



### **A Study Of Euclidean K-Ary Gcd Algorithm**

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#### **ABSTRACT**

The problem of computing the greatest common divisor GCD of natural numbers is one of the most common problems that is solved in modern computational mathematics and its applications. At the moment, many algorithms are known to solve this problem, but every day the requirements for the effectiveness of such algorithms become more stringent. As a result, there is a need to create new algorithms that are more efficient in time and number of the operations performed. Besides, they must allow the possibility of their transformation into modern programming languages while maintaining the efficiency of the algorithm.

This article presents an analysis of the realization features and the results of testing the speed of three GCD computation algorithms: the classical Euclidean algorithm, the Sorenson k-ary algorithm, and the approximating k-ary algorithm developed by the second of the authors in MicrosoftVisualStudio in C#. Qualitative and quantitative data on the effectiveness of these algorithms in terms of time and number of the steps within the main cycle are have been obtained.

The concluding part of the article contains the analysis of the results obtained, their representation in the diagrams, and gives the recommendations on the choice of the parameters of the methods.

**Keywords:**GCD of natural numbers, Euclidian GCD algorithm, k-ary GCD algorithm, approximating k-ary algorithm.



## **1. INTRODUCTION**

As a rule, modern cryptographic systems work with long numbers and provide a procedure for calculating their greatest common divisor GCD (Ishmukhametov S.T. 2011), (Ishmukhametov S., Mubarakov B., Mochalov A. 2015). The most common of these procedures is the classical Euclidean algorithm. However, other algorithms are also known, the overall goal of which was to reduce the complexity of GCD computation, and to reduce the search time.

One of the most promising algorithms is the k-ary algorithm of the GCD search, published in the 1990s by Jonathan Sorenson (Sorenson J. 2004, Sorenson J. 1990, Sorenson J. 1994). In comparison with the classical Euclidean algorithm, this algorithm makes it possible to significantly reduce the complexity of calculations with a careful choice of the k parameter. Sorenson and other scientists subsequently proposed several variants to improve and modify the k-ary algorithm ((Weber K. 1995, Jebelean T. 1993, Wang X., Pan V. 2003)).

One of them is the approximating k-ary algorithm proposed by S. T. Ishmukhametov (Ishmukhametov S.T. 2016).

The aim of this paper is to realize and study the three mentioned algorithms for computing GCD. The next three parts deal with the classical Euclidean algorithm, the original version of k-ary algorithm and its modification - an approximating k-ary algorithm. For each of the algorithms, a theoretical justification is given, an estimate of the complexity is indicated, and a realization of the algorithm using the programming language C # is performed. Next, the results of testing each of the algorithms by runtime using the numbers of different lengths are presented. The final part of the work compares the algorithms according to the efficiency of the time of work and the number of steps within the main cycle.

## **2. METHODS**

The tables below present the results of testing of the classical Euclidean algorithm, the Sorenson k-ary algorithm and the approximating k-ary algorithm. All algorithms were realized using the C # programming language in Microsoft Visual Studio 2010.

In the tables compiled from the results of testing the original and approximating k-ary algorithms the “arity” of the algorithm is reserved vertically, and horizontally - the



length of the numbers in decimal positions. Each cell of the table is divided into two parts. At the top of the cell, the average search time of the GCD algorithm of a given “arity” on one of a thousand pairs of randomly generated numbers of a given length is indicated. By analogy, at the bottom of the cell there is shown the number of iterations of the main loop done by the algorithm of a given “arity” also by one of a thousand pairs of randomly generated numbers of a given length. It should be noted that the classical Euclidean algorithm has no “arity”, so the results of its testing are tabulated, in the upper line of which, horizontally, the length of the numbers in decimal places is postponed, and below, the corresponding results are given for the average working time of the algorithm realization by thousand pairs randomly generated numbers and the average number of steps made within the main loop.

### 3. RESULTS

**Table 1. Classical Euclidean Algorithm**

<i>D</i> (numbers' length in decimal digits)	<i>10</i>	<i>25</i>	<i>50</i>	<i>100</i>	<i>250</i>	<i>500</i>	<i>1000</i>
<i>Working time (μs)</i>	4,5	16	35	97	313	760	2650
<i>Average number of iterations (iter)</i>	19	48	96	194	485	969	1940

**Table 2. Sorenson K-ary Algorithm**

<i>K\D</i>		<i>10</i>	<i>25</i>	<i>50</i>	<i>100</i>	<i>250</i>	<i>500</i>	<i>1000</i>
<i>16</i>	<i>μs</i>	17	70	175	450	1001	5763	20744
	<i>iter</i>	17	70	94	200	483	959	1900
<i>64</i>	<i>μs</i>	12	61	133	367	918	4641	15437
	<i>iter</i>	15	40	80	160	403	806	1615
<i>256</i>	<i>μs</i>	8	48	120	315	1161	3701	13583
	<i>iter</i>	13	32	66	133	331	664	1331
<i>421</i>	<i>μs</i>	15	41	111	257	1018	3243	11367
	<i>iter</i>	13	31	65	129	329	661	1328
<i>853</i>	<i>μs</i>	10	43	93	244	969	3047	10493



