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Notes on f(R) Cosmological Models with Yang-Mills and Scalar Fields

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Abstract

The present study deals with the Friedmann-Robertson-Walker cosmological models of the Yang-Mills fields and $f(R)$ generalization of the Einstein cosmology. We investigate the corresponding relation between $f(R)$ gravity and interacting Yang-Mills and scalar fields. A method of conformal transformation is proposed, which allows to transform a cosmological theory with nonlinearity induced by the Yang-Mills field into $f(R)$ generalization of Einstein's theory. A specific generalization of Einstein theory on the basis of exact solution for the Yang-Mills equation obtained by the author early is presented in explicit form. Moreover, for the $f(R)$ theory we have found the explicit form of potential of induced scalar field, which can be identified with the effective potential of the $f(R)$ cosmology.

***Keywords*:** Yang-Mills and scalar fields, Conformal transformation, $f(R)$ cosmology.

1. Introduction

It is well known that numerous attempts of a theoretical explanation of cosmological inflation and accelerated expansion of the Universe [1, 2] are currently associated with a modification of either the right (material) part of the gravity equation, or the left (geometric) part of this equation. Several attempts have been made to justified the current accelerated expansion of the universe by means of simultaneous modification of both sides of the gravity equation.

The standard model of cosmology is based on the Einstein's theory of relativity, or General Relativity (GR). The modified gravity theories attract a great attention because of the failure of GR to explain the cosmic acceleration and other problems in the early universe evolution. In this context, the most accepted hypothesis is that dark energy, a fluid which does not interact with photons, with a negative pressure, is responsible for this accelerated expansion of the universe. But some papers suggest that using a modification of GR, one can create a cosmic history with the help of dark energy. On the other hand, the recent detection of gravitational waves puts GR on a level that generates various modifications of this theory. Some of the modified gravity theories currently having a wide attention are different scalar-tensor theories of gravity and $f(R)$ gravity in which $f(R)$ is an arbitrary function of the scalar curvature R [3-6].

However, one can note that there is an interesting feature of such modifications. Indeed, in the modified Einstein theory of gravity, in its simplest version, the action $\int R\sqrt{-g}d^4x$ is replaced by a generalized action $\int f(R)\sqrt{-g}d^4x$ [7], where $f''(R) \neq 0$, and R is a Ricci curvature scalar. At the same time, as it noted in [7], a $f(R)$ theory through conformal transformation of gravity

$$\bar{g}_{\mu\nu} = \Psi g_{\mu\nu}, \quad \text{where} \quad \Psi = f'(R), \tag{1}$$

can be transformed into the usual theory of gravity with quintessence scalar field ϕ :

$$f(R)\sqrt{-g} = \sqrt{-\bar{g}}[\bar{R} + \frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)] \tag{2}$$

where

$$\phi = \sqrt{\frac{6}{\chi}}\ln\Psi, \qquad V(\phi) = \frac{\Psi - f(R(\Psi))}{2[\Psi(\phi)]^2}, \tag{3}$$

and χ is the Einstein's gravitational constant. Note that if the Yang-Mills field Lagrangian

$$L_{YM} = \frac{1}{16\pi}\sqrt{-g}F^a_{\mu\nu}F^{a\mu\nu}, \tag{4}$$

is added to the gravity Lagrangian $L_G = R\sqrt{-g}$, then, due to its conformal invariance which follows from the masslessness of the field, the same Lagrangian can be added to the right-hand side of (2) in the conformal metric (1). On the other hand, if we consider the theory with Yang-Mills fields with induced non-linearity of the scalar field (as it was done in [8]), that is

$$L = \sqrt{-\bar{g}}[\frac{\bar{R}}{2\chi} + \frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \Psi(\phi)F^a_{\mu\nu}F^{a\mu\nu}] \tag{5}$$

then using the conformal transformation (1) with the proper choice of the interaction function $\Psi(\phi)$, i.e. in the case of equality $\Psi(\phi)I_{YM} = V(\phi)$, where $I_{YM} = F^a_{\mu\nu}F^{a\mu\nu}$ is the Yang-Mills field invariant, it is possible to bring the General Relativity theory with a non-Abelian field into $f(R)$ - theory. In this case, the Yang-Mills field is "absorbed" by the nonlinear $f(R)$ Lagrangian. Conversely, if we proceed from $f(R)$ theory, then we can define a class of cosmological models with Yang-Mills fields defined by the Lagrangian of the form (4), provided

$$V(\phi) = \Psi(\phi) I_{YM} . \tag{6}$$

In this paper, we, first of all, realize the last possibility with the example of one of the solutions of the Yang-Mills equations obtained in [8, 9], and, secondly, we consider the model [4] with the Yang-Mills fields, in the framework of the direct conformal transformation (1).

