution is unstable (ref 30), as shown numerically in the above example. Further, an improvement of a natural frequency can not be found if the precision of the arithmetic operations is fixed.

The inverse iteration method is iterative in nature. This means that the rounding off error will not affect the overall accuracy of the eigen-solution. As shown in the above example, if the approximated eigenvalue is reasonably close to a real eigenvalue, then the process will approach to the corresponding eigenvector. The eigenvalue can be improved by the stationary property of the Rayleigh's Quotient of the system. The algorithm will be summarized in next section.

5.11 SUMMARY OF THE METHODS OF SOLUTION

The chapter is concerned with the determination of natural frequencies and corresponding modes from the matrix equation

$$[K(\omega)]\{q\} = \omega^2 [M(\omega)]\{q\} \qquad (5.11.1)$$

or

$$[D(\omega)]\{q\} = \{0\} \qquad (5.11.2)$$

We summarize the methods of solution as following:

Since equation (5.11.1) may not be solved without an initial approximation, the natural frequencies are first determined approximately by solving an equivalent determinant frequency equation

$$\det [D(\omega)] = 0 \qquad (5.11.3)$$

This includes the isolation of natural frequencies by means of the Sturm sequence technique and the extraction of natural frequencies by means of the Newton's method or the interpolation methods. These methods were discussed in sections (5.5), (5.7), and (5.8) respectively. Every determinant evaluation in equation (5.11.3) involves a routine of Gauss elimation as shown in section (5.6). Since the elimination process suffers seriously from rounding off errors in the neighbourhood of a natural frequency the resulting natural frequencies are considered as approximations in this stage. The matrix iteration technique mentioned in section (5.9) by using these approximations gives the corresponding eigenvectors. Then the improvement of the natural frequencies are obtained by the application of Rayeigh's Quotient. Further refinement of the eigenvectors may be acheived by performing the inverse iteration again using the improved eigenvalues.

Upon determining the eigenvalues and the corresponding modes, the diaplacements and stress distributions over the complete region of the system are obtained in terms of generalized coordinates. This is achieved by the internal displacements and generalized coodinates relationship

$$\{u(x,y,z)\} = [a(x,y,z)]\{q\} \qquad (5.11.4)$$

for every element.

The decision diagram for the methods of solution of natural vibration problems in a simplified form is given in fig (5.11.1).

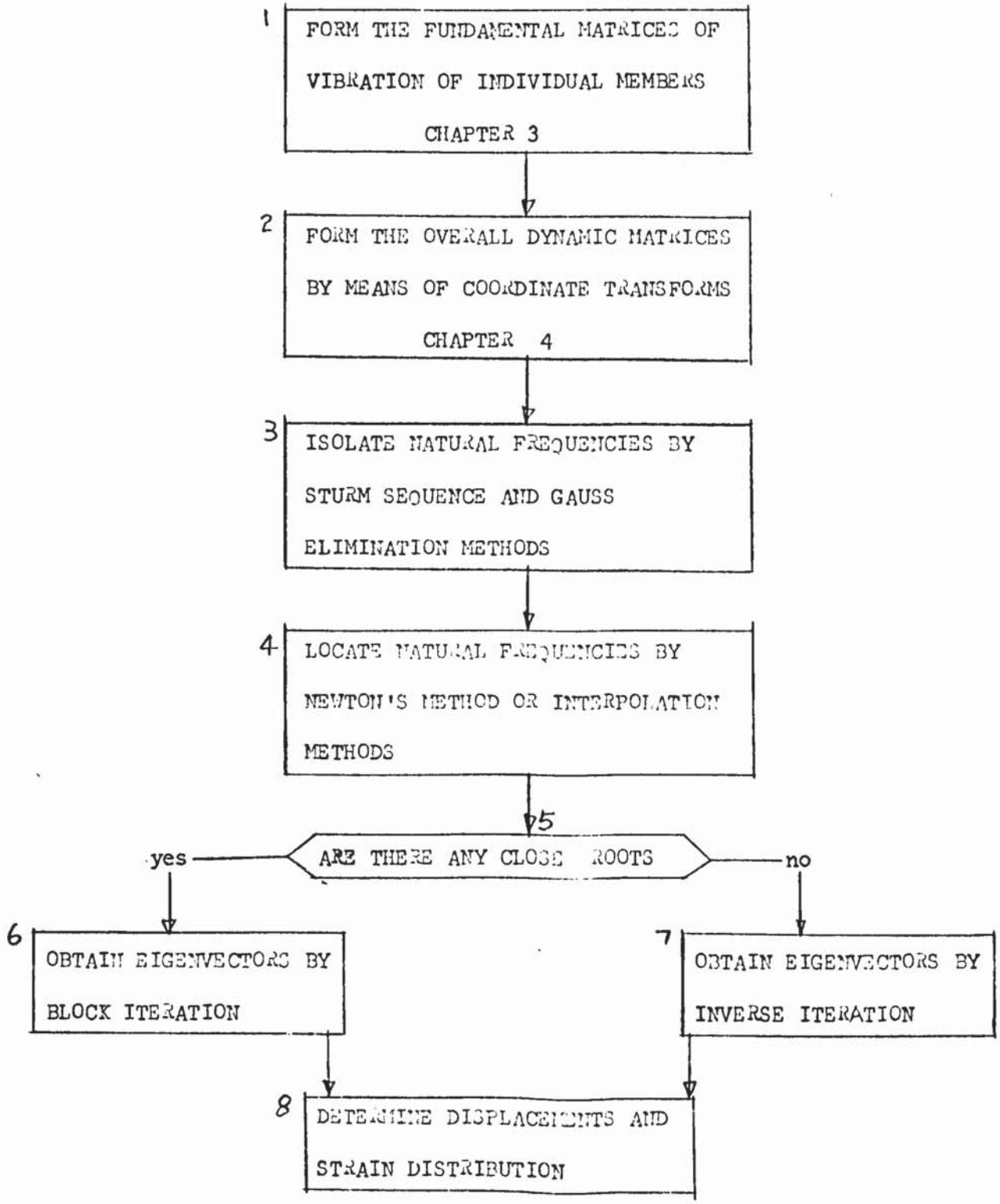


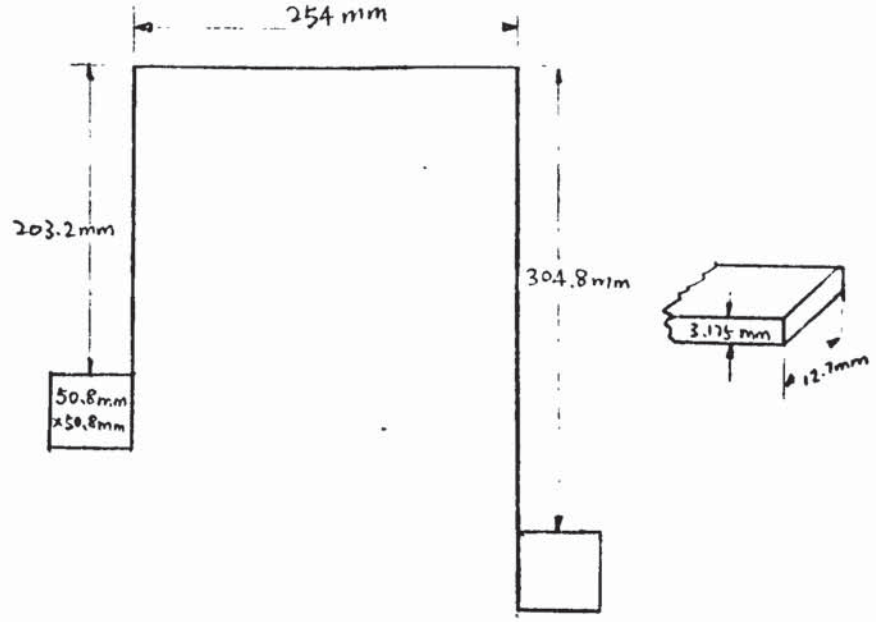
Fig (5.11.1) Decision diagram of the natural frequency problem

5.12 EXPERIMENT

An experiment was carried out to verify the computed natural modes of a plane frame structure.

A steel strip of cross section 12.7mm x 3.175mm and length 863.6mm was bent into an orthorgonal frame as shown in Fig(5.12.1). Two cast iron blocks of size 50.8mm x 50.8mm x12.7mm were welded to each end of the frame. The elastic modulus of the steel frame is 0.2119×10^{12} N/m^2; then shear modulus is 0.822×10^{11}N/m^2; and the mass density is 0.79×10^4 Kg/m^3. The whole structure was clamped to a solid rigid foundation as shown in Fig(5.12.2). Three holes of diameter 4mm were drilled at position A,B and C so that harmonic displacement excitation was applied by means of an electrical excitor E, to the frame at these positions, one at a time. An accelernometer D, was attached to the frame at various position to measure the response. The frequencies at which the response reached its maximum were the natural frequencies of the structure. These natural frequencies were also calculated by the computer programme presented in section (6.2) example 2.

The measured natural frequencies and the calculated natural frequencies were compared to Bishop and Johnson (ref 10) and listed in Fig (5.12.3) for the in plane vibration. The calculated natural frequencies and the measured natural frequencies are differed by less than four percents.



Fig(5.12.1) The frame model

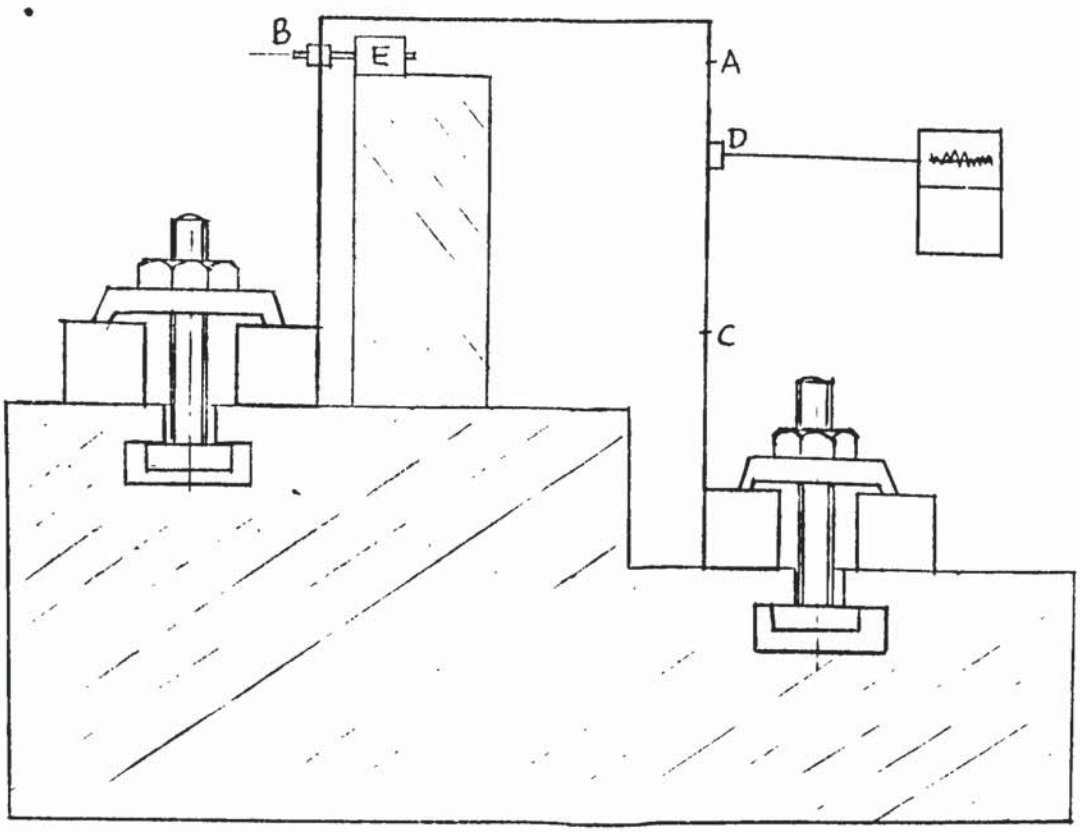


Fig (5.12.2) The setting up of the experiment

results of ref. 10	results computed in chapter 6	measured results at positions		
		A	B	C
42.5	40.57	42 (+3.52)	42 (+3.52)	42 (+3.52)
147	139.39	143 (+2.59)	139 (-0.03)	140 (+0.06)
215	201.42	198 (-1.69)	195 (-3.18)	194 (-3.68)
377	358.41	354 (-1.23)	356 (-0.70)	358 (-0.01)
475	448.43	458 (2.13)	458 (2.13)	450 (0.35)
598	605.71	620 (2.36)	622 (2.67)	—

Fig (5.12.3) The comparison of results

The listed natural frequencies are in unit of Hertz

The figures in Brackets are the % differences from the computed results

CHAPTER SIX

DEVELOPMENT OF COMPUTER PROGRAMMES

6.1 INTRODUCTION

Most of the programmes developed during the period of research were concerning with the following areas:

(i) the computation of natural frequencies and the associated modal shapes of arbitrary space frame structures;

(ii) the calculation of dynamic responses of such structures when the natural frequencies and the associated modes are given; and

(iii) the operations of linear arithmetics.

Therefore, the chapter is in five main sections. The first section is the introduction. The second section describes the various subroutines of natural frequencies and modal shapes computation and the main driving programme. The methods used are based on the methods of chapter five. The third section concerns the modal analysis technique. Although the modal analysis is to be presented in chapter seven, the programmes are given here for the convenience of reference. This involves mainly the integration process and the calculation of interior beam deflections when the nodal displacements are given. The fourth section contains a package of programmes of linear analysis. Particular interest is given to the matrix manipulation when the matrices are stored in different forms, e.g. full matrices, fixed banded matrices, and matrices of variable bandwidths. This is to speed up the programming efficiency. In the last section, all other programmes developed will be presented. These include

include the integration of beam functions, special beam elements, plate elements, etc..

All programmes are written in FORTRAN IV to be used for computers of ILC1900 series George 3 system and ICL system 4. In case there are differences, programmes for ICL system 4 only are listed. However, all computer programmes submitted in card forms with the these are ready to be used for ICL1900 George 3 system. The only major difference in these two systems are the application of NAG subroutine package.

6.2 PROGRAMMES FOR NATURAL VIBRATIONS OF FRAME SYSTEMS

A FORTRAN programme was written to compute the natural frequencies and the associated modal shapes of a space frame structure. The programme consists of a main driving programme, called MAINP and eleven subroutines called FINFQ, HALF, JJ, DYMAT, ARRANG, SELECZ, TNMAT, SPMA2, DECFL, ASSEM and FQFN1. These programmes are decribed in detail in the following sections.

6.2.1 Main programme MAINP

MAINP contains the declarations of variables and arrays, the readings of input data, the convertion of input data to a form suitable to store, the calculation of partial frequencies in the interested frequency range, and the call of FINFQ to find the natural frequencies and the associated modes between two neighbouring partial frequencies. Variables are read by statements 1 to 7. Since FORTRAN does not have the device of "variable dimensions" as in ALGOL, the arrays are declared explicity in statements 5 to 7. Input data are read by statements 10, 13, 35, 56, 67, 74, and the data are printed as read. Statement 10 reads one card which contains the following parameters:

IPRINT: an integer parameter specifying the output requirement.

IPRINT=1 gives full information for checking, IPRINT=2 only natural frequencies and modal shapes will be printed, IPRINT=3 output will be stored on magnetic tape

ORTH: a variable of integer type specifying the orthogonality of the structure to make the handling of data easier in the case of orthogonal structures. Set ORTH=1 if the structure is orthogonal, set ORTH=0 otherwise.

NNG: number of element member groups. The elements in the same group will have the same physical properties although the positions or the orientations may be different. The grouping of the individual members is to reduce the book keeping work of input data and to speed up the computer programme efficiency.

NES: total number of elements of the structure.

NDF: total number of degrees of freedom of the system.

NBD: bandwidth parameter of the overall stiffness and mass matrices, such that the overall bandwidth = 2*NBD-1. Set NBD=0 if full matrices are used.

NPE: number of nodes per element, NPE=2 for beam members.

NPD: number of degrees of freedom per node, NPD=6 for space beam members

NK: the first dimension declared of matrix KOV.

NR: the largest number of multiplicity of natural frequencies expected. If there happens that a natural frequency of multicity greater than NR exists, then the programme will print:
THERE ARE (M) MODES IN THE RANGE (A),(B), THEREFORE ONLY (NR) MODES ARE VALID, where M= actual number of multicity and A,B are the frequency bounds for the natural frequency. Further, if NR is greater than 30, the declaration of FINEQ needs modification.

NND: the total number of degrees of freedom per element = NPE*NPD. NND= 12 for space frame members.

NV: the first dimension declared for matrix VEC

NM: the first dimension declared for matrix MOV

NYW: the first dimension declared for matrix YW.

Statement 13 reads n cards into the array NNG(NNG), where n= (the largest integer less than NNG/16) + 1. Each card contains 16 items of integer type. There are NNG items totally. The ith item is equal to the number of elements of group i. The elements in the same elastic properties and dimensions although the positions and orientations may be different.

Statement 35 reads n cards, where n= NNG, the total number of element groups. The ith card contains the elastic properties and dimensions of the elements in group i. The items read are in the following order:

HL: length og the beam in meter

BY: width of the beam in meter

BZ: depth of the beam in meter

DN: density of the beam material in Kg/m^3

YM: Young's modulus in N/m^2

TM: Shear modulus in N/m^2

TI: torsional inertia of the whole beam, i.e. the polar moment of inertia.

These data are converted and stored in the matrix PG(7,NNG) in the following manner . For the Ith group,

PG(1,I) = length

PG(2,I) = axial rigidity

PG(3,I) = torsional rigidity

PG(4,I) = flexural rigidity in the XZ plane

PG(5,I) = flexural rigidity in the XY plane

PG(6,I) = total mass of the beam

PG(7,I) = total torsional inertia of the beam.

Note that the conversion is based on rectangular beam elements. For other types of beam crossections, the conversion part will need modification.

Statement 56 reads the Euler angles of rotation for each beam. The Euler angles specify the orientations of the beam member and are stored in the matrix TM(3,NES). The number of data cards equals the number of elements (NES). If ORTH was set to 1 for orthogonal structure, then the Euler angles are either 1,0, or -1. Otherwise the angles should be given in units of radian.

Statement 67 reads the code numbers for each beam into the matrix IC(2,NES). Each card contains NPE code numbers.

The last read statement 74 reads the value RG. This is the upper bound of the frequency range interested. The lower bound is assumed to be zero. All the partial frequencies below RG are first calculated and arranged in ascending order by the subroutine ARRANG and then all the natural frequencies between two neighbouring partial frequencies and the associated normal modes are computed by the subroutine FINDX. The resulting modal shapes are normalized such that the generalized mass associated to every mode is unity.

The arrays declared in the statements 5,6,7 have the following meanings:

NMG(NNG): such that NMG(i) is the number of elements in group i.

TM(3,NES): such that TM(J,I) is the Jth Euler angle of member I.

PG(7,NNG): such that PG(7,I) are the elastic properties and dimensions of the elements in group I.

IC(2,NES): such that IC(2,I) are the code numbers of member I.

WU(200): contains the partial frequencies.

KOV(NDF,NDF): contains the overall dynamic stiffness matrix.

MOV(NDF,NDF): contains the overall mass matrix.

VEC(NDF,NR): contain the eigenvectors.

LN(NDF): a working array of type LOGICAL.

ID(NDF): a working array of type integer.

YW(NDF,11): a working array of type REAL.

6.2.2 SUBROUTINE FINFQ

The subroutine FINFQ calculates all the natural frequencies and the associated modes within frequency bounds AW and BW by the dynamic stiffness method and inverse iteration. The parameters enter into the subprogramme are:

AW: lower frequency bound.

BW: upper frequency bound.

KOV: working space for the dynamic stiffness matrix.

MOV: working space for the mass matrix.

NDF: the total number of degrees of freedom of the structure.

NMG: an array containing the number of elements in each beam group.

NNG: number of element groups.

PG: a matrix containing the elastic properties and dimensions of each beam group.

TN: a matrix containing the Euler angles for each beam.

IC: an integer matrix containing the code numbers for each beam.

NES: total number of elements.

VEC: working space for eigenvectors.

NV: the first dimension declared for VEC.

YW: working space for inverse iteration.

NYW: the first dimension declared for YW.

ID: working space for row interchanges in the inverse iteration.

LN: working storage for counting frequency numbers.

NR: as defined in section(6.2.1).

The subroutine calls FUNCTION JJ, and SUBROUTINES DYMAT, SELECZ, and HALF. JJ counts the frequency number; DYMAT forms the dynamic stiffness matrix; SELECZ performs the inverse iteration ; and HALF bisects the interval when the distribution of natural frequencies is very irregular. The organization of the subroutine is to be described below.

Statements 1 to 9 are declarations.

Statements 10 to 38 check the input data and subdivide the interval (AW,BW) into convenient size for inverse iteration.

Statements 39 to 214 consist the main body and statements 215 to 220 check the results see if there are any natural frequencies missing. The performance of the main body is separated in the following four steps:

1. Statements 39 to 55 calculate the frequency numbers corresponding to the two frequency bounds of the interval and print the results if needed, i.e. when IPRINT=1. If there is no natural frequency in the interval, start the test for the next interval of frequencies.

2. Statements 56 to 91 scan the interval for initial approximations of every natural frequency lying in the interval. This is done by forming the overall mass matrix and stiffness matrix in statements 56 to 65; extracting the approximate natural frequencies between two bounds in statements 66; and checking that if these are really approximations in statements 68 to 91. When the distribution of natural frequencies are regular, i.e., the predicted approximations are really approximations to the corresponding frequencies of natural vibration, then goto step 4, otherwise goto step 3 for halving the interval. If enough natural frequencies are found, goto return.

3. Statements 92 to 149 extract natural frequencies and modal shapes by the method of bisection and inverse iteration. The bisection of frequency interval is performed by the subroutine HALF in statement 99. The upper and lower bounds resulted are stored in arrays VR and VAL respectively and the corresponding frequency numbers are stored in JVR and JVAL repectively. When the natural frequencies are well separated the accurate natural frequencies and the associated modes are calculated by the subroutine SELECZ in statement 130 and print out by statements 133-146. Goto step 2 for testing next interval of frequencies.

4. Statements 150 to 214 extract natural frequencies and the associated modes by means of Rayleigh's Quotient and inverse iteration. Rayleigh's Quotient improvements for all the initial approximated natural frequencies are done in statements 153 to 169 and printed in statements 170 to 176 if required. Inverse iteration is carried out in statements 178 to 213. The computation consists of setting the frequency bounds for the natural frequencies in statements 179 to 186; forming the overall dynamic matrices in statements 187 to 193; extracting the accurate natural frequencies and the corresponding modes in statements 194 and 195; and, finally, checking and printing the results in statements 196 to 211 according to the requirement. Goto to step 2 for next frequency interval.

6.2.3 SUBROUTINE HALF

The subroutine HALF reduces the bounds for natural frequencies by the method of bisection, i.e., halving the interval. The parameters entering to the subroutine are explained as follows:

A: original lower bound of the natural frequency interval.

B: original upper bound of the natural frequency interval.

JA: frequency number corresponding to A.

JB: frequency number corresponding to B.

EPS: the accuracy required.

KOV, NK, NDF, NMG, NMG, PG, TN, IC, NES, as in section(6.2.1).

IAGIN: an indicator. Set to 0 when a decomposition of the overall dynamic stiffness matrix may be saved by the information given in the previous bisection process. Otherwise or not sure, set IAGAIN=1.

Output from the subroutine are

X: new lower bound of natural frequency interval .

Z: new upper bound of natural frequency interval.

JX: frequency number associated with X.

JZ: frequency number associated with Z.

RZ,JR: information for next bisection process.

6.2.4 FUNCTION JJ

JJ calculates the frequency number of a specified frequency Y. The input parameters are

Y: the specified frequency.

KOV, NK, NDF, NMG, NMG, PG, TN, IC, NES: as in section(6.2.1).

The subroutine calls DYMAT and DECFL. DYMAT forms the dynamic stiffness matrix and DECFL decomposes the dynamic stiffness matrix by Gauss elimination without interchanges. The information of the number of partial frequencies below Y is obtained through a common block JJDYM accessing to DYMAT.

6.2.5 SUBROUTINE DYMAT

DYMAT forms the dynamic stiffness matrix or the mass matrix of the whole structure. The input parameters are

Y: the frequency of vibration.

MK: an indicator specifying whether the dynamic stiffness matrix or mass matrix is needed. For dynamic stiffness matrix set MK=1 and for mass matrix set MK=-1.

NK, NDF, NMG, MNG, PG, TN, IC, NES: as in section(6.2.1).

Output from the subroutine is the matrix KOV, which is the dynamic stiffness matrix or the mass matrix depending on the parameter MK.

The subroutine calls SPMA2, TNMAT, and ASSEM. SPMA2 gives the element dynamic matrix or the element mass matrix. TNMAT transforms these individual element matrices from the local coordinates to the global coordinates and ASSEM assembles the transformed matrices into the overall matrix.

6.2.6 SUBROUTINE ARRANG

ARRANG calculates all the partial frequencies below a specified frequency RG and arranges them into ascending order into the array WU. The input parameters are

RG: the specified frequency.

PG,MNG: as in section(6.2.1).

The output from the subroutine are

WU: an array containing the partial frequencies in ascending order.

NR: the total number of partial frequencies below the frequency Y.

The subroutine calls MO1AMF of NAG package.

6.2.7 SUBROUTINE SELEC2

SELEC2 calculates all the eigenvalues and eigenvectors of the eigenvalue problem $Ax=\lambda Bx$ within two eigenvalue bounds. The matrices A and B are full symmetric matrices and B is positive definite. Since A and B are full, they are transformed to tridiagonal forms before the inverse iteration process starts. This is to increase the computing efficiency because after transformation a decomposition of a tridiagonal matrix is required for each iteration, and the decomposition of a full matrix is avoided. The input parameters are

ALB: lower bound of eigenvalue

UB: upper bound of eigenvalue

N: the order of the matrices

F(nf,n): containing the matrix A

G(NG,N): containing the matrix B

NF,NG the first dimensions declared for F and G respectively.

K: the estimated number of eigenpairs in the interval.

The output parameters are

M: the actual number of eigenpairs calculated

RT(K): containing the eigenvalues

VEC(NV,K):containing the eigenvectors.

and D, E, E2, DG, IC, X, IN are working spaces.

This subroutine calls F01AEF, F01AGF, F02ASF, F01AHF, F01AFF of NAG subroutine package.

6.2.8 SUBROUTINE TNMAT

TNMAT transforms the element matrix KE in local coordinates to EK in global coordinates of a beam member when the Euler angles of the orientation of the local coordinates relative to the global coordinates, T1, T2, T3 are given. The method of transformation is according to chapter four. The input parameters are:

KE(12,12): matrix in local coordinates before transformation

T1, T2, T3: Euler angles of orientation.

The output from the subroutine are:

EK(12,12):the transformed matrix. The original matrix is kept unchanged.

6.2.9 SUBROUTINE SPMA2

SPMA2 calculates the element dynamic stiffness matrix or the mass matrix of a beam member when the physical properties and dimensions are given. This subroutine calls FQFM1 to obtain the frequency functions and access to FQFM1 through the common block BLK1.

The iuput parameters are:

W: a circular frequency where the dynamic matrix is calculated at.

L: length of the element beam

EA: axial rigidity.

GJ: torsional rigidity.

EZ: flexural rigidity in XZ plane.

EY: flexural rigidity in XY plane.

XM: total mass of the beam member.

TI: total torsional inertia ofthe beam member.

EZ and EV are set negative when mass matrix is required. MS is set negative if the dynamic stiffness matrix is required.

The output parameters from the subroutine are:

KE: the element dynamic stiffness matrix or the mass matrix depending on the input values of EZ, EV, MS.

JE: the number of partial frequencies below the frequency V.

6.2.10 SUBROUTINE DECFL

DECFL decomposes a symmetric matrix A into a product of lower and upper triangular matrices L and U repectively by Gauss elimination method without interchanges.

Input parameters are:

A(NA,N): symmetric matrix of order N, the strict lower triangle of the matrix is not used and unchanged.

NA: the first dimension declared for A.

N: the order of A.

The output parameters from the subroutine are:

A(NA,N): such that the upper triangle of A contains the matrix U.

6.2.11 SUBROUTINE ASSEM

ASSEM assembles all the transformed element matrices into an overall matrix when the code numbers of each element are given.

The input parameters entering into the subroutine are:

IC(NPE,NEL): an integer array containing the code numbers of every beam.

EM: element beam number.

NPE: number of nodes of element EM.

DPN: degrees of freedom per node of element EM.

BD: a bandwidth parameter of the overall matrix, such that the total bandwidth = 2*BD-1. Set BD=0 when full matrix is required. The bandwidth is obtained by inspection.

KE: transformed element matrix resulted from TRMAT.

NES: the total number of elements.

DOF: the total number of degrees of freedom of the structure.

ND: the total number of degrees of freedom per element.

NK: the first dimension declared of the overall matrix.

The output from the subroutine is KOV the overall matrix.

6.2.12 SUBROUTINE FQFN1

FQFN1 calculates the frequency functions of an Euler beam element. Singularity of the function at frequency equals to zero is removed by expanding the function in Taylor series and the overflow of the these functions are avoided by asymptotic expansion for large values of frequency. These have been studied in chapter three. The number of partial frequencies of the element below the specified frequency is computed at the same time.

The input parameters are:

W: the specified circular frequency.

MS: the total mass of the element

L: the length of the element.

RZ: the flexural rigidity.

EA: the axial rigidity.

TI: the total torsional inertia of the beam.

The output from the subroutine is JE the number of partial frequencies below W. The frequency functions access to CPMA2 through the block PLT1.

```
      INTEGER ORTH
      IMPLICIT REAL*8 (A-H,O-Z)
      LOGICAL LN,O1,O2,O3,OO
      REAL*8 KOV,MOV
      DIMENSION NNG(10),TN(3,30),PG(7,10),IC(2,30),WU(200),KOV(60,60),
     *MOV(60,60),VEC(60,12),LN(60),YW(60,11),ID(60)
      COMMON/PRINT/O1,O2,O3,OO
C
C
      READ (5,501)
     *     IPRINT,ORTH,NNG,NES,NDF,NBD,NPE,NPD,NK,NR ,NND,NM,NV,NYW
  501 FORMAT (14I5)
      READ (5,3) (NNG(I),I=1,NNG)
    3 FORMAT (10I5)
      O1=IPRINT.EQ.1
      O2=IPRINT.EQ.2
      O3=IPRINT.EQ.3
      OO=O1.OR.O2
      IF (OO)
     *WRITE (6,6) ORTH,NNG,NES,NDF,NBD,NPE,NPD,NK,NR ,NND,NM,NV,NYW
    6 FORMAT (//'ORTH,NNG,NES,NDF,NBD,NPE,NPD,NK,NR,NND,NM,NV,NYW'
     *,//14I5)
      P1=3.14159265359
      P2=3.14159265359*0.5
      IF(NNG.GT.10.OR.NES.GT.30.OR.NDF.GT.60) GOTO 98
      IF (OO)
     *WRITE (6,61)
   61 FORMAT (//'NUMBER OF ELEMENTS IN GROUPS'//)
      IF (OO)
     *WRITE (6,3) (NNG(I),I=1,NNG)
      DO 50 I=1,NNG
      IF (OO)
     *WRITE (6,49) I
   49 FORMAT (/'ELEMENT PROPERTY IN GROUP NUMBER ',I5/)
      READ (5,51) HL,BY,BZ,DN,YM,TM
      IF (OO)WRITE (6,51) HL,BY,BZ,DN,YM,TM
      AR=BY*BZ
      C=DMAX1(BY,BZ)
      B=DMIN1(BY,BZ)
      PG(1,I)=HL
      PG(2,I)=AR*YM
      R=B*B*B
      S=DEXP(-PI*C/B)
      PG(3,I)=(C*R/3.-64.*R*B/PI**5* (1-S)/(1+S))*TM
      R=BY*BY**3/12.
      S=BZ*BZ**3/12.
      PG(7,I)=DN*HL*(R+S)
      PG(4,I)=YM*R
      PG(5,I)=YM*S
      PG(6,I)=DN*HL*AR
   50 CONTINUE
   51 FORMAT (6D13.4)
      IF (OO)
     *WRITE (6,53) ((PG(I,J),I=1,7),J=1,NNG)
   53 FORMAT (/' CONVERTED QUANTITIES'//20(7D16.8//))
      READ (5,4) ((TN(J,I),J=1,3),I=1,NES)
    4 FORMAT (3F10.5)
      IF (OO)
     *WRITE (6,5)
    5 FORMAT (//'ANGLES OF ROTATION FOR EACh MEMBER'//)
      IF (OO)
     *WRITE (6,4) ((TN(J,I),J=1,3),I=1,NES)
      IF (O3)GO TO 9
      DO 10 I=1,NES
      DO 10 J=1,3
   10 TN(J,I)=TN(J,I)*P2
    9 READ (5,7) ((IC(J,I),J=1,2),I=1,NES)
    7 FORMAT (2I5)
      IF (OO)
     *WRITE (6,8)
    8 FORMAT (//'CODE NUMBERS'//)
      IF (O3)
     *WRITE (6,7) ((IC(J,I),J=1,2),I=1,NES)
      READ (5,11) SB
   11 FORMAT (F20.10)
      IF (OO)
```

```
      *WRITE (6,12) RG
   12 FORMAT (//'FREQ RANGE=',F20.10,'RAD/SEC'//)
      CALL ARRANG(RG,PG,WU,NS,NNG)
      BW=0.0
      IF (00)
     *WRITE (6,109) NS
  109 FORMAT (/' NUMBER OF PARTIAL FREQ.= ',I5/)
   48 FORMAT (/'PARTIAL FREQ. OF THE STRUCTURE='//100(5F20.5//))
      IF (00)
     *WRITE (6,48)  (WU(I),I=1,NS)
      IEG=0
      EPS=5.D-6
      OBW=EPS
      DO 13 NN=1,NS
      AW=OBW
      BW=WU(NN)*(1.-EPS)
      OBW=BW*(1.+2.*EPS)
      IF (BW.GT.RG) BW=RG
      IF (BW-AW.LT.BW*EPS) GOTO 13
      CALL      FINFQ(AW,BW,     KOV,NK,HOV,NM,NDF,NNG,NNG,PG,TN,IC,
     *NES,VEC,NV,YW,NYW,ID,LN,NR)
   13 CONTINUE
      GOTO 99
   98 WRITE(6,666)
  666 FORMAT('********DECLARATION OF THE MAIN PROG NEEDS MODIFIED'//)
   99 STOP
      END
```

```
      SUBROUTINE FINFQ(AW,BW,     KOV,NK,NOV,NM,NDF,NMG,NNG,PG,TN,IC,
     *NES,VEC,NV,YW,NYW,ID,LN,NR)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 KOV,NOV
      LOGICAL LN,O1,O2,O3,O0
      DIMENSION NMG(NNG),PG(7,NNG),TN(3,NES),KOV(NK,NDF) ,IC(2,NES)
      DIMENSION NOV(NM,NDF),YW(NYW,11),ID(NDF),LN(NDF),VEC(NV,NR)
      DIMENSION VAL(30),DIV(11),VR(30)  ,JVR(30),JVAL(30)
      COMMON/PRINT/O1,O2,O3,O0
      IF (NR.LE.30) GOTO 1000
      WRITE (6,1001)
 1001 FORMAT ('****NR.GT.30,     DECLARATION OF FINFQ NEEDS MODIFICATIO
     *****')
      CALL EXIT
 1000 IF (BW.LE.AW) GOTO 999
   16 FORMAT (/'**** FREQ. MISSED AT S=',F20.10,'BOUNDS=',2F20.10//)
   63 FORMAT('THE CORRESPONDING FREQUENCY NUMBER ARE',2I5/)
      BW2=BW*BW
      JA=      JJ(AW,KOV,NK,NDF,NMG,NNG,PG,TN,IC,NES)
      JB=      JJ(BW,KOV,NK,NDF,NMG,NNG,PG,TN,IC,NES)
      JL=JB-JA
      JJB=JA
      IEQ=JA
      JEQ=IEQ
      IF (O1)
     *WRITE (6,61) AW,BW,JA,JB
   61 FORMAT(/'****FREQ. RANGE=',2F15.5,
     *'     CORR. FREQ.NUMBER=',2I5,'****'//)
      IF (JL.EQ.0) GOTO 999
      NDV=MIN0( 8,2*JL)
      ST=(BW-AW)/DFLOAT(NDV)
      DIV(1)=AW
      DO 1 I=1,NDV
    1 DIV(I+1)=DIV(I)+ST
      DIV(NDV+2)=(DIV(NDV+1)+BW)*0.5
      DIV(NDV+3)=BW
      VJ=AW
      NDVP2=NDV+2
      DO 100 I=1,NDVP2
      IF (JJB.EQ.JB) GOTO 998
      ALB=DIV(I)
      UB=DIV(I+1)
      Y=(ALB+UB)*0.5
      JLB=JJB
      JJB=JJ(UB,KOV,NK,NDF,NMG,NNG,PG,TN,IC,NES)
      NJJ=JJB
      AAW=ALB
      BBW=UB
      P=ALB
      Q=UB
      NJJ=JUB-JLB
      IF (O1)
     *WRITE (6,64) ALB,UB,JLB,JUB
   64 FORMAT(/'FREQ. BOUNDS',2F20.10,'   FREQ. NUMBERS',2I5/)
      IF(NJJ.EQ.0) GOTO 100
    9 NK=1
      CALL DYHAT(Y,NK,KOV,NK,NDF,NMG,NNG,PG,TN,IC,NES)
      NM=1
      CALL DYHAT(Y,NK,NOV,NM,NDF,NMG,NNG,PG,TN,IC,NES)
      YY=Y*Y
      ALB=ALB*ALB
      UB=UB*UB
      DO 14 J=1,NDF
      DO 14 K=1,NDF
   14 KOV(J,K)=KOV(J,K)+YY*NOV(J,K)
      CALL SLEBZ(ALB,UB,NDF,M,KOV,NK,NOV,NM,VAL,VEC,NV,YW(1,8),
     *YW(1,9),YW(1,10),YW(1,11),ID,YW(1,1),LN,NR)
      IF (M.EQ.0) GOTO 46
      DO 40 J=1,M
   40 VR(J)=DSQRT(VAL(J))
      IF (O1)
     *WRITE (6,79)
   79 FORMAT(/'INITIAL APPROX. OF NAT. FREQ.'/)
      IF (O1)
     *WRITE (6,77 ) (VR(J),J=1,M)
      DO 41 J=1,M
```

```
      VRJ=VR(J)
      IF (VRJ.GE.P.AND.VRJ.LE.Q) GOTO 41
      VR(J)=-1.
   41 CONTINUE
      J=0
      MM=M
   42 J=J+1
      IF (J.GT.M) GOTO 43
      IF (VR(J).GT.0.) GOTO 42
      DO 44 K=J,MM
      DO 45 L=1,NDF
   45 VEC(L,K)=VEC(L,K+1)
   44 VR(K)=VR(K+1)
      MM=MM-1
      GOTO 42
   43 M=MM
      IF (M.EQ.MJJ) GOTO 134
   46 M=0
      P=AAW
      Q=BBW
      ESPP=0.01*P
      IAGAIN=1
   35 CALL HALF(P,Q,JLB,JUB,ESPP,X0,Z0,JX,JZ,KOV,NK,NDF,NMG,NNG,PG,
     *TN,IC,NES,IAGAIN,JM,RZ)
      AZ=(X0+Z0)*0.5
      IF (O1)
     *WRITE (6,65) X0,Z0,JX,JZ
   65 FORMAT(/' BISECTION BOUNDS',2F20.10,' FREQ.NUMBERS',2I5/)
      M=M+1
      VR(M)=X0
      JVR(M)=JX
      VAL(M)=Z0
      JVAL(M)=JZ
      IF (JZ.EQ.JUB) GOTO 39
      P=Z0
      JLB=JZ
      GOTO 35
   39 M1=M
      IF (M.EQ.0) GOTO 100
      DO 50 J=1,M1
      S=(VR(J)+VAL(J))*0.5
      CALL DYMAT(S,-1,KOV,NK,NDF,NMG,NNG,PG,TN,IC,NES)
      CALL DYMAT(S,-1,MOV,NM,NDF,NMG,NNG,PG,TN,IC,NES)
      R=S*S
      DO 51 K=1,NDF
      DO 51 L=1,NDF
   51 KOV(K,L)=KOV(K,L)+R*MOV(K,L)
      VRJ=VR(J)
      VALJ=VAL(J)
      Q=VRJ*VRJ
      T=VALJ*VALJ
      JVV=JVR(J)-JVAL(J)
      IF (JVV.EQ.0) GOTO 50
      CALL SELECZ(R  ,T  ,NDF,M,KOV,NK,MOV,NM,VR  ,VEC,NV,YW(1,8),
     *YW(1,9),YW(1,10),YW(1,11),ID,YW(1,1),LN,NR)
      IF(M.GT.0) GOTO 52
      IF (O0)
     *WRITE (6,16) S,VRJ,VALJ
      IF (O0)
     *WRITE (6,63) JVR(J),JVAL(J)
      GOTO 50
   52 DO 53 K=1,M
      R=VR(K)
      S=DSQRT(R)
      IEG=IEG+1
      S1=S*.5/3.14159265359
      IF (O0)
     *WRITE (6,13) IEG,R,S,S1,(VEC(L,K),L=1,NDF)
      IF (O3) WRITE (7,713) IEG,S,(VEC(L,K),L=1,NDF)
  713 FORMAT(I5,D20.13/20(3D20.10/))
   53 CONTINUE
   50 CONTINUE
      GOTO 100
  134 DO 32 J=1,M
      VAL(J)=VR(J)*VR(J)
   32 FORMAT(20(6F20.10//))
```

```
      DO 20 J=1,M
      R=VAL(J)
      S=DSQRT(R)
      P=0.
      Q=0.
      NK=1
      CALL DYNAT(S,NK,KOV,NK,NDF,NMG,NNG,PG,TN,IC,NES)
      NK=-1
      CALL DYNAT(S,NK,MOV,NM,NDF,NMG,NNG,PG,TN,IC,NES)
      DO 21 K=1,NDF
      DO 22 L=1,NDF
      P=P+VEC(K,J)*VEC(L,J)*KOV(K,L)
      Q=Q+VEC(K,J)*VEC(L,J)*MOV(K,L)
   22 CONTINUE
   21 CONTINUE
      IF (DABS(P/Q/R).LT.0.01) VAL(J)=R+P/Q
   20 CONTINUE
      DO 33 J=1,M
   33 VR(J)=DSQRT(VAL(J))
      IF (O1)
     *WRITE (6,72)
   72 FORMAT(/'RAYLEIGH''S IMPROVEMENTS'/)
   47 IF(O1)
     *WRITE(6,30) (VR(J),J=1,M)
      M1=M
      DO 10 J=1,M1
      IVT=0
      R=VAL(J)
      S=DSQRT(R)
      ALB=DMAX1(R*0.95,VJ)
      UB=R*1.01
  102 IF (UB.GT.BW2) UB=BW2
      VJ=UB
      IF((UB-ALB).LE.5.D-4*UB) GOTO 10
      NK=1
      CALL DYNAT(S,NK,KOV,NK,NDF,NMG,NNG,PG,TN,IC,NES)
      NK=-1
      CALL DYNAT(S,NK,MOV,NM,NDF,NMG,NNG,PG,TN,IC,NES)
      DO 11 K=1,NDF
      DO 11 L=1,NDF
   11 KOV(K,L)=KOV(K,L)+R*MOV(K,L)
      CALL SELECZ(ALB,UB,NDF,M,KOV,NK,MOV,NM,VR ,VEC,NV,YW(1,&),
     *YW(1,2),YW(1,3),YW(1,11),IC,YW(1,1),LS,NR)
      IF (M.GT.0) GOTO 15
      IF (IVT.NE.0) GOTO 46
      IVT=1
      ALB=DMAX1(R*0.9,VJ)
      UB=R*1.05
      GOTO 102
   15 DO 12 K=1,M
      R=VR(K)
      S=DSQRT(R)
      IEG=IEG+1
      S1=S*.5/3.14159265359
      IF (O2)
     *WRITE (6,13) IEG,R,S,S1,(VEC(L,K),L=1,NDF)
   13 FORMAT(/'EIGEN NUMBER',I3,F20.10,'  NAT.FREQ=',F20.10,
     *'  P/S OR',F20.10,'  HERTZ'//'EIGENVECTOR='//200(6F20.15//))
      IF (O2) WRITE (7,713) IEG,S,(VEC(L,K),L=1,NDF)
   12 CONTINUE
   10 CONTINUE
  100 CONTINUE
  999 I=NUL-IEG+NEG
      IF(I.LE.0) GOTO 999
      IF (O2)
     *WRITE (6,62) I,AW,BW
   62 FORMAT (//'****',I5,'NAT. FREQ. MISSED IN FREQ. RANGE',//'      ',
     *2F20.10,'  RAD/SEC  ****'//)
  999 RETURN
      END


      SUBROUTINE HALF(A,P,JA,JB,EPS,X,Z,JX,JZ,KOV,NK,NDF,NMG,NNG,PG,
     *TN,IC,NES,IAGAIN,JC,RZ)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 KOV(NK,NDF)
```

```
      LOGICAL 01,02,03,00
      DIMENSION NNG(NNG),PG(7,NNG),TN(3,NES),IC(2,NES)
      COMMON/PRINT/01,02,03,00
      IF (IAGAIN.EQ.1) GOTO 5
      IAGAIN=1
      JX=JZ
      JZ=JM
      X=Z
      Z=BZ
      GOTO 4
    5 X=A
      Z=B
      JX=JA
      JZ=JB
      IF (JZ-JX) 4,4,3
    3 Y=(X+Z)*0.5
      IF (Z-X.LE.EPS) GOTO 4
      JY=JJ(Y,KOV,NK,NDF,NMG,NNG,PG,TN,IC,NES)
      JM=JZ
      BZ=Z
      IF (01)
     *WRITE (6,6) JX,JY,JZ
    6 FORMAT('JX,JY,JZ=',3I5/)
      IF (JY-JX) 1,1,2
    1 JX=JY
      X=Y
      IAGAIN=1
      GOTO 3
    2 IF (JZ.NE.JY) IAGAIN=0
      JZ=JY
      Z=Y
      GOTO 3
    4 RETURN
      END
```

```
      SUBROUTINE SELECZ(ALB,UB,N,M,F,NF,G,NG,RT,VEC,NV,D,E,E2,DG,IC,X,
     *IN,K)
      IMPLICIT REAL*8 (A-H,O-Z)
      LOGICAL IN
      DIMENSION F(NF,N),G(NG,N),RT(K),VEC(NV,K),D(N),E(N),
     *E2(N),DG(N),IC(N),X(N,7),IN(N)
      CHEP=16.**(-13)
      I=1
      CALL F01AEF(N,F,NF,G,NG,DG,I)
      CALL F01AGF(N,16.0**(-53),F,NF,D,E,E2)
      I=1
      M=1
  101 CALL F02ASF(N,D,ALB,UB,CHEP,0.,E,E2,M,RT,VEC,NV,IC,X,IN,I)
      IF (M.GT.0) GOTO 100
      GOTO 99
  100 IF (I.EQ.1) GOTO 101
      IF (I.EQ.0) GOTO 102
      WRITE (6,601) I
  601 FORMAT('FAIL IN F02ASF',I5)
  102 CALL F01AHF(N,1,M,E,NF,E,VEC,NV)
      CALL F01AEF(N,1,M,G,NG,DG,VEC,NV)
      IF (M.GT.K) GOTO 98
      GOTO 99
   98 A=DSQRT(ALB)
      B=DSQRT(UB)
      WRITE (6,60) M,A,B,K
   60 FORMAT('THERE ARE ',I5,'MODES IN THE RANGE ',2D10.4//
     * 'THEREFORE ONLY    ',I5,'MODES ARE VALID'///)
      CALL EXIT
   99 RETURN
      END
C
C
      SUBROUTINE TNHAT (KE,T1,T2,T3,EK)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 EK(12,12),KE(12,12),TN(3,3)
      C1=DCOS(T1)
      C2=DCOS(T2)
      C3=DCOS(T3)
      S1=DSIN(T1)
      S2=DSIN(T2)
      S3=DSIN(T3)
      TN(1,1)=C2*C1-S2*S1*C3
      TN(1,2)=-S2*C1-C2*S1*C3
      TN(1,3)=S3*S1
      TN(2,1)=C2*S1+S2*C1*C3
      TN(2,2)=-S2*S1+C2*C1*C3
      TN(2,3)=-S3*C1
      TN(3,1)=S3*S2
      TN(3,2)=S3*C2
      TN(3,3)=C3
      DO 9 I1=1,10,3
      DO 10 J1=1,10,3
      DO 11 I=1,3
      DO 12 J=1,3
      R=0.
      I3=I+I1-1
      J3=J+J1-1
      IF (I3.GT.J3) GOTO 13
      DO 14 I2=1,3
      DO 14 J2=1,3
      I8=I2+I1-1
      J8=J2+J1-1
   14 R=R+TN(I,I2)*KE(I8,J8)*TN(J,J2)
      EK(I3,J3)=R
      GOTO 12
   13 EK(I3,J3)=EK(J3,I3)
   12 CONTINUE
   11 CONTINUE
   10 CONTINUE
    9 CONTINUE
      RETURN
      END
C
C
      SUBROUTINE SPHA2(N,L,EA,GJ,EZ,EY,MS,TI,KE,JE)
```

```
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 L,KE(12,12)
      COMMON/BLK17/G1,G2,G3,G4,G5,G6,G7,G8,G9,G10,G11,G12,G13
      PI=3.14159265358979D
      DO 10 I=1,12
      DO 10 J=1,12
   10 KE(I,J)=0.0
C
      CALL FQFN1(W,MS,L,EZ,MS,EA,J1)
      KK=J1+J1
      KK(1,1)=G11
      KK(1,2)=G12
      KK(2,2)=G11
      KK(5,5)=G11
      KK(2,5)=G12
      KK(2,12)=G4
      KK(6,6)=G11
      KK(6,12)=G8
      KK(12,12)=G7
      KK(8,8)=G11
      KK(12,12)=G9
      KK(10,10)=G10
      KK(7,7)=G13
      CALL FQFN1(W,MS,L,EY,TI,GJ,J1)
      KK=J1+J1
      KK(3,3)=G11
      KK(3,9)=G12
      KK(4,4)=G11
      KK(4,10)=G12
      KK(5,5)=G11
      KK(5,11)=G7
      KK(9,9)=G11
      KK(11,11)=G10
      KK(10,10)=G13
      DO 11 I=1,12
      DO 11 J=I,12
   11 KE(J,I)=KE(I,J)
      RETURN
      END
C
C
      SUBROUTINE DECEL(A,NA,N)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(NA,N)
      N1=N-1
      DO 3 I=1,N1
      S=1./A(I,I)
      I1=I+1
      DO 1 J=I1,N
      R=A(I,J)*S
      DO 2 K=J,N
    2 A(J,K)=A(J,K)-R*A(I,K)
    1 CONTINUE
    3 CONTINUE
      RETURN
      END
C
C
      SUBROUTINE ASSEM(IC,EN,NPE,DPN,BD,KOV,NK,KE,NES,DOF,ND)
      IMPLICIT REAL*8 (A-H,O-Z)
      INTEGER IC(NPE,NES),EN,NPE,DPN,BD,NES,DOF,ND,P,Q,R,S,U,
     1B,C,V,X,Y,Z,A,J,K,L
      REAL*8 KOV(NK,DOF),KE(ND,ND)
C     ND=NPE*DPN
      C=1
      IF (BD.EQ.0) C=0
      DO 1   J=1,NPE
      A=IC(J,EN)
      IF (A.LE.0) GOTO 1
      B=(J-1)*DPN
      B=(A-1)*DPN
      DO 2   K=J,NPE
```

```
      BESIC(K,EN)
      IF (B.LE.0) GOTO 2
      VE=(K-1)*DPN
      SE=(B-1)*DPN
      DO 101 L=1,DPN
      G=1
      IF (A.EQ.B) G=L
      DO 101 P=Q,DPN
      IF (A.LE.B) GOTO 102
      X=S+P
      Y=R+L-X*C+C
      GOTO 103
  102 X=R+L
      Y=S+P-X*C+C
  103 L1=L+U
      L2=P+V
      KOV(X,Y)=KOV(X,Y)+KE(L1,L2)
  101 CONTINUE
    2 CONTINUE
    1 CONTINUE
      RETURN
      END
C
C
      SUBROUTINE EQFN1(W,MS,L,RZ,TI,EA,JE)
      IMPLICIT REAL*8 (A-H,L,O-Z)
      REAL*8 MS
      DIMENSION ZZ(7)
      COMMON /BLK1/G1,G2,G3,G4,G5,G6,G7,G8,G9,G10,G11,G12,G13
      DATA ZZ/4.230047448627050,7.853204624958350,10.99560783800169
     *14.137165401257465O,17.278759657399472O,20.420352245626073O,
     *23.561944902040200/
      DATA PI,JA,JB/3.14159265358979,0,0/
      LOGICAL B1,B2,B3
      JA=0
      JB=0
      B1=MS.LE.0.0
      B2=RZ.LE.0.0
      B3=EA.LE.0.0
      IF (B1.AND.B2) GOTO 13
      IF (B1.OR.B2) GOTO 11
   13 WRITE (6,12) W
   12 FORMAT (//'FAILED IN EQFN1 WHEN FREQ=',F15.5,//'MASS OR STIFFNESS
     1      'MATRIX IS WANTED?')
      WRITE (6,14) MS,L,RZ,EA
   14 FORMAT (//'MS,L,RZ,EA='/4D25.10/)
      CALL EXIT
   11 Z=DSQRT(DSQRT(DABS(MS)/DABS(RZ)/L)*W)*L
      Y=DSQRT(DABS(TI)/DABS(EA)/L)*W*L
      ZP=Z/PI
      ZS=Z*Z
      ZC=ZS*Z
      ZQ=ZC*Z
      IF (Z.GT.1.) GOTO 1
      F1= 2.+(0.571428571+(0.000015704+0.000000032*ZQ)*ZQ)*ZQ
      F2= 4.-(0.666523810+(0.000016262+0.000000032*ZQ)*ZQ)*ZQ
      F3= 6.+(0.030952381+(0.000072193+0.000000148*ZQ)*ZQ)*ZQ
      F4= 16.+(0.052380952+(0.000076617+0.000000149*ZQ)*ZQ)*ZQ
      F5=-12.-(0.128571429+(0.000329571+0.000000664*ZQ)*ZQ)*ZQ
      F6= 12.-(0.371428571+(0.000364873+0.000000693*ZQ)*ZQ)*ZQ
      GOTO 2
    1 IF(Z.GT.25.6) GOTO 3
      DO 4 I=1,7
      IF (Z.GE.ZZ(I)) JB=I
    4 CONTINUE
      SB=DSIN(Z)
      CB=DCOS(Z)
      EB=DEXP(Z)
      SH=0.5*(EB-1./EB)
      CH=0.5*(EB+1./EB)
      C=Z/(C*CB-1.)
      F1=R*(SH-SB)
      F2=-R*(CH*SB-SH*CB)
      F3=-R*Z*(CH-CB)
      F4=R*Z*SH*SB
      F5=R*ZS*(SH+SB)
```

```
      F6=-R*ZS*(CH*SB+SH*CB)
      GOTO 2
    3 HR=ZP-0.5
      JB=IDINT(HR)
      CB=1./DCOS(Z)
      SB=DSIN(Z)
      F1=-Z*CB
      F2=Z*(1-CB*SB)
      F3=-ZS*CB
      F4=ZS*CB*SB
      F5=ZC*CB
      F6=-ZC*(1+CB*SB)
    2 IF (B3) GOTO 5
      IF (Y.LT.0.01) GOTO 9
      SB=DSIN(Y)
      CB=DCOS(Y)
      HR=Y/PI
      JA=IDINT(HR)
      F0=-Y/SB
      F7=CB/SB*Y
      GOTO 5
    9 Y2=Y*Y
      Y4=Y2*Y2
      F0=1.-Y2/3.-Y4/45.
      F7=1.-Y2/6.-Y4*7./360.
    5 JB=JA+JB
      IF (B2) GOTO 6
      B=RZ/(L*L*L)
      G7=F1
      G10=F2
      G5=F2
      G4=F3
      G8=-F3
      G9=F4
      G2=-F4
      G3=F5
      G1=F6
      G6=F6
      IF (B3) GOTO 7
      S=EA/L
      G11=F7
      G13=F7
      G12=F8
      GOTO 7
    6 B=0.25*US/ZQ
      G10=F1*F2-F3*F1
      G4=F1*F1-F2
      G5=G10
      G2=F1*F3+2*F4
      G8=-G2
      G6=F1*F4+2*F3
      G4=-G6
      G1=F3*F3-3*F6
      G9=G1
      G3=F3*F4-3*F5
      IF (B3) GOTO 7
      S=0.5*US/Y/Y
      G11=F8*F8-F7
      G13=G11
      G12=-F8*(1-F7)
    7 L2=L*L
      G1=G1*R
      G2=G2*R*L
      G3=G3*R
      G4=G4*R*L
      G5=G5*R*L2
      G6=G6*R
      G7=G7*R*L2
      G8=G8*R
      G9=G9*R*L
      G10=G10*R*L2
      IF (B3) GOTO 8
      G11=G11*S
      G12=G12*S
      G13=G13*S
    8 RETURN
```