	<i>iter</i>	13	30	65	131	327	655	1313
<b>1307</b>	$\mu s$	14	49	103	69	894	2947	10267
	<i>iter</i>	12	28	58	116	291	583	1167
<b>1801</b>	$\mu s$	10	48	101	104	1041	2832	1026
	<i>iter</i>	9	23	47	99	240	481	959

**Table 3.**ApproximatingK-aryAlgorithm

<i>K\D</i>		<b>10</b>	<b>25</b>	<b>50</b>	<b>100</b>	<b>250</b>	<b>500</b>	<b>1000</b>
<b>16</b>	$\mu s$	60	189	479	1152	4257	12946	44579
	<i>iter</i>	8	20	41	82	207	414	829
<b>64</b>	$\mu s$	63	194	475	1108	3824	11306	37393
	<i>iter</i>	6	16	33	67	167	335	671
<b>256</b>	$\mu s$	62	201	517	1046	3566	10290	32783
	<i>iter</i>	5	13	27	55	139	279	557
<b>422</b>	$\mu s$	62	212	552	1201	3871	10237	33000
	<i>iter</i>	5	13	27	55	138	277	551
<b>852</b>	$\mu s$	56	281	591	1175	3767	10165	32460
	<i>iter</i>	5	13	26	53	133	266	534
<b>1506</b>	$\mu s$	61	207	501	1155	3655	9848	30785
	<i>iter</i>	4	12	24	49	123	247	494
<b>3026</b>	$\mu s$	78	243	524	1195	3897	10078	29698
	<i>iter</i>	4	11	22	45	114	230	461

#### 4. DISCUSSION

CLASSICAL EUCLIDIANALGORITHM. The natural numbers  $A$  and  $B$ ,  $A > B$  are transmitted to the input of the algorithm. The classical Euclidean GCD search algorithm is based on the following recurrence equation:

$$GCD(A, B) = GCD(B, A \bmod B).$$

The algorithm is realized successively, called iterations, in the course of each of which the remainder is calculated  $C = A \bmod B$ , a transition to a new pair  $(B, C)$  takes place and the



calculation continues with the new pair. The calculation stops when the second argument of the pair is equal to 0, then the first argument is the sought GCD. The very algorithm can be written in the C # as follows:

```
public BigInteger Method(BigInteger number1, BigInteger number2)
{ if (number1 <= 1 || number2 <= 1)
return 1;
BigInteger R;
if (number2 > number1) { R = number1;
number1 = number2;
number2 = R; }
while (number2 != 0)
{ R = number1 % number2;
number1 = number2;
number2 = R; }
return number1;
}
```

Estimating the complexity of this algorithm, we note that for every two iterations of the cycle, the dividend decreases at least twice. This means that the number of iterations is  $O(n)$ , where  $n$  is the length of the input numbers. The worst estimate is achieved by the neighbor numbers of the Fibonacci series, and the average number of iterations is estimated by the value (Dixon (DixonJ. 1970)):

$$\text{Number of iteration} \sim \frac{12}{\pi^2} \log_2 B$$

The operation of calculating the remainder of  $A \bmod B$  can be performed within the time  $O(n \log_2 n)$ , so that the overall performance of the algorithm is estimated by  $O(n^2 \log_2 n)$ .

**K-ARY ALGORITHM OF SORENSON.** First, we consider a binary algorithm for calculating GCD for a pair of natural numbers. The input data of this algorithm are odd natural numbers. We designate them by  $u, v$ . The binary algorithm can be represented as follows:



```

while(u!= 0) and (v!=0) do:
if (u is even) u:=u/2
else    if (v is even) v:=v/2
else t:= abs(u - v)/2;
if (u > v) then u:=t else v:=t;
if (u == 0) then t:=v else t:=u;
output(t);

```

Generalizing this algorithm, we will obtain the k-ary GCD algorithm. The main idea of the iteration of the k-ary algorithm is to search for a given pair of numbers  $(A, B)$  of the integers  $x, y$  for a small fixed  $k$ :

$$0 < x < \sqrt{k} \quad -\sqrt{k} < y < \sqrt{k},$$

such that the identity

$$ux + vy \equiv 0 \pmod{k}.$$

is performed

Then, at the step of the algorithm, we can pass from the pair  $(u, v)$  to the new pair  $(v, w)$ , where  $w = (ux + vy)/k$  (it is also necessary to reduce  $w$  by the divisor  $k$ ).