2. Exponential Lagrangian for the gravitational field

For the Lagrangian (4) where $F^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + e\varepsilon_{abc} W^b_\mu W^c_{nu}$ is the stress tensor of the Yang-Mills fields W^a_μ , one can easily get the following energy-momentum tensor of of Yang-Mills fields by varying this Lagrangian with respect to metric:

$$T^\nu_\mu = -\frac{1}{4\pi} F^a_{\mu\alpha} F^{a\nu\alpha} + \frac{1}{16\pi} \delta^\nu_\mu F^a_{\alpha\beta} F^{a\alpha\beta} . \tag{7}$$

The exact solutions for the Einstein equation,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \chi T_{ik} , \tag{8}$$

the Yang-Mills equation

$$D_\nu \left(\sqrt{-g} F^{a\nu\mu} \Psi(\varphi) \right) = 0 \tag{9}$$

Here the symbol D_ν means a covariant derivative, and the scalar field equations

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} \left(\sqrt{-g} g^{\nu\mu} \frac{\partial \varphi}{\partial x^\mu} \right) + \frac{1}{16\pi} F^a_{\alpha\beta} F^{a\alpha\beta} \Psi_\varphi = 0 \tag{10}$$

are obtained in the metric of a homogeneous isotropic model of the universe with the Friedmann-Robertson-Walker line element

$$ds^2 = dt^2 - a^2(t)[dr^2 + \xi^2(r) d\Omega^2], \tag{11}$$

where $a(t)$ is a scale factor of the Universe, and $\xi(r) = \sinh(r),\ r,\ \sin(r)$ for the curvature sign $k = -1,\ 0,\ +1$, respectively. With the help of the Wu-Yang *ansatz*

$$W^a_i = (er)^{-2} \varepsilon_{iab} x^b [K - 1] + (er)^{-1} [\delta^a_i - r^{-2} x^a x_i] S,$$
$$W^a_0 = (er)^{-1} x^a W, \tag{12}$$

of the equation (8) are found in the following form

$$W = \frac{d\alpha}{dt},\ K = P(r)\cos\alpha,\ S = P(r)\sin\alpha, \tag{13}$$

where $\alpha(t)$ is an arbitrary differentiable function, and $P(r)=\xi'(r)$. For these solutions, $I_{YM}=e^{-2}a^{-4}(t)$, where the scale factor $a(t)$ is a solution to the Einstein equation (8). Let us consider the solution with $k=-1$ and $a(t)=a_0 t$ [9].

$$\phi=\pm\frac{1}{\lambda}\ln t,\quad \Psi=\Psi_0\exp(\pm 2\lambda\phi) \tag{14}$$

where $\lambda=\sqrt{\frac{\chi a_0^2}{6(1-a_0^2)}}$ and $\Psi_0=\chi^{-1}16\pi e^2 a_0^2(a_0^2-1)$. Comparing potential term in Lagrangian (2) for this solution with $V(\phi)$ from (3), we can obtain

$$\begin{aligned}&\frac{\Psi-f(R(\Psi))}{2[\Psi(\phi)]^2}=V_0 e^{\mp 2\lambda\phi},\\&\phi=\pm\sqrt{\frac{6}{\chi}}\ln[f'(R)],\end{aligned} \tag{15}$$

where $V_0=a_0^{-2}(1-a_0^2)$. Thus, from (15) we obtain the equations for the function $f(R)$ as follows

$$f(R)-f'(R)=2V_0[f'(R)]^{\mp 2\beta}, \tag{16}$$

where $\beta=a_0^2/\sqrt{1-a_0^2}$. The general solution of equation (16) in the case $\mp 2\beta\neq 1$ can be easily obtained in parametric form as follows.

$$f(R)=p-2V_0 p^{\mp 2\beta},\quad R=Cp\exp\left(\mp\frac{4V_0\beta}{1\pm 2\beta}p\right), \tag{17}$$

where p is a real parameter. In the case $\mp 2\beta=1$, the solution of equation (16) can be obtained explicitly as

$$f(R)=Ce^{\gamma R}, \tag{18}$$

where $\gamma=a_0^2(3a_0^2-2)^{-1}$, $V_0=\beta^{-1}$, and C is a constant of integration. So, we obtain the most natural form of the generalized gravitational Lagrangian considered earlier. As it proposed in Refs. [11, 12], while the Einstein-Hilbert (EH) action is a linear term in R, expanding $f(R)$ in a series, we would have, in principle, to fix an infinite number of coefficients. If we accept the point of view that only one characteristic length fixes the dynamics, then we can fix the explicit function $f(R)$ in equation (18) by means of two constants, C and γ. Comparing the first two terms in this expansion which is valid in the region $\gamma R\langle\langle 1$ with the EH action included a cosmological term Λ, i. e. $L_{EH}=R+2\lambda$, one can identify $C=2\Lambda$ and $\gamma=1/(2\Lambda)$ [12].