3L.YT51 (F0582Φ)/N FOR USER RAECOL 220

3Φ5ΦΦ END

3L.YT51 (F0582Φ)/N FOR USER RAECOL COMPLETED AT 10:06:00

```
      FUNCTION JJ(Y,KOV,NK,NDF,NMG,NNG,PG,TN,IC,NES)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 KOV(NK,NDF)
      DIMENSION NMG(NNG),PG(7,NNG),TN(3,NES),IC(2,NES)
      COMMON/JJDYN/JD
      JD=0
      CALL DYHAT(Y,1,KOV,NK,NDF,NMG,NNG,PG,TN,IC,NES)
      CALL DECFL(KOV,NK,NDF)
      J=0
      DO 1 I=1,NDF
      IF (KOV(I,I).LT.0.0) J=J+1
    1 CONTINUE
      JJ=J+JD
      RETURN
      END
C
C
      SUBROUTINE DYHAT(Y,NK,KOV,NK,NDF,NMG,NNG,PG,TN,IC,NES)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 KE(12,12),EK(12,12),KOV
      DIMENSION NMG(NNG),PG(7,NNG),TN(3,NES),KOV(NK,NDF) ,IC(2,NES)
      COMMON/JJDYN/JD
      R1=DFLOAT(NK)
      N2=0
      JD=0
      DO 3 I=1,NDF
      DO 3 J=1,NDF
    3 KOV(I,J)=0.0
      DO 1 NG=1,NNG
      N1=N2+1
      N2=NMG(NG)+N2
      CALL SPHA2(Y,PG(1,NG),PG(2,NG),PG(3,NG),R1*PG(4,NG),
     *           R1*PG(5,NG),-R1*PG(6,NG),PG(7,NG),KE,JE)
      JD=JD+NMG(NG)*JE
      DO 2 N=N1,N2
      CALL TNHAT(KE,TN(1,N),TN(2,N),TN(3,N),EK)
      CALL ASSEM(IC,N,2,6,0,KOV,NK,EK,NES,NDF,12)
    2 CONTINUE
    1 CONTINUE
      DO 4 I=1,NDF
      DO 4 J=I,NDF
    4 KOV(J,I)=KOV(I,J)
      RETURN
      END
C
C
      SUBROUTINE ARRANG(RG,PG,WU,NR,NNG)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION PG(7,NNG),ZO(7),WU(200)
      DATA ZO/4.7300407448627036,7.8532046240958350,10.995607838001669O
     *,14.137165491257464O,17.27875953739472O,20.420352245626073O,
     *23.561944901923204200/
      PI=3.14159265358979
      J=0
      DO 3 K=1,2
      DO 1 N=1,NNG
      IF (K.EQ.1) B=DSQRT(PG(4,N)/PG(6,N)/PG(1,N)**3)
      IF (K.EQ.2) B=DSQRT(PG(5,N)/PG(6,N)/PG(1,N)**3)
      I=0
    2 I=I+1
      IF (I.LE.7) R=ZO(I)
      IF (I.GT.7) R=(DFLOAT(I)+0.5)*PI
      J=J+1
      WU(J)=B*R**3
      IF (WU(J).GT.RG) GOTO 11
      GOTO 2
   11 J=J-1
    1 CONTINUE
    3 CONTINUE
      DO 4 N=1,NNG
      DO 5 K=1,2
      IF (K.EQ.1) B=PI*DSQRT(PG(2,N)/PG(6,N)/PG(1,N))
      IF (K.EQ.2) B=PI*DSQRT(PG(3,N)/PG(7,N)/PG(1,N))
      I=0
    6 I=I+1
      J=J+1
```

```
      WU(J)=DFLOAT(I)*8
      IF (WU(J).GT.RG) GOTO 51
      GOTO 6
   51 J=J-1
    5 CONTINUE
    4 CONTINUE
      J=J+1
      I=J
      WU(J)=RG
      NR=J
      CALL NO1ANF(WU,1,J,I)
      RETURN
      END
C
C
```

6.2.13 EXAMPLES

The programme is capable to calculate all the natural frequencies and the associated modes in a specified range of frequencies. These include close natural frequencies and equal natural frequencies of any multiplicity. In this section, three examples are given to illustrate the application of the programme. The first one is a space frame with equal frequencies of natural vibration. The second one is a plane frame whose natural frequencies have been obtained experimentally in chapter five. The last one involves inclining members in space.

EXAMPLE 1

As a first example, we consider the space frame made from mild steel as shown in fig(6.2.1). It consists of eight beam members of identical elastic properties and dimensions. The four clamped ends are having code number 0 and the four upper ends are denoted 1,2,3,4 as shown. The space frame will have repeated natural frequencies of multiplicity of order two in flexural vibration and unrepeated modes for torsional vibration and axial vibration. It is chosen to test the numerical performance of the programme when repeated natural frequencies are presented. We want to calculate the natural frequencies and the associated modes below frequency 200.0 rad/sec.

The input cards are shown in fig (6.2.2). The explanation is given below:

THE FIRST LOT CONTAINING ONE CARD

We want the output to be stored on file for later use in the response analysis therefore set IPRINT = 3. Because the space frame is an orthogonal structure, we set ORTH=1 to reduce the work of preparing data.

Number of groups NNG = 1. Total number of elements NES = 8. Total number of degrees of freedom NDF = 24. Because band structure of matrices has no advantages for this structure, full matrix arithmetic is used, set NBD = 0. Number of nodes for each element NPE = 2. Number of degrees of freedom for each node NPD = 6. The first dimension of the matrix KOV declared NK = 60. Maximum number of repeated frequencies expected NR = 12. Total number of degrees of freedom for each member NED = 12. The first dimension of the matrix MOV declared NM = 60. The first dimension of the matrix VEC declared NV = 60. The first dimension of the matrix YU declared NYU = 60.

THE SECOND LOT CONTAINS n CARDS

Where n=1+INT(no of groups/16). Since there is only one group of eight elements, NNG(1) = 8.

THE THIRD LOT CONTAINS n CARDS

Where n = number of groups. Since there is one group presented, there is only one card in this lot. The order of data is as follows: length = 10 meter, width = 0.5 meter, depth = 0.5 meter, density of mild steel = $7.9*10^{3}$ Kg/meter^{3}, Young's modulus = $0.2119*10^{12}$ N/ meter^{2}, shear modulus = $0.822*10^{11}$ N/meter^{2}.

THE FOURTH LOT CONTAINS NES CARDS

Since the total number of elements NES= 8, we have eight card specifying the Euler's angles of rotations for each element. Because the structure is orthogonal, and we have set ORTH=1, all Euler angles will be 1, 0, or -1.

THE FIFTH LOT CONTAINS NES CARDS

Since the total number of elements NES = 8, we have eight cards specifying the code number for each element.

THE FIFTH LOT CONTAINS ONE CARD

0 → 200.0 rad/sec is the frequency range interested.

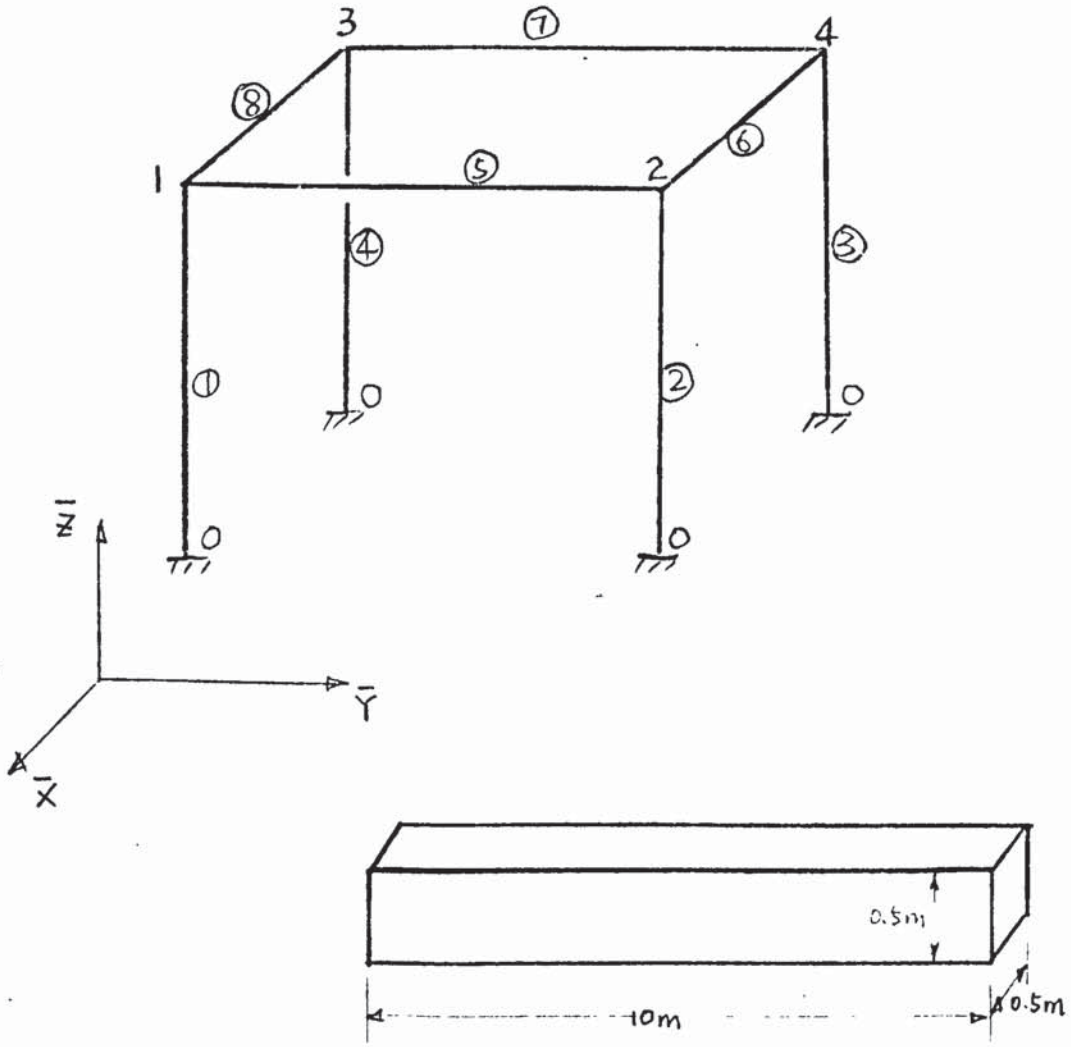


Fig (6.2.1) The space frame structure of example 1

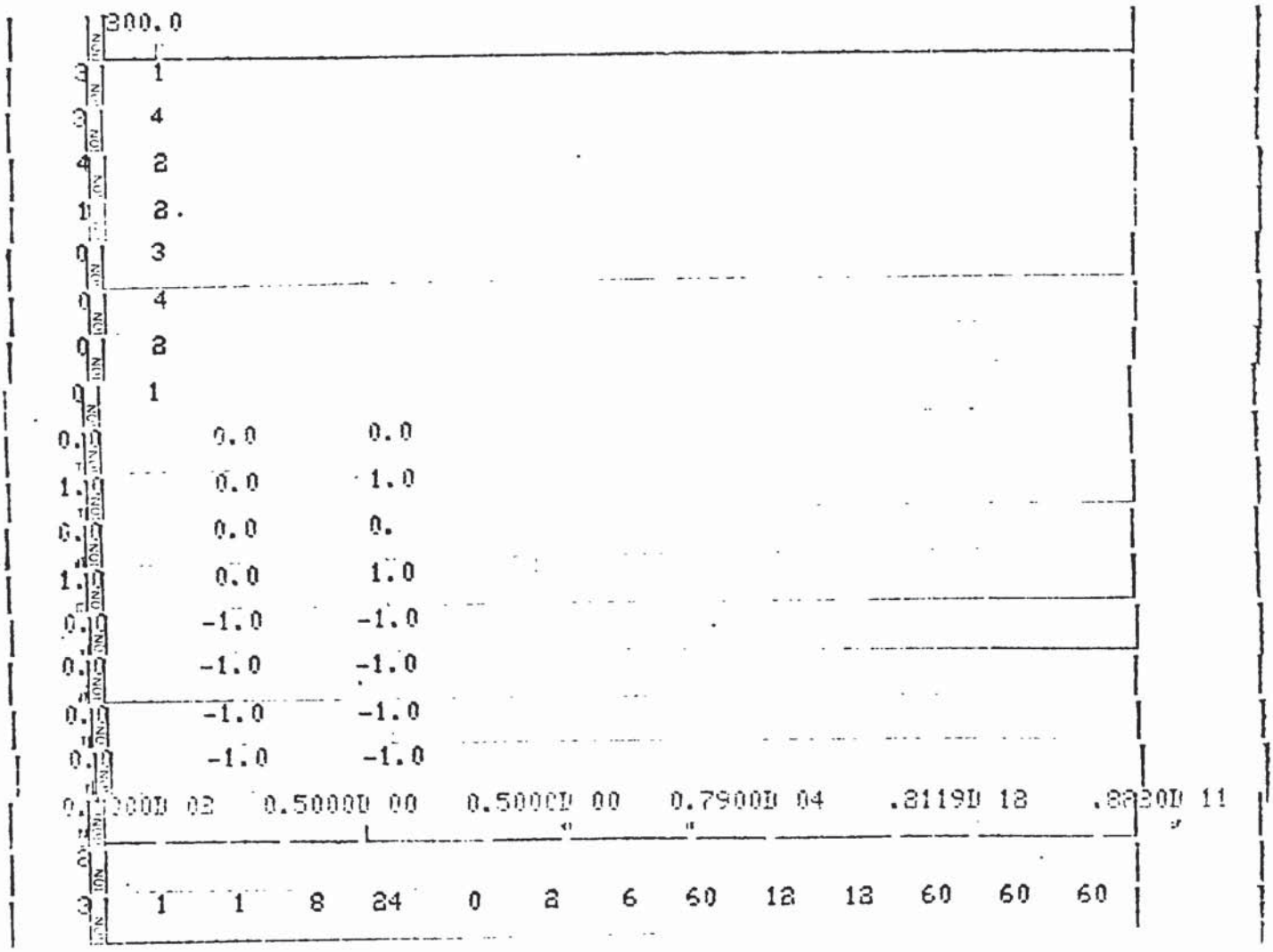


Fig (6.2.2) The input data cards for example 1

Note 1. Start reading from bottom.

Note 2: The Euler angles and the order of code numbers are chosen in such a way that the absolute value of each angle is less than 180°.

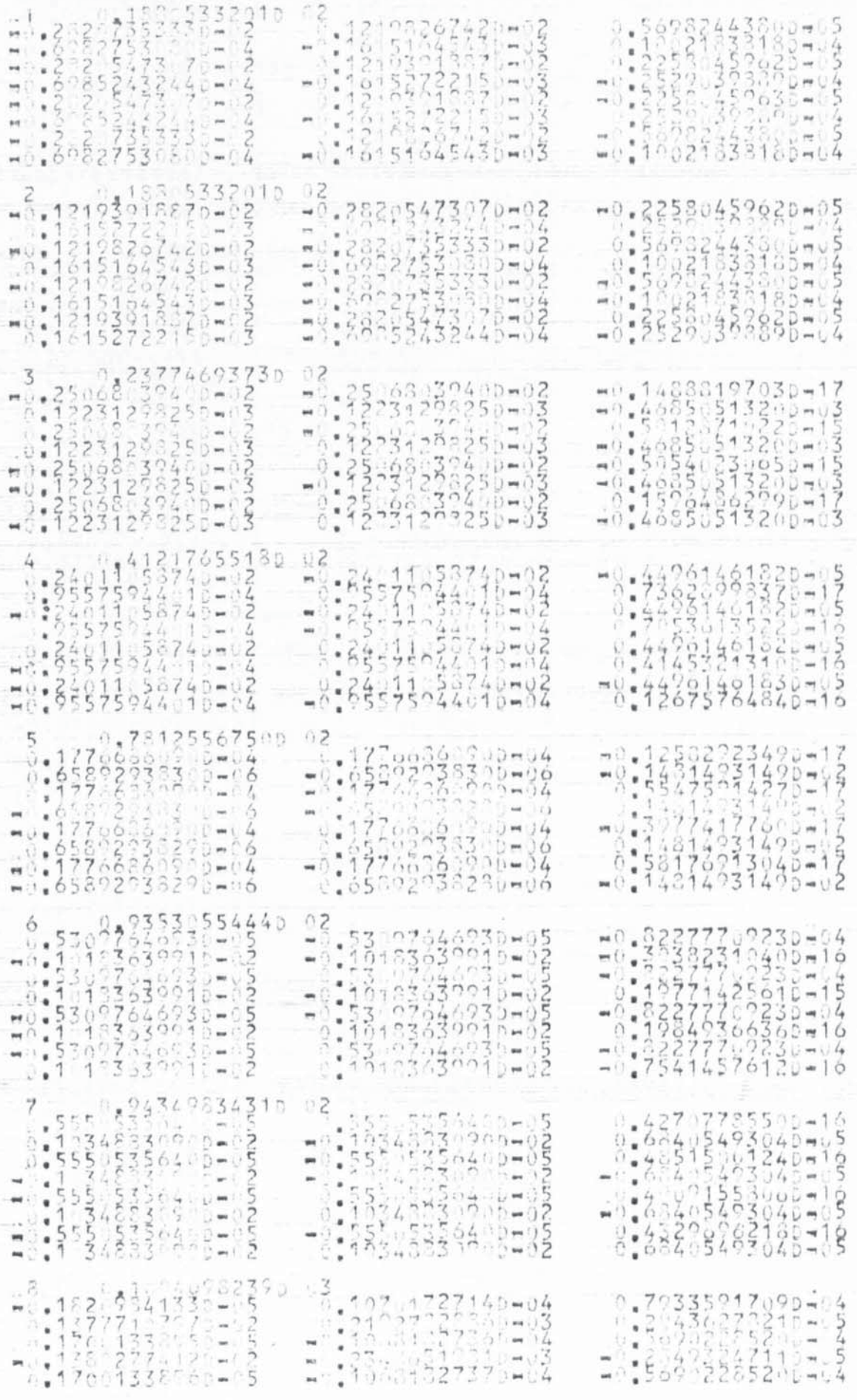


Fig (6.2.3) Computer output of example 1

MULTIJOB PRINT OF RAE00L:RAE03L.DT50 (S00004) FOR USER RAE00L

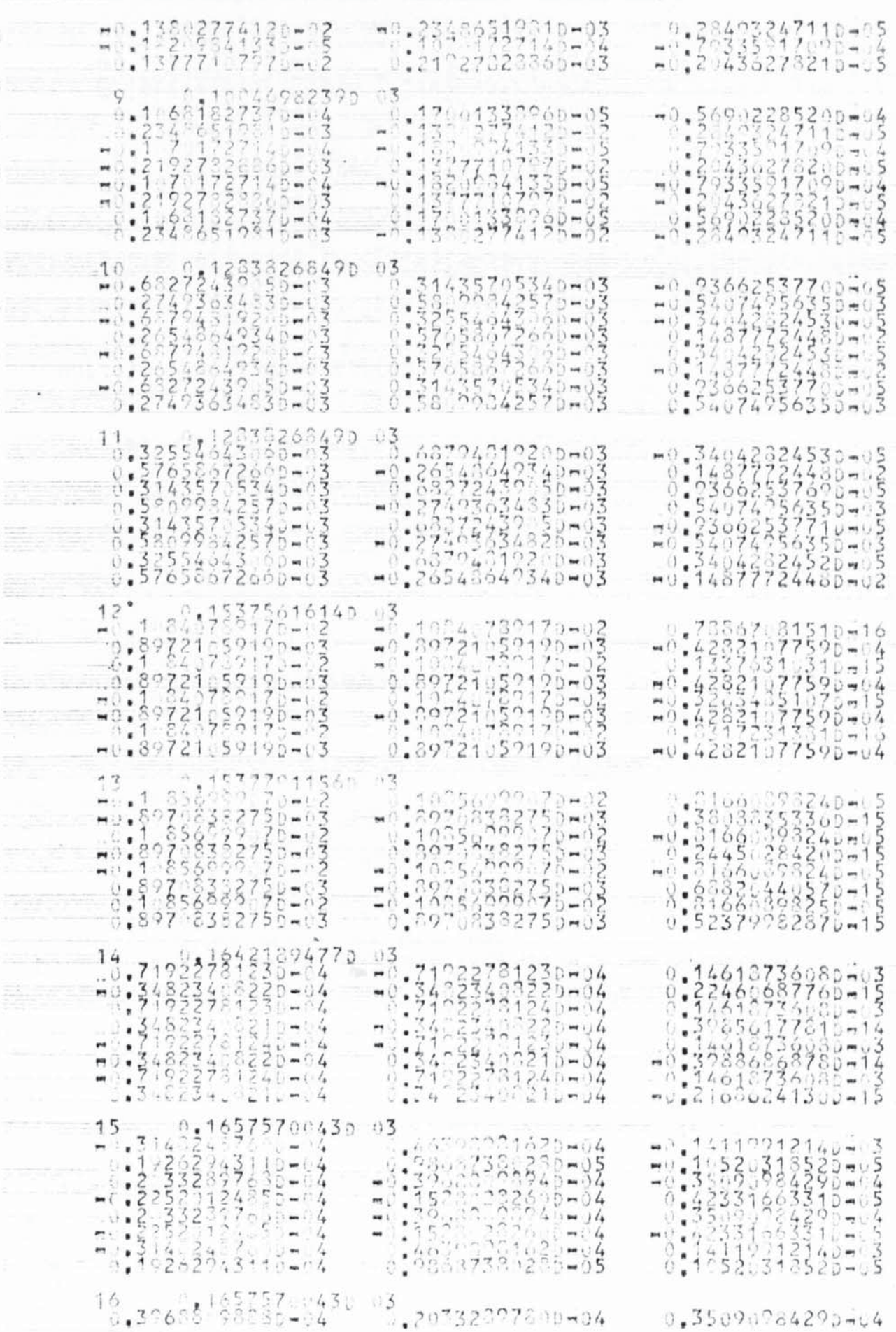


Fig (6.2.3) Continue

MULTIJOB PRINT OF RAE00L:RAE03L.DT50 (S00004) FOR USER RAE00L

MULTIJOB PRINT OF RAE00L:RAE03L.DT50 (S00004) FOR USER RAE00L

Fig (6.2.3) Continue

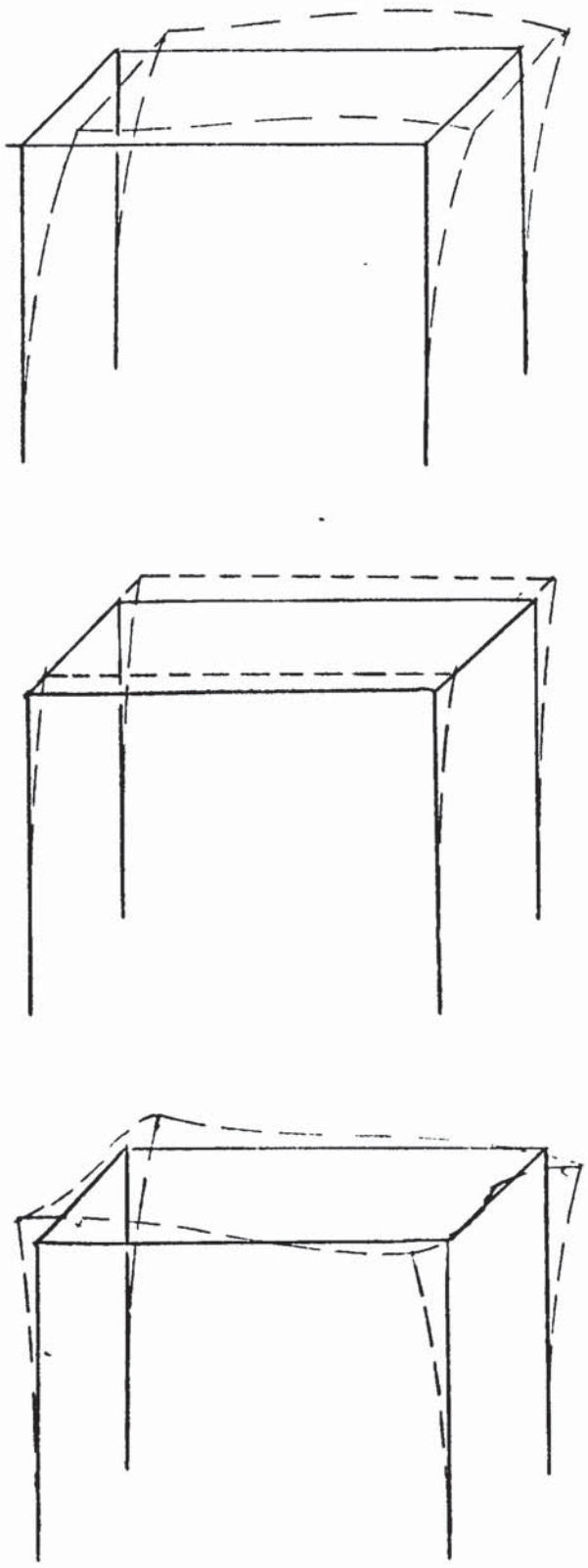


Fig (6.2.4) The first three modes of natural vibration for example 1

The computer output for the natural frequencies and the associated modes are printed in fig (6.2.3). The first line specifies the first natural frequency, i.e. 38.805332 rad/sec. The next two lines contains the displacements of the point 1 in the first mode. The first line of these contains the three translational displacement in the directions $\bar{X},\bar{Y},\bar{Z}$, respectively and the second lines contains the three angular displacements of this point in the direction $\bar{X},\bar{Y},\bar{Z}$ respectivly. Similarly for the other modes. All these modes are orthogonalized and normalized, and are ready to be used for the modal analysis. The first three modes are depicted in fig(6.2.4).

EXAMPLE 2

As a second example, we consider the plane frame as shown in fig(6.2.5). The iuput cards are presented in fig(6.2.6). The reader should refer to section (6.2.1) and the example 1 for the details of the input data. IPRINT was set to 2 in order to get output from line printer. The line printer output for all the natural modes are shown in fig(6.2.7). The upper frequency bound for the calculation was set to 6000 rad/sec. Note that the out of plane vibration modes are also calculated. These can be verified by the modal shapes.

EXAMPLE 3

As a third example, we consider the space frame as shown in fig (6.2.8). This space frame is made from mild steel as the last two examples. The input cards are presented in fig(6.2.9). The manners of inputing data is similar to example 1. IPRINT was set to 1 to get detailed information about the computation, e.g. which modes are obtained by root extraction and which modes are calculated by bisection, etc. The outputs are printed out in fig(6.2.10). The modal parameters are also stored on file for the used of modal analysis in next section.

```
000.0
 0   2
 1   2
 0   1
0.00000   -1.00000   -1.00000
1.00000    0.00000    1.00000
0.00000   -1.00000   -1.00000
0.3048D 00   0.31750-03   0.0187D 00   0.7900D 04   .2119D 18   .9930D 11
0.3540D 00   0.0187D 00   0.31750-03   0.7900D 04   0.2119D 18   0.9930D 11
0.3022D 00   0.31750-03   0.0187D 00   0.7900D 04   .2119D 18   .9930D 11
 1   1   1
 2   1   3   3   18   0   2   6   60   18   18   60   60   60
```

Fig (6.2.6)* The input cards for example 2

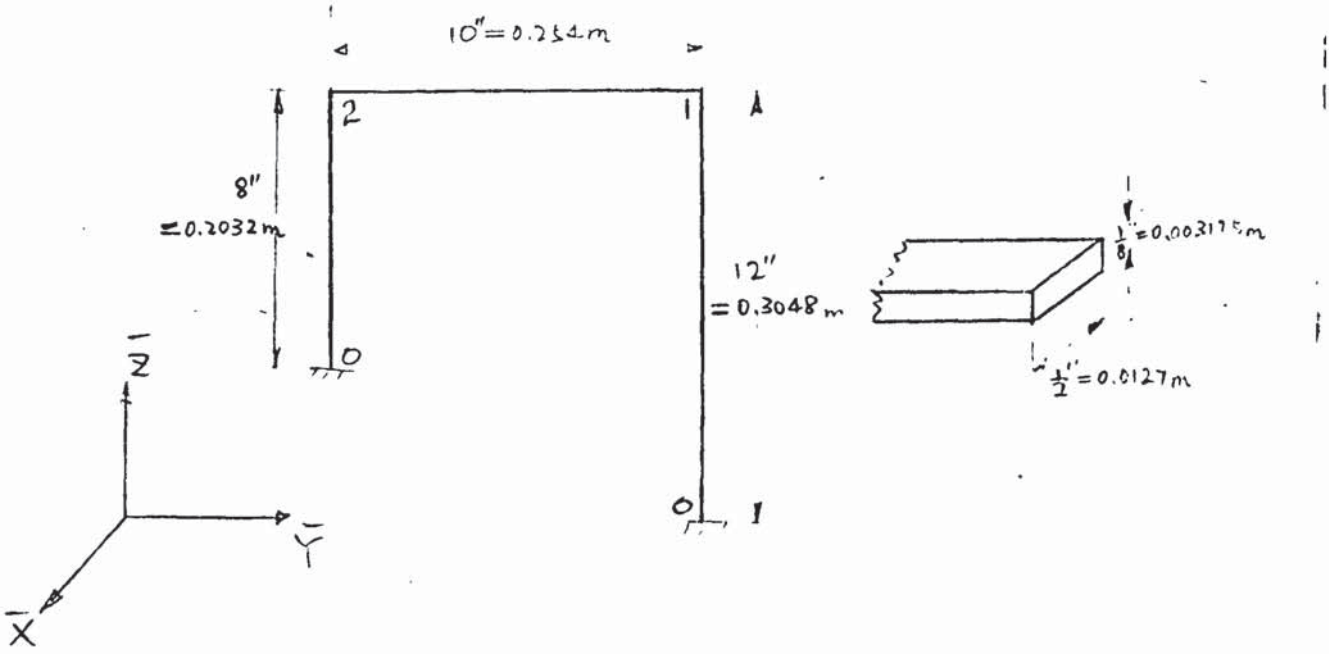


Fig (6.2.5) The plane frame of mild steel for example 2

* See foot notes of Fig (6.2.2)

Fig (6.2.7) Computer Print out for example 2

```
FORTRAN IV PROGRAM          (COMPOSED AS DYTL51PROG  ) STARTED 31/03/76 20:24:

ORTH,NNG,NES,NDF,NDD,NPE,NPD,NK,NR,NHD,NH1,NV,NYW
    1    3    3   12    0    2    0   60   12   12   60   60   60

NUMBER OF ELEMENTS IN GROUPS

    1    1    1

ELEMENT PROPERTY IN GROUP NUMBER =    1
  0.2032D 00  0.3175D-02  0.1270D-01  0.7900D 04  0.2119D 12  0.8220D 1

ELEMENT PROPERTY IN GROUP NUMBER =    2
  0.2540D 00  0.1270D-01  0.3175D-02  0.7900D 04  0.2119D 12  0.8220D 1

ELEMENT PROPELTY IN GROUP NUMBER =    3
  0.3048D 00  0.3175D-02  0.1270D-01  0.7900D 04  0.2119D 12  0.8220D 1

CONVERTED QUANTITIES
  0.2030 00  0.8540 07  0.930D 01  0.115D 03  0.710D 01  0.647D-01  0.924D-0
  0.2540 00  0.8540 07  0.930D 01  0.718D 01  0.115D 03  0.809D-01  0.116D-0
  0.3050 00  0.8540 07  0.930D 01  0.115D 03  0.710D 01  0.971D-01  0.139D-0

ANGLES OF ROTATION FOR EACH MEMBER

    0.00000  -1.00000  -1.00000
    1.00000   0.00000   1.00000
    0.00000  -1.00000  -1.00000

CODE NUMBERS

    0    1
    1    2
    0    2

FREQ RANGE=      0000.0000000000RAD/SEC

NUMBER OF PARTIAL FREQ.=     7

PARTIAL FREQ. OF THE STRUCTURE=
 1143.1543 1646.1421 2572.0972 3151.1469 4537.6523
 4572.6174 0000.0000
EIGEN NUMBER  1  64982.2628  NAT.FREQ=  254.9162  R/S OR  40.5712  HERTZ
EIGENVECTOR=
   0.00000  -2.70597  -0.00018  10.40275   0.00000  -0.00000
   0.00000  -2.70615   0.00050   1.80677   0.00000  -0.00000

EIGEN NUMBER  2  257991.1725  NAT.FREQ=  507.9283  R/S OR  80.8393  HERTZ
EIGENVECTOR=
  -0.15380  -0.00000  -0.00010   0.00000  -1.17145   2.99145
  -0.84783  -0.00000  -0.00000   0.00000  -3.93520   2.46265
```

EIGEN NUMBER 3 767932.7225 NAT.FREQ= 675.8040 R/S OR 139.3885 HERTZ
EIGENVECTOR=
-0.00000 -3.87262 -0.00173 -22.59641 -0.00000 -0.00000
0.00000 -0.87168 -0.00177 40.16066 0.00000 -0.00000