Sorenson in the article (3) proposed to perform k-ary algorithm in four stages: the stage of precomputations, the first “pilot” division, the main part, the second “pilot” division.

**The main part** of the algorithm is performed in the following way :

```

while(v!= 0){
if (gcd(u,k) > 1) u=u/gcd(u,k);
else (if gcd(v,k) > 1) v=v/ gcd(v,k);
else
find integers x, y,  $ux + vy \equiv 0 \pmod{k}$ ;
 $w = |ux + vy|/k$ ;
if ( v > w){u=v ;v=w ;} else u= w;
}
return u ;

```



When trying to realize the k-ary algorithm, it becomes clear that the way to choose integers  $x$  and  $y$  in many ways affects its performance. According to the Sorenson theorem, the coefficients  $x$  and  $y$  can be chosen  $0 < |x| + |y| \leq 2\sqrt{k}$ .

For the maximum performance of the algorithm, it is also advantageous to choose the number  $x$  being small and positive, the number  $y$  being sufficiently large negative.

**The stage of preliminary calculations:** if the base of algorithm  $k$  is already chosen, then for all natural numbers less than  $k$ , the algorithm involves calculating their greatest commonness with the number  $k$  of the divisor and searching for the inverse element by the modul  $k$  (the corresponding tables will be designated with the letters  $G, I$ ).

In addition, one compiles a special table for rapid calculation of the numbers  $x, y$  and a table for holding some small common divisors of the numbers  $u$  and  $v$ , which are cut off at the pilot division stage. All these calculations do not depend on the numbers  $u$  and  $v$  transferred to the algorithm, and therefore can be performed in advance and only once.

**The first "trial" division:** as it was mentioned earlier, at this stage all common dividers of the given numbers  $u$  and  $v$  are firstly cut off. Later they are stored in a separate table, up to the stage of the second test division. Also at this stage, those numbers being the dividers of one of the numbers  $u$  and  $v$  and not the divisors of the other are cut off. Such divisors do not participate in the formation of GCD, so one does not need to store them. Then one executes the main cycle during which the number  $M$  is calculated. This number is not necessarily equal to the desired GCD, but is equal to their multiple, that is, it can contain extraneous factors. To exclude them, the second test division is performed.

**The second "trial" division:** this stage of the algorithm is the final one and uses the result obtained after the realization of the main part of the algorithm. To get rid of extraneous factors, divide  $M$  by all small divisors in a pre-compiled table, and then  $M$  is multiplied by the result of the first test division. As a result, we obtain GCD of the given numbers  $u$  and  $v$ .

The most significant part of the algorithm is in the main loop, which represents the main stage of the algorithm. In C #, it can be written as follows:

```
private void MainLoop()
{
```



```
int u1, v1;
int a, b, x;
BigInteger t;
while ((u != 0) && (v != 0))
{ u1 = (int)(u % k); v1 = (int)(v % k);
if (G(u1) > 1) u /= G(u1);
else
{ if (G(v1) > 1) v /= G(v1);
else
{ x = (u1 * I(v1)) % k;
if (x < 0) x += k;
a = A(x);
b = ((-1) * a * x) % k;
t = BigInteger.Abs(a * u + b * v) / k;
if (u > v) u = t;
else v = t; } }
}
```

Variables G, I, A denote the arrays the dimension of which is equal to the base (“arity”) of the algorithm “k”. These arrays realize the tables of preliminary calculations.

Let us add the realization of the stages of the first and second trial divisions and obtain the function that realizes the k-ary algorithm:

**// First trial division**

```
Private void TrialDivision1()
{
    BigInteger g = 1;
    for (int i = 1; i < P.Length; i++)
        while (u % P(i) == 0 && v % P(i) == 0)
            { u /= P(i); v /= P(i); g *= P(i); }
}
```