3. An example of $R+\beta R^2$ cosmology

For the Lagrangian of the form

$$L_G = \sqrt{-g}\,(R+\beta R^2), \tag{19}$$

gravitational field equations have the form [12]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \beta B_{\mu\nu} = \chi T_{\mu\nu}, \tag{20}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor, $T_{\mu\nu}$ is the energy momentum tensor of matter, and the tensor $B_{\mu\nu}$ is as follows

$$B_{\mu\nu} = 2RG_{\mu\nu} + 2(R_{,\mu\nu} - g_{\mu\nu}\Delta R), \tag{21}$$

where $R_{,\mu\nu} = \partial^2 R / \partial x^\mu \partial x^\nu$ and $\Delta R = g^{\mu\nu} R_{,\mu\nu}$. Then from equations (1) and (20) we get

$$\Psi = 1 + 2\beta R, \quad \bar{g}_{\mu\nu} = (1+2\beta R) g_{\mu\nu}. \tag{22}$$

Using the latter and equation (3), one can obtain

$$\phi = \sqrt{\frac{6}{\chi}} \ln(1+2\beta R), \tag{23}$$

$$V(\phi) = \frac{1+(2\beta-1)R - \beta^2 R^2}{2(1+2\beta R)^2}. \tag{24}$$

Eliminating R from the last two equations, we obtain the explicit expression for the potential as follows

$$V(\phi) = \frac{1}{8\beta}\left(e^{-2\sqrt{\frac{\chi}{6}}\phi} + 4\beta e^{-\sqrt{\frac{\chi}{6}}\phi} - 1 \right), \tag{25}$$

which represents the early dark energy potential considered in [Barreiro]. It was emphasized that a remarkable property of exponential potentials is that they lead to large-scale solutions which can either mimic a background fluid or dominate the dynamics of the background depending on the slope of the potential.

If we assume that the metric $\bar{g}_{\mu\nu}$ describes the linear element (11), then we can add the Lagrangian L_{YM} of the Yang-Mills field to the Lagrangian (20), as it was done in (5), and the conformal invariance of L_{YM} means that it can be added to both sides of equation (2). Moreover, solutions of the Yang-Mills equations (9) will be the same in conformal metrics $\bar{g}_{\mu\nu}$ and $g_{\mu\nu}$, and will give the same value of the field invariant I_{YM}.

4. Conclusions

Thus, in the present work, a method of conformal transformation is proposed for the $f(R)$ cosmological models with the Yang-Mills fields, which allows transforming a cosmological theory with nonlinearity induced by the Yang-Mills field into $f(R)$ generalization of Einstein's theory. Based on the well-known solution for such a model [12], an exponential generalization is obtained in explicit form (18). For the ($R+\beta R^2$) –theory, we have found the explicit form of potential (25), which can be identified with the effective potential of the theory with the Lagrangian (5), from which the interaction function can be found according to the formula (6). We are going to present a more detailed investigation of the problem in our subsequent publications.

References

[1] A. G. Riess, et al., "Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant", The Astronomical Journal, **116**, 1009-1038 (1998)

[2] S. Perlmutter, et al., "Measurements of Ω and Λ from 42 High-Redshift Supernovae", The Astrophysical Journal, **517**, 565-586 (1999)

[3] T. P. Sotiriou and V. Faraoni, " $f(R)$ Theories Of Gravity", Rev. Mod. Phys., **82**, 451-497 (2010)

[4] A. De Felice and S. Tsujikawa, " $f(R)$ theories", Living Rev. Rel., **13**, 3, (2010)

[5] S. Nojiri and S.D. Odintsov, "Unified cosmic history in modified gravity: from $F(R)$ theory to Lorentz non-invariant models", Phys. Rept., **505**, 59-144 (2011)

[6] S. Nojiri, S. D. Odintsov and V. K. Oikonomou, "Modified Gravity Theories on a Nutshell: Inflation, Bounce and Late-time Evolution", Phys. Rept., **692**, 1-104 (2017)

[7] R. Durrer, R. Maartens, "Dark Energy and Dark Gravity: Theory Overview", Gen. Relativ. Gravit., **40**, 301-328 (2008)

[8] V. K. Shchigolev, "Cosmological model of interacting phantom and Yang–Mills fields", Pramana – J. Phys., **78**, No. 5, 819–825 (2012)

[9] V. K. Shchigolev, K. Samaroo, "Generalized Exact Cosmologies with Interacting Yang-Mills and Nonlinear Scalar Fields", Gen. Relativ. Gravit., **36**, No 7, 1661 (2004)

[10] O. M. Lecian, G. Montani, C. L. Bianco, & S.-S. Xue, "Scalar-tensor analysis of an exponential Lagrangian for the Gravitational Field", AIPConf.Proc.966:264-271, (2008)

[11] O. M. Lecian, G. Montani, "Exponential Lagrangian for the Gravitational Field and the Problem of Vacuum Energy", International Journal of Modern Physics A, 23(08), 1286–1289, (2008)

[12] A. I. Arbab, "Cosmological Models in Generalized Einstein Action", Abraham Zelmanov Journal, **1**, 3 (2009)

[13] T. Barreiro, E. Copeland, and N. J. Nunes, "Quintessence arising from exponential potentials", Phys. Rev. D, **61**, 127301 (2000)