EIGEN NUMBER 41177309.6691 NAT.FREQ= 1085.0390 R/S OR 172.6893 HERTZ
EIGENVECTOR=
-0.59203 0.00000 -0.00000 0.00000 -4.12490 -2.76041
0.25980 0.00000 0.00000 -0.00000 1.85803 -3.05337

EIGEN NUMBER 51593015.4703 NAT.FREQ= 1262.1472 R/S OR 200.8770 HERTZ
EIGENVECTOR=
-0.63542 0.00000 -0.00000 -0.00000 -4.00691 -2.89509
0.29370 0.00000 0.00000 0.00000 1.03971 -3.05837

EIGEN NUMBER 61601703.7523 NAT.FREQ= 1265.5844 R/S OR 201.4240 HERTZ
EIGENVECTOR=
0.00000 0.72892 -0.00314 -39.25803 0.00000 0.00000
0.00000 0.72715 -0.00541 5.07294 -0.00000 0.00000

EIGEN NUMBER 65071366.2414 NAT.FREQ= 2251.9681 R/S OR 358.4119 HERTZ
EIGENVECTOR=
-0.00000 0.90275 0.00139 -05.29404 0.00000 0.00000
0.00000 0.90983 0.00725 -32.18798 0.00000 -0.00000

EIGEN NUMBER 75072843.7922 NAT.FREQ= 2252.2974 R/S OR 358.4643 HERTZ
EIGENVECTOR=
0.12774 -0.00000 -0.00000 0.00000 1.59839 3.66735
0.08280 -0.00000 -0.00000 0.00000 1.02544 -3.82719

EIGEN NUMBER 77938549.0950 NAT.FREQ= 2817.5432 R/S OR 448.4259 HERTZ
EIGENVECTOR=
-0.00000 -0.48978 0.00000 18.73152 -0.00000 -0.00000
-0.00000 -0.47975 -0.00193 09.65717 0.00000 0.00000

EIGEN NUMBER 87951892.0976 NAT.FREQ= 2819.9101 R/S OR 448.8026 HERTZ
EIGENVECTOR=
0.19798 -0.00000 0.00000 0.00000 0.79013 4.43168
0.00969 -0.00000 -0.00000 0.00000 -2.94720 -3.29899

EIGEN NUMBER 97950100.4819 NAT.FREQ= 2820.0571 R/S OR 448.9215 HERTZ
EIGENVECTOR=
-0.00486 0.00000 -0.00000 -0.00000 -1.28417 -0.97715
-0.12243 0.00000 0.00000 -0.00000 -4.57947 2.28938

```
IGEN NUMBER  8********** NAT.FREQ= 3409.2784  R/S OR  541.1711  HERTZ
IGENVECTOR=
  0.17644   0.00000   0.00000  -0.00000   3.62333   3.69967
 -0.08511   0.00000  -0.00000  -0.00000  -4.04610  -2.09642

IGEN NUMBER  9********** NAT.FREQ= 3403.1850  R/S OR  541.6337  HERTZ
IGENVECTOR=
  0.68261   0.00000   0.00000  -0.00000   1.21974   2.53482
  0.14362   0.00000   0.00000  -0.00000   3.22945  -3.40138

IGEN NUMBER 10********** NAT.FREQ= 3805.6196  R/S OR  605.7150  HERTZ
IGENVECTOR=
 -0.00000   0.37740   0.02000  55.86563  -0.00000  -0.00000
  0.00000   0.37568  -0.03237  47.45887   0.00000   0.00000

IGEN NUMBER 11********** NAT.FREQ= 3806.7550  R/S OR  606.1822  HERTZ
IGENVECTOR=
  0.18568   0.00000  -0.00000   0.00000  -1.03183   3.50046
 -0.14352   0.00000  -0.00000   0.00000  -5.04076  -1.76973

IGEN NUMBER 12********** NAT.FREQ= 3814.0337  R/S OR  607.0223  HERTZ
IGENVECTOR=
 -0.09208  -0.00000  -0.00000  -0.00000  -2.97613  -2.74811
 -0.15662  -0.00000   0.00000  -0.00000  -3.04176   3.55009

IGEN NUMBER 10********** NAT.FREQ= 4996.3427  R/S OR  795.1926  HERTZ
IGENVECTOR=
 -0.10474   0.00000  -0.00000  -0.00000   0.24166  -3.36909
 -0.20200   0.00000  -0.00000   0.00000  -2.64008   3.77397

IGEN NUMBER 11********** NAT.FREQ= 5001.7453  R/S OR  796.0525  HERTZ
IGENVECTOR=
  0.02642   0.00000   0.00000  -0.00000   5.88498   0.53416
 -0.08647   0.00000  -0.00000   0.00000  -0.72008   1.35948

IGEN NUMBER 12********** NAT.FREQ= 5005.3130  R/S OR  796.6203  HERTZ
IGENVECTOR=
  0.02516   0.00000  -0.00000  -0.00000   1.98773  -4.94831
 -0.10884  -0.00000  -0.00000   0.00000  -1.53231  -2.99398
```

```
EIGEN NUMBER 13********** NAT.FREQ= 5663.7153  R/S OR  901.4083  HERTZ
EIGENVECTOR=
  -0.11122   0.00000   0.00000   0.00000   1.14759  -3.81569
  -0.23910   0.00000   0.00000  -0.00000  -2.25430   3.67468

EIGEN NUMBER 14********** NAT.FREQ= 5668.9995  R/S OR  902.2493  HERTZ
EIGENVECTOR=
  -0.05391   0.00000   0.00000   0.00000  -5.52592  -0.31514
   0.05787   0.00000   0.00000  -0.00000  -2.20622  -0.36434

EIGEN NUMBER 15********** NAT.FREQ= 5671.5241  R/S OR  902.6511  HERTZ
EIGENVECTOR=
   0.00000   0.27713   0.01344  87.32735   0.00000  -0.00000
  -0.00000   0.24633   0.00498 -92.91736  -0.00000  -0.00000

EIGEN NUMBER 16********** NAT.FREQ= 5672.2993  R/S OR  902.7745  HERTZ
EIGENVECTOR=
   0.00027  -0.00000  -0.00000  -0.00000   0.99494  -4.82991
  -0.07081  -0.00000  -0.00000   0.00000  -0.99625  -3.24371

**FORTRAN ** STOP
MULTIJOB PRINT OF RAE00L:[YTL51.USET99(53425)]D  FOR USER RAE00L, TASK YTL51
```

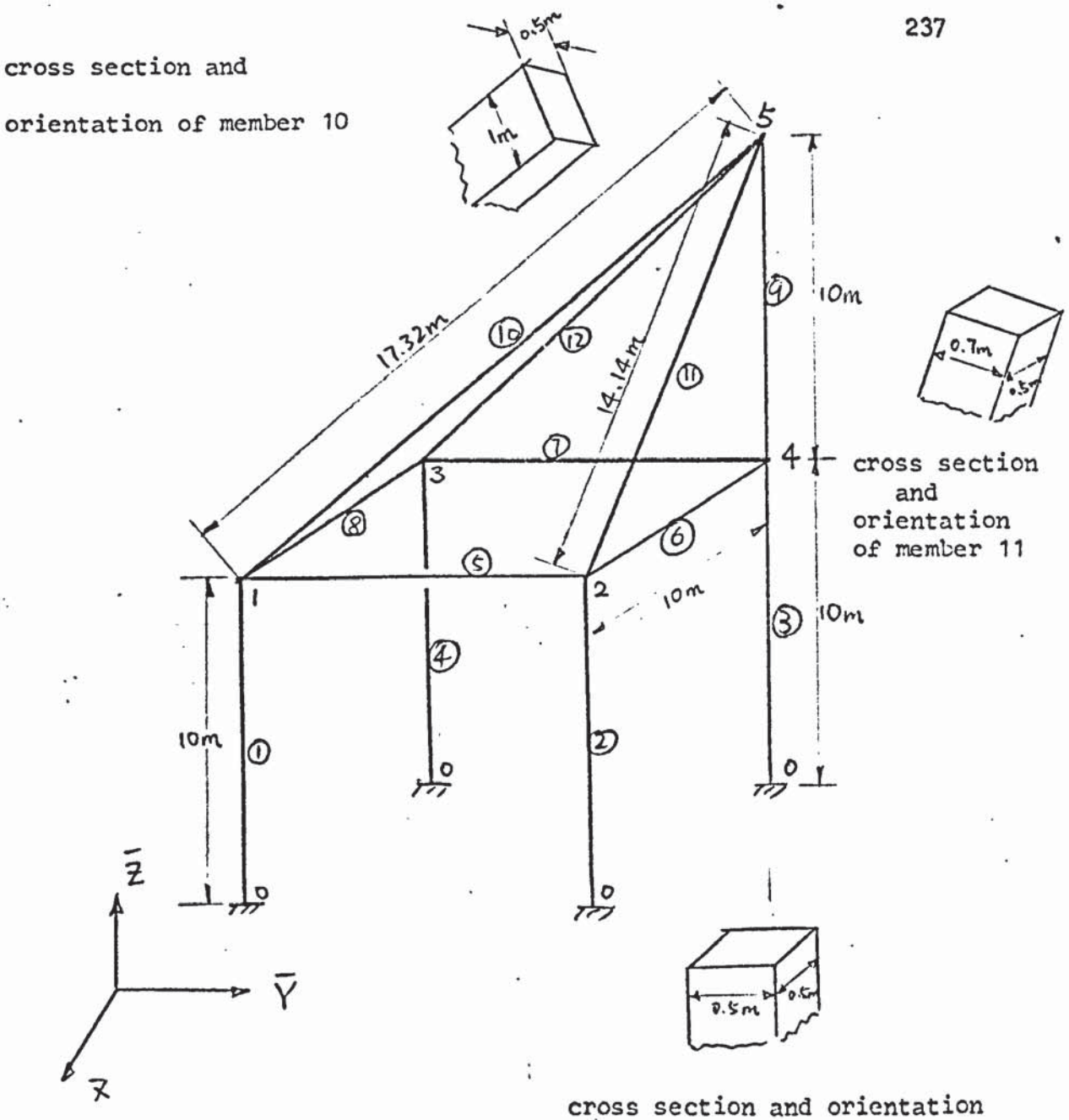
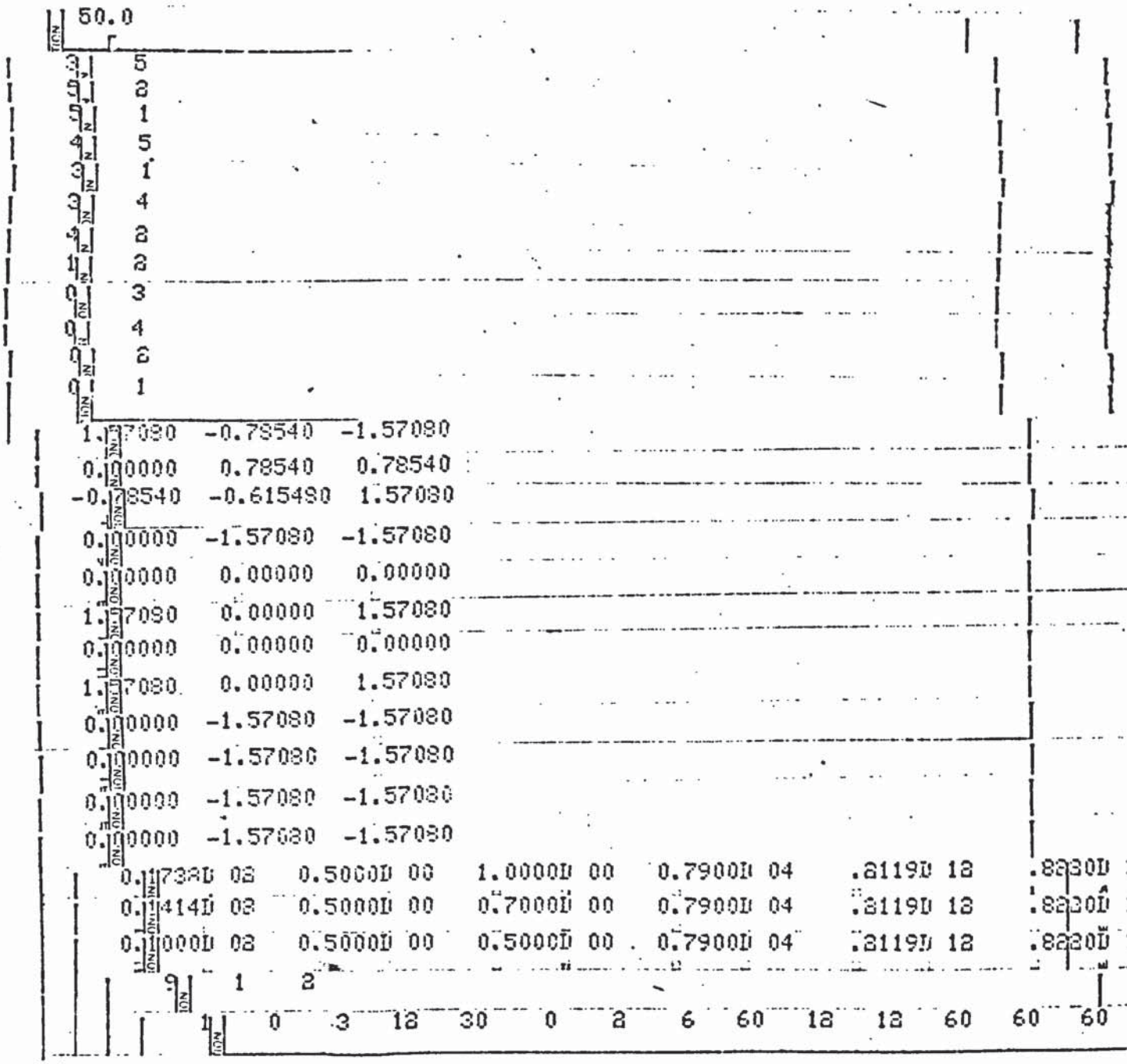


Fig (6.2.8) The space frame for example 3



```
50.0
5
2
1
5
1
4
2
2
3
4
2
1
1.57080   -0.78540   -1.57080
0.00000    0.78540    0.78540
-0.78540  -0.615480   1.57080
0.00000   -1.57080   -1.57080
0.00000    0.00000    0.00000
1.57080    0.00000    1.57080
0.00000    0.00000    0.00000
1.57080    0.00000    1.57080
0.00000   -1.57080   -1.57080
0.00000   -1.57080   -1.57080
0.00000   -1.57080   -1.57080
0.00000   -1.57080   -1.57080
0.11738D 03   0.5000D 00   1.0000D 00   0.7900D 04   .8119D 12   .8830D
0.11414D 03   0.5000D 00   0.7000D 00   0.7900D 04   .8119D 12   .8830D
0.10000D 03   0.5000D 00   0.5000D 00   0.7900D 04   .8119D 12   .8830D
1   2
1   0   3   18   30   0   2   6   60   18   18   60   60   60
```

Fig (6.2.9) The input cards for example 3

See foot notes of Fig(6.2.2)

Fig (6.2.10) Computer Output for example (3)

```
FORTRAN IV PROGRAM        (COMPOSED AS 0L51  P51   ) STARTED 07/06/76 18:31:3

ORTH,NNG,NES,NDF,NBD,NPE,NPD,NK,NR,NND,NM,NV,NYW
   0   3   12   30   0   2   6   72   12   12   72   72   72

NUMBER OF ELEMENTS IN GROUPS

   9   1   2

ELEMENT PROPERTY IN GROUP NUMBER     1
  0.1000D 02   0.5000D 00   0.5000D 00   0.7900D 04   0.2119D 12   0.8220D 11
ELEMENT PROPERTY IN GROUP NUMBER     2
  0.1414D 02   0.5000D 00   0.7000D 00   0.7900D 04   0.2119D 12   0.8220D 11
ELEMENT PROPERTY IN GROUP NUMBER     3
  0.1732D 02   0.5000D 00   0.1000D 01   0.7900D 04   0.2119D 12   0.8220D 11
CONVERTED QUANTITIES
  0.100D 02  0.530D 11  0.727D 09  0.110D 10  0.110D 10  0.198D 05  0.823D 03
  0.141D 02  0.742D 11  0.135D 10  0.303D 10  0.155D 10  0.391D 05  0.241D 04
  0.173D 02  0.106D 12  0.235D 10  0.883D 10  0.221D 10  0.684D 05  0.713D 04

ANGLES OF ROTATION FOR EACH MEMBER

  0.00000  -1.57080  -1.57080
  0.00000  -1.57080  -1.57080
  0.00000  -1.57080  -1.57080
  0.00000  -1.57080  -1.57080
  1.57080   0.00000   1.57080
  0.00000   0.00000   0.00000
  1.57080   0.00000   1.57080
  0.00000   0.00000   0.00000
  0.00000  -1.57080  -1.57080
 -0.78540  -0.61548   1.57080
  0.00000   0.78540   0.78540
  1.57080  -0.78540  -1.57080

CODE NUMBERS

   0   1
   0   2
   0   4
   0   3
   1   2
   4   2
   3   4
   3   1
   4   5
   5   1
   5   2
   3   5

FREQ RANGE=     50.000000000000RAD/SEC

NUMBER OF PARTIAL FREQ.=     1

PARTIAL FREQ. OF THE STRUCTURE=
  50.0000
```

```
****FREQ. RANGE=       0.00      50.00    CORR. FREQ.NUMBER=  0  5****

FREQ. BOUNDS       0.00       6.25    FREQ. NUMBERS  0  0

FREQ. BOUNDS       6.25      12.50    FREQ. NUMBERS  0  2

INITIAL APPROX. OF NAT. FREQ.
      11.3973093006       11.5041587546
RAYLEIGH'S IMPROVEMENTS
      11.3916502749       11.5006296559
EIGEN NUMBER  1    129.7695  NAT.FREQ=   11.3916  R/S OR    1.8130  HERTZ
EIGENVECTOR=
   0.00186  -0.00011  -0.00001  -0.00001   0.00003   0.00001
   0.00163  -0.00011   0.00000   0.00002   0.00006   0.00004
   0.00186  -0.00025   0.00000   0.00001   0.00010   0.00001
   0.00163  -0.00025   0.00000   0.00001   0.00008   0.00002
   0.00174  -0.00026   0.00001  -0.00000  -0.00002   0.00003

EIGEN NUMBER  2    132.2645  NAT.FREQ=   11.5006  R/S OR    1.8304  HERTZ
EIGENVECTOR=
  -0.00002  -0.00161  -0.00000   0.00008  -0.00001   0.00003
  -0.00039  -0.00161   0.00000   0.00007  -0.00001   0.00004
  -0.00002  -0.00183  -0.00001   0.00005  -0.00000   0.00003
  -0.00039  -0.00182   0.00001   0.00009  -0.00001   0.00002
  -0.00024  -0.00185   0.00001  -0.00003   0.00001   0.00000

FREQ. BOUNDS      12.50      18.75    FREQ. NUMBERS  2  2

FREQ. BOUNDS      18.75      25.00    FREQ. NUMBERS  2  3

INITIAL APPROX. OF NAT. FREQ.
      19.9944519436
RAYLEIGH'S IMPROVEMENTS
      19.9587578282
EIGEN NUMBER  3    398.3271  NAT.FREQ=   19.9581  R/S OR    3.1764  HERTZ
EIGENVECTOR=
   0.00199   0.00171  -0.00000  -0.00011   0.00009   0.00035
  -0.00216   0.00171  -0.00001  -0.00002  -0.00006   0.00018
   0.00199  -0.00075   0.00001   0.00003   0.00014   0.00027
  -0.00216  -0.00075   0.00000   0.00004  -0.00002   0.00035
  -0.00045  -0.00074   0.00001  -0.00003   0.00011   0.00033

FREQ. BOUNDS      25.00      31.25    FREQ. NUMBERS  3  3

FREQ. BOUNDS      31.25      37.50    FREQ. NUMBERS  3  3

FREQ. BOUNDS      37.50      43.75    FREQ. NUMBERS  3  4
```

```
INITIAL APPROX. OF NAT. FREQ.
       40.4102254333
RAYLEIGH'S IMPROVEMENTS
       40.4046761130
EIGEN NUMBER  4   1632.5300  NAT.FREQ=   40.4046  R/S OR    6.4306  HERTZ
EIGENVECTOR=
   0.00029   0.00065   0.00001   0.00010   0.00000  -0.00005
  -0.00047   0.00064   0.00002  -0.00045   0.00028   0.00027
   0.00029   0.00029  -0.00003  -0.00014  -0.00001   0.00002
  -0.00046   0.00029   0.00004  -0.00006   0.00002  -0.00002
  -0.00016   0.00015   0.00008   0.00030  -0.00042  -0.00035

FREQ. BOUNDS      43.75      50.00    FREQ. NUMBERS  4  5

INITIAL APPROX. OF NAT. FREQ.
       47.2604017547
RAYLEIGH'S IMPROVEMENTS
       47.2532809370
EIGEN NUMBER  5   2232.8691  NAT.FREQ=   47.2532  R/S OR    7.5206  HERTZ
EIGENVECTOR=
   0.00145   0.00082  -0.00000  -0.00005   0.00007  -0.00008
  -0.00008   0.00081  -0.00002  -0.00003  -0.00014   0.00035
   0.00143  -0.00013   0.00000   0.00001  -0.00037   0.00058
  -0.00008  -0.00013   0.00000   0.00000  -0.00001  -0.00007
   0.00053  -0.00012   0.00001  -0.00000   0.00025  -0.00032

7248 END OF FILE ENCOUNTERED, NO END ADDRESS SPECIFIED.

CURRENT DATA SET REF. NO. IS 97

CURRENT FORMAT STATEMENT IS:      (2413
        POINTER IS AT CHARACTER  .1.
                                  1

RECORD BEING PROCESSED IS:
        POINTER IS AT CHARACTER  .1.
                                  0

CONTROL WAS IN I/O 03
(LEVEL  1)  AT LOCATION 031590,= PROG. ORIGIN +000938 = I/O 03 +024350
 WAS CALLED BY      , COMPILED 07/06/76 BY VERSION 30 .  STATEMENT CARD NUME

MULTIJOB PRINT OF RAEOOL:OL51  .DSET99(S3741)/D  FOR USER RAEOOL, TASK L51
```

Fig (6.a.11) Output from the computer when IPRINT=3 for example 3


```
 1      0.113916430200D+02
 0.166055417400D-02   -0.114132730010D-03   -0.559751618100D-05
 0.114013157400D-04   -0.300621367000D-04    0.149946879500D-04
 0.163045696300D-02   -0.111013939000D-03    0.248338074800D-04
 0.166004981300D-04   -0.901349742500D-04    0.330053710570D-04
 0.185948431000D-02   -0.254220267000D-03    0.166703199800D-05
 0.639998153000D-05   -0.175677740000D-03    0.925561612000D-05
 0.162859361100D-02   -0.254611292000D-03    0.337855433330D-05
 0.128283506700D-04   -0.325230098500D-04    0.193328164400D-04
 0.173569275400D-02   -0.250861426100D-03    0.575338416380D-05
 0.242782066400D-05   -0.174504666600D-04    0.334695830900D-04

 2      0.115406295100D+02
 0.178925427500D-04   -0.161276662000D-02   -0.285150407600D-05
 0.771558725100D-04   -0.140042323300D-04    0.326029468940D-04
 0.392275524200D-03   -0.113213042500D-02    0.837081554000D-04
 0.668255277100D-04   -0.588316423300D-03    0.356274760000D-04
 0.164279735300D-04   -0.155135311400D-03    0.355724185370D-05
 0.538891366500D-04   -0.155345031400D-03    0.253006230000D-04
 0.390095535700D-03   -0.188503502590D-02    0.774539015000D-05
 0.897468670500D-04   -0.111351240600D-03    0.203985205700D-04
 0.236422668390D-03   -0.185153586600D-03    0.144199622680D-04
 0.306554041900D-04   -0.560012971000D-05    0.354148229400D-05

 3      0.199581213380D+02
 0.199239633300D-02   -0.171460493120D-02   -0.451453222910D-05
 0.109394819000D-03   -0.899824600300D-04    0.346673715700D-03
 0.216424572400D-02   -0.174662904000D-02    0.522033381380D-05
 0.242551066300D-04   -0.564574316000D-04    0.175932182800D-03
 0.199323318600D-02   -0.743224218230D-03    0.290926802400D-05
 0.232425220900D-04   -0.154344704000D-03    0.364197617000D-03
 0.215819344000D-02   -0.747604705500D-03    0.364856867810D-05
 0.357641324100D-04   -0.233349460130D-04    0.352915588900D-03
 0.445333522500D-03   -0.745567216000D-03    0.641926069000D-05
 0.284830563500D-04   -0.102956560000D-03    0.334663683590D-03

 4      0.424046342700D+02
 0.288404257000D-03   -0.647925554400D-03   -0.134255016600D-04
 0.181016926400D-03   -0.187336623800D-05    0.537499915800D-04
 0.467034188000D-03   -0.632921413700D-03   -0.192236510000D-04
 0.450744931200D-03   -0.276660330620D-03    0.274331281600D-03
 0.271395342200D-03   -0.662432240000D-03   -0.555000000000D-04
 0.143845980500D-03   -0.113922368670D-04   -0.207729107400D-04
 0.461837357700D-03   -0.200586011200D-03    0.400921357000D-04
 0.527336925500D-04   -0.108041220000D-04   -0.101231500000D-04
 0.164524324400D-03   -0.140041944400D-03   -0.704229148270D-04
 0.295583433400D-03   -0.415530727100D-03    0.553199409900D-03

 5      0.472532555500D+02
 0.145526200000D-04   -0.812550505500D-03    0.192022177500D-05
 0.528989324200D-04   -0.677443001800D-04    0.790555491000D-04
 0.795015645000D-04   -0.000422110000D-03    0.190164229100D-04
 0.264471658100D-04   -0.143024742175D-03   -0.352607929370D-03
 0.145379396800D-02   -0.312511501700D-03    0.400337043000D-03
 0.126136917000D-04   -0.307115524530D-03    0.378454603400D-03
 0.849480039420D-04   -0.134455823510D-03    0.372821387300D-05
 0.440089837670D-05   -0.636234847900D-05    0.743710508700D-04
 0.534522330000D-03   -0.124700015300D-03    0.250000075400D-03
 0.316147273700D-05   -0.247067331600D-03    0.315198443900D-03

 6      0.628445792400D+02
 0.238335460200D-03   -0.884077003610D-04    0.277745159900D-04
 0.364080074100D-03   -0.445614315900D-03   -0.921307050000D-04
 0.579593432100D-03   -0.111804734200D-03    0.205708930000D-04
 0.234816617000D-03   -0.255000135500D-03   -0.327156043500D-03
 0.550247022000D-03   -0.000000220000D-03    0.000423335000D-04
 0.563737612200D-03   -0.200572196000D-03    0.247061461500D-04
 0.598082784000D-04   -0.015548340000D-03    0.120446867300D-03
 0.535393478600D-04   -0.591063730000D-04    0.504852360100D-04
 0.237163300000D-03   -0.355000059000D-03    0.223184141700D-03
 0.540225072200D-03   -0.000024730000D-03   -0.430919612200D-05

 7      0.660169965700D+02
 0.481355220000D-03   -0.115420021000D-02   -0.130021656400D-04
 0.542433977200D-03   -0.000549067000D-03   -0.052755957500D-03
 0.308723545900D-03   -0.110911664300D-02    0.108890709300D-04
```

```
 4          -0.3351445289D-03     -0.4151503157D-03     -0.2507757911D-04
            -0.4326953150D-03     -0.7614349681D-03     -0.3867226055D-05
            -0.2477109452D-03     -0.1090009666D-03      0.2512663667D-03
             0.5669261610D-03     -0.8000019945D-03     -0.9053337377D-04
             0.6266908592D-04     -0.1436652330D-03     -0.1257032495D-03
             0.2546784327D-04     -0.6966757531D-03     -0.1232298901D-03
             0.2910825376D-03      0.1120502201D-03      0.7041391012D-03

 8          0.7590776642D 02
            -0.7089514135D-03      0.3459238432D-03     -0.5658119869D-04
            -0.3445915142D-03      0.1307413747D-03     -0.5567243933D-03
            -0.2762299390D-03      0.3160920630D-03     -0.4520951310D-05
            -0.4428[illegible]     0.[illegible]        -0.[illegible]
            -0.7207088513D-03      0.4553183434D-03      0.5337922432D-04
            -0.5795372455D-03      0.4754503451D-03      0.2970211390D-03
            -0.3984604950D-03      0.4553034577D-03      0.4431312790D-04
            -0.1642294647D-03     -0.1668122724D-03      0.3997624368D-03
            -0.4104152977D-03      0.6412741830D-03     -0.7470877405D-04
            -0.2565286245D-03      0.5005235212D-03      0.1256108538D-03

 9          0.3766478464D 02
-0.6706252309D-03      0.1416644061D-03     -0.5102695246D-04
-0.5979547177D-03      0.7026059542D-03      0.5167519973D-03
-0.5345556570D-03      0.1403803558D-03      0.3677277033D-05
-0.1349322030D-03      0.4934434322D-03      0.3413460338D-03
 0.7063893190D-03      0.2051855537D-03      0.4300435957D-04
 0.1926675656D-03      0.9507771740D-03      0.2243220533D-03
-0.8870992134D-03      0.1921363652D-03      0.1077176205D-04
-0.1130814855D-03      0.2741214404D-03      0.5666476201D-03
 0.7069416653D-03      0.4604814964D-04     -0.4165918444D-04
 0.7317215026D-04      0.5404732712D-03      0.4066643480D-03

10          0.9668013027D 02
 0.5775654742D-03      0.9329267613D-06      0.6070557514D-05
 0.6351066773D-03      0.[illegible]         0.[illegible]
 0.3297822114D-03      0.[illegible]         0.[illegible]
-0.1008367325D-03      0.[illegible]         0.[illegible]
-0.5462210[illegible]  0.[illegible]         0.[illegible]
-0.1317298[illegible]  0.[illegible]         0.[illegible]
-0.3[illegible]        0.[illegible]         0.[illegible]
 0.1496298[illegible]  0.[illegible]         0.[illegible]
-0.4370[illegible]     0.[illegible]         0.[illegible]
-0.8542006389D-04      0.1620894129D-03      0.3526228052D-03

11          0.[illegible]
-0.1985[illegible]     0.[illegible]         0.[illegible]
-0.8[illegible]        0.[illegible]         0.[illegible]
 0.1639716523D-03      0.[illegible]         0.[illegible]
-0.3[illegible]        0.[illegible]         0.[illegible]
-0.[illegible]         0.[illegible]         0.[illegible]
 0.5572[illegible]     0.[illegible]         0.[illegible]
-0.1[illegible]        0.[illegible]         0.[illegible]
-0.4[illegible]        0.[illegible]         0.[illegible]
 0.2122[illegible]     0.[illegible]         0.[illegible]
 0.1745540196D-03     -0.8335745335D-03     -0.3076408881D-03
```

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6.3 PROGRAMMES FOR THE RESPONSE ANALYSIS

The dynamic response analysis based on the modal analysis technique is to be discussed in chapter seven. When the modes of natural vibration are given, a programme for the analysis was designed and is presented in this section. The programme consists of a main driving programme and four subprogrammes, namely, INMOD, CFORCE, PCOORD, and GDISPL. INMOD integrates the differential equation $\ddot{\rho}_i + 2\omega_i\zeta_i\dot{\rho}_i + \omega_i^2\rho_i = 0$; CFORCE supplies the different forcing functions at different points on the structure; PCOORD superimposes all the responses to the individual forcing functions; and GDISPL calculates the generalized displacements and velocities at points of interest. The functions of the various segments of the programme are described in the following sections.

6.3.1 THE MAIN DRIVING PROGRAMME

The main driving programme consists of statements in three parts, which are the declaration statements (1 to 4), the input and print statements (5 to 29), and the execution statements (30 to 38).

The declaration statements declare the following arrays:

QG: contains the normal modes of the structure.

KC: contains the generalized coordinates which are subjected to forcing functions.

XX: contains the displacements of the generalized coordinates which are subjected to excitations.

XD: contains the velocities of the generalized coordinates which are subjected to excitations.

FQ: contains the natural frequencies in rad/ sec.

DM: contains the damping ratios for each natural mode.

RX: contains principal displacements of response p_i.

RD: contains principal velocities of response $\dot{p}_i$

U: contains the displacements of the generalized coordinates q_i.

V: contains the velocities of the generalized coordinates $\dot{q}_i$.

The read statement number 6 reads the following parameters:

NQ: the number of generalized coordinates.

NFORCE: the number of excitation forces.

NMOD: the number of modes taken in the analysis.

NC1: the lower bound of the generalized coordinates where the responses are of interest.

NC2: the upper bound of the generalized coordinates where the responses are of interest.

NSTEP: the number of time increments assumed.

TEND: the termination of the time interval of interest in sec.

Statement 14 reads the generalized coordinate numbers where the excitations are presented into the array KC. Statement 17 reads the damping ratios for each mode. Statements 20 to 24 read the natural frequencies and the associated modes as resulted from MAINP by setting IPRINT=3.

The execution statement number 25 works out the time increment DT. The subroutine GDISPL is called for every time step and every generalized coordinate whose displacements and velocities of reponses are of interest.

6.3.2 SUBROUTINE INMOD

The subroutine INMOD was designed to calculate the reponse x(t) to an arbitrary forcing function f(t) with the consideration of damping effects as described in section (7.8). The differential equation to be solved is $\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = f(t)$

where ζ is the damping ratio and ω is the natural frequency.

The solutions are given by equations (7.8.11) and (7.8.12):

$$x(t) = \frac{1}{a}\int_0^t f(\tau)e^{b(t-\tau)}\sin a(t-\tau)\,d\tau + x_o e^{bt}\cos at + (\dot{x}_o - bx_o)\frac{1}{a}e^{bt}\sin at$$

$$\dot{x}(t) = \int_0^t e^{b(t-\tau)}\cos a(t-\tau)f(\tau)\,d\tau + \frac{b}{a}\int_0^t e^{b(t-\tau)}f(\tau)\sin a(t-\tau)\,d\tau + \dot{x}_o e^{bt}\cos at + (b\dot{x}_o - \omega^2 x_o)\frac{1}{a}e^{bt}\sin at$$

where $a = \omega\sqrt{1-\zeta^2}$, $b = -\zeta\omega$

The input data to the subroutine are

XX: initial displacement x_o.

XD: initial velocity $\dot{x}_o$.

FQ: the natural frequency.

S: the modal damping ratio.

DT: the increment of time interval.

T: the termination of the time interval.

The output parameters from the subroutine are:

XX: the final displacement, x(t).

XD: the final velocity, $\dot{x}(t)$.

FQ,S,DT,T unchanged.

6.3.3 REAL FUNCTION CFORCE

CFORCE returns the forcing functions at different points on the structure. The number of excitations is obtained through the common block PCOCFO. This subprogramme should be changed for each new forcing function.

6.3.4 SUBROUTINE PCOORD

PCOORD superimposes all the responses to the individual forcing functions. This subroutine calls INMOD n times, where n= the number of the excitations presented.

6.3.5 SUBROUTINE GDISPL

GDISPL calculates the generalized displacements and velocities at points of interest. This subroutine calls PCOORD n times, where n= the number of modes taken in the modal analysis.

6.3.6 EXAMPLE

As an example, we consider the space frame structure shown in fig(6.2.8), whose natural modes calculated in example 3 of section (6.2.13). The input data are shown in fig (6.3.1). Where 30 is the number of generalized coordinates which consist of the whole structure; 5 is the number of excitation forces; 10 is the number of modes taken; 25 and 30 are read such that the displacements and velocities at generalized coordinates 25 to 30 inclusive are required; 10 is the number of time

steps assumed and 1.000 sec is the upper bound of the time interval interested. For the second line, 1,8,10,25 and 26 are the generalized coordinate numbers which are subjected to excitations as supplied by the subprogramme CFORCE in that order. The remaining part of the input data contains the natural frequencies and the associated modes of the structure.

The output from the programme is printed in fig (6.3.2).

```
 TRIALS RUN 3227   DATE 03/03/76   TIME 10:22:59                    FROM PROCEDUR
 LOGIN USERNAME:  RAE00L      GROUPNAME:  RAE03L

// FTRAN1 RAE00L:@L122.YT122(F3227),K                               SOURCE FILE :
// OPTION LIST,DEBUG
EXIT TO SYSTEM:TRPROG.FTRAN1(P90224)   AT 10:23:04


FTRN4 800S30  SOURCE PROGRAM                                        03/03/76   PAG
    1        DOUBLE PRECISION DD(60),DF
    2        DIMENSION QG(60,10),KC(10),XX(10),XD(10),FQ(20),DM(20),RX(20),
    3       *          RD(20),U(60),V(60)
    4        EQUIVALENCE (DD(1),U(1)),(DD(30),V(1))
    5        DATA XX,XD/20*0./
    6        NGQ=60
    7        READ (5,1) NQ,NFORCE,NMOD,NC1,NC2,NSTEP,TEND
    8        WRITE(6,1) NQ,NFORCE,NMOD,NC1,NC2,NSTEP,TEND
    9      1 FORMAT (6I5,F10.4)
   10        IF (NQ.LE.60.AND.NFORCE.LE.10.AND.NMOD.LE.20) GOTO 6
   11        WRITE (6,7) NQ,NFORCE,NMOD
   12      7 FORMAT (/'NQ,NFORCE,NMOD=',3I5//'DIMENSION STATEMENT NEEDS',
   13       *' MODIFIED')
   14        CALL EXIT
   15      6 READ (5,2) (KC(I),I=1,NFORCE)
   16        WRITE(6,2) (KC(I),I=1,NFORCE)
   17      2 FORMAT (16I5)
   18        READ (5,3) (DM(I),I=1,NMOD)
   19        WRITE(6,3) (DM(I),I=1,NMOD)
   20      3 FORMAT(10F8.4)
   21        DO 4 K=1,NGQ
   22        READ (5,5)I,DF    ,(DD(L)   ,L=1,NQ)
   23        WRITE(6,5)I,DF    ,(DD(L)   ,L=1,NQ)
   24      5 FORMAT (I5,D20.10/200(3D20.10/))
   25        FQ(K)=SNGL(DF)
   26        DO 8 L=1,NQ
   27      8 QG(L,K)=SNGL(DD(L))
   28      4 CONTINUE
   29        DT=TEND/FLOAT(NSTEP)
   30        CALL GDISPL(QG,NQ,NGQ,NMOD,KC,NFORCE,XX,XD,FQ,DM,DT,NSTEP,RX,RD,
   31       *     U,V,NC1,NC2)
   32        STOP
   33        END


FTRN4 800S30  SOURCE PROGRAM  CFORCE   FUNCTION                     03/03/76   PAG
   34        FUNCTION CFORCE(T)
   35        COMMON /PCOCF0/ KGC
   36        PI=3.14159265358979
   37        GOTO (1,2,3,4,5), KGC
   38      1 CFORCE= SIN(T/PI)*10000.
   39        GOTO 99
   40      2 CFORCE= SIN(T*0.1)*10000.
   41        GOTO 99
   42      3 CFORCE=0.
   43        IF(T.GT.10) CFORCE=50000.
   44        IF(T.GT.20) CFORCE=0.
   45        GOTO 99
   46      4 CFORCE=T*1000.
   47        GOTO 99
   48      5 CFORCE=T*T*20000.
   49     99 RETURN
   50        END
```

```
FTRN4 S00S3C  SOURCE PROGRAM  GDISPL   SUBROUTINE                    03/03/76   PAG

   51         SUBROUTINE GDISPL(QG,ND,NOG,NM,KC,NF,XX,XD,FQ,DM,DT,T,RX,RD,U,V,
   52        *       NC1,NC2)
   53         DIMENSION QG(NOG,NM),KC(NF),XX(NF),XD(NF),FQ(NM),DM(NM),RX(NM),
   54        *       RD(NM),U(ND),V(ND),XI(6),WI(6)
   55         COMMON /PCOCFO/ X
   56         DATA
   57        *XI/.1252334,.3678315,.5873180,.7699027,.9041173,.9815606/
   58         DATA
   59        *WI/.2491470,.2334925,.2031674,.1600783,.1069393,.0471753/
   60      10 FORMAT('T=',F10.5,'  SEC')
   61      20 FORMAT('DISP=',100(6E15.5/5X))
   62      21 FORMAT('VECLO',100(6E15.5/5X))
   63         NPNT=6
   64         DO 100 IDT=1,NSTEP
   65         T=DT*FLOAT(IDT)
   66         WRITE(6,10) T
   67         DO 102 L=NC1,NC2
   68         R=0.
   69         S=0.
   70         DO 201 I=1,NM
   71         X0=0.
   72         D0=0.
   73         DMI=DM(I)
   74         FQI=FQ(I)
   75         DO 301 K=1,NF
   76         A=FQI*SQRT(1.-DMI*DMI)
   77         B=-DMI*FQI
   78         F=XX(K)
   79         G=XD(K)
   80         Y=0.
   81         Z=0.
   82         P=0.5*DT
   83         Q=T-P
   84         DO 5 J=1,NPNT
   85         X=XI(J)
   86         W=WI(J)
   87         Y1=P+P*X
   88         Y2=P-P*X
   89         X1=Y1+T-DT
   90         X2=Y2+T-DT
   91         C=DT-Y1
   92         D=DT-Y2
   93         E2= EXP(B*D)
   94         E1= EXP(B*C)
   95         C1= COS(A*C)
   96         C2= COS(A*D)
   97         S1= SIN(A*C)
   98         S2= SIN(A*D)
   99         F1=CFORCE(X1)*E1
  100         F2=CFORCE(X2)*E2
  101         G1=F1*C1
  102         G2=F2*C2
  103         H1=F1*S1
  104         H2=F2*S2
  105         Y=Y+W*(G1+G2)
  106         Z=Z+W*(H1+H2)
  107       5 CONTINUE
  108         Y=Y*DT*.5
  109         Z=Z*DT*.5
  110         BD=B*DT
  111         AD=A*DT
  112         E= EXP(BD)
  113         D= SIN(AD)/A
  114         C= COS(AD)
  115         XX(K)=Z/A+F*E*C+(G-B*F)*E*D
  116         XD(K)=Y+B*Z/A+G*E*C+(B*G-FQI*FQI*F)*E*D
  117         J=KC(K)
  118         RQ=QG(J,I)
  119         X0=X0+RQ*XX(K)
  120         D0=D0+RQ*XD(K)
  121     301 CONTINUE
  122         RX(I)=X0
  123         RD(I)=D0
  124         R=R+QG(L,I)*RX(I)
  125         S=S+QG(L,I)*RD(I)
  126     201 CONTINUE

FTRN4 S00S3C  SOURCE PROGRAM  GDISPL   SUBROUTINE                    03/03/76   PAG

  127         U(L)=R
  128         V(L)=S
  129     102 CONTINUE
  130         WRITE (6,20) (U(L),L=NC1,NC2)
  131         WRITE (6,21) (V(L),L=NC1,NC2)
  132     100 CONTINUE
  133         RETURN
  134         END
```

```
 FORTRAN IV PROGRAM          (COMPOSED AS @L121 YT121 ) STARTED 01/03/76

          30     5    10    25    30    10     5.0000
           1     8    10    25    26
       0.1000   0.0100   0.0100   0.0100   0.0100   0.0100   0.0100   0.0100   0.0
           1     0.1139164302D 02
        0.1860504174D-02    -0.1141327301D-03    -0.5597516181D-05
       -0.1140131574D-04     0.8066021387D-04     0.1299468795D-04
        0.1630456963D-02    -0.1116183098D-03     0.2483390748D-06
        0.1680049813D-04     0.6013467425D-04     0.3802371057D-04
        0.1859484810D-02    -0.2542020700D-03     0.1667031993D-05
        0.6389801530D-05     0.1048577493D-03     0.9725806102D-05
        0.1628590611D-02    -0.2542001299D-03     0.3878843333D-05
        0.1282835067D-04     0.8252300935D-04     0.1933281644D-04
       -0.1735692754D-02    -0.2588614261D-03     0.5753341638D-05
       -0.2427828684D-05    -0.1745646668D-04     0.3348958809D-04

           2     0.1150062951D 02
       -0.1789254275D-04    -0.1612766690D-02    -0.2851504076D-05
        0.7715587251D-04    -0.1400423233D-04     0.3260294894D-04
       -0.3922755242D-03    -0.1613284251D-02     0.9087981554D-06
        0.6682552771D-04    -0.5808164283D-05     0.3562747660D-04
       -0.1842767353D-04    -0.1825103311D-02    -0.5872418537D-05
        0.5389186653D-04    -0.1522439314D-05     0.2530628303D-04
       -0.3909503557D-03    -0.1822858259D-02     0.7974539015D-05
        0.8974807654D-04    -0.1113585486D-04     0.2039852057D-04
       -0.2364208839D-03    -0.1851533968D-02     0.1441966268D-04
       -0.3065540419D-04     0.5680129710D-05     0.3541482940D-05

           3     0.1995812138D 02
       -0.1992398333D-02     0.1714849312D-02    -0.4514582291D-05
       -0.1093948196D-03     0.8628246903D-04     0.3466737157D-03
       -0.2164245724D-02     0.1714662204D-02    -0.5283838138D-05
       -0.2425510683D-04    -0.5645704316D-04     0.1750321828D-03
        0.1993233186D-02    -0.7473221823D-03     0.5609268024D-05
        0.2824252961D-04     0.1363449964D-03     0.2741070170D-03
       -0.2158193944D-02    -0.7476947095D-03     0.3085688781D-05
        0.3570413241D-04    -0.2333049813D-04     0.3529155889D-03
       -0.4453385925D-03    -0.7420567218D-03     0.6410206890D-05
       -0.2846305835D-04     0.1092095956D-03     0.3346868359D-03

           4     0.4040463427D 02
       -0.2884042567D-03    -0.6479025504D-03    -0.1342550166D-04
       -0.1010169284D-03    -0.1873686238D-05     0.5374299158D-04
        0.4670341806D-03    -0.6379214137D-03    -0.1923051900D-04
        0.4507449302D-03    -0.2780663062D-03    -0.2743312816D-03
       -0.2903954220D-03    -0.2862450024D-03     0.2540808530D-04
        0.1438459805D-03     0.1139228867D-04    -0.2077991074D-04
        0.4618373577D-03    -0.2900830102D-03    -0.4009213570D-04
        0.5973382953D-04    -0.1649813293D-04     0.1912315060D-04
        0.1645243244D-03    -0.1486419444D-03    -0.7842804827D-04
       -0.2955884334D-03     0.4105307271D-03     0.3531994099D-03
```

```
 5     0.47253255555D 02
-0.14528269266D-02   -0.81825505056D-03    0.19292217756D-05
 0.52898932426D-04   -0.67744300186D-04    0.79055549106D-04
 0.79591564586D-04   -0.80720210696D-03    0.13616042916D-04
 0.26447066516D-04    0.14395421756D-03   -0.35260769376D-03
-0.14307373846D-02    0.13125159176D-03   -0.48033764306D-05
-0.12613691766D-04    0.36711524536D-03   -0.57845480346D-03
 0.84948039426D-04    0.13445355516D-03   -0.37282138736D-05
-0.44089837676D-05    0.63623484796D-05    0.74371050876D-04
-0.53452230096D-03    0.12476088156D-03   -0.95000675486D-05
 0.31614727376D-05   -0.24706733166D-03    0.31519844396D-03

 6     0.62844579246D 02
-0.23833546926D-03    0.98487903616D-04    0.27774515996D-04
 0.36468074106D-03    0.44541431596D-03   -0.98913670506D-04
 0.57959343216D-03    0.11080473426D-03    0.20576889306D-04
-0.23481061706D-03    0.25754218556D-03   -0.32715604356D-03
-0.25624082986D-03    0.49089538426D-03    0.63664233356D-04
 0.56373761226D-03   -0.29950721966D-03    0.24706146156D-04
 0.59808278406D-03    0.49731848346D-03    0.12044686736D-03
-0.53539347866D-04    0.59066370396D-04    0.58485236016D-04
 0.23716330966D-03    0.35590059806D-03    0.22318414176D-03
-0.54092507226D-03   -0.48908024736D-03   -0.43091961226D-05

 7     0.66016996576D 02
-0.40813559296D-03   -0.11548298106D-02   -0.13069165646D-04
-0.54243597726D-03    0.30634096706D-03   -0.65275505756D-03
 0.30872354596D-03   -0.11691166436D-02    0.10882070936D-04
 0.33514459896D-03    0.41515081576D-03   -0.25077579116D-04
-0.43295315886D-03   -0.78214545816D-03    0.83036872266D-05
-0.24771994526D-03   -0.10900296666D-03    0.25126036876D-03
 0.30689261616D-03   -0.80099199456D-03   -0.60865337376D-04
 0.62669859566D-04   -0.14363523996D-03   -0.19278324956D-03
 0.25467843276D-04   -0.60690575316D-03   -0.12322989016D-03
 0.29198023766D-03    0.10126502816D-03    0.76415910126D-03

 8     0.75907768426D 02
-0.70895141356D-03    0.34592384326D-03   -0.56581198696D-04
-0.34459151426D-03   -0.30579137476D-03   -0.55672439336D-03
-0.27622993096D-03    0.31699206306D-03   -0.45209513106D-05
-0.44295603796D-04   -0.22774818586D-04    0.58611245946D-03
-0.72078851636D-03    0.45316543466D-03    0.53379204326D-04
 0.57953724956D-03    0.47048255176D-03    0.23970911396D-03
-0.30846049566D-03    0.45336345776D-03    0.44313127906D-04
-0.16422946476D-03   -0.16681207246D-03   -0.39976343686D-03
-0.41041529976D-03    0.44097418396D-03    0.74768774656D-04
-0.25652862456D-03    0.50052352126D-03    0.12561085386D-03

 9     0.87664784646D 02
-0.67062523096D-03   -0.14166408616D-03   -0.51026952466D-04
-0.59795471776D-03   -0.70269595426D-03    0.51675190736D-03
 0.83455565476D-03   -0.14603635046D-03    0.63677277636D-05
 0.10498229306D-03    0.49344343926D-03   -0.34134633806D-03
 0.70638393196D-03   -0.20216505376D-03    0.43804359576D-04
 0.19266756566D-03   -0.67297771746D-04   -0.27243029536D-03
 0.88709921316D-03   -0.19821363656D-03   -0.10971762056D-04
-0.11308104856D-03   -0.27412144046D-03    0.56664762616D-03
 0.70694186536D-03   -0.46848149646D-04   -0.41659184446D-04
 0.78172150266D-04    0.34647827126D-03    0.46666434806D-03

10     0.96680130276D 02
 0.57756547496D-03    0.93292676136D-06    0.60705575146D-05
 0.63510697396D-03   -0.63355454336D-03   -0.56824270786D-03
 0.32982211436D-03   -0.27017464906D-04    0.86391032296D-04
-0.10063673256D-03    0.75147336286D-03    0.56608404406D-03
-0.54682164846D-03   -0.51796175236D-04   -0.41563650186D-04
-0.13172983086D-03    0.51094596026D-03    0.11136951016D-02
 0.30556615906D-03   -0.90732488966D-04   -0.22301468106D-04
 0.14962983546D-03   -0.45870373196D-03   -0.11376388486D-02
 0.43761125906D-03   -0.19536720626D-04   -0.65710369876D-04
-0.85420063896D-04    0.16203941296D-03    0.35262280526D-03
```