**// Second trial division**

```
Private void TrialDivision2()
```

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```
{
BigInteger t;
if (v == 0) t = u;
else t = v;
for (inti = 0; i<P.Length; i++)
while (t % P(i) == 0)
t /= P(i);
g = t * g;
}
// k-ary algorithm
PublicBigInteger Method(BigInteger number1, BigInteger number2)
{
if (number1 <= 1 || number2 <= 1) return 1;
TrialDivision1();
MainLoop();
TrialDivision2();
return GetGCD();
}
```

The complexity of the k-ary algorithm for the numbers  $u$  and  $v$  with length  $n$  bits in the optimal case of the choice of the parameter  $k$  is equal to Sorenson's estimate  $O(n^2/\log k)$ .

APPROXIMATING K-ARY ALGORITHM. Let there be given natural numbers  $A > B > 0$ . For simplicity, suppose that  $A, B$  are odd. We choose a parameter  $k$  to be equal to the power of two  $k = 2^s$ . It is assumed that  $A, B$  are sufficiently long, therefore, to simplify further calculations, we find

$$a = A \bmod k, \quad b = B \bmod k.$$

Now, by analogy with Sorenson's k-ary algorithm, it is required to find such numbers  $x$  and  $y$  that

$$Ax + By \equiv 0 \pmod{k},$$



.and then calculate  $C = (Ax + By)/k$ , replace  $(A, B)$  with the new pair  $(B, C)$  and pass on to the next iteration.

The approximating k-ary algorithm offers a new way of calculating the numbers  $x, y$ , using the approximate value of  $r$ , calculated by the formula

$$r = \left[ \frac{r' - q}{k} \right],$$

where  $r' = A/B, q = AB^{-1} \bmod k, [s]$  – integer parts.

First, an approximate value  $r' = A/B$  is calculated to  $1/k$ . Then the accuracy of computing  $r$  is  $1/k^2$ . Then approximate  $r$  accurate within  $3/2k^2$  by the fraction of Farey ((Hardy G. H., Wright E. M. 1959), chapter 2). The Farey fraction  $m/n$  is a proper fraction with a numerator and denominator smaller than  $k$ . Then, the difference between  $m/n$  and  $r$  does not exceed  $5/2k^2$ .

The most significant part of the algorithm is the calculation of the approximating fraction  $m/n$ . The required values of the coefficients  $x, y$  are chosen by  $m$  and  $n$  as follows:  $x = n, y$  is chosen so as to minimize the sum  $Ax + By$ . Obviously,  $y$  must be a negative number, then the terms  $Ax$  and  $By$  have different signs and mutually cancelled. The peculiarity of this algorithm is that it operates not only with the very numbers, but also with the length of the binary representation of each of the numbers. To apply the approximation to the above relation, it is necessary that the numbers  $A, B$  be sufficiently approximate in length. Otherwise, the usual iteration of the classical Euclidean algorithm is performed, which, in the absence of approximation, speeds up the algorithm (the same substitution also accelerates the Sorenson algorithm).

Let the input data of the algorithm be odd numbers  $A, B$ , the basis of the algorithm  $k = 2^s$ , and the lengths of the numbers  $A, B, k$  in the binary representation are  $L1, L2$  and  $L$ , respectively. Then the realization of one step of the approximating k-ary algorithm can have the following form:

**PublicBigInteger Step(BigInteger A, BigInteger B)**

```
{  
// Item 1  
int d = L1 - L2;
```



```
BigInteger q;  
if (d >= 2 * L){  
q = BigInteger.Remainder(A * ReverseElement(B,  
BigInteger.Pow(2, d)), BigInteger.Pow(2, d));  
C = (A - q * B) / BigInteger.Pow(2, d);  
if (C == 0) C = B;  
while (C % 2 == 0) C /= 2;  
return BigInteger.Abs(C); }  
  
// Item 2  
int a = (int)(A % k);  
int b = (int)(B % k);  
q = (int)((a * ReverseElement(b, k)) % k);  
  
// Item 3  
int t = L2 - L;  
if (t <= 0){ B = BigInteger.GreatestCommonDivisor(A, B); return B; }  
BigInteger A1 = BigInteger.Divide(A, BigInteger.Pow(2, t));  
BigInteger B1 = BigInteger.Divide(B, BigInteger.Pow(2, t));  
  
// Item 4  
BigInteger n0 = B1 * k;  
BigInteger m1 = A1 - ((int)q) * B1;  
if (m1 < 0) { q -= k; m1 = (int)(A1 - ((int)q) * B1); }  
  
// Item 5  
BigInteger m0 = m1 % n0;  
BigInteger s1 = (m1 - m0) / n0;  
  
// Item 6  
BigInteger x = 1;  
BigInteger y = -(((int)q) + s1 * k);  
BigInteger m = 1, n = 1;  
bool flag = false;  
if (m0 * k >= n0){ FareyApproximation(m0, n0, out m, out n); flag = true; }  
  
// Item 7  
if (flag){ x = n; BigInteger s = -(s1 * x + m); y = s * k - ((int)q) * x; }
```