FORTRAN IV PROGRAM (COMPOSED AS BL122 YT122) STARTED 04/03/76 19:

```
   30     5    10    25    30    10     5.0000
    1     8    15    25    26
 0.100  0.010  0.010  0.010  0.010  0.010  0.010  0.010  0.000  0.000
    1     0.113916430200 02
  0.186050417400-02    -0.114132738010-03    -0.559751618100-05
  0.114001315740-04    -0.000000213000-04    -0.000000687950-04
  0.163045696300-02    -0.111613300000-03    -0.248330907480-06
  0.135943481900-02    -0.545000000000-04    -0.106731319000-04
  0.638980153000-05    -0.577400000000-04    -0.375500610200-05
  0.162000596110-02    -0.255000000000-04    -0.333300164400-04
  0.126000635067-04    -0.225000000000-03    -0.575334410300-05
  0.173560927540-02    -0.255000614261-03    -0.334000588000-04
  0.242782864840-05    -0.174564600000-04    -0.334000588000-04

   2     0.115006295100 02
  0.178025427500-04    -0.161277066000-02    -0.225152407600-05
  0.771555872510-04    -0.140000333333-04    -0.320000489400-04
  0.392275524200-03    -0.000000000000-04    -0.000000155430-06
  0.668255277100-04    -0.558000316420300-05    -0.356627476600-04
  0.184276735300-04    -0.155000000000-03    -0.555724185375000-05
  0.538918665300-04    -0.152240000314000-05    -0.253062830800-04
  0.390055355700-03    -0.188000000000-02    -0.797453990150000-05
  0.897488876540-04    -0.111350000000-03    -0.203905520570000-04
  0.236420839000-03    -0.100015330000-02    -0.144129662000000-04
  0.365540419000-04    -0.560012071000-05    -0.354140029400-05

   3     0.199581213380 02
  0.129239333300-02    -0.171400493120-02    -0.451456622910-05
  0.193904610000-03    -0.000000240000-04    -0.000000715700-03
  0.216424572400-02    -0.171400000000-02    -0.000000000000-03
  0.242551760000-04    -0.564000000000-04    -0.000000000000-03
  0.199323310000-02    -0.134000000000-02    -0.000000000000-03
  0.282425209610-04    -0.744000064000-05    -0.000000000000-03
  0.215691939440-02    -0.744000000000-03    -0.000000000000-03
  0.357041324100-04    -0.234000000000-03    -0.000000000000-03
  0.445330000000-03    -0.000000000000-03    -0.000000000000-03
  0.284630583500-04    -0.100020050560-03    -0.334686835950-03

   4     0.400000000000 02
  0.288404256700-03    -0.000000000000-03    -0.134255016600-04
  0.131016922400-03    -0.000000000000-03    -0.000000915800-04
  0.467034130000-03    -0.000000000000-03    -0.102305190000-04
  0.450744930000-03    -0.000000000000-03    -0.274331221000-03
  0.290395422000-03    -0.000000000000-03    -0.254000000000-04
  0.143845980000-03    -0.000000000000-03    -0.207291074000-04
  0.461037757700-03    -0.000000000000-03    -0.000000000000-04
  0.597338200000-04    -0.164001100000-04    -0.191231506000-04
  0.164000000000-03    -0.000000000000-03    -0.000000000000-04
  0.295588434340-03    -0.418530727100-03    -0.353199400900-03

   5     0.472532555500 02
  0.145200000000-02    -0.000000000000-03    -0.192922177500-05
  0.528000000000-04    -0.000000000000-04    -0.700000000000-04
  0.795915645600-04    -0.000000000000-04    -0.186164291000-04
  0.264470000000-04    -0.000000000000-04    -0.352000000000-03
  0.143000000000-02    -0.000000000000-03    -0.000000000000-05
  0.126136000000-04    -0.000000000000-04    -0.000000000000-04
  0.849480000000-05    -0.000000000000-05    -0.000000000000-03
  0.440890000000-05    -0.000000000000-05    -0.000000000000-05
  0.534522000000-03    -0.000000000000-03    -0.000000000000-05
  0.316147273700-05    -0.247000733160-03    -0.315100844390-03

   6     0.628445792400 02
  0.238335460000-03    -0.000000000000-04    -0.277745159000-04
  0.364000000000-03    -0.000000000000-03    -0.000000000000-04
  0.579593432100-03    -0.000000000000-03    -0.000000000000-04
  0.234000000000-03    -0.000000000000-03    -0.000000000000-03
  0.556240000000-03    -0.000000000000-03    -0.000000000000-04
  0.563737600000-03    -0.000000000000-03    -0.000000000000-04
  0.598000000000-03    -0.000000000000-03    -0.000000000000-03
  0.535300000000-04    -0.000000000000-04    -0.000000000000-04
  0.237163300000-03    -0.000000000000-03    -0.223100414170000-03
  0.540925072200-03    -0.400000824730-03    -0.430919612200-05
```

```
 7     0.66016896570 02
 0.40813559220D-03    -0.11740290310D-02    -0.13560165640D-04
 0.54243577750D-03    -0.15610541060D-02    -0.18038556750D-03
 0.33871233450D-03    -0.97511115150D-03    -0.11263093850D-03
 0.33514470070D-03    -0.96504115640D-03    -0.11147722110D-03
 0.43295331500D-03    -0.12465146660D-02    -0.14402722030D-03
 0.24771004520D-03    -0.71309666030D-03    -0.82377027110D-04
 0.30689261910D-03    -0.88373145370D-03    -0.10214737220D-03
 0.62669885560D-04    -0.18042249500D-03    -0.20855191900D-04
 0.25467043270D-03    -0.73312310210D-03    -0.84737911200D-04
 0.29198257600-03     -0.84063020110D-03    -0.97041591120D-03

 8     0.75907768420 02
 0.78895141350D-03    -0.34559330430D-03    -0.55531103690D-04
 0.34459151420D-03    -0.35770137470D-03    -0.55672439330D-03
 0.21762200D-03       -0.62660D-03          -0.35131D-03
 0.44295613790D-04    -0.12749401050D-03    -0.14741D-04
 0.72078851130D-03    -0.20744531050D-03    -0.23970D-03
 0.57953724050D-03    -0.16679D-03          -0.19277D-03
 0.38461405560D-03    -0.11069D-03          -0.12795D-03
 0.16422946470D-03    -0.47260D-03          -0.54631D-04
 0.41041522070D-03    -0.11809D-03          -0.13648D-03
 0.25652386245D-03    -0.73824D-03          -0.12561D-03

 9     0.87664784640 02
 0.67062523090D-03    -0.14166403610D-03    -0.51026952460D-04
 0.59795477170D-03    -0.12630D-03          -0.45510D-03
 0.83455565470D-03    -0.17633D-03          -0.63522D-03
 0.10498222030D-03    -0.22100D-03          -0.79618D-04
 0.70638399190D-03    -0.14873D-03          -0.53588D-03
 0.19266756560D-03    -0.40574D-04          -0.14619D-03
 0.88709921310D-03    -0.18677D-03          -0.67302D-03
 0.11308194650D-03    -0.23810D-04          -0.85777D-04
 0.70694180530D-03    -0.14884D-03          -0.53619D-03
 0.78172150200D-04    -0.16457D-04          -0.59280D-04

10     0.96660130270 02
 0.57756547600D-03    -0.12161376130D-03    -0.43805575140D-04
 0.63513697390D-03    -0.13376D-03          -0.48191D-03
 0.32082211D-03       -0.67541D-04          -0.24331D-03
 0.10063673250D-03    -0.21189D-03          -0.76351D-04
 0.54632164540D-03    -0.11503D-03          -0.41456D-03
 0.13172983D-03       -0.27735D-04          -0.99911D-04
 0.30556615D-03       -0.64341D-04          -0.23181D-03
 0.14982900D-03       -0.31550D-04          -0.11365D-03
 0.43761125D-03       -0.92123D-04          -0.33189D-03
 0.85420D-03          -0.17987D-03          -0.65D-04

 T=    0.50000  SEC
 DISP=
  0.3649E-04 -0.6178E-05  0.1354E-06 -0.5714E-07 -0.4109E-06  0.7882E-06

 VECLO
  0.4355E-02 -0.7098E-03  0.1551E-04 -0.6550E-05 -0.4710E-04  0.9035E-04

 T=    1.00000  SEC
 DISP=
  0.8125E-04 -0.1302E-04  0.2807E-06 -0.1221E-06 -0.8782E-06  0.1685E-05

 VECLO
  0.4623E-03 -0.2050E-03  0.4443E-05 -0.1864E-05 -0.1342E-04  0.2574E-04

 T=    1.50000  SEC
 DISP=
  0.5092E-04 -0.8351E-05  0.1843E-06 -0.8530E-07 -0.4679E-06  0.8977E-06

 VECLO
 -0.3213E-02  0.5304E-03 -0.1160E-04  0.4946E-05  0.3552E-04 -0.6815E-04

 T=    2.00000  SEC
 DISP=
 -0.6872E-04  0.1263E-04 -0.2781E-06  0.1172E-06  0.8432E-06 -0.1613E-05

 VECLO
 -0.8399E-02  0.1513E-02 -0.3296E-04  0.1391E-04  0.1000E-03 -0.1912E-03

 T=    2.50000  SEC
 DISP=
 -0.2798E-03  0.4590E-04 -0.1015E-05  0.4279E-06  0.3077E-05 -0.5903E-05

 VECLO
 -0.1513E-01  0.2723E-02 -0.5945E-04  0.2508E-04  0.1804E-03 -0.3461E-03
```

```
 T=    3.00000   SEC
 DISP=
  =0.5844E-03   0.9310E-04 -0.2062E-05  0.8606E-06  0.6253E-05 -0.1200E-04

 VECLO
  =0.2344E-01   0.4172E-02 -0.9123E-04  0.3840E-04  0.2768E-03 -0.5311E-03

 T=    3.50000   SEC
 DISP=
  =0.9842E-03   0.1545E-03 -0.3425E-05  0.1445E-05  0.1039E-04 -0.1993E-04

 VECLO
  =0.3333E-01   0.5862E-02 -0.1294E-03  0.5415E-04  0.3894E-03 -0.7471E-03

 T=    4.00000   SEC
 DISP=
  =0.1481E-02   0.2304E-03 -0.5102E-05  0.2156E-05  0.1550E-04 -0.2974E-04

 VECLO
  =0.4481E-01   0.7793E-02 -0.1703E-03  0.7208E-04  0.5183E-03 -0.9944E-03

 T=    4.50000   SEC
 DISP=
  =0.2075E-02   0.3209E-03 -0.7110E-05  0.3003E-05  0.2160E-04 -0.4143E-04

 VECLO
  =0.5791E-01   0.9966E-02 -0.2187E-03  0.9227E-04  0.6636E-03 -0.1273E-02

 T=    5.00000   SEC
 DISP=
  =0.2768E-02   0.4261E-03 -0.9455E-05  0.3989E-05  0.2868E-04 -0.5503E-04

 VECLO
  =0.7260E-01   0.1238E-01 -0.2710E-03  0.1147E-03  0.8250E-03 -0.1583E-02

 **FORTRAN ** STOP
 MULTIJOB PRINT OF RAEOOL:QL122 .DSET99(S3243)/D  FOR USER RAEOOL, TASK L1
```

6.4 THE LINEAR ALGEBRA PACKAGE

In the structure engineering analysis by computer, linear matrix operations appear very often and perhaps are the most computing time consuming operations. A package for linear matrix operations was designed to speed up the programming efficiency. The package consists of the following subroutines, SET1, SET0, DEC1, DEC2,DEC3, FOR1, FOR2, FOR3, BAC1, BAC2, BAC3, MULT1, MULT2, MULT3, MULT0, MULT21, MULT22, and MULT23. The functions of these subroutines are summarized as follows:

SET1: sets up an unit matrix

SET0: sets up a zero matrix

DEC1: decomposes a symmetric matrix in full form

DEC2: decomposes a symmetric matrix stored in fixed band form

DEC3: decomposes a symmetric matrix stored in variable band form

FOR1: performs the forward substitution of a system of linear equations stored in full form.

FOR2: performs the forward substitution of a system of linear equations stored in fixed band form.

FOR3: performs the forward substitution of a system of linear equations stored in variable band form.

BAC1: performs the backward substitution of a system of linear equations stored in full form.

BAC2: performs the backward substituion of a system of linear equations stored in fixed band form.

BAC3: performs the backward substituion of a system of linear equations stored in variable band form.

MULT1: performs the matrix multiplication $\{D\} = [A]\{C\}$, where $[A]$ is a symmetric matrix in full form.

MULT2: performs the matrix multiplication $\{D\} = [A]\{C\}$, where A is a symmetric matrix in fixed band form.

MULT3: performs the matrix multiplication $\{D\} = [A]\{C\}$, where A is a symmetric matrix stored in variable band form.

MULT0: performs the matrix product $[A] = [B][C]$ or $[A] = [B]^T[C]$ with options to overwrite the result [A] onto [B] or [C].

MULT21: performs the matrix product $[A] = [X]^T[B][X]$, where [B] is a symmetric matrix stored in full form.

MULT22: performs the matrix product $[A] = [X]^T[B][X]$, where [B] is a symmetric matrix stored in fixed band form.

MULT23: performs the matrix product $[A] = [X]^T[B][X]$, where [B] is a symmetric matrix stored in variable band form.

The individual subroutines are described in the following sections.

6.4.1 SUBROUTINE SET1

SET1 sets up a matrix [A] of size (NA,N), such that the diagonal elements are unities and the off diagonal elements are zeros. The input parameters are:

NA: the first dimension declared for the matrix A.

N: the second dimension of A.

The output parameter from the subroutine is the unit matrix A.

6.4.2 SUBROUTINE SET0

SET0 sets up a matrix [A] of size (NA,N), such that all elements are equal to zeros. The input parameters are:

NA: the first dimension declared for the matrix A.

N: the second dimension of A.

The output from the subroutine is the zero matrix A.

6.4.3 SUBROUTINE DEC1

DEC1 decomposes a symmetric matrix [A] when it is stored in full form into a product of an upper triangle and a lower triangle, i.e. [A]=[L][U] by Gauss elimination without interchanges. The resulting upper triangle [U] is overwritten on [A]. The matrix [U] is stored in the same form as A. The reciprocals of the diagonal elements are stored instead of the ordinary diagonal elements. The input parameters are:

A: the original symmetric matrix of size (NA,N), where NA N. Only the upper triangular elements are used and the strict lower triangular elements are unchanged after the call.

NA: the first dimension declared for A.

N: the actual order of the matrix A.

The output parameter is the upper triangular matrix overwritten on A. The reciprocals of the diagonal elements are stored instead of the ordinary ones. This is to minimize the number of divisions.

6.4.4 SUBROUTINE DEC2

DEC2 decomposes a symmetric matrix [A] stored in fixed band form into a product of an upper and a lower triangular matrix according to [A]=[L][U] by Gauss elimination method with out interchanges. The resulting upper triangular matrix [U] is overwritten on [A] and the reciprocals of the diagonal elements are stored instead of the ordinary diagonal elements.

The elements are stored in such a way that the Ith diagonal element is A(I,1). The input parameters are:

A: the original real symmetric matrix stored in fixed banded form with the Ith diagonal element stored in A(I,1).

NA: the first dimension declared for A.

IBW: the bandwidth parameter, such that the overall bandwidth = 2*IBW-1.

N: the actual order of matrix A.

The output is the upper triangular matrix overwritten on A in the same form as A and the reciprocals of the diagonal elements are stored instead of the ordinary diagonal elements.

6.4.5 SUBROUTINE DEC3

DEC3 decomposes a symmetric matrix [A] stored in variable band form in to a product of an upper triangular matrix and a lower triangular matrix according to [A] = [L][U]by Gauss elimination method without interchanges. The resulting matrix [U] is overwrittn on [A] and the reciprocals of the diagonal elements are stored instead of the ordinary diagonal elements. The matrix [A] is stored in one dimensional form such that the Ith diagonal element is stored in A(IU(I)), where IU(I) is an addressing array of integer type.

The input parameters to the subroutine are:

A: the input one dimensional array of size IUN, such that the Ith diagonal element of the original matrix is stored in A(IU(I)).

IU: a one dimensional integer addressing array of size N, such that the Ith element of the main diagonal of the original matrix is stored in A(IU(I)).

IUN: = IU(N).

MAX: a one dimensional integer array of size N, such that MAX(1)=1,

and MAX(K)=K-IU(K) + IU(K-1) + 1.

N: the actual order of the original matrix.

The output from the subroutine is the matrix U overwritten on A such that the reciprocals of the diagonal elements are stored instead of the ordinary diagonal elements.

6.4.6 SUBROUTINE FOR1

FOR1 solves $[U]\{Z\} = \{B\}$, when the upper triangular matrix $[U]$ and $\{B\}$, the coefficient vector, are given. $[U]$ is of the form output from DEC1, such that U(I,I) contains the reciprocal diagonal element instead of the ordinary diagonal element. The input to the subroutine are:

U: the upper triangular matrix as result from DEC1.

NU: the first declared dimension of matrix U.

N: the actual order of the matrix U.

B: the coefficient vector.

The output from the subroutine is the solution vector Z . The vector Z can be overwritten on B .

6.4.7 SUBROUTINE BAC1

BAC1 solves $[L]\{X\} = \{Z\}$, when the upper triangular matrix $[U]$ and the coefficient vector $\{Z\}$ are given. The organization of $[U]$ should be the same as the output from DEC1. A combined use of DEC1, FOR1, BAC1 will solve a set a linear equations $[A]\{X\} = \{B\}$ in the following manner :

Decomposition: $[A] = [L][U]$

Forward substitution: $[U]\{Z\} = \{B\}$

Backward substitution: $[L]\{X\} = \{Z\}$

The input parameters are:

U: the upper triangular matrix as result from DEC1.

NU: the first declared dimension for U.

N: the actual order of the matrix U.

Z: the coefficient vector.

The output from the subroutine is the solution vector X. The vector X can be overwritten on Z.

6.4.8 SUBROUTINE FOR2

FOR2 solves $[U]\{Z\} = \{B\}$, when the upper triangular matrix [U] and the coefficient vector $\{B\}$ are given. [U] is of the form as output from DEC2, such that U(I,1) contains the reciprocals of the diagonal elements instead of the ordinary diagonal elements. The input parameters are:

U: the upper triangular matrix stored in band form as result from DEC2.

NU: the first dimension of the declared matrix U.

N: the actual order of the matrix U.

B: the coefficient vector.

The output is the solution vector Z. The vector Z can be overwritten on B. IBW is the bandwidth parameter of U.

6.4.9 SUBROUTINE BAC2

BAC2 solves $[L]\{X\} = \{Z\}$, when the upper triangular matrix [U] is given in fixed band form. The organization of [U] should be the same as the output from DEC2. The input parameters are:

U: the upper triangular matrix stored in band form as resulted from DEC2.

NU: the first dimension declared for U.

N: the actual order of U.

IBW: the bandwidth parameter of U.

Z: the coefficient vector Z.

The output from the subroutine is the solution vector X. The vector X can be overwritten on Z.

6.4.10 SUBROUTINE FOR3

FOR3 solves $[U]\{Z\} = \{B\}$, when the upper triangular matrix [U] is given in variable band form. [U] is of the form as output from DEC3, such that U(IU(I)) contains the reciprocal of the Ith diagonal element instead of the ordinary diagonal element. The input parameters are:

U: the upper triangular matrix stored in one dimensional form as output form DEC3.

IU: an integer addressing array as in DEC3.

IUN: = IU(N).

MAX: an integer array as in DEC3.

N: the actual order of U.

B: the coefficient vector.

The output from the subroutine is the solution vector Z. The vector Z can be overwritten on B.

6.4.11 SUBROUTINE BAC3

BAC3 solves $[L]\{X\} = \{Z\}$, when the upper triangular matrix [U] is given

in variable band form. [U] is of the from as output from DEC3.

The input parameters are:

U: the upper triangular matrix stored in one dimensional form as output from DEC3.

IU: an integer addressing array as in DEC3.

IUN: = IU(N).

MAX: an integer array as in DEC3.

N: the actual order of U.

Z: the coefficient vector.

The output is the solution vector X which can be overwritten on Z.

6.4.12 MULT1

MULT1 performs the matrix product $\{D\} = [A]\{C\}$

where [A] is a full matrix with symmetry.

Input parameters are:

A: the symmetric matrix of size (NA,N). The upper triangular elements are required only.

NA: the first dimension declared for the matrix A.

N: the actual order of A.

C: a vector of size N.

The output is D the product required. D can be overwritten on C.

6.4.13 SUBROUTINE MULT2

The function of MULT2 is the same as MULT1 except that the symmetric matrix[A]is stored in fixed band form with bandwidth parameter IBW.

Input parameters are:

A: the symmetric matrix stored in band form.

NA: the first declared dimension of A.

N: the actual order of A.

IBW: the bandwidth parameter of A.

C: a vector of constants.

The output from the subroutine is the product vector D. D can be overwritten on C.

6.4.14 SUBROUTINE MULT3

The function of MULT3 is the same as MULT1 except that the symmetric matrix is stored in a one dimensional array and takes the advantage of variable bandwidth. The organization of [A] is the same as DEC3. The input parameters are:

A: the symmetric matrix stored in one dimensional array as the input matrix of DEC3.

IU: an integer addressing array as in DEC3.

IUN: =IU(N).

MAX: an integer array as in DEC3.

N: the actual dimension of A.

C: the constant vector.

The output is the product vector D. D can be overwritten on C.

6.4.15 SUBROUTINE MULTO

MULTO performs the matrix multiplications

$$[A] = [B][C] \quad \text{or} \quad [A] = [B]^T[C]$$

with options to overwritten the results [A] onto [B] or [C].

The input parameters are:

B: a two dimensional matrix of size (NB,M) containing the matrix [B].

C: a two dimension matrix of size (NC,N) containing the elements of [C].

L: the first dimension of B

M: the second dimension of B

N: the second dimension of C

Z: working vector of size N.

OPT: an integer indicator, such that when

OPT=1 replaces [A] by [B][C]

OPT=2 replaces [B] by [B][C]

OPT=3 replaces [C] by [B][C]

OPT=4 replaces [A] by [B][C]

OPT=5 replaces [B] by [B][C]

OPT=6 replaces [C] by [B][C]

NA: the first dimension declared for A

NB: the first dimension declared for B

NC: the first dimension declared for C

The output is depending on the value of OPT. Note that the dimensions of the matrices must be able to match so that the product can be performed.

6.4.16 SUBROUTINE MULT21

MULT21 performs the matrix product $[A] = [X]^T[B][X]$

where B is a symmetric matrix in full form.

The input parameters are:

X: a two dimensional array of size(NX,N) containing the elements of X.

The input parameters are:

X: as in MULT21

B: as the matrix A in DEC3 containing the elements of [B].

NA: the first dimension declared for A.

NX: the first dimension declared for X.

N: the actual order of matrix [B].

IU: the addressing array for B as in DEC3.

IUN: =IU(N).

MAX: an integer array as in DEC3.

NP: the actual order of A.

B: a two dimensional matrix of size (NB,N) containing the symmetrix matrix B , the upper triangular elements are required only.

NA: the first dimension declared for A.

NX: the first dimension declared for X.

NB: the first dimension declared for B.

N: the actual order of B.

NP: the actual order of A.

The output parameter is A containing the product $[X]^T[B][X]$.

6.4.17 SUBROUTINE MULT 22

The function of MULT22 is the same as MULT21 except that the symmetric matrix [B] is in band form.

The input parameters are:

X: as in MULT21

B: as the matrix A in DEC2 containing the elements of [B].

NA: the first declared dimension of A

NX: the first declared dimension of X

NB: the first declared dimension of B

N: the actual dimension of B

IBW: the bandwidth parameter of B

NP: the actual order of A.

The output is the product $[X]^T[B][X]$ being stored in[A].

6.4.18 SUBROUTINE MULT23

The function of MULT23 is the same as MULT22 except that the matrix [B] is stored in a one dimensional array and is taking advantage of variable band form.

```
TRIALS RUN  457   DATE 04/12/75   TIME 1-:32:42                    FROM PROCEDUR
LOGIN USERNAME:  RAE03L        GROUPNAME:  RAE03L                      268

// FTRAN1 RAE03L:RAE03L.YTOO(F),K                                  SOURCE FILE :
// OPTION LIST,DEBUG
EXIT TO SYSTEM:TRPROG.FTRAN1(P.0229)  AT 1-:34:10

FTRN4 8.0S30   SOURCE PROGRAM  SET1      SUBROUTINE                 04/12/75   PAG
    1          SUBROUTINE SET1(A,NA,N)
    2          IMPLICIT REAL*8 (A-H,O-Z)
    3          DIMENSION A(NA,N)
    4          DO 1 I=1,NA
    5          DO 2 J=1,N
    6          A(I,J)=0.
    7          IF (I.EQ.J) A(I,I)=1.
    8        2 CONTINUE
    9        1 CONTINUE
   10          RETURN
   11          END
   12  C
   13  C

FTRN4 8.0S30   SOURCE PROGRAM  SET0      SUBROUTINE                 04/12/75   PAG
   14          SUBROUTINE SET0(A,NA,N)
   15          IMPLICIT REAL*8 (A-H,O-Z)
   16          DIMENSION A(NA,N)
   17          DO 1 I=1,NA
   18          DO 2 J=1,N
   19        2 A(I,J)=0.
   20        1 CONTINUE
   21          RETURN
   22          END
   23  C
   24  C

FTRN4 8.0S30   SOURCE PROGRAM  FOR1      SUBROUTINE                 04/12/75   PAG
   25          SUBROUTINE FOR1(U,NU,N,B,Z)
   26          IMPLICIT REAL*8 (A-H,O-Z)
   27          DIMENSION U(NU,N),B(N),Z(N)
   28          Z(1)=B(1)
   29          DO 1 I=2,N
   30          R=B(I)
   31          I1=I-1
   32          DO 2 K=1,I1
   33        2 R=R-U(K,I)*Z(K)*U(K,K)
   34        1 Z(I)=R
   35          RETURN
   36          END
   37  C
   38  C

FTRN4 8.0S30   SOURCE PROGRAM  BAC1      SUBROUTINE                 04/12/75   PAG
   39          SUBROUTINE BAC1(U,NU,N,Z,X)
   40          IMPLICIT REAL*8 (A-H,O-Z)
   41          DIMENSION U(NU,N),Z(N),X(N)
   42          X(N)=Z(N)/U(N,N)
   43          NM1=N-1
   44          DO 1 I1=1,NM1
   45          I=N-I1
   46          R=Z(I)
   47          IP1=I+1
   48          DO 2 K=IP1,N
   49        2 R=R-U(I,K)*X(K)
   50        1 X(I)=R*U(I,I)
   51          RETURN
   52          END
   53  C
```

```
FTRN4 80OS3   SOURCE PROGRAM  FOR2      SUBROUTINE                    04/12/75   PAG

      SUBROUTINE FOR2(U,NU,IBW,N,B,Z)                                269
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION U(NU,IBW),B(N),Z(N)
      Z(1)=B(1)
      IB=IBW-1
      DO 1 I=2,N
      R=B(I)
      I1=I-1
      IT=MAX0(1,I-IB)
      DO 2 K=IT,I1
      M=I-K+1
    2 R=R-U(K,M)*Z(K)*U(K,1)
    1 Z(I)=R
      RETURN
      END
C
C

FTRN4 80OS3   SOURCE PROGRAM  BAC2      SUBROUTINE                    04/12/75   PAG

      SUBROUTINE BAC2(U,NU,IBW,N,Z,X)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION U(NU,IBW),Z(N),X(N)
      X(1)=Z(1)*U(1,1)
      NM1=N-1
      IB=IBW-1
      DO 1 I1=1,NM1
      IMN=I1
      R=Z(I)
      IP1=I+1
      I2=MIN0(N,I+IB)
      DO 2 K=IP1,I2
      M=K-I+1
    2 R=R-U(I,M)*X(K)
    1 X(I)=R*U(I,1)
      RETURN
      END
C
C

FTRN4 80OS3O  SOURCE PROGRAM  FOR3      SUBROUTINE                    04/12/75   PAG

      SUBROUTINE FOR3(U,IU,IUN,MAX,N,B,Z)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION U(IUN),IU(N),MAX(N),B(N),Z(N)
      Z(1)=B(1)
      DO 1 I=2,N
      R=B(I)
      I1=MAX(I)
      I2=I-1
      IF (I2.LT.I1) GOTO 1
      M=IU(I)-I
      DO 2 K=I1,I2
      M=M+1+K
      L=IU(M)
    2 R=R-U(M)*Z(K)*U(L)
    1 Z(I)=R
      RETURN
      END
C
C

FTRN4 80OS3   SOURCE PROGRAM  BAC3      SUBROUTINE                    04/12/75   PAG

      SUBROUTINE BAC3(U,IU,IUN,MAX,N,Z,X)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION U(IUN),IU(N),MAX(N),Z(N),X(N)
      M=IU(1)
      X(1)=Z(1)*U(M)
      NM1=N-1
      DO 1 I1=1,NM1
      I=I1+1
      R=Z(I)
      IP1=I+1
      DO 2 K=IP1,N
      IF (MAX(K).GT.I) GOTO 2
```

```
      1=I(N)=N+1
      R=R-U(I)*X(K)
    2 CONTINUE
      R=IU(I)
    1 X(I)=R*U(N)
      RETURN
      END
C                                                                    270
C
```

FTRN4 80OS30 SOURCE PROGRAM DEC1 SUBROUTINE 04/12/75 PAG

```
      SUBROUTINE DEC1(A,NA,N)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(NA,N)
      N1=N-1
      DO 3 I=1,N1
      S=1./A(I,I)
      A(I,I)=S
      I1=I+1
      DO 1 J=I1,N
      R=A(I,J)*S
      DO 2 K=J,N
    2 A(J,K)=A(J,K)-R*A(I,K)
    1 CONTINUE
    3 CONTINUE
      RETURN
      END
C
C
```

FTRN4 80OS30 SOURCE PROGRAM DEC2 SUBROUTINE 04/12/75 PAG

```
      SUBROUTINE DEC2(A,NA,IBW,N)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(NA,IBW)
      N1=N-1
      IB=IBW-1
      DO 3 I=1,N1
      S=1./A(I,1)
      A(I,1)=S
      I1=I+1
      I2=MINO(N,I+IB)
      DO 1 J=I1,I2
      M=J-I+1
      R=A(I,M)*S
      DO 2 K=J,I2
      M1=K-J+M
      M2=K-I+1
    2 A(J,M1)=A(J,M1)-R*A(I,M2)
    1 CONTINUE
    3 CONTINUE
      RETURN
      END
C
C
```

FTRN4 80OS30 SOURCE PROGRAM DEC3 SUBROUTINE 04/12/75 PAG

```
      SUBROUTINE DEC3(A,NA,IUP,MAX,N)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(IUP),IU(N),MAX(N)
      N1=N-1
      DO 3 I=1,N1
      M=IU(I)
      S=1./A(M)
      A(M)=S
      I1=I+1
      DO 1 J=I1,N
      IF (MAX(J).GT.I) GOTO 1
      M=IU(J)-J+I
      R=A(M)*S
      DO 2 K=J,N
      IF (MAX(K).GT.I) GOTO 2
      M1=IU(K)-K+J
      M=M1-J+I
      A(M)=A(M)-R*A(L)
    2 CONTINUE
    1 CONTINUE
    3 CONTINUE
      RETURN
      END
C
```

```
FTRN4 800S30  SOURCE PROGRAM  MULT1    SUBROUTINE                    04/12/75  PAG
                                                                      271
      SUBROUTINE MULT1 (A,NA,N,C,D)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(NA,N),C(N),D(N)
      DO 1 I=1,N
      R=0.
      DO 2 J=1,N
      A=A(I,J)
      C=C(J)
    2 R=R+A(I,K)*C(J)
    1 D(I)=R
      RETURN
      END
C
C

FTRN4 800S30  SOURCE PROGRAM  MULT2    SUBROUTINE                    04/12/75  PAG
      SUBROUTINE MULT2(A,NA,IBW,N,C,D)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(NA,IBW),C(N),D(N)
      IB=IBW-1
      DO 1 I=1,N
      R=0.
      I1=MAXO(1,I-IB)
      I2=MINO(IB+I,N)
      DO 2 J=I1,I2
      K=MAX(I,J)
      L=MIN0(I,J)
      A=A(K,L+1
    2 R=R+A(L,M)*C(J)
    1 D(I)=R
      RETURN
      END
C
C

FTRN4 800S30  SOURCE PROGRAM  MULT3    SUBROUTINE                    04/12/75  PAG
      SUBROUTINE MULT3(A,IU,IUN,MAX,N,C,D)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(IUN),C(N),D(N),IU(N),MAX(N)
      DO 1 I=1,N
      R=0.
      I1=MAX(I)
      DO 2 J=I1,N
      IF (J.GT.I.AND.MAX(J).GT.I) GOTO 2
      K=MAXO(I,J)
      L=MIN0(I,J)
    3 M=IU(K)-K+L
      R=R+A(M)*C(J)
    2 CONTINUE
    1 D(I)=R
      RETURN
      END
C
C

FTRN4 800S30  SOURCE PROGRAM  MULTU    SUBROUTINE                    04/12/75  PAG
      SUBROUTINE MULTU(A,B,C,L,M,N,Z,OPT,NA,NB,NC)
      IMPLICIT REAL*8 (A-H,O-Z)
      INTEGER P,Q,OPT
      DIMENSION A(NA,N),B(NB,N),C(NC,M),Z(N)
      GOTO (10,20,30,40,50),OPT
   10 DO 11 I=1,L
      DO 13 J=1,M
      R=0.
      DO 12 K=1,
   12 R=R+A(I,K)*C(K,J)
   13 A(I,J)=R
   11 CONTINUE
      GOTO 99
   20 DO 21 I=1,L
      DO 22 J=1,N
```

```
      DO 23 K=1,
   23 =+B(I,K)*C(K,J)
   22 Z( )=
      DO 24 J=1,
   24 (I,J)=Z(J)
   21 CONTINUE
      GOTO 99
   30 DO 41 I=1,L
      DO 33 J=I,N
      =0.
      DO 32 K=1,
   32 R=R+B(K,I)*C(K,J)
   33 (I, )=R
   31 CONTINUE
      GOTO 99
   40 DO 41 I=1,L
      DO 42 J=1,
      =0.
      DO 43 K=1,
   43 =+B(K,I)*C(K,J)
   42 Z(J)=R
      DO 44 J=1,N
   44 A(I,J)=Z(J)
   41 CONTINUE
   99 RETURN
      END
C
C
```

```
FTRN4 8.0S30   SOURCE PROGRAM  MULT21   SUBROUTINE          04/12/75   PAG

      SUBROUTINE MULT21 (A,X,B,NA,NX,NB,N,NP)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(NA,NP),X(NX,NP),B(NB,N)
      INTEGER P,Q
C           B IS A SYMMETRICAL MATRIX
      DO 1 I=1,NP
      DO 3 J=I,NP
      R=0.
      DO 2 K=1,N
      DO 4 L=1,N
      P=MAXO(K,L)
      Q=MINO(K,L)
    4 R=R+B(Q,P)*X(K,I)*X(L,J)
    2 CONTINUE
      IF (I.NE.J) A(J,I)=R
    3 A(I,J)=R
    1 CONTINUE
      RETURN
      END
C
C
```

```
FTRN4 8.0S30   SOURCE PROGRAM  MULT22   SUBROUTINE          04/12/75   PAG

      SUBROUTINE MULT22 (A,X,B,NA,NX,NB,N,IBW,NP)
      IMPLICIT REAL*8 (A-H,O-Z)
      INTEGER P,Q
      DIMENSION A(NA,NP),X(NX,NP),B(NB,IBW)
      IB=IBW-1
      DO 1 I=1,NP
      DO 3 J=I,NP
      R=0.
      DO 2 K=1,N
      L1=MAXO(1,K-IB)
      L2=MINO(N,K+IB)
      DO 4 L=L1,L2
      P=MAXO(K,L)
      Q=P+1-K
    4 R=R+B(Q,K)*X(K,I)*X(L,J)
    2 CONTINUE
      IF (I.NE.J) A(J,I)=R
    3 A(I,J)=R
    1 CONTINUE
      RETURN
      END
C
C
```

```
FTRN4 8.0S30   SOURCE PROGRAM  MULT23   SUBROUTINE          04/12/75   PAG
```

```
      SUBROUTINE SOLVE2(A,X,B,NX,NP,IU,IJ,IU1,MAX,MP)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(NX,NP),X(NX,NP),B(10N),IU(N),MAX(N)
      INTEGER P,Q
      DO 1 I=1,NP
      DO 3 J=1,NP                                                    273
      P=0
      DO 2 K=1,N
      L1=MAX(K)
      DO 4 L=L1,
      IF (L.GT.K.AND.MAX(L).GT.K) GOTO 4
      P=MAX0(K,L)
      Q=IU(K,L)
      IU(P)=P+0
      R=R+B( )*X(K,I)*X(L,J)
    4 CONTINUE
    2 CONTINUE
      IF (I.NE.J) A(J,I)=R
    3 A(I,J)=R
    1 CONTINUE
      RETURN
      END
C
C
```

```
FTRN4 8-0S3-  OBJECT SUMMARY  SET1      SUBROUTINE                    04/12/75    PAG

      OBJECT DECK ORIGIN  FDFE30
      INSTRUCTIONS                          00108
      ARRAYS                                00000
      FORMATS, NAMELISTS, LITCONS           00000
      VARIABLES, CONSTANTS, TEMPORARIES     00138
      SET1        SD  01  00308       ITF#MPI   ER  02                  ITF#BUG
```

```
FTRN4 8-0S3-  OBJECT SUMMARY  SET0      SUBROUTINE                    04/12/75    PAG

      OBJECT DECK ORIGIN  FDFE68
      INSTRUCTIONS                          00198
      ARRAYS                                00000
      FORMATS, NAMELISTS, LITCONS           00000
      VARIABLES, CONSTANTS, TEMPORARIES     00120
      SET0        SD  01  00288       ITF#MPI   ER  02                  ITF#BUG
```

```
FTRN4 8-0S3-  OBJECT SUMMARY  FOR1      SUBROUTINE                    04/12/75    PAG

      OBJECT DECK ORIGIN  FDFF08
      INSTRUCTIONS                          00238
      ARRAYS                                00000
      FORMATS, NAMELISTS, LITCONS           00003
      VARIABLES, CONSTANTS, TEMPORARIES     00164
      FOR1        SD  01  00390       ITF#MPI   ER  02                  ITF#BUG
```

```
FTRN4 8-0S3-  OBJECT SUMMARY  SAC1      SUBROUTINE                    04/12/75    PAG

      OBJECT DECK ORIGIN  FDFF98
      INSTRUCTIONS                          00268
      ARRAYS                                00000
      FORMATS, NAMELISTS, LITCONS           00004
      VARIABLES, CONSTANTS, TEMPORARIES     00148
      SAC1        SD  01  00308       ITF#MPI   ER  02                  ITF#BUG
```

```
FTRN4 8-0S3-  OBJECT SUMMARY  FOR2      SUBROUTINE                    04/12/75    PAG

      OBJECT DECK ORIGIN  FDFF78
      INSTRUCTIONS                          00288
      ARRAYS                                00000
      FORMATS, NAMELISTS, LITCONS           00004
      VARIABLES, CONSTANTS, TEMPORARIES     00178
      FOR2        SD  01  00408       ITF#MPI   ER  02                  ITF#BUG
      ITF#17      EV  04
```

```
FTRN4 8-0S3-  OBJECT SUMMARY  SAC2      SUBROUTINE                    04/12/75    PAG

      OBJECT DECK ORIGIN  FDFF48
      INSTRUCTIONS                          00288
      ARRAYS                                00000
```

6.5 MISCELLANEOUS PROGRAMMES

6.5.1 SUBROUTINE GBMS

GBMS calculates the integrals $\int_0^1 x^{\gamma} \xi_m(x) \phi_k(x) dx$ where γ is a positve integer, $\xi_m(x)$ are polynomials of order m, $\phi_k(x)$ is the kth mode of beam function with two ends clamped.

Input parameters:

GM an integer constant equals to γ

A the kth frequency parameter of a clamped beam.

EQ a two dimensional array such that EQ(M,I) contains the Ith coefficient of the Mth polynomial in accending orders.

The output parameters are the integrals $\int_0^1 x^{\gamma} \xi_m(x) \phi_k(x) dx$ and are shown in Fig(3.5.5).

6.5.2 PROGRAMME YT15

This programme calculates the natural frequencies of a clamped-clamped beam by Newton's method for eigenvalue problems as described in section (5.7). Finite element method is employed. Two subroutines DECSYM and DECS1 are used in this programme. These subroutines are described as follows:

SUBROUTINE DECSYM

This decomposes the matrix A into a product of lower and a upper triangular matrices stored in LU for the use of Newton's method for eigenvalue problems as described in section (5.7).

Input parameters:

A(NA,N) a positive definite matrix

NA the first dimension declared for A

N the order of A.

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION GS(4,8),X(5)
      INTEGER GN,K
      DATA X/4.730041,7.853205,10.995608,14.137165,17.278760/
      K=0
  101 K=K+1
      A=X(K)
      WRITE (6,901) K
  901 FORMAT ('K=',I2)
      CALL GBMS(7,A,GS,11)
      IF (K.LT.5) GOTO 101
      DO 102 K=6,50
      WRITE (6,901) K
      A=(DFLOAT(K)+0.5)*3.14159265359
      CALL GBMS(7,A,GS,11)
  102 CONTINUE
      STOP
      END
```

FTRN4 800530 SOURCE PROGRAM GBMS SUBROUTINE 26/09/75 PAG

```
      SUBROUTINE GBMS(GN,A,GS,N4)
      INTEGER GN,N4,G
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION GS(4,8),EP(4,20),U(4),V(20)
      B=A
      EA=DEXP(A)/A
      EB=DEXP(B)/B
      CB=DCOS(A)/A
      SB=DSIN(A)/A
      EP(3,1)=EA-1/A
      EP(2,1)=1/A-CB
      EP(4,1)=EB-1/B
      EP(1,1)=SB
      I=GN+3
      DO 101 G=1,I
      EP(3,G+1)=EA-G*EP(3,G)/A
      EP(4,G+1)=EB-G*EP(4,G)/B
      EP(2,G+1)=-CB+G*EP(1,G)/A
  101 EP(1,G+1)=SB-G*EP(2,G)/A
      B=EA*A
      CB=CB*A
      SB=SB*A
      S=(.5*(B+1./B)-CB)/(.5*(B-1./B)-SB)
      U(1)=-1.
      U(2)=S
      U(3)=(-1./B+CB-SB)/(.5*(B-1./B)-SB)*0.5
      U(4)=0.5*(1+S)
      DO 104 G=1,N4
      R=0.
      DO 105 I=1,4
  105 R=R+U(I)*EP(I,G)
  104 V(G)=R
      EP(1,1)=1.
      EP(1,2)=0.
      EP(1,3)=-3.
      EP(1,4)=2.
      EP(2,1)=0.
      EP(2,2)=1.
      EP(2,3)=-2.
      EP(2,4)=1.
      EP(3,1)=0.
      EP(3,2)=0.
      EP(3,3)=3.
      EP(3,4)=-2.
      EP(4,1)=0.
      EP(4,2)=0.
      EP(4,3)=-1.
      EP(4,4)=1.
      DO 102 I=1,4
      I1=GN+1
      DO 102 J=1,I1
      R=0.
      DO 103 G=1,4
      J1=G+J-1
  103 R=R+V(J1)*EP(I,G)
      GS(I,J)=R
  102 CONTINUE
      WRITE (6,902)((GS(I,J),J=1,I1),I=1,4)
  902 FORMAT(8F10.7)
      RETURN
      END
```

FTRN4 800530 OBJECT SUMMARY 26/09/75 PAG

OBJECT DECK ORIGIN FDFD30

Output parameters:

LU(NA,N) a matrix storing the decomposed matrices for the use of DECS1

A(NA,N) unchanged.

SUBROUTINE DECS1

This calculates the first derivatives of the upper and lower triangular matrices LU when the first derivative of matrix A is given for the use of Newton's method in solving eigenvalue problems as mentioned in section (5.7).

Input parameters:

DA(NA,N) a matrix derivative of A

N the order of DA

NA the first declared dimension of DA

LU the output from DECSYM

Output parameters:

LU1 a matrix storing the first derivatives of LU.

D the derivative of the determinant of A.

```
FTRN4 8B0S30  SOURCE PROGRAM                                    13/01/7
      1       REAL ED1(20,20),EM1(20,20),D(20,20),E(4,2),F(4,2),
      2      1U(20,20),U1(20,20),V(3),ED(20,20),EM(20,20)
      3       DATA V/4.,7.,9.,11./
      4       DATA ED/400*0./
      5       DATA EM/400*0./
      6       DATA ED1/400*0./
      7       DATA EM1/400*0./
      8       DATA E/24.,0.,-12.,-6.,18.,-6.,2.,0./
      9       DATA F/312.,0.,54.,-13.,8.,13.,-3.,0./
     10       NA=20
     11       DO 1 I=1,4
     12       DO 1 J=1,2
     13       F(I,J)=F(I,J)/420.
     14     1 CONTINUE
     15       DO 2 I=1,10
     16       I1=I*2-2
     17       DO 2 J=1,4
     18       J1=I1+J
     19       ED1(I1+1,J1)=E(J,1)
     20       ED1(I1+2,J1+1)=E(J,2)
     21       EM1(I1+1,J1)=F(J,1)
     22       EM1(I1+2,J1+1)=F(J,2)
     23     2 CONTINUE
     24       DO 3 N=3,11
     25    60 FORMAT('NUMBER OF ELEMENTS=',I5)
     26       WRITE(6,60) N
     27       H=1./FLOAT(N)
     28       R=H*H*H
     29       N1=(N-1)*2
     30       DO 4 I=1,N1
     31       DO 4 J=I,N1
     32       ED(I,J)=ED1(I,J)/R
     33       ED(J,I)=ED1(I,J)/R
     34       EM(I,J)=EM1(I,J)*H
     35       EM(J,I)=EM1(I,J)*H
     36     4 CONTINUE
     37       DO 5 I=2,N1,2
     38       DO 5 J=1,N1
     39       EM(I,J)=EM(I,J)*H
     40       EM(J,I)=EM(J,I)*H
     41       ED(I,J)=ED(I,J)*H
     42       ED(J,I)=ED(J,I)*H
     43     5 CONTINUE
     44       DO 108 IV=1,3
     45       W=V(IV)*V(IV)
     46       K=0
     47   102 W2=W*W
     48       DO 103 I=1,N1
     49       DO 103 J=1,N1
     50   103 D(I,J)=ED(I,J)-W2*EM(I,J)
     51       CALL DECSYM(D,U,N1,NA)                                  !
     52       DO 104 I=1,N1
     53       DO 104 J=1,N1
     54   104 D(I,J)=-2*W*ED(I,J)
     55       CALL DECSI(N1,NA,D,U,U1,R)
     56       S=1.0
     57       DO 105 I=1,N1
     58   105 S=S*U(I,I)
     59       W=W-S/R
     60       K=K+1
     61       IF (W.LT.0) GOTO 3
     62       T=SQRT(W)
     63       WRITE(6,601) W,T,S,R
     64   601 FORMAT (4E20.6)
     65       IF (K.LE.10.AND.ABS(S/R/W).GT.1.E-3) GOTO 102
     66   108 CONTINUE
     67     3 CONTINUE
     68       STOP
     69       END
```

```
FTRN4 800S30  SOURCE PROGRAM  DECSYM  SUBROUTINE                    13/01/7

      70          SUBROUTINE DECSYM(A,LU,N,NA)
      71          REAL A(NA,N),LU(NA,N)
      72          S=0.
      73          DO 101 I=1,N
      74          R=0.
      75          DO 102 J=1,N
      76    102   R=R+ABS(A(I,J))
      77          IF (S.LT.R) S=R
      78    101   CONTINUE
      79          S=S*1.E-7
      80          DO 103 I=1,N
      81          DO 103 J=I,N
      82          R=A(I,J)
      83          I1=I-1
      84          IF (I1.LT.1) GOTO 105
      85          DO 104 K=1,I1
      86    104   R=R-LU(I,K)*LU(K,J)
      87    105   LU(I,J)=R
      88          IF (I.EQ.J.AND.S.GT.ABS(R)) LU(I,J)=S
      89          IF (I.NE.J) LU(J,I)=LU(I,J)/LU(I,I)
      90    103   CONTINUE
      91          RETURN
      92          END
```

```
FTRN4 800S30  SOURCE PROGRAM  DECS1  SUBROUTINE                     13/01/7

      93          SUBROUTINE DECS1(N,NA,DA,LU,LU1,D)
      94          REAL DA(NA,N),LU(NA,N),LU1(NA,N),D
      95          DO 101 I=1,N
      96          DO 101 J=I,N
      97          R=DA(I,J)
      98          I1=I-1
      99          IF (I1.LT.1) GOTO 105
     100          DO 102 K=1,I1
     101    102   R=R-LU(I,K)*LU1(K,J)-LU1(I,K)*LU(K,J)
     102    105   LU1(I,J)=R
     103          IF (I.NE.J) LU1(J,I)=(R-LU(J,I)*LU1(I,I))/LU(I,I)
     104    101   CONTINUE
     105          D=0.
     106          DO 103 I=1,N
     107          R=LU1(I,I)
     108          DO 106 J=I,N
     109          IF (I.NE.J) R=R*LU(J,J)
     110    106   CONTINUE
     111          D=D+R
     112    103   CONTINUE
     113          RETURN
     114          END
```

```
FTRN4 800S30  OBJECT SUMMARY                                        13/01/7

          OBJECT DECK ORIGIN  FDCC38
          INSTRUCTIONS                                  00790
          ARRAYS                                        02C10
          FORMATS, NAMELISTS, LITCONS                   00028
          VARIABLES, CONSTANTS, TEMPORARIES             00224
                    SD  01  035EC          ITF#UPI     ER  02
          DECSYM    EV  04                 DECS1       EV  05
          ITF#OI    EV  07                 ITF#OH      EV  08
          ITF#ST    EV  0A                 ITF#30      EV  0B
```

```
FTRN4 800S30  OBJECT SUMMARY  DECSYM  SUBROUTINE                    13/01/7

          OBJECT DECK ORIGIN  FDF8F8
          INSTRUCTIONS                                  00408
          ARRAYS                                        00000
          FORMATS, NAMELISTS, LITCONS                   00000
          VARIABLES, CONSTANTS, TEMPORARIES             00174
          DECSYM      SD  01  0057C          ITF#UPI     ER  02
```

```
FTRN4 800S30  OBJECT SUMMARY  DECS1  SUBROUTINE                     13/01/7

          OBJECT DECK ORIGIN  FDFC10
```

```
RAN IV PROGRAM          (COMPOSED AS @YTL17PROG  ) STARTED 13/01/76

ER OF ELEMENTS=     3
      0.224653E 02          0.473031E 01          -0.215701E 07          -0.573937E 07
      0.224648E 02          0.473071E 01          -0.563669E 04          -0.576942E 07
      0.629131E 02          0.793178E 01          -0.627835E 07           0.124784E 08
      0.629041E 02          0.793121E 01           0.116139E 00           0.129406E 08
ER OF ELEMENTS=     4
      0.224436E 02          0.473324E 01          -0.171930E 12          -0.548295E 12
      0.224030E 02          0.473310E 01          -0.294960E 09          -0.550050E 12
      0.622442E 02          0.788950E 01           0.179363E 12           0.106161E 13
      0.622433E 02          0.788944E 01           0.894198E 09           0.107084E 13
      0.123724E 03          0.111231E 02          -0.147052E 14          -0.539896E 13
      0.123487E 03          0.111125E 02          -0.154719E 13          -0.654890E 13
      0.123406E 03          0.111124E 02          -0.120872E 11          -0.644590E 13
ER OF ELEMENTS=     5
      0.223861E 02          0.473139E 01          -0.429914E 17          -0.145204E 18
      0.223857E 02          0.473125E 01          -0.552293E 14          -0.145532E 18
      0.622512E 02          0.788994E 01           0.132260E 18           0.833102E 18
      0.619222E 02          0.786907E 01           0.900668E 17           0.276195E 18
      0.619191E 02          0.786887E 01           0.841417E 15           0.271227E 18
      0.122657E 03          0.110750E 02          -0.207280E 19          -0.125119E 19
      0.122577E 03          0.110714E 02          -0.110671E 18          -0.138534E 19
ER OF ELEMENTS=     6
      0.223789E 02          0.473063E 01           0.249375E 23          -0.863198E 23
      0.223794E 02          0.473069E 01           0.438795E 20          -0.865308E 23
      0.618039E 02          0.783155E 01           0.989613E 23           0.163291E 24
      0.617937E 02          0.783080E 01           0.162000E 22           0.157917E 24
      0.121787E 03          0.110357E 02          -0.565731E 24          -0.718401E 24
      0.121770E 03          0.110349E 02          -0.133333E 23          -0.752216E 24
ER OF ELEMENTS=     7
      0.223772E 02          0.473046E 01           0.288382E 29          -0.100403E 30
      0.223779E 02          0.473053E 01           0.714220E 26          -0.100640E 30
      0.617511E 02          0.785811E 01           0.124046E 30           0.188274E 30
      0.617390E 02          0.785742E 01           0.219864E 28           0.181663E 30
      0.121309E 03          0.110177E 02           0.317472E 30          -0.816097E 30
      0.121393E 03          0.110178E 02           0.378037E 28          -0.984792E 30
ER OF ELEMENTS=     8
      0.223765E 02          0.473038E 01          -0.594073E 35          -0.207338E 36
      0.223754E 02          0.473026E 01          -0.239408E 33          -0.207766E 36
      0.617249E 02          0.785652E 01           0.264963E 36           0.386747E 36
      0.617118E 02          0.785569E 01           0.485261E 34           0.372745E 36
      0.121191E 03          0.110087E 02           0.316694E 36          -0.165966E 37
      0.121192E 03          0.110087E 02           0.176014E 34          -0.167740E 37
ER OF ELEMENTS=     9
      0.223771E 02          0.473044E 01           0.203010E 42          -0.707118E 42
      0.223782E 02          0.473056E 01           0.775327E 39          -0.708852E 42
      0.617114E 02          0.785508E 01           0.916152E 42           0.131439E 43
      0.616982E 02          0.785402E 01           0.168006E 41           0.126616E 43
      0.121025E 03          0.110038E 02          -0.473555E 42          -0.559924E 43
ER OF ELEMENTS=    10
      0.223790E 02          0.473032E 01           0.107679E 49          -0.376513E 49
      0.223773E 02          0.473046E 01           0.489846E 46          -0.377340E 49
      0.617036E 02          0.785510E 01           0.493255E 49           0.698493E 49
      0.616903E 02          0.785432E 01           0.910162E 47           0.673161E 49
      0.121023E 03          0.110010E 02           0.667197E 48          -0.296029E 50
ER OF ELEMENTS=    11
      0.223752E 02          0.473024E 01          -0.853704E 55          -0.292312E 56
      0.223767E 02          0.473040E 01           0.451083E 53          -0.300033E 56
      0.616987E 02          0.785485E 01           0.394207E 56           0.554244E 56
      0.616851E 02          0.785398E 01          -0.726237E 54           0.533666E 56
      0.120986E 03          0.109994E 02          -0.321144E 55          -0.233983E 57

RTRAN ** STOP

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```

6.5.5 GENERATING THE COEFFICIENTS OF BEAM FUNCTIONS

Once the natural frequencies and the corresponding modes in terms of nodal displacements are determined by the programme of section(6.2), it is necessary to know the internal displacements of individual beam members between nodes, if the forcing functions in response analysis happen to be on the internal region of a member. It is because the pricipal generalized forces P_i are calculated according to, c.f. chapter two,

$$P_i(t) = \int_{vol} \{x\}^T \{\phi_i\} dvol + \int_S \{\Phi\}^T \{\phi_i\} dS$$

where $\{x\}$ is the body force vector and $\{\Phi\}$ the surface force vector. For a general beam member, the internal modal displacement vector $\{\phi_i(x)\}$ consists of six function components of spactial coordinates. These are

(i) axial displacement function $\phi_{i1}(x) = A_1 \cos\psi\xi + A_2 \sin\psi\xi$

(ii) flexural displacement functions,

$$\phi_{i2}(\xi) = A_3 \cos\lambda_z\xi + A_4 \sin\lambda_z\xi + A_5 \cosh\lambda_z\xi + A_6 \sinh\lambda_z\xi$$
$$\phi_{i3}(\xi) = A_7 \cos\lambda_y\xi + A_8 \sin\lambda_y\xi + A_9 \cosh\lambda_y\xi + A_{10} \sinh\lambda_y\xi$$

(iii) torsional displacement function

$$\phi_{i4}(\xi) = A_{11} \cos\gamma\xi + A_{12} \sin\gamma\xi$$

(iv) flexural rotation or angular displacement functions

$$\phi'_{i5}(\xi) = A_{13} \cos\lambda_y\xi + A_{14} \sin\lambda_y\xi + A_{15} \cosh\lambda_y\xi + A_{16} \sinh\lambda_y\xi$$
$$\phi'_{i6}(\xi) = A_{17} \cos\lambda_z\xi + A_{18} \sin\lambda_z\xi + A_{19} \cosh\lambda_z\xi + A_{20} \sinh\lambda_z\xi$$

The constants A_i, i=1,2,...,20 are expressed in terms of the nodal displacements q_i, i=1,2,...,12 in section(3.6). The following two subroutines BMDFFI and BMDFTN are designed to determine A_i's.

A third subroutine BMDFTN was designed to give the deflections.

SUBROUTINE BMDFFI

The subroutine calculates the beam function coefficients A_i's when supplying the vibration frequency, the elastic properties of the beam, and the nodal displacements. The subroutine uses BMDFCN to compute the frequency functions.

Input parameters:

W frequency of vibration

PG(7) a one dimensional array containing the properties of the beam member as described in section (6.2).

Q(12) a one dimensional array containing the nodal displacements.

Output parameters:

A(20) the beam function coefficients.

SUBROUTINE BMDFCN

The subroutine calculates the frequency functions to be used in BMDFFI.

Input parameters:

W frequency of vibration

MS total mass of the beam member

H length of the beam

RZ moment of inertia of cross sectional area about the Z axis

TI total polar moment of inertia of cross sectional area

EA axial rigidity

U1,U7,U2,U6,U8,U12 nodal displacements

A1(2) coefficient of beam function of axial displacement

A2(4) coefficient of beam function of flexural displacement

A3(4) coefficient of beam function of flexural angular displacement

SUBROUTINE BMDFTN

The subroutine calculates the six beam deflection components $\phi_i(x)$ i=1,...,6 at cross section X.

Input parameters:

W frequency of vibration

X the distance of the cross section from the left hand end

PG(7) an array of beam properties as in section (6.2)

A(20) an array containing the beam coefficients output from BMDFFI

Output parameters:

FF(6) the six components of the beam deflection at X

A simply supported beam was taken as checking example. The nodal displacements Q and member properties PG are input by the data statements. The calculated flexural deflections are compared with a sine curve and no real difference was found.

```
TRIALS RUN 3118   DATE 10/02/76   TIME 13:11:56                    FROM PROCED
LOGIN USERNAME:  RAE00L      GROUPNAME:  RAE03L
                                                    283
// FTRAN1 RAE00L:BL110.YT110(F3118),K                          SOURCE FILE
// OPTION LIST,DEBUG
EXIT TO SYSTEM:TPPROG.FTRAN1(P90220)  AT 13:12:12


FTRN4 800S30  SOURCE PROGRAM                                   10/02/76   P
    1         IMPLICIT REAL*3 (A-H,O-Z)
    2         DIMENSION Q(12),A(20),PG(7),FF(6)
    3         DIMENSION U(4),B(4)
    4         DATA Q/.1,.0,.0,-.1,-1.,1.,.1,.0,.0,-0.0,1.,-1./
    5         DATA PG/1.,10.,10.,1.,1.,1.,1./
    6         DATA U/0.,1.,0.,-1./
    7         PI=3.1415926535B979
    8         P2=PI*PI
    9         CALL BNDFFI(P2,PG,Q,A)
   10         WRITE (6,3) (A(I),I=1,20)
   11       3 FORMAT (//'A='//2(2D20.10/4D20.10/)//)
   12         DO 1 I=1,11
   13         X=DFLOAT(I-1)*.1
   14         P=DSIN(X*PI)
   15         P=P*0.0933631643/0.309016994
   16         CALL BNDFTN(P2,X,PG,A,FF)
   17         WRITE (6,2) P,(FF(J),J=1,6)
   18       2 FORMAT(7D18.9)
   19       1 CONTINUE
   20         STOP
   21         END


FTRN4 800S30  SOURCE PROGRAM  BNDFCN   SUBROUTINE             10/02/76   P
   22         SUBROUTINE BNDFCN(W,WS,R,RZ,TI,EA,U1,U7,U2,U6,U8,U12,A1,A2,A3)
   23         IMPLICIT REAL*8 (A-H,O-Z)
   24         REAL*8 NS
   25         DIMENSION A1(2),A2(4),A3(4),U(4),V(4,4)
   26         Z=DSQRT(DSQRT(WS/RZ/R)*W)*R
   27         Y=DSQRT(TI/EA/R)*W*R
   28         PI=3.1415926535B979
   29         ZS=Z*Z
   30         ZC=ZS*Z
   31         ZQ=ZC*Z
   32         IF (Z.GT.1.) GOTO 1
   33         F1= 2.+(0.007142857+(0.000015704+0.000000032*ZQ)*ZQ)*ZQ
   34         F2= 4.-(0.009523810+(0.000016262+0.000000032*ZQ)*ZQ)*ZQ
   35         F3= 6.+(0.030952381+(0.000072193+0.000000148*ZQ)*ZQ)*ZQ
   36         F4= -6.+(0.052380952+(0.000076617+0.000000149*ZQ)*ZQ)*ZQ
   37         F5=-12.-(0.128571429+(0.000329571+0.000000684*ZQ)*ZQ)*ZQ
   38         F6= 12.-(0.371428571+(0.000954873+0.000000693*ZQ)*ZQ)*ZQ
   39         GOTO 2
   40       1 IF(Z.GT.25.0) GOTO 3
   41         SB=DSIN(Z)
   42         CB=DCOS(Z)
   43         R=DEXP(Z)
   44         SH=0.5*(R-1./R)
   45         CH=0.5*(R+1./R)
   46         R=Z/(CH*CB-1.)
   47         F1=-R*(SH-SB)
   48         F2=-R*(CH*SB-SH*CB)
   49         F3=-R*Z*(CH-CB)
   50         F4=R*Z*SH*SB
   51         F5=R*ZS*(SH+SB)
   52         F6=-R*ZS*(CH*SB+SH*CB)
   53         GOTO 2
   54       3 CB=1./DCOS(Z)
   55         SB=DSIN(Z)
   56         F1=-Z*CB
   57         F2=Z*(1-CB*SB)
   58         F3=-ZS*CB
   59         F4=ZS*CB*SB
   60         F5=ZC*CB
   61         F6=-ZC*(1+CB*SB)
   62       2 ZS=0.5/ZS
   63         ZC=0.5/ZC
```

```
      F4=F4*ZS
      F2=F2*ZS*H
      F3=F3*ZS
      F1=F1*H*ZS
      F6=F6*ZC
      F5=F5*ZC
      R=H/Z                                                    284
      V(1,1)=0.5-F4
      V(1,2)=F2
      V(1,3)=-F3
      V(1,4)=F1
      V(2,1)=-F6
      V(2,2)=R*(0.5+F4)
      V(2,3)=-F5
      V(2,4)=-F3*R
      V(3,1)=0.5+F4
      V(3,2)=-F2
      V(3,3)=F3
      V(3,4)=-F1
      V(4,1)=F6
      V(4,2)=R*(0.5-F4)
      V(4,3)=F5
      V(4,4)=F3*R
      U(1)=U2
      U(2)=U6
      U(3)=U8
      U(4)=U12
      DO 21 I=1,4
      S=0.
      DO 22 J=1,4
   22 S=S+V(I,J)*U(J)
   21 A2(I)=S
      R=Z/H
      A3(1)=A2(2)*R

FTRN4 800S3C  SOURCE PROGRAM  BNDFCN   SUBROUTINE              10/02/76

      A3(2)=-A2(1)*R
      A3(3)=A2(4)*R
      A3(4)=A2(3)*R
      A1(1)=U1
      S=DSIN(Y)
      C=DCOS(Y)
      A1(2)=U7/S-U1*C/S
      RETURN
      END

FTRN4 800S3C  SOURCE PROGRAM  BNDFFI   SUBROUTINE              10/02/76

      SUBROUTINE BNDFFI(U,PG,Q,A)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 MS
      DIMENSION A1(2),A2(4),A3(4),Q(12),A(20),PG(7)
      H=PG(1)
      EA=PG(2)
      GJ=PG(3)
      RZ=PG(4)
      RY=PG(5)
      MS=PG(6)
      TI=PG(7)
      CALL BNDFCN (U,MS,H,RZ,MS,EA,Q(1),Q(7),Q(2),Q(6),Q(8),Q(12),
     *A1,A2,A3)
      A(1)=A1(1)
      A(2)=A1(2)
      DO 1 I=1,4
      A(I+2)=A2(I)
    1 A(I+6)=A3(I)
      CALL BNDFCN (U,MS,H,RY,TI,GJ,Q(4),Q(10),Q(3),-Q(5),Q(9),-Q(11)
     *A1,A2,A3)
      A(11)=A1(1)
      A(12)=A1(2)
      DO 2 I=1,4
      A(I+12)=A2(I)
    2 A(I+16)=A3(I)
      RETURN
      END

FTRN4 800S3C  SOURCE PROGRAM  BNDFTN   SUBROUTINE              10/02/76

      SUBROUTINE BNDFTN(U,X,PG,A,FF)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION PG(7),FF(6),A(20),V(4)
```

```
      REAL*8 HS
      R=PG(1)
      EA=PG(2)
      GJ=PG(3)
      RZ=PG(4)
      RY=PG(5)
      HS=PG(6)
      TI=PG(7)                                                    285
      Y=DSQRT(HS/EA/R)*U*X
      FF(1)=DCOS(Y)*A(1)+DSIN(Y)*A(2)
      Z=DSQRT(DSQRT(HS/RZ/R)*U)*X
      V(1)=DCOS(Z)
      V(2)=DSIN(Z)
      E=DEXP(Z)
      V(3)=0.5*(E+1./E)
      V(4)=0.5*(E-1./E)
      R=0.
      DO 1 I=1,4
    1 R=R+V(I)*A(I+2)
      FF(2)=R
      R=0.
      DO 2 I=1,4
    2 R=R+V(I)*A(I+6)
      FF(6)=R
      Y=DSQRT(TI/GJ/R)*U*X
      FF(4)=DCOS(Y)*A(11)+DSIN(Y)*A(12)
      Z=DSQRT(DSQRT(HS/RY/R)*U)*X
      V(1)=DCOS(Z)
      V(2)=DSIN(Z)
      E=DEXP(Z)
      V(3)=0.5*(E+1./E)
      V(4)=0.5*(E-1./E)
      R=0.
      DO 3 I=1,4
    3 R=R+V(I)*A(I+12)
      FF(3)=R
      R=0.
      DO 4 I=1,4
    4 R=R+V(I)*A(I+16)
      FF(5)= R
      RETURN
      END
```

```
FTRN4 800S30  OBJECT SUMMARY                                  10/02/76   I

      OBJECT DECK ORIGIN  FDFC00
      INSTRUCTIONS                          00170
      ARRAYS                                00188
      FORMATS, NAMELISTS, LITCONS           00030
      VARIABLES, CONSTANTS, TEMPORARIES     00198
                SD  01  004C0          ITF#HPI    ER  02              ITF#N
      BMDFFI    EV  04                 BMDFTN     EV  05              ITF#C
      ITF#0H    EV  07                 ITF#0D     EV  08              ITF#1
      ITF#13    EV  0A                 ITF#ST     EV  0B
```

```
FTRN4 800S30  OBJECT SUMMARY  BMDFCH   SUBROUTINE               10/02/76   I

      OBJECT DECK ORIGIN  FDF640                                          /
      INSTRUCTIONS                          00920
      ARRAYS                                000A0
      FORMATS, NAMELISTS, LITCONS           00000
      VARIABLES, CONSTANTS, TEMPORARIES     0042C
      BMDFCH    SD  01  00DEC          ITF#HPI    ER  02              ITF#I
      ITF#13    EV  04                 ITF#25     EV  05              ITF#.
      ITF#01    EV  07
```

```
FTRN4 800S30  OBJECT SUMMARY  BMDFFI   SUBROUTINE               10/02/76   I

      OBJECT DECK ORIGIN  FDFCA0
      INSTRUCTIONS                          00310
      ARRAYS                                00050
      FORMATS, NAMELISTS, LITCONS           00000
      VARIABLES, CONSTANTS, TEMPORARIES     001FC
      BMDFFI    SD  01  0055C          ITF#HPI    ER  02              ITF#I
      BMDFCH    EV  04
```

```
FTRN4 800S30  OBJECT SUMMARY  BMDFTN   SUBROUTINE               10/02/76   I

      OBJECT DECK ORIGIN  FDFA88
      INSTRUCTIONS                          00558
```

6.5.3 PROGRAMME PLATE

This is an ALGOL programme to calculate the natural frequencies and modal shapes of an anisotropic retangular plate as dexcribed in section(3.5.2). The equation (3.5.12) is used to find the natural modes. The product integrals of beam functions are stored in a data file ORPTDT. This data is to be read by the programme. The first line of the file ORPTDT contains

DIF an integer specifying the boundary conditions. DIF is equal to 0 for a plate with all edges clamped and no other types of boundary conditions are being implemented.

Q is the number of beam modes used in the expansion.

DN is the density per unit area

H is the thichness of the plate

D11,D12,D22,D16,D26,D66 are the flexural rigidities of the plate.

NX,NY,NXY are the axial compressions in X and Y directions and the shearing load respectively

U is a parameter controlling the intensity of axial and shear loads.

The vectors the product integrals of beam functions,

The results are - Fig(3.7.1)

```
RRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRR
RRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRR
RRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRR
RRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRR
RRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRR

STING OF :EMP0708,MOP273(1/B1B0) PRODUCED ON 27JAN75 AT 21.43.07

.7D  AT ASTON     IN ':EMP0708.MOP273' ON 27JAN75 AT 21.43.37

UMENT      :EMP0708,MOP273(/B1B0)


RTED :EMP0708,MOP273,27JAN75  21.26.06 TYPE:BACK
26.07+ RJ MOP273,ORPTPROG,JD(JT 200,MZ 32000)
26.07+ ORPTPROG
26.07+ UAALGOL PROG ORPTLD,DATA ORPTDT,LIB NAGA,EXIT
26.09+          TA CE,DP,OL,LS

27/07          27/01/75             COMPILED BY XALV MK.  3A
E STATEMENT
      0          'PROGRAM'(AXXX)
      0          'INPUT' 0=CR0
      0          'INPUT' 3=TR0
      0          'OUTPUT'0=LP0
      0          'OUTPUT'4=LP1
      0          'COMPACTDATA'
      0          'COMPACT'
      0          ****
      0          'BEGIN'
      1             'INTEGER'J,I,Q,K,L,N,M,DIF,P;
      1             'REAL'N1,N2,NX,NY,NXY,A,B,AB,BA,D,DN,H,A1,A2,B2,D11,D12,
      2                  D22,D16,D26,D66,T,U;
      2             DIF:=READ; Q:=READ; A:=READ; B:=READ; DN:=READ; H:=READ;
      9             D11:=READ; D12:=READ; D22:=READ; D16:=READ; D26:=READ;
     14             D66:=READ; NX:=READ; NY:=READ; NXY:=READ; P:=Q*Q; U:=READ;
     20             A1:=1/(A*B); A2:=1/(A*A); B2:=1/(B*B); BA:=B/A↑3;
     24             AB:=A/B↑3; N1:=NX*B/A; N2:=NY*A/B; D:=DN*H*A*B;
     28             'BEGIN''ARRAY'X,Y[1:6,1:Q,1:Q],C,E,F,R,S[1:P,1:P],V[1:P];
     28                'PROCEDURE'F02AFA(A,B,N,R,V,IFAIL); 'VALUE'N;
     31                'INTEGER'N,IFAIL; 'ARRAY'A,B,R,V;
     33                'ALGOL';
     33                'FOR'L:=1,2,3,4,6'DO'
     35                'FOR'K:=1,4,7,10'DO''FOR'I:=0'STEP'1'UNTIL'12'DO'
     37                'FOR'J:=-1,0,1,2'DO'
     38                'BEGIN''IF'I=0'OR'J=-1'THEN'PRINT(READ,2,0)
     39                   'ELSE''IF'I>Q'OR'K+J>Q'THEN'H:=READ'ELSE'
     39                   X[L,I,K+J]:=READ;
     40                'END';
     41                'FOR'I:=1'STEP'1'UNTIL'Q'DO''FOR'J:=1'STEP'1'UNTIL'Q'DO'
     43                Y[5,I,J]:=-X[2,I,J];
     44
     44                'COMMENT'THE ABOVE TWO STATEMENTS SHOULD BE MODIFIED
     44                   FOR NON-SELF ADJOINT BOUNDARY CONDITIONS;
     44
     44                'IF'DIF=1'THEN''BEGIN'
     44                'FOR'L:=1,2,3,4,6'DO'
     46                'FOR'K:=1,4,7,10'DO''FOR'I:=0'STEP'1'UNTIL'12'DO'
     48                'FOR'J:=-1,0,1,2'DO'
     49                'BEGIN''IF'I=0'OR'J=-1'THEN'PRINT(READ,2,0)
     50                   'ELSE''IF'I>Q'OR'K+J>Q'THEN'H:=READ'ELSE'
```

```
32   50                Y[L,I,K+J]:=READ;'
33   51             'END';
34   52             'FOR'I:=1'STEP'1'UNTIL'Q'DO''FOR'J:=1'STEP'1'UNTIL'Q'DO'
35   54             Y[5,I,J]:=-Y[2,I,J];
36   55
37   55             'COMMENT'THE ABOVE TWO STATEMANTS SHOULD BE MODIFIED
38   55                FOR NON-SELF ADJOINT BOUNDARY CONDITIONS;
39   55
40   55                                'END''ELSE'
41   55             'BEGIN''FOR'I:=1'STEP'1'UNTIL'Q'DO'
42   57                'FOR'J:=1'STEP'1'UNTIL'Q'DO''FOR'K:=1,2,3,4,5,6'DO'
43   59                Y[K,I,J]:=X[K,I,J];
44   60             'END';
45   61             'FOR'I:=1'STEP'1'UNTIL'Q'DO''FOR'K:=1'STEP'1'UNTIL'Q'DO'
46   63             'FOR'M:=1'STEP'1'UNTIL'Q'DO''FOR'N:=1'STEP'1'UNTIL'Q'DO'
47   65             'BEGIN' J:=(I-1)*Q+K; L:=(M-1)*Q+N;
48   68                F[J,L]:=D11*X[3,I,M]*Y[1,K,N]*BA

49   68                +D12*(X[5,I,M]*Y[5,N,K]+X[5,M,I]*Y[5,K,NJ)*A1
50   68                +D22*X[1,I,M]*Y[3,K,N]*AB
51   68                +2*D26*(X[4,I,M]*Y[6,N,K]+X[4,M,I]*Y[6,K,NJ)*B2
52   68                +2*D16*(X[6,M,I]*Y[4,K,N]+X[6,I,M]*Y[4,N,K])*A2
53   68                +4*D66*(X[2,I,M]*Y[2,K,N])*A1;
54   69                E[J,L]:=N1*X[2,I,M]*Y[1,K,N]+N2*X[1,I,M]*Y[2,K,N]
55   69                +NXY*(X[4,I,M]*Y[4,N,K]+X[4,M,I] *Y[4,K,N]);
56   70                R[J,L]:=D*X[1,I,M]*Y[1,K,N];
57   71             'END';
58   72             'FOR'T:=0'DO'
59   73             'BEGIN'H:=T;
60   75                'FOR'I:=1'STEP'1'UNTIL'P'DO''FOR'J:=1'STEP'1'UNTIL'P'DO
61   77                S[I,J]:=F[I,J]+H*E[I,J];
62   78                J:=1; FO2AEA(S,P,P,V,C,I);
63   80                WRITETEXT('('''('P')'H=')'); PPINT(H,0,4);
64   82          'IF'P>36'THEN'P:=36;
65   83                'FOR'K:=1,3+K'WHILE'P>K'DO'
66   84                'BEGIN'
67   84                   WRITETEXT('('''('1C')'EIGENVALUE')');
68   86                   'FOR'J:=0,1,2'DO''IF'K+J'LE'P'THEN'
69   87                   'BEGIN'PRINT(V[K+J],0,9)'END';
70   89                   NEWLINE(2);
71   90                   'FOR'I:=1'STEP'1'UNTIL'P'DO'
72   91                   'BEGIN'PRINT(I,2,0)'
73   93                      'FOR'J:=0,1,2'DO''IF'K+J'LE'P'THEN'
74   94                      PRINT(C[J+K,I],5,9);
75   95                      NEWLINE(1);
76   96                   'END'I;
77   97                'END'K;
78   98             'END'T;
79   99          'END'ARRAY;
80  100       'END'PROG;
MENT AXXX          LENGTH  1373
OF BUCKETS USED    33
IPILED   #AXXX          EC


GRAM AXXX
PACT DATA (15AM)
PACT PROGRAM (DHM)
E              10437
```

0, 6, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 1

0	1	2	3
1	1.000000000	0.000000000	-0.000000000
2	0.000000000	1.000000000	-0.000000000
3	-0.000000000	-0.000000000	1.000000000
4	0.000000000	-0.000000000	0.000000000
5	-0.000000000	0.000000000	-0.000000000
6	-0.000000000	0.000000000	0.000000000
7	0.000000000	-0.000000000	0.000000000
8	0.000000000	0.000000000	0.000000000
9	0.000000000	0.000000000	-0.000000000
10	0.000000000	0.000000000	0.000000000
11	-0.000000000	0.000000000	-0.000000000
12	0.000000000	-0.000000000	0.000000000

0	4	5	6
1	0.000000000	-0.000000000	-0.000000000
2	-0.000000000	0.000000000	0.000000000
3	0.000000000	-0.000000000	0.000000000
4	1.000000000	0.000000000	-0.000000000
5	0.000000000	1.000000000	-0.000000000
6	-0.000000000	-0.000000000	1.000000000
7	-0.000000000	0.000000000	-0.000000000
8	0.000000000	-0.000000000	0.000000000
9	0.000000000	-0.000000000	0.000000000
10	-0.000000000	-0.000000000	0.000000000
11	-0.000000000	0.000000000	-0.000000000
12	-0.000000000	-0.000000000	-0.000000000

0	7	8	9
1	0.000000000	-0.000000000	0.000000000
2	-0.000000000	0.000000000	0.000000000
3	0.000000000	0.000000000	-0.000000000
4	-0.000000000	0.000000000	0.000000000
5	0.000000000	-0.000000000	-0.000000000
6	-0.000000000	0.000000000	0.000000000
7	1.000000000	0.000000000	-0.000000000
8	0.000000000	1.000000000	0.000000000
9	-0.000000000	0.000000000	1.000000000
10	0.000000000	-0.000000000	-0.000000000
11	-0.000000000	0.000000000	0.000000000
12	0.000000000	-0.000000000	0.000000000

0	10	11	12
1	0.000000000	-0.000000000	0.000000000
2	0.000000000	0.000000000	-0.000000000
3	0.000000000	-0.000000000	0.000000000
4	-0.000000000	-0.000000000	-0.000000000
5	-0.000000000	0.000000000	-0.000000000
6	0.000000000	-0.000000000	-0.000000000
7	0.000000000	-0.000000000	0.000000000
8	-0.000000000	0.000000000	-0.000000000
9	-0.000000000	0.000000000	0.000000000
10	1.000000000	-0.000000000	-0.000000000
11	-0.000000000	1.000000000	0.000000000
12	-0.000000000	0.000000000	1.000000000

0	1	2	3
1	-12.302618625	-0.000000001	9.730792214
2	-0.000000001	-46.053120145	0.000000003
3	9.730792214	0.000000003	-28.904301404
4	-0.000000001	17.128920873	-0.000000007
5	7.615435410	-0.000000003	24.349871975
6	-0.000000006	15.194569863	-0.000000007
7	6.108043878	-0.000000000	22.984217479
8	-0.000000002	13.156673173	-0.000000005
9	5.066528018	0.000000004	20.852897944
10	-0.000000003	11.430116096	-0.000000007
11	4.318182720	-0.000000003	18.785424519
12	-0.000000004	10.065501451	-0.000000008

0	4	5	6
1	-0.000000001	7.615435409	-0.000000001
2	17.128920873	-0.000000002	15.194569860
3	-0.000000009	24.349371974	-0.000000006
4	-171.585655705	-0.000000015	31.276449266
5	-0.000000015	-263.997980742	0.000000015
6	31.276449289	0.000000015	-376.150083495
7	0.000000008	38.030180032	0.000000067
8	30.578550230	0.000000021	44.669148571
9	-0.000000002	37.960697384	-0.000000003
10	28.664289448	-0.000000001	45.157137484
11	-0.000000007	36.396419295	-0.000000003
12	26.502745188	-0.000000006	43.991876324

0	7	8	9
1	6.108043872	-0.000000000	5.066528012
2	0.000000001	13.136473170	-0.000000000
3	22.984217474	-0.000000002	20.852897936
4	0.000000009	30.578550226	-0.000000003
5	38.030180032	0.000000019	37.960697389
6	0.000000070	44.669148572	-0.000000007
7	-508.041357665	-0.000000047	51.229846105
8	-0.000000050	-659.671842884	-0.000000101
9	51.229846165	-0.000000096	-831.041536803
10	-0.000000061	57.735089048	-0.000000013
11	52.202717575	-0.000000041	64.199765365
12	-0.000000062	59.127870536	-0.000000040

0	10	11	12
1	-0.000000000	4.318182721	-0.000000000
2	11.430116083	-0.000000001	10.065501452
3	-0.000000004	18.785424526	-0.000000004
4	28.664289443	-0.000000002	26.502745184
5	0.000000002	36.396419300	-0.000000001
6	45.157137481	-0.000000001	43.991876320
7	-0.000000054	52.202717572	-0.000000050
8	57.735089045	-0.000000037	59.127870536
9	-0.000000007	64.199765365	-0.000000034

10	-1022.150439539	0.000000024	70.633940436
11	0.000000023	-1232.998551049	-0.000000145
12	70.633940439	-0.000000146	-1463.585871370

0	1	2	3
1	500.563901827	0.000000004	0.000000150
2	0.000000010	3803.557080784	-0.000000376
3	0.000000134	-0.000000358	14617.630132306
4	0.000000054	-0.000000224	0.000001125
5	-0.000000473	0.000001289	-0.000003114
6	0.000000283	-0.000000827	0.000002176
7	0.000000674	-0.000001460	0.000003651
8	-0.000000130	0.000000298	-0.000000307
9	-0.000000834	0.000002950	-0.000004441
10	0.000001140	-0.000002518	0.000006139
11	-0.000001140	0.000003129	-0.000007451
12	0.000002459	-0.000006914	0.000013441

0	4	5	6
1	0.000000054	-0.000000443	0.000000285
2	-0.000000209	0.000001281	-0.000000842
3	0.000001125	-0.000003159	0.000002235
4	39943.799006505	0.000006646	-0.000003070
5	0.000006676	89135.407652065	0.000001281
6	-0.000002891	0.000001252	173881.315475227
7	-0.000005841	0.000014305	-0.000033200
8	0.000001846	-0.000003338	0.000007155
9	0.000009388	-0.000014186	0.000018695
10	-0.000009894	0.000014186	-0.000021338
11	0.000009894	-0.000016749	0.000020266
12	-0.000021994	0.000032907	-0.000046965

0	7	8	9
1	0.000000667	-0.000000123	-0.000000854
2	-0.000001490	0.000000268	0.000002965
3	0.000003517	-0.000000507	-0.000004560
4	-0.000005697	0.000001758	0.000009507
5	0.000014275	-0.000003159	-0.000014544
6	-0.000033140	0.000007153	0.000019252
7	308208.452152030	0.000020862	-0.000053167
8	0.000019550	508481.543278321	0.000092149
9	-0.000053287	0.000089884	793403.134513937
10	0.000040412	-0.000064135	0.000035644
11	-0.000038981	0.000049353	-0.000045061
12	0.000069499	-0.000105381	0.000106096

0	10	11	12
1	0.000001132	-0.000001125	0.000002466
2	-0.000002474	0.000003114	-0.000006899
3	0.000006139	-0.000007510	0.000013530
4	-0.000009894	0.000009716	-0.000021994
5	0.000014603	-0.000016699	0.000032007
6	-0.000021219	0.000020742	-0.000047207
7	0.000041723	-0.000039697	0.000069380

8	-0.000064969	0.000048757	-0.000104904
9	0.000039458	-0.000045061	0.000107268
10	1184013.569581809	0.000016928	-0.000154018
11	0.000020027	1703691.090206120	0.000317524
12	-0.000153780	0.000315428	2376151.636385155

0	1	2	3
1	0.000000000	3.342016046	0.000000000
2	-3.342016046	0.000000000	5.516100834
3	-0.000000000	-5.516100834	-0.000000000
4	-0.906926112	-0.000000000	-7.632794028
5	-0.000000000	-1.726226988	-0.000000000
6	-0.430472416	0.000000000	-2.532454194
7	0.000000000	-0.899817481	0.000000000
8	-0.251252213	0.000000000	-1.396557351
9	0.000000000	-0.556575201	0.000000000
10	-0.164560293	0.000000000	-0.900005630
11	0.000000000	-0.378842807	-0.000000000
12	-0.116089088	-0.000000000	-0.631082863

0	4	5	6
1	0.906926112	0.000000000	0.430472416
2	0.000000000	1.726226088	0.000000000
3	7.632796028	0.000000000	2.532454194
4	0.000000000	9.703954989	0.000000000
5	-9.703954989	0.000000000	11.752299125
6	0.000000000	-11.752299125	0.000000000
7	-3.308822649	0.000000001	-13.787164597
8	0.000000000	-4.061440228	0.000000000
9	-1.890174102	-0.000000000	-4.796305029
10	0.000000000	-2.373709985	0.000000000
11	-1.254561557	-0.000000000	-2.846259152
12	-0.000000000	-1.609111039	-0.000000000

0	7	8	9
1	0.000000000	0.251252213	0.000000000
2	0.899817481	0.000000000	0.556575201
3	0.000000000	1.396557350	0.000000000
4	3.308822649	0.000000000	1.890174102
5	-0.000000000	4.061440228	-0.000000000
6	13.787164596	-0.000000000	4.796305030
7	0.000000000	15.813472765	0.000000001
8	-15.813472766	0.000000000	17.834017094
9	-0.000000001	-17.834017095	-0.000000000
10	-5.518018019	-0.000000000	-19.850498752
11	-0.000000000	-6.229869601	-0.000000000
12	-3.308823529	-0.000000001	-6.934200013

0	10	11	12
1	0.164560293	0.000000000	0.116089088
2	0.000000000	0.378842807	0.000000000
3	0.900005629	0.000000000	0.631082863
4	0.000000000	1.254361557	0.000000000
5	2.373709984	-0.000000000	1.609111038
6	0.000000000	2.846259152	0.000000000
7	5.518018018	0.000000001	3.308823529
8	0.000001001	6.229869600	0.000000001
9	19.850498752	-0.000000000	6.934200012
10	0.000000000	21.864011248	0.000000001

11	-21.864011247	-0.000000000	23.875288851
12	0.000000000	-23.875288852	0.000000000

0	1	2	3
1	0.000000001	122.065044925	0.000000002
2	-122.065044928	0.000000000	476.673745921
3	0.000000000	-476.673745928	0.000000014
4	-59.584499766	-0.000000005	-1186.455079150
5	0.000000006	-234.419954405	0.000000003
6	-40.851403176	0.000000025	-568.598972815
7	0.000000026	-166.628746528	0.000000030
8	-31.180072069	0.000000011	-410.045683858
9	0.000000034	-130.551448452	0.000000076
10	-25.226827854	0.000000099	-326.429251541
11	0.000000013	-107.569988999	-0.000000112
12	-21.186036041	0.000000019	-272.490235784

0	4	5	6
1	59.584499763	0.000000003	40.851403151
2	0.000000007	234.419954400	0.000000015
3	1186.455079096	0.000000023	568.598972815
4	0.000000023	2370.414426222	0.000000075
5	-2370.414426259	-0.000000093	4146.661683953
6	0.000000061	-4146.661683917	0.000000125
7	-1102.167491708	-0.000000017	-6633.600709756
8	0.000000037	-1873.964679598	-0.000000112
9	-797.513894453	-0.000000175	-2923.098859391
10	0.000000063	-1352.942579176	0.000000043
11	-640.666572115	-0.000000127	-2099.856031646
12	0.000000093	-1091.838581132	0.000000097

0	7	8	9
1	0.000000006	31.180072059	0.000000006
2	166.628746328	0.000000006	130.551448493
3	0.000000036	419.045683797	0.000000036
4	1102.167491708	0.000000045	797.513894453
5	0.000000030	1873.964479534	-0.000000056
6	6633.600709756	-0.000000002	2923.098859464
7	-0.000000186	9949.635925149	0.000000000
8	-9949.635925149	0.000000037	14213.186509005
9	-0.000000071	-14213.186509186	0.000000112
10	-4288.776575049	0.000000156	-19542.677843310
11	0.000000067	-6010.290559497	-0.000000276
12	-3061.573056802	-0.000000015	-8126.990052041

0	10	11	12
1	25.226827648	0.000000002	21.186036028
2	0.000000028	107.569989005	0.000000007
3	326.429251486	-0.000000024	272.490235725
4	0.000000043	640.666571977	0.000000037
5	1352.942578996	-0.000000030	1091.838580967
6	0.000000104	2099.856031538	0.000000075
7	4288.776574467	0.000000142	3061.573056183
8	0.000000294	6010.290558915	0.000000056
9	19542.677842582	0.000000019	8126.990051168
10	-0.000000355	26056.538851344	0.000000505
11	-26056.538852935	0.000000246	33873.200538801
12	0.000000283	-33873.200538801	0.000000164

6.5.4 TIMOSHENKO COLUMN ELEMENT

Two subroutines SPTMM and TMFQB were designed to calculate the dynamic stiffness matrix for a Timoshenko column member. These two subroutines are of the same organization as DYMAT and FQFN1 of the programme for Euler beam as in section (6.2). If these two subroutines are to be incooperated with MAINP, cares must give to the declarations of arrays, since for a Timoshenko column element, as discussed in section (3.6.5), the shear factor and the constant axial load will contribute to the property matrix PG and make the first dimension of PG as 9 instead of 7 for Euler beams.