```
BigInteger A2 = (A - a) / k;  
BigInteger B2 = (B - b) / k;  
BigInteger c = (a * x + b * y) / k;  
C = A2 * x + B2 * y + c;  
// Item 8  
if (C == 0) C = B;  
else { while (C % 2 == 0) C = C / 2; }  
return BigInteger.Abs(C); }
```

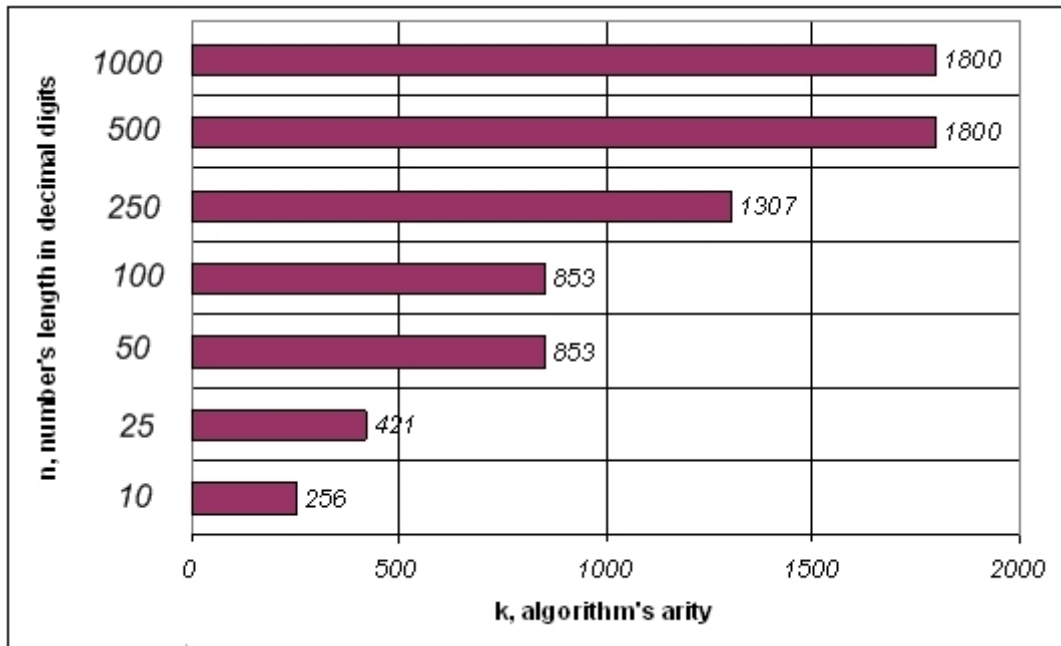
According to the theorem on the asymptotic complexity of the approximating k-ary algorithm, the number of iterations in each stage is estimated by  $O(n/\log_2 k)$ , where  $n$  is the length of the original numbers in bits. The complexity of the whole algorithm is estimated by

$$O(n^2/L + nL),$$

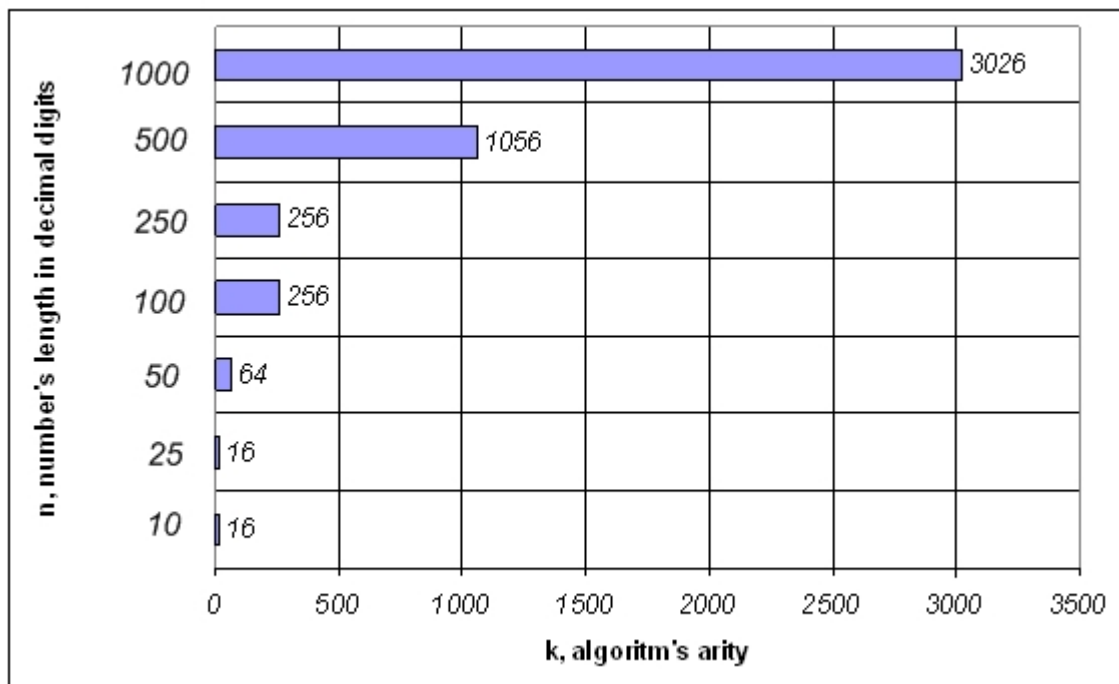
where  $L$  – binary length  $k$ .

## 5. SUMMARY

Analyzing the results of the original and approximating K-ary algorithms, one can note that the optimal value of their “arity” of  $k$ , i.e. the value at which the algorithm is realized by the numbers of a given length in the shortest time varies with the length of the given numbers. This observation is more clearly illustrated in the diagrams below: Diagram 1. The K-ary Sorensen Algorithm