SUBROUTINE SPTMM

This subroutine calculates the dynamic stiffness matrix of a Timoshenko column member as described in section (3.6.5).

Input parameters:

W frequency of vibration

H total length of the member

EA axial rigidity

GJ torsional rigidity

EZ flexural rigidity about Z axis

EY flexural rigidity about Y axis

MS total mass of the member

TI total polar moment of inertia

FI the shear parameter as decribed in equation(3.6.41) as GA_s

PP the axial force .

Output parameters:

KE the dynamic stiffness matrix

JE the number of partial frequencies which have been exceeded.

This subroutine calls the following subroutine for frequency functions.

SUBROUTINE TMFQB

This subroutine calculates the frequency functions of a Timoshenko column member for the use in subroutine SPTMM.

Input parameters:

W frequency of vibration

MS total mass of the member

H total length of the member

RZ flexural rigidity about Z axis

TI total polar moment of inertia

EA axial rigidity

GJ torsional rigidity

FI the factor GA_s in equation (3.6.41)

PP the constant axial force suffered by the element

JB

Output parameters:

G1,G2,G3,G4,G5,G6,G7,G8,G9,G10,G11,G12,G13 are the frequency functions

JB the number of partial frequencies exceeded

```
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      00000100      SUBROUTINE SPTHN(W,H,EA,GJ,EZ,EY,HS,TI,FI,PP,KE,JE)
      00000200      IMPLICIT REAL*8 (A-H,O-Z)
      00000300      REAL*8 L,HS,KE(12,12)
      00000400      COMMON/THF1/G1,G2,G3,G4,G5,G6,G7,G8,G9,G10,G11,G12,G13
      00000500      JE=0
      00000600      DO 10 I=1,12
      00000700      DO 10 J=1,12
      00000800   10 KE(I,J)=0.0
      00000900      CALL THFQ8(W,HS,H,EZ,HS,EA,EA,FI,PP,JB)
      00001000      JE=JE+JB
      00001100      KE(1,1)=G11
      00001200      KE(1,7)=G12
      00001300      KE(2,2)=G1
      00001400      KE(2,6)=G2
      00001500      KE(2,8)=G3
      00001600      KE(2,12)=G4
      00001700      KE(6,6)=G5
      00001800      KE(6,8)=G6
      00001900      KE(6,12)=G7
      00002000      KE(8,8)=G8
      00002100      KE(8,12)=G9
      00002200      KE(12,12)=G10
      00002300      KE(7,7)=G13
      00002400      CALL THFQ8(W,HS,H,EY,TI,EA,GJ,FI,PP,JB)
      00002500      JE=JE+JB
      00002600      KE(3,3)=G1
      00002700      KE(3,5)=-G2
      00002800      KE(3,9)=G3
      00002900      KE(3,11)=-G4
      00003000      KE(4,4)=G11
      00003100      KE(4,10)=G12
      00003200      KE(5,5)=G5
      00003300      KE(5,9)=-G6
      00003400      KE(5,11)=G7
      00003500      KE(9,9)=G8
      00003600      KE(9,11)=-G9
      00003700      KE(11,11)=G10
      00003800      KE(10,10)=G13
      00003900      DO 11 I=1,12
      00004000      DO 11 J=I,12
      00004100   11 KE(J,I)=KE(I,J)
      00004200      RETURN
      00004300      END
      00004400      SUBROUTINE THFQ8(W,HS,H,RZ,TI,EA,GJ,FI,PP,JB)
      00004500      IMPLICIT REAL*8 (A-H,O-Z)
      00004600      REAL*8 HU,MU,HS
      00004700      COMMON/THF1/G1,G2,G3,G4,G5,G6,G7,G8,G9,G10,G11,G12,G13
      00004800      PI=3.14159265358979
      00004900      H2=H*H
      00005000      H3=H*H2
      00005100      B=HS*W*W*H3/RZ
      00005200      R=RZ/EA/H2
      00005300      S=RZ/FI/H2
      00005400      P=PP*H2/RZ
      00005500   13 IF (B.LE.0.0001) GOTO 1
      00005600      Q=P/B+R*(1.-S*P)
      00005700      DT=DSQRT((Q-S)*(Q-S)+4./B)
      00005800      Q1=1.-S*P
      00005900      Q2=B/(2.*Q1)
      00006000      AF=DSQRT((Q2)*(DT-Q-S))
      00006100      BT=DSQRT((Q2)*(DT+Q+S))
      00006200      DB=Q1*AF+B*S/AF
      00006300      AB=Q1*BT-B*S/BT
      00006400      RR=DEXP(AF)
      00006500      CH=.5*(RR+1./RR)
      00006600      SH=.5*(RR-1./RR)
      00006700      CB=DCOS(BT)
      00006800      SB=DSIN(BT)
      00006900      E0=CH*CB-1.
      00007000      E1=AB*SH-DB*SB
      00007100      E2=DB*CH*CB-AB*SH*CB
      00007200      E3=CH-CB
      00007300      E4=SH*SB
      00007400      E5=DB*SH+AB*SB
      00007500      E6=AB*CH*SB+DB*SH*CB
      00007600      HU=AF*CB+BT*AB
```

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```
00007700        HU=AB*DB*HU
00007800        F0=2.*AB*DB*E0+(AB*AB-DB*DB)*E4
00007900        F0=1./F0
00008000        F1=-E1*HU*F0
00008100        F2=-E2*HU*F0
00008200        F3=-E3*HU*F0
00008300        F4=AB*DB*((AF*DB-BT*AB)*E0+(AF*AB+BT*DB)*E4)*F0
00008400        F5=E5*HU*F0
00008500        F6=-E6*HU*F0
00008600        GOTO 2
00008700      1 U=S/(1.+12.*S)
00008800        F1=2.-36.*U+B*(1./140.+R/30.)+P/30.
00008900        F2=4.-36.*U-B*(1./105.
00009000     *                          +2.*R/15.)-2.*P/15.
00009100        F3=0.-72.*U+B*(13./420.-R*.1)-P*.1
00009200        F4=0.+72.*U+B*(11./210.+R*.1)+P*.1
00009300        F5=-12.+144.*U-B*(9./70.-1.2*R)+1.2*P
00009400        F6=12.-144.*U-B*(13./35.+1.2*R)-1.2*P
00009500      2 R=RZ/R3
00009600        G7=F1*R*H2
00009700        G10=F2*R*H2
00009800        G5=G10
00009900        G4=F3*R*H
00010000        G6=-G4
00010100        G9=F4*R*H
00010200        G2=-G9
00010300        G3=F5*R
00010400        G1=F6*R
00010500        G8=F6*R
00010600        JC=IDINT(BT/PI)
00010700        J1=1
00010800        J2=1
00010900        R=F2-F1*F1/F2
00011000        IF(R.LT.0.00) J1=-1
00011100        IF(F2.LT.0.00) J2=-1
00011200        JS=1-(J1+J2)/2
00011300        JC=JC-JS
00011400        Y=DSQRT(TI/GJ/H)*W*H
00011500        IF (Y.LT..01) GOTO 9
00011600        SB=DSIN(Y)
00011700        CB=DCOS(Y)
00011800        HR=Y/PI
00011900        JA=IDINT(HR)
00012000        F6=-Y/SB
00012100        F7=CB/SB*Y
00012200        GOTO 5
00012300      9 Y2=Y*Y
00012400        Y4=Y2*Y2
00012500        F7=1.-Y2/3.-Y4/45.
00012600        F8=-1.-Y2/6.-Y4*7./360.
00012700      5 S=LA/H
00012800        G11=F7*S
00012900        G13=G11
00013000        G12=F8*S
00013100        JB=JB+JA
00013200        RETURN
00013300        END
```

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CHAPTER SEVEN

DYNAMIC RESPONSE AND MODAL ANALYSIS

7.1 INTRODUCTION

The dynamic response of a structure subjected to loadings or excitations is governed by a set of differential equations in time and space variables. These equations are called the Lagrange's equations of motion if the generalized displacements are taken as unknowns, see section (2.5). The modal analysis is characterized by the fact that these equations are decoupled when the principal coordinates are taken as generalized coordinates for linear systems, see section (2.6). Therefore, in a system having n generalized coordinates, we may deal with n uncoupled differential equations rather than a system of n simultaneous differential equations. Thus, the computational effort required for the solution is considerably reduced. This is called the technique of modal analysis.

The technique applies only to linear systems. When the linearity is violated by the large amplitude of vibrations or the elasticity of the materials of the system, the application of the modal analysis will result in a set of nearly uncoupled differential equations. The advantages of applying the modal analysis in this nonlinear case will include the ease of obtaining first approximation and fast convergent properties of the numerical results to the solution (ref 73).

The modal analysis simplifies the computing effort for the response of the structure even when the excitations are just nearly periodic. In the

dynamic stability study of a structure, one of the most important cases is when the mean frequency of excitations is very close to a natural frequency of the structure. Then the reponse can be calculated with good accuracy by using the corresponding mode only (ref 95). This technique has advantages over the step-by-step integration (ref 94) by the fact that the modal analysis gives asymptotic approximation to the reponse and the results from step-by-step integration method are subjected to accumulative errors. The modal analysis for dynamic reponse should always be used when the natural modes are available.

The chapter will start with a general study of the modal analysis. A plane frame and a space frame are used as examples to several classical applications of the technique in sections (7.3) to (7.7). These include the response calculations for harmonic and nonharmonic, periodic and transient excitations. Damping effects are considered in section (7.8). The reponse to random loadings will be discussed in sections (7.9) to (7.11).

7.2 MODAL ANALYSIS

A brief mathematical account of modal analysis of various types of loadings to a structure is given in this section. Linear theory is assumed. If the motion of a linear structure is described sufficiently by a set of N generalized coordinates, q_i, i=1,2,..,N, and the associated generalised forces are Q_i, i=1,2,..,N, then the governing equations of motion in Lagrangian form are given by, see section (2.5),

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_r} - \frac{\partial L}{\partial q_r} = Q_r \qquad r=1,2,..,N \quad (7.2.1)$$

where L=T-U is the Lagrangian of the system,

$$T = \frac{1}{2}\int_{vol} \{\dot{u}\}^T [\rho] \{\dot{u}\}\, dvol \qquad (7.2.2)$$

is the kinetic energy,

$$U = \frac{1}{2}\int_{vol} \{u\} [C] \{u\}\, dvol \qquad (7.2.3)$$

is the strain energy stored in the system, and the displacement function vector $\{u\}$ and the strain distribution vector $\{\varepsilon\}$ are related to the generalized coordinate vector $\{q\}$ by

$$\{u\} = [a]\{q\} \qquad \text{and} \qquad \{\varepsilon\} = [b]\{q\} \qquad (7.2.4)$$

The generalized forces are obtained from equation (2.5.15), i.e.

$$Q_r = \int_{vol} \{X\}^T \frac{\partial\{u\}}{\partial q_r}\, dvol + \int_S \{\Phi\}^T \frac{\partial\{u\}}{\partial q_r}\, dS \qquad r=1,2,\ldots,N \qquad (7.2.5)$$

where $\{X\}$ and $\{\Phi\}$ are the vectors of body forces and surface forces respectively.

If, instead of the generalized coordinate system q_i, i=1,2,...,N, we use principal coordinates p_i, i=1,2,...,n, n may not be equal to N, such that, the distributions of displacement and strain are represented by

$$\{u\} = \sum_{i=1}^{n} \{\phi_i\} p_i(t) \qquad (7.2.6)$$

and

$$\{\varepsilon\} = \sum_{i=1}^{n} \{\varepsilon_i\} p_i(t) \qquad (7.2.7)$$

where $\{\phi_i\}$ and $\{\varepsilon_i\}$ are the ith modal displacement and strain distributions respectively, equations (7.2.2) and (7.2.3) become

$$T = \frac{1}{2}\sum_{i,j=1}^{n} \dot{p}_i \dot{p}_j \int_{vol} \{\phi_i\}^T [\rho] \{\phi_j\}\, dvol$$

$$U = \frac{1}{2}\sum_{i,j=1}^{n} p_i p_j \int_{vol} \{\varepsilon_i\}^T [C] \{\varepsilon_j\}\, dvol$$

When the conditions of orthogonality, section (2.7), are introduced, we have

$$T = \frac{1}{2}\sum_{i=1}^{n} \dot{p}_i^2 M_i \qquad (7.2.8)$$

and

$$U = \frac{1}{2}\sum_{i=1}^{n} p_i^2 \omega_i^2 M_i \qquad (7.2.9)$$

where

$$M_i = \int_{vol} \{\phi_i\}^T [\rho] \{\phi_i\}\, dvol \qquad (7.2.10)$$

is the generalized mass corresponding to the ith mode.

The generalized forces in principal coordinates P_i are given by equation (2.7.10), i.e.

$$P_i = \int_{vol} \{X\}^T \{\phi_i\} dvol + \int_S \{\Phi\}^T \{\phi_i\} dS \qquad i=1,2,\ldots,n \qquad (7.2.11)$$

By means of equations (7.2.8) to (7.2.11), the Lagrange's equations become

$$\ddot{p}_i + p_i \omega_i^2 = P_i \qquad i=1,2,\ldots,n \qquad (7.2.12)$$

with initial conditions

$$p_i(0) = \frac{1}{M_i} \int_{vol} \{\phi_i\}^T [\rho] \{u_0\} dvol$$

and

$$\dot{p}_i(0) = \frac{1}{M_i} \int_{vol} \{\phi_i\}^T [\rho] \{\dot{u}_0\} dvol \qquad (7.2.13)$$

for the unknowns $p_i(t)$. Where $\{u_0\}$ and $\{\dot{u}_0\}$ are the values of $\{u\}$ and $\{\dot{u}\}$ at $t=0$.

Therefore, when the normal modes of a structure are known, the dynamic response of some given excitations can be obtained by (i) expressing the displacement functions in terms of normal modes according to equation (7.2.6); (ii) calculating the generalized forces from equation (7.2.11); and (iii) solving the differential equations (7.2.12) for principal displacements P_i.

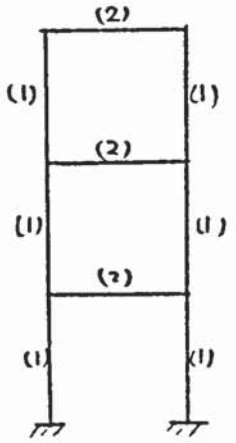
The body force vector $\{X\}$ represents the total body force per unit volume. This includes Electromagnetic forces, hydrodynamic forces, inertia forces caused by displacement excitation, etc.. The surface force per unit area is represented by $\{\Phi\}$ which consists of distributed and concentrated loads, travelling loads, external spring forces, viscously damping forces, and wind forces, etc.. We shall give some particular examples of the modal analysis technique in the next few sections.

7.3 NORMAL MODES OF A PLANE FRAMED STRUCTURE

A framed structure as shown in fig (7.3.1) will be used to illustrate the various numerical methods. The structure consists of six identical vertical beams marked by (1) and three horizontal ones marked by (2). The geometry and the elastic properties of each kind of beams are tabulated in fig (7.3.2). The first three symmetrical modes of natural vibration are depicted in fig (7.3.3) and the numerical values are given in fig (7.3.4). The first three anti-symmetrical modes are also depicted in fig (7.3.5). In the calculation of these natural modes, the longitudinal vibration of each beam was neglected.

Suppose the end displacements of the ith mode for a typical element are denoted as $q_1^{(i)}$, $q_2^{(i)}$, $q_3^{(i)}$, $q_4^{(i)}$ shown in fig (7.3.1). Then the ith normal mode of vibration of the structural element is obtained from equation (3.3.9) as

$$\phi_i(x) = \left[\cos\frac{\lambda x}{l} \;\; \sin\frac{\lambda x}{l} \;\; \cosh\frac{\lambda x}{l} \;\; \sinh\frac{\lambda x}{l}\right] \begin{bmatrix} \frac{1}{2} - \frac{F_4}{2\lambda^2} & \frac{F_2 l}{2\lambda^2} & -\frac{F_3}{2\lambda^2} & \frac{F_1 l}{2\lambda^2} \\ -\frac{F_6}{2\lambda^3} & \frac{1}{2\lambda} + \frac{F_4 l}{2\lambda^3} & -\frac{F_5}{2\lambda^3} & -\frac{F_3 l}{2\lambda^3} \\ \frac{1}{2} + \frac{F_4}{2\lambda^2} & -\frac{F_2 l}{2\lambda^2} & \frac{F_3}{2\lambda^2} & -\frac{F_1 l}{2\lambda^2} \\ \frac{F_6}{2\lambda^3} & \frac{1}{2\lambda} - \frac{F_4 l}{2\lambda^3} & \frac{F_5}{2\lambda^2} & \frac{F_3 l}{2\lambda^3} \end{bmatrix} \begin{Bmatrix} q_1^{(i)} \\ q_2^{(i)} \\ q_3^{(i)} \\ q_4^{(i)} \end{Bmatrix} \tag{7.3.1}$$



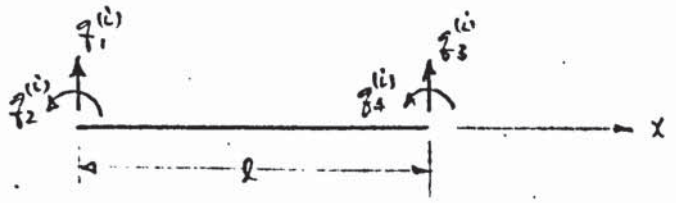


Fig (7.3.1) A plane frame

	Beam (1)	Beam (2)
Mass per unit length ρA (kg m s)	20.4	51.0
Young's Modulus $E(kg_f m^{-2})$	21×10^9	21×10^9
2nd moment of area $I(m^{-4})$	1.0×10^{-4}	2.0×10^{-4}
length (m)	6.0	5.0
frequency parameter λ	$0.334968 \times \omega^{\frac{1}{2}}$	$0.295155 \times \omega^{\frac{1}{2}}$

Fig (7.3.2) Geometry and Elasticity of each beam member

mode no.	NAT. freq. s^{-1}	λ_1	λ_2	$q_1^{(i)}$	$q_2^{(i)}$	$q_3^{(i)}$	v_d	M_i
1	103.703	3.41061	3.00524	0.58375	-0.96152	1	1.52519	$1.573 \times 10^3 Kg_f m s^2$
2	126.246	3.76310	3.31583	-1.01255	0.11273	1	1.70024	$1.388 \times 10^3 Kg_f m s^2$
3	155.437	4.17555	3.67926	0.77571	1.17364	1	2.06710	$2.011 \times 10^3 Kg_f m s^2$

Fig (7.3.3) Symmetrical modes of free vibration

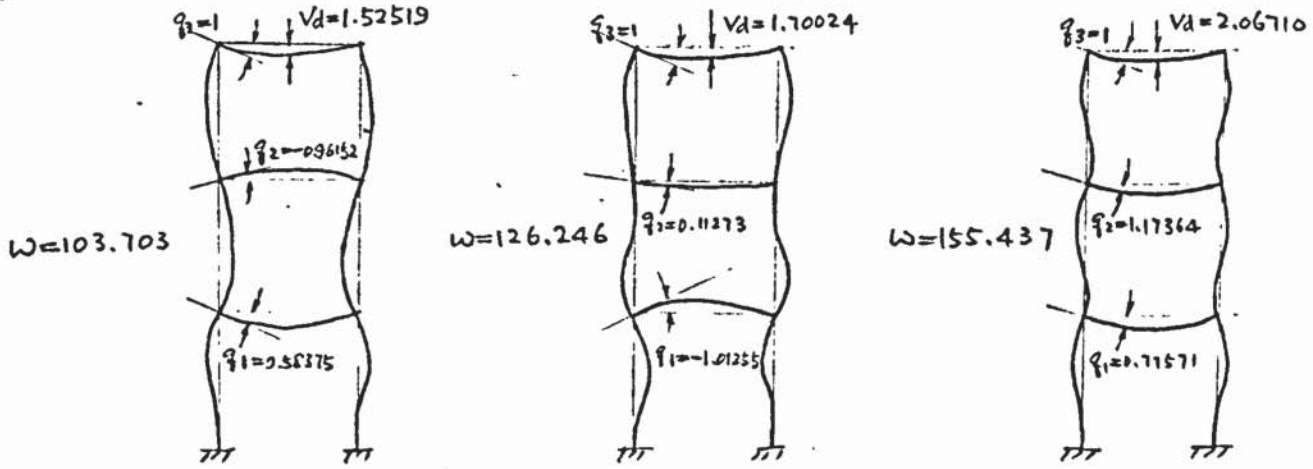


Fig (7.3.4) Symmetrical modes of free vibration

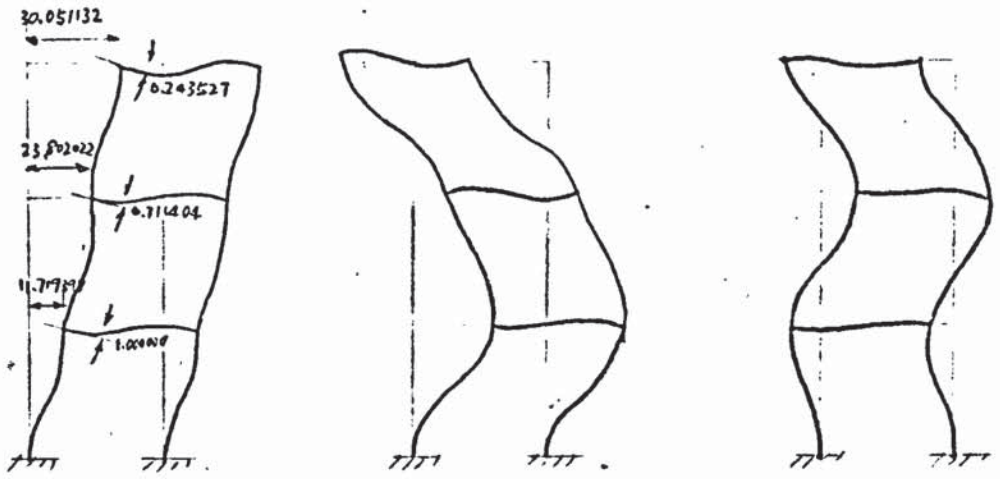


Fig (7.3.5) Antisymmetrical modes of free vibration

7.4 HARMONIC EXCITATION

Suppose the midpoint of the topmost span of the frame is subjected to a harmonic force of magnitude $P=10^3\ kg_f = 9810\ N$ and with frequency $\omega = 10 \times 2\pi s^{-1}$, as shown in fig (7.4.1). We are going to find the displacement response of the point of excitation. In this case, only symmetric vibration will be excited.

Three methods are presented here. They are the modal analysis, the method of dynamic stiffness and the combined method of dynamic stiffness and partial vibration. We shall discuss the advantages and disadvantages of each method afterwards.

MODAL ANALYSIS

In the modal analysis, the coordinate functions used are the normal modes of free vibration of the overall system $\phi_i(x)$. We first express the deflection amplitude as $u \doteq \sum_{i=1}^{3} p_i \phi_i(x)$. The generalized masses are given by $M_i = \{q\}^T [m(\omega_i)] \{q\}$.

The generalized forces are given by $P_i = \frac{1}{M_i} \sum_{\text{All members}} \int_l P(x)\, \phi_i(x)\, dx$.

For the first three modes, we have

$$M_1 = 1.573 \times 10^3 \ \ kg_f\ m\ s^2,$$

$$M_2 = 1.388 \times 10^3 \ \ kg_f\ m\ s^2,$$

$$M_3 = 2.011 \times 10^3 \ \ kg_f\ m\ s^2;$$

$$P_1 = \frac{1}{M_1} \sum \int_L p(x)\, \phi_1(x)\, dx = \frac{1}{M_1} P\, v_d = \frac{1.525}{1.573} = 0.9695\ s^{-2},$$

$$P_2 = \frac{1}{M_2} P\, v_d = \frac{1.700}{1.388} = 1.2248\ s^{-2},$$

$$P_3 = \frac{1}{M_3} P\, v_d = \frac{2.067}{2.011} = 1.0278\ s^{-2}.$$

The Lagrange's equations of motion in terms of principal coordinates are

$$p_i + \omega_i^2 p_i = P_i .$$

If steady state vibration is considered, $p_i = P_i / (\omega_i^2 - \omega^2)$, and we have

$$p_1 = 0.9695 / (103.703^2 - 62.832^2) = 1.4244 \times 10^{-4},$$

$$p_2 = 1.2248 / (126.246^2 - 62.832^2) = 1.0215 \times 10^{-4},$$

$$p_3 = 1.0278 / (155.437^2 - 62.832^2) = 0.50849 \times 10^{-4}.$$

The resultant amplitudes of rotations at the generalized coordinates are given by

$$q_1 \doteq \sum_{i=1}^{3} q_1^{(i)} p_i = (1.4244 \times 0.58375 - 1.0215 \times 1.01255 + 0.50849 \times 0.77571) \times 10^{-4} = 0.19161 \times 10^{-4} \ \text{rad},$$

$$q_2 \doteq \sum_{i=1}^{3} q_2^{(i)} p_i = -0.65765 \times 10^{-4} \ \text{rad},$$

$$q_3 \doteq \sum_{i=1}^{3} q_3^{(i)} p_i = 2.95439 \times 10^{-4} \ \text{rad},$$

and the amplitude of deflection at the midpoint of the topmost span is

$$v_d = \sum_{i=1}^{3} v_d^{(i)} p_i = (1.52519 \times 1.4244 + 1.70024 \times 1.0215 + 2.06710 \times 0.50849) \times 10^{-4} = 4.96038 \times 10^{-4}\ \text{m}.$$

DYNAMIC STIFFNESS METHOD

If the dynamic stiffness matrix is available for a system, the amplitudes of harmonic response can be solved directly from the dynamic stiffness equations. No modal information is needed. The generalized coordinates are arranged as shown in fig (7.4.1) and the dynamic stiffness equation is tabulated in fig (7.4.2). The frequency parameters are found to be

$$\lambda_1 = 0.334968\,\omega^{\frac{1}{4}} = 2.65519$$

$$\lambda_2 = 0.295155\,\omega^{\frac{1}{4}} = 2.33963;$$

and the frequency functions are calculated as

$$F_1(\lambda_1) = 2.39819$$

$$F_2(\lambda_1) = 3.48206$$

$$F_1(\lambda_2) = 2.22904$$

$$F_2(\lambda_2) = 3.69912$$

$$F_3(\lambda_2/2) = 6.05824$$

$$F_6(\lambda_2/2) = 11.30282.$$

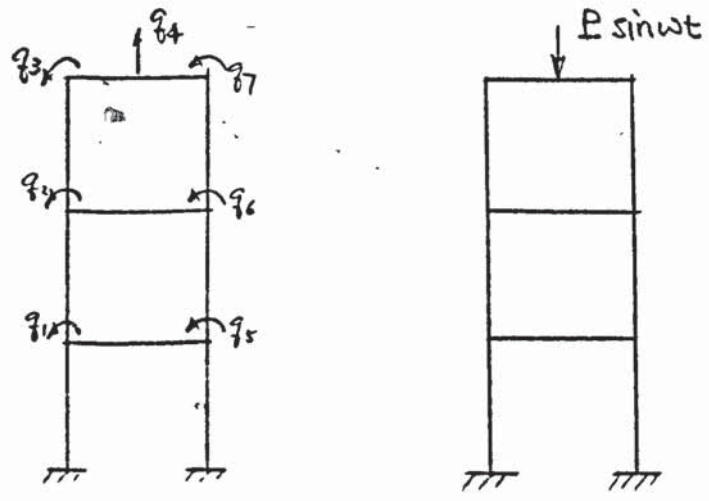


Fig (7.4.1) Generalized coordinates for the dynamic stiffness method

q_1	q_2	q_3	q_4	q_5	q_6	q_7	P
$\frac{2EI_1}{\lambda_1}F_2(\lambda_1) + \frac{EI_3}{l_2}F_2(\lambda_2)$	$\frac{EI_1}{\lambda_1}F_1(\lambda_1)$	0	0	$\frac{EI_2}{l_2}F(\lambda_2)$	0	0	0
	$\frac{2EI_1}{\lambda_1}F_2(\lambda_1) + \frac{EI_2}{l_2}F_2(\lambda_2)$	$\frac{EI_1}{\lambda_1}F_1(\lambda_1)$	0	0	$\frac{EI_2}{l_2}F_1(\lambda_2)$	0	0
		$\frac{EI_1}{\lambda_1}F_2(\lambda_1) + \frac{2EI_2}{l_1}F_2(\frac{\lambda_2}{2})$	$-\frac{EI_2}{(\frac{l_2}{2})^2}F_3(\frac{\lambda_2}{2})$	0	0	0	0
			$\frac{2EI_2}{(\frac{l_2}{2})^2}F_2(\frac{\lambda_2}{2})$	0	0	$\frac{EI_2}{(\frac{l_1}{2})^2}F_3(\frac{\lambda_1}{2})$	10^3 Kgf
				$\frac{2EI_1}{\lambda_1}F_2(\lambda_1) + \frac{EI_2}{l_2}F_2(\lambda_2)$	$\frac{EI_1}{\lambda_1}F_1(\lambda_1)$	0	0
					$\frac{2EI_1}{\lambda_1}F_2(\lambda_1) + \frac{EI_2}{l_2}F_2(\lambda_2)$	$\frac{EI_1}{\lambda_1}F_1(\lambda_1)$	0
						$\frac{2EI_2}{l_1}F_2(\frac{\lambda_2}{2}) + \frac{EI_1}{\lambda_1}F_2(\lambda_1)$	0

symmetrical

Fig (7.4.2) The dynamic stiffness equations

The solution of these equations is found to be:

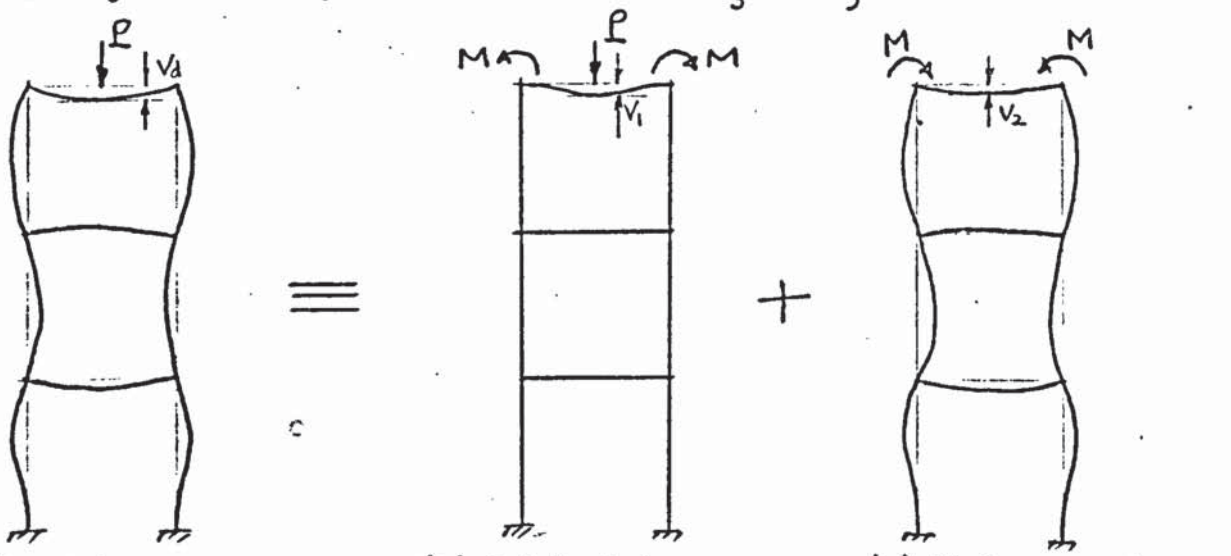
$q_1 = -q_5 = 0.16397 \times 10^{-4}$ rad., $q_2 = -q_6 = -0.71738 \times 10^{-4}$ rad.,

$q_3 = -q_7 = 2.97458 \times 10^{-4}$ rad., $v_d = 5.62803 \times 10^{-4}$ m.

METHOD OF PARTIAL VIBRATION

By the principle of superposition of linear structures, the loadings may be resolved into two systems of forces, as shown in fig (7.4.3), and the actual deflections may be obtained by summing the deflections resulting from these two individual systems. The first system consists of the original vertical force P and two fictious moments of magnitude M at the generalized coordinates q_3 and q_5 so that the deflections at these coordinates are zeros and the top span of the frame will be excited only. The other consists of the loadings which are needed to balance the fictious loadings of the first system, i.e.

two fixing moments at generalized coordinates q_3 and q_5.



(a) Original force system

(b) Original force system plus fixing moments

(c) Fixing moments at the generalized coordinates

Fig (7.4.3) Resolution of force system according to principle of superposition

The analysis of the frame work shown in (c) can be acheived by the dynamic stiffness method. The equations of dynamic stiffness for the symmetrical vibration are

$$\begin{aligned}
&\left\{\frac{2EI_2}{L_1}F_2(\lambda_1)+\frac{EI_2}{L_2}[F_2(\lambda_2)-F_1(\lambda_2)]\right\}q_1+\frac{EI_2}{L_1}F_1(\lambda_1)q_2=0,\\
&\frac{EI_2}{L_1}F_1(\lambda_1)q_1+\left\{\frac{2EI_2}{L_1}F_2(\lambda_1)+\frac{EI_2}{L_2}[F_2(\lambda_2)-F_1(\lambda_2)]\right\}q_2+\frac{EI_2}{L_1}F(\lambda_1)q_3=0,\\
&\frac{EI_2}{L_1}F_1(\lambda_1)q_2+\left\{\frac{EI_2}{L_1}F_2(\lambda_1)+\frac{EI_2}{L_2}[F_2(\lambda_2)-F_1(\lambda_2)]\right\}q_3=0.
\end{aligned} \qquad (7.4.1)$$

For the partial vibration of a single beam, one of the well known methods is the method of interior receptance (ref 56). We describe the method briefly as follows. If a harmonic force $F\sin\omega t$ is applied to the beam, shown in fig (7.4.4), at the section x=h, the deflection v satisfies the governing equation of motion of the beam. After solving the governing equation of motion, the deflection v may be expressed in terms of F by a receptance α_{xh} such that

$$v(x)=\alpha_{xh}F\sin\omega t. \qquad (7.4.2)$$

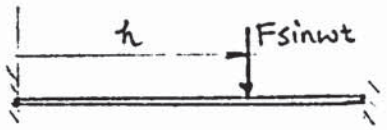


Fig (7.4.2) A clamped-clamped beam subjected to concentrated load.

Similarly, we can obtain the following relations for a clamped-clamped beam,

$$V'(x) = F \sin\omega t\, \alpha_{x'h}$$

$$V(x) = \alpha_{xh'} H \sin\omega t$$

$$V'(x) = \alpha_{x'h'} H \sin\omega t$$

where Hsinωt is an applied couple or bending moment at x=h. The receptances are symmetrical, i.e. $\alpha_{xh} = \alpha_{hx}$ etc.. The receptance for a clamped-clamped beam are given by , ($0 \le x \le h \le l$) , (ref 82),

$$\alpha_{xh} = \frac{l^3}{4EI\lambda^3 \mathcal{F}_3}\left\{-\mathcal{F}_1\mathcal{F}_8\left(\frac{\lambda x}{l}\right)\mathcal{F}_8\left[\frac{\lambda}{l}(l-h)\right] - \mathcal{F}_{10}\mathcal{F}_{11}\left(\frac{\lambda x}{l}\right)\mathcal{F}_8\left[\frac{\lambda}{l}(l-h)\right] - \mathcal{F}_{10}\mathcal{F}_{11}\left(\frac{\lambda x}{l}\right)\mathcal{F}_{11}\left[\frac{\lambda}{l}(l-h)\right] + \mathcal{F}_8\mathcal{F}_{10}\left(\frac{\lambda x}{l}\right)\mathcal{F}_{10}\left[\frac{\lambda}{l}(l-h)\right]\right\}$$

$$\alpha_{x'h} = \frac{l^2}{4EI\lambda^2 \mathcal{F}_3}\left\{-\mathcal{F}_1\mathcal{F}_{10}\left(\frac{\lambda x}{l}\right)\mathcal{F}_8\left[\frac{\lambda}{l}(l-h)\right] + \mathcal{F}_{10}\mathcal{F}_{11}\left(\frac{\lambda x}{l}\right)\mathcal{F}_8\left[\frac{\lambda}{l}(l-h)\right] - \mathcal{F}_{10}\mathcal{F}_{11}\left(\frac{\lambda x}{l}\right)\mathcal{F}_{11}\left[\frac{\lambda}{l}(l-h)\right] - \mathcal{F}_8\mathcal{F}_{11}\left(\frac{\lambda x}{l}\right)\mathcal{F}_{10}\left[\frac{\lambda}{l}(l-h)\right]\right\}$$

$$\alpha_{xh'} = \frac{l^2}{4EI\lambda^2 \mathcal{F}_3}\left\{\mathcal{F}_1\mathcal{F}_8\left(\frac{\lambda x}{l}\right)\mathcal{F}_{10}\left[\frac{\lambda}{l}(l-h)\right] + \mathcal{F}_{10}\mathcal{F}_{11}\left(\frac{\lambda x}{l}\right)\mathcal{F}_{10}\left[\frac{\lambda}{l}(l-h)\right] - \mathcal{F}_{10}\mathcal{F}_8\left(\frac{\lambda x}{l}\right)\mathcal{F}_{11}\left[\frac{\lambda}{l}(l-h)\right] + \mathcal{F}_8\mathcal{F}_{10}\left(\frac{\lambda x}{l}\right)\mathcal{F}_{11}\left[\frac{\lambda}{l}(l-h)\right]\right\}$$

$$\alpha_{x'h'} = \frac{l}{4EI\lambda\mathcal{F}_3}\left\{\mathcal{F}_1\mathcal{F}_{10}\left(\frac{\lambda x}{l}\right)\mathcal{F}_{10}\left[\frac{\lambda}{l}(l-h)\right] - \mathcal{F}_{10}\mathcal{F}_{11}\left(\frac{\lambda x}{l}\right)\mathcal{F}_{10}\left[\frac{\lambda}{l}(l-h)\right] - \mathcal{F}_{10}\mathcal{F}_{10}\left(\frac{\lambda x}{l}\right)\mathcal{F}_{11}\left[\frac{\lambda}{l}(l-h)\right] - \mathcal{F}_8\mathcal{F}_{11}\left(\frac{\lambda x}{l}\right)\mathcal{F}_{11}\left[\frac{\lambda}{l}(l-h)\right]\right\}$$

where the frequency functions are given by fig (3.6.5) and $\mathcal{F}_i = \mathcal{F}_i(\lambda)$.

We turn back to the top span of fig (7.4.3 b) now. The vertical displacement of the midspan is calculated by interior receptance for x=h=0.5 :

$$v_1 = \alpha_{xh} P$$

$$= \frac{-5.85995 \times 0.53550^2 + 5.93294 \times 1.37618 \times 0.53550 \times 2 - 4.42280 \times 1.37618^2}{4 \times 4200 \times 2.3396^3 / 5^3 \times 4.64251}$$

$$= 1.64209 \times 10^{-4} \text{ m.}$$

The bending moments at supports are given by

$$M = v_1/\alpha_{xh'} = 0.669992 \times 10^3 \text{ kg}_f\text{m.}$$

The stiffness equations of equation (7.4.1) become

$$3675\, q_1 + 840 q_2 = 0$$

$$840\, q_1 + 3675\, q_2 + 840 q_3 = 0$$

$$840\, q_2 + 2455\, q_3 = 0.66999.$$

The solution for joint rotations is

$$q_1 = 0.1640 \times 10^{-4} \quad q_2 = -0.7174 \times 10^{-4}, \quad q_3 = 2.9746 \times 10^{-4}.$$

Applying the reciprocal theorem to the states as shown in fig(7.4.3b) and fig (7.4.3c), we have

$$P \times v_2 - 2 \times M \times q_3 = 0,$$

or $$v_2 = \frac{2Mq}{P}3 = 2 \times 0.66999 \times 2.9746 \times 10^{-4} = 3.9859 \times 10^{-4} \text{ m.}$$

From the principle of superposition, the total deflection at the midspan is $$v_d = v_1 + v_2 = 5.6280 \times 10^{-4} \text{ m.}$$

COMPARISON

The numerical results of deflections by the above three methods are tab lated in fig (7.4.5).

Deflections	Modal analysis	Dynamic stiffness	Partial vibration	% error for 3 modes approx. of modal analysis
q_1 ($\times 10^{-4}$)	0.19161	0.16397	0.1640	+16.86
q_2 ($\times 10^{-4}$)	-0.65765	-0.71738	-0.7174	+ 8.33
q_3 ($\times 10^{-4}$)	2.95439	2.97458	2.9746	- 0.68
v_d ($\times 10^{-4}$)	4.96038	5.62803	5.6280	-11.86

Fig (7.4.5) Comparison of the resultant deflections

Among these three methods for the particular example, the dynamic stiffness method is the simplest in application. But in general, if the excitation is not at a fixed point or is transient, the method is inapplicable. Moreover, if the number of points of excitation on the structure is large or the loadings are continuous, the method will involve a large matrix which may not be easy to handle. For nonharmonic but periodic response analysis

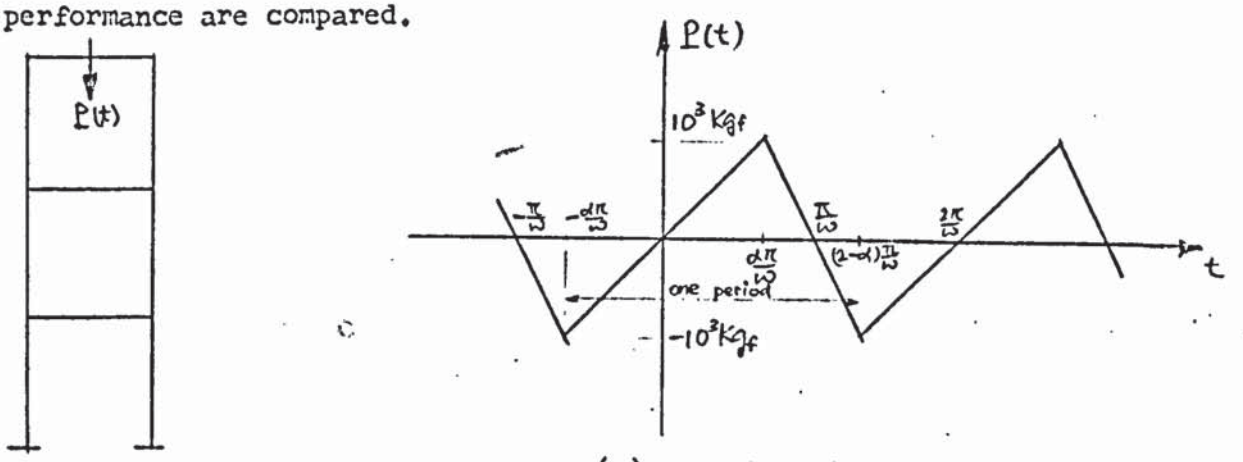
the excitation may be resolved into its harmonic components, the principle of superposition of linear structures enables the method to carry out a simple, direct and accurate analysis of the response. Examples will be given in next section.

The modal analysis is the most general and widely used method in forced vibrations. Since the Lagrange's equations of motion can be set up in terms of principal coordinates to almost all types of forcing functions, the modal analysis is capable to handle many types of response problems. In this particular example, the series does not converge very rapidly. The reason is that the excitating frequency is not near to any natural frequency of the structure. When the excitating frequency is near enough to one of the natural frequencies then even one term of the modal analysis will give very good approximation to the responses. Since the method involves the solution of free vibration modes, the integration process for generalized forces and the solution of a set of differential equations, it may not be recommended when-ever other methods are available.

Finally, the method of partial vibration is basically a mixture of the dynamic stiffness method and the method of receptance. The global vibration responses are solved by the dynamic stiffness equations and the local vibration response are calculated by the interior receptance. The loadings may be concentrated or distributed, may be forces or couples. The method is suitable for harmonic loads with complicated distribution.

7.5 PERIODIC NON-HARMONIC VIBRATIONS

We study the periodic vibration of the same structure subjected to the loadings as shown in fig (7.5.1). The time history of the excitation has the waveform of saw tooth type with angular frequency ω. All the three methods mentioned in last section will be used and their results and performance are compared.



(a) The applied force

(b) The time history of the applied force

Fig(7.5.1) Forced vibration

The forcing function P(t) is defined over the period $(-\frac{\alpha\pi}{\omega}, (2-\alpha)\frac{\pi}{\omega})$ by the equations

$$P(t)= \frac{\omega t}{\alpha\pi} \times 10^3 \text{ Kgf} \qquad -\frac{\alpha\pi}{\omega} \le t \le \frac{\alpha\pi}{\omega}$$

$$= \frac{\pi-\omega t}{\pi(1-\alpha)} \times 10^3 \text{ Kgf} \qquad \frac{\alpha\pi}{\omega} \le t \le (2-\alpha)\frac{\pi}{\omega}$$

where $0<\alpha<1$. The forcing function is firstly expressed as Fourier series as

$$P(t)=\frac{2}{\alpha(1-\alpha)\pi^2}\sum_{j=1}^{\infty}\frac{1}{j^2}\sin j\alpha\pi \sin j\omega t \times 10^3 \text{ Kgf}.$$

When $\alpha=\frac{1}{2}$, then

$$P(t) = \frac{8}{\pi^2}\left(\sin\omega t - \frac{1}{3^2}\sin 3\omega t + \frac{1}{5^2}\sin 5\omega t - \ldots\right) \times 10^3 \text{ Kg}_f$$

$$= (0.810566 \sin\omega t - 0.090063 \sin 3\omega t + 0.032423 \sin 5\omega t) \times 10^3 \text{ kg}_f \qquad (7.5.1)$$

The idea is that we calculate the response for each harmonic component of frequency ω, 3ω, 5ω respectively, and the resulting response is obtained by combining these components according to equation (7.5.1). In the following calculation, we assume ω=100 rad/sec. The transient vibration is neglected.

MODAL ANALYSIS

The first method used is the modal analysis. Similar to the last section, the generalized forces associated with each mode are

$$P_1 = \frac{1}{M_1} \sum_{\text{all members}} \int_L p(x)\, \phi(x) dx = 0.9695\ s^{-2}$$

$$P_2 = 1.2248\ s^{-2}, \quad P_3 = 1.0278\ s^{-2}.$$

The generalized forces can be resolved into harmonic components according to equation (7.5.1):for the periodic vibration:

$$P_1(t) = (0.810566 \times 0.9695 \sin\omega t - 0.090063 \times 0.9695 \sin 3\omega t + 0.032423 \times 0.9695 \sin 5\omega t)\ s^{-2}$$

$$= (0.7858434 \sin\omega t - 0.0873159 \sin 3\omega t + 0.0314337 \sin 5\omega t)\ s^{-2}.$$

The harmonic components of the generalized forces are tabelated in fig(7.5.2).

natural frequency		harmonic components		
i	ω_i	ω = 100	300	500
1	103.703	0.7858434	-0.0873159	0.0314337
2	126.246	0.9927809	-0.1103090	0.0397112
3	155.437	0.8330994	-0.0925666	0.0333240

Fig (7.5.2) The harmonic components of the generalized forces

The harmonic components of the generalized displacements for each mode are calculated according to $\rho_i = \frac{P_i}{\omega_i^2 - \omega^2}$, and are listed in fig (7.5.3).

natural frequency		harmonic components of the principal displacements			
i	ω_i	ω= 100	300	500	
1	103.703	10.41801	0.0110184	-0.001314	
2	126.246	1.67190	0.0148942	-0.001697	x 10^{-4}
3	155.437	0.58832	0.0140595	-0.001476	

Fig (7.5.3) The harmonic components of the generalized displacements

The resultant displacements are given by

$$q_j(t) = \sum_{i,n} q_j^{(i)}\, p_i^{(n)} \sin n\omega t$$

which are calculated as:

$q_1(t)$ =[(0.58375 x 10.41801 -1.01255 x1.67190 + 1 x 0.58832)sin100t

+(0.58375 x 0.0110184 -1.01255 x 0.0148942 + 1 x 0.0140595)sin300t

+(-0.58375 x 0.001314 +1.01255 x 0.001697 - 1 x 0.001476) sin500t]

x 10^{-4} rad

=4.84500 sin100t + 0.00226 sin300t - 0.00019 sin 500t x 10^{-4} rad

$q_2(t)$ =-9.13817 sin100t + 0.00759 sin300t - 0.00066 sin500t x 10^{-4} rad

$q_3(t)$ =12.67823 sin100t + 0.03997 sin300t - 0.00449 sin500t x 10^{-4} rad

$v_d(t)$ =19.85825 sin100t + 0.07119 sin300t - 0.00794 sin500t. x 10^{-4} m.

These are the responses required. We can see from the results that the higher harmonics are decaying very rapidly and the the first mode is predominant over the others. This is because that the fundamental frequency of the excitation is very near the first natural frequency of the structure.

DYNAMIC STIFFNESS METHOD

We study the same problem by the dynamic stiffness method. The generalized coordinates are arranged are listed in fig (7.4.1). The frequency parameters are found for the first three harmonic components as:

$\lambda_1(100) = 3.3497$, $\lambda_1(300) = 5.80185$, $\lambda_1(500) = 7.49016$,

$\lambda_2(100) = 2.9516$, $\lambda_2(300) = 5.11232$, $\lambda_2(500) = 6.59998$.

The frequency functions corresponding to these parameters are tabulated in fig (7.5.4).

frequency $\omega =$	frequency parameter	$F_1(\lambda)$	$F_2(\lambda)$	$F_3(\lambda)$	$F_6(\lambda)$
100	$\lambda_1 = 3.3497$	3.23326	2.45782		
	$\lambda_2 = 2.9516$	2.64908	3.16692	6.14849	10.22947
300	$\lambda_1 = 5.80185$	-6.60917	8.89283		
	$\lambda_2 = 5.11232$	-13.70682	17.76333	7.46507	-4.58235
500	$\lambda_1 = 7.49016$	-21.04805	-12.17994		
	$\lambda_2 = 6.59998$	-6.94556	-13.53302	11.00902	-38.69359

Fig (7.5.4) Frequency functions

Three sets of dynamic stiffness equations can be set up in a form similar to those in Fig (7.4.2). The generalized forces associated with q_4 are given according to equation (7.5.1) as:

$$P(100) = 0.810566 \times 10^3 \text{ kg}_f$$

$$P(300) = -0.090063 \times 10^3 \text{ kg}_f$$

$$P(500) = 0.032423 \times 10^3 \text{ kg}_f.$$

From these three sets of equations, the generalized displacements for each

harmonic component can be calculated. The results are listed in the fig (7.5.5) below.

	sin100t	sin300t	sin500t	units
$q_1=-q_5=$	4.87845	0.00031	0.00975	$\times 10^{-4}$ rad.
$q_2=-q_6=$	9.29209	0.00444	-0.01860	$\times 10^{-4}$ rad.
$q_3=-q_7=$	12.82039	0.06242	0.02574	$\times 10^{-4}$ rad.
$q_4=$	20.73848	-0.06189	-0.03391	$\times 10^{-4}$ m.

Fig (7.5.5) Responses calculated by the method of dynamic stiffness

In this method, three sets of equations of order seven must be solved. Although this method is direct, the deflections of other points than the generalized coordinates are difficult to calculate.

METHOD OF PARTIAL VIBRATION

The first step of the method of partial vibration is to resolve the original system in two parts according to the principle of superposition as shown in fig(7.4.3). The displacement amplitudes of the midspan of the top beam in fig (7.4.3b) for frequence 100,300, and 500 are obtained from interior receptance. When we take $x=h=\frac{1}{2}l_2$, the receptances are calculated as:

$$\alpha_{xh}(100) = 1.818396 \times 10^{-7},$$
$$\alpha_{xh}(300) = 4.059270 \times 10^{-7},$$
$$\alpha_{xh}(500) = -0.480832 \times 10^{-7}.$$

Therefore the displacement at this point is given by

$$v_1 = (0.810566 \times 1.818396 \sin 100t - 0.090063 \times 4.059270 \sin 300t - 0.032423 \times 0.480832 \sin 500t) \times 10^{-4} \text{ m.}$$

$$= (1.47393 \sin 100t - 0.36559 \sin 300t - 0.01559 \sin 500t) \times 10^{-4} \text{ m.}$$

The harmonic components of the fixing moment are obtained from $M = v_1 / \alpha_{xh}$, where $x = \frac{1}{2}\lambda$ and $h = 0$. Therefore

$M(t) = (0.608995\sin 100t + 0.183417\sin 300t - 0.011531\sin 500t) \times 10^3$ $kg_f m$.

Three sets of dynamic stiffness equations are obtained for the system of fig (7.4.3c):

$$\begin{aligned} 2155.46q_1 + 1131.64q_2 &= 0 \\ 1131.64q_1 + 2155.46q_2 + 1131.64q_3 &= 0 \\ 1131.64q_2 + 1295.22q_3 &= 0.608995 \end{aligned}$$

$$\begin{aligned} 32659.9q_1 - 2313.21q_2 &= 0 \\ -2313.21q_1 + 32659.9q_2 - 2313.21q_3 &= 0 \\ -2313.21q_2 + 29547.4q_3 &= 0.183417 \end{aligned}$$

$$\begin{aligned} 14059.4q_1 + 7366.8q_2 &= 0 \\ 7366.8q_1 + 14059.4q_2 + 7366.8q_3 &= 0 \\ 7366.8q_2 + 9796.4q_3 &= -0.011531 \end{aligned}$$

The solutions are tabulated in fig (7.5.6).

	sin100t	sin300t	sin500t	
$q_1 =$	4.87845	0.0003147	0.009751	
$q_2 =$	9.29209	0.0044436	-0.018609	$\times 10^{-4}$ rad.
$q_3 =$	12.82039	0.0624235	0.025765	

Fig (7.5.6) Harmonic components of generalized coordinates by method of partial vibration

The deflection v_2 is obtained by the reciprocal theorem when applying to the states as shown in fig(7.4.3b) and fig(7.4.3c) for each harmonic component, i.e.

$$v_2 = \frac{2 \times M \times q_3}{P}$$

Therefore,

$v_2 = v_2^{(1)} \sin 100t + v_2^{(2)} \sin 300t + v_2^{(3)} \sin 500t$

$= (19.26445 \sin 100t - 0.254256 \sin 300t - 0.0183265 \sin 500t) \times 10^{-4}$ m.

The total deflection of the midspan of the top beam is

$v_d = v_1 + v_2 = (20.73848 \sin 100t - 0.619846 \sin 300t - 0.0339165 \sin 500t) \times 10^{-4}$m.

When comparing the results from these three methods, the responses for the first harmonic components are coincident. For the higher harmonics, the results from the modal analysis are a bit out of way from the other two methods. It is because the natural frequencies of the representating modes are not near any of the frequencies of the higher harmonics.

7.6 TRANSIENT RESPONSE

In this section, examples are given to the vibration problems of determining the response of a structure when the forcing functions are applied during a finite interval of time, and are removed hereafter. The response is called transient.

Now, the domain of interest is time variable. The methods such as dynamic stiffness which are based on frequency variable are not as convenient as the method of modal analysis. We shall discuss only the method of modal analysis to the transient response.

In the analysis, the displacement response $\{u(x,t)\}$ is first expressed in terms of principal coordinates as:

$$\{u(x,t)\} = \sum_{i=1}^{n} p_i(t)\{\phi_i(x)\} \qquad (7.6.1)$$

where $p_i(t)$ is the principal displacement of the ith mode of the natural vibration, $\{\phi_i(x)\}$ is the corresponding modal displacement, n is the number of terms taken.

For the generalized forces $P_i(t)$ of the ith mode, we have, from equation(7.2.11),

$$P_i = \int_{vol} \{\ddot{x}\}^T \{\phi_i\}\, dvol + \int_S \{\bar{T}\}^T \{\phi_i\}\, dS$$

$$i=1,2, \ldots ,n \qquad (7.6.2)$$

Then the principal coordinates are governed by equations(7.2.12), i.e.

$$\ddot{p}_i + \omega_i^2 p_i = P_i \qquad (7.6.3)$$

with initial conditions

$$p_i(0) = \frac{1}{M_i} \int_{vol} \{\phi_i\}^T [\rho] \{u_0\}\, dvol$$

$$\dot{p}_i(0) = \frac{1}{M_i} \int_{vol} \{\phi_i\}^T [\rho] \{\dot{u}_0\}\, dvol \qquad (7.6.4)$$

Return to our typical frame shown in fig (7.3.1). Suppose now a concentrated force of magnitude $F(t)x10^3 kg_f$ is acting at the midspan of the topmost beam vertically. The vibration is symmetrical again. For the first three symmetrical modes, we have

$$p_1 + 103.703^2\, p_1 = 0.9695\, F(t)$$
$$p_2 + 126.246^2\, p_2 = 1.2248\, F(t)$$
$$p_3 + 155.437^2\, p_3 = 1.0278\, F(t). \qquad (7.6.5)$$

If the initial conditions are such that the structure is at rest initially, then the solution for the principal coordinates are found to be

$$p_1(t) = 0.9695 \int_o^t \sin\,(103.703(t-\tau))\, F(\tau)\, d\tau$$
$$p_2(t) = 1.2248 \int_o^t \sin\,(126.246(t-\tau))\, F(\tau)\, d\tau$$
$$p_3(t) = 1.0278 \int_o^t \sin\,(155.437(t-\tau))\, F\,(\tau)\, d\tau \qquad (7.6.6)$$

The integrals in equations(7.6.6) are evaluated for some typical functions F(t), (ref 8), and they are listed in fig (7.6.1) for reference purposes.

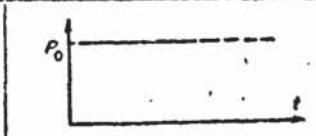
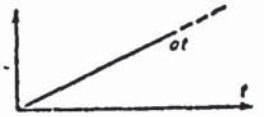
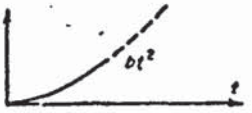
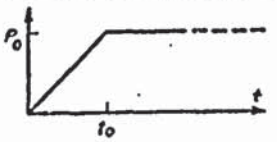
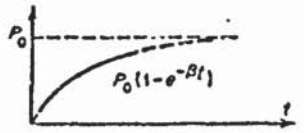
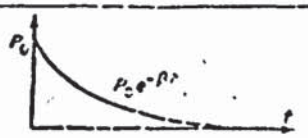
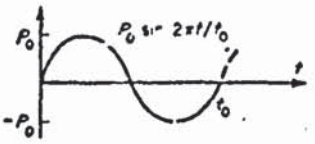
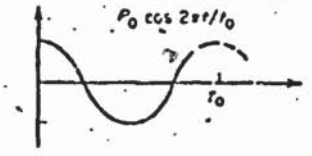
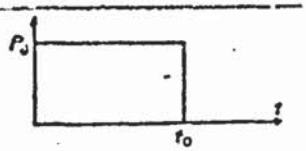
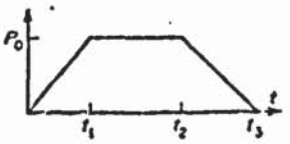
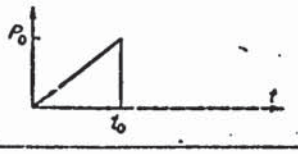
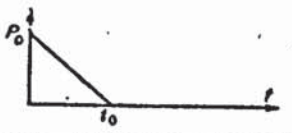
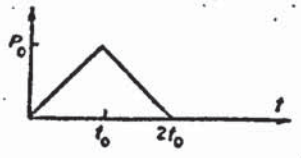
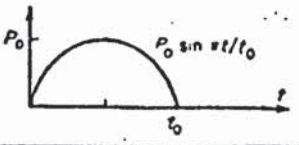
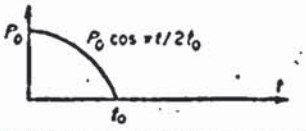
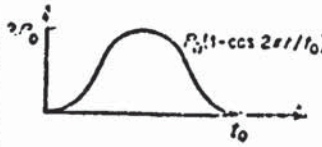
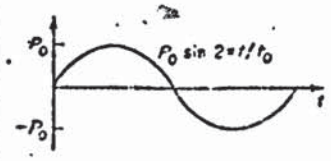
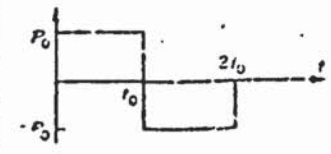
1		$\frac{P_0}{\omega}(1-\cos\omega t)$
2		$\frac{a}{\omega}\left(t-\frac{\sin\omega t}{\omega}\right)$
3		$\frac{b}{\omega}\left(t^2+\frac{2\cos\omega t}{\omega^2}-\frac{2}{\omega^2}\right)$
4		$\frac{P_0}{\omega t_0}\left(t-\frac{\sin\omega t}{\omega}\right)$ $t<t_0$ $\frac{P_0}{\omega t_0}\left[t_0+\frac{\sin\omega(t-t_0)}{\omega}-\frac{\sin\omega t}{\omega}\right]$ $t>t_0$
5		$\frac{P_0}{\omega}(1-\cos\omega t)+\frac{P_0\omega}{\omega^2+\beta^2}\left(-e^{-\beta t}+\cos\omega t-\frac{\beta\sin\omega t}{\omega}\right)$
6		$\frac{P_0\omega}{\omega^2+\beta^2}\left(e^{-\beta t}-\cos\omega t+\frac{\beta\sin\omega t}{\omega}\right)$
7		$\frac{P_0t_0}{\omega^2t_0^2-4\pi^2}\left(\omega t_0\sin 2\pi\frac{t}{t_0}-2\pi\sin\omega t\right)$
8		$\frac{P_0\omega t_0^2}{\omega^2t_0^2-4\pi^2}\left(\cos 2\pi\frac{t}{t_0}-\cos\omega t\right)$
9		$\frac{P_0}{\omega}(1-\cos\omega t)$ $t<t_0$ $\frac{P_0}{\omega}[\cos\omega(t-t_0)-\cos\omega t]$ $t>t_0$
10		$t<t_2$; $t=t_0$ $\frac{P_0}{\omega^2t_1}[\omega t_1+\sin\omega(t-t_1)-\sin\omega t]-\frac{P_0}{\omega^2(t_3-t_2)}[\omega(t-t_2)-\sin\omega(t-t_2)]$ $t_2<t<t_3$ $\frac{P_0}{\omega}\left[\frac{\sin\omega(t-t_1)}{\omega t_1}-\frac{\sin\omega t}{\omega t_1}-\frac{\sin\omega(t-t_2)}{\omega(t_3-t_2)}+\frac{\sin\omega(t-t_2)}{\omega(t_3-t_2)}\right]$ $t>t_3$

FIG (7.6.1) Duhamel Integrals for some special functions

11		$\frac{P_0}{\omega t_0}\left(t-\frac{\sin\omega t}{\omega}\right)$	$t<t_0$
		$\frac{P_0}{\omega t_0}\left[t_0\cos\omega(t-t_0)+\frac{\sin\omega(t-t_0)}{\omega}-\frac{\sin\omega t}{\omega}\right]$	$t>t_0$
12		$\frac{P_0}{\omega}\left(1-\cos\omega t-\frac{t}{t_0}+\frac{\sin\omega t}{\omega t_0}\right)$	$t<t_0$
		$\frac{P_0}{\omega}\left[-\cos\omega t-\frac{\sin\omega(t-t_0)}{\omega_0 t_0}+\frac{\sin\omega t}{\omega_0 t_0}\right]$	$t>t_0$
13		$\frac{P_0}{\omega t_0}\left(t-\frac{\sin\omega t}{\omega}\right)$	$t<t_0$
		$\frac{P_0}{\omega t_0}\left[2t_0-t+\frac{2\sin\omega(t-t_0)}{\omega}-\frac{\sin\omega t}{\omega}\right]$	$t_0<t<2t_0$
		$\frac{P_0}{\omega^2 t_0}[2\sin\omega(t-t_0)-\sin\omega(t-2t_0)-\sin\omega t]$	$t>2t_0$
14		$\frac{P_0 t_0}{\omega^2 t_0^2-\pi^2}\left(\omega t_0\sin\pi\frac{t}{t_0}-\pi\sin\omega t\right)$	$t<t_0$
		$\frac{-P_0\pi t_0}{\omega^2 t_0^2-\pi^2}[\sin\omega(t-t_0)+\sin\omega t]$	$t>t_0$
15		$\frac{4P_0\omega t_0^2}{4\omega^2 t_0^2-\pi^2}\left(\cos\frac{\pi t}{2t_0}-\cos\omega t\right)$	$t<t_0$
		$\frac{-4P_0 t_0^2}{4\omega^2 t_0^2-\pi^2}\left[\frac{\pi}{2t_0}\sin\omega(t-t_0)+\omega\cos\omega t\right]$	$t>t_0$
16		$\frac{P_0}{\omega}(1-\cos\omega t)-\frac{P_0\omega t_0^2}{\omega^2 t_0^2-4\pi^2}\cdot\left(\cos\frac{2\pi t}{t_0}-\cos\omega t\right)$	$t<t_0$
		$\frac{P_0}{\omega}\left\{\cos\omega(t-t_0)-\cos\omega t-\frac{\omega^2 t_0^2}{\omega^2 t_0^2-4\pi^2}\right.$	
		$\left.\times[\cos\omega(t-t_0)-\cos\omega t]\right\}$	$t>t_0$
17		$\frac{P_0 t_0}{\omega^2 t_0^2-4\pi^2}\left(\omega t_0\sin\frac{2\pi t}{t_0}-2\pi\sin\omega t\right)$	$t<t_0$
		$-\frac{2\pi P_0 t_0}{\omega^2 t_0^2-4\pi^2}[\sin\omega(t-t_0)-\sin\omega t]$	$t>t_0$
18		$\frac{P_0}{\omega}(1-\cos\omega t)$	$t<t_0$
		$\frac{P_0}{\omega}[2\cos\omega(t-t_0)-\cos\omega t-1]$	$t_0<t<2t_0$
		$\frac{P_0}{\omega}[2\cos\omega(t-t_0)-\cos\omega t-\cos\omega(t-2t_0)]$	$t>2t_0$

FIG(7.6.1) Continue

7.7 EXTENSION TO THREE DIMENSIONAL STRUCTURES

In the previous sections, the effects of flexural vibration were considered only. Now the calculation is extended to three dimensional frames where the effects of axial and torsional vibrations as well as flexural are considered. We take the frame work shown in Fig(7.7.1) as example. The detailed information about the frame work can be found in section (6.2.13) example 3. The first eleven modes of natural vibration are given in Fig(6.2.11). When a torsional moment of magnitude Q=100,000 sin10t N is acting at point 1, and the translational displacement response at point 5 in $\bar{X}$ direction is required, then the modal analysis using the first five modes is carried out as follows.

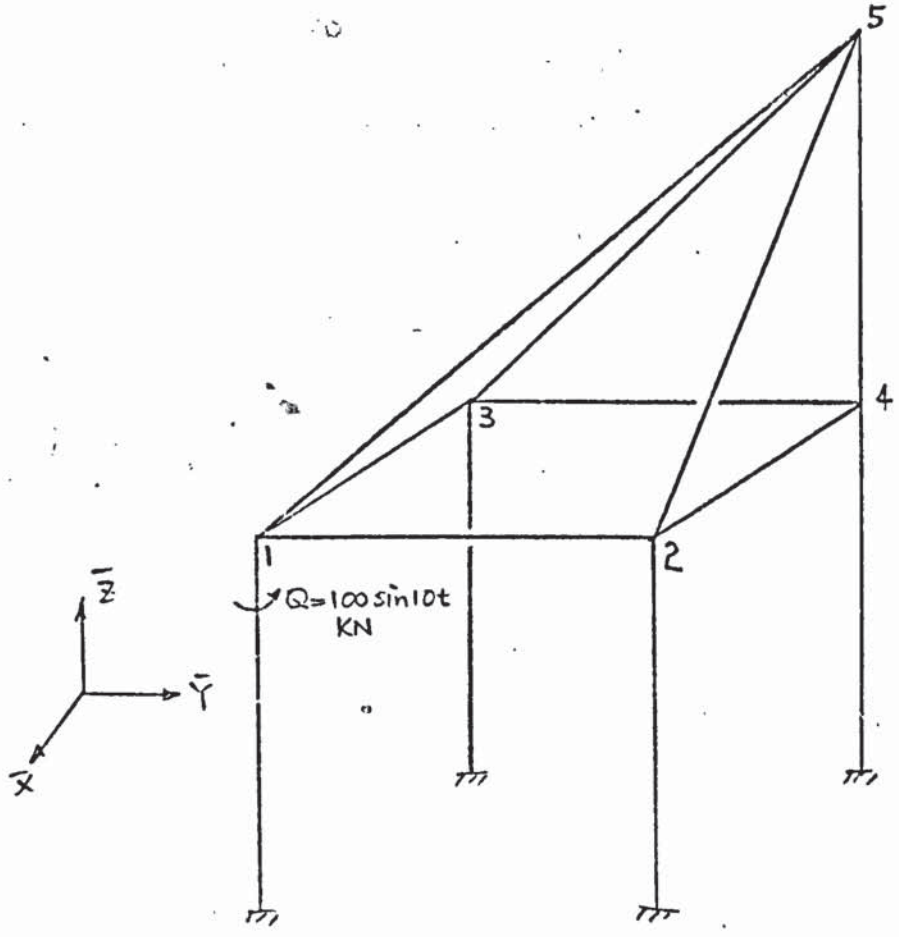


Fig (7.7.1) The reponse analysis of a space frame

The deflection of point 5 in $\bar{X}$ direction is expressed as

$$u(at\,5) = \sum_{i=1}^{5} P_i \Phi_i(at\,5) \qquad (7.7.1)$$

The translational modal displacements ϕ_i(at5) in $\bar{X}$ direction and the rotational modal displacements ϕ_i(at 1) in $\bar{Z}$ direction are tabulated in the third and fourth columns of Fig (7.7.2). The amplitudes of the generalized forces, $P_i = Q\,\phi_i$(at 1), are listed in Fig(7.7.2) column five. The amplitudes of the generalized coordinates, $P_i = \hat{P}_i/(\omega_i^2-\omega^2)$, where $\omega = 10$ are calculated in column six. The amplitude of the translational displacement reponse at point 5 in X direction is given by equation (7.7.1) as:

$$u = (17.36*4.369 - 2.364*10.11 - 4.453*11.62 + 1.645*.3753 - 5.345*.3707)*10^{-2}\ m$$

$$= -1.162*10^{-2}\ m.$$

The negative sign denotes 180° out of phase.

For more complicated forcing functions, calculation by hand seems very tedious. Therefore a general computer programme for the modal analysis of deterministic responses was designed. This programme is taking account of modal damping as well. The programme was presented in section (6.3).

mode number	nat. freq. c/s	tran. disp. at pt.5 in $\bar{X}$ dirn. $*10^{-4}$	rot. disp. at pt. 1 in $\bar{Z}$ dirn. $*10^{-4}$	gen. forces	gen. coord. $*10^{-2}$
1	11.39	17.36	.1299	1.299	4.369
2	11.50	-2.364	.3260	3.260	10.11
3	19.96	-4.453	3.467	34.67	11.62
4	40.40	1.645	.5375	5.375	.3753
5	47.25	-5.345	.7906	7.906	.3707

Fig (7.7.2) The harmonic analysis of a space frame

7.8 DAMPED VIBRATIONS

Any real structure will subject to some degree of damping effects. Internal damping will be considered in the section. The internal energies of an elastic system in terms of modal components have the forms,

$$T=\tfrac{1}{2}\sum_{i=1}^{n} p_i^{\,2}\, M_i$$
$$U=\tfrac{1}{2}\sum_{i=1}^{n} p_i^{\,2}\, {}_i^{\,2}\, M_i \qquad (7.8.1)$$

where M_i is the generalized mass.

The energy loss because of the damping effects can be expressed in the form (ref 10), $D=-\frac{1}{2}\sum_{i,j=1}^{n} b_{ij}p_i p_j$ (7.8.2)

where b_{ij} are damping coefficients. In general, the damping coefficients are functions of frequency and are coupling all the modes. In practice, the damping coefficients are difficult to determine. For metallic material the damping effects are small. In this case, the coupling effects are negligible. Therefore the consideration of the modal damping only will give a good estimation of the effects of damping. When modal damping is consider, the energy lost is given by

$$D=-\sum_{i=1}^{n} \omega_{bi} p_i^{\,2} \qquad (7.8.3)$$

where ω_{bi} is the damping frequency of mode i. The Lagrange's eqs of motion are;

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{p}_i}-\frac{\partial L}{\partial p_i}=\frac{\partial D}{\partial \dot{p}_i}+P_i \qquad i=1,2,\cdots n. \qquad (7.8.4)$$

Substituting equations (7.8.1) and (7.8.3) into (7.8.4), we have

$$\ddot{p}_i+2\omega_{bi}\omega_i\dot{p}_i+\omega_i^2 p_i=P_i \qquad i=1,2,\cdots n. \qquad (7.8.5)$$

When the loading is a harmonic one,

$$P_i(t)=Q_i\sin\omega t$$

the steady-state solution is

$$p_i(t)=Q_i\sin(\omega t+\varphi_i)/[(\omega_i^2-\omega^2)+4\omega^2\omega_{bi}^2]^{\frac{1}{2}} \qquad (7.8.6)$$

where the angle of lead φ_i is defined as

$$\varphi_i=-\tan^{-1}\frac{2\omega\omega_{bi}}{\omega_i^2-\omega^2} \qquad (7.8.7)$$

When the forcing functions $P_i(t)$ are not harmonic , then the following set of ordinary differential equations must be solved with some specified initial conditions,

$$\ddot{p}_i + 2\omega_i\omega_i\dot{p}_i + \omega_i^2 p_i = P_i \qquad i=1,2,\cdots n . \tag{7.8.8}$$

An ordinary differential equation of the form

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = f(t) \tag{7.8.9}$$

can be solved by means of Laplace's method. Take the Laplace transform of equation (7.8.9)

$$s^2\bar{x} + 2\zeta\omega\bar{x}s + \omega^2\bar{x} = \bar{f}(s) + sx_0 + x_0' + 2\zeta\omega x_0$$

where $x_0 = x(0)$, $x_0' = x'(0)$

and $\mathcal{L}\{x\} = \bar{x}$ etc.

Therefore,

$$\bar{x} = [\bar{f}(s) + (s+\zeta\omega)x_0 + (x_0' + \zeta\omega x_0)]/(s^2 + 2\zeta\omega s + \omega^2) \tag{7.8.10}$$

Take the inverse transform of equation (7.8.10),

$$x(t) = \frac{1}{a}\int_0^t f(\tau)e^{b(t-\tau)}\sin a(t-\tau)d\tau + x_0 e^{bt}\cos at + (x_0' - bx_0)\frac{1}{a}e^{bt}\sin at \tag{7.8.11}$$

where $a = \omega\sqrt{1-\zeta^2}$,

and $b = -\zeta\omega$.

For velocity $x'(t)$, by the formula , $\mathcal{L}\{x'(t)\} = s\bar{x} - x_0$,

$$\mathcal{L}\{x'\} = [(s+\zeta\omega)\bar{f}(s) - \zeta\omega\bar{f}(s) + (s+\zeta\omega)x_0' - (\zeta\omega x_0' + \omega^2 x_0)]/(s^2 + 2\zeta\omega s + \omega^2),$$

$$\therefore \quad x'(t) = \int_0^t e^{b(t-\tau)}\sin a(t-\tau)f(\tau)d\tau + \frac{b}{a}\int_0^t e^{b(t-\tau)}\sin a(t-\tau)f(\tau)d\tau + x_0' e^{bt}\cos at + (bx_0' - \omega^2 x_0)\frac{1}{a}e^{bt}\sin at. \tag{7.8.12}$$

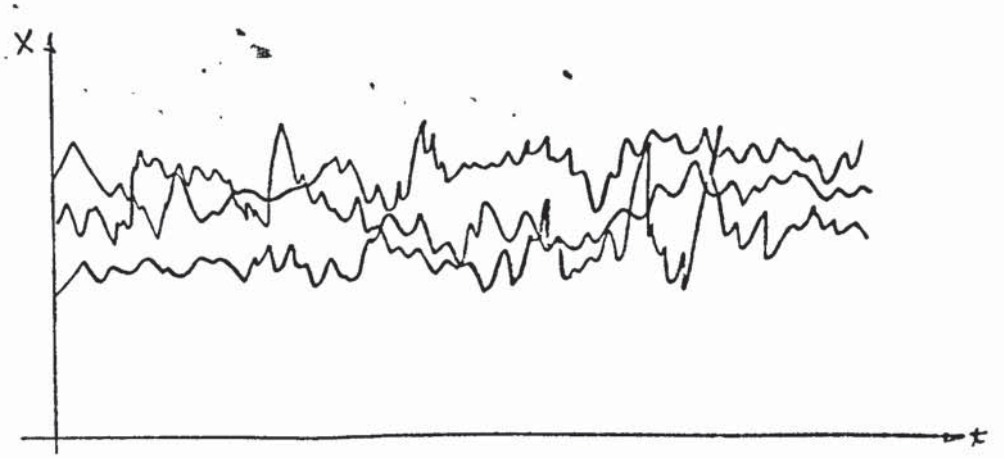
A computer programme was designed for computing the damped reponse of a three dimensional structure by arbitrary excitations. This can be found in section (6.3), where numerical examples are also given.

7.9 RANDOM VIBRATION

The examples of the forced vibrations in the previous sections are termed deterministic since the magnitudes of the loadings are predetermined functions of time and space coordinates. In the following sections we shall discuss the topics under the heading of random vibrations. The forcing functions are in statistical forms and the problem is to determine the correlation between the forcing functions and the reponse. For a comprehensive study of random vibrations, the text by Robson (ref 109) should be recommended. A brief account of the study by modal analysis, i.e. the spectral method, is our concern here.

We begin with some necessary definitions in statistics and then the spectral method is introduced by an example of single-degree-of-freedom system. Then the method is extended to complicated elastic structural systems by means of the modal analysis. A space frame is taken as numerical example.

Function, X(t), whose value is a random variable for fixed values of the argument, t, is called a random function, as shown in Fig (7.9.1).



Fig(7.9.1) Random function

The pressure from a gusty wind might be an example of such a function. Random functions of time are called stochastic processes. Velocity and pressure variations in a turbulant flow, the noise of a jet engine and jolts experienced by a transport mechanism moving an a rough path are all stochastic processes.

The average of a random variable, $\langle X(t)\rangle$, is defined as

$$\langle X(t)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} X(t)\,dt \qquad (7.9.1)$$

By forming the products of the random variable at different time, and averaging these products over the set of all possibilities, we obtain a sequence of functions

$$\langle X(t)\rangle\ ,\ \langle X(t_1)X(t_2)\rangle\ ,\ \langle X(t_1)X(t_2)X(t_3)\rangle\ ,\ \text{etc.} \qquad (7.9.2)$$

The first of these functions is the mathematical expectation, i.e. the average, of the stochastic function $X(t)$, depending on time as a parameter. The second function becomes the mean square $\langle X^2(t)\rangle$ at $t_1=t_2=t$. The functions

$$\begin{aligned} R_{XX}(t_1,t_2) &= \langle X(t_1)X(t_2)\rangle\ , \\ R_{XXX}(t_1,t_2,t_3) &= \langle X(t_1)X(t_2)X(t_3)\rangle\ , \qquad (7.9.3) \\ &\cdots \end{aligned}$$

describe the statistical connection, or correlation, between the values at different times and are called correlation functions. To obtain a complete picture of a given stochastic process it is necessary to know the mathematical expection and a complete system of correlation functions.

Now, let us examine the set of several stochastic functions $X_1(t)$, $X_2(t),\ldots,X_n(t)$ instead of a single function. Their correlation function are defined by mean of the expressions

$$\begin{aligned} R_{X_jX_k}(t_1,t_2) &= \langle X_j(t_1)X_k(t_2)\rangle \\ R_{X_jX_kX_l}(t_1,t_2,t_3) &= \langle X_j(t_1)X_k(t_2)X_l(t_3)\rangle \qquad (7.9.4) \\ &\cdots \end{aligned}$$

In all there are n^2 second order functions, n^s functions of order s, etc..

If the stochastic functions $Y_1(t)$, $Y_2(t)$, ..., $Y_m(t)$ are obtained by a linear transformation of the stochastic functions $X_1(t), X_2(t), \ldots, X_n(t)$,

$$Y_j(t) = \sum_{k=1}^{n} a_{jk} X_k(t) + C_j \quad j=1,2,\ldots,m \qquad (7.9.5)$$

we obtain a relationship for the second order correlation functions:

$$R_{Y_j Y_k}(t_1,t_2) = \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} a_{j\alpha} a_{k\alpha} R_{X_\alpha X_\beta}(t_1,t_2) \qquad (7.9.6)$$

The second order correlation functions for the first derivatives $\dot{X}_j(t)$ and $\dot{X}_k(t)$ of the differentiable stochastic functions $X_j(t)$ and $X_k(t)$ are determined from

$$R_{\dot{X}_j \dot{X}_k}(t_1,t_2) = \langle \dot{X}_j(t_1) \dot{X}_k(t_2) \rangle = \partial^2 R_{X_j X_k}(t_1,t_2)/\partial t_1 \partial t_2 \qquad (7.9.7)$$

Since the mathematical expection and the mean square of the stochastic functions will give an approximate picture of the random process, we shall consider the average and the second order correlation functions only.

A stochastic process is called stationary if its statistical properties are independent of time. A stationary stochastic process is a system of irregular variations around some mean value. The power distribution, i.e. the rate of change of energy, of the process in the different frequencies plays an essential part. The spectral characteristics are the basis of the spectral method of describing stationary stochastic processes. Correlation functions for stationary stochastic functions depend only on the intervals t_2-t_1, t_3-t_1, etc.. Thus the second reder correlation functions become

$$R_{X_j X_k}(\tau) = \langle X_j(t) X_k(t+\tau) \rangle \qquad (7.9.8)$$

These correlation functions possess the property of symmetry, i.e.

$$R_{X_j X_k}(\tau) = R_{X_j X_k}(-\tau) \qquad (7.9.9)$$

For j=k, they are even functions of τ , and

$$R_{X_jX_j}(0) = \langle X_j^2(t) \rangle \tag{7.9.10}$$

is the mean square. Since $R_{XX}(\tau)$ is an even function, it can be represented by using a Fourier cosine transform

$$R_{XX}(\tau) = \int_0^\infty \Phi_{XX}(\omega) \cos\omega\tau d\omega \tag{7.9.11a}$$

The inverse transform is

$$\Phi_{XX}(\omega) = \frac{2}{\pi} \int_0^\infty R_{XX}(\tau) \cos\omega\tau d\tau \tag{7.9.11b}$$

The newly introduced function $\Phi_{XX}(\omega)$, dependent on the frequency ω , is called the spectral density of $X(t)$. If the spectral density is known, the mean square value of $X(t)$ is given by

$$\langle X^2(t) \rangle = \int_0^\infty \Phi_{XX}(\omega) d\omega \tag{7.9.12}$$

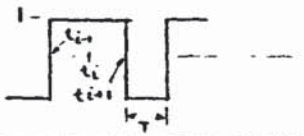
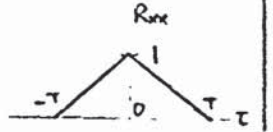
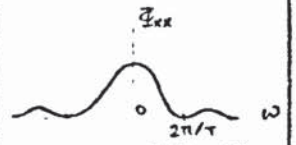
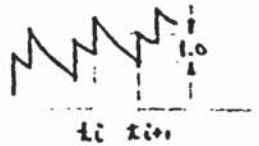
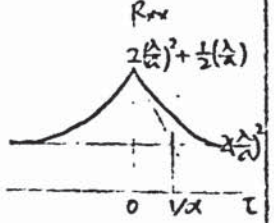
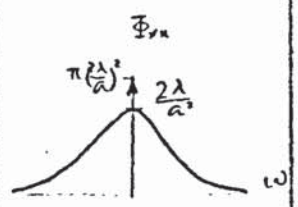
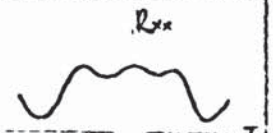
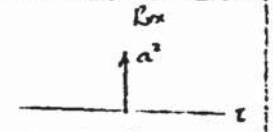
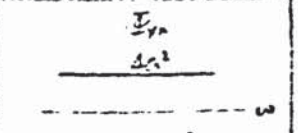
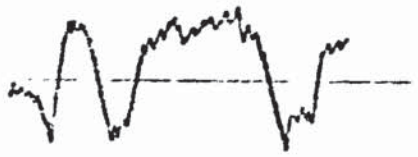
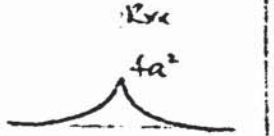
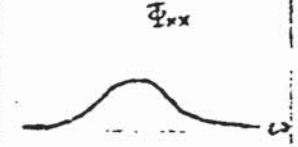
To establish the meaning of the spectral density, let us note that the quantity $\langle X^2(t) \rangle$ is proportional to the mean power of the stochastic process. Therefore the product $\Phi_{XX}(\omega) d\omega$ corresponding to that fraction of power which is included with the frequency range from ω to $\omega + d\omega$. Hence, the function $\Phi_{XX}(\omega)$ is sometimes called the spectral power, the energy spectral density, etc..

Extending to the case of several mutually correlated stochastic processes $X_1(t), X_2(t), \ldots, X_n(t)$, we have

$$R_{X_jX_k}(\tau) = \int_0^\infty \Phi_{X_jX_k}(\omega) \cos\omega\tau d\omega \tag{7.9.13}$$

$$\Phi_{X_jX_k}(\omega) = \frac{2}{\pi} \int_0^\infty R_{X_jX_k}(\tau) \cos\omega\tau d\omega \tag{7.9.14}$$

The practical determination of spectral densities by means of empirical data usually reduces to the numerical determination of the correlation functions and the execution of the Fourier transformation. The operations may be mechanized. Some spectral density curves are constructed as shown in Fig (7.9.2).

TYPICAL ENSEMBLE MEMBER DEFINITION		EXPECTATION		AUTOCORRELATION	POWER SPECTRAL DENSITY
		MEAN	VARIANCE		
1. Random Binary transmission 	$f(t)=\pm 1$; $t_i<t<(t_i+T)$ depending on the flip of a fair coin at t_i. (Ensemble members randomly phased-uniform distribution in 0,T)	0	1	 $R_{xx}(\tau)=\begin{cases}1-\lvert\tau\rvert/T\ ;\ \lvert\tau\rvert<T\\ 0\ ;\text{elsewhere}\end{cases}$	 $\Phi_{xx}(\omega)=2T\left[\frac{\sin^2(\omega T/2)}{(\omega T/2)^2}\right]$
2. Shot Noise (special case) 	$f(t)=\sum_i h(t-t_i)$; $h(t)=e^{-\alpha t}$ Where t_i are Poisson distributed points, that is $P_{t_i}(n,T)=\frac{(\lambda T)^n}{n!}e^{-\lambda T}$ $1/\lambda$ =average interval between T =some time interval n =the number of t_i in T	0	$\sqrt{2}\left(\frac{\lambda}{\alpha}\right)$	 $R_{xx}(\tau)=2\left(\frac{\lambda}{\alpha}\right)^2+\frac{1}{2}\left(\frac{\lambda}{\alpha}\right)e^{-\alpha\lvert\tau\rvert}$	 $\Phi_{xx}(\omega)=\pi\left(\frac{2\lambda}{\alpha}\right)^2\delta(\omega)+\frac{2\lambda}{\alpha}\frac{\alpha}{\omega^2+\alpha^2}$
3. Sum of Random Sine waves 	$f(t)=A_0+\sum_{k=1}^{N}A_k\cos(\omega_k t+\varphi_k)$ A_k arbitrary ω_k noncommensurate φ_k random phase angle, independent and uniformly distributed in	A_0	$A_0^2+\sum_{k=1}^{N}\frac{A_k^2}{2}$	 $R_{xx}(\tau)=A_0^2+\sum_{k=1}^{N}\frac{A_k^2}{2}\cos(\omega_k\tau)$	 $\Phi_{xx}(\omega)=\pi\sum_{k=1}^{N}A_k^2\,\delta(\omega-\omega_k)$
4. Gaussian white noise 		0	∞	 $R_{xx}(\tau)=a^2\delta(\tau)$	 $\Phi_{xx}(\omega)=4a^2$
5. Gaussian colour noise-first order system 		0	a^2	 $R_{xx}(\tau)=a^2e^{-\beta\lvert\tau\rvert}$	 $\Phi_{xx}(\omega)=4a^2\left(\frac{\beta}{\omega^2+\beta^2}\right)$

FIG(7.9.2) Representing Random functions by their autocorrelation and spectral densities

7.10 STOCHASTIC VIBRATIONS OF A SINGLE-DEGREE-OF-FREEDOM SYSTEM

Before turning to an analysis of the behaviour of elastic systems under the effect of stochastic forces, let us consider the intensively studied case of a linear vibrational system with one degree of freedom. Let f be a generalized coordinate, ω_0 the natural frequency, and ζ the damping characteristic equal to the ratio of the logarithmic damping decrement to 2π. Denoting the generalized random force by $Q(t)$, we obtain the equation of motion

$$\ddot{f} + 2\zeta\omega_0\dot{f} + \omega_0^2 f = Q(t) \qquad (7.10.1)$$

Its solution may be written by using a pulse transient function $h(t-\tau)$, equal to the value of the generalized coordinate f at time t due to a unit impulse applied at time τ,

$$f(t) = \int_{-\infty}^{t} h(t-\tau)Q(\tau)\, d\tau \qquad (7.10.2)$$

The correlation function of f is

$$R_{ff}(t_1,t_2) = \langle f(t_1) f(t_2) \rangle \qquad (7.10.3)$$

Substituting eq.(7.10.2) into (7.10.3), we have

$$R_{ff}(t_1,t_2) = \int_{-\infty}^{t_1}\int_{-\infty}^{t_2} h(t_1-\tau_1)h(t_2-\tau_2)R_{QQ}(\tau_1,\tau_2)\, d\tau_1 d\tau_2 \qquad (7.10.4)$$

where $R_{QQ}(\tau_1,\tau_2)$ is the correlation function for the generalized force. If the external loading starts to act at time t=0 and the system is at rest at t less than 0, then $t_1=t_2=0$ should be taken as the lower limits of integration in equation (7.10.4).

Let us consider a stationary stochastic process in detail. Making the change of variable in equation (7.10.4),

$$t_1-\tau_1=\tau_1', \quad t_2-\tau_2=\tau_2', \quad t_2-t_1=\tau$$

we have,

$$R_{ff}(\tau) = \int_0^\infty\int_0^\infty h(\tau_1')h(\tau_2')R_{QQ}(\tau-\tau_2'+\tau_1')\, d\tau_1' d\tau_2' \qquad (7.10.5)$$

Henceforth, we shall omit the primes in τ_i' and τ_i' .

Let us form the Fourier transforms of the left and right sides of equation (7.10.5), then

$$\int_{-\infty}^{\infty} \Phi_{ff}(\omega) e^{i\omega\tau} d\omega$$

$$= \int_{-\infty}^{\infty} d\omega \int_0^{\infty}\int_0^{\infty} h(\tau_1) h(\tau_2) \Phi_{QQ}(\omega) e^{i\omega(\tau-\tau_1+\tau_2)} d\tau_1 d\tau_2 \qquad (7.10.6)$$

Let us now introduce the transfer function of the system $F(i\omega)$, such that

$$F(i\omega) = \int_0^{\infty} h(\tau)\, e^{-i\omega\tau}\, d\omega. \qquad (7.10.7)$$

The transfer function is defined most simply as the ratio of the steady state solution of a linear system under a harmonic action $\exp(i\omega t)$ to this action. It is sometimes called the mechanical receptance.. Thus for equation (7.10.1),

$$F(i\omega) = 1/(\omega_0^2 - \omega^2 + 2i\zeta\omega\omega_0) \qquad (7.10.8)$$

This can also be obtained by substuting

$$h(\tau) = \begin{cases} 0 & \text{if } \tau \le 0 \\ \dfrac{1}{\omega_0\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_0\tau} \sin(\omega_0\sqrt{1-\zeta^2}\,\tau) & \text{if } \tau > 0 \end{cases} \qquad (7.10.9)$$

into equation (7.10.7).

Taking into account of equation (7.10.7) into equation (7.10.6), we have

$$\int_{-\infty}^{\infty} \Phi_{ff}(\omega) e^{i\omega\tau}\, d\omega = \int_{-\infty}^{\infty} F(i\omega) F(-i\omega) \Phi_{QQ}(\omega) e^{i\omega\tau} d\omega \qquad (7.10.10)$$

This relationship should be satisfied for all τ , hence

$$\Phi_{ff}(\omega) = F(i\omega) F(-i\omega) \Phi_{QQ}(\omega) \qquad (7.10.11)$$

The equation (7.10.11) establishes a very simple connection between the spectral density of the generalized force $Q(t)$ and the spectral density of the generalized coordinate $f(t)$. The simplicity of this relationship indicates the great advantages of the spectral approach as applied to stationary stochastic processes. Here, the structure of the linear system itself is not specified, it is just necessary that it be described by equations with constant coefficients.

7.11 VIBRATIONS OF AN ELASTIC SYSTEM UNDER A STATIONARY STOCHASTIC INPUT

The investigation of the behavior of an elastic system is more difficult than the simplest case presented in the last section. This is mainly because the vibrations of elastic systems are described by partial differential equations in time and spatial variables. However, the problem of solving the stochastic response of an elastic system may be reduces to systems of ordinary differential equations by expanding the desired solution as series of normal modes of the system. If we expand the displacement response $\{u(x,t)\}$ in terms of principal coordinates $p_i(t)$ and the corresponding normal modes $\{\phi_i(x,$, then

$$\{u(x,t)\} = \sum_{i=1}^{n} p_i(t)\{\phi_i(x)\} \qquad (7.11.1)$$

For the generalized forces $P_i(t)$, we have

$$P_i(t) = \int_S \{q\}^T \{\phi_i\} dS \qquad i=1,2,\cdots n \qquad (7.11.2)$$

where $\{q(x,y,z,t)\}$ is the surface force vector. The principal coordinates are governed by equations (7.8.8), which are,

$$\ddot{p}_i + 2\omega_{bi}\omega_i \dot{p}_i + \omega_i^2 p_i = P_i \qquad i=1,2,\cdots n \qquad (7.11.3)$$

where ω_i is the ith natural frequency of the system and ω_{bi} is the ith modal damping ratio.

Because the system of equations of motion are uncoupled now, it becomes possible to investigate the behavior of each generalized coordinate independently. Hence, equations (7.101) to (7.10.11) which were derived for a system with one degree of freedom remain valid for each generalized coordinate. Nevertheless, for a complete

description of the behavior of an elastic system it is generally necessary to also take account of the correlation between the various generalized coordinates. Let us examine this by considering the mean-square of the deflection $\{u(x,t)\}$, where x is a vector quantity of the three spatial variables. Now,

$$\langle \{u(x,t)\}^T \{u(x,t)\} \rangle = \sum_{j=1}^{n} \sum_{k=1}^{n} R_{p_j p_k}(t,t) \{\phi_j(x)\}^T \{\phi_k(x)\} \qquad (7.11.4)$$

where $R_{p_j p_k}(t,t)$ are second order correlation functions of the generalized coordinates $p_j(t)$. Equations for the correlation functions $R_{p_j p_k}(t_1,t_2)$ are obtained analogously to the manner in which the equations for systems with one degree of freedom have been derived. The solution of equations (7.11.3) in terms of pulse transient functions $h_k(t-\tau)$ is given as follows:

$$p_k(t) = \int_{-\infty}^{t} h_k(t-\tau)\, P_k(\tau)\, d\tau$$

Substituting this into equations (7.11.3), we obtain,

$$R_{p_j p_k}(t_1,t_2) = \int_{-\infty}^{t_1}\int_{-\infty}^{t_2} h_j(t_1-\tau_1)\, h_k(t_2-\tau_2)\, R_{P_j P_k}(\tau_1,\tau_2)\, d\tau_1 d\tau_2 \qquad (7.11.5)$$

If the external loading is a stationary stochastic function of time, we then have, by introducing the loading spectral densities $\Phi_{P_j P_k}(\omega)$ and the spectral densities of the generalized coordinates $\Phi_{p_j p_k}(\omega)$,

$$R_{p_j p_k}(\tau) = \frac{1}{2}\int_{-\infty}^{\infty} \Phi_{p_j p_k}(\omega)\, e^{i\omega\tau} d\omega \qquad (7.11.6)$$

where $$\Phi_{p_j p_k}(\omega) = F_j(-i\omega)\, F_k(i\omega)\, \Phi_{P_j P_k}(\omega) \qquad (7.11.7)$$

and $$F_j(i\omega) = 1/(\omega_j^2 - \omega^2 + 2i\omega_{bj}\omega_j\omega) \qquad (7.11.8)$$

According to the definition $R_{P_j P_k}(t_1,t_2) = \langle P_j(t_1) P_j(t_2) \rangle$

we find,

$$R_{P_j P_k}(t_1,t_2) = \iint_S \iint_S \langle \{q(x_1,t)\}^T \{q(x_2,t)\} \rangle \{\phi_j(x_1)\}^T \{\phi_k(x_1)\}\, dS_1 dS_2 \qquad (7.11.9)$$

Taking the Fourier transforms, we have,

$$\Phi_{P_j P_k}(\omega) = \iint_S \iint_S \Phi_{qq}(x_1,x_2,\omega) \{\phi_j(x_1)\}^T \{\phi_k(x_2)\}\, dS_1 dS_2 \qquad (7.11.10)$$

By substituting equations (7.11.7) to (7.11.10) into equation (7.11.6), we find the correlation functions of the generalized coordinates.

To illustrate some of the theoretical points discussed in this section let us consider a uniform beam governed by the equation,

$$EI\frac{\partial^4 w}{\partial x^4} + c\dot{w} + m\ddot{w} = q(x,t) \qquad (7.11.11)$$

where EI is the flexural rigidity, c the damping factor, m the mass density. Further assume that both ends of the beam are simply supported. Then the natural modes are given by

$$\phi_j(x) = \sin\frac{j\pi x}{l} \qquad (7.11.12)$$

where l is the span length of the beam. Note that ϕ_j is a scalar in this example since we consider the flexural deflection only. The generalized mass, generalized mass, modal damping ratio, and the natural frequency in the jth mode are, respectively,

$$m_j = \frac{ml}{2}\ ; \quad c_j = \frac{cl}{2}\ ; \quad \omega_j = \sqrt{\frac{EI}{m}}\left(\frac{j\pi}{l}\right)^2 \qquad (7.11.13)$$

Then the tranfer function F_j is given by

$$F_j(i\omega) = \frac{2}{l}\left[EI\left(\frac{j\pi}{l}\right)^4 - \omega^2 m + ic\omega\right]^{-1} \qquad (7.11.14)$$

Now we assume that the random load q(x,t) is a white noise with respect to the time parameter t, and it has an exponential-decay type of correlation function with respect to the spatial coordinate x. Then the cross spectral density of the random load is

$$\Phi_{qq}(x_1 - x_2, \omega) = \frac{\sigma^2}{2\pi}\exp\{-a|x_1 - x_2|\} \qquad (7.11.15)$$

where σ and a are constants.

Substituting equations (7.11.15), (7.11.10), (7.11.7) and (7.11.6) into equation (7.11.4), we have, the variance of deflection,

$$\sigma_w^2(x) = \langle w^2(x,t)\rangle$$

$$= \frac{32c\sigma^2}{\pi^6 m^2 EI}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sin\frac{j\pi x}{l}\sin\frac{k\pi x}{l}\left\{\frac{al}{2}\left[\frac{1}{a^2+(\frac{j\pi}{l})^2} + \frac{1}{a^2+(\frac{k\pi}{l})^2}\right]\delta_{jk} + \right.$$

$$\left. + \frac{j\pi}{l}\frac{k\pi}{l}\frac{2+e^{-al}[(-1)^{j+1}+(-1)^{k+1}]}{[a^2+(\frac{j\pi}{l})^2][a^2+(\frac{k\pi}{l})^2]}\right\} \Big/ \left[\left(\frac{EI}{m}\right)\left(\frac{\pi}{l}\right)^4(j^4 - k^4) + 2\left(\frac{c}{m}\right)^2(j^4 + k^4)\right]$$

If the standard deviation of loading, σ ,is unity, then the root mean square values of deflection along the beam are plotted in Fig(7.11.1) for different damping ratios c and decay constants a. We see that the standard deviation of response decreases in amplitude with increasing damping ratio and increasing damping ratio.

The magnitudes of E,I,m,l are assumed to be unities. The computer programme for the plots is listed in Fig(7.11.2).

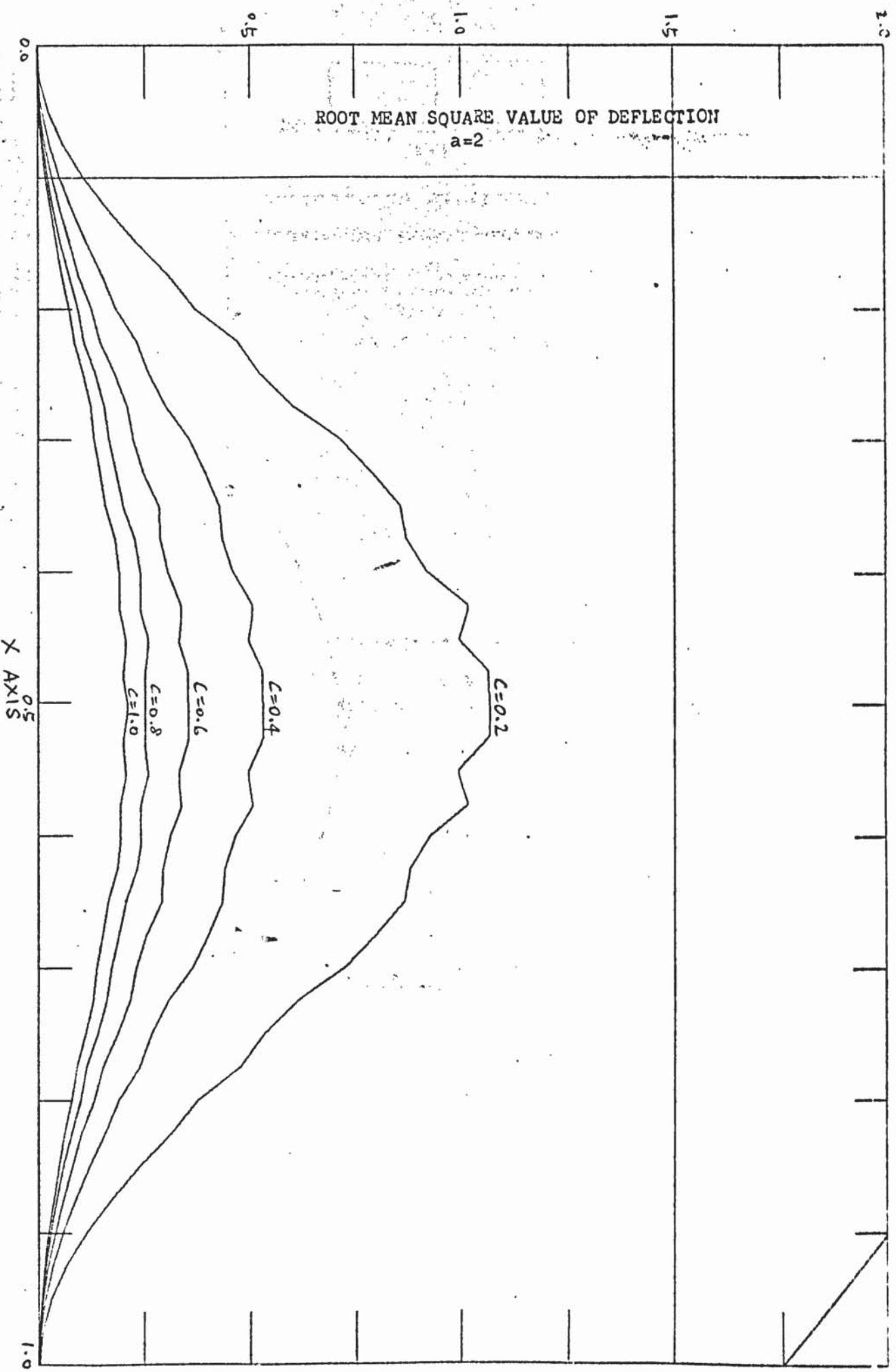


Fig (7.11.1) Random vibration of a simply supported beam

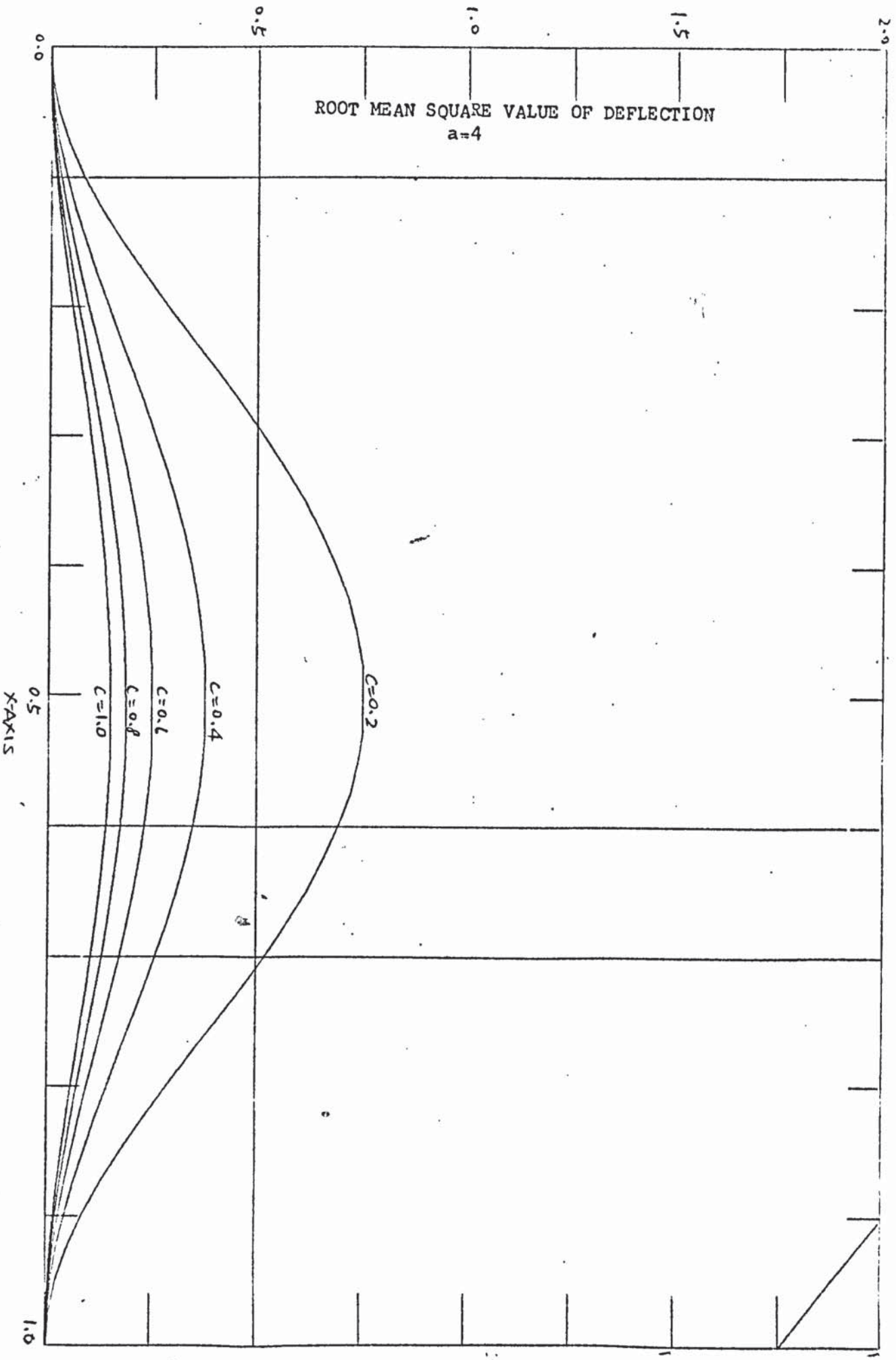


Fig (7.11.1) Continued

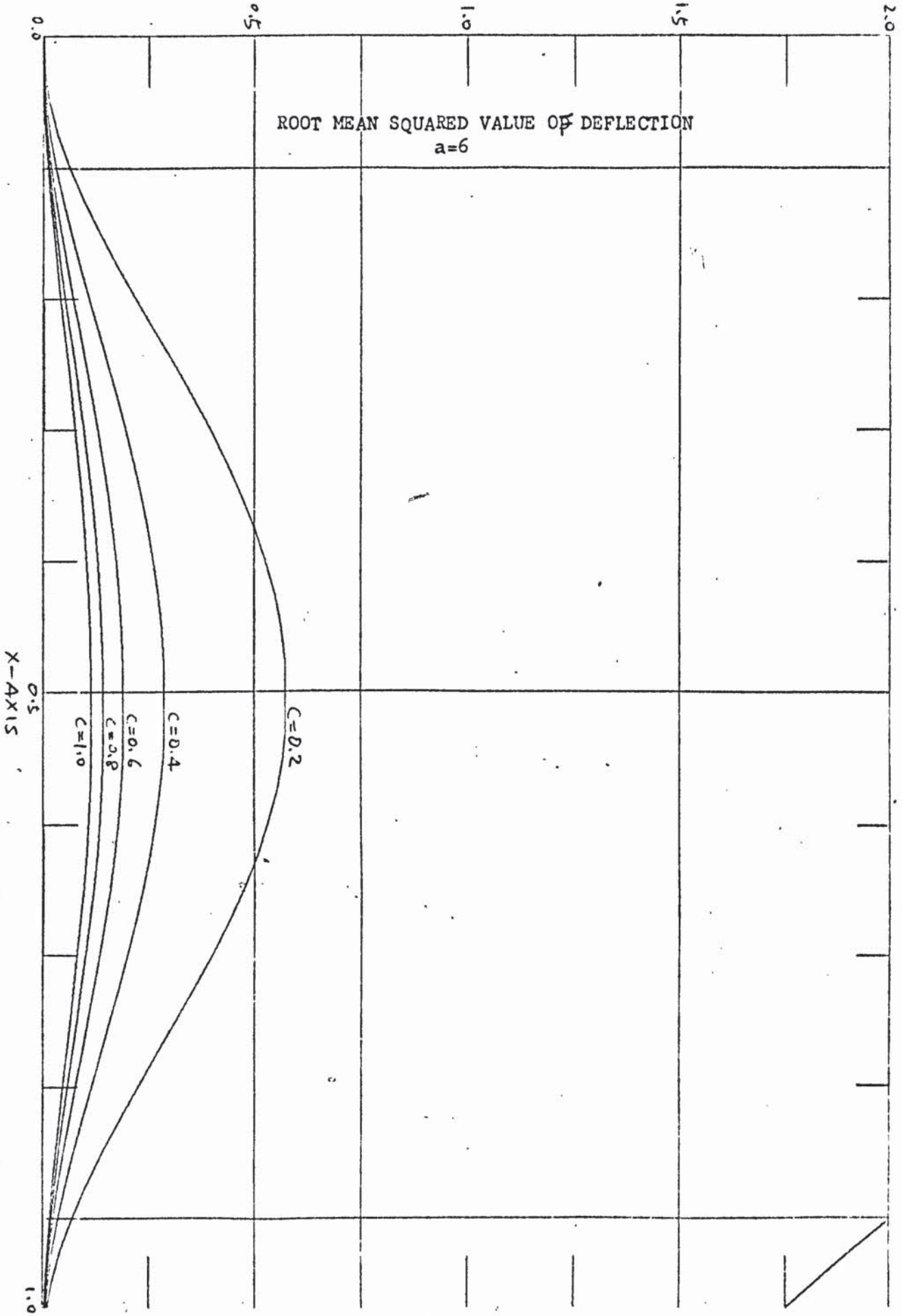


Fig (7.11.1) Continued.