**Diagram 2.**Approximating K-ary algorithm



When investigating the dependence of the number of steps performed by the algorithm in the search for GCD, on the length of the given numbers, we can conclude that when searching for the GCD of two given numbers, the number of steps made by the approximating k-ary algorithm is the smallest. Following this indicator is k-ary



algorithm by Sorenson, and in this case, the “arities” of the algorithms are considered to be equal. The classical Euclidean algorithm takes the most number of steps for the GCD search. The difference in this parameter is noticeable even with a small base of the k (“arity”). Let’s visualize this fact in the diagrams, choosing “arity” of the original and approximating algorithms equal to 64. We can note one more fact connected with the change in the algorithm “arity” in calculating GCD numbers of the given length. With increasing “arity” of the original or approximating algorithm, the GCD search is performed in fewer steps.

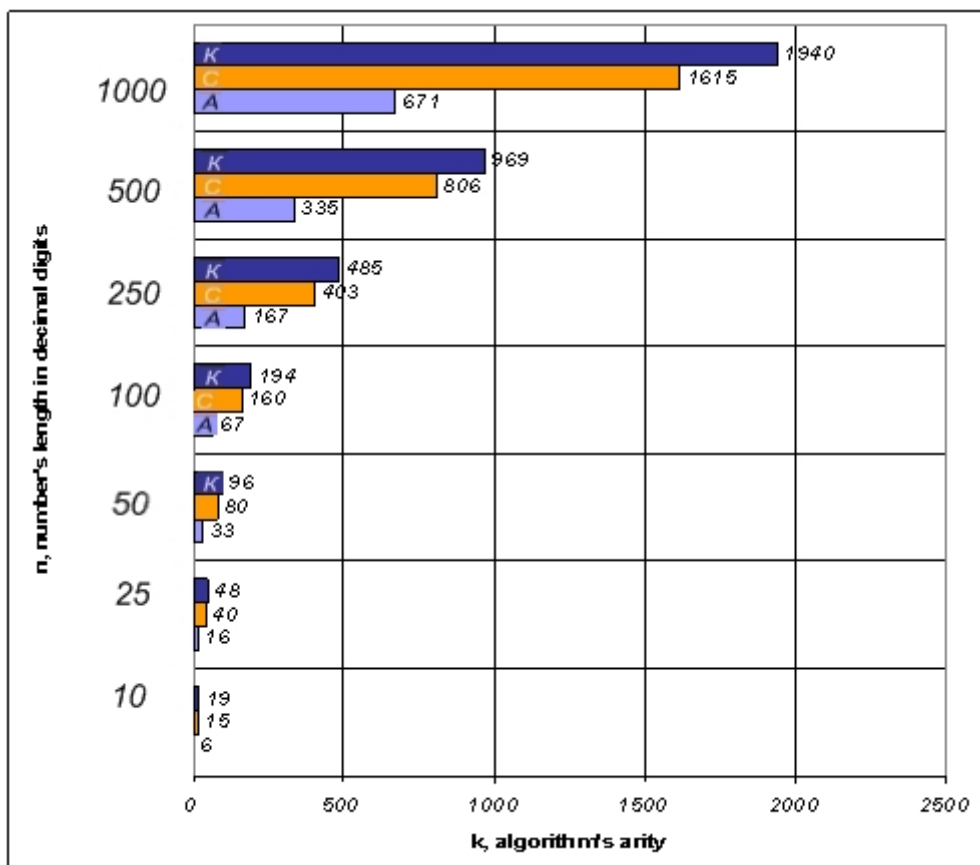
Below, the following is designated in diagrams 3,4,5:

K – the classical Euclidean algorithm,

C – Sorensen’s k-ary algorithm (in brackets “arity”)

A – Approximating k-ary algorithm (in brackets “arity”)

Diagram 3. Number of steps within the main cycle



**Diagram 4.** 64-ary and 408-ary algorithms of Sorenson

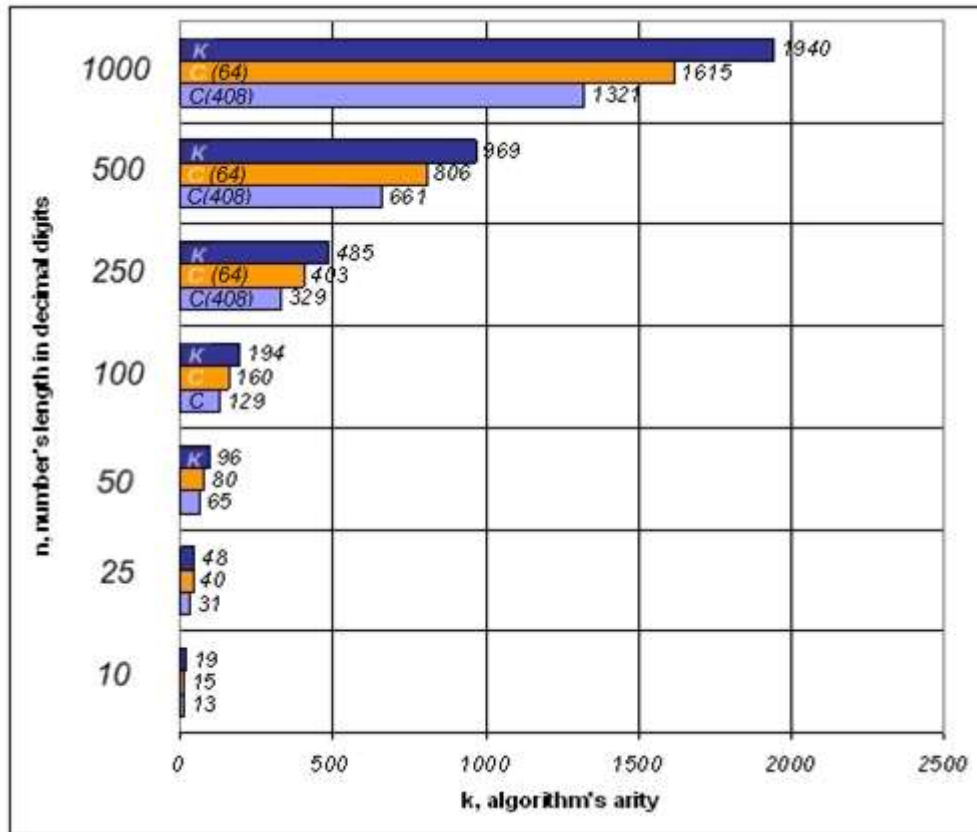
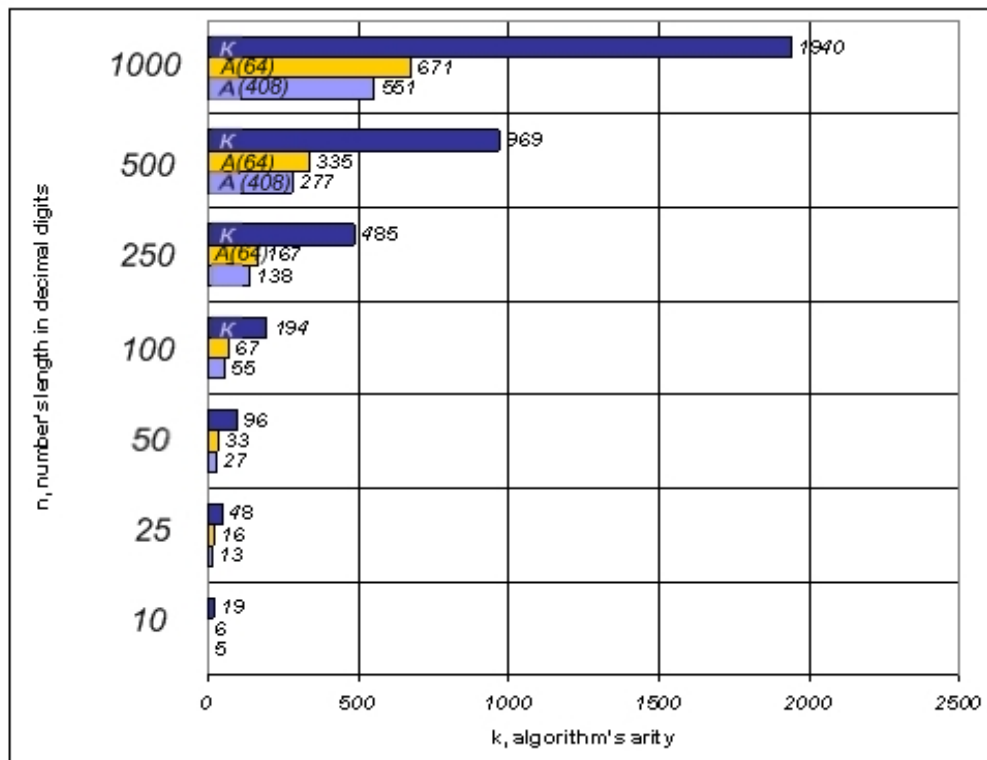


Diagram 5. 64-ary and 408-ary approximating algorithms





However, this circumstance is not a reason for always taking the maximum possible basis to calculate GCD by means of the k-ary algorithm. From a direct analysis of the tables based on the results of k-ary algorithms, it follows that an increase in the basis of the algorithm, without detriment to its effectiveness on time, can occur only up to some boundary (Diagrams 6 and 7)

Diagram 6.

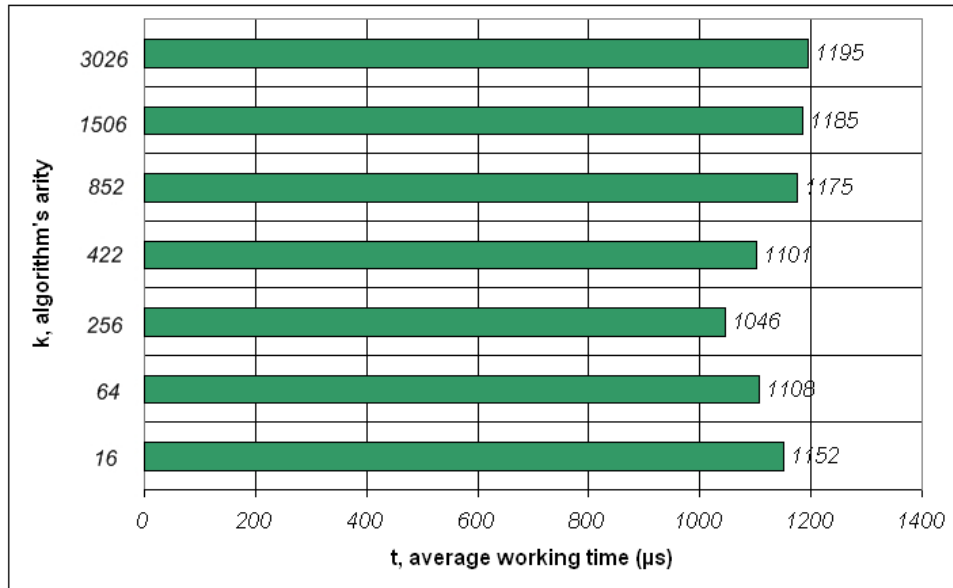
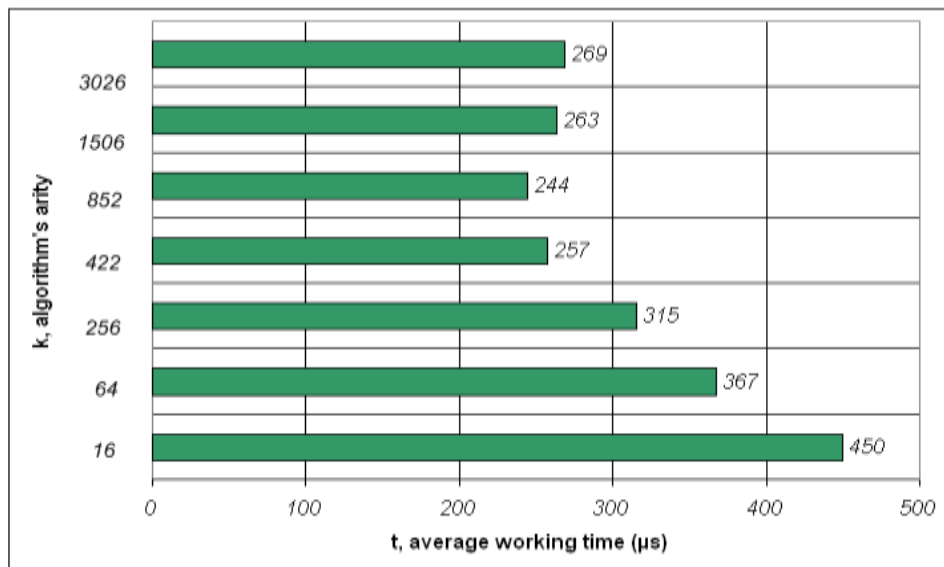


Diagram 7.



## 6. CONCLUSIONS

Taking into account the peculiarities of the original and approximating k-ary algorithms, as well as the classical Euclidean algorithm, it is possible to propose using





them sequentially, at that, first, several iterations of the approximating k-ary algorithm are applied, and the result is input to the k-ary algorithm of Sorenson or the classical Euclidean algorithm. After applying the approximating algorithm in the first stage, the length of the pair of obtained numbers with the equivalent GCD is already substantially smaller than the length of the original numbers. After a few iterations, a pair of relatively small length numbers will be obtained, and now the GCD calculation can be completed with the help of two other algorithms that work much faster than the approximating algorithm *начислах* with small numbers.

## **7. ACKNOWLEDGEMENTS**

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