```
      DOUBLE PRECISION R,S
      DIMENSION Y(41),Z(41),W(12)
      REAL L,M
      T=0.000001
      L=1.
      M=1.
      P0=3.14159265
      E1=1.
      Y(1)=0.
      Y(41)=L
      Z(1)=0.
      Z(41)=0.
      CALL PLOTTE
      CALL DEVICE(1,50)
      CALL WINDOW(2)
      CALL DEVPAP(762.,355.,)
      S0=1.
      DO 100 IA=1,6
      I=IA
      A=FLOAT(IA)
      IF (I.LE.3) Y0=105.
      IF (I.GT.3) Y0=22.
      IF (I.EQ.1.OR.I.EQ.4) X0=0.
      IF (I.EQ.2.OR.I.EQ.5) X0=260.
      IF (I.EQ.3.OR.I.EQ.6) X0=520.
      CALL GRAPAP(250.,100., X0,Y0 ,1,0.01,0.99,1,0.01,.199,2,W)
      A1=A*L
      P=P0/L
      A2=A*A
      E0=EXP(-A1)
      E2=E1/M*P*P*P*P
      DO 1 IC=2,10,2
      C=FLOAT(IC)*0.1
      C2=C/M
      C2=C2*C2
      J0=-1
      DO 2 IX=1,20
      X=FLOAT(IX)*0.025*L
      R=0.D0
      DO 3 J=1,100
      S=0.D0
      J0=-J0
      P1=P*J
      A3=1./(A2+P1*P1)
      J4=J*J*J*J
      K0=-1
      S1=SIN(P1*X)
      DO 4 K=1,10
      K4=K*K*K*K
      I=0
      IF (J.NE.K) GOTO 320
      I=1
  320 K0=-K0
      P2=P*K
      A4=1./(A2+P2*P2)
      E=E0*(J0+K0)
      S2=SIN(P2*C)
      D=S1*S2*(.5*A1*(A3+A4)*I+P1*P2*(2.+E)*A3*A4)
      D=D/(E2*(J4-K4)+2.*C2*(J4+K4))
      S=S+D
      T0=SNGL(DABS(D/S))
      IF (T0.LT.T) GOTO 420
    4 CONTINUE
  420 R=R+S
      T0=SNGL(DABS(S/R))
      IF (T0.LT.T) GOTO 450
    3 CONTINUE
  450 R=E+32.*C*S0*S0/(P0**4*M*M*E1)
      J=41-IX
      Y(J)=L-X
      Z(J)=R
      Y(IX+1)=X
      Z(IX+1)=R
    2 CONTINUE
      CALL GRAPLO(41,Y,Z,1,W)
    1 CONTINUE

FTN14 3.0033  SOURCE PROGRAM                                17/03/76  PA

  100 CONTINUE
      CALL DEVEND
      STOP
      END
```

Fig(7.11.2) Plotter programme for Random reponse of the simply supported beam

CHAPTER EIGHT

DISCUSSION

8.1 INTRODUCTION

We have presented a method to determine the natural frequencies and the associated modes of a linearly elastic structural system by means of frequency dependent fundamental matrices. The method of solution could be summarized in Fig(8.1.1). Along this flow diagram, the new studies developed during the period of research are given below:

1. The formulation of the frequency dependent mass and stiffness matrices for an elastic member of distributed mass and elasticity by means of the solution of the gorvening equations of motion.
2. The formulation of the fundamental matrices by means of the eigenfunction expansion of the elastic member with all generalized displacements set equal to zero.
3. The theorem 3.4.1 which enables one to separate the dynamic stiffness matrix into mass and stiffness matrices.
4. The proof of the Sturm sequence properties of the dynamic stiffness matrix with respect to the natural frequencies by means of the theories of determinant and the concepts of substructures.

The theorem 3.4.1 is of particularly useful when the dynamic stiffness is obtained by experiment and the equations of motion are not known or are impossible to solve. The new computer programmes

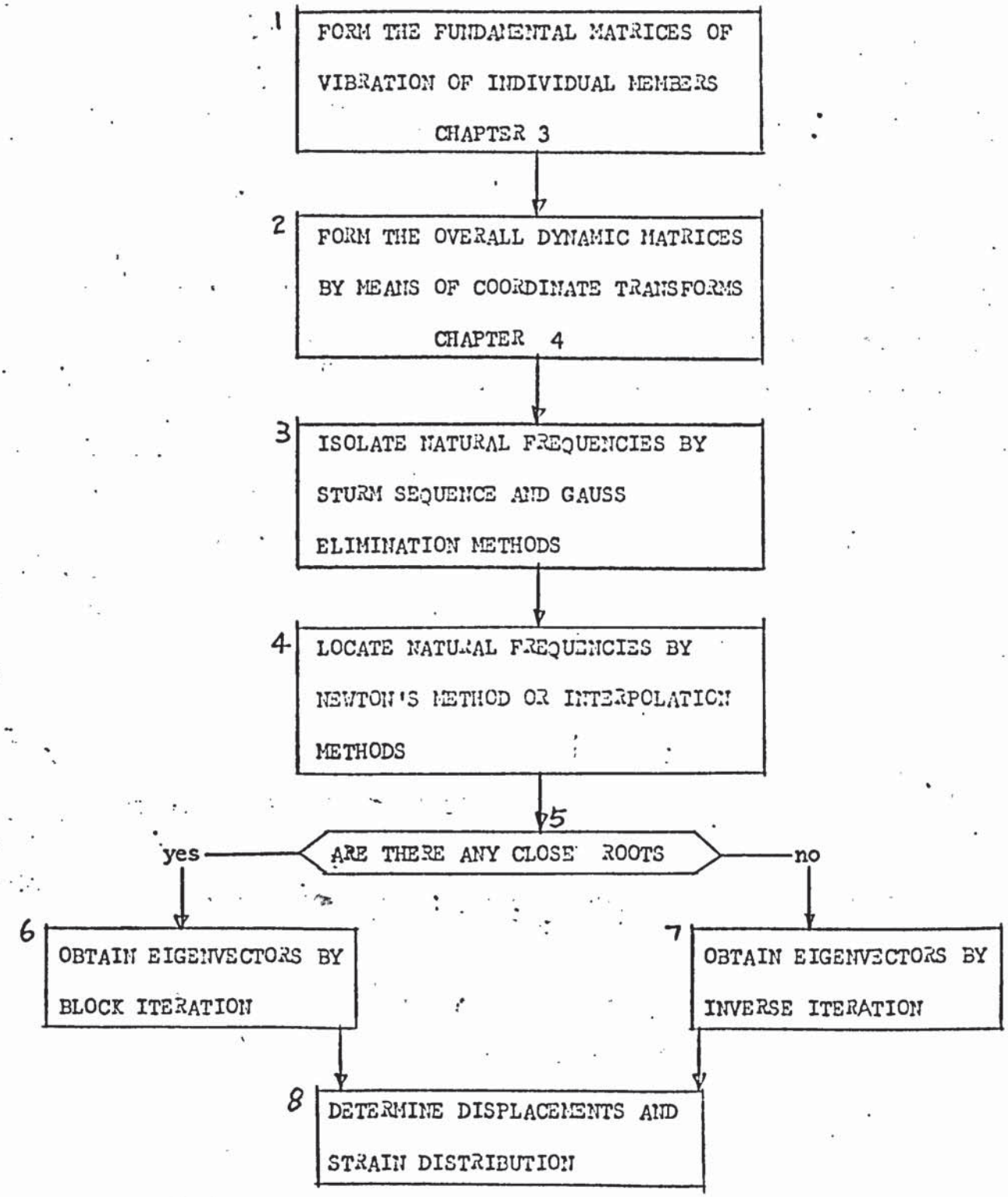


Fig (8.1.1) The overall solution of the free vibration problem

developed include an overall programme to calculate the normal modes of an arbitrary space frame, a programme making use of the normal modes thus obtained to determine the structural response to arbitrary excitations, and a package of numerical linear analysis of different forms of matrices. An experiment was conducted to verify the results obtained from the main programme for a plane frame. In the following sections we shall discuss the advantages and disadvantages of the method of solution in natural vibration problems and the possibility of developing these studies to some practical engineering problems.

8.2 COMPARISON OF THE METHODS OF SOLUTION

The frequency dependent matrices approach to the solution of natural vibration problems is like other methods and will have its advantages and disadvantages. If exact solution of the gorvening equations of motion of each member of the structure is used to contruct the fundamental matrices than exact results of the normal modes are expected. There is no restriction to the size and shape of an elemental member as long as the solution of the equations of motion with general boundary conditions can be obtained. Although we have constructed the fundamental matrices for some members such as uniform beams, tapered beams, Timoshenko columns, there is not always the case that the equations of motion may be solved. Therefore we developed another method to form the fundamental matrices by expanding the solution of equations of motion with general boundary displacement conditions by the normal modes of the member with all

generalized coordinates set to zero. The eigenfunction expansion ensures the convergence of the series solution to the real solution. Again, there is no theorectical restriction to the size and shape of an elemental member as long as the normal modes can be obtained. Because the normal modes are easier to obtain than the general solution of the equations of motion, therefore, we were able to construct the fundamental matrices for some more structural members of one- , two-, and three-dimension. It is not difficult to obtain the fundamental matrices of the frame works shown in section (6.2) examples 1, 2 and 3 when regarding these structures as single members because we have calculated the normal modes already. Of course, we can not extend the argument indefinitely, since the normal modes of a complicated structure is difficult to obtain. If it is the case, the method of finite element with constant mass and stiffness matrices may be suitable, because there does not arise the problem of solving the governing equations of motion.

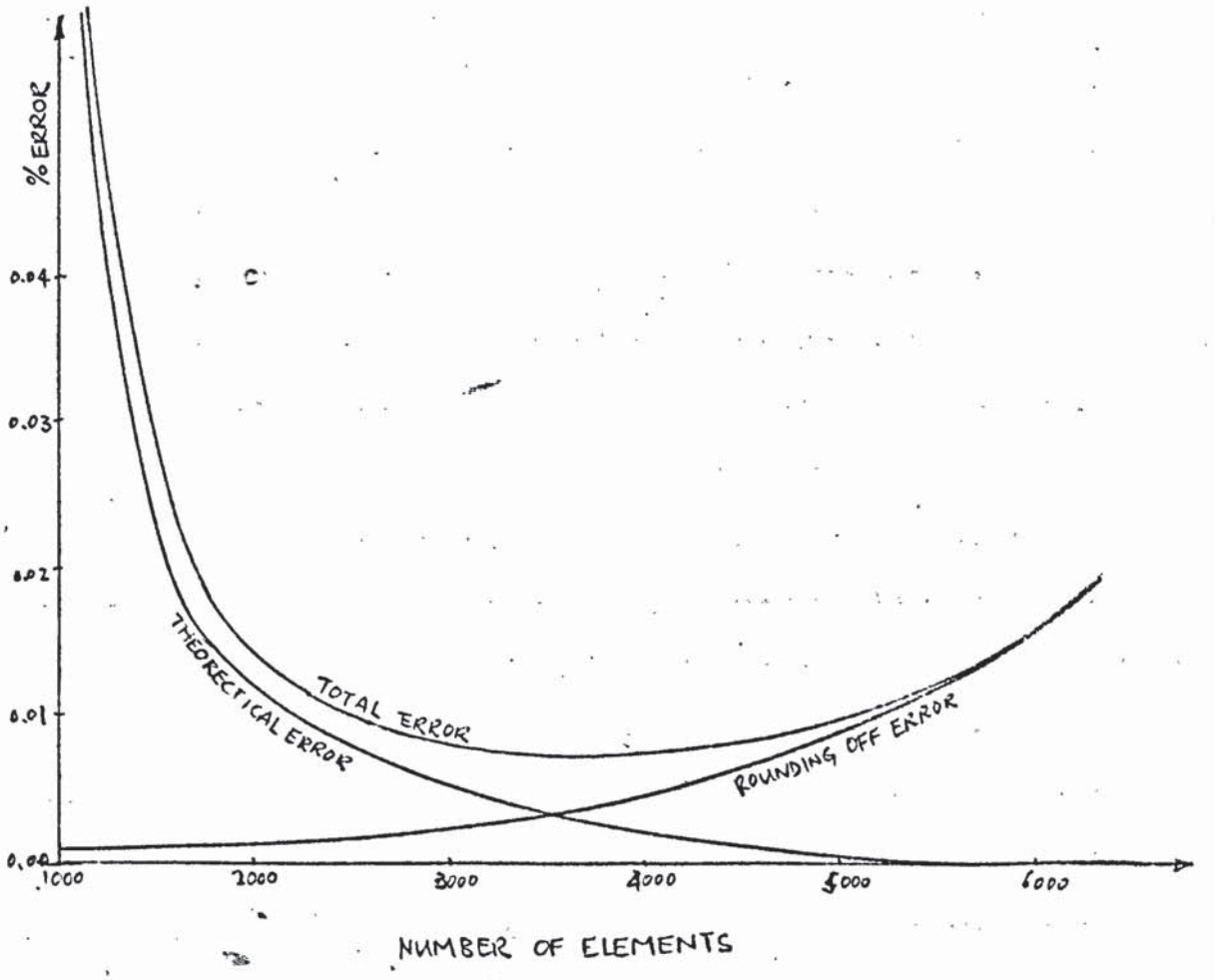
On the other hand, if we calculate the normal modes of a structure by finite element or finite difference or lumped parameters we will end up with some large overall matrices if a large number of elements are necessary to increase the accuracy or if the higher modes of vibration are required. It is very common in engineering structure that the number of elements are large, and the accurate higher modes are unavoidable during the analysis of the sensitivity of the normal modes or the rate of change of the normal modes with respect to some design parameters such as wind pressure and temperature. An ordinary computer of 32k can solve a set of 100 linear equations

in core, and a set of 5000 linear equations by means of sequential backing storage. Solving eigenvalue problems required at least three times more storage as solving linear problems and the sequential method is not convenient for eigenvalue problems. The other difficulty is that the total numerical error in these methods consists of two branches: the theorectical error which decreases with increasing number of elements and the rounding off error which is proportional to the cube of the order of the overall matrices, as shown in Fig(8.2.1). To solve the problem of computer storage, the method of substructuring may be used. However, since the method of substructuring is equivalence to linear elimination of redundent equations, the rounding off error remains, and the computing time which is proportional to the cube of the order of the overall matrices can not be reduced.

The method of frequency dependent matrices will not suffer these two difficulties of storage and rounding off error, even when higher modes are required, because the size of the overall matrices needs not to increase in order to obtain high accuracy. Giant member may be used for complicated structures as indicated in section(3.5.3). Finally, if the structure is too complicated, the overall dynamic stiffness matrix may be constructed experimentally for each frequency, and the theorem 3.4.1 may be used to separate it into two positive definite matrices.

Therefore we see that for some particular types of structures, the method of frequency dependent matrices may be suitable and for the other structures, other methods may be more advantageous. In

the following section, we shall discuss some engineering structures which may be analyzed by the present method.



Fig(8.2.1) The numerical errors of finite element methods

8.3 ENGINEERING STRUCTURES

The present method is very suitable to analyze the following types of engineering structures: structures consist of skeletal members, such as space frames; structures with repeated identical members, such as folded plate structures; and some three dimensional structures with one of the overall dimension much larger than the other two, such as long bridges and tall buildings. In case of space frames, a complete computer programme was presented in chapter six. A specific natural mode can be calculated accurately without subdividing the individual members, hence, the data preparation is kept to a minimum. For structures consisting of repeated members, the establishment of the frequency dependent fundamental matrices is needed for a few member types only, and therefore the effort spent in developing these matrices and the calculation of frequency functions are reduced. As far as a"long" structure is concerned, subdividing the whole structure into a number of convenient substructures and computing the normal modes for each substructure will not be difficult. As an example, a tall building may be considered as an assemblage of some core members where the lifts are presented, some supporting columns and shear walls, and some floor levels. All these elements are connected in the floor levels only and these floor levels may be considered as rigid in their own planes, and there are six generalized coordinates only for each floor. Therefore, if we consider the cores and shear walls as openthin wall beams or box beams which are considered in finite element method as assemblages of plate elements, tremendous savings in data preparation, in computer storage, in computer time and in the analyzing process of results.

8.4 PROBABILITY FOR FURTHER DEVELOPMENT

Since the method of frequency dependent matrices in structural vibration analysis is very useful for many common engineering structures, to develop the method in a wider sense is seemed to be worthwise. The first probability is to develop the fundamental matrices for various common structural members such as box beams,torsion beams with open sections, etc., and store them in a library. Since the method of eigenfunction expansion for fundamental matrices may be applied to more general structural members, therefore there is a need to develop a computer programme to construct these matrices from the normal modes of the member. General plate and shell elements are not recommended to develop by the present method because of the complicating boundary conditions.

The solution method for the the resulting overall nonlinear eigenvalue problem for natural vibration analysis if very stable both for natural frequencies and the corresponding modes of any multiplicity. The modal analysis method has been used to analyze the deterministic structural response to arbitrary excitations. For random response analysis, with the fact that the response spectrum is the bilinear transformation of the loading spectrum with respect to the dynamic receptance of the structure, to develop a method by which the dynamic receptance could be constructed by means of eigenfunction expansion can be profitable. Finally, the dynamic receptance may be separated into two positive

definite matrices associated with the strain and kinetic energies. However, there is no need of doing so, since for natural vibration analysis, the method of dynamic stiffness is more suitable because its assembling process from elements to the overall structure is much simpler.

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